Approximate Controllability and Observability measures in Control Systems Design

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Declaration

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Abstract

The selection of systems of inputs and outputs (input and output structure) forms part of early system design, which is important since it preconditions the potential for control design. Existing methodologies for input, output structure selection rely on criteria expressing distance from uncontrollability (unobservability). The thesis introduces novel measures for evaluating and estimating the distance to uncontrollability and relatively unobservability. At first, the modal measuring approach is studied in detail, providing a framework for the "best" structure selection. Although controllability (observability) is invariant under state feedback (output injection), the corresponding degrees expressing distance from uncontrollability (unobservability) are not.

Hence, the thesis introduces new criteria for the distance problem from uncontrollability (unobservability) which is invariant under feedback transformations. The approach uses the restricted input-state (state-output) matrix pencil and then deploys exterior algebra that reduces the overall problem to the standard problem of distance of a set of polynomials from non-coprimeness. Results on the distance of the Sylvester Resultants from singularity provide the new measures.

Since distance to singularity of the corresponding Sylvester matrix is the key in evaluating the distance to uncontrolability it is of the particular interest in the present work. In order to find the solution two novel methods are introduced in the thesis, namely *the alternating projection algorithm* and *a structured singular value approach.* A least-squares alternating projection algorithm, motivated by a factorisation result involving the Sylvester resultant matrix, is proposed for calculating the "best" approximate GCD of a coprime polynomial set. The properties of the proposed algorithm are investigated and the method is compared with alternative optimisation techniques which can be employed to solve the problem.

It is also shown that the problem of an approximate GCD calculation is equivalent to the solution of a structured singular value (μ) problem arising in robust control for which numerous techniques are available. Motivated by the powerful concept of the structured singular values, the proposed method is extended to the special case of an implicit system that has a wide application in the behavioural analysis of complex systems. Moreover, μ-value approach has a potential application for the general distance problem to uncontrollability that is numerically hard to obtain.

Overall, the proposed framework significantly simplifies and generalises the input-output structure selection procedure and evaluates alternative solutions for a variety of distance problems that appear in Control Theory.

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Nomenclature

Symbols

$\overline{\text{vec}}(A) \in \mathbb{R}^r$	denotes the vector vec(A), where $r = \frac{n(n+1)}{2}, A \in \mathbb{R}^{n \times n}, A = A'$ with all the elements of A below the main diagonal eliminated (and $\overline{\text{vec}}^{-1}(\cdot)$ denotes the inverse operation)
vec(A)	denotes the vectorisation operation of stacking the columns of $A \in \mathbb{R}^{m \times n}$ into a vector (and vec$^{-1}(\cdot)$ denotes the inverse operator)
$S_{\mathcal{P}}$	denotes Sylvester resultant matrix of a set of polynomials $\mathcal{P}$
$\lambda(A)$	defines the spectrum of $A \in \mathbb{R}^{n \times n}$
$\mathbb{C}^{n \times m}$	denotes the set of $n \times m$ complex matrices
$\mathbb{R}$[s]	defines the set of polynomials with real coefficients
$\mathbb{R}^{n \times m}$	denotes the set of $n \times m$ real matrices
$\mathbb{R}_{+0}, \mathbb{R}_{+}$	denotes the set of non-negative,positive numbers
$\mathbb{S}^n$	denotes the set of $n \times n$ symmetric matrices
$\mathcal{G}(r, \mathbb{R}^v)$	denotes the r-Grassmaniann as the set of r-dimensional subspaces of $\mathbb{R}^v$
$\mathcal{N}(A), \mathcal{R}(A)$	corresponds to the right null-space (kernel) and the range (column span) of $A \in \mathbb{R}^{n \times m}$ respectively
$\Phi_n^k(\underline{z})$	corresponds to the Grassmann matrix
$\sigma(A)$	the singular values of $A \in \mathbb{R}^{n \times m}$, such that $\sigma(A) = \{\sigma_i(A), i = 1, \ldots, min(n, m)\}$, listed in the non-increasing order $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{min(n,m)} \geq 0$
$\underline{h} \wedge^r$	denotes the r-th exterior product of $\underline{h}$
A'	corresponds to the transpose of matrix $A \in \mathbb{R}^{n \times m}$

$A \otimes B$	denotes the Kronecker product of two matrices A and B
A^*	denotes the complex conjugate transpose of $A \in \mathbb{C}^{n \times m}$
$C_k(M)$	denotes the k-the compound matrix of $M \in \mathbb{R}^{n \times m}, k \leq \{m, n\}$
$P_\Psi^\perp$	denotes the projection operator onto the orthogonal complement of Φ
$Q_{k,n}$	defines the lexicographically ordered, increasing sequences of k integers, $1, 2, \ldots n$

Acronyms / Abbreviations

DAP	Determinantal Assignment Problem
GCD	Greatest Common Divisor
GIDP	General Invariant Distance Problem
ICP	Invariant Controllability Polynomial
IO	Input-Output
IOP	Invariant Observability Polynomial
IQC	Integral Quadratic Constraints
LCMFD	Left Coprime Matrix Fraction Description
LFT	Linear Fractional Transformation
LIDP	Linear Invariant Distance Problem
LMI	Linear Matrix Inequality
LTI	Linear Time-Invariant
MIMO	Multi-Input Multi-Output
QPR	Quadratic Plücker Relations
RCMFD	Right Coprime Matrix Fraction Description
RSIP	Restricted State-Input Pencil
RSOP	Restricted State-Output Pencil
SISO	Single-Input Single-Output

Chapter 1

Introduction

1.1 Overview

In the last decades there has been significant progress in the area of Control Theory, with Control Systems being analysed from various perspectives. Various methods and techniques are used to determine the optimal location of actuators and sensors to meet predefined systems performance requirements. A fundamental issue of this type is the assessment of systems' properties, namely stability, controllability, observability, reachability, etc., applied for open-loop as well as for closed-loop systems.

Fundamental methods and procedures, developed by Kalman, Moore, Tarokh, etc. have recently been modified and extended, according to input/output dimensionality, model type and design specifications. Another important consideration is the way the systems' performance can be analysed in terms of actuation and sensoring structure. The concept of Global Instrumentation proposed in [1], [2], [3] falls in this category of research, where the overall system is observed from the global perspective of selecting appropriate input and output variables. In contrast, the main focus in local instrumentation is placed on optimisation of given physical variables. The concept of Global Instrumentation aims to develop a unified systematic approach for selecting systems' structures in order to achieve the best possible performance criteria. For input and output selection methodologies, significant progress has been made to the problem of deciding what is the best number of actuators and sensors, taking also into consideration the specifics of their placement and the corresponding problem type. In [4] the list of proposed Input-Output (IO) selection methods is reviewed and the desired properties of IO placement and configuration are summarised. Ideally, the procedures of optimal actuator and sensor placement should rely on quantitative measures that provide precise information of how strong the influence of the selected variables on the system is. The importance of quantitative measures has been addressed in [2] where an evaluation of structural framework of nonlinear plants at the early stage of the control design is discussed. The importance of quantitative measures for the structure selection problem

has been studied from various perspectives, e.g. using modal analysis [5], [6], [7], interval approach based on the state space sequence [8], novel eigenvector scaling approach [9], etc.

Input-Output Selection may be used prior to controller design to define appropriate controller constraints and specifications. Thus, the selection of appropriate structures of actuators and sensors should rely on the system's controllability and observability properties. However, as highlighted in [4], controllability and observability measures may not always provide strong conditions for the selection of these structures due to the binary nature of these concepts. Instead, methodologies based on Gramians, Lyapunov equations, matrix pencils, and others can provide a wider perspective for the optimal placement of sensors and actuators. Note that powerful approaches for identifying the relative strength of these two system properties involve the minimum and maximum singular value, condition number and eigenvalues and eigenvectors of the system's characteristic matrix. A similar set of criteria proposed in [4] for IO selection is based on the concept of "balancing" the modes of the system in terms of input and output energy, which is achieved when the system is in "balanced realisation" specified by Moore [10] and later studied in [11], [12] and others.

The alternative point of view in [2] has given rise to a novel research area linked to a new framework of input-output design. In this area the main emphasis is placed on the notions of energy and gain requirements, as well as the robustness property and simultaneous design. The review in [2] identifies gaps in this area of work, and outlines the main challenges for further studies.

In many modern Control Design methods significant attention is placed on optimal placement of sensors and actuators, where new methods of measuring system properties are introduced. For example, in [12], [13], [14], [15] the measure of modal controllability is analysed with respect to the angles between the range space of the actuator (input) matrix and the dual (left) eigenvectors of the state matrix (the angles between the row space of the sensor (output) matrix and the right eigenvectors of the state matrix for observability, respectively for observability). Gross measures and the relation with the residue matrix are also defined. In addition, a novel control method has been presented in [16] that deals with orthogonal eigenstructure control leading to an algorithm for determining the best location for actuators. In [12] new measures of modal controllability and observability have been proposed with respect to balanced coordinate systems. This approach is based on the notions introduced in [13], where controllability and observability properties are studied with respect to the distance between two subspaces defined by the dual eigenvectors and the range of the input matrix.

