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Split spin factor algebras

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Abstract

Motivated by Yabe's classification of symmetric 2-generated axial algebras of Monster type [10], we introduce a large class of algebras of Monster type $(\alpha, \frac{1}{2})$, generalising Yabe's $\mathrm{III}(\alpha, \frac{1}{2}, \delta)$ family. Our algebras bear a striking similarity with Jordan spin factor algebras with the difference being that we asymmetrically split the identity as a sum of two idempotents. We investigate the properties of this algebra, including the existence of a Frobenius form and ideals. In the 2-generated case, where our algebra is isomorphic to one of Yabe's examples, we use our new viewpoint to identify the axet, that is, the closure of the two generating axes.

1 Introduction

The class of axial algebras was introduced by Hall, Shpectorov and Rehren [4, 5]. Recently there has been much research into the class of algebras of Jordan type, which includes the classical Jordan algebras and also Matsuo algebras arising from 3-transposition groups, but also into the wider class of axial algebras of Monster type (α, β) (as defined by the fusion law in Table 1). This class adds algebras for some sporadic finite simple groups including the Griess algebra for the Monster M.

Recently Yabe [10] classified symmetric 2-generated axial algebras of Monster type. In doing so, he introduced several new families of 2-generated algebras, in addition to those found by Rehren in [9] and found by Joshi using the double axis construction [6] (see also [3]). The starting point of this article is an attempt to understand one of Yabe's families, $\mathrm{III}(\alpha, \frac{1}{2}, \delta)$. These algebras have the puzzling property that while they are finite dimensional, they have potentially (depending on the field) infinitely many axes. We were looking for a description of these algebras which exhibits their full symmetry and makes them easy to work with by hand.

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We succeeded in doing so and, as a bonus, our description generalises to a much richer family with any number of generators. Our description is also reminiscent of the spin factor Jordan algebras. These are extensions of a quadratic space by an identity. In our new family, we also start with a quadratic space, but instead expand by a 2-dimensional piece with an identity asymmetrically split as the sum of two idempotents. Because of this similarity, we call these algebras *split spin factors*.

Definition 1. Let E be a vector space with a symmetric bilinear form $b\colon E \times E \to \mathbb{F}$ and $\alpha \in \mathbb{F}$. The *split spin factor* $S(b, \alpha)$ is the algebra on $E \oplus \mathbb{F}z_1 \oplus \mathbb{F}z_2$ with multiplication given by

$$
\begin{gathered}
z_1^2 = z_1, \quad z_2^2 = z_2, \quad z_1 z_2 = 0, \\
e z_1 = \alpha e, \quad e z_2 = (1-\alpha)e, \\
ef = -b(e,f)z,
\end{gathered}
$$

for all $e, f \in E$, where $z := \alpha(\alpha-2)z_1 + (\alpha-1)(\alpha+1)z_2$.

As can be seen, z_1 and z_2 are two idempotents and their sum $\mathbb{1} = z_1 + z_2$ is the identity for the algebra. If $\alpha = 1, 0$, then $S(b, \alpha)$ is a direct sum of $\mathbb{F}$ and the spin factor algebra coming from the quadratic space E. Also, when $\alpha = \frac{1}{2}$, the algebra is isomorphic to the spin factor algebra for an extended quadratic space $\hat{E} = E \oplus \mathbb{F}(z_1 - z_2)$. *So we will assume in our statements that* $\alpha \neq 1, 0, \frac{1}{2}$.

We classified all additional non-zero idempotents in this algebra and showed that they fall into two classes:

(a) $\frac{1}{2}\left(e + \alpha z_1 + (\alpha+1)z_2\right)$,

(b) $\frac{1}{2}\left(e + (2-\alpha)z_1 + (1-\alpha)z_2\right)$,

where $e \in E$ of length $b(e,e) = 1$.

Theorem 2. (1) *z_1 and z_2 are primitive axes of Jordan type α and $1-\alpha$, respectively.*

(2) *Idempotents from families* (a) *and* (b) *are primitive axes of Monster type $(\alpha, \frac{1}{2})$ and $(1-\alpha, \frac{1}{2})$, respectively.*

In particular, when E is spanned by vectors of length 1, the algebra is generated by axes and so it becomes an axial algebra of Monster type $(\alpha, \frac{1}{2})$, or $(1-\alpha, \frac{1}{2})$ depending on our choice. In the former case, we can take family (a) with or without z_1 as axes, similarly in the second case, family (b) with or without z_2. Note the symmetry between α and $\alpha' = 1-\alpha$. In particular, $\alpha = 1-\alpha'$ and similarly for the coefficients of the vector z and for the families (a) and (b).

The algebra admits a Frobenius form, that is, a non-zero symmetric bilinear form that associates with the algebra product. In our case, this is just an extension of the form b. The existence of the Frobenius form allows us to decide the simplicity of the algebra using the theory from [7].

Theorem 3. *$S(b,\alpha)$ is simple if and only if b is non-degenerate and $\alpha \notin \{-1, 2\}$.*

When $\alpha = -1, 2$, the algebra is baric, that is, there is an ideal of codimension one, which is also the radical of the Frobenius form. If $\alpha = -1$, then family (a) and z_1 are in the radical and the other way round for $\alpha = 2$.

One of the main features of an axial algebra is that, when the fusion law is C_2-graded, we can associate to each axis x an automorphism τ_x called the Miyamoto involution. In our case, the Miyamoto involution associated to an axis from family (a), or (b) fixes z_1 and z_2 and acts on E as $-r_e$, where r_e is the reflection in the vector e. The full automorphism group of the algebra is the orthogonal group $O(E, b)$.

