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Extending limits for wave power absorption by axisymmetric devices

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The theoretical limit for absorption of energy in monochromatic water waves of wavelength λ by axisymmetric wave energy converters operating in rigid-body motion was established in the 1970s. The maximum mean power generated by a device absorbing due to heave motion is equivalent to that contained in $\lambda/2\pi$ length of incident wave crest. For devices absorbing through surge and/or pitch motions the so-called capture width doubles to λ/π . For devices absorbing in both heave and surge/pitch the capture width increases further to $3\lambda/2\pi$. In this paper it is demonstrated it is theoretically possible to extend the capture width for axisymmetric wave energy converters without bound through the use of generalised (non-rigid body) modes of motion. This concept will be applied to vertical cylinders whose surface is surrounded by an array of narrow vertical absorbing paddles. A continuum approximation is made to the paddle motion which simplifies the problem and allows strategies to be developed for setting the springs and dampers that control the power absorption. Results demonstrate that a cylinder of fixed size can absorb as much power as demanded from a plane incident wave although the practical limitations of linear theory are rapidly breached as that demand increases unless the size of the cylinder increases in proportion. In this paper we do not explore these limits in detail or further practical design considerations, such as imposing motion constraints. The continuum approximation is tested against a discrete paddle simulation for accuracy. [doi:10.1017/jfm.2021.645]

1. Introduction

Ocean waves offer an abundant source of clean energy, but the reality of designing and operating an economically viable, efficient and robust solution for harnessing that energy has proved immensely challenging. There are many reasons for this which are well documented (Yemm *et al.* 2012; Garrad 2012; Salter 2016; Cruz 2008). The biggest current challenge to continued interest and investment in the development of ocean wave energy renewables stems from the recent fall in the cost of production of energy from alternative renewable sources, principally wind and solar, now the cheapest form of energy production in many parts of the world. For example wind and solar in the UK is 30-50% cheaper in 2020 than the UK government's previous estimate made just 4 years earlier (UK Department for Business, Energy & Industrial Strategy 2020). On the other hand, it has been anticipated (UK Department of Energy & Climate Change 2011) that a carbon neutral future will require renewable ocean energy to contribute a significant and vital part of the energy mix. Thus, in addition to existing challenges there is an even sharper focus on developing wave energy converters (WECs) which are underpinned by

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42 high efficiency. Practically this requires developing WECs with the capacity to produce
 43 large amounts of energy from a single installation.

44 This demand presents a fundamental problem since it has long been known that there
 45 are theoretical limits on power absorption for certain types of WEC. For long so-called
 46 terminator devices which are aligned broadside to the oncoming wave direction it is
 47 theoretically possible, under classical linearised water wave theory, to absorb up to 100%
 48 of the incident wave energy along most of their length (e.g. Salter Duck, Bristol Cylinder
 49 – see Cruz (2008)). Once regarded as the most promising solution, the scaling up of
 50 capacity requires additional device length with its associated costs.

51 However, for axisymmetric devices (which tend to be classified as point absorbers) it
 52 is theoretically possible to absorb all of the wave energy from a length of incident wave
 53 crest which exceeds the physical dimensions of the device. Specifically the power available
 54 to a rigid axisymmetric wave absorber depends only on the wavelength, λ , in the manner
 55 described in the abstract. Practically, it is hard to exploit since device motions increase
 56 as the device size reduces and eventually must become constrained (Evans 1981; Pizer
 57 1993). For attenuator devices aligned with the incoming wave direction (e.g. Pelamis)
 58 theoretical limits are less clear although a similar principle applies: it is possible to absorb
 59 energy from a much greater length of incident wave crest than the slender width of the
 60 device. There are sound arguments (see Mei (1983)) that the amount of energy captured
 61 can increase with the number of absorbing mechanisms placed along the length of the
 62 attenuator (articulations between Pelamis raft sections, for example). Again there are
 63 practical considerations which imply that attenuators either need to be of considerable
 64 length and/or require constraints to be applied on the motion as in Newman (1979),
 65 Ancellin *et al.* (2020) to ensure predictions remain within the limitations of the underlying
 66 theory.

67 A comprehensive study carried out by Babarit (2015) (see Babarit’s Fig. 16) catalogu-
 68 ing many of the different types of wave energy converter design highlights the role of
 69 these limits.

70 In this paper we return to axisymmetric devices and, instead of allowing them to
 71 operate and absorb energy in the usual rigid-body modes of motion, consider devices
 72 which operate in “generalised modes” of motion, reminiscent of ideas developed in
 73 Newman (1979), Newman (1994). This involves allowing the surface of the device to
 74 move with more degrees of freedom than would be afforded if the surface of the device
 75 were rigid. In this paper we imagine that this effect is created by placing a large array of
 76 narrow paddles around the surface of a vertical cylinder. There may be other approaches
 77 which produce a similar effect through hydroelasticity, for example. Indeed, Garnaud
 78 & Mei (2009) have previously shown that a compact array of floating buoys extracting
 79 power in heave and distributed over a circular region of the surface can absorb more
 80 than the equivalent size of a rigid cylinder. Zheng *et al.* (2020) have demonstrated how
 81 a structured porous cylinder can be capable of exceeding the equivalent rigid body
 82 absorption limits. Very recently, Michele *et al.* (2020) have used a distributed power
 83 take-off system connecting a floating elastic plate to the bed to generate power.

84 2. General theory and motivation

There are a number of different ways of developing the theoretical framework which
 describes the capacity of a WEC to absorb power from an incoming plane wave. One
 such approach (see Mei (1983)) is summarised below. A plane monochromatic wave of
 wavelength $\lambda = 2\pi/k$, angular frequency ω and amplitude A travelling in the positive

x -direction on water of depth h is described by the velocity potential

$$\phi_{pw}(x, y, z) = -\frac{iAg}{\omega} e^{ikx} \psi_0(z) \quad (2.1)$$

where $\omega = \sqrt{gk \tanh kh}$ is the assumed radian frequency of motion, related to the wavenumber k and $\psi_0(z) = \cosh k(z+h)/\cosh kh$ is the depth eigenfunction associated with propagating waves. Thus, inviscid incompressible linearised water wave theory is in operation and a time factor of $e^{-i\omega t}$ has been suppressed so that ϕ_{pw} is a solution of the governing equations

$$\nabla^2 \phi = 0, \quad \text{in the fluid} \quad (2.2)$$

with

$$\phi_z = 0, \quad \text{on } z = -h \quad (2.3)$$

and

$$\phi_z - (\omega^2/g)\phi = 0, \quad \text{on } z = 0. \quad (2.4)$$

The mean (time-averaged over a period) flux of energy per unit length of wave crest contained in the plane wave is calculated from

$$\mathbb{P}_{pw} = \frac{1}{2} \text{Re} \left\{ \int_{-h}^0 i\omega \rho \phi_{pw} \frac{\partial \phi_{pw}^*}{\partial x} dx \right\} = \frac{1}{2} \rho g |A|^2 c_g \quad (2.5)$$

85 where the asterisk denotes complex conjugation and $c_g = d\omega/dk = \frac{1}{2}(\omega/k)(1 +$
 86 $2kh/\sinh 2kh)$ is the group velocity.

The incident plane wave defined by (2.1) can be expressed as the sum of incoming and outgoing circular waves by writing (e.g. Mei (1983))

$$\phi_{pw}(r, \theta, z) = \phi_{in}(r, \theta, z) + \phi_{out}(r, \theta, z) \quad (2.6)$$

where

$$\phi_{in} = -\frac{iAg}{2\omega} \psi_0(z) \sum_{n=0}^{\infty} \epsilon_n i^n H_n^{(2)}(kr) \cos n\theta \quad (2.7)$$

and

$$\phi_{out} = -\frac{iAg}{2\omega} \psi_0(z) \sum_{n=0}^{\infty} \epsilon_n i^n H_n^{(1)}(kr) \cos n\theta \quad (2.8)$$

87 where $\epsilon_0 = 1$ and $\epsilon_n = 2$ for $n \geq 1$. The mean flux of energy to/from infinity
 88 attributed to the n th circular component of (2.8)/(2.7) has the value $P_n = (\epsilon_n \lambda / 2\pi) \mathbb{P}_{pw}$.
 89 Contrasting font styles indicate different dimensions of \mathbb{P}_{pw} and P_n (units kW/m and
 90 kW respectively).

Consider plane waves incident upon a device which we assume for simplicity is symmetric with respect to the incident wave heading. Then far away from the device

$$\phi(r, \theta, z) \sim \phi_{pw}(r, \theta, z) - \frac{iAg}{\omega} \psi_0(z) \sum_{n=0}^{\infty} \epsilon_n i^n a_{n,0} H_n^{(1)}(kr) \cos n\theta \quad (2.9)$$

where $a_{n,0}$ are coefficients determined by the shape and dynamics of the device as well as the wave frequency. When written as

$$\phi = \phi_{in} - \frac{igA}{2\omega} \psi_0(z) \sum_{n=0}^{\infty} \epsilon_n i^n (2a_{n,0} + 1) H_n^{(1)}(kr) \cos n\theta \quad (2.10)$$

it can be seen that the power lost to the device is

$$P = \frac{\mathbb{P}_{pw}\lambda}{2\pi} \sum_{n=0}^{\infty} \epsilon_n (1 - |2a_{n,0} + 1|^2). \quad (2.11)$$

It follows that a non-absorbing device (including fixed, freely floating, or those constrained to move with sprung mooring lines) must have scattering coefficients, $a_{n,0} \equiv a_{n,0}^S$, say, satisfying $|2a_{n,0}^S + 1| = 1$.