The majority of classical methods and existing numerical stable algorithms for measuring systems properties provide only a yes/no answer to the question of a system's controllability. An early approach of measuring the distance to the nearest uncontrollable (A, B) pair was proposed in [17] as a more useful tool. Paige proposed the concept of quantifying the strength

of controllability by estimating the minimum norm perturbation in the pair (A, B) which makes the system uncontrollable, an approach clearly linked to robustness. The work was met with considerable research interest and initiated further studies which focused on the numerical treatments of the matrix pencil $[sI - A, \ B]$ [18], [19], [20], [21], [22], etc. It was also shown in [19], [23] that the distance to uncontrollability is closely related to the notion of the smallest singular value of a matrix pencil, however its maximisation may be numerically inefficient due to the lack of convexity and hence search methods may not always converge to the global optimum. Although the controllability and observability properties of a system are defined in specific state-space coordinates, they remain invariant (as binary quantities) under feedback transformations, while their strength is not. In [24], [25], [26] the problem of defining and evaluating a distance measure that is invariant under feedback has been addressed. In [24] a new framework for calculating a relative measure of controllability (observability) is introduced as an invariant distance from state feedback (output injection) orbits.

The problem of evaluating the best structural framework for control design will be addressed throughout the research presented in this thesis. Moreover, techniques for estimating the distance to uncontrollability (unobservability), linking this notion to modal system characteristics and developing further the novel approach introduced in [24] of feedback-invariant distance measures to the nearest set of uncontrollable pairs is studied in this work.

1.2 Aims and Objectives

The current research aims to develop a general framework and a systematic methodology for optimal Control Structure Selection. The overall philosophy that is deployed here involves the following main objectives:

1. Evaluating methodology, based on the integration of structural diagnostic methodologies, in order to predict properties, namely controllability and observability, of the resulting system;

2. Estimating modal measures of the distance to the nearest uncontrollable (unobservable) sets of a system;

3. Evaluating optimisation methodologies and tools that provide effective selection of the input and output structures satisfying a multitude of design criteria;

4. Defining and estimating measures of system properties (controllability and observability) that remain invariant under feedback transformations, leading to the effective selection of sensors and actuators;

5. Developing a computational framework for the distance problems that arise in a variety of Control Design problems.

1.3 Research contribution

The thesis is focused on the analysis of system properties and evaluating alternative approaches for structure selection. The basic properties of systems, namely controllability and observability, are studied throughout the work. The importance of establishing the generalised framework for evaluating invariant measures of these properties is demonstrated. The proposed approach uses the restricted input-state (state-output) matrix pencil and then deploys exterior algebra that reduces the overall problem to the standard distance problem of a set of polynomials from non-coprimeness.

A huge variety of Control Problems lead to the analysis of the polynomial matrices and coprimeness of a set of polynomials. Such a description requires robust procedures for the computation of the distance to singularity, Greatest Common Divisors (GCD) or, more commonly, "approximate" Greatest Common Divisors, since this is clearly a non-generic problem.

One of the main contributions of the work is to evaluate computational procedures of the "approximate" common root calculations of a set of polynomials that have applications in a variety of Control Theory, Linear Algebra and Robust Control problems. The main challenge of the "approximate" common divisor calculations is that the matrices representing uncertainty due to perturbations in the polynomial coefficients are highly structured. Motivated by this challenge two alternative algorithms for the computation of the structured distance to the nearest common divisor of a set of polynomials are proposed in the current work. One applies a nonlinear least-squares optimisation procedure, while the other is based on the notion of the structured singular value (or μ-value). Moreover, the later approach has a wide range of applications in the Robust Control area and might be beneficial for Robust Control analysis.

The proposed structured singular value methodology has been extended to the analysis of implicit systems. Significance and simplicity of the μ-value approach for the case of overconstrained systems has been demonstrated. Such a description can be defined as a special case of the distance problems and also can be applied to the general problem of uncontrollability, unobservability. In the later part of the work the future research perspectives based on the proposed computation framework are presented which also contributes to the broader area of Robust Control and Systems Design.

1.4 Publications

The following papers have been presented and published during the PhD study:

1. Limantseva, O., Halikias, G. and Karcanias, N., (2020). Nearest common root of a set of polynomials: A structured singular value approach. *Linear Algebra and its Applications*, 584, pp.233 – 256.

2. Limantseva, O., Halikias, G. and Karcanias, N., (2020). An alternating projection algorithm for the "approximate" GCD calculation. *IFAC-PapersOnLine*, 53(2), pp.5837 – 5842.

3. Limantseva, O., Halikias, G., and Karcanias, N. (2020). Structured singular value of implicit systems. *Mathematical Methods in the Applied Sciences*.

4. Karcanias, N., Limantseva, O. and Halikias, G., (2020). The Feedback Invariant Measures of distance to Uncontrollability and Unobservability. *International Journal of Control*, 1 – 10.

5. Karcanias, N., Limantseva, O. and Halikias, G., Distance to Uncontrollability: Feedback Invariant Measures. Paper submitted for the 24$^{\text{th}}$ International Symposium on Mathematical Theory of Networks and Systems (MTNS 2020), **(Accepted)**.

6. Limantseva, O., Halikias, G. and Karcanias, N., (2019). Nearest common root of many polynomials. Structured Singular Value approach. The work presented at the 22$^{\text{nd}}$ Conference of the International Linear Algebra Society (ILAS).

1.5 Thesis structure

Aside from the introductory and summary sections covered in Chapter 1 and Chapter 7 respectively, the main body of the thesis consists of five chapters. The main concept of system properties and specifics of distance problems are developed throughout the thesis. Similar notations are used for all presented materials. Some additional findings as well as the extension of the methodologies for the different cases of analysed data sets is combined in Appendices A and B. The structure of the work is illustrated in the figure 1.1 below.

Background results and open research areas are presented in Chapter 2. The material is divided into three parts corresponding to the problem of measuring system properties, i.e. controllability and observability, as well as criteria and strategy of evaluating controllability and observability conditions that remain invariant under feedback transformations. The later chapters rely again on the background material as seen in figure 1.1.

Chapter 3 deals with the problem of developing modal measures of controllability and observability. The proximity of the invariant subspaces of the state matrix A to the range space of the input matrix B is important for the geometric characterisation of approximate controllability (and similarly for the dual problem of approximate observability) [13]. As

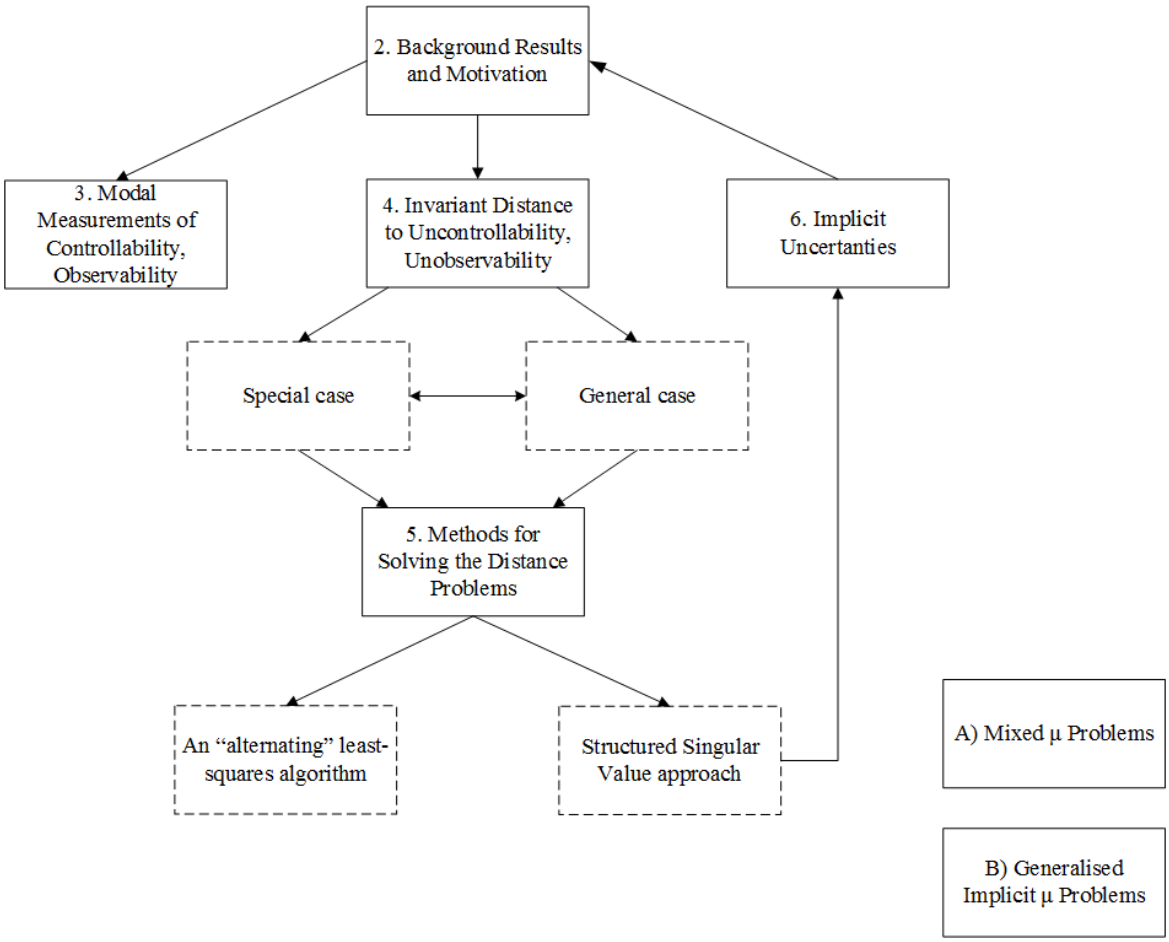

Fig. 1.1 Structure of the thesis

stated in [13] for a Single-Input Single-Output (SISO) system, for example, to be controllable the input vector $\underline{b}$ has to avoid all invariant subspaces in A, which is a requirement that will be related to the modal controllability of the system. The procedure for the optimal selection of input vectors is developed in this chapter.