The last part of the paper is about the 2-generated case. Here E is necessarily 2-dimensional spanned by vectors e and f of length 1 and hence the form b is fully determined by $\mu := b(e, f)$.

In view of the symmetry between α and $\alpha' = 1 - \alpha$, we can focus on two axes x and y from family (a).

Theorem 4. *Let $x = \frac{1}{2}(e + \alpha z_1 + (\alpha+1)z_2)$ and $y = \frac{1}{2}(f + \alpha z_1 + (\alpha+1)z_2)$. If $\alpha \neq -1$ and $\mu \neq 1$ then $S(b,\alpha) = \langle\langle x, y\rangle\rangle$ is isomorphic to Yabe's algebra $\mathrm{III}(\alpha, \frac{1}{2}, \delta)$ with $\delta = -2\mu - 1$.*

We have already mentioned the case of $\alpha = -1$ above. If $\mu = 1$, then $xy = \frac{1}{2}(x + y)$ and so x and y do not generate the whole algebra. In the corresponding case of Yabe's algebra $\mathrm{III}(\alpha, \frac{1}{2}, -3)$, the identity of the algebra turns into a nil element.

Finally, we investigate the closure $X = x^D \cup y^D$ of axes x and y under the action of the Miyamoto group $D := \langle \tau_x, \tau_y \rangle$.

In Section 5, we formulate precise statements for the size of X depending on the value μ. The final statement, Lemma 5.5, is quite technical, but the outcome can be summarised here as follows: In characteristic 0, $|X|$ is generically infinite, but can have any finite size $n \geq 2$ for specific values of μ. Whereas in characteristic p, X can have infinite size if μ is transcendental over the prime subfield, size p or $2p$ for $\mu = \pm 1$, or size coprime to p otherwise.

2 Axial algebras

Throughout this paper we are considering commutative algebras that are not necessarily associative.

Definition 2.1. A *fusion law* is a set $\mathcal{F}$ with a binary operation $*\colon \mathcal{F}\times\mathcal{F}\to 2^{\mathcal{F}}$, where $2^{\mathcal{F}}$ is our notation for the set of all subsets of $\mathcal{F}$. Just like we only consider commutative algebras, we will only consider *commutative* fusion laws, i.e., we will have $\lambda * \mu = \mu * \lambda$ for all $\lambda, \mu \in \mathcal{F}$.

In this paper we are concerned with the fusion laws shown in Table 1. The full fusion law on $\{1, 0, \alpha, \beta\}$ is called the *fusion law of Monster type*

$*$	1	0	α	β
1	1		α	β
0		1	α	β
α	α	α	$1, 0$	β
β	β	β	β	$1, 0, \alpha$

Table 1: Fusion law of Monster type (α, β)

and the sublaw on $\{1, 0, \alpha\}$ is the *fusion law of Jordan type*.

Suppose A is an algebra defined over a field $\mathbb{F}$. For $a \in A$, let ad_a be the adjoint map and $A_\lambda(a)$ be the λ-eigenspace of ad_a (note that we allow this to be trivial). For a set $N \subset \mathbb{F}$, we write $A_N(a) := \bigoplus_{\nu\in N} A_\nu(a)$.

Definition 2.2. Suppose $1 \in \mathcal{F} \subset \mathbb{F}$ is a fusion law. An $\mathcal{F}$-*axis* is a non-zero idempotent $a \in A$ such that

(1) $A = \bigoplus_{\lambda\in\mathcal{F}} A_\lambda(a)$

(2) $A_\lambda(a)A_\mu(a) \subseteq A_{\lambda*\mu}(a)$, for all $\lambda, \mu \in \mathcal{F}$.

We say that a is *primitive* if $A_1(a) = \langle a\rangle$.

Definition 2.3. A commutative algebra A together with a collection X of $\mathcal{F}$-axes which generate A, is called an $\mathcal{F}$-*axial algebra*. The algebra is called *primitive* if all axes in X are primitive.

In this paper, we are focussing on axial algebras of Jordan and Monster type, which are primitive algebras with the fusion laws in Table 1. Both these laws are C_2-graded. This means that for every axis a, the algebra admits a C_2-grading $A = A_+(a) \oplus A_-(a)$, where $A_+(a) = A_1(a) \oplus A_0(a) \oplus A_\alpha(a)$ and $A_-(a) = A_\beta(a)$ for the Monster type fusion law and $A_+(a) = A_1(a) \oplus A_0(a)$ and $A_-(a) = A_\alpha(a)$ for the Jordan type fusion law. Correspondingly, we have an involution $\tau_a \in \mathrm{Aut}(A)$, known as the *Miyamoto involution* defined by

$$v \mapsto \begin{cases} v & \text{if } v \in A_+(a) \\ -v & \text{if } v \in A_-(a) \end{cases}$$

and extended linearly to A. The *Miyamoto group* of A is the subgroup $\mathrm{Miy}(X) \leq \mathrm{Aut}(A)$ generated by τ_a for all $a \in X$.

Often the algebra admits a bilinear form with a nice property.

Definition 2.4. A *Frobenius form* on an axial algebra A is a non-zero symmetric bilinear form $(\cdot,\cdot)\colon A \times A \to \mathbb{F}$ such that

$$(a, bc) = (ab, c)$$

for all $a, b, c \in A$.

