System of Spheres-based Multiple Contractions

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Abstract We propose a new class of multiple contraction operations — the *system of spheres-based multiple contractions* — which are a generalization of Grove's system of spheres-based (singleton) contractions to the case of contractions by (possibly non-singleton) sets of sentences. Furthermore, we show that this new class of functions is a subclass of the class of the partial meet multiple contractions.

1 Introduction

The logic of theory change became a major subject in philosophical logic and artificial intelligence in the middle of the 1980's. The most important model of belief change is known as the AGM model and has been proposed by Alchourrón et al. [2]. The AGM model is a formal framework to characterize the dynamics and state of belief of a rational agent. In the AGM framework,

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the beliefs are represented by belief sets (i.e. deductively closed sets of sentences). A change consists in adding or removing a specific sentence from a belief set to obtain a new belief set. The AGM model has been characterized in at least five different equivalent ways: postulates, partial meet functions, epistemic entrenchment, safe/kernel contraction and sphere-systems.

AGM has acquired the status of a standard model, which inspired many researchers to propose extensions and generalizations. One of those generalizations is *multiple belief change* which consists of the study of change operations for which the input is a (possibly non-singleton) set of sentences instead of a single sentence. In particular, in what concerns operations dealing with the removal of sets with more than one element from a theory we can find three different models in the literature: *package contraction* [4–6, 8] (the output of the contraction does not contain any of the elements of the input set), *choice contraction* [6] (at least one of the elements of the input set is not contained in the output of the contraction), and *set contraction* [12, 13] (the output of the contraction is consistent with the input set). In the present paper we will only consider package contraction and we will use the expression *multiple contraction* to refer to an operation of that kind. More precisely, we will only consider *multiple contractions* by finite sets. Hence, throughout this paper whenever we refer to the multiple contraction of a belief set **K** by a set of sentences B, such set B is assumed to be finite. We will use the expression *singleton contraction* to designate contractions by a single sentence.

The first constructive model of multiple contractions to be presented has been the *partial meet multiple contraction* [6, 8]. Later in [3] kernel contraction has been generalized to the case of multiple contraction and in [10] a semantics for partial meet multiple contraction in terms of possible worlds has been presented.

In this work we aim to propose a way of defining *system of spheres-based multiple contractions*, assuring that such operations: (1) generalize Grove's system of spheres-based (singleton) contractions to the multiple contraction level, and (2) are partial meet multiple contractions.

2 Background

2.1 Formal Preliminaries

We will assume a language $\mathcal{L}$ that is closed under truth-functional operations. We shall make use of a consequence operation Cn that takes sets of sentences to sets of sentences and which satisfies the standard Tarskian properties, namely *inclusion, monotony* and *iteration*. Furthermore we will assume that Cn satisfies *supraclassicality, compactness* and *deduction*. We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. The letters $\alpha, \alpha_i, \beta, \ldots$ (except for γ) will be used to denote sentences. $\top$ stands for an arbitrary tautology and $\bot$ for an arbitrary contradiction. $A, A_i, B, \ldots$ shall denote subsets of sentences of $\mathcal{L}$.

K is reserved to represent a set of sentences that is closed under logical consequence (i.e. $\mathbf{K} = Cn(\mathbf{K})$) — such a set is called a *belief set* or *theory*. The conjunction of all elements of a finite and non-empty set A shall be denoted by $\&A$. We shall denote the set of all maximal consistent subsets of $\mathcal{L}$ by $\mathcal{M}_{\mathcal{L}}$. We will use the expression *possible world* (or just *world*) to designate an element of $\mathcal{M}_{\mathcal{L}}$. $\mathcal{M}, \mathcal{N}_i, \mathcal{W}, \ldots$, (except for $\mathcal{L}$ and $\mathcal{P}$), shall be used to denote subsets of $\mathcal{M}_{\mathcal{L}}$. Such sets are called *propositions*. Given a set of sentences R, the set consisting of all the possible worlds that contain R is denoted by $\|R\|$. The elements of $\|R\|$ are the R-worlds. $\|\varphi\|$ is an abbreviation of $\|\{\varphi\}\|$ and the elements of $\|\varphi\|$ are the φ-worlds. To any set of possible worlds $\mathcal{V}$ we associate a belief set $Th(\mathcal{V})$ given by $Th(\mathcal{V}) = \bigcap \mathcal{V}$ — under the assumption that $\bigcap \emptyset = \mathcal{L}$. $\mathbb{M}, \mathbb{N}_i, \mathbb{W}, \ldots$ shall be used to denote subsets of $\mathcal{P}(\mathcal{M}_{\mathcal{L}})$.

2.2 Partial Meet Contractions

We start by recalling the basic concepts necessary for the definition of the partial meet contractions introduced in [2].

Given a belief set **K** and a set of sentences B, the *remainder set* of **K** by B is the set of maximal subsets of **K** that do not imply any element of B and is denoted by $\mathbf{K} \perp B$. The elements of $\mathbf{K} \perp B$ are the *remainders* (of **K**, by B) [1, p. 128]. It is also convenient to notice here that, according to [1, Observation 2.2], since we are assuming that the consequence operation Cn is compact, it holds that $\mathbf{K} \perp B \neq \emptyset$ if and only if $B \cap Cn(\emptyset) = \emptyset$.

For any sentence α, $\mathbf{K} \perp \alpha$ is an abbreviation of $\mathbf{K} \perp \{\alpha\}$ and is called the *remainder set* of **K** by α.

Definition 2.1 [2] Let **K** be a belief set. A *selection function* for **K** is a function γ such that for all sentences α: if $\mathbf{K} \perp \alpha \neq \emptyset$ then $\emptyset \neq \gamma(\mathbf{K} \perp \alpha) \subseteq \mathbf{K} \perp \alpha$, and if $\mathbf{K} \perp \alpha = \emptyset$, then $\gamma(\mathbf{K} \perp \alpha) = \{\mathbf{K}\}$.

An operation $-$ is a *partial meet contraction* on **K** if and only if there is a selection function γ for **K** such that for all sentences α: $\mathbf{K} - \alpha = \bigcap \gamma(\mathbf{K} \perp \alpha)$.

2.3 System of Spheres-based Contractions

Now we recall the definitions of a *system of spheres* and of the *system of spheres-based contractions* presented in [7].

Definition 2.2 [7] Let $\mathcal{X}$ be a subset of $\mathcal{M}_{\mathcal{L}}$. A *system of spheres* (abrev. S.S.), or *spheres' system*, centred on $\mathcal{X}$ is a collection $\mathbb{S}$ of subsets of $\mathcal{M}_{\mathcal{L}}$, i.e. $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$, that satisfies the following conditions:

(S1) $\mathbb{S}$ is totally ordered with respect to set inclusion; that is, if $\mathcal{U}, \mathcal{V} \in \mathbb{S}$, then $\mathcal{U} \subseteq \mathcal{V}$ or $\mathcal{V} \subseteq \mathcal{U}$.

(S2) $\mathcal{X} \in \mathbb{S}$, and if $\mathcal{U} \in \mathbb{S}$ then $\mathcal{X} \subseteq \mathcal{U}$.

(S3) $\mathcal{M}_\mathcal{L} \in \mathbb{S}$ (and so it is the largest element of $\mathbb{S}$).

(S4)   For every $\varphi \in \mathcal{L}$, if there is any element in $\mathbb{S}$ intersecting $\|\varphi\|$ then there is also a smallest element in $\mathbb{S}$ intersecting $\|\varphi\|$.

The elements of $\mathbb{S}$ are called *spheres*.

For any consistent sentence $\varphi \in \mathcal{L}$, the smallest sphere in $\mathbb{S}$ intersecting $\|\varphi\|$ is denoted by $\mathbb{S}_\varphi$ and $f_\mathbb{S}(\varphi)$ denotes the set consisting of the φ-worlds *closest* to $\mathcal{X}$, i.e.,

$$f_\mathbb{S}(\varphi) = \|\varphi\| \cap \mathbb{S}_\varphi.$$

Definition 2.3 [7] Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The $\mathbb{S}$-*based contraction on* $\mathbf{K}$ is the contraction operation $-_\mathbb{S}$ defined, for any $\varphi \in \mathcal{L}$, by:

$$\mathbf{K} -_\mathbb{S} \varphi = \begin{cases} Th(\|\mathbf{K}\| \cup f_\mathbb{S}(\neg\varphi)) , & \text{if } \nvdash \varphi \\ \mathbf{K} & , \quad \text{if } \vdash \varphi. \end{cases}$$

An operation $-$ on $\mathbf{K}$ is a *system of spheres-based contraction on* $\mathbf{K}$ if and only if there is a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, such that, for all sentences $\varphi \in \mathcal{L}$, $\mathbf{K} - \varphi = \mathbf{K} -_\mathbb{S} \varphi$.

To close this subsection we remind that Grove [7] has shown that every system of spheres-based contraction is a partial meet contraction. More precisely, in [7] it has been shown that the class of *system of spheres-based contractions* coincides with the class of *transitively relational partial meet contractions* (TRPMCs), introduced in [2], which is a (proper) subclass of the class of partial meet contractions.

2.4 Partial Meet Multiple Contractions

The *partial meet multiple contractions* are a generalization of the partial meet contraction functions to the case of contractions by (possibly non-singleton) sets.

Definition 2.4 [6, 8] Let $\mathbf{K}$ be a belief set. A *package selection function* for $\mathbf{K}$ is a function γ such that for all sets of sentences B: if $\mathbf{K} \perp B \neq \emptyset$, then $\emptyset \neq \gamma(\mathbf{K} \perp B) \subseteq \mathbf{K} \perp B$, and if $\mathbf{K} \perp B = \emptyset$ then $\gamma(\mathbf{K} \perp B) = \{\mathbf{K}\}$.

An operation $\div$ is a *partial meet multiple contraction* on $\mathbf{K}$ if and only if there is some package selection function γ for $\mathbf{K}$, such that for all sets of sentences B: $\mathbf{K} \div B = \bigcap \gamma(\mathbf{K} \perp B)$.

An operation $\div$ on $\mathbf{K}$ is a *maxichoice multiple contraction* if and only if it is a partial meet multiple contraction generated by a package selection function γ such that for all sets B, the set $\gamma(\mathbf{K} \perp B)$ has exactly one element. The *full meet multiple contraction* on $\mathbf{K}$ is the partial meet multiple contraction $\dot{\sim}$ that is generated by the package selection function γ such that for all sets B,

if $\mathbf{K}\bot B \neq \emptyset$, then $\gamma(\mathbf{K}\bot B) = \mathbf{K}\bot B$, i.e., $\mathbf{K}\mathbin{\dot{\sim}} B = \bigcap \mathbf{K}\bot B$, if $B \cap Cn(\emptyset) = \emptyset$, and $\mathbf{K}\mathbin{\dot{\sim}} B = \mathbf{K}$, otherwise.