Chapter 4 evaluates preliminary results for the framework of measuring the invariant distance to uncontrollability, unobservability. The novel approach of [24] motivated the development of a general methodology of feedback-invariant system properties and of effective algorithms for their calculations, presented in Chapter 5. The invariant distance problem uses the notions of Exterior Algebra and separates the problem into two cases: a special case, where the relationships of the multivectors are satisfied, and the general case, where an additional optimisation procedure has to be imposed in order to satisfy Quadratic Plücker Relations (later denoted as QPRs). It is demonstrated that both cases analyse the invariant distance as the distance to the GCD variety. The smallest singular value of the corresponding Sylvester Resultant matrix defines the lower bound of the distance.

The problem addressed in Chapter 4 is equivalent to finding the nearest common root of the polynomials that characterises the distance of the Sylvester Resultant matrix to singularity. Two different solutions of the problem are presented in Chapter 5. First, the approximate factorisation of the Sylvester matrix motivated the definition of an alternating least-squares projection algorithm that minimises the Frobenious norm of the error matrix of the factorisation. It is shown that a general nonlinear problem can be divided into two linear sub-problems and solved iteratively. Such an approach avoids significant complexity in numerical calculations of the "best" approximate GCD of a coprime set of polynomials. The second approach is mainly focused on the minimum-magnitude perturbations in the coefficients of the polynomials in order for them to have a common root. It is demonstrated that the problem is equivalent to the calculations of the standard structured singular value of a matrix. However, the problem is more complicated for an arbitrary number of polynomials and the fact that the uncertainty structure has to be redefined during the solution process.

As a byproduct of the structured singular value method a new algorithm for solving a non-standard structured approximation problem arising in the theory of implicit uncertain systems and the general problem of the distance to uncontrollability, unobservability is derived in Chapter 6. The proposed methodology based on the powerful notion of μ-value, and provides solutions for the variety of problems that appear in Robust Control. Moreover, it is demonstrated that the general problem of uncontrollability (unobservability) can be also formulated as a non-standard structured singular value problem.

Appendix A expands the the structured singular value method for the complex and mixed data sets, while Appendix B collects solutions for the generalised constrained μ-problems. All the results in the current work are illustrated with simple numerical examples.

Chapter 2

Background Results and Motivation

The problem of measuring system properties, namely controllability and observability, has been of interest for many years. Several tools and quantitative methodologies have been developed providing criteria for control design and input-output structure selection. In order to emphasise the research directions and highlight the open research problems, background results will be presented first.

For ease of exposition, the literature survey presented here is divided into three sections: Section 2.1 reviews various traditional and modal measures of controllability and observability; analysis of relative controllability (observability) via the solution of minimum distance problems to the nearest uncontrollable (unobservable) pair is discussed in Section 2.2; while Section 2.3 focuses on the feedback-invariant relative controllability (observability) measures.

2.1 Measures of controllability and observability

One of the main concepts in Control Theory involves the properties of a system. These are important for control design since they impose limits on what can be achieved. Control design starts with the system description, including input and output structures that identify preliminary characteristics of the system, whether it is possible to transfer the system from a given state to the given final state (controllability) or whether it is possible to determine the states from the outputs for any given input vectors (observability). Thus, the selection of input and output structures is one of the key topics in Control Theory that has appeared in many research areas of systems design, such as process systems [27], acoustics [28], noise suppression [28], chemical industry [27], aerospace [29] and many others.

The problem of sensor and actuator placement has been extensively studied for a long time [16], [30], [31], [12], [4], [32], [33] and many others. Traditional methods of measuring

system properties tend to be based on the analysis of $(A, B), (A, C)$ pairs for controllability and observability respectively.

The nature of the measures varies. Some straightforward controllability (observability) tests are based on the rank properties of a given matrix [34]. However, such approaches are only capable of evaluating whether or not a system is controllable and/or observable, while determining the sensitivity of the system properties is a more complex task that requires stable numerical algorithms and procedures. The alternative approach introduced by Paige C. [17] opened a new research area on quantifying the strength of the relative property by introducing the distance to the property loss. This is discussed in more detail later in this Chapter.

Another important area of control focuses on computing the zeros of a given system [35], [31], [36], [24], [33], [37], [38]. The study of dynamic and constant pole-zero assignment problems of linear systems, denoted as Determinantal Assignment Problems (DAP), established an alternative technique for control design and optimal structure selection [39]. Such a framework studies centralised or decentralised control structures and frequency assignment problems from two perspectives: in affine space as the intersection of linear spaces [40], [41]; in projective space that deals with notions of algebraic-geometry in order to obtain solutions on the intersection of linear spaces in projective space [42], [43], [44], [45], [39].

The notion of modal controllability and observability has been extensively studied for many years [13], [12], [9], [46], [14], [47], [12], and references therein. Estimating the modal distance to uncontrollability is an important problem for input-output framework selection. This can be formulated as a distance problem between two subspaces defined by the left eigenvectors and the range of the input matrix. To begin with, two definitions are given:

Definition 2.1.1 ([13])**.** The i-th mode of the Linear Time-Invariant (LTI) system is said to be controllable from the j-th input, $\underline{b}_j$, if

$$\underline{v}_i'\{s_i I - A,\ \ \underline{b}_j\} = \underline{X}' \neq \underline{0} \tag{2.1}$$

where A is the state matrix, s_i is a simple eigenvalue of A and $\underline{v}_i$ is the corresponding left eigenvector.

Definition 2.1.2 ([13])**.** The i-th mode of the LTI system is said to be observable in the k-th output if and only if

$$\underline{X} = \left\{ \begin{array}{c} \underline{c}_k' \\ s_i I - A \end{array} \right\} \underline{u}_i \neq \underline{0} \tag{2.2}$$

where $\underline{c}_k'$ defines the k-th row of the output matrix C and $\underline{u}_i$ is the right eigenvector of s_i. It is assumed again for simplicity that s_i is a non-repeated eigenvalue of A.

Lemma 2.1.1 ([13])**.** In SISO systems the pair (A, b) is controllable if and only if the input vector $\underline{b}$ is not orthogonal to the left eigenvectors of A.

The above Lemma says that in the SISO case the input vector $\underline{b}$ should not be contained in any invariant subspace of A for the system to be controllable. Intuitively, we expect that the closer $\underline{b}$ is to an A-invariant subspace the closer to being uncontrollable the system is.

Consider now the case of Multi-Input Multi-Output (MIMO) systems and assume that $B \in \mathbb{R}^{n\times m}(C \in \mathbb{R}^{p\times n})$, where $B = [\underline{b}_1, \underline{b}_2, \ldots, \underline{b}_m]$ and $C' = [\underline{c}'_1, \underline{c}'_2, \ldots, \underline{c}'_p]'$. Taking the above analysis into consideration it is reasonable to use $\|\underline{v}'_i B\|$ as the controllability measure of the i-th mode (and $\|C\underline{u}_i\|$ for the observability measure of the i-th mode). So far, different approaches have been suggested in the literature, proposing normalisation of the eigenvalues and scaling of the modal matrix in order to define the optimal quantitative measure of systems properties [46], [13], [8], [9]. Let

$$V'B = \begin{pmatrix} \underline{v}'_1\underline{b}_1 & \underline{v}'_1\underline{b}_2 & \ldots & \underline{v}'_1\underline{b}_m \\ \underline{v}'_2\underline{b}_1 & \underline{v}'_2\underline{b}_2 & \ldots & \underline{v}'_2\underline{b}_m \\ \vdots & \vdots & & \vdots \\ \underline{v}'_n\underline{b}_1 & \underline{v}'_n\underline{b}_2 & \ldots & \underline{v}'_n\underline{b}_m \end{pmatrix} \tag{2.3}$$