3 The algebra

Definition 3.1. Suppose E is a vector space over the field $\mathbb{F}$ with a symmetric bilinear form $b\colon E \times E \to \mathbb{F}$ and let $\alpha \in \mathbb{F}$. Define a commutative algebra $A = S(b, \alpha)$ on the vector space $E \oplus \mathbb{F}z_1 \oplus \mathbb{F}z_2$ by

$$\begin{gathered} z_1^2 = z_1, \quad z_2^2 = z_2, \quad z_1 z_2 = 0, \\ e z_1 = \alpha e, \quad e z_2 = (1-\alpha)e, \\ ef = -b(e,f)z, \end{gathered}$$

for all $e, f \in E$, where $z := \alpha(\alpha-2)z_1 + (\alpha-1)(\alpha+1)z_2$. We call this algebra the *split spin factor* algebra.

Note that the algebra always has an identity given by $\mathbb{1} = z_1 + z_2$. Note also the symmetry between z_1 and z_2: if we set $\alpha' = 1-\alpha$ then $\alpha = 1-\alpha'$ and hence $S(b,\alpha) = S(b,\alpha')$, where in the right side the rôles of z_1 and z_2 are switched.

Let us first dispose of the special cases $\alpha = 0, 1$.

Proposition 3.2. *$S(b, 0)$ and $S(b, 1)$ are both the direct product of a spin factor Jordan algebra by a copy of the field.*

Proof. We just do the case where $\alpha = 0$. In this case, $z = -z_2$ and so $B := E \oplus \mathbb{F}z_2$ is the spin factor Jordan algebra. Since $\alpha = 0$, $z_1 b = 0$ for all $b \in B$, and so A is the direct sum of B with $\mathbb{F}z_1 \cong \mathbb{F}$. □

Another special situation is where $\alpha = \frac{1}{2}$, in which case, clearly, $\mathrm{char}(\mathbb{F}) \neq 2$.

Proposition 3.3. *$S(b, \frac{1}{2})$ is a spin factor Jordan algebra.*

Proof. Since $\alpha = \frac{1}{2}$, we have that $z = -\frac{3}{4}z_1 - \frac{3}{4}z_2 = -\frac{3}{4}\mathbb{1}$. Let $u = z_1 - z_2$ and note that $eu = ue = e(z_1 - z_2) = \frac{1}{2}e - \frac{1}{2}e = 0$ and $u^2 = z_1^2 + z_2^2 = z_1 + z_2 = \mathbb{1}$. Set $\hat{E} = E \oplus \mathbb{F}u$. Then for $v = e + \gamma u$ and $w = f + \delta u$, where $e, f \in E$ and $\gamma, \delta \in \mathbb{F}$, we have that $vw = ef + \gamma uf + \delta eu + \gamma\delta u^2 = (\frac{3}{4}b(e,f) + \gamma\delta)\mathbb{1}$. The expression in the brackets is a symmetric bilinear form on $\hat{E}$. Hence, indeed, $S(b, \frac{1}{2})$ is a spin factor algebra. □

In particular, in all three special cases, $\alpha = 1, 0, \frac{1}{2}$, the algebra is a Jordan algebra. Note that we could also use the language of [1] and view these algebras as axial decomposition algebras for the full Monster type fusion law.

In view of Propositions 3.2 and 3.3, *in the remainder of the paper we will assume that* $\alpha \neq 1, 0, \frac{1}{2}$.

Let us denote the subalgebra $\mathbb{F}z_1 \oplus \mathbb{F}z_2$ by Z; this is isomorphic to $\mathbb{F}^2$. Let us also note that the orthogonal group $G = O(E, b)$, extended to the entire A by letting all its elements fix z_1 and z_2, preserves the algebra product and hence is a subgroup of $\mathrm{Aut}(A)$. We will show later in the paper that G is the full group of automorphisms of A.

The next step is to determine all idempotents in A.

Proposition 3.4. *Let $A = S(b, \alpha)$ be the split spin factor algebra. Then a non-zero idempotent in A is one of $\mathbb{1}$, z_1, z_2, or is in one of the following two families:*

(a) $\frac{1}{2}\left(e + \alpha z_1 + (\alpha+1)z_2\right)$,

(b) $\frac{1}{2}\left(e + (2-\alpha)z_1 + (1-\alpha)z_2\right)$,

where $e \in E$ such that $b(e, e) = 1$. Note that the two families require $\mathrm{char}(\mathbb{F}) \neq 2$.

Proof. Let $x = e + \gamma z_1 + \delta z_2$ be a non-zero idempotent where $\gamma, \delta \in \mathbb{F}$. If $e = 0$, then x is an idempotent in $Z \cong \mathbb{F}^2$ and so it is clearly one of the three idempotents $\mathbb{1}$, z_1, or z_2. Now suppose that $e \neq 0$. We deduce the relations:

$$\begin{aligned}
x^2 &= (e + \gamma z_1 + \delta z_2)^2 \\
&= -b(e,e)z + 2\gamma z_1 e + 2\delta z_2 e + 2\gamma\delta z_1 z_2 + \gamma^2 z_1 + \delta^2 z_2 \\
&= -b(e,e)\left(\alpha(\alpha-2)z_1 + (\alpha-1)(\alpha+1)z_2\right) \\
&\quad + 2\gamma\alpha e + 2\delta(1-\alpha)e + \gamma^2 z_1 + \delta^2 z_2 \\
&= \left(2\gamma\alpha + 2\delta(1-\alpha)\right)e \\
&\quad + \left(\gamma^2 - \alpha(\alpha-2)b(e,e)\right) z_1 \\
&\quad + \left(\delta^2 - (\alpha-1)(\alpha+1)b(e,e)\right) z_2
\end{aligned}$$

