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Construction of system of spheres-based transitively relational partial meet multiple contractions: An impossibility result

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ABSTRACT

In this paper we show that, contrary to what is the case in what concerns contractions by a single sentence, there is not a system of spheres-based construction of multiple contractions which generates each and every transitively relational partial meet multiple contraction.

Before proving the general result, we consider the class of system of spheres-based multiple contractions introduced in [17,5] and show that this class neither subsumes nor is subsumed by the class of transitively relational partial meet multiple contractions.

Furthermore, we propose two system of spheres-based constructions of multiple contractions which generate (only) transitively relational partial meet multiple contractions. Therefore we can conclude that, although it is impossible to obtain a system of spheres-based definition of all the transitively relational partial meet multiple contractions, there are classes of system of spheres-based multiple contractions which are subsumed by the class of transitively relational partial meet multiple contractions.

1. Introduction

In the belief revision literature the *partial meet contraction*, introduced in the seminal paper [1], constitutes the standard model of belief contraction functions. The main purpose of such framework — which is commonly known as AGM contraction — is modelling the dynamics of the set of beliefs of an agent. More precisely, the AGM model essentially provides a definition for a class of *contraction functions* that receive a belief set — a logically closed set of sentences — and a sentence, and return a belief set which is a subset of the original one that does not contain the received sentence. A possible worlds semantics for partial meet contractions (i.e. a characterization of those functions in terms of possible worlds) was proposed in [8,12]. Furthermore, based on such semantics, Grove [8] presented a way of defining contraction function by means of a system of spheres — the so-called *system of spheres-based contractions*. In that same paper it was shown that such class of functions coincides with the class of *transitively relational partial meet contractions*, a special (proper) subclass of *partial meet contractions* which was also introduced in [1].

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A natural generalization of the above mentioned contraction functions is to allow the epistemic input to be *a set of sentences* rather than a *single* sentence. In this case, the new belief set is a subset of the original belief set that does not contain the set of sentences given as input. In [7], Fuhrmann and Hansson remarked that two kinds of such functions may be considered, namely, on the one hand, *package contractions* which are operations that account for the removal of all the sentences of a given set from the original belief set and, on the other hand, *choice contractions* which are functions that, receiving as input a belief set and a set of sentences, return as output a belief set which is contained in the input belief set and does not contain at least one of the sentences of the received set. In that same paper, it was shown that when the set to be contracted is finite, the choice contraction can be reduced to a contraction by a single sentence (namely by the conjunction of all the elements from that set). Since in this paper we study only contractions by finite sets of sentences, we will only consider package contractions, which from now on we shall call *multiple contractions*. We shall often use the expression *singleton contraction* to refer to an operation of contraction by a single sentence.

Multiple contraction is an important type of belief change that has been studied extensively in the literature since the early '90s [14,7,23,16,18,5,6,19]. Nowadays its significance has increased even further, partially due to the emergence of intelligent agents (softbots, robots, etc.) which typically receive input from more than one source simultaneously (for example, through several sensors). We note that such scenarios are outside the scope of classical belief change operators that can only handle changes by a single input at a time.

The generalization of partial meet singleton contractions was originally presented in [9,10] where the class of *partial meet multiple contractions* was introduced. Afterwards, in [17,18] the possible worlds semantics for such functions was provided (which can be seen as a generalization to the multiple contraction level of the possible worlds semantics for partial meet singleton contraction). In [17,5] the class of *spheres filtration-based multiple contractions* was presented,[1] which is a generalization to the multiple contraction case of Grove's class of system of spheres-based singleton contractions. Later, in [6,19] three different axiomatic characterizations for this class of multiple contractions were presented.

Furthermore, in [17,5] it was shown that every spheres filtration-based multiple contraction is a partial meet multiple contraction. However, in order to verify whether there is a complete analogy between the proposed class of spheres filtration-based multiple contractions and Grove's class of system of spheres-based (singleton) contractions it remains to analyse if the class of spheres filtration-based multiple contractions coincides with the class of transitively relational partial meet multiple contractions (as it is the case regarding their singleton contraction counterparts).

In this paper we will show that this does not hold and, more generally, we will prove that it is in fact impossible to obtain a system of spheres-based definition of multiple contraction functions which encompasses all the transitively relational partial meet multiple contractions.

More precisely, we will start by showing that the class of spheres filtration-based multiple contractions (proposed in [17,5]) neither contains nor is contained in the class of transitively relational partial meet multiple contractions. This conclusion naturally leads to question if there is a (different) system of spheres-based definition of multiple contractions which generates precisely the class of transitively relational partial meet multiple contractions or at least a class of functions which is contained in it. We shall address this issue as follows. First we will prove that it is not possible to construct all the transitively relational partial meet multiple contractions by means of a system of spheres-based method. Afterwards we will present two system of spheres-based methods for constructing multiple contractions and show that both those methods give rise (only) to transitively relational partial meet multiple contractions.

The paper is organized as follows: In Section 2 we recall all the background needed for the rest of the paper. In Section 3 we show, by means of two counterexamples that neither all spheres filtration-based multiple contractions are transitively relational partial meet multiple contractions nor the converse holds. Afterwards, in Section 4 we show that, in general, using a system of spheres-based approach it is not possible to obtain all the transitively relational partial meet multiple contractions. Then, in Section 5 we propose two system of spheres-based definitions of partial meet multiple contraction functions, which give rise (only) to transitively relational partial meet multiple contractions. Subsequently, in Section 6 we briefly describe some works that can be found in the literature and whose topics are closely related to the one of the present paper. Finally, in Section 7 we summarize the main contributions of the paper and identify some open questions related to the topic of this paper.

1.1. Basic notations and conventions

We will assume a language $\mathcal{L}$ that is built from a finite set of propositional symbols and the Boolean connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$. We shall make use of a consequence operation *Cn* that takes sets of sentences to sets of sentences and which satisfies the standard Tarskian properties, namely *inclusion, monotony* and *iteration*. Furthermore we will assume that *Cn* satisfies *supraclassicality, compactness* and *deduction*. We will sometimes use $Cn(\alpha)$ for $Cn(\{\alpha\})$, $A \vdash \alpha$ for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$, $A \nvdash \alpha$ for $\alpha \notin Cn(A)$, $\nvdash \alpha$ for $\alpha \notin Cn(\emptyset)$. The letters $\alpha, \alpha_i, \beta, \ldots$ (except for γ) will be used to denote sentences. $\top$ stands for an arbitrary tautology and $\bot$ for an arbitrary contradiction. $A, A_i, B, \ldots$ shall denote sets of sentences of $\mathcal{L}$. $\mathbf{K}$ is reserved to represent a set of sentences that is closed under logical consequence (i.e. $\mathbf{K} = Cn(\mathbf{K})$) — such a set is called

[1] In [17,5] these functions were designated by *system of spheres-based multiple contractions*, however here it is convenient to use this less general denomination.

a *belief set* or *theory*. We shall denote the set of all theories of $\mathcal{L}$ by $\mathcal{T}_{\mathcal{L}}$ and the set of all consistent complete theories (i.e. maximal consistent subsets) of $\mathcal{L}$ by $\mathcal{M}_{\mathcal{L}}$. We will use the expression *possible world* (or just *world*) to designate an element of $\mathcal{M}_{\mathcal{L}}$. Given a possible world W, we shall denote by $\bigwedge W$ the conjunction of all literal in W. $\mathcal{M}, \mathcal{N}_i, \mathcal{W}, \ldots$ (except for $\mathcal{L}$ and $\mathcal{P}$), shall be used to denote subsets of $\mathcal{M}_{\mathcal{L}}$. Such sets are called *propositions*. Given a set of sentences R, the set consisting of all the possible worlds that contain R is denoted by $\|R\|$. The elements of $\|R\|$ are the R-worlds. $\|\varphi\|$ is an abbreviation of $\|\{\varphi\}\|$ and the elements of $\|\varphi\|$ are the φ-worlds. To any set of possible worlds $\mathcal{V}$ we associate a belief set $Th(\mathcal{V})$ given by $Th(\mathcal{V}) = \bigcap \mathcal{V}$ – under the convention that $\bigcap \emptyset = \mathcal{L}$. $\mathbb{M}, \mathbb{N}_i, \mathbb{W}, \ldots$ shall be used to denote subsets of $\mathcal{P}(\mathcal{M}_{\mathcal{L}})$.

Now we introduce the following useful definitions and notations.

Definition 1.1. Let $\mathbb{W}$ be a (possibly empty) collection of sets of worlds. A *hitting set* for $\mathbb{W}$ is any subset $\mathcal{Q}$ of $\bigcup \mathbb{W}$ that intersects every element of $\mathbb{W}$. $\mathcal{Q}$ is a *minimal hitting set* for $\mathbb{W}$ if no other hitting set for $\mathbb{W}$ is a proper subset of $\mathcal{Q}$. The set of all minimal hitting sets for $\mathbb{W}$ is denoted by $\mathscr{H}(\mathbb{W})$.

Notation 1.2. For any set of sentences A, we denote by $\langle A \rangle$ the following collection of sets of worlds:

$$\langle A \rangle = \{\|\neg\alpha\| : \alpha \in A\}.$$

In what follows the above introduced notations shall be essentially used in order to refer to the set $\mathscr{H}(\langle B \cap \mathbf{K} \rangle)$,[2] where $\mathbf{K}$ is a belief set and B is a set of sentences. This set is formed by (all) the minimal hitting sets for the set $\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}$ (i.e. the set formed by the sets of $\neg\alpha_i$-worlds, for (all) sentences $\alpha_i \in B \cap \mathbf{K}$). For convenience we present here the following explicit definition of that set:

Definition 1.3. (See [17,18].) Let $\mathbf{K}$ be a belief set and B be a set of sentences. The set $\mathscr{H}(\langle B \cap \mathbf{K} \rangle)$ is the subset of $\mathcal{P}\left(\bigcup\{\|\neg\alpha_i\| : \alpha_i \in B \cap \mathbf{K}\}\right)$ such that $\mathcal{W} \in \mathscr{H}(\langle B \cap \mathbf{K} \rangle)$ if and only if:

1. $\mathcal{W} \cap \|\neg\alpha_i\| \neq \emptyset$, for all $\alpha_i \in B \cap \mathbf{K}$.
2. If $\mathcal{W}' \subset \mathcal{W}$ then there is some $\alpha_j \in B \cap \mathbf{K}$ such that $\mathcal{W}' \cap \|\neg\alpha_j\| = \emptyset$.

The following example clarifies the concept introduced in the above definition.

Example 1.4. Let $\mathcal{L}$ be the propositional language that is built from the two propositional symbols p and q. Define the worlds W_0, W_1, W_2 and W_3 as follows:

$$\begin{array}{ll} W_0 = Cn(p \wedge q) & W_2 = Cn(\neg p \wedge \neg q) \\ W_1 = Cn(p \wedge \neg q) & W_3 = Cn(\neg p \wedge q) \end{array}$$

Let $\mathbf{K}$ be the theory $\mathbf{K} = W_0$, consider the sentences $\alpha = q$, $\beta = p$ and $\delta = p \vee \neg q$ and set $B = \{\alpha, \beta, \delta\}$.

In these circumstances, $\alpha, \beta, \delta \in \mathbf{K}$ and $\|\neg\alpha\| = \{W_1, W_2\}$, $\|\neg\beta\| = \{W_2, W_3\}$ and $\|\neg\delta\| = \{W_3\}$.

Therefore the set $\mathscr{H}(\langle B \cap \mathbf{K} \rangle)$ is composed by all the subsets of $\{W_1, W_2, W_3\}$ which contain one element of each one of the sets $\|\neg\alpha\|$, $\|\neg\beta\|$ and $\|\neg\delta\|$ and whose proper subsets do not fulfil that requirement. Hence, it holds that

$$\mathscr{H}(\langle B \cap \mathbf{K} \rangle) = \{\{W_1, W_3\}, \{W_2, W_3\}\}.$$

2. Background

In this section we recall the main definitions and results which we shall need in the remaining of the paper.

2.1. Singleton contraction

In what follows we recall two of the most well known models of singleton contraction (i.e. contraction of a belief set $\mathbf{K}$ by a *single* sentence α).

[2] In [17,18] such set is denoted by $\mathbb{W}_{\mathbf{K} \perp B}$.

2.1.1. Partial meet contractions

A Partial Meet Contraction function [3,1] is built upon a selection from the maximal subsets of $\mathbf{K}$ that do not imply the sentence to be contracted. In what follows we present the formal definition of such functions.