The magnitude of the (i, j)-th elemnent of $V'B$ can be written as [12]:

$$|\underline{v}'_i\underline{b}_j| = \|\underline{v}'_i\|\|\underline{b}_j\| \cos\theta_{i,j}, \;\; i = 1, 2, \ldots n, \;\; j = 1, 2, \ldots, m \tag{2.4}$$

where $\theta_{i,j}$ corresponds to the angle between the j-th input vector and the i-th left eigenvector. The value of $|\underline{v}'_i\underline{b}_j|$ can be taken as a controllability measure of the i-th mode through the j-th input, as stated in [12], [15], [13], [14]. Note that for a single-input system if $\beta_i = \underline{v}'_i\underline{b} = 0$ and $\underline{v}'_i(s_iI - A) = \underline{0}$, then s_i is an uncontrollable mode. Thus, for controllability in this case it is necessary that $\beta_i \neq 0$ for all i. Generally modal controllability can be quantified by taking into account all input vector directions, i.e. the degree of controllability of the i-th mode can be obtained with respect to the i-th left eigenvector and the j-th input vector. In the relevant literature the problem of measuring the distance to uncontrollability (unobservability) is formulated with respect to the angles between the range of the input matrix and the corresponding dual eigenvectors of the state matrix [12], [13], [14]. In the literature, controllability (observability) properties are typically studied from the properties of the corresponding controllability (observability) matrices and their corresponding (A, B) ((A, C) respectively) pairs [48], [8], [49], modal controllability [46], [9], [50], properties of Gramians [51], [32], [52], [53], eigenstructure assignment [16], pole-zero placement [54], [36], [35], eigenvalue mobility and eigenvalue structure [6], [7], [55], minimum energy perspectives [30] etc. However, not all these measures are invariant under feedback transformations, a topic that is reviewed later in the Chapter. Before this point, however, the background results related to the distance problems to the nearest uncontrollable (unobservable) pair are reviewed.

2.2 Distance problems to uncontrollability, unobservability

Analysing relative controllability from the perspective of the distance to the nearest uncontrollable set is more informative compared to standard tests of the controllability property. The distance, denoted as $d(A, B)$, also referred to as the distance to the set of uncontrollable systems, was defined in [17] as the minimum norm perturbation $[\Delta A,\ \Delta B]$ in the entries of the matrix $[A,\ B]$ that makes the pair $(A+\Delta, B+\Delta)$ uncontrollable. It was later proved that the distance is equal to the minimal singular value of a matrix, i.e. $d(A, B) = \min \sigma_n([sI - A, B])$ as s varies over the set of complex numbers [18], [19], [23].

A comprehensive review of the early estimation algorithms for solving the distance problem is given in [56]. The concept of the distance to uncontrollability has been widely adopted and developed further in [57], [58], [59], [60], [61], [20], [21], [62], [22], [63], [64]. Note that some algorithms deal with complex perturbations [64], [56], [65], [66], [67], while others [68], [69], [70], [71] focus on the estimation of the real controllability radius.

Numerical techniques developed in this area have identified a number of difficulties for evaluating the exact distance of $d(A, B)$ in the general case and have proposed the estimation of upper and lower bounds as an alternative. Gao and Neumann [22] argue that the distance between the lower and upper bounds of $d(A, B)$ resulting from earlier work, e.g. [17] and [61], can be significant and propose a new algorithm which reduces the gap. Exact optimisation algorithms are not always successful [57], [58], [20], [62]. Algorithms with guaranteed convergence to the global optimum are reported in references [72], [22], [64] and [56].

Numerical methodologies proposed for the solution of the distance to uncontrollability problem have been applied from a number of different perspectives. Early studies proposed the use of hybrid algorithms [62], methodologies based on the Kronecker structure [59], the Staircase algorithms [17], [61] and Newton's method [20] among others. Some methodologies consider using techniques based on the solution of an Algebraic Riccati equation [68], [21], [73]. For example, Gahinet and Laub established a link between a lower bound of $d(A, B)$ and the "nearest-to-singularity" solution of the corresponding Algebraic Riccati equation. A bisection method is presented in [72], while [67] also suggested the use of Kronecker structures for tackling the distance problem. A trisection method was studied in [74] and [67].

An analysis of structured distance to uncontrollability problems and a discussion on the difference between structured and unstructured distance measures have been reported in [75], [66] and later in [76] and [65]. In [77] Demmel established a relation between the condition number and the problem of nearest-singularity perturbation. This may be used to provide useful insights when estimating the distance to uncontrollability of a controllable pair (A, B).

2.3 The need for feedback invariant measures

The selection of system inputs and outputs (input and output structure) forms part of early system design. This is important since it preconditions the potential for control design. Existing methodologies for input, output structure selection rely on criteria expressing distance to uncontrollability (unobservability). The notion of the distance to uncontrollability (unobservability) of a pair (A, B) $((A, C))$ has been studied extensively over recent years, see [76], [78], [65], [26] , [66], [79], [73], [70], [67] and references therein. Although controllability (observability) is invariant under state feedback (output injection), their corresponding degrees expressing distance from uncontrollability (unobservability) are not. To illustrate this consider a closed-loop description of a system with applied state feedback and output injection respectively, i.e.

$$\begin{aligned} \dot{\underline{x}}(t) &= (A + BL)\underline{x}(t) + B\underline{u}(t) \\ \dot{\underline{x}}(t) &= (A + KC)\underline{x}(t), \;\; \underline{y}(t) = C\underline{x}(t) \end{aligned} \tag{2.5}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{m\times n}, L \in \mathbb{R}^{p\times n}, K \in \mathbb{R}^{n\times m}$. The controllability pencils with and without the transformations are related as:

$$(i) \;\; (sI - A - BL, \;\; B) = (sI - A, \;\; B) \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix}, \;\; (ii) \;\; (sI - A, \;\; B)$$

and therefore the singular values of the two pencils are not invariant under the state-feedback transformations. A similar argument applies for the two observability pencils with and without output injection.

Consider a strictly-proper LTI system $S(A, B, C)$ that is represented by the state space model:

$$\begin{aligned} \dot{\underline{x}}(t) &= A\underline{x}(t) + B\underline{u}(t) \\ \underline{y}(t) &= C\underline{x}(t) \end{aligned} \tag{2.6}$$

where the state matrix is $A \in \mathbb{R}^{n\times n}$, the input (actuator) matrix is $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{m\times n}$ represents the output (sensor) matrix. Time invariant system representations are widely used for control synthesis problems [80], [3], especially in the areas of large scale systems design and input-output pairing problems. From the point of view of analysis, this class of systems offers the advantage that certain important properties remain invariant under certain transformations [48]. An important aim of the research is to develop strategies for the effective control design of complex large-scale systems. Integration Methodologies [3] propose the use of models that best describe system properties. Due to the presence of model uncertainties an important problem arises, i.e. the need to ensure the applicability

of approximate solutions without distorting the characteristics of the real system. In order to assess the feasibility of control design specifications specific system properties should be ensured. In order to identify how far the system is from achieving certain performance criteria, analysis often involves the notion of matrix pencils. Matrix pencils' properties have been the focus of studies in the recent decades [26], [3], [81], [82], [83], [36], [84], [85], [86], [24] etc.

In the literature on linear systems theory the notions of matrix pencils and rational vector spaces are extensively used [3], [86], [42], [87]. This approach is also crucial for the study of input-output structures, which is one of the main aims of the proposed research. This is because rational vector spaces can be linked to the useful mathematical properties of Grassmann representatives, Plücker matrices, compound matrices, etc. The Smith and Kronecker forms of the system matrix also provide important information on the zero structure, which influences significantly the performance of the system.

In general, the theory of invariants and its relation to system properties is central in Control Systems Design. One methodology relies on Multilinear and Exterior Algebra in order to study the complex nature of linear systems. The Determinantal Assignment Problem (DAP), introduced in [86], [42], [24], generalises the problem of pole and zero placements via feedback. The study of DAP combines zero assignment problems and general problems of multilinear algebra, where the solution is derived using the notion of Quadratic Plücker Relations and Grassmann varieties [42]. As was highlighted in [24], one of the DAP sub-problems is related to the identification of the structural invariants of multivariable systems under feedback which is also the focus of the present research (also see [88], [42]).

A useful representation of system $S(A, B, C)$ is the following matrix pencil form [89]

$$P(s) = \begin{pmatrix} sI - A & -B \\ -C & 0 \end{pmatrix} \tag{2.7}$$

Various sub-pencils of $P(s)$ may be used to define important input-state and state-output properties of the system:

Definition 2.3.1 ([89])**.** Let $S(A, B, C)$ be a minimal realisation and consider the matrix pencils

$$C(s) = (sI - A, \ -B), \ K(s) = \begin{pmatrix} sI - A \\ -C \end{pmatrix} \tag{2.8}$$

Then, the system controllability and observability properties are characterised by the absence of zeros in the Smith form of pencils (2.8), respectively, i.e. by the absence of input decoupling zeros and output decoupling zeros, respectively.

Remark 2.3.1. Note that matrix pencils $C(s), K(s)$ are modified under state feedback $L \in \mathbb{R}^{p\times n}$ and output injection $K \in \mathbb{R}^{n\times m}$, respectively. Although the rank of the two

pencils is invariant under these two transformations, the numerical rank may be considerably different.

In order to demonstrate the sensitivity analysis of the controllability property (duality holds for observability) consider the following example.