For example, consider a rigid vertical cylinder extending through the depth of the fluid for which the potential everywhere in the fluid domain may be written (e.g. MacCamy & Fuchs (1954))

$$\phi(r, \theta, z) = -\frac{iAg}{\omega} \psi_0(z) \sum_{n=0}^{\infty} \epsilon_n i^n \left(J_n(kr) - \frac{J'_n(ka)}{H_n^{(1)'}(ka)} H_n^{(1)}(kr) \right) \cos n\theta \quad (2.12)$$

wherein $a_{n,0}^S = -J'_n(ka)/H_n^{(1)'}(ka)$ and it is confirmed that $|2a_{n,0}^S + 1| = 1$.

More importantly, (2.11) tells us a device with the capacity to absorb energy can extract up to the maximum mean power, P_n , from the n th circular component of the wave field if its dynamics can be orchestrated to meet the condition $a_{n,0} = -\frac{1}{2}$. For this to happen the device must have the capacity to radiate waves through motions responsible for absorbing wave energy in the n th circular mode, i.e., in proportion to $\cos n\theta$. For example, rigid-body heave motion of an axisymmetric device radiates waves in the zeroth circular mode, and so its maximum power absorption is limited to $P_{max} = P_0$, whilst surge and pitch motions radiate in the $n = 1$ circular mode giving rise to a maximum of $P_{max} = P_1$; combined heave and surge/pitch provide a maximum of $P_{max} = P_0 + P_1$. Thus we recover the well-known theoretical limits derived independently by Newman (1976), Evans (1976), Budal & Falnes (1977) and summarised in the abstract.

The capacity to absorb energy in excess of these limits thus lies in the ability to radiate in multiple circular modes. This is a well-understood concept and approaches to exploit this have been made by Newman (1979), Haren & Mei (1979), Ancellin *et al.* (2020) for elongated attenuator WEC devices and when WECs are comprised of multiple distinct absorbers such as Garnaud & Mei (2009), Wolgamot *et al.* (2012). In both cases the operation is characterised by multiple degrees of freedom.

In this paper we apply the principle to axisymmetric devices by imagining that a WEC device is fitted with a large number (N , say) of narrow vertical paddles across its surface which oscillate normal to that surface. These paddles could be hinged along a level below the water surface or perhaps operate with a linear piston-like motion directed from the vertical axis. We suppose the paddles have the capacity to convert hydrodynamic forces into useful power.

The N paddles could be connected to their own springs and dampers and operate independently from one another. However, for the moment, let us imagine that the paddle operation can be designed to oscillate as a superposition of $M + 1$ (say) modes which, when absorbing, radiate in the far-field with a variation of $\cos n\theta$ for $0 \leq n \leq M$. For example, the $n = 0$ mode corresponds to the paddles operating synchronously and, in the $n = 1$ mode, the paddle oscillation is modulated by $\cos \theta$. Then it is possible, in principle at least, to design the paddle springs and dampers such that

$$P_{max} = \frac{\mathbb{P}_{pw}\lambda}{\pi} \left(M + \frac{1}{2} \right). \quad (2.13)$$

In this paper we focus on a circular cylinder extending through the depth covered with narrow vertical paddles with the capacity to absorb, but do not suppose the type of

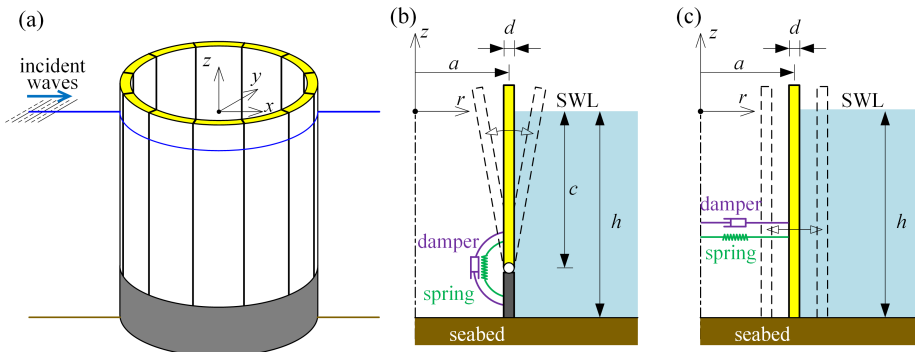


FIGURE 1. Sketch of an axisymmetric device: (a) bird's-eye view of the device with hinged paddles; (b) section of the device with hinged paddles; (c) section of the device with piston-like paddles.

120 complicated engineering solutions or control theory suggested above is needed to operate
 121 the paddles (see figure 1). Instead, each paddle is supposed to operate independently with
 122 their own spring and damper and the paper explores strategies to design the springs
 123 and damper characteristics with a view to developing power beyond that available to
 124 an equivalent cylinder operating in rigid body motion thereby showing that (2.13) is
 125 theoretically attainable. This investigation is assisted by the development of a continuum
 126 approximation to the arrangement of narrow paddles across the surface of the cylinder.
 127 The accuracy of this approximation is assessed against an exact description of the
 128 hydrodynamic/mechanical problem for a finite number of paddles.

129 The aim of the current work is to highlight the potential for a single axisymmetric
 130 device fitted with multiple paddles to absorb power in excess of the power from rigid body
 131 motion. It does not, however, address the important issue of adding motion constraints
 132 in order that the underlying linearised water wave framework is not compromised.

133 3. A cylindrical wave energy converter: governing equations

134 A vertical cylinder of radius a centred on the z -axis extends through a fluid of density
 135 ρ and depth h with a mean free surface on $z = 0$. An array of $N \gg 1$ identical narrow
 136 vertical paddles are attached to the surface of the cylinder having width $2\pi a/N$ assumed
 137 to be much smaller than their length c (no larger than the fluid depth, h) and the
 138 wavelength λ . The angular coordinate of the centre of the n th paddle is denoted $\theta_n =$
 139 $(2n - 1)\pi/N$, $n = 1, 2, \dots, N$. Each rigid paddle can move in a radial direction along
 140 its central axial plane and the motion of the n th paddle is resisted by a linear spring
 141 with spring constant κ_n and a linear damper with damping rate γ_n through which power
 142 is extracted. In motion, the n th paddle oscillates through a small displacement (linear
 143 or angular) $S_n(t) = \text{Re}\{\sigma_n e^{-i\omega t}\}$ where the time dependence of radian frequency ω has
 144 been assumed.

The motion of the fluid is governed by a potential $\phi(r, \theta, z)$ which satisfies (2.2), (2.3)
 and (2.4). Additionally, the kinematic condition connecting the velocity of the fluid to
 that of the paddles normal to the cylinder surface is written

$$\frac{\partial \phi}{\partial r} \Big|_{r=a} = -i\omega \sigma_n f(z) \cos(\theta - \theta_n), \quad -h < z < 0, \quad \theta_n - \pi/N < \theta < \theta_n + \pi/N \quad (3.1)$$

for $n = 1, 2, \dots, N$ and $\cos(\theta - \theta_n)$ is a geometric factor due to the curvature of the

paddle surface. In (3.1), $f(z)$ encodes the spatial variation of the displacement along the length of the paddle. For example, a paddle operating in a radial piston-like motion along a submerged extent $c \leq h$ will be defined by $f(z) = 1$, $-c < z < 0$ and $f(z) = 0$, $-h < z < -c$ whereas a paddle operating as a hinged flap pivoted along its bottom edge along $z = -c$ ($c < h$) would be defined by

$$f(z) = \begin{cases} z + c, & -c < z < 0, \\ 0, & -h < z < -c. \end{cases} \quad (3.2)$$

The equation of motion for the n th paddle is expressed by

$$-\omega^2 \mathcal{M}(2\pi a/N)\sigma_n = -(\kappa_n + \mathcal{C}(2\pi a/N))\sigma_n + i\omega\gamma_n\sigma_n + X_n \quad (3.3)$$

where \mathcal{M} is the mass (or moment of inertia) per unit width, \mathcal{C} accounts for any buoyancy restoring force (or moment) per unit width present and

$$X_n = -i\omega\rho \int_{-h}^0 \int_{\theta_n - \pi/N}^{\theta_n + \pi/N} \phi(a, \theta, z) f(z) \cos(\theta - \theta_n) a d\theta dz \quad (3.4)$$

is the hydrodynamic wave force (or moment). The cosine terms appearing in (3.1) and (3.4) are geometrical factors arising from the component normal to the assumed curved surface of the paddles.