We start by recalling the basic concepts necessary for the definition of the partial meet contractions introduced in [1]. Given a belief set $\mathbf{K}$ and a set of sentences B, the *remainder set* of $\mathbf{K}$ by B is the set of maximal subsets of $\mathbf{K}$ that do not imply any element of B and is denoted by $\mathbf{K}\perp B$. Its elements are the *remainders* (of $\mathbf{K}$, by B) [2]. It is also convenient to notice here that, according to [2, Observation 2.2], since we are assuming that the consequence operation Cn is compact, it holds that $\mathbf{K}\perp B \neq \emptyset$ if and only if $B \cap Cn(\emptyset) = \emptyset$.

For any sentence α, $\mathbf{K}\perp\alpha$ is an abbreviation of $\mathbf{K}\perp\{\alpha\}$ and is called the *remainder set* of $\mathbf{K}$ by α.

Definition 2.1. (See [1].) Let $\mathbf{K}$ be a belief set. A *selection function* for $\mathbf{K}$ is a function γ such that for all sentences α: if $\mathbf{K}\perp\alpha \neq \emptyset$ then $\emptyset \neq \gamma(\mathbf{K}\perp\alpha) \subseteq \mathbf{K}\perp\alpha$, and if $\mathbf{K}\perp\alpha = \emptyset$, then $\gamma(\mathbf{K}\perp\alpha) = \{\mathbf{K}\}$.

An operation $-$ is a *partial meet contraction* on $\mathbf{K}$ if and only if there is a selection function γ for $\mathbf{K}$ such that for all sentences α: $\mathbf{K} - \alpha = \bigcap \gamma(\mathbf{K}\perp\alpha)$.

In the following definition we present a special class of partial meet contractions.

Definition 2.2. (See [1].) A selection function γ for a belief set $\mathbf{K}$ is *transitively relational* over $\mathbf{K}$ if and only if there is a reflexive transitive relation $\sqsubseteq$ over $\bigcup_{\varepsilon\in\mathcal{L}} \mathbf{K}\perp\varepsilon$ such that for all $\alpha \in \mathcal{L} \setminus Cn(\emptyset)$:

$$\gamma(\mathbf{K}\perp\alpha) = \{B \in \mathbf{K}\perp\alpha : B' \sqsubseteq B \text{ for all } B' \in \mathbf{K}\perp\alpha\}.$$

The above formula is called the *marking-off identity* and $\sqsubseteq$ is the *marking-off relation*.

A partial meet contraction function $-$ is *transitively relational* over $\mathbf{K}$ if and only if it is determined by some selection function that is so.

2.1.2. Possible worlds semantics for partial meet contractions

In this subsubsection we present the possible worlds semantics for partial meet multiple contraction that was presented in [12]. We start by recalling the concept of *propositional selection function*.

Definition 2.3. (See [12].) Let $\mathcal{M}$ be a proposition. A *propositional selection function* for $\mathcal{M}$ is a function f such that for all sentences α: (1) $f(\|\alpha\|) \subseteq \|\alpha\|$, (2) If $\|\alpha\| \neq \emptyset$ then $f(\|\alpha\|) \neq \emptyset$ and (3) If $\mathcal{M} \cap \|\alpha\| \neq \emptyset$, then $f(\|\alpha\|) = \mathcal{M} \cap \|\alpha\|$.

Now we are in a position to present the characterization of the partial meet contractions in terms of possible worlds.

Observation 2.4. *(See [8,12].) Let* $\mathbf{K}$ *be a belief set. An operation* $-$ *on* $\mathbf{K}$ *is a* partial meet contraction *if and only if there is a propositional selection function* f *for* $\|\mathbf{K}\|$ *such that for all sentences* α*:* $\mathbf{K} - \alpha = Th(\|\mathbf{K}\| \cup f(\|\neg\alpha\|))$.

2.1.3. System of spheres-based contractions

Now we recall the definitions of a *system of spheres* and of the *system of spheres-based contractions* presented in [8].

Definition 2.5. (See [8].) Let $\mathcal{X}$ be a subset of $\mathcal{M}_{\mathcal{L}}$. A *system of spheres* (abrev. *S.S.*) centred on $\mathcal{X}$ is a collection $\mathbb{S}$ of subsets of $\mathcal{M}_{\mathcal{L}}$, i.e. $\mathbb{S} \subseteq \mathcal{P}(\mathcal{M}_{\mathcal{L}})$, that satisfies the following conditions:

($\mathbb{S}1$) $\mathbb{S}$ is totally ordered with respect to set inclusion; that is, if $\mathcal{U}, \mathcal{V} \in \mathbb{S}$, then $\mathcal{U} \subseteq \mathcal{V}$ or $\mathcal{V} \subseteq \mathcal{U}$.
($\mathbb{S}2$) $\mathcal{X} \in \mathbb{S}$, and if $\mathcal{U} \in \mathbb{S}$ then $\mathcal{X} \subseteq \mathcal{U}$.
($\mathbb{S}3$) $\mathcal{M}_{\mathcal{L}} \in \mathbb{S}$ (and so it is the largest element of $\mathbb{S}$).
($\mathbb{S}4$) For every $\varphi \in \mathcal{L}$, if there is any element in $\mathbb{S}$ intersecting $\|\varphi\|$ then there is also a smallest element in $\mathbb{S}$ intersecting $\|\varphi\|$.[3]

The elements of $\mathbb{S}$ are called *spheres*. For any consistent sentence $\varphi \in \mathcal{L}$, the smallest sphere in $\mathbb{S}$ intersecting $\|\varphi\|$ is denoted by $\mathbb{S}_\varphi$ and $f_{\mathbb{S}}(\varphi)$ denotes the set consisting of the φ-worlds *closest* to $\mathcal{X}$, i.e.,

$$f_{\mathbb{S}}(\varphi) = \|\varphi\| \cap \mathbb{S}_\varphi.$$

[3] Notice that when $\mathcal{L}$ is a propositional language with finitely many propositional symbols (as we assume to be the case throughout this paper), this condition is vacuous.

Definition 2.6. (See [8].) Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The $\mathbb{S}$*-based contraction on* $\mathbf{K}$ is the contraction operation $-_{\mathbb{S}}$ defined, for any $\varphi \in \mathcal{L}$, by:

$$\mathbf{K}-_{\mathbb{S}}\varphi = \begin{cases} Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\varphi)), & \text{if} \nvdash \varphi \\ \mathbf{K}, & \text{if} \vdash \varphi. \end{cases}$$

An operation $-$ on $\mathbf{K}$ is a *system of spheres-based contraction on* $\mathbf{K}$ if and only if there is a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, such that, for all sentences $\varphi \in \mathcal{L}$, $\mathbf{K} - \varphi = \mathbf{K}-_{\mathbb{S}}\varphi$.

To close this subsection we remind that Grove [8] has shown that the class of *system of spheres-based contractions* coincides with the class of *transitively relational partial meet contractions*.

Observation 2.7. *(See [8].) Let* $\mathbf{K}$ *be a belief set and* $-$ *be a (singleton) contraction function on* $\mathbf{K}$*. Then* $-$ *is a system of spheres-based contraction if and only if it is a transitively relational partial meet contraction.*

2.2. Multiple contraction

Below we recall two models of multiple contraction (i.e. contraction of a belief set $\mathbf{K}$ by a set of sentences B) which result of generalizing the models presented in the previous subsection.

2.2.1. Partial meet multiple contractions

The *partial meet multiple contractions* are a generalization of the partial meet contraction functions to the case of contractions by (possibly non-singleton) sets.

Definition 2.8. (See [9,7].) Let $\mathbf{K}$ be a belief set. A *package selection function* for $\mathbf{K}$ is a function γ such that for all sets of sentences B: if $\mathbf{K}\perp B \neq \emptyset$, then $\emptyset \neq \gamma(\mathbf{K}\perp B) \subseteq \mathbf{K}\perp B$, and if $\mathbf{K}\perp B = \emptyset$ then $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$.

An operation $\div$ is a *partial meet multiple contraction* (abrev. *PMMC*) on $\mathbf{K}$ if and only if there is some package selection function γ for $\mathbf{K}$, such that for all sets of sentences B: $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$.

We close this subsubsection with the following definition which introduces the class of *transitively relational partial meet multiple contraction* that is a the natural generalization to the case of multiple contraction of the class of functions presented in Definition 2.2.

Definition 2.9. Let $\mathbf{K}$ be a belief set. An operation $\div$ is a *transitively relational partial meet multiple contraction* (abrev. *TRPMMC*) on $\mathbf{K}$ if and only if there is a transitive relation $\sqsubseteq$ on the set $\bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$ such that, for all sets of sentences B, $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$, where γ is a *(transitively) relational package selection function* defined by $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$ if $B \cap Cn(\emptyset) \neq \emptyset$, and by the *marking-off identity*:

$$\gamma(\mathbf{K}\perp B) = \{X \in \mathbf{K}\perp B : X' \sqsubseteq X \text{ for all } X' \in \mathbf{K}\perp B\},$$

if $B \cap Cn(\emptyset) = \emptyset$.

2.2.2. Possible worlds semantics for partial meet multiple contractions

The possible worlds semantics for partial meet multiple contraction was presented in [17,18]. In this subsubsection we recall the concepts and results (as well as some immediate consequences of some results) of the mentioned references which we will need further ahead.

We start by recalling the following result concerning the composition of the set $\mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ (cf. Definition 1.3).

Observation 2.10. *(See [17,18].) Let* $\mathbf{K}$ *be a belief set and* B *be a set of sentences. Then the following statements hold:*

1. $B \cap Cn(\emptyset) \neq \emptyset$ *if and only if* $\mathscr{H}(\langle B \cap \mathbf{K}\rangle) = \emptyset$.
2. $B \cap \mathbf{K} = \emptyset$ *if and only if* $\mathscr{H}(\langle B \cap \mathbf{K}\rangle) = \{\emptyset\}$.
3. $B \cap Cn(\emptyset) = \emptyset$ *and* $B \cap \mathbf{K} \neq \emptyset$ *if and only if* $\mathscr{H}(\langle B \cap \mathbf{K}\rangle) \neq \emptyset$ *and* $\emptyset \notin \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$.

The following lemma shall be used further ahead.

Lemma 2.11. *(See [17,18].) Let* $\mathbf{K}$ *be a belief set,* B *be a finite set of sentences and* $\mathcal{N}$ *be a subset of* $\mathcal{M}_{\mathcal{L}}$*. If* $\mathcal{N} \cap \|\neg\alpha_i\| \neq \emptyset$*, for all* $\alpha_i \in B \cap \mathbf{K}$*, then there is some set* $\mathcal{N}'$ *such that* $\mathcal{N}' \subseteq \mathcal{N}$ *and* $\mathcal{N}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$*.*

Now we present the relation between the sets $\mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ and $\mathbf{K}\perp B$.

Observation 2.12. *(See [17,18].) Let* $\mathbf{K}$ *be a belief set and* B *be a finite set of sentences. Then* $X \in \mathbf{K} \perp B$ *if and only if there is some* $\mathcal{W} \in \mathscr{H}(\langle B \cap \mathbf{K} \rangle)$ *such that* $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$.

Corollary 2.13. *Let* $\mathbf{K}$ *be a belief set and* B *be a set of sentences. Then the function*

$$\begin{aligned} t : \mathscr{H}(\langle B \cap \mathbf{K} \rangle) &\rightarrow \mathbf{K} \perp B \\ \mathcal{W} &\mapsto Th(\|\mathbf{K}\| \cup \mathcal{W}) \end{aligned}$$

is a bijection.

Proof. It follows immediately from Observation 2.12 that t is a (well defined) surjective function. On the other hand, since we are working under the assumption that the underlying language $\mathcal{L}$ is based on a finite alphabet, it holds, moreover, that t is injective and, therefore, it is a bijection. □

Next we recall the concept of propositional package selection function and a couple of results which involve that concept, and which will be necessary further ahead.

Definition 2.14. Let $\mathbf{K}$ be a belief set. A *propositional package selection function* for $\|\mathbf{K}\|$ is a function f such that for all sets of sentences B: (1) $f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle)) \subseteq \mathscr{H}(\langle B \cap \mathbf{K} \rangle)$, and (2) if $\mathscr{H}(\langle B \cap \mathbf{K} \rangle) \neq \emptyset$ then $f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle)) \neq \emptyset$.