2.5 Possible Worlds Semantics for Partial Meet Multiple Contractions

In [10] we have presented the possible worlds semantics for partial meet multiple contraction. More precisely, in that paper we have shown how the remainders can be defined in terms of possible worlds and, making use of that alternative way of defining the remainders, we have shown how Partial Meet Multiple Contractions can be defined as intersections of (appropriate) sets of possible worlds.

In the present subsection we will recall the concepts and results of the mentioned paper which we will need further ahead.

We start by recalling the definition of the set $\mathbb{W}_{\mathbf{K}\bot B}$, that allows us to define a remainder set in terms of possible worlds.

Definition 2.5 [10] Let $\mathbf{K}$ be a belief set and B be a set of sentences. We denote by $\mathbb{W}_{\mathbf{K}\bot B}$ the subset of $\mathcal{P}\left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}\right)$ such that $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\bot B}$ if and only if:

1. $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$.
2. If $M \in \mathcal{W}$ then there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W} \cap \|\neg\alpha_j\| = \{M\}$.

The relation between the set $\mathbb{W}_{\mathbf{K}\bot B}$ and the remainder set $\mathbf{K}\bot B$ is the following:

Observation 2.6 [10] *Let $\mathbf{K}$ be a belief set and B be a finite set of sentences. Then:*

1. *If $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\bot B}$ then $X = Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \mathbf{K}\bot B$.*
2. *If $X \in \mathbf{K}\bot B$ then there is some $\mathcal{W} \in \mathbb{W}_{\mathbf{K}\bot B}$ such that $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$.*
3. *$\mathbf{K}\bot B = \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in \mathbb{W}_{\mathbf{K}\bot B}\}$.*

The first statement of the observation means that the sets in $\mathbb{W}_{\mathbf{K}\bot B}$ are minimal sets of worlds in the sense that the addition of each of those sets to $\|\mathbf{K}\|$ corresponds to a maximal subset of $\mathbf{K}$ that does not imply any element of B. The second statement yields that each of the remainders can be constructed by means of a set of $\mathbb{W}_{\mathbf{K}\bot B}$. The third shows how to construct a remainder set in terms of possible worlds.

Using the set $\mathbb{W}_{\mathbf{K}\bot B}$ it is possible to define a partial meet multiple contraction in terms of possible worlds:

Observation 2.7 [10] *Let $\mathbf{K}$ be a belief set. An operation $\div$ is a partial meet multiple contraction on $\mathbf{K}$ if and only if for any set of sentences B:*

$$\mathbf{K}\div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathbb{W}_{\mathbf{K}\bot B})\right)\right),$$

where f is a propositional package selection function for $\|\mathbf{K}\|$ such that for all sets of sentences B: (1) $f(\mathbb{W}_{\mathbf{K}\perp B}) \subseteq \mathbb{W}_{\mathbf{K}\perp B}$, *and* (2) *if* $\mathbb{W}_{\mathbf{K}\perp B} \neq \emptyset$ *then* $f(\mathbb{W}_{\mathbf{K}\perp B}) \neq \emptyset$.

The lower bound of partial meet multiple contraction, called full meet multiple contraction corresponds to the situation when $f(\mathbb{W}_{\mathbf{K}\perp B}) = \mathbb{W}_{\mathbf{K}\perp B}$ [10, Observation 3.6]. In the case of singleton contraction, the full meet contraction $\mathbf{K} \sim \alpha$ corresponds to the addition of the whole set $\|\neg\alpha\|$ to $\|\mathbf{K}\|$. However, in the multiple case it may not hold that $\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ [10, Observation 3.7]. For instance, if $B = \{\alpha, \alpha \wedge \beta\}$ and $\alpha \nvdash \beta$, then since every $\neg\alpha$-world is also a $\neg(\alpha \wedge \beta)$-world the addition of a $\neg\alpha$-world to $\|\mathbf{K}\|$ immediately leads to the removal of both α and $\alpha \wedge \beta$ from the new (resulting) belief set. Therefore none of the $\alpha \wedge \neg\beta$-worlds (i.e. the worlds in $\|\neg(\alpha \wedge \beta)\| \setminus \|\neg\alpha\|$) belongs to a set $\mathcal{W}$ of $\mathbb{W}_{\mathbf{K}\perp B}$. Based on this fact in [10, Observation 3.9] the following equality has been presented

$$\bigcup \mathbb{W}_{\mathbf{K}\perp B} = \bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\},$$

where

$$B_N = \{\alpha_i \in B : \text{ For all } \alpha_j \in B \text{ it holds that } \|\neg\alpha_j\| \not\subset \|\neg\alpha_i\|\}.[1]$$

Furthermore, making use of the above equality the following observation has been obtained in the mentioned paper:

Observation 2.8 [10] *Let $\mathbf{K}$ be a belief set and B be a set of sentences and $\dot{\sim}$ be the full meet multiple contraction on $\mathbf{K}$. Then*

$$\|\mathbf{K}\dot{\sim}B\| = \|\mathbf{K}\| \cup \left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B_N \cap \mathbf{K}\}\right).$$

3 How (and how not) to Construct Partial Meet Multiple Contractions Using Systems of Spheres

In this section we will generalize Grove's construction of a S.S.-based contraction to the multiple case. More precisely, given a belief set $\mathbf{K}$ and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$ we will define a multiple contraction operation $\div_{\mathbb{S}}$ on $\mathbf{K}$ — which we will designate by $\mathbb{S}$-*based multiple contraction on* $\mathbf{K}$ — such that (1) it is a partial meet multiple contraction, and (2) it is such that for all sentences $\alpha \in \mathcal{L}$ it holds that $\mathbf{K}\div_{\mathbb{S}}\{\alpha\} = \mathbf{K}-_{\mathbb{S}}\alpha$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction.

We will start by showing that the most intuitive proposals for the definition of the $\mathbb{S}$-*based multiple contraction* fail to be partial meet multiple contractions,

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[1]Notice that the set B_N can, alternatively, be defined without making use of the notion of possible world, in the following (equivalent) way:

$$B_N = \{\alpha_i \in B : \alpha_i \nvdash \alpha_j \text{ for all } \alpha_j \in B \text{ s.t. } Cn(\alpha_i) \neq Cn(\alpha_j)\}.$$

then we find sufficient conditions for a $\mathbb{S}$-*based multiple contraction on* **K** be a partial meet multiple contraction and finally we present a method to construct functions that satisfy those conditions.

Recalling that if $\div$ is a partial meet multiple contraction and B contains tautologies (i.e. $B \cap Cn(\emptyset) \neq \emptyset$) then $\mathbf{K} \div B = \mathbf{K}$, we immediately conclude that the operation $\div_{\mathbb{S}}$ must satisfy $\mathbf{K} \div_{\mathbb{S}} B = \mathbf{K}$, for all such sets B.

Having seen this, it only remains to investigate how shall $\mathbf{K} \div_{\mathbb{S}} B$ be defined for sets B such that $B \cap Cn(\emptyset) = \emptyset$. However, as we will see below, it is not trivial to find out an appropriate way of defining $\mathbf{K} \div_{\mathbb{S}} B$ (in order to assure that $\div_{\mathbb{S}}$ fulfils the desired properties (1) and (2) introduced above). Indeed, in what follows we show that the most intuitive proposals for the definition of the $\mathbb{S}$-*based multiple contraction on* **K** actually do not fulfil the above mentioned requirements.

3.1 Unsuccessful Proposals

3.1.1 First Unsuccessful Proposal

Consider a belief set **K** and a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$. Recalling that Grove's $\mathbb{S}$-based singleton contraction is such that if $\nvdash \varphi$ then $\mathbf{K} -_{\mathbb{S}} \varphi = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi))$, it follows that the operation $\div_{\mathbb{S}}^1$ defined by:

$$\mathbf{K} \div_{\mathbb{S}}^1 B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}$$

is the most obvious proposal for the definition of the $\mathbb{S}$-*based multiple contraction on* **K**.[2,3] However, the following simple counterexample, shows that $\div_{\mathbb{S}}^1$ is not a partial meet multiple contraction.

Counterexample 3.1 *Consider Fig.* 1 *and let* $B = \{\alpha_1, \alpha_2, \alpha_3\}$. *Notice that, in these conditions, it follows from Definition 2.2 that* $f_{\mathbb{S}}(\neg\alpha_1) = \|\neg\alpha_1\|$, $f_{\mathbb{S}}(\neg\alpha_2) = \|\neg\alpha_2\|$ *and* $f_{\mathbb{S}}(\neg\alpha_3) = \|\neg\alpha_3\|$. *Then, according to the definition of* $\div_{\mathbb{S}}^1$ *proposed above, we have that* $\mathbf{K} \div_{\mathbb{S}}^1 B = \bigcap(\|\mathbf{K}\| \cup \|\neg\alpha_1\| \cup \|\neg\alpha_2\| \cup \|\neg\alpha_3\|)$, *that is,* $\mathbf{K} \div_{\mathbb{S}}^1 B$ *consists of the intersection of all the worlds included in the shaded region of Fig.* 1. *Then, in particular,* $\|\neg\alpha_1\| \subseteq \|\mathbf{K} \div_{\mathbb{S}}^1 B\|$. *On the other hand, in*

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[2]Notice that, since, whenever $B \cap Cn(\emptyset) = \emptyset$ it holds that $\bigcap_{\alpha_i \in B} \mathbf{K} -_{\mathbb{S}} \alpha_i = \bigcap_{\alpha_i \in B} (Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_i))) = \bigcap_{\alpha_i \in B} (\bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_i))) = \bigcap\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B} f_{\mathbb{S}}(\neg\alpha_i)\right)\right)$, it follows that operation $\div_{\mathbb{S}}^1$ on **K** can be equivalently defined in the following (alternative) way:

$$\mathbf{K} \div_{\mathbb{S}}^1 B = \begin{cases} \bigcap_{\alpha_i \in B} \mathbf{K} -_{\mathbb{S}} \alpha_i & \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}$$

where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction.

[3]Based on a ranking theoretic approach Spohn [11] suggested a multiple contraction function very similar to operation $\div_{\mathbb{S}}^1$.