When N is large and the width of the paddle, $2\pi a/N$, is small with respect to the wavelength λ and the length of the paddle, c , we assume that σ_n may be replaced by discrete evaluations, $\sigma(\theta_n)$, of a continuous function $\sigma(\theta)$ allowing (3.1) to be approximated by

$$\left. \frac{\partial \phi}{\partial r} \right|_{r=a} = -i\omega\sigma(\theta)f(z), \quad -h < z < 0, \quad 0 < \theta \leq 2\pi. \quad (3.5)$$

Similarly, we let $\kappa_n = \kappa(\theta_n)(2\pi a/N)$ and $\gamma_n = \gamma(\theta_n)(2\pi a/N)$ where κ and γ are continuous functions representing the spring force (or torque) and damping rate per unit width whilst (3.4) becomes

$$X_n = \frac{2a\pi}{N} X(\theta_n) \approx -i\omega\rho \frac{2a\pi}{N} \int_{-h}^0 \phi(a, \theta_n, z) f(z) dz. \quad (3.6)$$

Then the N discrete equations of motion for the N paddles in (3.3) are approximated by the θ -continuous equation of motion

$$[\kappa(\theta) + \mathcal{C} - \omega^2 \mathcal{M} - i\omega\gamma(\theta)]\sigma(\theta) = X(\theta), \quad 0 < \theta \leq 2\pi. \quad (3.7)$$

It follows that the combined dynamic and kinematic boundary condition on $r = a$ is

$$[\kappa(\theta) + \mathcal{C} - \omega^2 \mathcal{M} - i\omega\gamma(\theta)] \left. \frac{\partial \phi}{\partial r} \right|_{r=a} = -\omega^2 \rho f(z) \int_{-h}^0 \phi(a, \theta, z) f(z) dz \quad (3.8)$$

for $-h < z < 0$ and $0 < \theta \leq 2\pi$. We write this as

$$A(\theta) h a \left. \frac{\partial \phi}{\partial r} \right|_{r=a} = f(z) \int_{-h}^0 \phi(a, \theta, z) f(z) dz \quad (3.9)$$

where

$$A(\theta) = \frac{\mathcal{M} - \omega^{-2}(\kappa(\theta) + \mathcal{C}) + i\omega^{-1}\gamma(\theta)}{\rho h a}. \quad (3.10)$$

148 **4. Solution for narrow paddles**

Following the description of the plane wave in (2.1) we can write the full depth-dependent potential satisfying (2.2), (2.3) and (2.4) as the expansion

$$\phi(r, \theta, z) = -\frac{igA}{\omega} \sum_{m=0}^{\infty} \varphi_m(r, \theta) \psi_m(z) \tag{4.1}$$

over all depth eigenfunctions

$$\psi_m(z) = \cos k_m(z + h) / \cos(k_m h) \tag{4.2}$$

that arise from separating variables: k_m are the increasing sequence of positive roots of $-\omega^2/g = k_m \tan k_m h$ and (see, e.g., Mei (1983)). The depth eigenfunctions defined in (4.2) alongside $\psi_0(z)$ defined after (2.1) with $k_0 = -ik$ satisfy the orthogonality relation

$$\frac{1}{h} \int_{-h}^0 \psi_n(z) \psi_m(z) dz = N_n \delta_{mn} \tag{4.3}$$

for all $n, m = 0, 1, 2, \dots$ where

$$N_n = \frac{1}{2} (1 + \sin(2k_n h) / (2k_n h)) / \cos^2(k_n h). \tag{4.4}$$

The functions $\varphi_m(r, \theta)$ are given by

$$\varphi_0(r, \theta) = \sum_{n=0}^{\infty} \epsilon_n i^n \left(J_n(kr) + a_{n,0} H_n^{(1)}(kr) \right) \cos n\theta \tag{4.5}$$

and

$$\varphi_m(r, \theta) = \sum_{n=0}^{\infty} \epsilon_n i^n a_{n,m} K_n(k_m r) \cos n\theta \tag{4.6}$$

149 for $m \geq 1$ and $K_n(\cdot)$ are modified Bessel functions.

We define

$$F_n = \frac{1}{h} \int_{-h}^0 \psi_n(z) f(z) dz, \quad n = 0, 1, \dots \tag{4.7}$$

as constants which can be calculated for a given $f(z)$. Using (4.1) in (3.9) gives

$$\Lambda(\theta) a \sum_{m=0}^{\infty} \frac{\partial \varphi_m}{\partial r}(a, \theta) \psi_m(z) = f(z) G(\theta) \tag{4.8}$$

where

$$G(\theta) = \frac{1}{h} \int_{-h}^0 \sum_{m=0}^{\infty} \varphi_m(a, \theta) \psi_m(z) f(z) dz = \sum_{m=0}^{\infty} F_m \varphi_m(a, \theta). \tag{4.9}$$

It follows after using (4.7) again, that

$$\Lambda(\theta) a N_m \frac{\partial \varphi_m}{\partial r}(a, \theta) = F_m G(\theta), \quad 0 < \theta \leq 2\pi \tag{4.10}$$

for all $m = 0, 1, \dots$ and so

$$\frac{\partial \varphi_m}{\partial r}(a, \theta) = \frac{N_0 F_m}{N_m F_0} \frac{\partial \varphi_0}{\partial r}(a, \theta). \tag{4.11}$$

Application of this relation to (4.5) and (4.6) gives

$$a_{n,m} = \frac{k F_m N_0}{k_m F_0 N_m K'_n(k_m a)} (J'_n(ka) + a_{n,0} H_n^{(1)'}(ka)) \tag{4.12}$$

150 for $m \geq 1$, after equating coefficients of $\cos n\theta$. This important relation illustrates that the
 151 dependence of the fluid motion through the depth is set by the function $f(z)$ describing
 152 the vertical displacement of the paddle motion.

In particular, using (4.12) in (4.5) and (4.6) allows us to express the general solution (4.1) in the form

$$\phi(r, \theta, z) = -\frac{igA}{\omega} \sum_{n=0}^{\infty} \epsilon_n i^n \phi_n(r, z) \cos n\theta \quad (4.13)$$

where

$$\begin{aligned} \phi_n(r, z) = & (J_n(kr) + a_{n,0} H_n^{(1)}(kr)) \psi_0(z) \\ & + (J'_n(ka) + a_{n,0} H_n^{(1)'}(ka)) \sum_{m=1}^{\infty} \frac{k F_m N_0 K_n(k_m r)}{k_m F_0 N_m K'_n(k_m a)} \psi_m(z) \end{aligned} \quad (4.14)$$

153 is expressed in terms of $a_{n,0}$ only.

We will also find it convenient to write

$$G(\theta) = \sum_{n=0}^{\infty} \epsilon_n i^n G_n \cos n\theta \quad (4.15)$$

where, from the definition implied by its introduction in (4.8),

$$G_n = F_0 (J_n(ka) + a_{n,0} H_n^{(1)}(ka)) + \frac{ka N_0}{F_0} (J'_n(ka) + a_{n,0} H_n^{(1)'}(ka)) E_n \quad (4.16)$$

and we have defined

$$E_n = \sum_{m=1}^{\infty} \frac{F_m^2 K_n(k_m a)}{k_m a N_m K'_n(k_m a)}. \quad (4.17)$$

154

4.1. Equal springs and dampers

We let $\kappa(\theta) = \kappa$ and $\gamma(\theta) = \gamma$ so that

$$\Lambda(\theta) = \frac{\mathcal{M} - (\kappa + \mathcal{C})/\omega^2 + i\gamma/\omega}{\rho h a} \equiv \Lambda_0, \quad (4.18)$$

say, is a constant and it follows that the boundary condition (3.9) applies to each circular wave component thus

$$\Lambda_0 h a \left. \frac{\partial \phi_n}{\partial r} \right|_{r=a} = f(z) \int_{-h}^0 \phi_n(a, z) f(z) dz. \quad (4.19)$$

Substituting in (4.14), multiplying through by $\psi_0(z)$ and integrating over $-h < z < 0$ gives

$$ka \Lambda_0 (J'_n(ka) + a_{n,0} H_n^{(1)'}(ka)) N_0 = F_0 G_n \quad (4.20)$$

155 where G_n is given by (4.16). Note that integrating over $-h < z < 0$ with other depth
 156 functions $\psi_m(z)$ for $m \geq 1$ does not provide any new information as the dependence on
 157 the vertical has already been incorporated into the solution.

Thus we can calculate $a_{n,0}$ explicitly from substituting (4.16) into (4.20) and rearranging to get

$$a_{n,0} = -\frac{\Gamma_n J'_n(ka) - J_n(ka)}{\Gamma_n H_n^{(1)'}(ka) - H_n^{(1)}(ka)} \quad (4.21)$$

where

$$\Gamma_n = \frac{kaN_0}{F_0^2} (\Lambda_0 - E_n). \quad (4.22)$$

The power generated by the paddles is subsequently calculated using (2.11). After some lengthy but routine algebra requiring the use of the following Wronskian identity for Bessel functions

$$J_n(x)Y_n'(x) - J_n'(x)Y_n(x) = 2/(\pi x) \quad (4.23)$$

(Abramowitz & Stegun (1964, §9.1.16)) we find that

$$P = \frac{\mathbb{P}_{pw}\lambda}{2\pi} \left(\frac{8N_0\gamma}{\pi\omega\rho ha F_0^2} \right) \sum_{n=0}^{\infty} \frac{\epsilon_n}{|\Gamma_n H_n^{(1)'}(ka) - H_n^{(1)}(ka)|^2}. \quad (4.24)$$

158 Although explicit, this expression above for the power is not particularly informative.
 159 For example, the maximum power available to each circular mode, $P_n = \epsilon_n \lambda / 2\pi$, is not
 160 evident in the form given in (4.24), nor is it easy to see how (4.24) could be used to
 161 optimise P with respect to the spring and damping parameters κ and γ .

162 We can, however, derive expressions for κ and γ which maximise the power absorbed in
 163 any individual circular mode. This can be done in one of two ways. The first is to isolate
 164 the m th component, P_m , from the sum in (4.24) and then set $\partial P_m / \partial \kappa = \partial P_m / \partial \gamma = 0$.

It is easier, though, to use the theoretical framework developed in §2 and impose $a_{m,0} = -\frac{1}{2}$ in (4.21) as a condition for maximum power absorption from the m th circular wave component and this yields the expression

$$\Gamma_m = \frac{H_m^{(2)}(ka)}{H_m^{(2)'}(ka)} = \frac{(J_m(ka)J_m'(ka) + Y_m(ka)Y_m'(ka)) + 2i/(\pi ka)}{|H_m^{(2)'}(ka)|^2} \quad (4.25)$$

165 using (4.23) once again. The coefficients $a_{n,m}$ for $n \neq m$ are subsequently defined by
 166 (4.12).