Lemma 2.15. *(See [17,18].) If* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$*, i.e.* $\div$ *is such that for all finite sets* B, $\mathbf{K} \div B = \bigcap \gamma(\mathbf{K} \perp B)$*, where* γ *is a package selection function for* $\mathbf{K}$*, then the function* f *defined for all sets* B *by* $f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle)) = \{\mathcal{W} \in \mathscr{H}(\langle B \cap \mathbf{K} \rangle) : Th(\|\mathbf{K}\| \cup \mathcal{W}) \in \gamma(\mathbf{K} \perp B)\}$[4] *is a propositional package selection function for* $\|\mathbf{K}\|$ *such that* $\mathbf{K} \div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle))\right)\right)$*, for all sets* B.

Proof. See the *left-to-right* part of the Proof of [18, Theorem 3.14]. □

Lemma 2.16. *(See [17,18].) If* $\div$ *is an operation such that, for all sets* B, $\mathbf{K} \div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle))\right)\right)$*, where* f *is a propositional package selection function for* $\|\mathbf{K}\|$*, then the function* γ *defined by* $\gamma(\mathbf{K} \perp B) = \{\mathbf{K}\}$ *if* $\mathbf{K} \perp B = \emptyset$*, and* $\gamma(\mathbf{K} \perp B) = \{Th(\|\mathbf{K}\| \cup \mathcal{W}) : \mathcal{W} \in f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle))\}$ *if* $\mathbf{K} \perp B \neq \emptyset$ *is a package selection function for* $\mathbf{K}$*, such that* $\mathbf{K} \div B = \bigcap \gamma(\mathbf{K} \perp B)$*, for all sets* B.

Proof. See the *right-to-left* part of the Proof of [18, Theorem 3.14]. □

Finally, the following observation, which is an immediate consequence of the two above lemmas, provides the characterization of partial meet multiple contractions in terms of possible worlds:

Observation 2.17. *(See [17,18].) Let* $\mathbf{K}$ *be a belief set. An operation* $\div$ *is a partial meet multiple contraction on* $\mathbf{K}$ *if and only if there exists a propositional package selection function* f *for* $\|\mathbf{K}\|$ *such that, for any set of sentences* B*:*

$$\mathbf{K} \div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle))\right)\right).$$

2.2.3. Spheres filtration-based multiple contractions

In this subsection we present the definition of *spheres filtration-based multiple contraction*[5] which was introduced in [17,5] and is a generalization to the case of multiple contraction of Grove's definition of system of spheres-based (singleton) contraction.

Such definition makes use of the concept of $\mathbb{S}$ *filtration* of a set of sentences B,[6] where $\mathbb{S}$ is a system of spheres centred on $\|\mathbf{K}\|$, for some given belief set $\mathbf{K}$:

Definition 2.18. (See [17,5].) Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$.

Consider a set of sentences $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathcal{L}$ such that $B \setminus Cn(\emptyset) \neq \emptyset$.

Denote by $C_1, \ldots, C_m$ the (different) equivalence classes in the quotient set of $(B \setminus Cn(\emptyset))$ by $\backsim$, i.e. $\{C_1, \ldots, C_m\} = (B \setminus Cn(\emptyset))/_{\backsim}$, where $\backsim$ is the equivalence relation on $B \setminus Cn(\emptyset)$ defined by:

$$\forall \alpha, \beta \in B \setminus Cn(\emptyset),\ \alpha \backsim \beta \text{ iff } \mathbb{S}_{\neg\alpha} = \mathbb{S}_{\neg\beta}.$$

Moreover, assume that the equivalence classes $C_1, \ldots, C_m$ are ordered according to the following condition:

---

[4] Notice that this definition for f yields that $f(\mathscr{H}(\langle B \cap \mathbf{K} \rangle)) = \emptyset$, for any set B such that $B \cap Cn(\emptyset) \neq \emptyset$.

[5] Please see Footnote 1.

[6] Here we use the terminology $\mathbb{S}$ *filtration* to designate the concept that was named $\mathbb{S}$*-based filtration* in [17,18].

If $1 \leq i < j \leq m$, $\alpha_r \in C_i$ and $\alpha_s \in C_j$, then $\mathbb{S}_{\neg\alpha_s} \subset \mathbb{S}_{\neg\alpha_r}$.

Now consider the following list of subsets of B:

$$B_0 = B \cap Cn(\emptyset);$$
$$C_1' = C_1;$$
$$C_1'' = \{\alpha_i \in C_1' : \forall \alpha_j \in C_1' \; f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\};$$
$$B_1 = C_1''.$$

Moreover, if $m > 1$ for all $l \in \{2, \ldots, m\}$, let C_l', C_l'' and B_l be the sets defined by:

$$C_l' = \{\alpha_i \in C_l : \forall \alpha_j \in B_{l-1} \; f_{\mathbb{S}}(\neg\alpha_j) \not\subset \|\neg\alpha_i\|\};$$
$$C_l'' = \{\alpha_i \in C_l' : \forall \alpha_j \in C_l' \; f_{\mathbb{S}}(\neg\alpha_j) \not\subset f_{\mathbb{S}}(\neg\alpha_i)\};$$
$$B_l = B_{l-1} \cup C_l''.$$

The set $B_{\mathbb{S}} = B_m$ is the $\mathbb{S}$ *filtration of* B.

If D is a set of sentences such that $D \subseteq Cn(\emptyset)$, then the $\mathbb{S}$ *filtration of* D is the empty set and is denoted by $D_{\mathbb{S}}$, i.e. $D_{\mathbb{S}} = \emptyset$.

Definition 2.19. (See [17,5].) Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$. The $\mathbb{S}$ *filtration-based multiple contraction on* $\mathbf{K}$ is the multiple contraction function $\div_{\mathbb{S}}$ defined by:

$$\mathbf{K} \div_{\mathbb{S}} B = \begin{cases} Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right), & \text{if } B \cap Cn(\emptyset) = \emptyset \\ \mathbf{K}, & \text{if } B \cap Cn(\emptyset) \neq \emptyset \end{cases}$$

for any set of sentences B and where $B_{\mathbb{S}}$ is the $\mathbb{S}$ filtration of B. An operator $\div$ on $\mathbf{K}$ is a *spheres filtration-based multiple contraction on* $\mathbf{K}$ (abrev. *S.F.-bMC*) if and only if there is a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$, such that $\mathbf{K} \div B = \mathbf{K} \div_{\mathbb{S}} B$, for any set of sentences B.

Next, in order to clarify the two definitions above — i.e. the concepts of $\mathbb{S}$ filtration and of spheres filtration-based multiple contraction —, we present an example.

Example 2.20. Let $\mathcal{L}$ be the propositional language that is built from the three propositional symbols p, q and r. Define the worlds $W_0, \ldots, W_7$ as follows:

$$\begin{array}{ll} W_0 = Cn(p \wedge q \wedge r) & W_4 = Cn(\neg p \wedge q \wedge r) \\ W_1 = Cn(p \wedge q \wedge \neg r) & W_5 = Cn(\neg p \wedge q \wedge \neg r) \\ W_2 = Cn(p \wedge \neg q \wedge r) & W_6 = Cn(\neg p \wedge \neg q \wedge r) \\ W_3 = Cn(p \wedge \neg q \wedge \neg r) & W_7 = Cn(\neg p \wedge \neg q \wedge \neg r) \end{array}$$

Set $\mathbf{K} = W_0$ and consider the sentences $\alpha = \neg(\bigwedge W_3)$, $\beta = \neg(\bigwedge W_1 \vee \bigwedge W_2)$, $\delta = \neg(\bigwedge W_2 \vee \bigwedge W_3)$ and $\epsilon = \neg(\bigwedge W_3 \vee \bigwedge W_4)$. Furthermore, let $\mathbb{S}$ be the following system of spheres centred on $\|\mathbf{K}\|$:

$$\mathbb{S} = \{\{W_0\}, \{W_0, W_1\}, \{W_0, W_1, W_2\}, \{W_0, W_1, W_2, W_3, W_4\}, \{W_0, W_1, W_2, W_3, W_4, W_5, W_6, W_7\}\}.$$

Fig. 1 contains a possible graphical representation of the system of spheres $\mathbb{S}$, where the sets $\|\neg\alpha\| = \{W_3\}$, $\|\neg\beta\| = \{W_1, W_2\}$, $\|\neg\delta\| = \{W_2, W_3\}$ and $\|\neg\epsilon\| = \{W_3, W_4\}$ are highlighted.

Let $\div_{\mathbb{S}}$ be the $\mathbb{S}$ filtration-based multiple contraction, and consider the sets $A = \{\alpha, \beta\}$ and $B = \{\beta, \delta, \epsilon\}$. In what follows we illustrate how to obtain the sets $\mathbf{K} \div_{\mathbb{S}} A$ and $\mathbf{K} \div_{\mathbb{S}} B$.

We start by obtaining the former. First of all we need to identify the set $A_{\mathbb{S}}$, i.e. the $\mathbb{S}$ filtration of A.

According to the process described in Definition 2.18 and based on the information exposed in Fig. 1 we obtain that:

- Since $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\alpha}$, the set $(A \setminus Cn(\emptyset))/\backsim$ is composed of the following two equivalence classes: $C_1 = \{\alpha\}$ and $C_2 = \{\beta\}$ (and, therefore, $m = 2$).
- $A_1 = C_1'' = C_1' = C_1 = \{\alpha\}$.
- $C_2' = C_2 = \{\beta\}$ (notice that $f_{\mathbb{S}}(\neg\alpha) \not\subset \|\neg\beta\|$).
- $C_2'' = C_2' = \{\beta\}$.
- $A_{\mathbb{S}} = A_2 = A_1 \cup C_2'' = \{\alpha, \beta\}$.

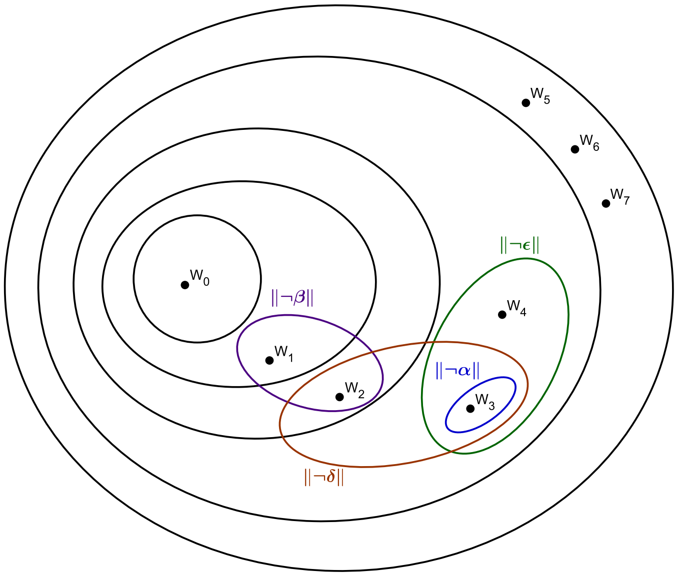

Fig. 1. Schematic representation of the system of spheres $\mathbb{S}$ and the sets $\|\neg\alpha\|$, $\|\neg\beta\|$, $\|\neg\delta\|$ and $\|\neg\epsilon\|$ described in Example 2.20.

Thus, according to Definition 2.19, we have that

$$\mathbf{K} \div_{\mathbb{S}} A = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in A_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\alpha) \cup f_{\mathbb{S}}(\neg\beta)) = Th(\{W_0, W_1, W_3\}).$$

Next we find $B_{\mathbb{S}}$ — the $\mathbb{S}$ filtration of the set B — and we obtain the set $\mathbf{K} \div_{\mathbb{S}} B$. So, proceeding as above, we obtain that:

- Since $\mathbb{S}_{\neg\beta} \subset \mathbb{S}_{\neg\delta} \subset \mathbb{S}_{\neg\epsilon}$, the set $(B \setminus Cn(\emptyset))/_{\backsim}$ is composed of the following three equivalence classes: $C_1 = \{\epsilon\}$, $C_2 = \{\delta\}$ and $C_3 = \{\beta\}$ (and, therefore, $m = 3$).
- $B_1 = C_1'' = C_1' = C_1 = \{\epsilon\}$.
- $C_2' = C_2 = \{\delta\}$ (notice that $f_{\mathbb{S}}(\neg\epsilon) \not\subset \|\neg\delta\|$).
- $C_2'' = C_2' = \{\delta\}$.
- $B_2 = B_1 \cup C_2'' = \{\epsilon, \delta\}$.
- $C_3' = C_3 \setminus \{\beta\} = \emptyset$ (notice that $\beta \notin C_3'$ because $f_{\mathbb{S}}(\neg\delta) \subset \|\neg\beta\|$).
- $C_3'' = C_3' = \emptyset$.
- $B_{\mathbb{S}} = B_3 = B_2 \cup C_3'' = \{\epsilon, \delta\}$.