Since $x^2 = x$ and $e \neq 0$, we have the following three equations

$$1 = 2\gamma\alpha + 2\delta(1-\alpha) \tag{1}$$

$$\gamma^2 - \gamma = \alpha(\alpha-2)b(e,e) \tag{2}$$

$$\delta^2 - \delta = (\alpha-1)(\alpha+1)b(e,e) \tag{3}$$

From Equation (1), we see that $\mathrm{char}(\mathbb{F}) \neq 2$ and $\alpha\gamma = \frac{1}{2} - \delta(1-\alpha)$. Hence,

$$\begin{aligned}
\alpha^2(\gamma^2 - \gamma) &= \alpha\gamma(\alpha\gamma - \alpha) \\
&= (\tfrac{1}{2} - \delta(1-\alpha))(\tfrac{1}{2} - \alpha - \delta(1-\alpha))) \\
&= \tfrac{1}{2}(\tfrac{1}{2} - \alpha) - (1-\alpha)(\tfrac{1}{2} - \alpha)\delta - \tfrac{1}{2}(1-\alpha)\delta + (1-\alpha)^2\delta^2 \\
&= \tfrac{1}{2}(\tfrac{1}{2} - \alpha) + (1-\alpha)^2(\delta^2 - \delta)
\end{aligned}$$

So we have a system of linear equations for the variables $u = \gamma^2 - \gamma$ and $v = \delta^2 - \delta$ given by $\alpha^2 u - (1-\alpha)^2 v = \frac{1}{2}(\frac{1}{2} - \alpha)$ and $(\alpha^2 - 1)u - \alpha(\alpha - 2)v = 0$. The determinant of the matrix on the left hand side is $-\alpha^3(\alpha-2) + (1-\alpha)^2(\alpha^2 - 1) = 2\alpha - 1 \neq 0$ since $\alpha \neq \frac{1}{2}$. Therefore, we find that the unique solution is $u = \frac{1}{4}\alpha(\alpha - 2)$ and $v = \frac{1}{4}(\alpha^2 - 1)$. Substituting these into Equations (2) and (3) we get that $\frac{1}{4}\alpha(\alpha-2) = \alpha(\alpha-2)b(e,e)$ and $\frac{1}{4}(\alpha^2 - 1) = (\alpha^2 - 1)b(e,e)$. Note that $\alpha(\alpha - 2)$ and $\alpha^2 - 1$ are equal to zero at the same time if and only if $\alpha = -1 = 2$, which means that $\mathrm{char}(\mathbb{F}) = 3$. However, then α is also equal to $\frac{1}{2}$, a contradiction. So, we see that $b(e,e) = \frac{1}{4}$. Using this, we deduce from Equation (2) that $0 = \gamma^2 - \gamma - \frac{1}{4}\alpha(\alpha - 2) = (\gamma - \frac{1}{2}\alpha)(\gamma + \frac{1}{2}(\alpha - 2))$ and so get the two solutions for γ. Then the corresponding values for δ come from Equation (1). $\square$

To clarify the relationship between families (a) and (b), note that if $x = \frac{1}{2}(e + \alpha z_1 + (\alpha+1)z_2)$ is in family (a) then $y = \mathbb{1} - x = z_1 + z_2 - x = \frac{1}{2}(-e + (2-\alpha)z_1 + (1-\alpha)z_2)$ is in family (b), and vice versa.

Next we investigate which fusion law is satisfied for each non-zero, non-identity idempotent. We start with z_1 and z_2.

Proposition 3.5. *The idempotents z_1 and z_2 are primitive and they satisfy the fusion law of Jordan type α and $1 - \alpha$, respectively.*

Proof. In view of symmetry between z_1 and z_2, we just deal with z_1. We have that $z_1 \in A_1(z_1)$, $z_2 \in A_0(z_1)$, and $E \subseteq A_\alpha(z_1)$. Since $\{z_1\} \cup \{z_2\} \cup E$ contains a basis of A, we have equalities: $A_1(z_1) = \mathbb{F}z_1$, $A_0(z_1) = \mathbb{F}z_2$, and $A_\alpha(z_1) = E$. In particular, z_1 is primitive. Manifestly, $A_1(z_1)A_1(z_1) = A_1(z_1)$, $A_0(z_1)A_0(z_1) = A_0(z_1)$, and $A_1(z_1)A_0(z_1) = 0$. Also, $(A_1(z_1) + A_0(z_1))A_\alpha(z_1) \subseteq A_\alpha(z_1)$. Finally, $A_\alpha(z_1)A_\alpha(z_1) = EE \subseteq \mathbb{F}z \subseteq A_1(z_1) + A_0(z_1)$. $\square$

Since the Jordan type fusion law is C_2-graded, A admits Miyamoto involutions $\tau_{z_i} \in \mathrm{Aut}(A)$. For $u = e + \gamma z_1 + \delta z_2 \in A$, where $e \in E$ and $\gamma, \delta \in \mathbb{F}$, we set $u^- = -e + \gamma z_1 + \delta z_2$. From the description of the eigenspaces of ad_{z_1} above, we see that $\sigma := \tau_{z_1}$ is the central involution of $G = O(E, b)$ negating all of E. It follows also that $\sigma = \tau_{z_2}$ and that $u^\sigma = u^-$ for all $u \in A$. In particular, u is an idempotent if and only if u^- is an idempotent and they are in the same family.