Fig. 1 Illustration of
Counterexample 3.1

M_L

$\|K\|$

$\|\neg\alpha_1\|$

$\|\neg\alpha_2\|$ $\|\neg\alpha_3\|$
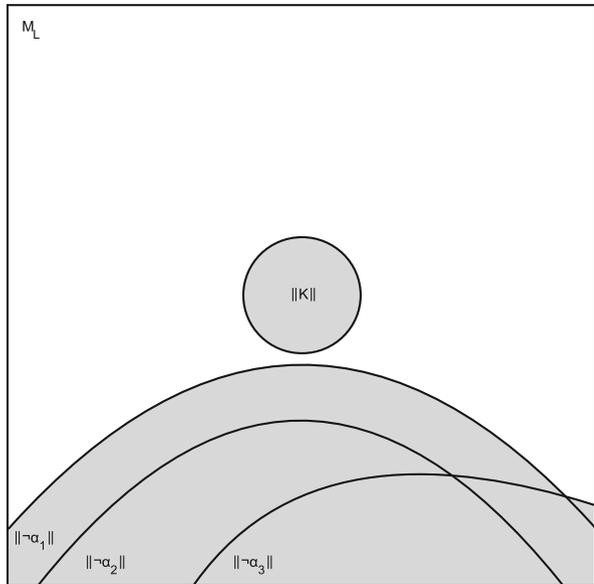
the above conditions, $B_N = \{\alpha_2, \alpha_3\}$, from which it follows (by Observation 2.8)
*that $\mathbf{K} \div^1_{\mathbb{S}} B$ is not a superset of the full meet multiple contraction, and therefore
$\div^1_{\mathbb{S}}$ is not a partial meet multiple contraction.*

3.1.2 Second Unsuccessful Proposal

Taking into account that $\div^1_{\mathbb{S}}$ failled to be a partial meet multiple contraction
because that $\mathbf{K} \div^1_{\mathbb{S}} B$ was not a superset of the full meet multiple contraction,
in our second proposal we just replace B by B_N:

$$\mathbf{K} \div^2_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}$$

Nevertheless, the following counterexample shows that $\div^2_{\mathbb{S}}$ is not a partial
meet multiple contraction either.

Counterexample 3.2 *Consider Fig.* 2 *and let* $B = \{\alpha_1, \alpha_2, \alpha_3\}$*. Assume, ad-
ditionally,   that $\|Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3))\| = \|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$
(which is the set represented by the shaded region of Fig.* 2*).*

*Notice that, in the above conditions, $B_N = \{\alpha_2, \alpha_3\}$. Therefore $\|\mathbf{K} \div^2_{\mathbb{S}} B\| =
\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$.*

*Now assume by reductio that the operation $\div^2_{\mathbb{S}}$ is a partial meet multiple con-
traction on $\mathbf{K}$. Then, according to Observation* 2.7*, there is some $\mathbb{M}_B \subseteq \mathbb{W}_{\mathbf{K} \perp B}$
such that $\mathbb{M}_B \neq \emptyset$ and $\mathbf{K} \div^2_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup \mathbb{M}_B\right)\right)$. Consequently, $\|\mathbf{K}\| \cup
\left(\bigcup \mathbb{M}_B\right) \subseteq \|\mathbf{K} \div^2_{\mathbb{S}} B\|$ which is to say that $\|\mathbf{K}\| \cup \left(\bigcup \mathbb{M}_B\right) \subseteq \|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup
f_{\mathbb{S}}(\neg\alpha_3)$. Moreover, noticing that from $\mathbb{M}_B \subseteq \mathbb{W}_{\mathbf{K} \perp B}$ it follows that $\left(\bigcup \mathbb{M}_B\right) \cap*

Fig. 2 Illustration of Counterexample 3.2
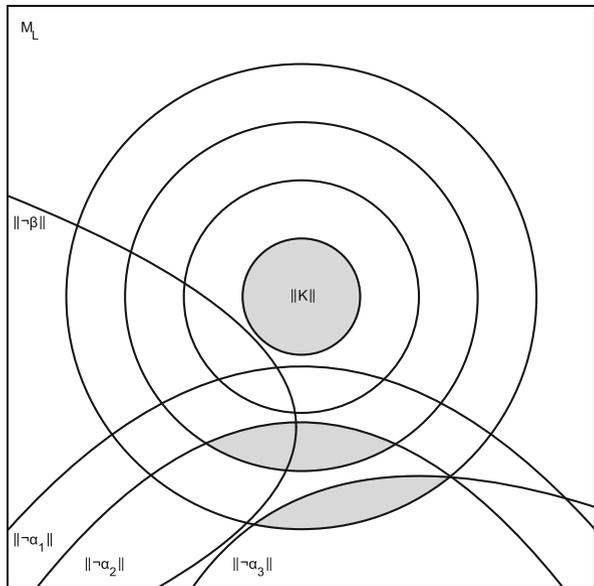
$\|K\| = \emptyset$ *(since, according to Definition 2.5, $\left(\bigcup \mathbb{W}_{\mathbf{K} \perp B}\right) \cap \|K\| = \emptyset$), we can additionally conclude that $\left(\bigcup \mathbb{M}_B\right) \subseteq f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$.*

Furthermore, an attentive observation of Fig. 2 allows us to conclude that if $M \in f_{\mathbb{S}}(\neg\alpha_2)$ then it follows immediately from Definition 2.5 that there is no $\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B}$ such that $\mathcal{W} \subseteq f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3)$ and $M \in \mathcal{W}$. Therefore $\left(\bigcup \mathbb{M}_B\right) \subseteq f_{\mathbb{S}}(\neg\alpha_3)$.

Then, observing Fig. 2, from $\mathbf{K} \div_{\mathbb{S}}^2 B = Th\left(\|K\| \cup \left(\bigcup \mathbb{M}_B\right)\right)$ we can conclude that $\beta \in \mathbf{K} \div_{\mathbb{S}}^2 B$. However, on the other hand, since $\mathbf{K} \div_{\mathbb{S}}^2 B = \bigcap(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha_2) \cup f_{\mathbb{S}}(\neg\alpha_3))$, also by analysing the mentioned figure, we can see that $\beta \notin \mathbf{K} \div_{\mathbb{S}}^2 B$. From this contradiction we can conclude that $\div_{\mathbb{S}}^2$ is not a partial meet multiple contraction.

3.1.3 Third Unsuccessful Proposal

Having in mind Counterexample 3.2, we may notice that, if $B = \{\alpha_i, \alpha_j\} \subset \mathbf{K}$ and α_i, α_j are such that $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i) \neq \emptyset$ then, in order to assure that an operation $\div_{\mathbb{S}}$ is a partial meet multiple contraction, we must prevent that the worlds in $f_{\mathbb{S}}(\neg\alpha_j) \setminus f_{\mathbb{S}}(\neg\alpha_i)$ are included in the set of possible worlds that we choose to intersect to originate $\mathbf{K} \div_{\mathbb{S}} B$.

The above paragraph naturally leads us to consider the following alternative operation $\div_{\mathbb{S}}^3$:

$$\mathbf{K} \div_{\mathbb{S}}^3 B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}$$

Fig. 3 Illustration of
Counterexample 3.3
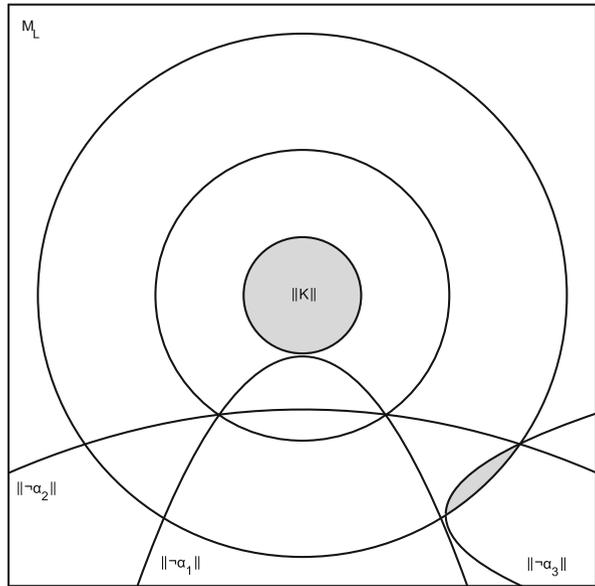
where, $B_{\mathbb{S},1} = \{\alpha_i \in B : \forall \alpha_k \in B \setminus Cn(\emptyset) \ (f_{\mathbb{S}}(\neg \alpha_k) = f_{\mathbb{S}}(\neg \alpha_i)$ or $f_{\mathbb{S}}(\neg \alpha_k) \not\subseteq \|\neg \alpha_i\|)\}.$[4]

Nonetheless, this construction also fails to be a partial meet multiple contraction, as we can conclude from the following counterexample:

Counterexample 3.3 *Consider Fig. 3 and let $B = \{\alpha_1, \alpha_2, \alpha_3\}$. Notice that, in the above conditions we obtain that $B_{\mathbb{S},1} = \{\alpha_3\}$, and consequently, $\mathbf{K} \div_{\mathbb{S}}^3 B = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg \alpha_3))$. Hence, we have that $\mathbf{K} \div_{\mathbb{S}}^3 B$ consists of the intersection of all the worlds included in the shaded region of Fig. 3. So, noticing that none of those worlds is a $\neg \alpha_1$-world we immediately conclude that $\alpha_1 \in \mathbf{K} \div_{\mathbb{S}}^3 B$ which contradicts the successful elimination of all the members of B from $\mathbf{K}$, a primary condition of partial meet multiple contraction.*

3.2 (Finding) The Way to a Successful Proposal

We will use the previous counterexamples in order to, finally, present a way of defining an operation $\div_{\mathbb{S}}$ which fulfils the desired properties.

We start by observing that all the above proposed operations $\div_{\mathbb{S}}^1$, $\div_{\mathbb{S}}^2$ and $\div_{\mathbb{S}}^3$ have the form $Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B'} f_{\mathbb{S}}(\neg \alpha_i)\right)\right)$ where B' is a subset of B. However we have shown that none of those operations is a partial meet multiple contraction.

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[4]Notice that, if $B \subseteq (\mathbf{K} \setminus Cn(\emptyset))$ and $\mathbb{S} = \{\|\mathbf{K}\|, \mathcal{M}_{\mathcal{L}}\}$ then $B_{\mathbb{S},1} = B_N$ (since, in that case, $\forall \alpha_k \in \mathbf{K} \setminus Cn(\emptyset) \ f_{\mathbb{S}}(\neg \alpha_k) = \|\neg \alpha_k\|$). However, in general, neither $B_{\mathbb{S},1} \subseteq B_N$ nor $B_N \subseteq B_{\mathbb{S},1}$.

In particular, in $\div_{\mathbb{S}}^2$ we have taken $B' = B_N$ and in $\div_{\mathbb{S}}^3$ we have considered $B' = B_{\mathbb{S},1}$, where the set $B_{\mathbb{S},1}$ has been defined with the specific goal of assuring that the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i) \right)$ is a subset of $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_N} f_{\mathbb{S}}(\neg\alpha_i) \right)$. However, it follows from Counterexample 3.3 that operation $\div_{\mathbb{S}}^3$ fails to be a partial meet multiple contraction because the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i) \right)$ has, in general, too few worlds.