Hence, according to Definition 2.19, we have that

$$\mathbf{K} \div_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup_{\alpha_i \in B_{\mathbb{S}}} f_{\mathbb{S}}(\neg\alpha_i)\right)\right) = Th(\|\mathbf{K}\| \cup f_{\mathbb{S}}(\neg\epsilon) \cup f_{\mathbb{S}}(\neg\delta)) = Th(\{W_0, W_2, W_3, W_4\}).$$

To close this subsection we remark that three alternative axiomatic characterizations for the class of spheres filtration-based multiple contraction can be found in [6,19]. Moreover, we recall that in [17,5] it was shown that all *spheres filtration-based multiple contractions* are *partial meet multiple contractions*, but the interrelation between the class of *spheres filtration-based multiple contractions* and the class of *transitively relational partial meet multiple contractions* has not been studied there.

3. Spheres filtration-based multiple contractions vs. TRPMMCs

In this section we show that not all spheres filtration-based multiple contractions are TRPMMCs and that, vice versa, not all TRPMMCs are spheres filtration-based multiple contractions.

We start by showing, in the following counterexample, that the spheres filtration-based multiple contraction function considered in Example 2.20 is different from any transitively relational partial meet multiple contraction on **K**.

Counterexample 3.1. Let $\mathcal{L}, W_0, \ldots, W_7, \mathbf{K}, \alpha, \beta, \delta, \epsilon$ and $\mathbb{S}$ be as stated in Example 2.20 and recall that Fig. 1 contains a possible graphical representation of the system of spheres $\mathbb{S}$, where the sets $\|\neg\alpha\| = \{W_3\}$, $\|\neg\beta\| = \{W_1, W_2\}$, $\|\neg\delta\| = \{W_2, W_3\}$ and $\|\neg\epsilon\| = \{W_3, W_4\}$ are highlighted.

Let $\div_{\mathbb{S}}$ be the $\mathbb{S}$ filtration-based multiple contraction. In what follows we show that $\div_{\mathbb{S}}$ is not a TRPMMC. Assume towards contradiction that there is a transitive binary relation $\sqsubseteq$ on the $\mathbf{K}$-remainders, i.e. on $\bigcup_{B \subseteq \mathcal{L}} \mathbf{K} \perp B$, such that $\div_{\mathbb{S}}$ coincides with the TRPMMC which is based on $\sqsubseteq$. Consider the set $A = \{\alpha, \beta\}$.

As we have seen in Example 2.20, it holds that $\mathbf{K} \div_{\mathbb{S}} A = Th(\{W_0, W_1, W_3\})$. Moreover, having in mind Observation 2.12, it is not hard to see that the only remainders in $\mathbf{K} \perp A$ are $Th(\{W_0, W_1, W_3\})$ and $Th(\{W_0, W_2, W_3\})$. Thus, the binary relation $\sqsubseteq$ must be such that:

$$Th(\{W_0, W_2, W_3\}) \sqsubset Th(\{W_0, W_1, W_3\}) \tag{1}$$

Similarly, let B be the set of sentences $\{\beta, \delta, \epsilon\}$.

It was shown in Example 2.20 that $\mathbf{K} \div_{\mathbb{S}} B = Th(\{W_0, W_2, W_3, W_4\})$. Moreover we have

$$\mathbf{K} \perp B = \{Th(\{W_0, W_1, W_3\}), Th(\{W_0, W_2, W_3\}), Th(\{W_0, W_2, W_4\})\}.$$

Thus, the binary relation $\sqsubseteq$ must be such that

$$Th(\{W_0, W_1, W_3\}) \sqsubset Th(\{W_0, W_2, W_3\}) = Th(\{W_0, W_2, W_4\}).^{7}$$

Since this contradicts (1) we can conclude that $\div_{\mathbb{S}}$ is not a transitively relational partial meet multiple contraction, as we wished to show. □

The above counterexample shows that not all spheres filtration-based multiple contractions are TRPMMCs.

Next, by means of another counterexample, we show that the converse inclusion does not hold either, i.e. that not all TRPMMCs are spheres filtration-based multiple contractions. Thus, in the following counterexample we present a transitively relational partial meet multiple contraction function $\div$ on a belief set $\mathbf{K}$ which is different from the $\mathbb{S}$ filtration-based multiple contraction $\div_{\mathbb{S}}$, for any system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$.

Counterexample 3.2. Let $\mathcal{L}$, W_0, W_1, W_2 and W_3 be as stated in Example 1.4.

Define $\mathbf{K}$ to be the theory $\mathbf{K} = W_0$ and let $\div$ be the transitively relational partial meet multiple contraction that is based on the preorder $\sqsubseteq$ on $\bigcup_{B \subseteq \mathcal{L}} \mathbf{K} \perp B$ — the set of all remainders of $\mathbf{K}$ — defined as follows (where, as usual, $\sqsubset$ denotes the strict part of $\sqsubseteq$):

$$\begin{aligned} &Th(\{W_0, W_1, W_2, W_3\}) \sqsubset Th(\{W_0, W_1, W_2\}) \sqsubset Th(\{W_0, W_1, W_3\}) \sqsubset \\ &\quad \sqsubset Th(\{W_0, W_2, W_3\}) \sqsubset Th(\{W_0, W_3\}) \sqsubset Th(\{W_0, W_2\}) \sqsubset \\ &\qquad \sqsubset Th(\{W_0, W_1\}) \sqsubset Th(\{W_0\}) \end{aligned}$$

Now consider the sentences $\alpha = q$, $\beta = p$ and $\delta = p \vee \neg q$. Hence, it holds that $\|\neg\alpha\| = \{W_1, W_2\}$, $\|\neg\beta\| = \{W_2, W_3\}$ and $\|\neg\delta\| = \{W_3\}$. Thus, having in mind Observation 2.12, it follows that

$$\mathbf{K} \perp \alpha = \{Th(\{W_0, W_1\}), Th(\{W_0, W_2\})\} \tag{2}$$

$$\mathbf{K} \perp \beta = \{Th(\{W_0, W_2\}), Th(\{W_0, W_3\})\} \tag{3}$$

$$\mathbf{K} \perp \{\alpha, \delta\} = \{Th(\{W_0, W_1, W_3\}), Th(\{W_0, W_2, W_3\})\}$$

Finally, according to the definition of $\sqsubseteq$, it follows that:

$$\mathbf{K} \div \{\alpha\} = Th(\{W_0, W_1\}) \tag{4}$$

$$\mathbf{K} \div \{\beta\} = Th(\{W_0, W_2\}) \tag{5}$$

$$\mathbf{K} \div \{\alpha, \delta\} = Th(\{W_0, W_2, W_3\}) \tag{6}$$

Now assume towards contradiction that there is a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\| = \{W_0\}$ such that $\div$ is the $\mathbb{S}$ filtration-based multiple contraction. Then it follows from (2) and (4) that $\mathbb{S}_{\bigwedge W_1} \subset \mathbb{S}_{\bigwedge W_2}$. Similarly, from (3) and (5) it follows that $\mathbb{S}_{\bigwedge W_2} \subset \mathbb{S}_{\bigwedge W_3}$. Hence,

$$\mathbb{S} = \{\{W_0\}, \{W_0, W_1\}, \{W_0, W_1, W_2\}, \{W_0, W_1, W_2, W_3\}\}$$

Fig. 2 contains a possible graphical representation of the system of spheres $\mathbb{S}$, where the sets $\|\neg\alpha\|$ and $\|\neg\delta\|$ are highlighted.

However, the $\mathbb{S}$ filtration-based multiple contraction $\div_{\mathbb{S}}$ is such that $\mathbf{K} \div_{\mathbb{S}} \{\alpha, \delta\} = Th(\{W_0, W_1, W_3\})$ and, therefore, according to (6), $\div$ differs from $\div_{\mathbb{S}}$. Thus we can conclude that $\div$ is not a spheres filtration-based multiple contraction. □

[7] Here, by $Th(\{W_0, W_2, W_3\}) = Th(\{W_0, W_2, W_4\})$ we mean $Th(\{W_0, W_2, W_3\}) \sqsubseteq Th(\{W_0, W_2, W_4\})$ and $Th(\{W_0, W_2, W_4\}) \sqsubseteq Th(\{W_0, W_2, W_3\})$.

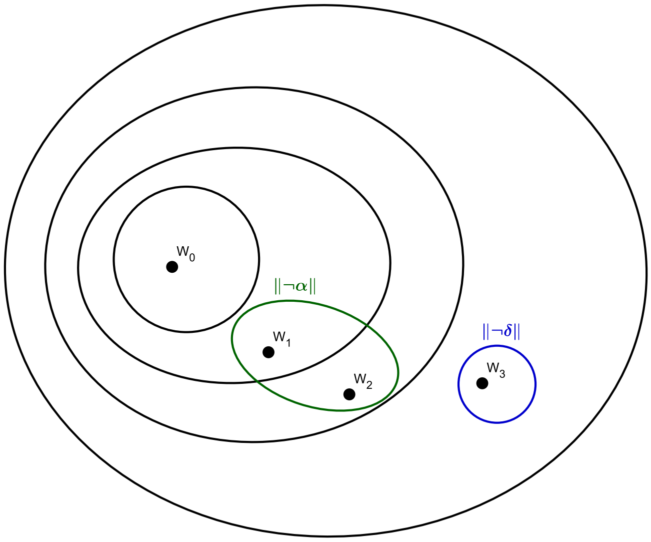

Fig. 2. Schematic representation of the system of spheres $\mathbb{S}$ and the sets $\|\neg\alpha\|$ and $\|\neg\delta\|$ described in Counterexample 3.2.

4. The impossibility of constructing (all the) TRPMMCs by means of systems of spheres

Next we show that not only the construction of spheres filtration-based multiple contractions proposed in [17,5] fails to generate all TRPMMCs (cf. Counterexample 3.2) but, furthermore, any (other) method for constructing multiple contractions which is based on systems of spheres shall equally fail to fulfil such goal.

Indeed, apart from confirming that not all TRPMMCs are S.F.-bMCs, Counterexample 3.2 allows us to conclude that there is no way of defining multiple contractions from systems of spheres which is such that the class of all the thus constructed functions subsumes the class of TRPMMCs.

The reason why there is not a method of constructing multiple contraction functions which is based on systems of spheres and which gives rise to the whole class of TRPMMCs can be informally exposed as follows: Let **K** be a belief set and $\div$ and $\div'$ be two multiple contractions on **K** induced (by the same method) from the systems of spheres $\mathbb{S}$ and $\mathbb{S}'$, respectively. Assume additionally that $\div$ and $\div'$ are such that for any sentence α of $\mathcal{L}$, $\mathbf{K}\div\{\alpha\} = \mathbf{K}-_{\mathbb{S}}\alpha$ and $\mathbf{K}\div'\{\alpha\} = \mathbf{K}-_{\mathbb{S}'}\alpha$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based contraction on **K** and $-_{\mathbb{S}'}$ is the $\mathbb{S}'$-based contraction on **K**. It follows that if $\div$ and $\div'$ agree on all contractions by singleton sets, then it must be the case that $\mathbb{S} = \mathbb{S}'$, and therefore $\div$ and $\div'$ have to agree on contractions by sets of arbitrary size (i.e. $\mathbf{K}\div A = \mathbf{K}\div' A$, for any set of sentences A), no matter what the method of producing $\div$ and $\div'$ from $\mathbb{S}$ (and $\mathbb{S}'$) might be. On the other hand, there exist partial orders $\sqsubset$ and $\sqsubset'$ on the set of remainders of **K**, that agree when restricted to remainders of **K** by singleton sets, but differ otherwise (and, therefore, induce TRPMMCs which are different from each other but, nevertheless, agree on all contractions by singleton sets).

In what follows, we clarify the above statement with the help of Counterexample 3.2.