We now turn to families (a) and (b) from Proposition 3.4. In particular, in this segment of the paper $\mathrm{char}(\mathbb{F}) \neq 2$. Note again that the symmetry between z_1 and z_2 switches the rôles of families (a) and (b). Indeed, if we again set $\alpha' = 1 - \alpha$, then $\frac{1}{2}(e + (2-\alpha)z_1 + (1-\alpha)z_2) = \frac{1}{2}(e + (\alpha' + 1)z_1 + \alpha' z_2)$. This means that it suffices to consider an arbitrary idempotent $x = \frac{1}{2}(e + \alpha z_1 + (\alpha+1)z_2)$ from family (a). Here $e \in E$ satisfies $b(e,e) = 1$.

We first observe the following.

Proposition 3.6. *For an idempotent $x = \frac{1}{2}(e+\alpha z_1+(\alpha+1)z_2)$, the subspace $B_x := \langle x, x^-, z_1\rangle$ is a subalgebra isomorphic to* $3\mathrm{C}(\alpha)$*.*

Proof. We have that $xz_1 = \frac{1}{2}(\alpha e + \alpha z_1) = \frac{\alpha}{2}(x + z_1 - x^-)$ since $e = x - x^-$. Applying σ, we also get $x^- z_1 = \frac{\alpha}{2}(x^- + z_1 - x)$. Finally, $xx^- = \frac{1}{4}(e + \alpha z_1 + (\alpha+1)z_2)(-e + \alpha z_1 + (\alpha+1)z_2) = \frac{1}{4}(-e^2 + (\alpha z_1 + (\alpha+1)z_2)^2) = \frac{1}{4}(z+\alpha^2 z_1+(\alpha+1)^2 z_2) = \frac{1}{4}(\alpha(\alpha-2)z_1+(\alpha-1)(\alpha+1)z_2+\alpha^2 z_1+(\alpha+1)^2 z_2) = \frac{1}{4}(2\alpha(\alpha-1)z_1 + 2\alpha(\alpha+1)z_2) = \frac{\alpha}{2}((\alpha-1)z_1 + (\alpha+1)z_2) = \frac{\alpha}{2}(x + x^- - z_1)$ since $x + x^- = \alpha z_1 + (\alpha+1)z_2$. ∎

Proposition 3.7. *Idempotents from families* (a) *and* (b) *from Proposition* 3.4 *are primitive and satisfy the fusion law of Monster type* $(\alpha, \frac{1}{2})$ *and* $(1 - \alpha, \frac{1}{2})$*, respectively.*

Proof. As discussed before the proposition, it suffices to consider $x = \frac{1}{2}(e + \alpha z_1 + (\alpha+1)z_2)$. We first determine the eigenspaces. Inside B_x, we see $\mathbb{F}x \subseteq A_1(x)$, $\mathbb{F}y \subseteq A_0(x)$, where $y = \mathbb{1} - x = \frac{1}{2}(-e + (2-\alpha)z_1 + (1-\alpha)z_2)$, and $\mathbb{F}(z_1 - x^-) \subseteq A_\alpha(x)$. Next, if $f \in e^\perp$ (a hyperplane of E) then $xf = \frac{1}{2}(-b(e,f)z+\alpha^2 f+(1-\alpha)(\alpha+1)f) = \frac{1}{2}f$, since $b(e,f) = 0$. This means that $A_{\frac{1}{2}}(x) \supseteq e^\perp$. Since, manifestly, $A = B_x \oplus e^\perp$, we conclude that $A_1(x) = \mathbb{F}x$, $A_0(x) = \mathbb{F}y$, $A_\alpha(x) = \mathbb{F}(z_1 - x^-)$, and $A_{\frac{1}{2}}(x) = e^\perp$.

Turning to the fusion law for x, since B_x is an algebra of Jordan type α, we know all the fusion rules on the set $\{1, 0, \alpha\}$. Note that the linear map $\tau\colon A \to A$ acting as identity on B_x and negating $e^\perp$ coincides with $-r_e \in G$ (r_e is the reflection in the hyperplane of E perpendicular to e) and hence it is an automorphism of A of order 2. Hence, the fusion law of x admits a C_2-grading with $A_+ = B_x = A_1(x) + A_0(x) + A_\alpha(x)$ and $A_- = e^\perp = A_{\frac{1}{2}}(x)$. This readily implies that x obeys the Monster fusion law of type $(\alpha, \frac{1}{2})$, as claimed. ∎

We note for the future that $\tau_x = \tau = -r_e \in G$. This gives us the following fact.

Proposition 3.8. *We have that $\tau_x = \tau_{x^-} = \tau_{\mathbb{1}-x} = \tau_{\mathbb{1}-x^-}$. Moreover, these are the only axes with this Miyamoto involution.*

Let us summarise.

Theorem 3.9. *The algebra $A = S(b, \alpha)$, where $E \neq 0$, is a primitive axial algebra of Monster type $(\alpha, \frac{1}{2})$ if and only if* $\mathrm{char}(\mathbb{F}) \neq 2$ *and E is spanned by vectors e with $b(e, e) = 1$.*

Proof. If $\mathrm{char}(\mathbb{F}) = 2$ then z_1 and z_2 are the only axes, and they only generate Z. Hence we assume that $\mathrm{char}(\mathbb{F}) \neq 2$. It is clear from the definition that, for every subspace $W \subseteq E$, $A_W := W \oplus Z$ is closed for products and hence is a subalgebra. Let $W = \langle e \in E \mid b(e, e) = 1\rangle$. If $W \neq E$ then according to Proposition 3.4, A_W contains all idempotents from A, and so A is not generated by idempotents, so it is not an axial algebra. On the other hand, if $W = E$ then it is immediate from Proposition 3.4 that A is generated (in fact, spanned) by z_1 and the idempotents from family (a). By Propositions 3.5 and 3.7, all these idempotents are primitive and satisfy the fusion law of Monster type $(\alpha, \frac{1}{2})$. □

In view of symmetry between z_1 and z_2, A is an algebra of Monster type $(\alpha, \frac{1}{2})$ if and only if it is an algebra of Monster type $(1 - \alpha, \frac{1}{2})$— we just need to switch from z_1 and family (a) to z_2 and family (b). Also note that, in both realisations, A is 1-closed, that is, it is spanned by the axes. Finally, A is also generated by family (a) (respectively, (b)) alone. However, with this set of generating axes, A is 2-closed, since family (a) (respectively, (b)) spans a subspace of A of codimension 1 (we assume that $E \neq 0$).