Example 2.3.1. Consider the state space description of a linearised two-mass model of a wind turbine from [90]

$$\begin{aligned} S(A,B,C,D): \ \underline{\dot{x}} &= A\underline{x} + B\underline{u}, \ \ A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p} \\ \underline{y} &= C\underline{x} + D\underline{u}, \ \ C \in \mathbb{R}^{m\times n}, D \in \mathbb{R}^{m\times p} \end{aligned}$$

having system matrices defined as:

$$A = \begin{pmatrix} \frac{1}{2J_r}\rho\pi R^4 \underline{v}\frac{\partial C_Q}{\partial \lambda}|_{(\underline{\omega_r},\underline{v})} - \frac{K_r}{J_r} & 0 & -\frac{1}{J_r} \\ 0 & -\frac{K_g}{J_g} & \frac{1}{n_g J_g} \\ \frac{K_{ls}}{2J_r}\rho\pi R^4 \underline{v}\frac{\partial C_Q}{\partial \lambda}|_{(\underline{\omega_r},\underline{v})} + B_{ls} - \frac{K_r K_{ls}}{J_r} & \frac{K_{ls}K_g - J_g B_{ls}}{n_g J_g} & -\frac{J_r K_{ls} + n_g^2 J_g K_{ls}}{n_g^2 J_g J_r} \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & -\frac{1}{J_g} & \frac{K_{ls}}{n_g J_g} \end{pmatrix}', \ C = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \ D = 0,$$

where K_g, J_g, ω_g, n_g correspond to the generator external damping, inertia, torque, speed and gearbox ratio respectively; B_{ls}, K_{ls} identify the low-speed shaft speed and damping; v is the wind speed, λ is tip-speed ratio and ρ is the air density; K_r, ω_r, J_r correspond to rotor external damping, speed and inertia respectively.

Let input parameters be defined as in [90] for the wind turbine model and consider numerical values of the state and input matrices as follows

$$A = \begin{pmatrix} -0.0467 & 0 & -9.104 \times 10^{-7} \\ 0 & 0 & 3.3673 \times 10^{-4} \\ 8.0215 \times 10^6 & -1.8583 \times 10^5 & 0 \end{pmatrix}, \ B = \begin{pmatrix} 0 \\ -0.014535 \\ 0 \end{pmatrix}$$

Let the distance of the open-loop system from uncontrollability be denoted by

$$d_o(A,B) = \min_{s\in\mathbb{C}} \sigma_{min}([sI - A, \ \ B])$$

Then for the closed-loop system (with the LQR controller denoted by $K = -kR^{-1}B'V$, where R is a positive-definite matrix, V is the unique positive definite solution of the corresponding Riccati equation and k is a gain parameter) the distance becomes $d_c(\hat{A}, B) = \min_{s\in\mathbb{C}} \sigma_{min}([sI - (A + BK), \ \ B])$. The optimal solution of the Riccati equation V verified

in [90] is

$$V = \begin{pmatrix} 0.7439 & 0.0017 & -2.2412 \times 10^{-8} \\ 0.0017 & 1.0230 \times 10^{-5} & -5.9486 \times 10^{-11} \\ -2.2412 \times 10^{-8} & -5.9486 \times 10^{-11} & 1.0973 \times 10^{-14} \end{pmatrix}$$

With $k = 1$ the closed loop state matrix is computed as

$$\hat{A} = \begin{pmatrix} -0.0467 & 0 & -9.1040^{-7} \\ -3.5915 \times 10^{-7} & -2.1613 \times 10^{-9} & 3.3673 \times 10^{-4} \\ 8.0215 \times 10^{6} & -1.8583 \times 10^{5} & 0 \end{pmatrix}$$

Using Gu's algorithm along the straight line [72] for estimation of the distance to uncontrollability we start increasing the value of the gain k in order to observe differences in the distance. With $k = 10^8$ the closed-loop distance to uncontrollability of the system (which is already close to uncontrollability in open-loop) demonstrates a rapid decrease. The results can be evaluated with given tolerance $tol = 10^{-8}$ as follows

$$d_o(A, B) = 2.8997 \times 10^{-7}, \;\; d_c(\hat{A}, B) = 1.9118 \times 10^{-8}$$

It should be noted that the higher the value of the gain, the smaller the value of $d_c(\hat{A}, B)$ (for this particular example).

The above simple example demonstrates that for the analysis of system properties an alternative measure of the distance to uncontrollability (respectively unobservability) that remains invariant under feedback transformations is required.

The implicit autonomous systems, described by $(sN - NA)$, $(sM - AM)$ are referred to as the Restricted Input, Restricted Output differential systems, respectively, whereas the corresponding pencils $R(s) = sN - NA$ and $Q(s) = sM - MA$ are defined as Restricted Input and Restricted Output pencils, respectively. The above restricted pencils are a powerful mathematical representation of the system and will be used extensively in this work to define optimal structural properties.

Theorem 2.3.1 ([54])**.** Consider a system $S(A, B, C)$ and let N, M be the left annihilators of B and C, respectively. Then the following properties hold true:

1. The differential system $(sN - NA)\underline{x}(s) = 0$ is state feedback invariant;
2. The differential system $\underline{x}'(s)(sM - AM) = 0$ is output injection invariant.

Proof. Since under state feedback the input pair (A, B) becomes $(A + BL, B)$, then the corresponding system can be defined as $(sN - N(A + BL))\underline{x}(s) = 0$. Note, that N is the left annihilator of B, satisfying $NB = 0$ by definition and hence:

$$(sN - NA - NBL)\underline{x}(s) = (sN - NA)\underline{x}(s) = 0$$

Similarly, it holds under output injection. If M is a right annihilator of C, i.e. $CM = 0$, then

$$\underline{x}'(s)(sM - (A + KC)M) = \underline{x}'(s)(sM - AM - KCM) = \underline{x}'(s)(sM - AM) = 0$$

which completes the proof. □

The proof of the theorem above is presented in order to improve readability of the work.

The numerical rank of the pencil $[sI - A, \ -B]$ has been used to study the relative degree of controllability of the system. Similarly, the numerical rank of $\begin{pmatrix} sI - A \\ -C \end{pmatrix}$ can be used to define the relative observability properties of the system. Note that these two measures are affected by state feedback and output injection respectively, and hence are not appropriate if we wish to define transformation-invariant properties. In this case it is more appropriate to use pencils $sN - NA$ and $sM - AM$ which are not affected by state feedback and output injection transformations. It will be seen that this approach to the problem is linked to the notions of characteristic zeros, Greatest Common Divisor and, in general, properties of Exterior Algebra.

The computation of GCD has been central in engineering problems that deal with complex algebraic synthesis methods [91], [85], [92], [24], [42] and is closely related to the notion of Grassmann Matrices and Matrix Representatives. The notion of GCD and "approximate" GCD, defined in [24], [93], [91], [94], [92], [95], [85], [96] can be extended to the problem of measuring the relative strength of system properties that are invariant under feedback.

Calculation of the GCD is a nontrivial computational problem and falls within the class of non-generic computations, requiring an approach based on generalised resultants and Toeplitz matrix representations [84], [85]. Some interesting facts of the Sylvester resultant are addressed in [94], [85], [24], [97], in order to establish procedures for calculating the GCD.

2.4 Summary

Based on the open issues in Control Theory and Control Design, the research is focused on two distinct parts. The first area, motivated by the notion of modal measurement of controllability, observability, is studied as an optimisation problem, providing a solution for the "best" vector selection. Such an approach can be extended and applied to vector spaces forming the basis of future work.

It has been illustrated that sensitivity of the distance to uncontrollability does not remain invariant under feedback transformations. Hence, there arises the important question of how to establish an invariant measurement of the system properties. The next Chapter is devoted to the analysis of the modal measures of controllability and observability based on the selections of the best sets of input and output vectors respectively. In addition,

has been illustrated that sensitivity of the distance to uncontrollability does not remain invariant under feedback transformations. Hence, there arises the important question of how to establish an invariant measurement of the system properties. This corresponds to another open research problem for the current work that is discussed and in chapter 4 in details. The problem of invariant controllability and observability measures is motivated by the notions of the restricted matrix pencils, namely Input-State Restriction and State-Output Restriction pencils, later denoted as $R(s)$ and $Q(s)$ respectively. Such an approach requires the use of geometric system theory and properties of Exterior algebra that is discussed later in the work.

Chapter 3

Modal Measurement of Controllability, Observability

In this chapter an alternative computational framework for optimal structure selection based on the modal measurement of controllability and observability respectively is presented.

In [15], [13] a novel quantitative measure of modal controllability and observability was introduced, based on the angles between the vector spaces defined by the left (right) eigenvectors of the state matrix and the column vectors of the input matrix B (row vectors of the output matrix C). This approach evaluates the strength of controllability and observability of each mode. As mentioned in [12] the proposed strategy of Hamdan and Nayfeh deals with generalised angles between the corresponding vectors. Thus, they suggest solving an optimisation problem using the balancing coordinate transformation as the way to achieving equally balanced controllability and observability system properties for each mode. This approach can lead to the effective selection of both sensors and actuators. Returning to the literature, the benefit of analysing controllability and observability properties of a system in balanced coordinates was described by Moore [10]. The main advantage is that in balanced coordinates, the two Gramians of the system are diagonal and equal. This effectively equalises the controllability and observability properties of each mode and hence makes their ordering in terms of input-output energy transfer meaningful [51].