So, let $\mathcal{L}$, W_0, W_1, W_2, W_3, **K**, $\sqsubseteq$, $\div$, α, β and δ be as described in Counterexample 3.2 and let $\div'$ be the transitively relational partial meet multiple contraction that is based on the transitive relation $\sqsubseteq'$ on the set of all remainders of **K** (i.e. on $\bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$) defined in the following way:[8]

$$\begin{aligned} &Th(\{W_0, W_1, W_2, W_3\}) \sqsubset' Th(\{W_0, W_1, W_2\}) \sqsubset' Th(\{W_0, W_2, W_3\}) \sqsubset' \\ &\sqsubset' Th(\{W_0, W_1, W_3\}) \sqsubset' Th(\{W_0, W_3\}) \sqsubset' Th(\{W_0, W_2\}) \sqsubset' \\ &\sqsubset' Th(\{W_0, W_1\}) \sqsubset' Th(\{W_0\}) \end{aligned}$$

In the above conditions we obtain:

$$\mathbf{K}\div'\{\alpha\} = \mathbf{K}\div\{\alpha\} = Th(\{W_0, W_1\}) \tag{7}$$

$$\mathbf{K}\div'\{\beta\} = \mathbf{K}\div\{\beta\} = Th(\{W_0, W_2\}) \tag{8}$$

$$\mathbf{K}\div\{\alpha, \delta\} = Th(\{W_0, W_2, W_3\}) \tag{9}$$

$$\mathbf{K}\div'\{\alpha, \delta\} = Th(\{W_0, W_1, W_3\}) \tag{10}$$

[8] Notice that the binary relation $\sqsubseteq'$ results from replacing in the binary relation $\sqsubseteq$ the pair $(Th(\{W_0, W_1, W_3\}), Th(\{W_0, W_2, W_3\}))$ by its symmetric $(Th(\{W_0, W_2, W_3\}), Th(\{W_0, W_1, W_3\}))$.

Thus, on the one hand, according to (9) and (10) we have that the multiple contraction functions $\div$ and $\div'$ are different from each other. On the other hand, reasoning as we did in the analogous part of Counterexample 3.2 from (7) and (8) we can conclude that if $\div'$ is a spheres filtration-based multiple contraction then the system of spheres from which it is obtained must be precisely the same system of spheres $\mathbb{S}$ that arose in Counterexample 3.2. In other words, we have that, in spite of the fact that the TRPMMCs $\div$ and $\div'$ are different, the two transitive relations $\sqsubseteq$ and $\sqsubseteq'$ on which they are (respectively) based induce the same system of spheres $\mathbb{S} = \{\{W_0\}, \{W_0, W_1\}, \{W_0, W_1, W_2\}, \{W_0, W_1, W_2, W_3\}\}$.

Now, assume there is a method for constructing all the transitively relational partial meet multiple contractions by means of systems of spheres and let $\mathbf{K} \ominus_{\mathbb{S}} A$ denote the result of contracting the belief set $\mathbf{K}$ by (the set of sentences) A using the above exposed system of spheres $\mathbb{S}$ and applying such method.[9]

It follows that, when considering the result of the multiple contraction of $\mathbf{K}$ by $\{\alpha, \delta\}$ we have that **one and only one** of the following identities holds:

(a) $\mathbf{K} \ominus_{\mathbb{S}} \{\alpha, \delta\} = Th(\{W_0, W_1, W_3\})$
(b) $\mathbf{K} \ominus_{\mathbb{S}} \{\alpha, \delta\} = Th(\{W_0, W_2, W_3\})$
(c) $\mathbf{K} \ominus_{\mathbb{S}} \{\alpha, \delta\} = Th(\{W_0, W_1, W_2, W_3\})$

However, the two transitively relational partial meet multiple contractions that we have presented above $\div$ and $\div'$ are such that, on the one hand, they induce the (same) system of spheres $\mathbb{S}$,[10] and, on the other hand, they return different results for the contraction of $\mathbf{K}$ by $\{\alpha, \delta\}$ (cf. (9) and (10)).

Therefore at most one (and eventually none) of the two TRPMMCs $\div$ and $\div'$ coincides with the multiple contraction function obtained from $\mathbb{S}$ by means of the method that we have assumed to exist. Thus, at least one of those two TRPMMCs cannot be obtained by any system of spheres through such method (independently of how it is defined).

And from all the above we can conclude that there is not any method for constructing (partial meet) multiple contractions which (simultaneously) is based on systems of spheres and generates each and every transitively relational partial meet multiple contraction. That is, there is not a method for constructing multiple contractions which is based on systems of spheres and is such that for any TRPMMC there is a system of spheres such that the multiple contraction obtained from that system of spheres by means of that method coincides with that TRPMMC. This fact is highlighted in the following observation.

Observation 4.1. *There is not a system of spheres-based definition of multiple contractions which generates each and every TRPMMC.*

In what follows we present a formal proof for the fact that we have exposed above by means of a counterexample. More precisely, below we show that no construction of a multiple contraction which is based on a system of spheres can cover the entire spectrum of transitively relational partial meet multiple contractions. The proof is basically a counting argument: loosely speaking, we prove that there are more transitive binary relations $\sqsubseteq$ on remainders than there are systems of spheres, and moreover, that each such relation $\sqsubseteq$ induces a different multiple contraction.

We start by introducing the following notation which shall be used in what follows:

Notation 4.2. We shall denote the set of all remainders of $\mathbf{K}$ not including $\mathbf{K}$ by $\mathbf{K}_{\mathcal{R}}$, i.e. $\mathbf{K}_{\mathcal{R}} = (\bigcup_{B \subseteq \mathcal{L}} \mathbf{K} \perp B) \setminus \mathbf{K}$.[11]

Now we present some results concerning the interrelation among systems of spheres and preorders on the set of remainders.

Theorem 4.3. *Let $\mathbf{K}$ be a theory such that $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$ contains at least two worlds. Then there are more preorders in $\mathbf{K}_{\mathcal{R}}$, than there are systems of spheres centred on $\|\mathbf{K}\|$.*

Proof. Let $\mathbf{K}$ be a theory and let n be the size of $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$. Furthermore, assume that $n > 1$. We must show that there are more preorders in $\mathbf{K}_{\mathcal{R}}$, than there are systems of spheres centred on $\|\mathbf{K}\|$. Given the one-to-one correspondence between systems of spheres centred on $\|\mathbf{K}\|$ and preorders in $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$, it suffices to show that there are less elements in $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$ than there are in $\mathbf{K}_{\mathcal{R}}$. If follows from Corollary 2.13 that there is a one-to-one correspondence between $\mathbf{K}_{\mathcal{R}}$ and non-empty subsets of $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$. Hence there are $2^n - 1$ remainders in $\mathbf{K}_{\mathcal{R}}$ while there are n worlds in $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$. Finally we notice that, since $n > 1$, it holds that $2^n - 1 > n$, and this concludes the proof. □

[9] Notice that, since one of the underlying goals of this paper is to study generalizations of Grove's system of spheres-based construction of (singleton) contraction functions, we are naturally additionally assuming here that $\ominus_{\mathbb{S}}$ is such that, for any sentence α of $\mathcal{L}$, $\mathbf{K} \ominus_{\mathbb{S}} \{\alpha\} = \mathbf{K} -_{\mathbb{S}} \alpha$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction on $\mathbf{K}$.

[10] Here by saying that (a multiple contraction function) $\div$ *induces* the system of spheres $\mathbb{S}$ we mean that $\mathbb{S}$ is a system of spheres such that, for any sentence α of $\mathcal{L}$, it holds that $\mathbf{K} \div \{\alpha\} = \mathbf{K} -_{\mathbb{S}} \alpha$, where $-_{\mathbb{S}}$ is the $\mathbb{S}$-based (singleton) contraction on $\mathbf{K}$.

[11] The reason why $\mathbf{K}$ is not included in the counting arguments that follow is the fact that its position according to $\sqsubseteq$ is irrelevant as far as the induced multiple contraction is concerned; i.e. two preorders on remainders that agree on their projection in $\mathbf{K}_{\mathcal{R}}$, will produce the same multiple contraction even if they disagree in the relative position of $\mathbf{K}$.

Theorem 4.4. *Let* $\mathbf{K}$ *be a theory and* $\sqsubseteq, \sqsubseteq'$ *be two preorders on the remainders of* $\mathbf{K}$ *such that* $\sqsubseteq$ *and* $\sqsubseteq'$ *disagree on their projection on* $\mathbf{K}_{\mathscr{R}}$*. Then the transitively relational partial meet multiple contractions* $\div$ *and* $\div'$ *induced from* $\sqsubseteq$ *and* $\sqsubseteq'$ *respectively, are different.*

Proof. Since $\sqsubseteq$ and $\sqsubseteq'$ disagree on their projection on $\mathbf{K}_{\mathscr{R}}$, we can assume without loss of generality that there are two $\mathbf{K}$-remainders H and H', both different from $\mathbf{K}$, such that $H \sqsubseteq H'$ and $H \not\sqsubseteq' H'$. Define $R_1, \ldots, R_m$ to be the worlds in $(\|H\| \cap \|H'\|) \setminus \|\mathbf{K}\|$, $V_1, \ldots, V_p$ to be the worlds in $\|H\| \setminus \|H'\|$, and $U_1, \ldots, U_q$ to be the worlds in $\|H'\| \setminus \|H\|$. That is, $\|H\| = \|\mathbf{K}\| \cup \{R_1, \ldots, R_m, V_1, \ldots, V_p\}$, and $\|H'\| = \|\mathbf{K}\| \cup \{R_1, \ldots, R_m, U_1, \ldots, U_q\}$. Notice that since both H and H' are different from $\mathbf{K}$ and from each other, $\{V_1, \ldots, V_p\} \neq \emptyset$ and $\{U_1, \ldots, U_q\} \neq \emptyset$. Define $H \times H'$ as follows:

$$
\begin{aligned}
H \times H' = \{ \quad & \neg(\textstyle\bigwedge R_1), \\
& \vdots \\
& \neg(\textstyle\bigwedge R_m), \\
& \neg(\textstyle\bigwedge V_1 \vee \bigwedge U_1), \\
& \vdots \\
& \neg(\textstyle\bigwedge V_1 \vee \bigwedge U_q), \\
& \vdots \\
& \neg(\textstyle\bigwedge V_p \vee \bigwedge U_1), \\
& \vdots \\
& \neg(\textstyle\bigwedge V_p \vee \bigwedge U_q) \quad \}
\end{aligned}
$$

It is not hard to verify that H and H' are the only two remainders of $\mathbf{K}$ with respect to $H \times H'$; i.e. $\mathbf{K}\perp(H \times H') = \{H, H'\}$. Since $H \sqsubseteq H'$, it follows that $\mathbf{K}\div(H \times H') \subseteq H'$. On the other hand, since $H \not\sqsubseteq' H'$ we derive that $\mathbf{K} \div' (H \times H') \nsubseteq H'$. Thus $\div \neq \div'$. □

The following corollary follows immediately from the two previous results:

Corollary 4.5. *Let* $\mathbf{K}$ *be a theory such that* $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$ *contains at least two worlds. Then there are more transitively relational partial meet multiple contractions on* $\mathbf{K}$*, than there are systems of spheres centred on* $\|\mathbf{K}\|$*.*

At this point we must emphasise that it follows from the above theorem that (given a theory $\mathbf{K}$ which is such that $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$ contains at least two worlds) independently of how we define it, any class of multiple contractions (on $\mathbf{K}$) defined by means of system of spheres does not subsume the class of transitively relational partial meet multiple contractions (on $\mathbf{K}$). This fact is more formally stated in the following observation:

Observation 4.6. *Let* $\mathbf{K}$ *be a belief set such that the set* $\mathcal{M}_{\mathcal{L}} \setminus \|\mathbf{K}\|$ *contains more than one possible world,* $\mathbb{S}$ *be an arbitrary system of spheres centred on* $\|\mathbf{K}\|$ *and* $\div^{\mathbb{S}}$ *be a multiple contraction on* $\mathbf{K}$ *whose definition is based on (i.e., depends on)* $\mathbb{S}$*. Then the class of all such contraction function does not contain the class of (all) transitively relational partial meet multiple contractions, i.e.*

$$\{\div : \div \textit{ is a TRPMMC on } \mathbf{K}\} \nsubseteq \{\div^{\mathbb{S}} : \mathbb{S} \textit{ is a S.S. centred on } \|\mathbf{K}\|\}.$$

It is also worth highlighting here that the above result exposes a major difference among singleton contractions and multiple contractions. Namely because in what concerns singleton contractions, as we have already mentioned before, the class of system of spheres-based contractions (cf. Definition 2.6) coincides with the class of transitively relational partial meet contractions (cf. Definition 2.2), which means that it is possible to obtain all the transitively relational partial meet (singleton) contractions by means of a system of spheres-based method for constructing singleton contractions. Indeed, moreover, in [8] the proof for the above mentioned fact is based on the explicit presentation of an one-to-one correspondence between the class of systems of spheres centred on $\|\mathbf{K}\|$ and the class of preorders on the remainders of $\mathbf{K}$, i.e. on the set $\bigcup_{\varepsilon\in\mathcal{L}} \mathbf{K}\perp\varepsilon$, for a given belief set $\mathbf{K}$.