Hence, holding on to the idea of defining $\mathbf{K} \div_{\mathbb{S}} B$ as an intersection of a set of possible worlds of the form $\|\mathbf{K}\| \cup (\bigcup_{\alpha_i \in B'} f_{\mathbb{S}}(\neg\alpha_i))$ with $B' \subseteq B$, our next goal is to find a systematic way of defining, a subset $B_{\mathbb{S}} \subseteq B$ such that: (i) $B_{\mathbb{S},1} \subseteq B_{\mathbb{S}} \subseteq B_N$ and (ii) the operation $\div_{\mathbb{S}}$ defined by $\mathbf{K} \div_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup (\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)) \right)$ is a partial meet multiple contraction on $\mathbf{K}$.

So, in order to find out how to define a set $B_{\mathbb{S}} \subseteq B$ such that the above conditions hold, it is convenient to start by understanding in more detail why the set $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i) \right)$ contains, in general, too few worlds (or, from another point of view, why the set $B_{\mathbb{S},1}$ contains, in general, too few sentences) for the operation $\div_{\mathbb{S}}^3$, defined in the previous subsection, to be a partial meet multiple contraction.

We may observe from its definition, that the set $B_{\mathbb{S},1}$ is, roughly speaking, obtained from B by discarding from it all the sentences α_r for which there is some other sentence α_s in B such that $f_{\mathbb{S}}(\neg\alpha_s) \subset \|\neg\alpha_r\|$ and $f_{\mathbb{S}}(\neg\alpha_s) \neq f_{\mathbb{S}}(\neg\alpha_r)$. However, as we have seen in Counterexample 3.3, for some sets B, such definition of $B_{\mathbb{S},1}$ leads to a situation in which for some $\alpha_t \in B \setminus B_{\mathbb{S},1}$ it holds that $\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i) \right) \right) \cap \|\neg\alpha_t\| = \emptyset$ (and, therefore, $\alpha_t \in \mathbf{K} \div_{\mathbb{S}}^3 B$, which yields that, in that case, $\div_{\mathbb{S}}^3$ is not a partial meet multiple contraction on $\mathbf{K}$). Indeed, as it can be seen in the mentioned counterexample (where we have shown that $\alpha_1 \in B$ but $\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S},1}} f_{\mathbb{S}}(\neg\alpha_i) \right) \right) \cap \|\neg\alpha_1\| = \emptyset$), such a situation occurs whenever $\alpha_t \in B \setminus B_{\mathbb{S},1}$ is such that (i) $\alpha_t \in B \cap K$, (ii) there is only one sentence in B, say α_q, which is, roughly speaking, responsible for the fact that $\alpha_t \notin B_{\mathbb{S},1}$, (iii) for all $\alpha_p \in B \setminus \{\alpha_t, \alpha_q\}$ it holds that $f_{\mathbb{S}}(\neg\alpha_p) \cap \|\neg\alpha_t\| = \emptyset$ and (iv) $\alpha_q \notin B_{\mathbb{S},1}$.

Hence, having the above in mind, it seems that, for any given set B not containing tautologies, the subset $B_{\mathbb{S}}$ of B that we wish to define should be constructed in a way that, if $\alpha_r, \alpha_s \in B \cap K$ are two sentences in the above described conditions then they do not simultaneously belong to $B_{\mathbb{S}}$ but, on the other hand, $B_{\mathbb{S}}$ must be such that for all sentences $\alpha_l \in (B \cap K) \setminus B_{\mathbb{S}}$ there is some $\alpha_k \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subset \|\neg\alpha_l\|$ and $f_{\mathbb{S}}(\neg\alpha_k) \neq f_{\mathbb{S}}(\neg\alpha_l)$. The following theorem asserts that the above analysis is correct:

Theorem 3.4 *Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. Let $\div_{\mathbb{S}}$ be an operation defined on $\mathbf{K}$ by:*

$$\mathbf{K} \div_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup (\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)) \right) & \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

where, for any finite set of sentences B satisfying $B \cap Cn(\emptyset) = \emptyset$, the set $B_{\mathbb{S}}$ is a subset of B such that the following conditions hold:

(i)   *If $\alpha_j \in B_{\mathbb{S}}$ then $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_j\|$, for all $\alpha_k \in B_{\mathbb{S}}$.*

(ii) *For all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ such that $f_{\mathbb{S}}(\neg\alpha_m) \subseteq \|\neg\alpha_l\|$.*

(iii) *For any finite set of sentences C satisfying $C \cap Cn(\emptyset) = \emptyset$, if every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C, then for all $\beta_i \in (B_N \cap \mathbf{K})_{\mathbb{S}}$ there is some $\varsigma_i \in (C_N \cap \mathbf{K})_{\mathbb{S}}$ such that $\|\beta_i\| = \|\varsigma_i\|$.*

(iv) *$B_{\mathbb{S}} \cap \mathbf{K} = (B \cap \mathbf{K})_{\mathbb{S}}$.*

(v)   *$(B_N)_{\mathbb{S}} = (B_{\mathbb{S}})_N$.*

(vi) *$\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$.*

Then the operation $\div_{\mathbb{S}}$ is a partial meet multiple contraction on $\mathbf{K}$.

Conditions (i) and (ii) in the statement of the above theorem are motivated by our previous analysis. It is also worth noticing that the second one of those conditions assures that the set $\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$ contains some $\neg\alpha_l$-world for all sentences $\alpha_l \in B \cap \mathbf{K}$ and, consequently, the operation $\div_{\mathbb{S}}$ defined as suggested in the statement of the above theorem is such that the set $\mathbf{K} \div_{\mathbb{S}} B$ does not contain any elements from B. Conditions (iv)−(vi) are technical constraints necessary for the proof.

3.3 Spheres' System-based Multiple Contractions

In this subsection we will start by presenting a way of systematically obtaining, for any finite set of sentences B, a subset $B_{\mathbb{S}}$ such that, whenever $B \cap Cn(\emptyset) = \emptyset$, the conditions (i)−(vi) stated in Theorem 3.4 are satisfied. After that we will make use of the constructed sets $B_{\mathbb{S}}$ to present a definition for the S.S.-based multiple contractions.

At this point we start by remarking that condition (i) of the mentioned theorem presents in some sense a necessary condition for an element α_j of B to belong to $B_{\mathbb{S}}$ (namely: for all $\alpha_k \in B_{\mathbb{S}}$ it holds that $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_j\|$). We will use that condition as our guideline towards the definition of a set $B_{\mathbb{S}}$ in the desired conditions. Taking this into account and observing that the condition imposed to be satisfied by all elements of $B_{\mathbb{S}}$ is *recurring* (in the sense that it makes reference to the set $B_{\mathbb{S}}$ itself) we are led to believe that the definition of $B_{\mathbb{S}}$ shall follow an iterative procedure.

Having this in mind we suggest the following construction for the set $B_{\mathbb{S}}$:[5]

Definition 3.5 Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$.

Consider a set of sentences $B = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{L}$ such that $B \setminus Cn(\emptyset) \neq \emptyset$.

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[5]Further bellow we provide a more detailed explanation of the intuition behind this definition.

Denote by $C_1, \ldots, C_m$ the (different) equivalence classes in the quotient set of $(B \setminus Cn(\emptyset))$ by $\backsim$, i.e. $\{C_1, \ldots, C_m\} = (B \setminus Cn(\emptyset))/\backsim$, where $\backsim$ is the equivalence relation on $B \setminus Cn(\emptyset)$ defined by:

$$\forall \alpha, \beta \in B \setminus Cn(\emptyset), \ \alpha \backsim \beta \ \text{iff} \ \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta},$$

Moreover, assume that the equivalence classes $C_1, \ldots, C_m$ are ordered according the following condition:

$$\text{If } 1 \leq i < j \leq m \text{ then } \forall \alpha_r \in C_i \forall \alpha_s \in C_j \ \mathbb{S}_{\neg\alpha_s} \subset \mathbb{S}_{\neg\alpha_r}.$$

Now consider the following list of subsets of B:

$$B_0 = B \cap Cn(\emptyset)$$

$$C_1' = C_1$$

$$C_1'' = \{\alpha_i \in C_1' : \forall \alpha_j \in C_1' \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\}$$

$$B_1 = C_1''$$

Moreover, if $m > 1$ for all $l \in \{2, \ldots, m\}$, let C_l', C_l'' and B_l be the sets defined by:

$$C_l' = \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1} \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|\};$$

$$C_l'' = \{\alpha_i \in C_l' : \forall \alpha_j \in C_l' \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\};$$

$$B_l = B_{l-1} \cup C_l''.$$

The set $B_{\mathbb{S}} = B_m$ is the $\mathbb{S}$-*based filtration of* B.

If D is a set of sentences such that $D \subseteq Cn(\emptyset)$, then the $\mathbb{S}$-*based filtration of* D is the empty set and is denoted by $D_{\mathbb{S}}$, i.e. $D_{\mathbb{S}} = \emptyset$.

The following observation presents an alternative (equivalent) way of defining the sets $C_1, \ldots, C_m, C_1', \ldots, C_m', C_1'', \ldots, C_m''$ introduced in the above definition.

Observation 3.6 *Let* **K** *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a finite set of sentences such that* $B \setminus Cn(\emptyset) \neq \emptyset$. *Furthermore, let* $C_1, \ldots, C_m, C_1', \ldots, C_m', C_1'', \ldots, C_m'', B_1, \ldots, B_m$ *be the subsets of* B *constructed as we described in the above definition. Then:*

(a) *For all* $l \in \{1, \ldots, m\}$ *it holds that* $C_l'' = \{\alpha_i \in C_l' : \forall \alpha_j \in C_l' \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i) \ or \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

(b) *If* $m > 1$ *then for all* $l \in \{2, \ldots, m\}$ *the following identity is satisfied:* $C_l' = \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1} \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i) \ or \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

We are now in a position to expose the intuition behind the above definition of the $\mathbb{S}$-*based filtration of* B. In order to do that we must start by remarking that our intention when proposing the above definition was that the set $B_{\mathbb{S}} \subseteq B$ should be such that, whenever B is a set that does not contain any tautology, conditions (i) and (ii) of Theorem 3.4 are satisfied.

If we were only interested in the fulfilment of condition (i) above, then it would be enough to define $B_{\mathbb{S}} = \{\alpha_i \in B : \forall \alpha_k \in B \setminus Cn(\emptyset) \ (f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_i\|)\}(= B_{\mathbb{S},1})$, however, as we have concluded with the help of Counterexample 3.3, if $B_{\mathbb{S}}$ was defined in that way then condition (ii) would not, in general, be satisfied.

Recalling the situation described in Counterexample 3.3 we can observe that, in those circumstances, condition (ii) is not satisfied essentially because, among other facts, it holds that $f_{\mathbb{S}}(\neg\alpha_2) \subset \|\neg\alpha_1\|$ and $f_{\mathbb{S}}(\neg\alpha_3) \subset \|\neg\alpha_2\|$, while, on the other hand, $f_{\mathbb{S}}(\neg\alpha_3) \not\subseteq \|\neg\alpha_1\|$. Roughly speaking, in those conditions, what happens is that the sentence $\alpha_2 \in B$, which is responsible for the non-inclusion of α_1 in $B_{\mathbb{S}}$, does not belong to the set $B_{\mathbb{S}}$ either, and that fact (together with the remaining circumstances) causes that for all $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ it holds that $f_{\mathbb{S}}(\neg\alpha_m) \not\subseteq \|\neg\alpha_l\|$.