Definition 3.0.1 ([10], [51])**.** A system is called balanced, if there exists a transformation matrix P, with respect to which controllability and observability Gramians are equal and have the following form

$$\hat{W}_c = \hat{W}_o = \Sigma = \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{pmatrix} \tag{3.1}$$

where $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n \geq 0$ are the singular values and the transformed Gramians are derived as

$$\hat{W}_c = PW_cP' \tag{3.2}$$

$$\hat{W}_o = (P^{-1})'W_oP^{-1} \tag{3.3}$$

where W_c, W_o are corresponding Gramians in the original coordinates and the state space transformation is denoted by P.

Such an interpretation of a system rests on the fact that under a balancing coordinate transformation each mode becomes "equally" controllable and observable, in the sense that the control energy required to reach the given state and the output energy released by the free trajectory starting at the given state are equalised. Note that in general these properties are not transformation-invariant [51]. Since in this framework the states of the system can be meaningfully ordered, the optimal selection of sensors and actuators is easier to tackle. (Another task, which is simplified, is the model-reduction of the system, e.g. by eliminating the less important states, a process that is referred to as model-truncation [98]). It should be pointed out that achieving the best location of actuators with the maximal degree of controllability is the dual problem to the problem of optimal placement of sensors (for more details see [12]). It was also shown that such modal measures can be applied to verify properties after the implementation of state transformations or scaling, which is attractive for control design. The general objective of the work is to evaluate criteria for the design of optimal sensing and actuating structures of a system in order to maximise the strength of the relevant system's properties, namely controllability and observability, in order to optimise the performance of the overall system.

In this Chapter a problem of modal measurement of controllability and observability is considered. Motivated by the work of [13] the problem of modal characterisation can be presented as a simple optimisation problem that aims to estimate the direction of the input (output) vectors. Hence, the study of modal criteria falls into two cases: a simple case of a Single-Input Single-Output system presented in Section 3.1 and the general case of a system with Multi-Input and Multi-Output structures investigated in Section 3.2.

3.1 Modal measurements for the SISO systems

Let us assume for simplicity that A has a simple structure with distinct eigenvalues and a set of linearly independent eigenvectors. Let $\underline{b}$ be the input vector normalised to unit length. According to [13], [99], [47] the modal controllability of the i-th mode of the system can be measured by $cos(\underline{v}_i, \underline{b})$, the cosine of the angle between the input vector and the i-th left eigenvector of the state matrix. If this measure is near zero, the two vectors are almost

orthogonal. Hence, if the inner product of any pair $|\langle \underline{v}_i, \underline{b} \rangle|$ is close to zero, then the system contains an almost uncontrollable mode. We can use these modal measures to determine the optimal vector $\underline{b}$, e.g. by requiring that the largest angle it makes with each of the n left eigenvectors in n-dimensional space is minimised. A similar procedure can be applied to characterise approximately unobservable modes.

One important property, studied in [14], [15], [13], is the measure of modal controllability (observability) when the system is in balanced coordinates. Note that under similarity transformations the eigenvalues of the state matrix remain invariant, but the eigenvectors change. References [13], [14], [15] show that for balanced realisations there is partial symmetry of the input and output matrices. Thus, in the SISO case, the angle between the left eigenvector and the input vector $\underline{b}$ is equal to the angle between the corresponding right eigenvector and the output vector $\underline{c}$, i.e.

$$\cos(\underline{v}_i', \underline{b}_j) = \cos(\underline{u}_j, \underline{c}_i) \tag{3.4}$$

The angle between the left eigenvector $\underline{v}_i'$ and the input vector $\underline{b}$ should lie in the range $[0, \pi/2)$ in order to keep the two vectors from being orthogonal, which would imply uncontrollability of the corresponding mode. Since a small deviation of orthogonality results in "an almost uncontrollability" condition, it makes sense to attempt to maximise the minimum deviation from orthogonality between $\underline{b}$ and all the left eigenvectors $\underline{v}_i$. Assuming that the direction of $\underline{b}$ is unconstrained, the optimal solution in the two dimensional case is clearly obtained when the two angles defined by vector $\underline{b}$ and each of the two left eigenvectors $\underline{v}_i$ are equal, as shown in the figure 3.1.

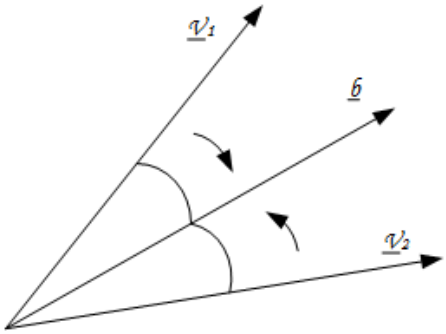

Fig. 3.1 Bisection of two vectors

Note that it can be assumed, without loss of generality, that the angle between any two left eigenvectors does not exceed $\pi/2$ (otherwise one of the two eigenvectors can be inverted). Hence, in this case the input vector should be placed along the direction which bisects the angle $\theta_{i,j}$ defined by the two eigenvectors. This computational procedure is simple, if we deal with a system of two modes. However, when the number of modes increases it becomes more difficult to generalise the approach.

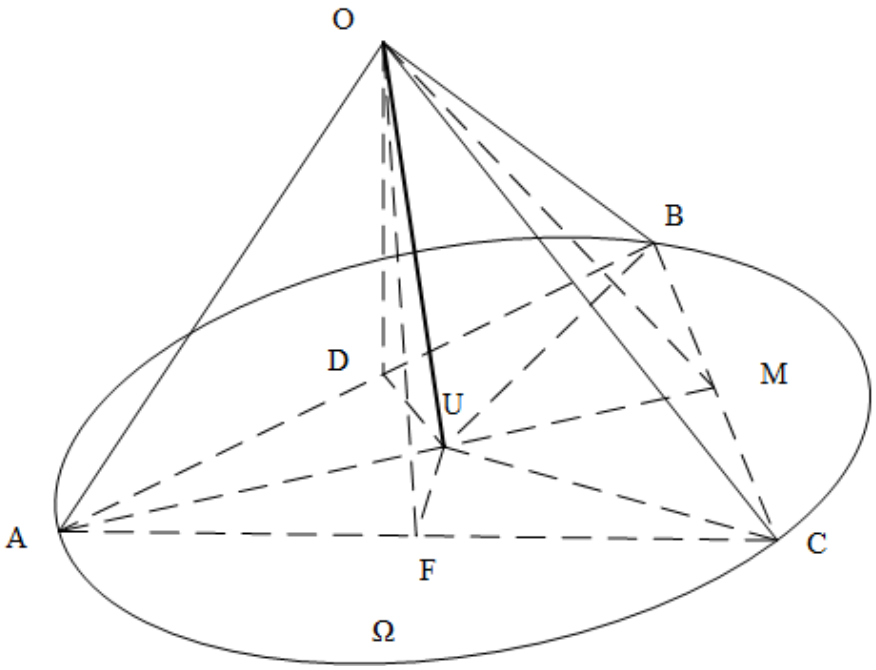

Fig. 3.2 Bisection of three vectors

Consider next a SISO system with a state matrix $A \in \mathbb{R}^{3\times 3}$. The three left eigenvectors in this case lie in a three dimensional space and the optimal input vector $\underline{b}$ still needs to be selected so that the minimum angle between $\underline{b}$ and each of the three eigenvectors is maximised. Clearly, this can be achieved by the following geometric construction indicated in figure 3.2.

Assume that OA, OC and OB are the left eigenvectors of the state matrix A. OU corresponds to the optimal placement of $\underline{b}$, where point U is the projection of O onto the plane ABC. If we analyse a cone (or tetrahedron) the shortest distance from the vertex O onto the base plane is a perpendicular OU, i.e. the angle from any cone generator will be less than $\pi/2$ (and in the case of a regular cone angles from a perpendicular to any cone generator are equal). Alternatively U is the centre of the circumscribed circle around triangle ABC, figure 3.3.