5. Two methods for constructing (some) TRPMMCs by means of systems of spheres

The definition of the spheres filtration-based multiple contractions (see Definition 2.19) is quite intricate and, at first sight, unintuitive. On the other hand, as we have clarified by means of Counterexample 3.1, not all spheres filtration-based multiple contractions are TRPMMCs. These two facts are enough to motivate the search for alternative constructions of multiple contractions from systems of spheres. And, in particular, the latter one raises the issue of finding out a way of defining multiple contraction functions from systems of spheres which generates TRPMMCs.

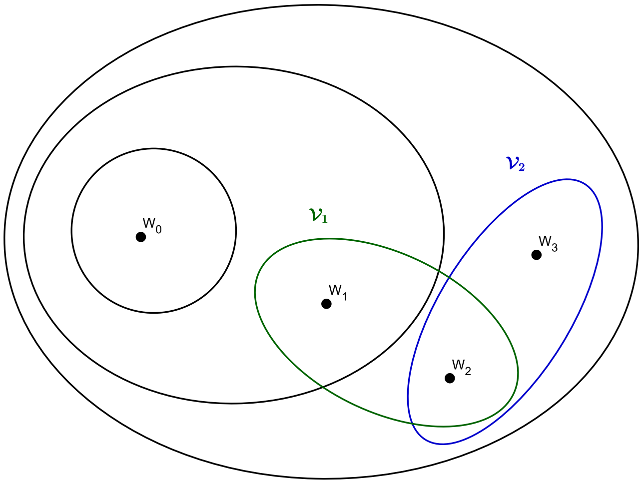

Fig. 3. Schematic representation of the system of spheres $\mathbb{S}$ and the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ described in Example 5.1.

Having the above in mind, in Subsection 5.1 we present two ways of constructing multiple contraction functions by means of systems of spheres (alternative to the one used in Definition 2.19) and afterwards, in Subsection 5.2, we investigate the interrelations among those constructions, the spheres filtration-based multiple contractions and the transitively relational partial meet multiple contractions.

5.1. Two (alternative) system of spheres-based definitions of multiple contraction functions

In what follows we present some definitions of multiple contraction functions on a belief set $\mathbf{K}$ which are based on a system of spheres $\mathbb{S}$ centred on $\|\mathbf{K}\|$.

In order to be in a position for proposing two alternative ways of defining multiple contraction functions by means of the system of spheres $\mathbb{S}$, we need to start by introducing two binary relations on $2^{\mathcal{M}_\mathcal{L}}$. We denote those two relations by $\sqsubseteq^1_\mathbb{S}$ and $\sqsubseteq^2_\mathbb{S}$, and the definitions we propose for each of them are as follows:

- $\mathcal{V}' \sqsubseteq^1_\mathbb{S} \mathcal{V}$ iff for all $W \in \mathcal{V}$ there is some $W' \in \mathcal{V}'$ such that $\mathbb{S}_{\wedge W} \subseteq \mathbb{S}_{\wedge W'}$.
- $\mathcal{V}' \sqsubseteq^2_\mathbb{S} \mathcal{V}$ iff for all $W \in \mathcal{V}$ there is some $W' \in \mathcal{V}'$ such that $\mathbb{S}_{\wedge W} \subseteq \mathbb{S}_{\wedge W'}$ and for all $R \in \mathcal{V}'$ there is some $S \in \mathcal{V}$ such that $\mathbb{S}_{\wedge S} \subseteq \mathbb{S}_{\wedge R}$.

In order to clarify the intuitions behind the above proposed binary relations we now present an example.

Example 5.1. Let $\mathcal{L}$, W_0, W_1, W_2 and W_3 be as stated in Example 1.4 and let $\mathbf{K} = W_0$ and $\mathbb{S}$ be the following system of spheres centred on $\|\mathbf{K}\|$:

$$\mathbb{S} = \{\{W_0\}, \{W_0, W_1\}, \{W_0, W_1, W_2, W_3\}\}.$$

Furthermore, consider the sentences $\alpha = q$ and $\delta = p \vee \neg q$ and set $A = \{\alpha, \delta\}$.

Since $\|\neg\alpha\| = \{W_1, W_2\}$ and $\|\neg\delta\| = \{W_3\}$, it holds that:

$$\mathscr{H}(\langle A \cap \mathbf{K}\rangle) = \{\mathcal{V}_1, \mathcal{V}_2\}, \text{ where } \mathcal{V}_1 = \{W_1, W_3\} \text{ and } \mathcal{V}_2 = \{W_2, W_3\}.$$

Fig. 3 contains a possible graphical representation of the system of spheres $\mathbb{S}$, where the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are highlighted. In these circumstances, $\mathbb{S}_{\wedge W_1} \subset \mathbb{S}_{\wedge W_2} = \mathbb{S}_{\wedge W_3}$. Therefore, having in mind the two above defined binary relations $\sqsubseteq^1_\mathbb{S}$ and $\sqsubseteq^2_\mathbb{S}$ we have that:

$$\mathcal{V}_2 \sqsubseteq^1_\mathbb{S} \mathcal{V}_1 \quad \text{and} \quad \mathcal{V}_1 \sqsubseteq^1_\mathbb{S} \mathcal{V}_2 \tag{11}$$

$$\mathcal{V}_2 \sqsubseteq^2_\mathbb{S} \mathcal{V}_1 \quad \text{and} \quad \mathcal{V}_1 \not\sqsubseteq^2_\mathbb{S} \mathcal{V}_2 \tag{12}$$

The following observation highlights a property that is satisfied by both relations $\sqsubseteq^1_\mathbb{S}$ and $\sqsubseteq^2_\mathbb{S}$.

Observation 5.2. *Let A be any set of sentences and $\mathcal{V}$ be any element of $\mathscr{H}(\langle A \cap \mathbf{K}\rangle)$. If $\mathcal{V} \subseteq \bigcup_{\alpha \in A \cap \mathbf{K}} f_\mathbb{S}(\neg\alpha)$ then for all $\mathcal{V}' \in \mathscr{H}(\langle A \cap \mathbf{K}\rangle)$, $\mathcal{V}' \sqsubseteq^1_\mathbb{S} \mathcal{V}$ and $\mathcal{V}' \sqsubseteq^2_\mathbb{S} \mathcal{V}$.*

At this point we remark that it is natural to require that the property mentioned in the above observation is satisfied by a binary relation to be used in the context of *choosing* the *best* elements of $\mathscr{H}(\langle A \cap \mathbf{K}\rangle)$ (which is precisely the use that we shall give to those two binary relations in what follows).

Now, making use of the two binary relations introduced above we can provide two new constructions of multiple contraction functions from $\mathbb{S}$.

Definition 5.3. Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$, the multiple contraction function $\div^1_{\mathbb{S}}$ is defined, for any set of sentences B, as follows:

$$\mathbf{K} \div^1_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup\{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle), \mathcal{V}' \sqsubseteq^1_{\mathbb{S}} \mathcal{V}\}\right)\right),$$

where $\sqsubseteq^1_{\mathbb{S}}$ is the binary relation on $2^{\mathcal{M}_{\mathcal{L}}}$ defined by:

$\mathcal{V}' \sqsubseteq^1_{\mathbb{S}} \mathcal{V}$ iff for all $W \in \mathcal{V}$ there is some $W' \in \mathcal{V}'$ such that $\mathbb{S}_{\wedge W} \subseteq \mathbb{S}_{\wedge W'}$.

Definition 5.4. Let $\mathbf{K}$ be a belief set and $\mathbb{S}$ be a system of spheres centred on $\|\mathbf{K}\|$, the multiple contraction function $\div^2_{\mathbb{S}}$ is defined, for any set of sentences B, as follows:

$$\mathbf{K} \div^2_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup\{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle), \mathcal{V}' \sqsubseteq^2_{\mathbb{S}} \mathcal{V}\}\right)\right),$$

where $\sqsubseteq^2_{\mathbb{S}}$ is the binary relation on $2^{\mathcal{M}_{\mathcal{L}}}$ defined by:

$\mathcal{V}' \sqsubseteq^2_{\mathbb{S}} \mathcal{V}$ iff for all $W \in \mathcal{V}$ there is some $W' \in \mathcal{V}'$ such that $\mathbb{S}_{\wedge W} \subseteq \mathbb{S}_{\wedge W'}$ and for all $R \in \mathcal{V}'$ there is some $S \in \mathcal{V}$ such that $\mathbb{S}_{\wedge S} \subseteq \mathbb{S}_{\wedge R}$.

5.2. Interrelations among the newly proposed multiple contraction functions, the S.F.-bMCs and the TRPMMCs

Throughout this subsection we assume that $\mathbf{K}$ is a belief set, $\mathbb{S}$ is a system of spheres centred on $\|\mathbf{K}\|$ and $\div^1_{\mathbb{S}}$ and $\div^2_{\mathbb{S}}$ are the multiple contraction functions whose definitions we have introduced in the previous subsection. In what follows we shall investigate the interrelations among those functions, the $\mathbb{S}$ filtration-based multiple contraction, denoted by $\div_{\mathbb{S}}$ (cf. Definition 2.19), and the transitively relational partial meet multiple contractions (cf. Definition 2.9).

We start by showing that the multiple contraction functions $\div^1_{\mathbb{S}}$ and $\div^2_{\mathbb{S}}$ are different from each other.

Observation 5.5. *It does not hold in general that, given a belief set* $\mathbf{K}$ *and a system of spheres* $\mathbb{S}$ *centred on* $\|\mathbf{K}\|$, *the multiple contraction function* $\div^1_{\mathbb{S}}$ *on* $\mathbf{K}$ *is identical to the multiple contraction function* $\div^2_{\mathbb{S}}$ *on* $\mathbf{K}$. *That is, for some belief set* $\mathbf{K}$ *there is a system of spheres* $\mathbb{S}$ *centred on* $\|\mathbf{K}\|$ *and a set of sentences* A, *such that* $\mathbf{K} \div^1_{\mathbb{S}} A \neq \mathbf{K} \div^2_{\mathbb{S}} A$, *i.e.*

$$\div^1_{\mathbb{S}} \neq \div^2_{\mathbb{S}}.$$

Proof. Let $\mathcal{L}$, W_0, W_1, W_2, W_3, $\mathbf{K}$, $\mathbb{S}$ and A be as stated in Example 5.1 and recall that, in such circumstances, it holds that $\mathscr{H}(\langle A \cap \mathbf{K}\rangle) = \{\mathcal{V}_1, \mathcal{V}_2\}$, with $\mathcal{V}_1 = \{W_1, W_3\}$ and $\mathcal{V}_2 = \{W_2, W_3\}$. For convenience recall also that Fig. 3 contains a possible graphical representation of the system of spheres $\mathbb{S}$, where the sets $\mathcal{V}_1$ and $\mathcal{V}_2$ are highlighted.

From (11), (12) and Definitions 5.3 and 5.4 (of the multiple contraction functions $\div^1_{\mathbb{S}}$ and $\div^2_{\mathbb{S}}$) it follows that:

- $\mathbf{K} \div^1_{\mathbb{S}} A = Th(\|\mathbf{K}\| \cup \mathcal{V}_1 \cup \mathcal{V}_2) = Th(\{W_0, W_1, W_2, W_3\})$.
- $\mathbf{K} \div^2_{\mathbb{S}} A = Th(\|\mathbf{K}\| \cup \mathcal{V}_1) = Th(\{W_0, W_1, W_3\})$.

Thus, we can conclude that $\mathbf{K} \div^1_{\mathbb{S}} A \neq \mathbf{K} \div^2_{\mathbb{S}} A$, as we wished to prove. □

Our next goal is to prove that the multiple contraction functions $\div^1_{\mathbb{S}}$ and $\div^2_{\mathbb{S}}$ are TRPMMCs. Having that in mind we start by introducing the concept of *transitively relational propositional package selection function* and a characterization for the TRPMMCs in terms of such concept.