Now notice that, given two non tautological sentences α and β, it follows immediately from Definition 2.2 that if $f_{\mathbb{S}}(\neg\alpha) \subset \|\neg\beta\|$ then $\mathbb{S}_{\neg\beta} \subseteq \mathbb{S}_{\neg\alpha}$. On the other hand, if $\mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}$ and $f_{\mathbb{S}}(\neg\alpha) \subset \|\neg\beta\|$ then $f_{\mathbb{S}}(\neg\alpha) \subseteq f_{\mathbb{S}}(\neg\beta)$.

Hence, we can conclude that conditions $f_{\mathbb{S}}(\neg\alpha_q) \subset \|\neg\alpha_p\|$, $f_{\mathbb{S}}(\neg\alpha_r) \subset \|\neg\alpha_q\|$ and $f_{\mathbb{S}}(\neg\alpha_r) \not\subseteq \|\neg\alpha_p\|$ can only hold simultaneously as long as $\mathbb{S}_{\neg\alpha_q} \subset \mathbb{S}_{\neg\alpha_r}$.

Therefore, from all the above we can observe that:

1. In order to assure that the set $B_{\mathbb{S}} \subseteq B$ satisfies conditions (i)−(ii) mentioned above, then it must be defined in a way which imposes that a sentence $\alpha_i \in B$ only belongs to $B_{\mathbb{S}}$ if $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_i\|$, for all sentences α_k which are, themselves, elements of $B_{\mathbb{S}}$ (but not necessarily for all sentences of B).
2. If C is a finite set such that $C \cap Cn(\emptyset) = \emptyset$ and for some $\mathcal{D} \in \mathbb{S}$ it holds that $\mathbb{S}_{\neg\alpha_i} = \mathcal{D}$, for all $\alpha_i \in C \cap \mathbf{K}$, and $C_{\mathbb{S}} = \{\alpha_i \in C : \forall \alpha_k \in C \ (f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_i\|)\}$ then the conditions obtained from (i) and (ii), by replacing, in each of those clauses, B by C, are satisfied.[6]
3. If α and β are two sentences such that $\mathbb{S}_{\neg\alpha} \subset \mathbb{S}_{\neg\beta}$ then $f_{\mathbb{S}}(\neg\alpha) \not\subset \|\neg\beta\|$.

Since it is based in the facts listed above, we are now in a position to explain the idea behind the definition of the $\mathbb{S}$-based filtration of a set B (Definition 3.5). For that purpose in what follows we assume that B is a finite set such that $B \setminus Cn(\emptyset) \neq \emptyset$ and describe how the set $B_{\mathbb{S}}$ is obtained as well as the motivations behind such procedure.

The first concern underlying such definition has to do with point 1. above and consists in constructing the set $B_{\mathbb{S}}$ a the result of a sequence of steps (rather than by a single step definition) in order to assure that at each stage of that construction we only prevent from being included in $B_{\mathbb{S}}$ those sentences $\alpha_i \in B$ such that $f_{\mathbb{S}}(\neg\alpha_k) \neq f_{\mathbb{S}}(\neg\alpha_i)$ and $f_{\mathbb{S}}(\neg\alpha_k) \subset \|\neg\alpha_i\|$, for some sentence $\alpha_k \in B$ which is somehow guaranteed to be such that it *will belong* to the set $B_{\mathbb{S}}$ when its *ongoing* construction is finished.

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[6]Notice that a rigorous proof that this indeed holds can be trivially obtained by combining Observation 3.6 and Lemma A.4.

So, as it can be seen from its definition above, in order to construct the set $B_{\mathbb{S}}$ we start by partitioning B in several classes $B_0, C_1, \ldots, C_m$, where $B_0 = B \cap Cn(\emptyset)$ and for each $i \in \{1, \ldots, m\}$ the class C_i is such that if $\alpha_r \in C_i$ then for all $\alpha_s \in B \setminus Cn(\emptyset)$ it holds that $\alpha_s \in C_i$ if and only if $\mathbb{S}_{\neg\alpha_s} = \mathbb{S}_{\neg\alpha_r}$ (hence $m \geq 1$ is uniquely determined by the set B and the system of spheres $\mathbb{S}$). Furthermore, given two such classes C_i and C_j, if $i < j$, $\alpha_r \in C_i$ and $\alpha_s \in C_j$ then $\mathbb{S}_{\neg\alpha_s} \subset \mathbb{S}_{\neg\alpha_r}$.

In the above conditions we have that, for each $i \in \{1, \ldots, m\}$ the class C_i is such that if $\alpha_r \in C_i$ and $i < j$, with $j \in \{1, \ldots, m\}$, then for all $\alpha_s \in C_j$ it holds that $f_{\mathbb{S}}(\neg\alpha_s) \not\subset \|\alpha_r\|$. Hence, if $\alpha_p \in C_1$ then $f_{\mathbb{S}}(\neg\alpha_q) = f_{\mathbb{S}}(\neg\alpha_p)$ or $f_{\mathbb{S}}(\neg\alpha_q) \not\subset \|\neg\alpha_p\|$, for all $\alpha_q \in B \setminus (B_0 \cup C_1)$. However, there can be some sentences $\alpha_l \in C_1$ for which it does not hold that $f_{\mathbb{S}}(\neg\alpha_m) = f_{\mathbb{S}}(\neg\alpha_l)$ or $f_{\mathbb{S}}(\neg\alpha_m) \not\subset \|\neg\alpha_l\|$, for all $\alpha_m \in C_1$.

So, naturally, the next step in the construction of the set $B_{\mathbb{S}}$ consists in obtaining the subset of C_1 defined by $C_1'' = \{\alpha_i \in C_1' : \forall \alpha_j \in C_1' \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\}$, where $C_1' = C_1$.[7] It follows immediately from its construction and some of our remarks above that the set C_1'' is such that, on the one hand, if $\alpha_p \in C_1''$ then $f_{\mathbb{S}}(\neg\alpha_q) = f_{\mathbb{S}}(\neg\alpha_p)$ or $f_{\mathbb{S}}(\neg\alpha_q) \not\subset \|\neg\alpha_p\|$, for all $\alpha_q \in C_1'' \cup (B \setminus (B_0 \cup C_1))$ and, on the other hand, for all $\alpha_l \in C_1 \setminus C_1''$ there is some $\alpha_m \in C_1''$ such that $f_{\mathbb{S}}(\neg\alpha_m) \neq f_{\mathbb{S}}(\neg\alpha_l)$ and $f_{\mathbb{S}}(\neg\alpha_m) \subset \|\neg\alpha_l\|$ (see footnote 6). At this point we define the set $B_1 = C_1''$, which is the set of sentences of C_1 that we will include in the set $B_{\mathbb{S}}$. Indeed, if $B \setminus (B_0 \cup C_1) = \emptyset$ (or, equivalently, if $m = 1$) then the set $B_{\mathbb{S}}$ is simply defined by $B_{\mathbb{S}} = B_1$.

If, instead, $B \setminus (B_0 \cup C_1) \neq \emptyset$ (or, equivalently, if $m > 1$), the next phase consists in choosing which sentences of C_2 shall be included in the set $B_{\mathbb{S}}$. We do this in two steps. First we consider the set $C_2' = \{\alpha_i \in C_2 : \forall \alpha_j \in B_1 \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|\}$[8] which consists of the subset of C_2 such that $\alpha_i \in C_2'$ if and only if $\alpha_i \in C_2$ and $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|$ (or $f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$), for all $\alpha_j \in B_1 \cup (B \setminus (B_0 \cup C_1 \cup C_2))$. Moreover, we also have that if $\alpha_l \in C_2 \setminus C_2'$ then there is some $\alpha_m \in B_1$ such that $f_{\mathbb{S}}(\neg\alpha_m) \neq f_{\mathbb{S}}(\neg\alpha_l)$ and $f_{\mathbb{S}}(\neg\alpha_m) \subset \|\neg\alpha_l\|$. However there may be some sentence $\alpha_k \in C_2'$ such that $f_{\mathbb{S}}(\neg\alpha_h) \subset \|\neg\alpha_k\|$ and $f_{\mathbb{S}}(\neg\alpha_h) \neq f_{\mathbb{S}}(\neg\alpha_k)$ for some $\alpha_h \in C_2'$. Hence, secondly we must obtain the subset $C_2'' \subseteq C_2'$ which represents to C_2' the same that the set C_1'' represented to C_1', i.e. $C_2'' = \{\alpha_i \in C_2' : \forall \alpha_j \in C_2' \ f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\}$. Afterwards we define $B_2 = B_1 \cup C_2''$. Therefore B_2 is a subset of $C_1 \cup C_2$ such that (a) if $\alpha_i \in B_2$ then $f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|$, for all $\alpha_j \in B_2 \cup (B \setminus (B_0 \cup C_1 \cup C_2))$ and (b) for all $\alpha_l \in (C_1 \cup C_2) \setminus B_2$ there is some $\alpha_m \in B_2$ such that $f_{\mathbb{S}}(\neg\alpha_m) \neq f_{\mathbb{S}}(\neg\alpha_l)$ and $f_{\mathbb{S}}(\neg\alpha_m) \subset \|\neg\alpha_l\|$. Now, if $B \setminus (B_0 \cup C_1 \cup C_2) = \emptyset$ (or, equivalently, if $m = 2$) then we just define $B_{\mathbb{S}} = B_2$. Otherwise, we obtain the set B_3 by applying to the class C_3 the same procedure (with the necessary obvious adaptations) that we have just described above regarding the class C_2, and we

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[7] Notice that, according to Observation 3.6, $C_1'' = \{\alpha_i \in C_1' : \forall \alpha_j \in C_1' \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

[8] Notice that, according to Observation 3.6, $C_2' = \{\alpha_i \in C_2 : \forall \alpha_j \in B_1 \ (f_{\mathbb{S}}(\neg\alpha_j) = f_{\mathbb{S}}(\neg\alpha_i)$ or $f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|)\}$.

go on repeating this process until we have obtained the set B_m. Then, finally, we define $B_{\mathbb{S}} = B_m$ and we are done.

The following observation asserts that the $\mathbb{S}$-based filtration of B satisfies the conditions (i)$-$(vi) of Theorem 3.4.