Equating (4.25) with the definition of Γ_n in (4.22) implies a complex condition to be satisfied by Λ_0 , defined here by (4.18) and equating real and imaginary parts gives the conditions

$$\frac{\gamma}{\omega\rho ha} = \frac{2F_0^2}{\pi k^2 a^2 N_0 |H_m^{(2)'}(ka)|^2} \quad (4.26)$$

and

$$\frac{\mathcal{M} - \omega^{-2}(\kappa + \mathcal{C})}{\rho ha} = \frac{F_0^2 (J_m(ka)J_m'(ka) + Y_m(ka)Y_m'(ka))}{kaN_0 |H_m^{(2)'}(ka)|^2} + E_m. \quad (4.27)$$

167 These two equations define κ and γ for absorption of the maximum power, P_m , from the
 168 m th circular wave component.

4.2. Unequal springs and dampers

Let us now assume that the springs and dampers can vary with position around the cylinder so that the boundary condition (3.9) remains as

$$\Lambda(\theta) ha \left. \frac{\partial \phi}{\partial r} \right|_{r=a} = f(z) \int_{-h}^0 \phi(a, \theta, z) f(z) dz \quad (4.28)$$

with

$$\Lambda(\theta) = \frac{\mathcal{M} - \omega^{-2}(\kappa(\theta) + \mathcal{C}) + i\omega^{-1}\gamma(\theta)}{\rho ha} = \sum_{m=0}^{\infty} \epsilon_m \Lambda_m \cos m\theta \quad (4.29)$$

once expressed as a Fourier series. After substituting in the partial wave decomposition (4.13), multiplying by $\cos p\theta$ and integrating over $0 < \theta \leq 2\pi$ the result can be expressed either as

$$\frac{1}{2} \sum_{m=0}^{\infty} \epsilon_m \Lambda_m a \left[i^{|p-m|} \frac{\partial \phi_{|p-m|}}{\partial r} + i^{p+m} \frac{\partial \phi_{p+m}}{\partial r} \right]_{r=a} = f(z) i^p G_p \quad (4.30)$$

where G_p is defined by (4.16) or as

$$\frac{1}{2} \sum_{n=0}^{\infty} \epsilon_n i^n \frac{\partial \phi_n}{\partial r} \Big|_{r=a} a (\Lambda_{|p-n|} + \Lambda_{p+n}) = f(z) i^p G_p \quad (4.31)$$

170 depending on how one chooses to eliminate the summation variables through the orthog-
171 onality of the product of three cosines.

As in the previous section, there are two ways of proceeding. One is to imagine that the setting for the springs and dampers have been made such that Λ_m are presumed known and then use the system above to determine $a_{n,0}$ and, subsequently, the power P . Substituting (4.14) and (4.16) into (4.31), multiplying by $\psi_0(z)$ and integrating over $-h < z < 0$ gives the system of equations

$$\begin{aligned} a_{p,0} \left[H_p^{(1)}(ka) + i^p \frac{kaN_0}{F_0^2} E_p H_p^{(1)'}(ka) \right] \\ - \frac{kaF_0^2}{2N_0} \sum_{n=0}^{\infty} a_{n,0} \epsilon_n i^n H_n^{(1)'}(ka) (\Lambda_{|p-n|} + \Lambda_{p+n}) = \\ - \left[J_p(ka) + i^p \frac{kaN_0}{F_0^2} E_p J_p'(ka) \right] + \frac{kaF_0^2}{2N_0} \sum_{n=0}^{\infty} \epsilon_n i^n J_n'(ka) (\Lambda_{|p-n|} + \Lambda_{p+n}) \end{aligned} \quad (4.32)$$

172 for $p = 0, 1, \dots$. When $\Lambda_n = 0$ for $n \geq 1$ and $\Lambda(\theta) = \Lambda_0$, a constant, (4.32) reduces to
173 (4.21).

However, we also have the opportunity to design the settings of springs and dampers to control the device performance and so we treat Λ_m as unknown and proceed as if $a_{n,0}$ are prescribed. Following the same procedure as above but with (4.30) replacing (4.31) leads to

$$\frac{kaN_0}{2F_0^2} \sum_{m=0}^{\infty} \epsilon_m \Lambda_m (Q_{|p-m|} + Q_{p+m}) = \frac{i^p G_p}{F_0} \quad p = 0, 1, \dots \quad (4.33)$$

where $Q_n = i^n (J_n'(ka) + a_{n,0} H_n^{(1)'}(ka))$. With a view to reaching the limit (2.13) set out in the introduction, albeit via a different route, we set $a_{n,0} = -\frac{1}{2}$ for $n \leq M$ and $a_{n,0} = -J_n'(ka)/H_n^{(1)'}(ka)$ for $n > M$ (corresponding to a non-absorbing cylinder – see Section 2) so that

$$Q_n = \begin{cases} \frac{1}{2} i^n H_n^{(2)'}(ka), & n \leq M \\ 0, & n > M \end{cases} \quad (4.34)$$

and the right-hand side of (4.33) is

$$\frac{i^p G_p}{F_0} = \begin{cases} \frac{1}{2} i^p (H_p^{(2)}(ka) + (kaN_0/F_0^2) E_p H_p^{(2)'}(ka)), & p \leq M \\ 2i^{p+1}/(\pi ka H_p^{(1)'}), & p > M. \end{cases} \quad (4.35)$$

174 The infinite system of equations (4.33) is then subject, for numerical purposes, to
175 truncation subject to suitable convergence for a given M .

176 This process above describes how to fix the values of Λ_m by tuning for maximum power
 177 from the first $M + 1$ circular modes at a specified frequency. At other frequencies $a_{n,0}$
 178 will need to be determined from (4.32) in terms of the fixed values of Λ_m .

179 Other design strategies could be adopted. For example, there may be benefits to
 180 distributing the capacity to absorbing the maximum power from different circular modes
 181 across a range of frequencies. This might mitigate against overloading the device at single
 182 frequency and could improve its overall performance in real sea states. It's not yet clear
 183 from the theory developed above how to design Λ_m for such an outcome, other than
 184 perhaps by brute force numerical optimisation.

185 5. A discrete paddle calculation

186 The previous sections have concentrated on a continuum description of the paddle
 187 motion and this has allowed us to develop particular strategies for selecting spring and
 188 damper settings. It is possible to construct solutions for the original arrangement of N
 189 discrete paddles. Although this does not lead to the same mathematical insight, it will
 190 allow the accuracy of the continuum description of the absorbing cylinder to be assessed.

What follows is a standard linear decomposition method (e.g. Mei (1983)) in which we write

$$\phi = \phi_S + \sum_{q=1}^N (-i\omega\sigma_q)\phi_R^{(q)} \quad (5.1)$$

where ϕ_S is the scattering problem, subject to an incident plane wave (2.1) and satisfying

$$\left. \frac{\partial\phi_S}{\partial r} \right|_{r=a} = 0, \quad 0 < \theta \leq 2\pi, \quad -h < z < 0 \quad (5.2)$$

whilst $\phi_R^{(q)}$ is the radiation potential associated with the forced motion of the q th paddle and satisfying

$$\left. \frac{\partial\phi_R^{(q)}}{\partial r} \right|_{r=a} = \begin{cases} f(z) \cos(\theta - \theta_q), & \theta_q - \pi/N < \theta < \theta_q + \pi/N \\ 0, & \text{otherwise.} \end{cases} \quad (5.3)$$

The solution to the scattering problem for ϕ_S is given in (2.12) with $a_{n,0} \equiv a_{n,0}^S = -J'_n(ka)/H_n^{(1)}(ka)$. We can take advantage of the earlier theory to write the general expansion for the radiation potential as

$$\phi_R^{(q)} = \sum_{n=0}^{\infty} \epsilon_n i^n b_{n,0}^{(q)} \left[H_n^{(1)}(kr) \psi_0(z) + H_n^{(1)'}(ka) \sum_{m=1}^{\infty} \frac{kN_0 F_m K_n(k_m r)}{k_m N_m F_0 K_n'(k_m a)} \psi_m(z) \right] \cos n\theta \quad (5.4)$$

which takes account of the depth dependence $f(z)$ of the paddle. Using (5.4) in (5.3) and the orthogonality of $\cos n\theta$ and $\psi_m(z)$ determines the expansion coefficients as

$$b_{n,0}^{(q)} = \frac{i^{-n} F_0 C_{qn}}{2\pi k N_0 H_n^{(1)'}(ka)} \quad (5.5)$$

where

$$\begin{aligned}
 C_{qn} &= \int_{\theta_q - \pi/N}^{\theta_q + \pi/N} \cos(\theta - \theta_q) \cos n\theta \, d\theta \\
 &= \begin{cases} \left(\frac{1}{2} \sin(2\pi/N) + \pi/N\right) \cos \theta_q, & n = 1 \\ \left(\frac{\sin((n+1)\pi/N)}{n+1} + \frac{\sin((n-1)\pi/N)}{n-1}\right) \cos(n\theta_q), & n \neq 1. \end{cases} \quad (5.6)
 \end{aligned}$$