Thus, under the assumption that eigenvectors are of unit length we can write:

$$AO = BO = CO = 1 \tag{3.5}$$

$$AB = 2AO \sin(\alpha/2) \tag{3.6}$$

$$AC = 2AO \sin(\phi/2) \tag{3.7}$$

$$BC = 2AO \sin(\beta/2) \tag{3.8}$$

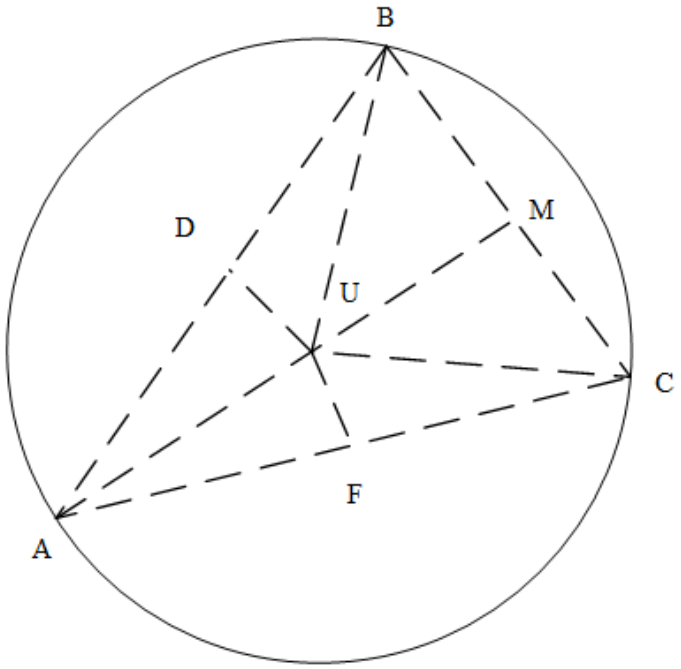

Fig. 3.3 Projection of the bisection of three vectors

where α, ϕ, β are the corresponding angles between the three eigenvector pairs. Then the angle between the perpendicular and each of the eigenvectors may be computed as

$$\theta = \arcsin \frac{R}{AO} = \arcsin R, \tag{3.9}$$

where R is the radius of the circumscribed circle. With respect to the basic properties of the circumscribed circle, the radius is defined as

$$R = \frac{AB \;\; AC \;\; BC}{4\sqrt{P(P-AB)(P-AC)(P-BC)}}, \;\; P = \frac{AB + AC + BC}{2} \tag{3.10}$$

A similar approach can be applied for higher order systems, however some approximations should be made when evaluating the hyper-sphere. Note that a circumscribed circle can be defined around an N-sided polygon only if it is regular.

If the order of the system is $n > 3$, the projection of the eigenvectors should be defined in the n-dimensional space. Assume without loss of generality that all eigenvectors are of unit length, then it is possible to construct a circumscribed sphere of unity radius, the centre of which is associated with the origin and the sphere's surface is structured by the directions of the eigenvectors, as shown in figure 3.4.

Hence, every direction normal to the surface of the sphere may be considered as a good choice of the input vector if it coincides with or is close to the direction of an eigenvector.

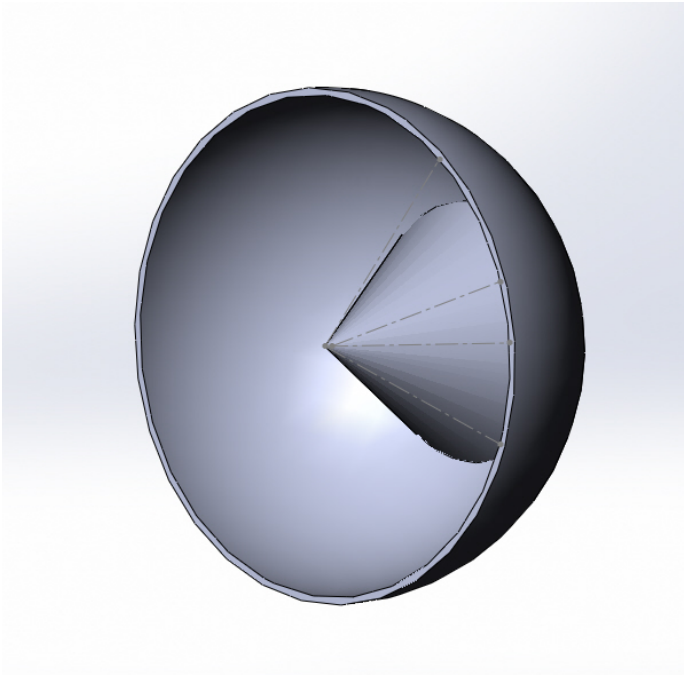

Fig. 3.4 Bisection of the n-dimensional space

Example 3.1.1. Consider a SISO system with a given randomly generated state matrix $A \in \mathbb{R}^{3\times 3}$ and a simple non-zero initial structure for the actuator vector.

Then, regarding the geometric approach (illustrated in figure 3.5) the estimated optimal placement of the input vector is calculated as follows

$$\underline{b} = (-0.4337,\ \ 0.0471,\ \ 0.1570)'$$

giving the optimal placement of the vector with the equal modal controllability measure of $\cos(\theta) = 0.4637$ for all modes.

The geometric solution to the problem is hard to apply when the size of the system is large, however it provides the exact optimal solution for the special case, where the size of the system is $n = 2, 3$. In order to identify a generalised approach for the problem of finding the best placement of actuators it is possible to define an optimisation problem, i.e. maximising the smallest measure of modal controllability.

Let us consider the system $\dot{\underline{x}} = A\underline{x} + \underline{bu}$, where $A \in \mathbb{R}^{n\times n}, \underline{b} \in \mathbb{R}^{n\times 1}$ and assume for simplicity that the eigenvalues of A are distinct and real, i.e. $\lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}, \lambda_i \neq \lambda_j, i \neq j$. Assume further that $\underline{v}_i', i = 1, 2, \ldots, n$ is the left eigenvector of A corresponding to λ_i. Assume without loss of generality that $\|\underline{v}_i\| = 1, i = 1, 2, \ldots, n$ (eigenvectors are normalised). Then n controllability measures associated with the system modes can be defined as:

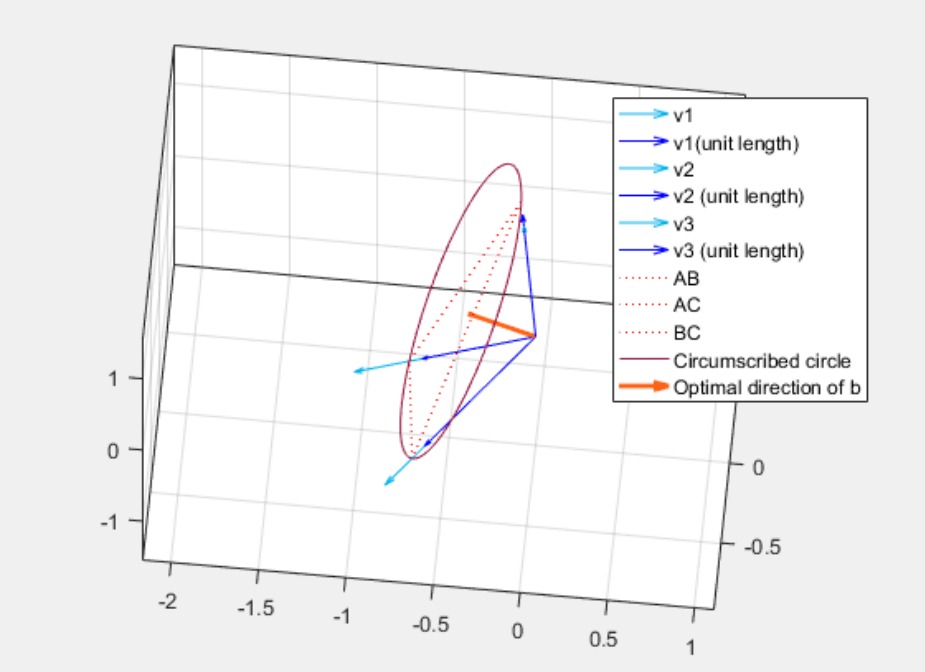

Fig. 3.5 Geometric solution (Example 3.1.1)

$$\gamma_i = \cos(\underline{v}_i, \underline{b}) = \frac{|\langle \underline{v}_i, \underline{b} \rangle|}{\|b\|}, i = 1, 2, \ldots, n \tag{3.11}$$

Let $\hat{\underline{b}} = \frac{\underline{b}}{\|\underline{b}\|}$, so that $\|\hat{\underline{b}}\| = 1$. We seek to maximise the controllability index in the most critical direction (i.e. for the most critical mode). Hence the problem can be formulated as follows:

$$\gamma^* = \max_{\hat{\underline{b}} \in \mathbb{R}^n, \|\hat{\underline{b}}\|=1} \min_{i \in \{1,2,\ldots,n\}} |\langle \underline{v}_i, \hat{\underline{b}} \rangle| \tag{3.12}$$

In order to obtain the solution for the problem we introduce the following proposition. Also, for convenience of calculations we take that $\|\hat{\underline{b}}\| = \|\hat{\underline{b}}\|_\infty = \max\{|b_1|, \ldots, |b_n|\}$.

Proposition 3.1.1. The optimisation problem defined in (3.11) is equivalent to:

$$\begin{gathered} \max \gamma \\ \text{s.t.} \quad \gamma \leq |v_{i1} b_1 + \cdots + v_{in} b_n|, \quad \forall i = 1, 2, \ldots, n \end{gathered} \tag{3.13}$$

This is further equivalent to the linear problem:

$$\begin{gathered} \min(-\gamma) \\ \text{s.t.} \quad \gamma - (v_{i1} b_1 + \cdots + v_{in} b_n) \leq 0, \quad i = 1, 2, \ldots, n \\ b_i \leq 1, \quad -b_i \leq 1, \quad i = 1, 2, \ldots n \end{gathered} \tag{3.14}$$

or in a standard form

$$\begin{aligned} &\min \left(\begin{array}{ccc|c} 0 & \dots & 0 & -1 \end{array} \right) \left(\begin{array}{c} b_1 \\ \dots \\ b_n \\ \hline \gamma \end{array} \right) \\ &\text{s.t.} \left(\begin{array}{ccc|c} -v_{11} & \dots & -v_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ -v_{n1} & \dots & -v_{nn} & 1 \end{array} \right) \left(\begin{array}{c} b_1 \\ \dots \\ b_n \\ \hline \gamma \end{array} \right) \leq \left(\begin{array}{c} 0 \\ \dots \\ 0 \end{array} \right) \\ &b_i \leq 1, -b_i \leq 1, \;\; i = 1, 2, \dots, n \end{aligned} \tag{3.15}$$

It can be observed that the solution to the problem in (3.11) is given by the solution of the linear problem. It can be easily solved using standard optimisation algorithms (e.g. Simplex method or interior-point).