Observation 3.7 *Let* **K** *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$. *If* $B_{\mathbb{S}}$ *is the* $\mathbb{S}$-*based filtration of* B, *then:*

(i)   *If* $\alpha_j \in B_{\mathbb{S}}$ *then* $f_{\mathbb{S}}(\neg\alpha_i) = f_{\mathbb{S}}(\neg\alpha_j)$ *or* $f_{\mathbb{S}}(\neg\alpha_i) \not\subset \|\neg\alpha_j\|$, *for all* $\alpha_i \in B_{\mathbb{S}}$.
(ii) *For all* $\alpha_l \in B \cap \mathbf{K}$ *there is some* $\alpha_k \in B_{\mathbb{S}} \cap \mathbf{K}$ *such that* $f_{\mathbb{S}}(\neg\alpha_k) \subseteq \|\neg\alpha_l\|$.
(iii) *For any set of sentences* C, *if every subset* X *of* **K** *implies some element of* B *if and only if* X *implies some element of* C, *then for all* $\beta_i \in (B_N \cap \mathbf{K})_{\mathbb{S}}$ *there is some* $\varsigma_i \in (C_N \cap \mathbf{K})_{\mathbb{S}}$ *such that* $\|\beta_i\| = \|\varsigma_i\|$.
(iv) $B_{\mathbb{S}} \cap \mathbf{K} = (B \cap \mathbf{K})_{\mathbb{S}}$.
(v) $(B_N)_{\mathbb{S}} = (B_{\mathbb{S}})_N$.
(vi) $\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i) = \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$.

Now, having in mind Theorem 3.4 and the above observation, we are naturally led to suggest the following definition for the *system of spheres-based multiple contractions*:

Definition 3.8 (System of Spheres-based Multiple Contractions)
Let **K** be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The $\mathbb{S}$-*based multiple contraction on* **K** is the multiple contraction function $\div_{\mathbb{S}}$ defined by:

$$\mathbf{K}\div_{\mathbb{S}}B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{, if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{, if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}$$

for any set of sentences B and where $B_{\mathbb{S}}$ is the $\mathbb{S}$-based filtration of B. An operator $\div$ on **K** is a *system of spheres-based multiple contraction on* **K** if and only if there is a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, such that $\mathbf{K}\div B = \mathbf{K}\div_{\mathbb{S}}B$, for any set of sentences B.

We are finally in a position to present the following main result of the present paper which confirms that every system of spheres-based multiple contraction is a partial meet multiple contraction.

Theorem 3.9 *Let* **K** *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *Then the* $\mathbb{S}$-*based multiple contraction on* **K** *is a partial meet multiple contraction on* **K**.

The next observation shows that the above defined system of spheres-based multiple contractions are indeed a generalization of Grove's system of spheres-based (singleton) contractions.

Fig. 4 Illustration
of Example 3.11
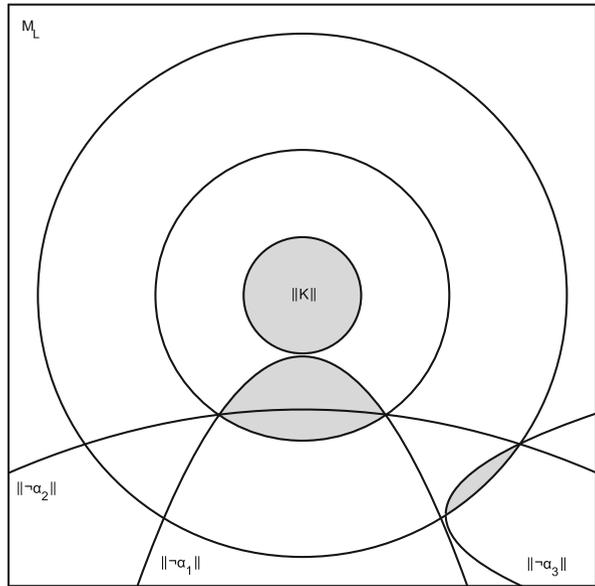
Observation 3.10 *Let* **K** *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *If* $\div_\mathbb{S}$ *is the* $\mathbb{S}$-*based multiple contraction and* $-_\mathbb{S}$ *is the* $\mathbb{S}$-*based (singleton) contraction, then:*

1. *For any set of sentences B such that* $B \cap Cn(\emptyset) = \emptyset$ *it holds that:*

$$\mathbf{K} \div_\mathbb{S} B = \bigcap_{\alpha_i \in B_\mathbb{S}} \mathbf{K} \div_\mathbb{S} \{\alpha_i\} = \bigcap_{\alpha_i \in B_\mathbb{S}} \mathbf{K} -_\mathbb{S} \alpha_i,$$

   where $B_\mathbb{S}$ *is the* $\mathbb{S}$-*based filtration of B.*
2. *The identity* $\mathbf{K} \div_\mathbb{S} \{\alpha\} = \mathbf{K} -_\mathbb{S} \alpha$ *is satisfied for any sentence* $\alpha \in \mathcal{L}$.

3.3.1 An Example of the Construction

Now, with the goal of clarifying the idea behind Definitions 3.5 and 3.8, we present an example illustrating the obtention of a set $\mathbf{K} \div_\mathbb{S} B$:

Example 3.11 Consider Fig. 4. Let **K** be a belief set and α_1, α_2 and α_3 be sentences such that the sets $\|\mathbf{K}\|$, $\|\neg\alpha_1\|$, $\|\neg\alpha_2\|$ and $\|\neg\alpha_3\|$ can be represented as in that figure. Furthermore, consider the set $B = \{\alpha_1, \alpha_2, \alpha_3\}$ and let $\mathbb{S}$ be the system of spheres centred on $\|\mathbf{K}\|$ whose spheres are $\mathcal{M}_\mathcal{L}$ and the sets of possible worlds represented by each of the circles present in the mentioned figure.

In what follows we obtain the $\mathbb{S}$-based filtration $B_\mathbb{S}$ of the set B and we clarify how the set $\mathbf{K} \div_\mathbb{S} B$ is obtained, where $\div_\mathbb{S}$ is the $\mathbb{S}$-based multiple contraction on **K**. So, according to the process of construction of $B_\mathbb{S}$ described

in Definition 3.5 and based on the information present in the mentioned figure we obtain that:

- $B_0 = B \cap Cn(\emptyset) = \emptyset$.
- $(B \setminus Cn(\emptyset))/_\backsim$ is composed of two equivalence classes (and, therefore, $m = 2$): $C_1 = \{\alpha_3\}$ and $C_2 = \{\alpha_1, \alpha_2\}$.
- $B_1 = C_1'' = C_1' = C_1 = \{\alpha_3\}$
- $C_2' = \{\alpha_1\}$ (notice that $\alpha_2 \notin C_2'$ because $f_\mathbb{S}(\neg\alpha_3) \subset \|\neg\alpha_2\|$).
- $C_2'' = C_2' = \{\alpha_1\}$.
- $B_\mathbb{S} = B_2 = B_1 \cup C_2'' = \{\alpha_1, \alpha_3\}$.

Hence, according to Definition 3.8, in the above conditions we have that $\mathbf{K} \dotdiv_\mathbb{S} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_\mathbb{S}} f_\mathbb{S}(\neg\alpha_i)\right)\right) = Th(\|\mathbf{K}\| \cup f_\mathbb{S}(\neg\alpha_1) \cup f_\mathbb{S}(\neg\alpha_3))$, that is, the set $\mathbf{K} \dotdiv_\mathbb{S} B$ is given by the intersection of all the worlds in the shaded region of Fig. 4.

4 Conclusions

We have introduced a new class of multiple contraction functions—the *S.S.-based multiple contractions*—which are a generalization to the case of contraction by (possibly non-singleton) sets of sentences of Grove's S.S.-based contraction functions. In particular we have shown that every S.S.-based multiple contraction is a partial meet multiple contraction.

In a paper which is currently in preparation we shall provide an axiomatic characterization for the newly introduced S.S.-based multiple contractions.

Having in mind that in the singleton contraction level the S.S.-based contractions are transitively relational partial meet contractions (in fact, those two classes of singleton contraction function are identical and constitute a subclass of the class of partial meet contractions), we are led to expect that the S.S.-based multiple contraction functions that we have presented are *transitively relational partial meet multiple contractions (TRPMMCs)*.

Nevertheless, in the present paper we have only focused in assuring that the new class of multiple contractions proposed was a subclass of the class of partial meet multiple contractions. We leave it as subject of future work to investigate if (as it is suggested by the analogy with the singleton case) it holds, additionally, that the S.S.-based multiple contractions are indeed TRPMMCs.

Appendix: Proofs

Previous Lemmas

Lemma A.1 [10] *Let $\mathbf{K}$ be a belief set and B and C be sets of sentences. Then the following statements hold:*

(i) $B \cap Cn(\emptyset) \neq \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K} \perp B} = \emptyset$.
(ii) $B \cap \mathbf{K} = \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K} \perp B} = \{\emptyset\}$.
(iii) $B \cap Cn(\emptyset) = \emptyset$ *and* $B \cap \mathbf{K} \neq \emptyset$ *if and only if* $\mathbb{W}_{\mathbf{K} \perp B} \neq \emptyset$ *and* $\emptyset \notin \mathbb{W}_{\mathbf{K} \perp B}$.
(iv) $\mathbf{K} \perp B = \mathbf{K} \perp C$ *iff* $\mathbb{W}_{\mathbf{K} \perp B} = \mathbb{W}_{\mathbf{K} \perp C}$.

Lemma A.2 [10] *Let $\mathbf{K}$ be a belief set and B and C be sets of sentences. Then:*

(i) *If $\alpha_l \in B$ and $\alpha_l \notin B_N$ then there is some $\alpha_k \in B_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_l\|$.*
(ii) *If every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C, then for all $\beta_i \in B_N \cap \mathbf{K}$ there is some $\varsigma_i \in C_N \cap \mathbf{K}$ such that $\|\beta_i\| = \|\varsigma_i\|$.*

Lemma A.3 [10] *Let $\mathbf{K}$ be a belief set, B be a set of sentences such that $B \cap Cn(\emptyset) = \emptyset$ and $f : \{\|\alpha\| : \alpha \in \mathcal{L} \text{ and } \not\vdash \neg\alpha\} \to \mathcal{P}(\mathcal{M}_{\mathcal{L}})$ be a function such that if $\alpha \in \mathcal{L}$ and $\not\vdash \neg\alpha$ then $\emptyset \neq f(\|\alpha\|) \subseteq \|\alpha\|$. If $B_f \subseteq B$ is such that:*

(i) *If $\alpha_j \in B_f$ then $f(\|\neg\alpha_k\|) = f(\|\neg\alpha_j\|)$ or $f(\|\neg\alpha_k\|) \not\subset \|\neg\alpha_j\|$, for all $\alpha_k \in B_f$,*
(ii) *For all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_f \cap \mathbf{K}$ such that $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$,*

then $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_f \cap \mathbf{K}} f(\|\neg\alpha_i\|)\} \neq \emptyset$ and $\bigcup \{\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_f \cap \mathbf{K}} f(\|\neg\alpha_i\|)\} = \bigcup_{\alpha_i \in B_f \cap \mathbf{K}} f(\|\neg\alpha_i\|)$.