The wave force upon the p th paddle is similarly decomposed as

$$X_p = X_{S,p} + \sum_{q=1}^N (-i\omega\sigma_q) X_{R,p}^{(q)} \quad (5.7)$$

where

$$\begin{aligned}
 X_{S,p} &= -i\omega\rho \int_{-h}^0 \int_{\theta_p - \pi/N}^{\theta_p + \pi/N} \phi_S(a, \theta, z) \cos(\theta - \theta_p) f(z) \, ad\theta dz \\
 &= -\frac{2i\rho gh AF_0}{\pi k} \sum_{n=0}^{\infty} \frac{\epsilon_n i^n C_{pn}}{H_n^{(1)'}(ka)} \quad (5.8)
 \end{aligned}$$

after use of a number of previous results. Similarly

$$\begin{aligned}
 X_{R,p}^{(q)} &= -i\omega\rho \int_{-h}^0 \int_{\theta_p - \pi/N}^{\theta_p + \pi/N} \phi_R^{(q)}(a, \theta, z) \cos(\theta - \theta_p) f(z) \, ad\theta dz \\
 &= -i\omega ha\rho \sum_{n=0}^{\infty} \epsilon_n i^n b_{n,0}^{(q)} \left[H_n^{(1)}(ka) F_0 + H_n^{(1)'}(ka) \sum_{m=1}^{\infty} \frac{k N_0 F_m^2 K_n(k_m a)}{k_m N_m F_0 K_n'(k_m a)} \right] C_{pn}. \quad (5.9)
 \end{aligned}$$

It is common practice to decompose complex-valued radiation forces into real added inertia and radiation damping components:

$$X_{R,p}^{(q)} = i\omega A_{pq} - B_{pq}. \quad (5.10)$$

The equation of motion for the n th paddle in (3.3) is now written

$$(\kappa_n + \mathcal{C}(2\pi a/N) - i\omega\gamma_n - \omega^2 \mathcal{M}(2\pi a/N))\sigma_n - \sum_{m=1}^N (\omega^2 A_{nm} + i\omega B_{nm})\sigma_m = X_{S,n} \quad (5.11)$$

191 for $n = 1, 2, \dots, N$. This represents an $N \times N$ system of equations for the unknown
192 complex-valued paddle displacement amplitudes σ_n .

Subsequently, the power generated by the device can be calculated in at least two independent ways. One is to see from (5.1), (5.4) that the total radiated wave potential is

$$\phi^R \sim \sum_{q=1}^N (-i\omega\sigma_q) \sum_{n=0}^{\infty} \epsilon_n i^n b_{n,0}^{(q)} H_n^{(1)}(kr) \cos n\theta, \quad \text{as } kr \rightarrow \infty \quad (5.12)$$

and use this with ϕ_S to calculate the power in outgoing circular waves and subtract it from the incoming circular waves.

$$\phi_{out} \sim \psi_0(z) \sum_{n=0}^{\infty} \epsilon_n i^n \left(\sum_{q=1}^N (-i\omega\sigma_q) b_{n,0}^{(q)} + \frac{iAg}{\omega} \frac{J_n'(ka)}{H_n^{(1)'}(ka)} \right) H_n^{(1)}(kr) \cos n\theta \quad (5.13)$$

as $kr \rightarrow \infty$, and so we can use the expression (2.11) for the power where

$$a_{n,0} = \frac{\omega^2}{Ag} \sum_{q=1}^N \sigma_q b_{n,0}^{(q)} - \frac{J'_n(ka)}{H_n^{(1)'}(ka)}. \quad (5.14)$$

The other method is to calculate the power generated by each of the paddles and sum over all N paddles to give which results in

$$P = \frac{\omega^2}{2} \sum_{q=1}^N \gamma_q |\sigma_q|^2. \quad (5.15)$$

Both expressions are calculated numerically to check the accuracy of the numerical code and produce graphically indistinguishable results.

6. Results

The power absorption of the cylinder will be measured by using the dimensionless capture factor, defined as

$$\eta = \frac{2\pi P}{\lambda \mathbb{P}_{pw}}. \quad (6.1)$$

A value of $\eta = 1$ thus represents the maximum power capable of being absorbed by a rigid axisymmetric device operating in heave; $\eta = 3$ is the maximum power that a rigid body can absorb in any combination of all rigid body motions. Values of $\eta > 3$ therefore indicate that the cylinder is absorbing power in excess of the capacity of a traditional axisymmetric wave energy absorbing device. Many of the results will involve plotting η against dimensionless wavenumber ka ($= 2\pi a/\lambda$) and we have chosen to fix the depth against the cylinder radius with $a/h = 1$ throughout the results (changing this value does not alter the qualitative nature of results). This means $ka \lesssim \frac{1}{2}$ represents long waves with respect to both the cylinder diameter and the water depth whereas $ka \simeq 5$ implies a wavelength comparable to the cylinder radius.

The paddles are given a uniform density, ρ_s , and thickness, d . For paddles hinged along the centre of the bottom edge $\mathcal{M} = \frac{1}{3}\rho_s d c(c^2 + d^2/4)$ represents the moment of inertia per unit width about the point of rotation $\mathcal{C} = \frac{1}{2}\rho g d^3$ is the buoyancy moment per unit width.

For paddles operating in piston-like motion we will assign values to dimensionless quantities

$$\bar{\mathcal{M}} = \mathcal{M}/(\rho a h), \quad \bar{\mathcal{C}} = \mathcal{C}/(\rho g a), \quad \bar{\kappa} = \kappa/(\rho g a), \quad \bar{\gamma} = \gamma/(\rho a g^{1/2} h^{1/2}). \quad (6.2)$$

and for hinged paddles each right-hand side above is additionally divided by c^2 .

Numerically, we shall consider values of $\bar{\mathcal{M}} = 0.1$, $\bar{\mathcal{C}} = 0$ for piston-like operation and $\bar{\mathcal{M}} = 0.034$, $\bar{\mathcal{C}} = 0.0003$ for hinged motion. Whilst we are not trying to prescribe exact engineering parameters we have based these values on reasonable estimates of $a = 10$ m, $h = 10$ m, $\rho = 1025 \text{ kg m}^{-3}$ and paddles with $c = 5$ m, $d = 1$ m and density of $\rho_s = 2\rho$.

Our principle interest will be in adjusting the spring and damper settings to assess the performance of the device in relation to the theory we have developed.

We start using the continuous paddle distribution approximation to assess the performance for a range of spring and damper constants, $\bar{\kappa}$ and $\bar{\gamma}$ in figures 2 and 3. It can be seen that the rigid-body limit of $\eta = 3$ is exceeded for values of $ka \gtrsim 1$ and one can see that generally softer springs provide better performance for lower values of ka and vice versa.

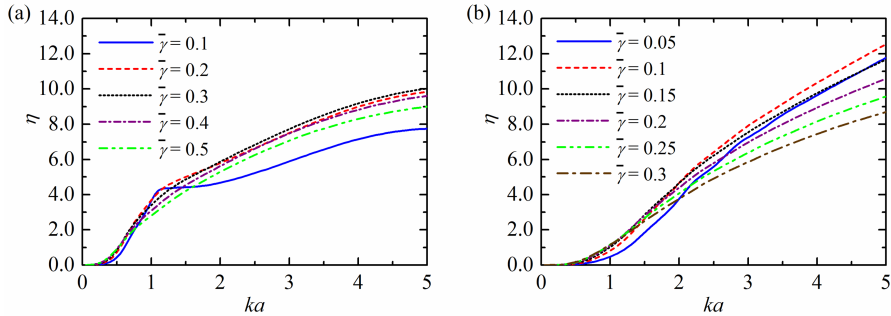


FIGURE 2. Capture factor against dimensionless wavenumber for $\bar{\kappa} = 0.3$: (a) piston-like paddles; (b) hinged paddles.

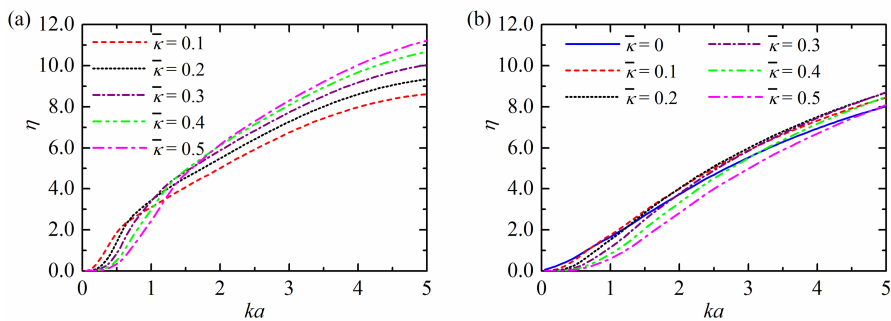


FIGURE 3. Capture factor against dimensionless wavenumber for $\bar{\gamma} = 0.3$: (a) piston-like paddles; (b) hinged paddles.

222 Instead of fixing the springs and dampers, we implement the optimisation outlined in
 223 §4.1 which provides a recipe for setting equal spring and damper settings to extract the
 224 maximum available power from any given circular mode component, m , in the incident
 225 wave. Results are illustrated in figure 4. In subfigures (b) and (c) the variation of the
 226 optimal values of $\bar{\kappa}$ and $\bar{\gamma}$ with frequency are shown alongside the resulting capture factor
 227 in figure 4(a). According to the optimisation strategy, when $m = 0$ the capture factor is
 228 guaranteed to exceed a value of unity and when $m > 0$ it must exceed $\eta = 2$. In practice,
 229 the amount by which the capture factor exceeds these minimum values can be large,
 230 since power is absorbed from circular wave components in the incident wave other than
 231 the one being targeted. Indeed, the capture factor appears to grow linearly with ka ,
 232 once $ka \simeq m$, and that growth is independent of the mode number, m .

233 The corresponding results for hinged paddles are shown in figure 5 and are qualitatively
 234 very similar to piston-like paddle motion.