At this point we are ready to determine the full automorphism group of $A = S(b, \alpha)$.

Theorem 3.10. *Assume that $E \neq 0$. Either $A \cong 3\mathrm{C}(\alpha)$, or $\mathrm{Aut}(A) = G = O(E, b)$.*

Proof. Suppose that $A \ncong 3\mathrm{C}(\alpha)$. We claim that every automorphism of A fixes z_1 and z_2. Indeed, the adjoints of z_1 and z_2 have different spectrum, so it suffices to show that they are not conjugate to idempotents from families (a) and (b). If $\dim(E) \geq 2$ then the latter idempotents have $\frac{1}{2}$ in the spectrum, so they are not conjugate to z_i. If $\dim(E) = 1$ and there exists $e \in E$ with $b(e, e) = 1$ then $A \cong 3\mathrm{C}(\alpha)$ by Proposition 3.6; a contradiction. In the remaining cases, families (a) and (b) are empty, and so the claim holds. We have shown that $\mathrm{Aut}(A)$ fixes z_1 and z_2.

Now, since $\mathrm{Aut}(A)$ fixes z_1, it stabilises every eigenspace of ad_{z_1}; in particular, $E = A_\alpha(z_1)$ is left invariant under $\mathrm{Aut}(A)$. Since $\mathrm{Aut}(A)$ acts as the identity on $Z = \langle z_1, z_2\rangle$ and since $A = E \oplus Z$, it follows that $\mathrm{Aut}(A)$ acts faithfully on E. Furthermore, since $b(e, f)z = (b(e, f)z)^\varphi = (-ef)^\varphi = -e^\varphi f^\varphi = b(e^\varphi, f^\varphi)z$ for all $e, f \in E$ and $\varphi \in \mathrm{Aut}(A)$, we see that $\mathrm{Aut}(A)$ preserves the form b and this means that $\mathrm{Aut}(A) = G = O(E, b)$, as claimed. □

The automorphism group of $A = 3\mathrm{C}(\alpha)$, $\alpha \neq \frac{1}{2}$, is known to be isomorphic to S_3, and it is strictly bigger than $O(E, b)$, which is of order 2 in this case. So here we have a true exception to the theorem.

Recall that a Frobenius form on an algebra A is a non-zero symmetric bilinear form that associates with the algebra product.

Theorem 3.11. *The algebra $A = S(b, \alpha)$ admits a Frobenius form $(\cdot, \cdot)$ given by*

$$(e, f) = (\alpha + 1)(2 - \alpha)b(e, f), \quad (e, z_1) = (e, z_2) = 0,$$
$$(z_1, z_1) = \alpha + 1, \quad (z_2, z_2) = 2 - \alpha, \quad (z_1, z_2) = 0,$$

for all $e, f \in E$ and extended linearly to A.

Proof. We begin by noting that the form is invariant under the symmetry which exchanges z_1 and z_2 and also exchanges α and $\alpha' = 1 - \alpha$. In light of this, we just need to check $(a, bc) = (ab, c)$ for the following triples (a, b, c): (z_1, z_1, z_1), (z_1, z_1, z_2), (z_1, z_1, e), (z_1, e, f) and (e, f, g), for all $e, f, g \in E$. Since the form is symmetric, $(z_1^2, z_1) = (z_1, z_1^2)$. Notice that z_2 and e are a 0- and α-eigenvector for z_1, respectively. Since $(z_1, z_2) = 0 = (z_1, e)$, we get $(z_1^2, z_2) = 0 = (z_1, z_1 z_2)$ and $(z_1^2, e) = 0 = (z_1, z_1 e)$. For the remaining two, we calculate: $(z_1, ef) = \big(z_1, -b(e, f)(\alpha(\alpha - 2)z_1 + (\alpha^2 - 1)z_2)\big) = -\alpha(\alpha + 1)(\alpha - 2)b(e, f)$ and $(z_1 e, f) = \alpha(e, f) = \alpha(\alpha + 1)(2 - \alpha)b(e, f)$ which are equal. Finally, $(e, fg) = (e, -b(f, g)z) = 0$ and hence $(e, fg) = 0 = (ef, g)$, as required. ☐

Since the Frobenius form is an extension of (a scaled version of) b, it is invariant under $G = O(E, b)$. One can also check that if $\dim(E) = 1$ and $A \cong$ 3A, then the Frobenius form is invariant under S_3.

Note that for the particular scaling of the form that we have chosen, idempotents in family (a) have length $\alpha + 1$, the same as z_1, and those in family (b) have length $2 - \alpha$, the same as z_2.

4 Simplicity

In this section we additionally assume that $\mathrm{char}(\mathbb{F}) \neq 2$, and E is spanned by vectors e with $b(e, e) = 1$.