Lemma A.4 *Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. Assume $C = \{\alpha_1, \ldots, \alpha_n\} \subseteq \mathcal{L} \setminus Cn(\emptyset)$, with $n \geq 1$ is a non-empty finite set of sentences such that $\mathbb{S}_{\neg\alpha_1} = \cdots = \mathbb{S}_{\neg\alpha_n}$, and, for each $\alpha_j \in C$ let C_{α_j} be the set defined by $C_{\alpha_j} = \{\alpha_k \in C : f_{\mathbb{S}}(\neg\alpha_k) \subset f_{\mathbb{S}}(\neg\alpha_j)\}$.*
 If $\alpha_j \in C$ and $C_{\alpha_j} \neq \emptyset$ then there is some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$.

Proof In what follows #S denotes the number of elements of S.
 Let $\alpha_j \in C$ be such that $C_{\alpha_j} \neq \emptyset$. It follows immediately from the definition of C_{α_i} that for any $\alpha_r \in C$, if $\alpha_r \in C_{\alpha_j}$ then $C_{\alpha_r} \subset C_{\alpha_j}$. Now we proceed according to the following (finite) sequence of steps:

Step 1:   Pick some $\alpha_{j1} \in C_{\alpha_j}$. Then $C_{\alpha_{j1}} \subset C_{\alpha_j}$, hence $0 \leq \#C_{\alpha_{j1}} < n - 1$.

Step 2:   If $\#C_{\alpha_{j1}} = 0$ we have that $C_{\alpha_{j1}} = \emptyset$ and then we can take $\alpha_l = \alpha_{j1}$ and this finishes the proof. Otherwise, pick some $\alpha_{j2} \in C_{\alpha_{j1}}$. Then $C_{\alpha_{j2}} \subset C_{\alpha_{j1}}$, hence $0 \leq \#C_{\alpha_{j2}} < n - 2$.

$$\vdots$$

Step i: If $\#C_{\alpha_{j(i-1)}} = 0$ we have that $C_{\alpha_{j(i-1)}} = \emptyset$ and then we can take $\alpha_l = \alpha_{j(i-1)}$ and this finishes the proof. Otherwise, pick some $\alpha_{ji} \in C_{\alpha_{j(i-1)}}$. Then $C_{\alpha_{ji}} \subset C_{\alpha_{j(i-1)}}$, hence $0 \leq \#C_{\alpha_{ji}} < n - i$.

$$\vdots$$

Suppose after $n - 2$ steps the above described process has not finished yet. Furthermore, assume that $\#C_{\alpha_{j(n-2)}} \neq 0$. Since, by construction, $0 \leq \#C_{\alpha_{j(n-2)}} < n - (n - 2) = 2$, we can conclude that $C_{\alpha_{j(n-2)}}$ is a singleton set. Then, at step $n - 1$ we pick the only element of $C_{\alpha_{j(n-2)}}$, which we denote by $\alpha_{j(n-1)}$, and from the fact that $C_{\alpha_{j(n-1)}} \subset C_{\alpha_{j(n-2)}}$ it must be the case that $C_{\alpha_{j(n-1)}} = \emptyset$. Hence, after at most n steps the process must have finished. That is, at most at step n we must find some $\alpha_l \in C_{\alpha_j}$ such that $C_{\alpha_l} = \emptyset$ as we wished to prove.   □

Lemma A.5 *Let* $\mathbf{K}$ *be a belief set,* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$ *and* B *be a set of sentences such that* $B \cap Cn(\emptyset) = \emptyset$. *If* $B_{\mathbb{S}}$ *is a subset of* B *such that:*

(i)   *If* $\alpha_j \in B_{\mathbb{S}}$ *then* $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$ *or* $f_{\mathbb{S}}(\neg\alpha_k) \not\subset \|\neg\alpha_j\|$, *for all* $\alpha_k \in B_{\mathbb{S}}$.
(ii)   *For all* $\alpha_l \in B \cap \mathbf{K}$ *there is some* $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ *such that* $f_{\mathbb{S}}(\neg\alpha_m) \subseteq \|\neg\alpha_l\|$.

then $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$, *and* $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$.

Proof Assume B is a set of sentences satisfying $B \cap Cn(\emptyset) = \emptyset$ and let $B_{\mathbb{S}}$ be a subset of B such that conditions (i) and (ii) in the statement of the observation are satisfied.

Consider the function $f : \{\|\alpha\| : \alpha \in \mathcal{L}$ and $\not\vdash \neg\alpha\} \to \mathcal{P}(\mathcal{M}_{\mathcal{L}})$ defined by $f(\|\alpha\|) = f_{\mathbb{S}}(\alpha)$. Then it follows from Definition 2.2 that if $\alpha \in \mathcal{L}$ and $\not\vdash \neg\alpha$ then $\emptyset \neq f(\|\alpha\|) \subseteq \|\alpha\|$.

Notice also that, it follows from the hypothesis that $B_{\mathbb{S}} \subseteq B$ and that (i) if $\alpha_j \in B_{\mathbb{S}}$ then $f(\|\neg\alpha_k\|) = f(\|\neg\alpha_j\|)$ or $f(\|\neg\alpha_k\|) \not\subset \|\neg\alpha_j\|$, for all $\alpha_k \in B_{\mathbb{S}}$; (ii) for all $\alpha_l \in B \cap \mathbf{K}$ there is some $\alpha_m \in B_{\mathbb{S}} \cap \mathbf{K}$ such that $f(\|\neg\alpha_m\|) \subseteq \|\neg\alpha_l\|$.

Hence, by Lemma A.3 we can conclude that $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} \neq \emptyset$ and $\bigcup\{\mathcal{W} \in \mathbb{W}_{\mathbf{K}\perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\} = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$.   □

Proofs

Proof of Theorem 3.4 Let $\div_{\mathbb{S}}$ be the operation defined by

$$\mathbf{K} \div_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) & \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K} & \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases},$$

where, for any finite set of sentences B satisfying $B \cap Cn(\emptyset) = \emptyset$, the set $B_{\mathbb{S}}$ is a subset of B such that conditions (i)−(vi) in the statement of the theorem are satisfied. We must show that $\dot{-}_{\mathbb{S}}$ is a partial meet multiple contraction.

According to Observation 2.7, in order to prove that, it is enough to show that there is a propositional package selection function h for $\|\mathbf{K}\|$ such that $\mathbf{K} \dot{-}_{\mathbb{S}} B = Th \left(\|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K} \perp B}) \right) \right)$, for all sets B.

Let h be such that $h(\mathbb{W}_{\mathbf{K} \perp B}) = \emptyset$ if $B \cap Cn(\emptyset) \neq \emptyset$, and $h(\mathbb{W}_{\mathbf{K} \perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i)\}$ if $B \cap Cn(\emptyset) = \emptyset$.

We start by showing that h is a function. Assume B and C are sets of sentences such that $\mathbb{W}_{\mathbf{K} \perp B} = \mathbb{W}_{\mathbf{K} \perp C}$. If $\mathbb{W}_{\mathbf{K} \perp B} = \mathbb{W}_{\mathbf{K} \perp C} = \emptyset$ then, according to Lemma A.1, $B \cap Cn(\emptyset) \neq \emptyset$ and $C \cap Cn(\emptyset) \neq \emptyset$ and, consequently, it follows from the definition of h that $h(\mathbb{W}_{\mathbf{K} \perp B}) = h(\mathbb{W}_{\mathbf{K} \perp C}) = \emptyset$. Now we consider the case $\mathbb{W}_{\mathbf{K} \perp B} = \mathbb{W}_{\mathbf{K} \perp C} \neq \emptyset$. Again by Lemma A.1 we have that $B \cap Cn(\emptyset) = C \cap Cn(\emptyset) = \emptyset$. Therefore, in order to demonstrate that $h(\mathbb{W}_{\mathbf{K} \perp B}) = h(\mathbb{W}_{\mathbf{K} \perp C})$ we must show that $\{\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i)\} = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp C} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in C_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i)\}$. To do that we start by proving that $\bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i) = \bigcup_{\alpha_i \in C_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i)$. It follows from Lemma A.1 that $\mathbf{K} \perp B = \mathbf{K} \perp C$. Therefore, according to [9, Observation 1.39], it holds that every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C.

Therefore, from (iii) we can conclude that $\bigcup_{\alpha_i \in (B_N \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_i) = \bigcup_{\alpha_i \in (C_N \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_i)$. On the other hand it follows immediately from the definition of normalization of a set of sentences and from (v) that $(B_N \cap \mathbf{K})_{\mathbb{S}} = ((B \cap \mathbf{K})_{\mathbb{S}})_N$ and $(C_N \cap \mathbf{K})_{\mathbb{S}} = ((C \cap \mathbf{K})_{\mathbb{S}})_N$.

Hence, we obtain that $\bigcup_{\alpha_i \in ((B \cap \mathbf{K})_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg \alpha_i) = \bigcup_{\alpha_i \in ((C \cap \mathbf{K})_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg \alpha_i)$. So, (vi) gives us that $\bigcup_{\alpha_i \in (B \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_i) = \bigcup_{\alpha_i \in (C \cap \mathbf{K})_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_i)$. Finally, we can conclude from (iv) that $\bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i) = \bigcup_{\alpha_i \in C_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i)$, as required.

To show that h is a propositional package selection function for $\|\mathbf{K}\|$ it only remains to prove that: (a) $h(\mathbb{W}_{\mathbf{K} \perp B}) \subseteq \mathbb{W}_{\mathbf{K} \perp B}$, and (b) if $\mathbb{W}_{\mathbf{K} \perp B} \neq \emptyset$ then $h(\mathbb{W}_{\mathbf{K} \perp B}) \neq \emptyset$.

(a) follows immediately from the definition of h. For (b), assume $\mathbb{W}_{\mathbf{K} \perp B} \neq \emptyset$. Then, by Lemma A.1, we have that $B \cap Cn(\emptyset) = \emptyset$. So, according to the definition of h, $h(\mathbb{W}_{\mathbf{K} \perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i)\}$ and (since $B_{\mathbb{S}} \subseteq B$ and conditions (i) and (ii) are satisfied) we can conclude from Lemma A.5 that $h(\mathbb{W}_{\mathbf{K} \perp B}) \neq \emptyset$.

Now it only remains to show that $\mathbf{K} \dot{-}_{\mathbb{S}} B = Th \left(\|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K} \perp B}) \right) \right)$ for any set of sentences B.