Definition 5.6. A *propositional package selection function* f for $\|\mathbf{K}\|$ is *(transitively) relational* if and only if there is a (transitive) relation $\preceq$ over $\bigcup_{B \subseteq \mathcal{L}} \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ such that for any set of sentences B:

$$f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle)) = \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \mathcal{V}' \preceq \mathcal{V} \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)\}.$$

Theorem 5.7. *Let* $\mathbf{K}$ *be a belief set. An operation* $\div$ *is a transitively relational partial meet multiple contraction on* $\mathbf{K}$ *if and only if there is a transitively relational propositional package selection function* f *for* $\|\mathbf{K}\|$ *such that:*

$$\mathbf{K} \div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle))\right)\right),$$

for any set of sentences B.

Proof. *Left-to-right:* Let $\div$ be a transitively relational partial meet multiple contraction on $\mathbf{K}$. Then there is a transitive relation $\sqsubseteq$ on the set $\bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$ such that $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$, where γ is a transitively relational package selection function defined, for all sets of sentences B, by $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$ if $B \cap Cn(\emptyset) \neq \emptyset$, and by the *marking-off identity*:

$$\gamma(\mathbf{K}\perp B) = \{X \in \mathbf{K}\perp B : X' \sqsubseteq X \text{ for all } X' \in \mathbf{K}\perp B\},$$

if $B \cap Cn(\emptyset) = \emptyset$.

Now let $\preceq$ be the binary relation over $\bigcup_{B\subseteq\mathcal{L}} \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ defined by:

$$\text{For all } \mathcal{V}, \mathcal{W} \in \bigcup_{B\subseteq\mathcal{L}} \mathscr{H}(\langle B \cap \mathbf{K}\rangle),\ \mathcal{V} \preceq \mathcal{W} \text{ iff } Th(\|\mathbf{K}\| \cup \mathcal{V}) \sqsubseteq Th(\|\mathbf{K}\| \cup \mathcal{W}).$$

It follows from the transitivity of $\sqsubseteq$ that $\preceq$ is also a transitive relation. Now let f be defined by $f(\mathbb{W}_{\mathbf{K}\perp B}) = \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \mathcal{V}' \preceq \mathcal{V} \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)\}$, for any set of sentences B.

Having in mind that, according to Observation 2.12, for all $X \in \mathbf{K}\perp B$ there is some $\mathcal{W} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ such that $X = Th(\|\mathbf{K}\| \cup \mathcal{W})$ we can conclude that, for any set of sentences B:

$$\begin{aligned} f(\mathbb{W}_{\mathbf{K}\perp B}) &= \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \mathcal{V}' \preceq \mathcal{V} \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)\} \\ &= \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : Th(\|\mathbf{K}\| \cup \mathcal{V}') \sqsubseteq Th(\|\mathbf{K}\| \cup \mathcal{V}) \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)\} \\ &= \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : Th(\|\mathbf{K}\| \cup \mathcal{V}) \in \gamma(\mathbf{K}\perp B)\} \end{aligned}$$

Thus, according to Lemma 2.15, we have that f is a propositional package selection function for $\|\mathbf{K}\|$ such that $\mathbf{K}\div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle))\right)\right)$, for all sets B. Furthermore, from the above we can also conclude that f is transitively relational, and this finishes the first part of the proof.

Right-to-left: Let $\div$ be such that, for any set of sentences B:

$$\mathbf{K} \div B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle))\right)\right),$$

where f is a transitively relational propositional package selection function defined by:

$$f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle)) = \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \mathcal{V}' \preceq \mathcal{V} \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)\},$$

for some transitive relation $\preceq$ over $\bigcup_{B\subseteq\mathcal{L}} \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$.

Now let $\sqsubseteq$ be the binary relation over $\bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$ defined by:

$$\text{For all } X, Y \in \bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B,\ X \sqsubseteq Y \text{ iff } \mathcal{V} \preceq \mathcal{W},$$

where $\mathcal{V}, \mathcal{W} \in \bigcup_{B\subseteq\mathcal{L}} \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ are such that $X = Th(\|\mathbf{K}\| \cup \mathcal{V})$ and $Y = Th(\|\mathbf{K}\| \cup \mathcal{W})$.

Notice that the relation $\sqsubseteq$ is well defined because it follows from Corollary 2.13 that, for any given remainder $X \in \bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$ there is one and only one $\mathcal{V} \in \bigcup_{B\subseteq\mathcal{L}} \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ such that $X = Th(\|\mathbf{K}\| \cup \mathcal{V})$. Furthermore, it follows from the transitivity of $\preceq$ that $\sqsubseteq$ is also a transitive relation.

To lighten the writing, in all that follows, given a set $X \in \bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$ we shall denote by $\mathcal{V}_X$ the only element of $\bigcup_{B\subseteq\mathcal{L}} \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ such that $X = Th(\|\mathbf{K}\| \cup \mathcal{V}_X)$.

Now let γ be defined by $\gamma(\mathbf{K}\perp B) = \{\mathbf{K}\}$ if $B \cap Cn(\emptyset) \neq \emptyset$, and $\gamma(\mathbf{K}\perp B) = \{X \in \mathbf{K}\perp B : X' \sqsubseteq X \text{ for all } X' \in \mathbf{K}\perp B\}$ if $B \cap Cn(\emptyset) = \emptyset$.[12]

Having in mind that, according to Corollary 2.13, there is a one-to-one correspondence between the sets $\mathbf{K}\perp B$ and $\mathscr{H}(\langle B \cap \mathbf{K}\rangle)$ we can conclude that, for any set of sentences B, such that $B \cap Cn(\emptyset) = \emptyset$:

$$\begin{aligned} \gamma(\mathbf{K}\perp B) &= \{X \in \mathbf{K}\perp B : X' \sqsubseteq X \text{ for all } X' \in \mathbf{K}\perp B\} \\ &= \{X \in \mathbf{K}\perp B : \mathcal{V}' \preceq \mathcal{V}_X \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)\} \\ &= \{X \in \mathbf{K}\perp B : \mathcal{V}_X \in f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle))\} \end{aligned}$$

Thus, it follows from Lemma 2.16 that γ is a transitively relational package selection function for $\mathbf{K}$, such that $\mathbf{K}\div B = \bigcap \gamma(\mathbf{K}\perp B)$, for all sets B. Therefore, $\div$ is a transitively relational partial meet multiple contraction. □

We are now finally in a position to prove that $\div^1_{\mathbb{S}}$ and $\div^2_{\mathbb{S}}$ are TRPMMCs.

[12] Notice that, according to [2, Observation 2.2], $\mathbf{K}\perp B \neq \emptyset$ if and only if $B \cap Cn(\emptyset) = \emptyset$.

Theorem 5.8. *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *Then the multiple contraction* $\div^1_{\mathbb{S}}$ *on* $\mathbf{K}$ *is a transitively relational partial meet multiple contraction.*

Proof. We start by recalling that $\div^1_{\mathbb{S}}$ is defined as follows:

$$\mathbf{K} \div^1_{\mathbb{S}} B = Th\left(\|\mathbf{K}\| \cup \left(\bigcup\{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle), \mathcal{V}' \sqsubseteq^1_{\mathbb{S}} \mathcal{V}\}\right)\right),$$

where $\sqsubseteq^1_{\mathbb{S}}$ is the binary relation on $2^{\mathcal{M}_{\mathcal{L}}}$ defined by:

$\mathcal{V}' \sqsubseteq^1_{\mathbb{S}} \mathcal{V}$ iff for all $W \in \mathcal{V}$ there is some $W' \in \mathcal{V}'$ such that $\mathbb{S}_{\wedge W} \subseteq \mathbb{S}_{\wedge W'}$.

Now let f be such that, for any set of sentences B, $f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle)) = \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle), \mathcal{V}' \sqsubseteq^1_{\mathbb{S}} \mathcal{V}\}$.

It follows from Theorem 5.7 that to prove that $\div^1_{\mathbb{S}}$ is a transitively relational partial meet multiple contraction it is enough to show that f is a transitively relational propositional package selection function for $\|\mathbf{K}\|$.

It follows immediately from its definition that f is a function, i.e. that if B and C are sets of sentences such that $\mathscr{H}(\langle B \cap \mathbf{K}\rangle) = \mathscr{H}(\langle C \cap \mathbf{K}\rangle)$ then $f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle)) = f(\mathscr{H}(\langle C \cap \mathbf{K}\rangle))$.

To prove that f is a propositional package selection function for $\|\mathbf{K}\|$ we must show that for any set of sentences B:

(i) $f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle)) \subseteq \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$.
(ii) If $\mathscr{H}(\langle B \cap \mathbf{K}\rangle) \neq \emptyset$ then $f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle)) \neq \emptyset$.

That condition (i) is satisfied follows immediately from the definition of f. Now we show that (ii) also holds. Assume $\mathscr{H}(\langle B \cap \mathbf{K}\rangle) \neq \emptyset$. Then it must hold that $B \cap Cn(\emptyset) = \emptyset$ and it follows from Lemma 2.11 that there exists a set $\mathcal{V}$ such that $\mathcal{V} \subseteq \bigcup_{\alpha \in B \cap \mathbf{K}} f_{\mathbb{S}}(\neg\alpha)$ and $\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle)$. Thus, according to Observation 5.2, it holds that $f(\mathscr{H}(\langle B \cap \mathbf{K}\rangle)) = \{\mathcal{V} \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle) : \text{ for all } \mathcal{V}' \in \mathscr{H}(\langle B \cap \mathbf{K}\rangle), \mathcal{V}' \sqsubseteq^1_{\mathbb{S}} \mathcal{V}\} \neq \emptyset$.

Finally, that f is transitively relational follows immediately from the fact that the binary relation $\sqsubseteq^1_{\mathbb{S}}$ over $2^{\mathcal{M}_{\mathcal{L}}}$ is transitive. □

Theorem 5.9. *Let* $\mathbf{K}$ *be a belief set and* $\mathbb{S}$ *be a system of spheres centred on* $\|\mathbf{K}\|$. *Then the multiple contraction* $\div^2_{\mathbb{S}}$ *on* $\mathbf{K}$ *is a transitively relational partial meet multiple contraction.*

Proof. A proof for this theorem can be obtained by making only a few small and obvious modifications to the above presented proof for the previous theorem. □

At this point it is worth to remark that both $\div^1_{\mathbb{S}}$ and $\div^2_{\mathbb{S}}$ differ from the multiple contraction function $\div_{\mathbb{S}}$, as it is stated in the following corollary:

Corollary 5.10. *Given a belief set* $\mathbf{K}$ *and a system of spheres* $\mathbb{S}$ *centred on* $\|\mathbf{K}\|$, *in general, none of the multiple contraction functions* $\div^1_{\mathbb{S}}$ *and* $\div^2_{\mathbb{S}}$ *coincides with the* $\mathbb{S}$ *filtration-based multiple contraction* $\div_{\mathbb{S}}$. *That is, it holds that:*

1. $\div^1_{\mathbb{S}} \neq \div_{\mathbb{S}}$.
2. $\div^2_{\mathbb{S}} \neq \div_{\mathbb{S}}$.

Proof. Follows trivially from the fact that $\div^1_{\mathbb{S}}$ and $\div^2_{\mathbb{S}}$ are TRPMMCs (cf. Theorems 5.8 and 5.9) whereas $\div_{\mathbb{S}}$ is not a TRPMMC (cf. Counterexample 3.1). □

6. Related works

In this section we shall make a very brief exposition of some works where multiple change operations are studied and which are related to the investigation reported in the present paper. Furthermore, some of the natural topics of future research (cf. Section 7) which arise from the study presented in this text consist in the investigation of the interconnections as well as of the possible combinations of the main contributions of this paper with some of the works referred in this section.

We start by mentioning paper [14], where an impossibility result concerning multiple contractions has also been presented. In that reference, firstly some intuitive properties are presented concerning the interrelation between the result of the multiple contraction of a belief set $\mathbf{K}$ by a set of sentences $\{\alpha, \beta\}$ and the results of the singleton contraction of $\mathbf{K}$ by α and by $\alpha \vee \beta$. Then, with the central goal of arguing against the recovery postulate, it is shown that, given a belief set $\mathbf{K}$ and a partial meet (singleton) contraction function, it is not possible to define a multiple contraction function such that the presented properties are all (simultaneously) satisfied. We notice that this result is quite different from the one underlying the present paper. In fact, very roughly speaking we can say that on the one hand, Niederée's result states that, given a certain contraction function, there is not any multiple contraction that satisfies certain conditions (concerning its interrelation with the given singleton contraction) while, on the other hand, our result states that there are some (transitively relational partial meet) multiple contraction functions which cannot be generated (in any way) from any (transitively relational partial meet) singleton contraction.