Recall that the algebra $S(b, \alpha)$ admits a Frobenius form. We first discuss the case where some of the axes have length 0 with respect to the form. Recall also that z_1 and the idempotents from family (a) have length $\alpha + 1$ while z_2 and the idempotents from family (b) have length $2 - \alpha$. Hence the special cases are $\alpha = -1$ and $\alpha = 2$. We note that $-1 = 2$ only when $\mathrm{char}(\mathbb{F}) = 3$, in which case α is also equal to $\frac{1}{2}$, which we assumed not to be the case. Thus, at least half of the idempotents are always non-singular.

From [7], the radical of the Frobenius form is an ideal.

Proposition 4.1. *If $\alpha = -1$ then the Frobenius form has rank 1 and its radical coincides with $E \oplus \mathbb{F}z_1$. Symmetrically, if $\alpha = 2$ then the Frobenius form also has rank 1 and its radical coincides with $E \oplus \mathbb{F}z_2$.*

Proof. Because of the definition of the Frobenius form, E is in the radical for $\alpha = -1$ and 2. Similarly, z_1 is in the radical for $\alpha = -1$ and z_2 is in the radical for $\alpha = 2$. □

Note that in these cases $S(b, \alpha)$ is baric. We will explore this in more detail in the next section.

For the remainder of the section, we assume that $\alpha \neq -1, 2$. Then all non-zero, non-identity idempotents are non-singular. Recall that, according to [7], ideals split into two kinds: the ones that do not contain axes, and the ones that do.

Ideals of the first kind are contained in the radical of the algebra, which is defined as the largest ideal not containing axes, and which in our case coincides with the radical of the Frobenius form (see [7, Theorem 4.9]).

Proposition 4.2. *If $\alpha \notin \{-1, 2\}$, the radical of $S(b, \alpha)$ coincides with the radical of the form b.*

Proof. Since z_1 and z_2 are non-singular and $\mathbb{F}z_1$ and $\mathbb{F}z_2$ split off as direct summands, the radical of the Frobenius form is contained in E and hence the claim holds. □

The ideals of the second kind are controlled by the projection graph, which in the present situation is the same as the non-orthogonality graph on the set of axes (cf. [7, Lemma 4.17]).

Proposition 4.3. *If $\alpha \notin \{-1, 2\}$ then there are no proper ideals in $S(b, \alpha)$ that contain axes.*

Proof. Without loss of generality, we view A as an algebra of type $(\alpha, \frac{1}{2})$, that is, we can assume that the set of axes is the union of family (a) with $\{z_1\}$. Now from the definition of the Frobenius form and from the description of the family (a), it is clear that z_1 is non-orthogonal to all idempotents in family (a), which means that the non-orthogonality graph is connected. According to [7, Corollary 4.15], this means that none of these idempotents lie in a proper ideal. □

We can now summarise when $S(b, \alpha)$ is simple.

Theorem 4.4. *The algebra $S(b, \alpha)$ is simple if and only if b is non-degenerate and $\alpha \notin \{-1, 2\}$.*

5 2-generated case

We write $\langle\langle x, y \rangle\rangle$ for the subalgebra generated by x and y. In this section, we assume that $A = \langle\langle x, y \rangle\rangle$ is generated by two axes $x = \frac{1}{2}(e + \alpha z_1 + (\alpha + 1)z_2)$ and $y = \frac{1}{2}(f + \alpha z_1 + (\alpha + 1)z_2)$ from family (a). Equivalently, $E = \langle e, f \rangle$, where $b(e, e) = 1 = b(f, f)$. We let $\mu := b(e, f)$.

Theorem 5.1. *If $\alpha \neq -1$ and $\mu \neq 1$ then $S(b,\alpha) = \langle\langle x,y\rangle\rangle$ is isomorphic to Yabe's algebra* $\mathrm{III}(\alpha, \frac{1}{2}, \delta)$ *with $\delta = -2\mu - 1$.*

Proof. Let $a_0 = x$, $a_1 = y$, and $a_{-1} = y^{\tau_x} = \frac{1}{2}(-f + 2\frac{b(e,f)}{b(e,e)}e + \alpha z_1 + (\alpha + 1)z_2) = \frac{1}{2}(-f + 2\mu e + \alpha z_1 + (\alpha+1)z_2)$. Assuming that $\alpha \neq -1$ and $\mu \neq 1$, let $q = \frac{\alpha(\alpha+1)(\mu-1)}{4}\mathbb{1}$. We verified in MAGMA that these four elements satisfy the relations for the basis of $\mathrm{III}(\alpha, \frac{1}{2}, -2\mu - 1)$ and $S(b,\alpha) = \langle\langle x,y\rangle\rangle = \langle x, y, a_{-1}, q\rangle$. $\square$

We have already discussed what happens when $\alpha = -1$. Namely, x and y are in the radical of the algebra and so they do not generate it. If $\mu = 1$ then x and y also do not generate $S(b,\alpha)$. In fact, in this case, $xy = \frac{1}{2}(x+y)$ and $\langle\langle x,y\rangle\rangle = \langle x,y\rangle$ is a 2-dimensional Jordan algebra.

In the remainder of this section, we investigate the *gonality* of the algebra $S(b,\alpha) = \langle\langle x,y\rangle\rangle$, that is, the cardinality of $x^D \cup y^D$, where $D = \langle \tau_x, \tau_y\rangle \leq G$ is the Miyamoto group of the 2-generated algebra. Note that this set may be a small part of all the available axes, just as D may be a proper subgroup of G.