235 To provide additional insight into how the cylinder device is operating we have plotted,
 236 in figure 6, a snapshot at an intermediate frequency, $ka = 2$, of the contribution of the
 237 capture factor from different circular wave components (n along the horizontal axis)
 238 when equal springs and dampers have been tuned to extract the maximum available in
 239 particular mode, m at this frequency. We can see that there is significant absorption
 240 across multiple modes. Taken with the previously observed linear trend in figures 4, 5
 241 it would appear that close to 100% of the energy flux available is being absorbed by all
 242 circular modes in the range $0 \leq n \lesssim ka/m$.

243 Figure 7 shows the maximum paddle amplitudes at $ka = 2$ under spring and damper
 244 tuning optimised to take all the available power from the m th mode. Here we see clearly

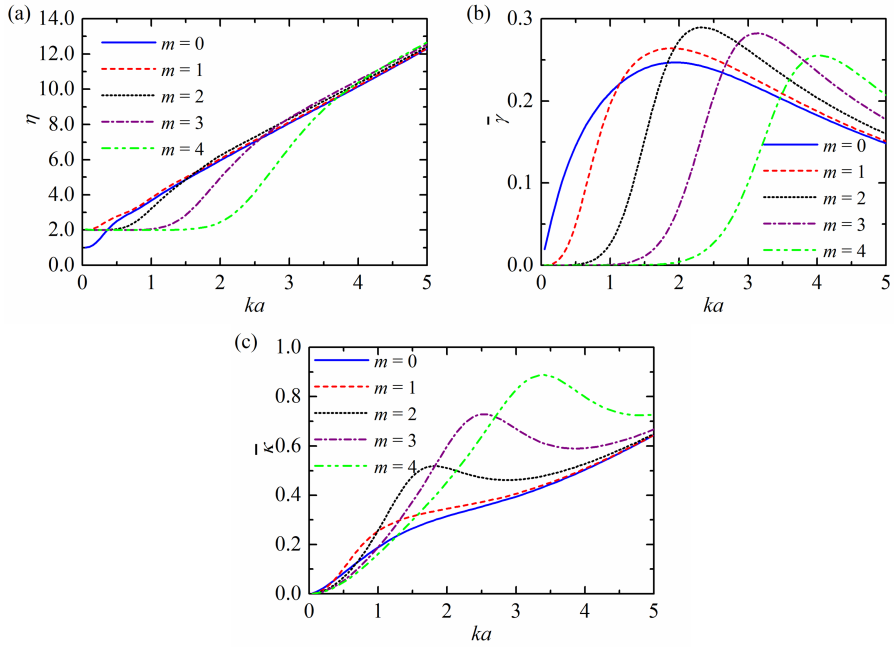


FIGURE 4. (a) Capture factor against dimensionless wavenumber for piston-like paddle motion, with corresponding damper and spring values in (b), (c) optimised in order to capture all the available power in the m th circular mode.

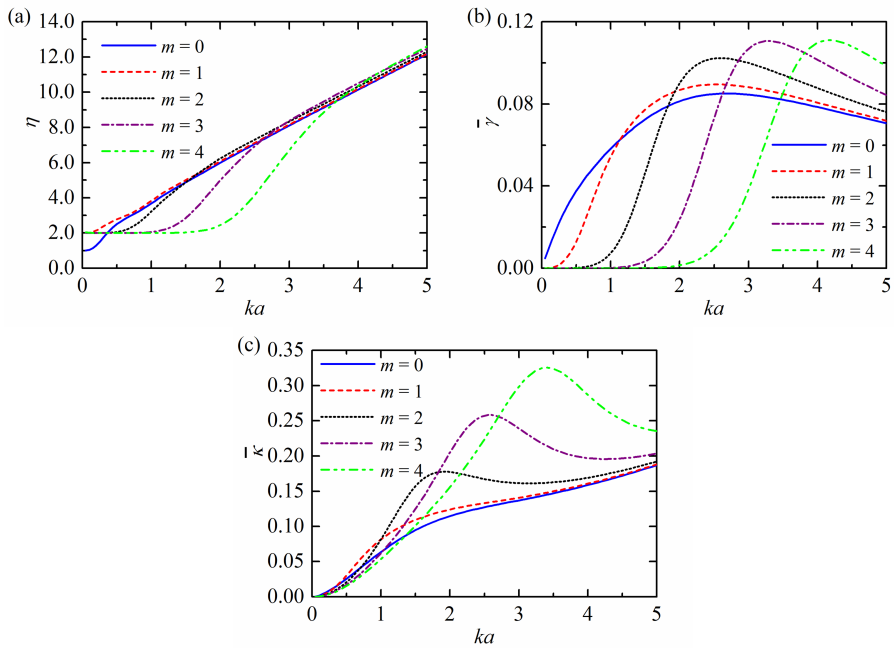


FIGURE 5. (a) Capture factor against dimensionless wavenumber for hinged paddle motion, with corresponding damper and spring values in (b), (c) optimised in order to capture all the available power in the m th circular mode.

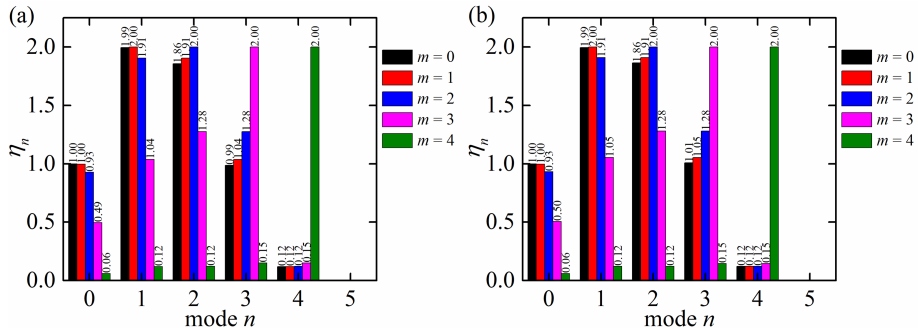


FIGURE 6. The partition of capture factor into contributions from the n th circular mode (along the horizontal axis) at $ka = 2$ for operation tuned to be optimal for mode m : (a) piston-like paddles, (b) hinged paddles.

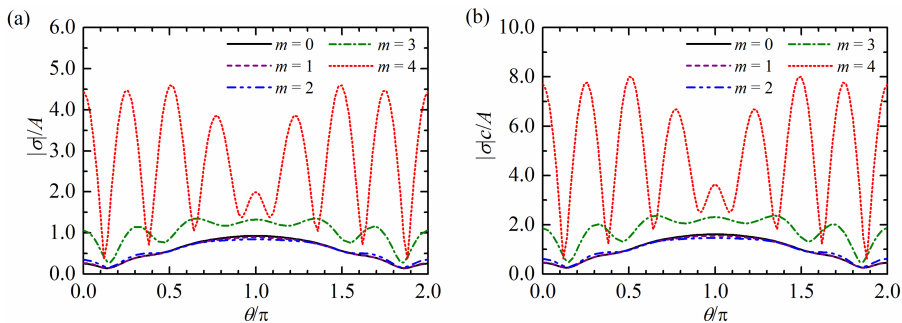


FIGURE 7. Dimensionless modal amplitudes of the paddles as a function of angle around the cylinder for springs and dampers tuned to absorb optimally in circular mode m at $ka = 2$: (a) piston-like paddles; (b) hinged paddles.

245 that the paddles are having to work harder to absorb power from higher modes both in
 246 terms the paddle amplitude and its variation around the cylinder. This is an indicator
 247 of the practical limitations for such a device. Note also that the hinged paddle requires
 248 roughly double the amplitude at the surface of the piston-like paddles. It can be seen that
 249 paddle amplitudes in excess of four times the incident wave amplitude are predicted for
 250 $m = 4$ and this would certainly violate the underlying linear assumptions. Indeed, this
 251 example serves to illustrate the important practical considerations which will impose
 252 quite severe limitations on how much additional predicted theoretical power one can
 253 actually exploit. The same comments apply to figure 12.

254 The maximum free surface elevation corresponding to the cases referenced in figures
 255 6(a), 7(a) is shown in figure 8 where it can be seen again how the paddles are
 256 working hard to absorb all of the available power for higher values of m where the $\cos m\theta$
 257 variation in the field becomes increasingly visible.