Another work where an impossibility result concerning multiple contraction has been presented is [13]. In that paper Sven Ove Hansson has shown that in general, given a partial meet multiple contraction $\div$ on a belief set $\mathbf{K}$ there is not a partial meet singleton contraction $-$ on $\mathbf{K}$ such that $\div$ can be reconstructed from $-$ in the sense that the following condition holds:

(M-S) For each set of sentences A there is a set B such that

$$\mathbf{K}\div A = \bigcap_{\beta \in B} \mathbf{K} - \beta. \tag{13}$$

Furthermore, Hansson proved also that the converse does not hold either, i.e. that, given a partial meet singleton contraction $-$ in general there is not a partial meet multiple contraction $\div$ such that:

(S-M) For each set of sentences B there is a set A such that equality (13) holds.

However, despite this strong impossibility result, in the mentioned paper some special situations where identified in which conditions (M-S) and/or (S-M) are satisfied.

We note that the impossibility result of the present paper is somehow akin to Hansson's one. In fact Corollary 4.5 yields that there are some transitively relational partial meet multiple contraction functions which cannot be obtained (neither as an intersection of certain singleton contraction results nor in any other way) from a transitively relational partial meet singleton contraction. Furthermore, this result leads to the question of identifying the conditions under which a transitively relational partial meet multiple contraction is such that it can be obtained by means of a system of spheres-based construction of multiple contractions, a question which is somehow similar to the one which underlays Hansson's paper.

Next we highlight some of the main existing works on two kinds of multiple change operations — namely, *set contraction* and *multiple revision* — which differ from but are strongly related with multiple contraction.

Thus, we start by introducing some notation and terminology concerning these kinds of operations. Given a belief set $\mathbf{K}$, generally speaking, by a *multiple revision on* $\mathbf{K}$ we mean a function, say $\oplus$ that associates to each set of sentences B a belief set $\mathbf{K}\oplus B$ which, if possible, (i) contains the set B, (ii) is consistent, and (iii) contains as many sentences of $\mathbf{K}$ as it is possible. On the other hand given a belief set $\mathbf{K}$, in general, the expression *set contraction on* $\mathbf{K}$ designates a function, say $\ominus$ whose output for an arbitrary argument B, denoted $\mathbf{K}\ominus B$, is a logically closed subset of $\mathbf{K}$ which is consistent with the set B (i.e. $(\mathbf{K}\ominus B) \cup B$ is a consistent set of sentences).

The class of *set contractions* was introduced in [21,22], as consisting of the family of functions that satisfied a certain set of eight postulates which, in [23, pp. 530–531], are divided in two sets: six *basic postulates for set contraction* and two *supplementary postulates for set contraction*. In [22,23], a partial meet model for set contraction was introduced and two representation theorems for those functions were obtained, one involving (only) the basic postulates for set contractions (see, e.g., [23, Theorem 3.13]) and another one involving the basic and the supplementary postulates for set contractions (see, e.g., [23, Theorem 3.26]). Additionally, in the mentioned papers, Zhang et al. introduced the *NOP-based contractions*, another class of set contraction functions whose definition is based on the concept of *nicely-ordered partition (NOP)* of a belief set. Moreover, in those references a representation theorem for such functions has been presented (e.g. [23, Theorem 4.19]) which asserts that a function $\ominus$ is a NOP-based contraction if and only if it satisfies the basic and the supplementary postulates for set contraction as well as an additional postulate — the so-called *Limit Postulate* (for set contraction) — which is essentially a principle concerning the relationship between the result of the set contraction by an infinite set and the results of the set contractions by the finite subsets of that infinite set and its precise formulation can be found in, e.g., [23, p. 548].

In [21], Zhang presented the definition of *multiple revision* by means of a list of eight postulates, namely postulates $(\otimes 1)$ – $(\otimes 8)$ listed in [23, Subsection 2.1], which are the result of the generalization to the case of revision by sets of sentences of the basic and the supplementary AGM postulates for revision. Furthermore, yet in [21], it was shown that the two mentioned kinds of operations are interdefinable by means of the following identities, which are analogous to the Levi identity and the Harper identity (that expose the relationship between AGM (singleton) contraction and revision operations):

$$\mathbf{K} \otimes B = Cn((\mathbf{K}\ominus B) \cup B) \tag{Def$\otimes$}$$

$$\mathbf{K}\ominus B = (\mathbf{K} \otimes B) \cap \mathbf{K} \tag{Def$\ominus$}$$

Now, to highlight the interconnection between multiple revision and multiple contraction we recall the generalization of the Levi Identity that was proposed by Sven Ove Hansson in [10,11]. Given a belief set $\mathbf{K}$, an arbitrary set of sentences B

and a multiple contraction $\div$ on $\mathbf{K}$, the first step towards the mentioned generalization was the definition of a set $\neg B$ — called the *negation of* B —, whose elements are, roughly speaking, the (finite) disjunctions of sentences that are negations of elements of B (e.g. [12, Definition 3.35]), and which is such that the result of the multiple contraction of $\mathbf{K}$ by $\neg B$ (i.e. $\mathbf{K}\div\neg B$) is (a subset of $\mathbf{K}$ which is) consistent with B. Afterwards, making use of the above mentioned concept of *negation of a set (of sentences)*, Hansson proposed the following straightforward generalization of the Levi identity for defining of a multiple revision function $\oplus$ on $\mathbf{K}$:

$$\mathbf{K}\oplus B = Cn((\mathbf{K}\div\neg B)\cup B) \tag{14}$$

Next we briefly describe the constructive models for multiple revision and for set contraction that were proposed by Peppas and which are strongly related with the above mentioned ones.

In [15], Peppas proposed a generalization of Grove's system of spheres-based definition of (singleton) revision functions to the case of multiple revision and showed that the resulting class of functions is axiomatically characterized by eight postulates for multiple revision — namely, postulates $(K\oplus 1)-(K\oplus 8)$ listed in [15, p. 365] — which are very similar to the above mentioned multiple revision postulates considered in [23]. The proposed generalization of Grove's construction requires that the systems of spheres considered are *well ranked* (where a system of spheres $\mathbb{S}$ is said to be *well ranked* if it holds that for every nonempty consistent set of sentences B, there exists a smallest sphere $\mathbb{S}_B$ in $\mathbb{S}$ intersecting $\|B\|$ and $\|B\|\cap\mathbb{S}_B = \|\bigcap(\|B\|\cap\mathbb{S}_B)\|$).

More recently, in [16], a constructive model for set contraction was presented. This model is based on a certain kind of binary relations on the set of nonempty sets of sentences — named *comparative possibility preorders* — which are a generalization of the epistemic entrenchment relations (in the sense of being preorders on *sets of sentences* rather than (simply) preorders on sentences). The class of *comparative possibility preorder-based set contractions* is axiomatically characterized by the basic and the supplementary postulates for set contraction. Thus, since the *limit postulate* is not included in its axiomatic characterization, we can conclude that this class of functions subsumes the one consisting of the NOP-based contractions. In this regard we mention yet that in [16], three properties were identified whose fulfilment by a comparative possibility preorder $\leqslant$ is necessary and sufficient to assure that the (corresponding) $\leqslant$-based set contraction satisfies the limit postulate.

Finally, to close this section, we mention reference [4], where Delgrande et al. have proposed two operators of belief change of logic programs: an expansion operator and a revision operator. The operators proposed there are the result of the adaptation to the context of logic programs of some well known operators of belief change developed in the context of classical propositional logic. The main idea behind the adaptations presented was the analogy that was made between a specific logic program and a formula or set of formulas in classical logic and also between sets of *SE models* ([20]) of a logic program and *propositions* (i.e. sets of possible worlds) in the context of propositional logic. We notice yet that, having the mentioned analogy in mind, the revision operator proposed there can be seen as a multiple revision operator.

7. Conclusions and future work

The most relevant outcome here reported is the proof that there is no system of spheres-based definition of multiple contraction functions which generates all the transitively relational partial meet multiple contractions. This fact has been thoroughly exposed in Section 4 and constitutes a noteworthy difference among the multiple and the singleton contractions since, in the case of the latter it is a very well-known result that Grove's class of system of spheres-based (singleton) contractions is identical to the class of transitively relational partial meet (singleton) contractions.

Before providing this general impossibility result, in Section 3 we considered the particular case of the class of spheres filtration-based multiple contractions proposed in [17,5]. There we presented two counterexamples clarifying that neither the class of spheres filtration-based multiple contractions is subsumed by the class of transitively relational partial meet multiple contractions, nor the converse inclusion is satisfied.

Finally, another significant contribution of this paper, is the proposal of two system of spheres-based constructive methods for defining multiple contractions and the proof that the two classes of functions that arise of those methods (differ from the class of spheres filtration-based multiple contractions presented in [17,5] and) are subsumed by the class of transitively relational partial meet multiple contractions.

Next we list some investigation topics that arise naturally from the research reported in the present paper and in some of the related works described in Section 6, and which deserve to be addressed in future studies:

- To find out whether the impossibility result presented in this paper remains valid when the underlying language is built from an infinite set of propositional symbols. The proofs we provided for Theorems 4.3 and 4.4 rely on the assumption that the underlying language $\mathcal{L}$ is based on a finite alphabet. To be more precise, the proof of Theorem 4.3 relies on the fact that there is a bijective relation between the sets $\mathscr{H}(\langle B\cap\mathbf{K}\rangle)$ and $\mathbf{K}\perp B$ (as proven in Corollary 2.13). On the other hand the proof of Theorem 4.4 makes use of the finiteness of the set of (all) possible worlds and, additionally, involves the construction of a set $(H\times H')$ of sentences, which can only be defined when there is only a finite number of literals in every possible world.[13] Furthermore, the sentences and structures (namely systems of spheres and the binary (transitive) relations on the set of all remainders of **K**) that were introduced in Counterexample 3.2 (which was used to prove the impossibility of defining all TRPMMCs by means of a system of spheres based construction), can only be defined as proposed, provided that the underlying language is finite.

- To obtain axiomatic characterizations for the two classes of multiple contraction functions proposed in Section 5. These characterizations would clarify the main differences concerning the behaviour of those two kinds of operation. Furthermore, since those two classes are subclasses of the class of TRPMMCs, their axiomatic characterizations might provide some insight towards an axiomatic characterization of the TRPMMCs — a problem that was originally raised by Fuhrmann and Hansson in [7] and which is still unsolved. These three axiomatic characterizations would possibly contribute to the identification of some necessary and sufficient properties for a TRPMMC to be constructable by means of a systems of spheres-based procedure.
- To identify the extra constraint(s) that a transitive relation $\sqsubseteq$ on the set $\bigcup_{B\subseteq\mathcal{L}} \mathbf{K}\perp B$ needs to satisfy in order for the TRPMMCs based on it to belong to each one of the two classes of multiple contraction functions introduced in Definitions 5.3 and 5.4.
- To investigate if there is a method for defining multiple contractions by means of total preorders over $2^{\mathcal{M}_{\mathcal{L}}}$ which is such that: (i) the class of functions thus obtained is identical with the class of TRPMMCs and, (ii) different total preorders over $2^{\mathcal{M}_{\mathcal{L}}}$ correspond to different TRPMMC functions.
- To obtain an explicit definition as well as an axiomatic characterization for the two classes consisting of functions $\ominus$ which are such that for any belief set **K** and any set of sentences B:

$$\mathbf{K}\ominus B = \mathbf{K}\div\neg B, \tag{15}$$

both when $\div$ is $\div^1_{\mathbb{S}}$ and when it is $\div^2_{\mathbb{S}}$ (cf. Definitions 5.3 and 5.4), and where $\neg B$ is the *negation of* B (e.g. [12, Definition 3.35]). Furthermore, to investigate the interrelation between the two thus obtained classes of functions and each one of the classes of set contraction functions proposed in [23] and in [16].
We notice that this investigation would also be a significant step towards the identification of a possible-worlds semantics for set contractions. Or, more precisely, towards the characterization of the set contractions in terms of systems of spheres.
- To obtain an explicit definition as well as an axiomatic characterization for the multiple revision functions that are obtained by means of Hansson's generalized Levi identity (equation (14)) from each one of the two classes of multiple contraction functions introduced in Definitions 5.3 and 5.4. And, additionally, to investigate the relationship between the thus obtained classes of multiple revision functions and the class of system of spheres-based multiple revisions introduced in [15].

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