Lemma 5.2. *The action of $\rho := \tau_x\tau_y$ on E is given by*

$$\rho = \begin{pmatrix} -1 & 2\mu \\ -2\mu & 4\mu^2 - 1 \end{pmatrix}$$

Proof. We have $\tau_x = -r_e$ and $\tau_y = -r_f$ and so the result follows from a calculation. $\square$

Note that since the determinant of ρ is 1, ρ have eigenvalues ζ and ζ^{-1} in (a suitable extension of) $\mathbb{F}$. Also, $\zeta + \zeta^{-1} = \mathrm{tr}(\rho) = 4\mu^2 - 2$ and hence $\mu^2 = \frac{\zeta+\zeta^{-1}+2}{4}$.

Lemma 5.3. (1) *If ζ is not a root of unity, then ρ has infinite order.*

(2) *If $\zeta \neq 1$ is of order n, then ρ has finite order n.*

(3) *If $\zeta = 1$, then ρ has order p if $\mathbb{F}$ has characteristic p and infinite order if $\mathbb{F}$ has characteristic* 0.

Proof. Since ρ^k has eigenvalues ζ^k and ζ^{-k}, the first claim is clear. Suppose first that $\zeta \neq 1$ has order n. Note that the only case where $\zeta = \zeta^{-1}$ is where $\zeta = -1$. In this case, $\mu = 0$ and $\rho = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ which has order 2, the same as the order of $\zeta = -1$. Now assume additionally that $\zeta \neq -1$ and so $\zeta \neq \zeta^{-1}$. Since ρ has two different eigenvalues, it is semisimple and conjugate to the diagonal matrix with entries ζ and ζ^{-1}, so ρ has the same order as ζ. Finally, let $\zeta = 1 = \zeta^{-1}$ and so $\mu = \pm 1$. Then $\rho = \left(\begin{smallmatrix} -1 & \pm 2 \\ \mp 2 & 3 \end{smallmatrix}\right)$. Since this is not the identity, its Jordan normal form is the 2×2 block $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. Hence, ρ has order p in characteristic p and infinite order otherwise. $\square$

Corollary 5.4. *Suppose that $\mathbb{F}$ is algebraically closed. In characteristic 0, ρ can have any finite order n as well as infinite order. In positive characteristic p, the order is finite as long as ζ is algebraic over $\mathbb{F}_p$ and it can be p, or any number n coprime to p.*

For a non algebraically closed field, the best we can say is that in positive characteristic p, if the order of ρ is finite, it is either p, or coprime to p.

In practical terms, there are two questions. First, suppose that we have μ and we want to know the order of ρ. We solve for ζ from the quadratic $\zeta^2 - (4\mu^2 - 2)\zeta + 1 = 0$ and then the order of ζ (possibly in a field extension) is the answer. But note the exception when $\mu = \pm 1$, where the order is $\mathrm{char}(\mathbb{F})$ or infinity. The other possible question is to list all values μ giving finite order n. Here it is best to start with all ζ of order n and find μ using $\mu^2 = \frac{\zeta + \zeta^{-1} + 2}{4}$. The number of different μ and ζ in an algebraically closed field is $\varphi(n)$ in all cases except where $n = \mathrm{char}(\mathbb{F}) = p$, in which case we have $\zeta = 1$ and so $\mu = \pm 1$.

In [8], we introduce the notion of an axet $X := (G, X, \tau)$ for an axial algebra. This encodes the action of the Miyamoto group G on the closed set of axes X together with the τ-map. We do not want to give all the details here, but just provide enough to identify the axet. Namely, we just want to know whether $D := \langle \tau_x, \tau_y \rangle$ has one or two orbits on $x^D \cup y^D$ and what the orbit lengths are.

Lemma 5.5. (1) *If ρ has infinite order, then x and y are in two different (infinite) orbits.*

(2) *If ρ has finite order n, then both x^D and y^D have length n.*

(3) *If n is even, then x^D and y^D are disjoint.*

(4) *If n is odd, then $x^- \notin x^D$ and either x lies in y^D, or x^- does. In the former case, $x^D = y^D$ and in the latter, they are disjoint.*

Proof. The centraliser in D of τ_x is either $\langle \tau_x \rangle$, or n is finite and even and $C_D(\tau) = \langle \tau_x, \rho^{\frac{n}{2}} \rangle$. In the latter case, $\rho^{\frac{n}{2}} = -\mathrm{id}$ and so it does not fix x. So in all cases, the stabiliser of x in D is just $\langle \tau_x \rangle$ and so the orbit length is always n, whether it is finite or infinite. If n is infinite or finite even, then τ_x and τ_y are not conjugate in the dihedral group D and so x and y cannot be in the same orbit. Finally, we consider the case where n is finite and odd. Then τ_x and τ_y are conjugate in D. By Proposition 3.8, y is conjugate either to x, or to x^-. So it remains to see that x and x^- are not conjugate. If they are conjugate, then there is some power ρ^k which conjugates x to x^- and so e to $-e$. Thus ρ^k has eigenvalue $\zeta = -1$ and hence the other eigenvalue is $\zeta^{-1} = -1$. Therefore ρ^k has order 2, contradicting ρ having odd order. $\square$

In terms of axets and using the notation from [8], this means that for ρ having infinite order we have the axet $X(\infty)$. For even n, we have $X(2n)$

and for odd n we have either $X(n)$, or $X(2n)$ depending on whether x and y are conjugate under D.

Finally, let us finish with the following question concerning the baric situation.

Question 5.6. When $A = \langle\langle x, y \rangle\rangle$ and $\alpha = 2$, is the baric algebra A a quotient of the highwater algebra $\mathcal{H}$ [2].

We expect that it is a quotient of $\mathcal{H}$, at least for some values of μ.

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