258 In figure 9(a) we show the proportion of power absorbed by each circular mode when
 259 paddles operating in piston-like motion are tuned to absorb 100% of the power available
 260 in the $m = 0$ mode. Each set of results comes from different values of ka . Of course 100%
 261 of power is taken from $n = 0$, but we again see that as ka increases, the device is taking
 262 close to 100% available power from modes n less than the integer part of ka . Figure 9(b)
 263 indicates the distribution of paddle amplitudes around the cylinder for these four sets
 264 of results. Optimising for total power absorption in mode $m = 0$ implies the paddle
 265 operation is well behaved for larger values of ka even though a significant proportion

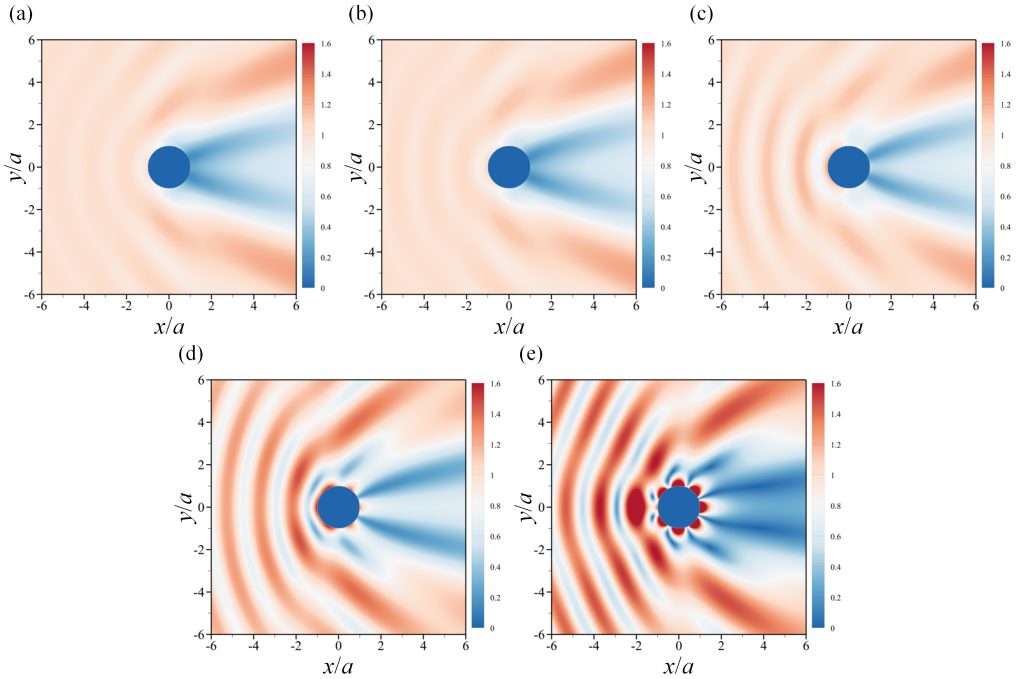


FIGURE 8. For piston-like paddle motion, the maximum free surface elevation at $ka = 2$ when springs and dampers are optimised to absorb 100% of the power available from modes: (a) $m = 0$; (b) $m = 1$; (c) $m = 2$; (d) $m = 3$; (e) $m = 4$.

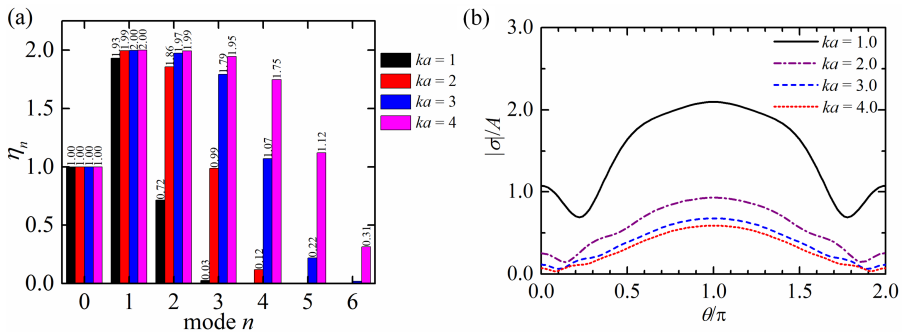


FIGURE 9. For piston-like paddle motion: (a) the partition of capture factor into different circular wave modes when springs and dampers optimised to absorb 100% in mode $m = 0$ at different wavenumbers; (b) the corresponding distribution of paddle amplitudes around the cylinder.

266 of the available power is being absorbed across a number of circular modes. For hinged
 267 paddle motion, the results are similar with roughly double the amplitudes of the piston-
 268 like motion. Figure 10 shows the maximum surface elevation corresponding to the cases
 269 referenced in figure 9.

270 In all the previous results, the springs and dampers have been equal around the
 271 cylinder and this means the device is omni-directional. We now consider the effect of
 272 tuning the springs and dampers to different values around the cylinder where the device
 273 operation becomes dependent on the wave heading. For simplicity however, we only
 274 consider operation under the designed wave heading. Following the recipe for selecting

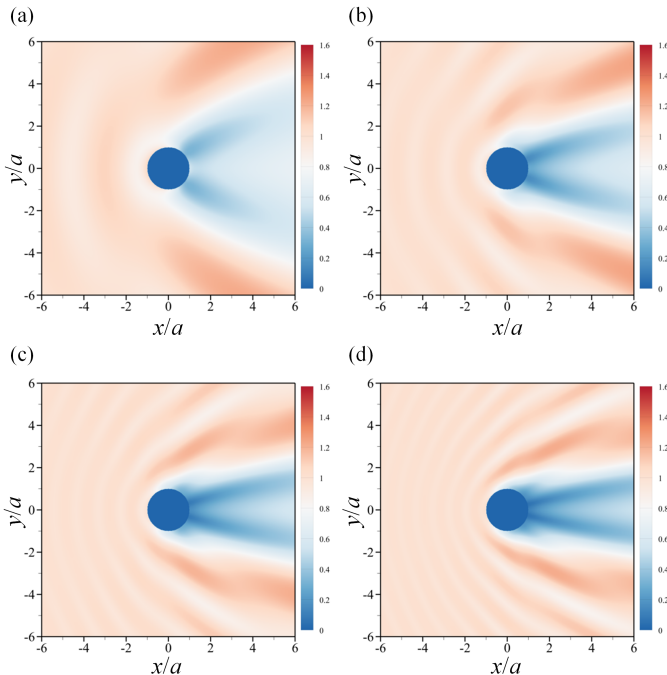


FIGURE 10. For piston-like paddle motion, the maximum free surface elevation for springs and dampers optimised to absorb 100% of the power available from the $m = 0$ mode: (a) $ka = 1$; (b) $ka = 2$; (c) $ka = 3$; (d) $ka = 4$.

275 the spring and damper settings in the main body of the paper we set up the system to
 276 absorb all of available power in the first $M + 1$ circular modes and nothing from higher
 277 modes.

278 Figure 11 shows the maximum free surface amplitude at $ka = 2$ associated with this
 279 system for $M = 1$ ($\eta = 3$) up to $M = 4$ ($\eta = 9$). For subfigure (d) the surface elevation has
 280 exceeded the displayed vertical scale and have been top-sliced in the plot. In that case,
 281 the paddles are working hard to absorb all the available power in the first $M + 1$ circular
 282 modes and undergoing large amplitude excursions dominated by a $\cos M\theta$ variation as
 283 highlighted by figure 12. Negative springs, where they exceptionally occur, can be offset
 284 to positive springs by an increase in paddle mass and this has been confirmed numerically.
 285 The specific strategy of tuning paddles to absorb 100% of the energy from the first $M + 1$
 286 modes at a specific frequency has also led to the prescription of negative dampers. In
 287 this case even though the net power is positive, some of the paddles must be driven
 288 and consume power, rather than absorb power. As can be seen in figure 12(d) this has
 289 undesirable consequences for fixed paddle parameters operating at wave frequencies for
 290 which they were not optimised including a net loss of power (illustrated by the curve for
 291 $M = 4$ dipping below 0).

292 The next set set of figures in this section consider optimising the distribution of springs
 293 and dampers for $M = 3$ ($\eta = 7$) for $ka = 2$ up to $ka = 4$. In figures 13, 14 it is illustrated
 294 that the paddles are forced to work at amplitudes well in excess of practical limits to
 295 absorb 100% of the power from the first four circular modes from low frequency waves
 296 (ka small), but becomes easier for higher frequency waves.

297 The final part of the results section compares continuous paddle theory against a
 298 discrete representation of the paddles. We show a single exemplary case in figure 15 in

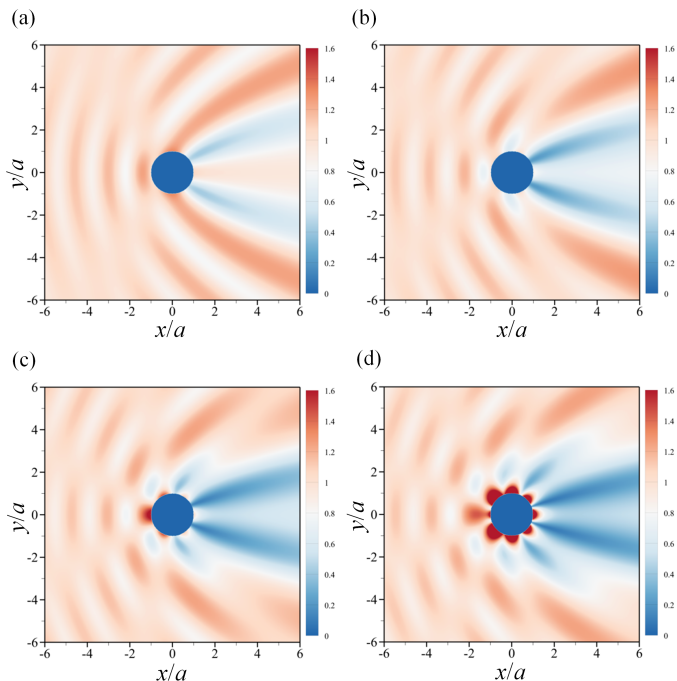


FIGURE 11. For piston-like paddle motion, the maximum free surface elevation at $ka = 2$ when springs and dampers are optimised to absorb 100% of the power available from the first $M + 1$ modes: (a) $M = 1$, (b) $M = 2$, (c) $M = 3$, (d) $M = 4$.

299 which we fix $\bar{\gamma} = \bar{\kappa} = 0.3$ and show the convergence of the results for N paddles placed
 300 around the cylinder towards the results from the continuum theory. As expected, as the
 301 wavelength-to-diameter ratio reduces (ka increases), larger values of N are required to
 302 resolve the variations around the cylinder captured by continuum theory. However, for
 303 the range of values of ka we have been interested in we can see that the continuum
 304 theory provides a good approximation to a discrete representation of $N \approx 24$ paddles.
 305 For example, for a 10m radius cylinder, a system of paddles of width 3m would be
 306 accurately predicted by the continuum description.