Case 1, $B \cap Cn(\emptyset) \neq \emptyset$. Then, on the one hand, according to the definition of $\dot{-}_{\mathbb{S}}$, it holds that $\mathbf{K} \dot{-}_{\mathbb{S}} B = \mathbf{K}$. And, on the other hand, it follows from the definition of h that $Th \left(\|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K} \perp B}) \right) \right) = Th(\|\mathbf{K}\|)$. Hence, noticing that, according to [7, Property (1), p. 158], since we are assuming that Cn is compact, it holds that $Th(\|\mathbf{K}\|) = \mathbf{K}$, we can conclude that $\mathbf{K} \dot{-}_{\mathbb{S}} B = Th \left(\|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K} \perp B}) \right) \right)$, as required.

Case 2, $B \cap Cn(\emptyset) = \emptyset$. Then $h(\mathbb{W}_{\mathbf{K} \perp B}) = \{\mathcal{W} \in \mathbb{W}_{\mathbf{K} \perp B} : \mathcal{W} \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg \alpha_i)\}$ and, according to the definition of $\dot{-}_{\mathbb{S}}$, we have that $\mathbf{K} \dot{-}_{\mathbb{S}} B = Th \left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg \alpha_i) \right) \right)$. Now, in order to show that the

required identity holds we will prove that, indeed, the following stronger identity is satisfied: $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})\right)$.

First of all we must note that if $\alpha_i \notin \mathbf{K}$ then $f_{\mathbb{S}}(\neg\alpha_i) \subseteq \|\mathbf{K}\|$. Hence $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)\right)$. Finally, observing that it follows from Lemma A.5 that $\bigcup h(\mathbb{W}_{\mathbf{K}\perp B}) = \bigcup_{\alpha_i \in B_{\mathbb{S}} \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha_i)$, we can conclude that $\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right) = \|\mathbf{K}\| \cup \left(\bigcup h(\mathbb{W}_{\mathbf{K}\perp B})\right)$ as we wished to prove.   □

Proof of Observation 3.6 (a) follows immediately from the fact that, for all $l \in \{1, \ldots, m\}$, given $\alpha_i, \alpha_j \in C'_l$, it holds that if $f_{\mathbb{S}}(\neg\alpha_i) \nsubseteq f_{\mathbb{S}}(\neg\alpha_j)$ then $f_{\mathbb{S}}(\neg\alpha_i) \not\subset \|\neg\alpha_j\|$ (because $\mathbb{S}_{\neg\alpha_i} = \mathbb{S}_{\neg\alpha_j}$).

On the other hand, (b) is an immediate consequence of the fact that, if $m > 1$ and $l \in \{2, \ldots, m\}$ then from $\alpha_i \in C_l$ and $\alpha_j \in B_{l-1}$ it follows that $\mathbb{S}_{\neg\alpha_i} \subset \mathbb{S}_{\neg\alpha_j}$ and, consequently, $f_{\mathbb{S}}(\neg\alpha_j) \neq f_{\mathbb{S}}(\neg\alpha_i)$.   □

Proof of Observation 3.7

   (i)   If $B \subseteq Cn(\emptyset)$ it follows trivially, since by Definition 3.5, $B_{\mathbb{S}} = \emptyset$. Let $B \setminus Cn(\emptyset) \neq \emptyset$ and $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m$, $B_1, \ldots, B_m$ be the subsets of B described in Definition 3.5. Let $\alpha_j \in B_{\mathbb{S}}$ and assume by reductio that there is some $\alpha_i \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_i) \neq f_{\mathbb{S}}(\neg\alpha_j)$ and $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$. From the latter condition and Definition 2.2 it follows that $\mathbb{S}_{\neg\alpha_j} \subseteq \mathbb{S}_{\neg\alpha_i}$.
         Case 1, $\mathbb{S}_{\neg\alpha_j} = \mathbb{S}_{\neg\alpha_i}$. Then, there is some class C_l such that $\alpha_j, \alpha_i \in C_l$. Since $\alpha_j, \alpha_i \in B_{\mathbb{S}}$ we must have $\alpha_j, \alpha_i \in C'_l$. But, in that case, from $f_{\mathbb{S}}(\neg\alpha_i) \neq f_{\mathbb{S}}(\neg\alpha_j)$ and $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$ we conclude that $\alpha_j \notin C''_l$ (cf. Observation 3.6-(a)), which contradicts $\alpha_j \in B_{\mathbb{S}}$.
         Case 2.2, $\mathbb{S}_{\neg\alpha_j} \subset \mathbb{S}_{\neg\alpha_i}$. Then, there are two classes C_l and C_m, with $l < m$ such that $\alpha_i \in C_l$ and $\alpha_j \in C_m$. Since $\alpha_i \in B_{\mathbb{S}}$ we have that $\alpha_i \in C''_l$. Hence, from $C''_l \subseteq B_l \subseteq B_{m-1}$ we can conclude that $\alpha_i \in B_{m-1}$. But then, from $f_{\mathbb{S}}(\neg\alpha_i) \neq f_{\mathbb{S}}(\neg\alpha_j)$ and $f_{\mathbb{S}}(\neg\alpha_i) \subset \|\neg\alpha_j\|$ it follows that $\alpha_j \notin C'_m$ (cf. Observation 3.6-(b)), which contradicts $\alpha_j \in B_{\mathbb{S}}$.
   (ii) We will prove that the following condition (stronger) than (ii) holds: If $\alpha_j \in B \setminus B_{\mathbb{S}}$, then there is some $\alpha_l \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$.
         Let $\alpha_j \in B \setminus B_{\mathbb{S}}$ and let $C_1, \ldots, C_m, C'_1, \ldots, C'_m, C''_1, \ldots, C''_m$, $B_1, \ldots, B_m$ be the subsets of B considered in the process of construction of the set $B_{\mathbb{S}}$ described in Definition 3.5.
         We will consider separately the two possibilities $\alpha_j \in C_1$ or $\alpha_j \in C_n$, with $n > 1$.
         Case 1, $\alpha_j \in C_1$. Then, since by construction $C_1 = C'_1$, $\alpha_j \in C'_1$. It follows from $\alpha_j \notin B_{\mathbb{S}}$ that $\alpha_j \notin C''_1$. Hence, there is some $\alpha_k \in C'_1$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subset f_{\mathbb{S}}(\neg\alpha_i)$. We have that for any $\alpha_r, \alpha_s \in C'_1$ it holds that $\mathbb{S}_{\neg\alpha_r} = \mathbb{S}_{\neg\alpha_s}$. Then, according to Lemma A.4, there is some $\alpha_l \in C'_1$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset f_{\mathbb{S}}(\neg\alpha_j)$ and for all $\alpha_m \in C'_1$ it holds that $f_{\mathbb{S}}(\neg\alpha_m) \not\subset f_{\mathbb{S}}(\neg\alpha_l)$. Hence $\alpha_l \in C''_1$ and, consequently,

$\alpha_l \in B_{\mathbb{S}}$. Finally we note that from $f_{\mathbb{S}}(\neg\alpha_l) \subset f_{\mathbb{S}}(\neg\alpha_j)$ it follows immediately that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$ as it was required.

Case 2, $\alpha_j \in C_n$, with $1 < n \leq m$. From $\alpha_j \notin B_{\mathbb{S}}$ it follows that $\alpha_j \notin C_n''$, and we have to consider the two possibilities $\alpha_j \notin C_n'$ or $\alpha_j \in C_n'$.

Case 2.1, $\alpha_j \notin C_n'$. Then there is some $\alpha_l \in B_{n-1}$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$. Since $B_{n-1} \subseteq B_{\mathbb{S}}$ we have that $\alpha_l \in B_{\mathbb{S}}$.

On the other hand, it follows from the construction of B_{n-1} and C_n that $\mathbb{S}_{\neg\alpha_j} \subset \mathbb{S}_{\neg\alpha_l}$ and, therefore, according to Definition 2.2 it holds that $f_{\mathbb{S}}(\neg\alpha_l) \cap \mathbb{S}_{\neg\alpha_j} = \emptyset$. Hence, since $\emptyset \neq f_{\mathbb{S}}(\neg\alpha_j) \subseteq \mathbb{S}_{\neg\alpha_j}$, we can conclude that $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$, and this finishes the proof.

Case 2.2, $\alpha_j \in C_n'$. Then, since $\alpha_j \notin C_n''$, there is some $\alpha_k \in C_n'$ such that $f_{\mathbb{S}}(\neg\alpha_k) \subset f_{\mathbb{S}}(\neg\alpha_j)$. Reasoning as we did in Case 1 above we can conclude that there is some $\alpha_l \in B_{\mathbb{S}}$ such that $f_{\mathbb{S}}(\neg\alpha_l) \subset \|\neg\alpha_j\|$ and $f_{\mathbb{S}}(\neg\alpha_l) \neq f_{\mathbb{S}}(\neg\alpha_j)$ as we wished to prove.

(iii) Assume every subset X of $\mathbf{K}$ implies some element of B if and only if X implies some element of C. According to Lemma A.2-(ii) it holds that for all $\beta_i \in B_N \cap \mathbf{K}$ there is some $\varsigma_i \in C_N \cap \mathbf{K}$ such that $\|\beta_i\| = \|\varsigma_i\|$. Hence it follows trivially from the definition of $\mathbb{S}$-based filtration of a set of sentences that for all $\beta_i \in (B_N \cap \mathbf{K})_{\mathbb{S}}$ there is some $\varsigma_i \in (C_N \cap \mathbf{K})_{\mathbb{S}}$ such that $\|\beta_i\| = \|\varsigma_i\|$, and we are done.

(iv) and (v) Follow immediately from the definitions of normalization and of $\mathbb{S}$-based filtration of a set of sentences.

(vi) That $\bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i) \subseteq \bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$ follows immediately from $(B_{\mathbb{S}})_N \subseteq B_{\mathbb{S}}$. For the converse inclusion, assume $M \in \bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)$ then there is some $\alpha_j \in B_{\mathbb{S}}$ such that $M \in f_{\mathbb{S}}(\neg\alpha_j)$. If $\alpha_j \in (B_{\mathbb{S}})_N$ it follows immediately that $M \in \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$ and we are done. Now assume $\alpha_j \notin (B_{\mathbb{S}})_N$. It follows from Lemma A.2-(i) that there is some $\alpha_k \in (B_{\mathbb{S}})_N$ such that $\|\neg\alpha_k\| \subset \|\neg\alpha_j\|$. Hence we have that $\alpha_j, \alpha_k \in B_{\mathbb{S}}$ and $f_{\mathbb{S}}(\neg\alpha_k) \subset \|\neg\alpha_j\|$. So we obtain from (i) (already proven above) that $f_{\mathbb{S}}(\neg\alpha_k) = f_{\mathbb{S}}(\neg\alpha_j)$. Therefore $M \in f_{\mathbb{S}}(\neg\alpha_k)$ and we can conclude that $M \in \bigcup_{\alpha_i \in (B_{\mathbb{S}})_N} f_{\mathbb{S}}(\neg\alpha_i)$. □

Proof of Theorem 3.9 Follows immediately from Theorem 3.4 and Observation 3.7. □

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