307 7. Conclusions

308 In this paper we have outlined a theoretical framework for extending rigid body limits
 309 on the capacity for an axisymmetric device to absorb power from a plane incident wave.
 310 This extends established limits to wave absorption by axisymmetric devices undergoing
 311 rigid body motion by allowing a generalised motion of the surface of the device. This
 312 general framework is developed into a WEC device by considering a circular cylinder
 313 extending throughout the fluid depth and surrounded by narrow submerged vertical
 314 paddles each attached to its own spring and damper. A continuum approximation for
 315 narrow paddles is presented and the power generated by the cylinder is determined from
 316 a system of equations which allow us to develop different strategies to determine spring
 317 and damper settings. Specifically, when all the springs and dampers are identical we can
 318 determine parameters allowing us to guarantee the absorption of 100% of the energy flux
 319 available in one circular component of the plane incident wave. Allowing the springs and
 320 dampers to have different settings as a function of position around the cylinder means we

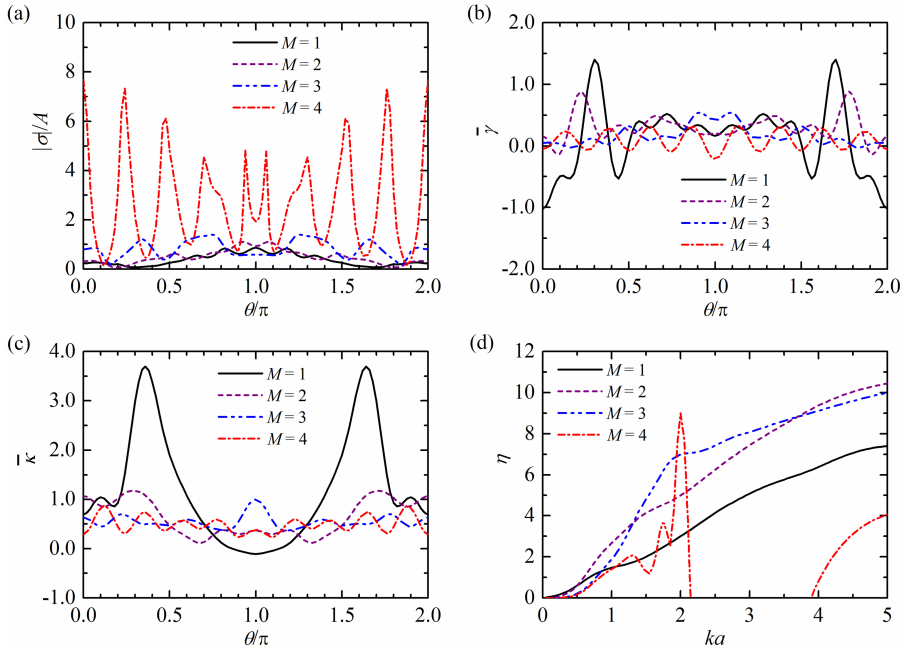


FIGURE 12. For piston-like paddle motion at $ka = 2$ optimised to absorb 100% power from first $M + 1$ circular modes showing the angular variation of: (a) maximum dimensionless paddle amplitude; (b) damping parameter, $\bar{\gamma}$; (c) spring constant $\bar{\kappa}$; (d) the corresponding frequency response of capture factor.

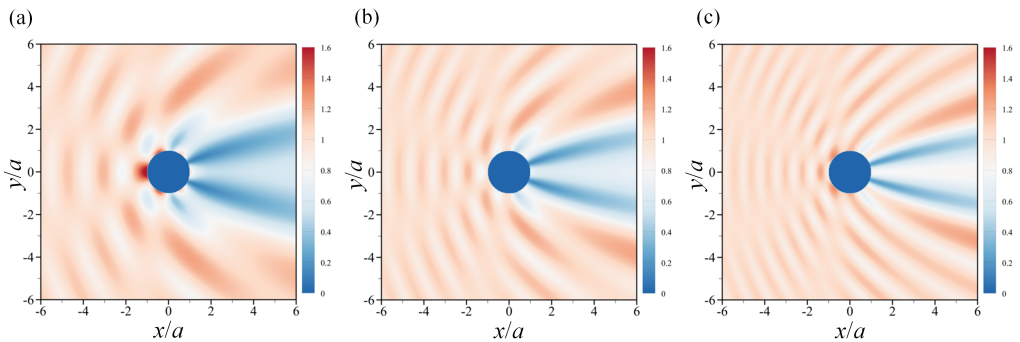


FIGURE 13. For piston-like paddle motion, the maximum free surface elevation when unequal springs and dampers are optimised to absorb 100% of the power available from the first 4 modes ($M = 3$, $\eta = 7$) at (a) $ka = 2$, (b) $ka = 3$, (c) $ka = 4$.

321 can extract 100% of the available flux of energy in the first $M + 1$ circular modes where M
 322 is theoretically as large as we choose. In both cases, results have shown how it is possible
 323 to achieve well in excess of the standard limit of a capture factor of $\eta = 3$ for rigid-body
 324 motion and capture factors in excess of $\eta = 8$ have been reported in computations in this
 325 paper.

326 Despite these claims, there are practical considerations which will limit the value of
 327 results from this theory. Unless the cylinder is large compared to the wavelength, paddle
 328 amplitudes exceed the limits of the underpinning linearised water wave theory as the
 329 demand for power is increased leading to a compromise between power and size of device.

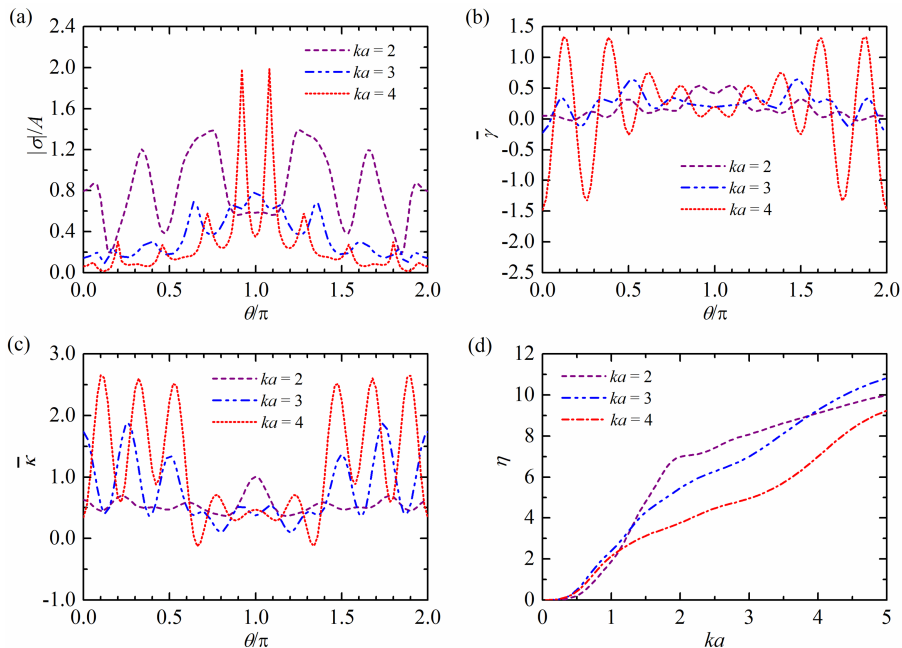


FIGURE 14. For piston-like paddle motion optimised to absorb 100% power from first four circular modes ($M = 3$, $\eta = 7$) showing the angular variation of: (a) maximum dimensionless paddle amplitude; (b) damping parameter, $\bar{\gamma}$; (c) spring constant $\bar{\kappa}$; (d) the corresponding frequency response of capture factor.

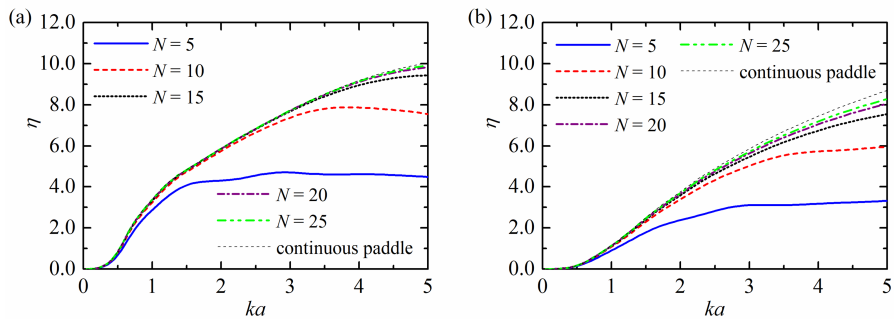


FIGURE 15. Capture factor against dimensionless wavenumber for different number of paddles, N , $\bar{\gamma} = \bar{\kappa} = 0.3$: (a) piston-like paddles, (b) hinged paddles.

330 To fully investigate this, motion constraints such as those used by Evans (1981) could be
 331 implemented.

332 The final part of the paper considers the exact description of N discrete paddles which
 333 is used to confirm that the continuum description of the paddle motion is converged to
 334 as N increases.

335 Paddles are just one means by which the general theory is implemented and other
 336 practical absorbing systems which provide the same effect such as distributing power
 337 absorption across the internal surface of a permeable axisymmetric device may work
 338 just as well (e.g. Zheng *et al.* (2020) or Garnaud & Mei (2009)).

339

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343

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