Consider a numerical example, where the corresponding eigenvectors are structured as an $n \times n$ matrix of randomly allocated values. The eigenvectors have been normalised and the problem has been solved using the MATLAB function *linprog.m*.

Example 3.1.2. Consider a normalised eigenvector structure, $V \in \mathbb{R}^{4\times 4}$

$$V' = \begin{pmatrix} -0.8558 & 0.1634 & 0.2828 & 0.4013 \\ -0.1712 & -0.7860 & -0.1586 & -0.5725 \\ -0.5954 & 0.1528 & -0.7887 & -0.0068 \\ -0.8353 & 0.1575 & -0.2895 & -0.4400 \end{pmatrix}$$

Solving the optimisation problem using the interior-point algorithm, the solution for the optimal cosine measures is given by the vector:

$$\gamma^* = (0.5466, \;\; 0.5466, \;\; 0.5466, \;\; 0.5466)'$$

while the optimal input vector is $\underline{b} = (-1.0000, \;\; -0.7734, \;\; -0.3078, \;\; 0.1905)$. In this example the selected eigenstructure results in equal controllability measures.

An interesting observation is made in the following Proposition which relates the controllability measures to the dimension of the problem.

Proposition 3.1.2. Consider a SISO system, where $V \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^{n\times 1}$ and $\|b\| = 1, \|v_i\| = 1, i = 1, 2, \dots, n$. Assume also that the eigenvalues of A are real and distinct. Assume without loss of generality that the eigenvectors are normalised. Then the optimal average value of the modal controllability measure for every pair $(\underline{v}_i, \underline{b}), i = 1, 2, \dots, n$ can be evaluated as $\gamma_i^* = \frac{1}{\sqrt{n}}$.

Previously it was assumed that all the eigenvalues of the state matrix are distinct and real meaning that the matrix structure is simple (i.e. matrix diagonalisable). Next non-simple matrix structures involving Jordan blocks are considered.

Consider a similarity transformation of a SISO system, i.e.

$$T^{-1}AT = J, \;\; T^{-1}b = \tilde{b}, \tag{3.16}$$

where T is a non-singular $n \times n$ matrix and the eigenvalues of A are real, but the Jordan canonical form contains Jordan block structures. In such a case the number of independent eigenvectors corresponding to λ_i will be denoted as the geometric multiplicity, i.e. $r_i = \dim\{\text{null}(\lambda_i I - A)\}$ that is equal to the number of Jordan blocks. It is known that a system is controllable if and only if there exists a Jordan block for every eigenvalue λ_i and every element of the input vector $\underline{b}$ that corresponds to the last row of the Jordan block is a nonzero element.

Since there is only one eigenvector per chain, the remaining vectors are generalised eigenvalues which just complete the basis of the space and differ only by a scalar, then it is possible to verify using the row-echelon transformations the non-vanishing elements that will be taken into account when computing the modal controllability measures.

It is clear that in order to measure system properties, calculated with respect to the angles between the corresponding eigenvectors and the input vector (for a single-input system), only those eigenvectors that are associated with the last eigenvalue in a Jordan block should be taken into consideration. Precisely, the optimisation of (3.15) can be stated in a similar way but now only the optimisation involves a reduced number of eigenvectors (equal in number to the sum of geometric multiplicities of all distinct eigenvalues).

$$\begin{aligned} &\min \left(\begin{array}{ccc|c} 0 & \dots & 0 & -1 \end{array} \right) \left(\begin{array}{c} b_1 \\ \dots \\ \hline b_n \\ \hline \gamma \end{array} \right) \\ &\text{s.t.} \left(\begin{array}{ccc|c} -v_{11} & \dots & -v_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ -v_{r1} & \dots & -v_{rn} & 1 \end{array} \right) \left(\begin{array}{c} b_1 \\ \dots \\ \hline b_n \\ \hline \gamma \end{array} \right) \leq \left(\begin{array}{c} 0 \\ \dots \\ 0 \end{array} \right) \\ &b_i \leq 1, -b_i \leq 1, \;\; i = 1, 2, \dots, n \end{aligned} \tag{3.17}$$

where $r \leq n$ and r is the geometric multiplicities of eigenvalues, identifying the number of linearly independent eigenvectors, associated with $\lambda_i, \forall i = 1, \dots, n$. It means that the solution to has $n - r$ less constraints leading to the faster convergence of the solution of the optimisation problem.

3.2 Modal measurements for MIMO systems

For the MIMO case the input matrix B is defined in terms of its columns, say $\underline{b}_j, j = 1, \ldots, m$. Then, due to the Cauchy-Schwartz inequality [100], $0 \leq |\underline{v}_i' \underline{b}_j| \leq \|\underline{v}_i\| \|\underline{b}_j\|$ and hence $0 \leq \theta_{i,j} \leq \frac{\pi}{2}$. If $cos(\theta_{i,j})$ is close to zero, then $\underline{v}_i'$ and $\underline{b}_j$ are nearly orthogonal and $|\underline{v}_i' \underline{b}_j|$ is close to zero, which implies that the i-th mode is a nearly-uncontrollable mode from the j-th input. The i-th mode will be almost uncontrollable through B if $\|\underline{v}_i' B\|$ is sufficiently small, i.e. if all $|\underline{v}_i' B|, j = 1, \ldots, m$ are sufficiently small.

Following similar arguments, it follows that the balanced system for the MIMO case satisfies the following equality

$$\|\underline{v}_i' \underline{b}_j\| = \|\underline{c}_i \underline{u}_j\| \tag{3.18}$$

in which $\underline{b}_j$ is the j-th column of the input matrix B and $\underline{c}_i$ is the i-th row of the output matrix C. The proof of the above is based on the properties of controllability and observability Gramians (see [10]). When applying coordinate transformations, the principal components are changed as

$$e^{\tilde{A}t} \tilde{B} = -P^{-1} e^{At} B, \;\; \tilde{C} e^{\tilde{A}t} = C e^{At} P \tag{3.19}$$

Let P be a balanced transformation. Consider the controllability and observability Gramians in balanced coordinates, i.e

$$\begin{aligned} \hat{W}_c^2(P) &= P^{-1} \left(\int_0^T e^{At} B B' e^{A't} dt \right) (P^{-1})' = \int_0^T e^{\tilde{A}t} \tilde{B} \tilde{B}' e^{\tilde{A}'t} dt \\ \hat{W}_o^2(P) &= P' \left(\int_0^T e^{A't} C' C e^{At} dt \right) P = \int_0^T e^{\tilde{A}'t} \tilde{C}' \tilde{C} e^{\tilde{A}t} dt \end{aligned} \tag{3.20}$$

respectively. If A is stable, the two Gramians in the limit $T \to \infty$, satisfy the algebraic Lyapunov equations:

$$\begin{aligned} \tilde{A} \hat{W}_c^2(P) + \hat{W}_c^2(P) \tilde{A}' &= -\tilde{B} \tilde{B}' \\ \tilde{A}' \hat{W}_o^2(P) + \hat{W}_o^2(P) \tilde{A} &= -\tilde{C}' \tilde{C} \end{aligned} \tag{3.21}$$

respectively. Since the two Gramians are diagonal and equal, (3.18) follows. In order to estimate the modal controllability measure for a MIMO system the smallest angle between all input vectors and eigenvectors should be maximised. The problem can be expanded for the multiple input case as

$$\underline{v}_i' [s_i I - A, \;\; B] = [0, \quad \underbrace{\underline{v}_i' B}_{\zeta_i \in \mathbb{R}^{1 \times m}} \;\;] \tag{3.22}$$

Here the $V'B$ matrix identifies measures for each ij-th modal controllability corresponding to the i-th mode with respect to the j-th input and can be structured as follows

$$V'B = \begin{pmatrix} \underline{v}_1'\underline{b}_1 & \underline{v}_1'\underline{b}_2 & \dots & \underline{v}_1'\underline{b}_m \\ \underline{v}_2'\underline{b}_1 & \underline{v}_2'\underline{b}_2 & \dots & \underline{v}_2'\underline{b}_m \\ \vdots & \vdots & & \vdots \\ \underline{v}_n'\underline{b}_1 & \underline{v}_n'\underline{b}_2 & \dots & \underline{v}_n'\underline{b}_m \end{pmatrix}$$

Consider that $\|\cdot\|_1$ is the norm of the j-th row, i.e. characterises the strength of the j-th mode controllability property for all the m inputs. Then, aiming to increase the measure of controllability introduced by the matrix $V'B$, a constrained optimisation problem can be formulated as:

$$\max_{\|B\|_\infty \le 1} \min_{i\in\{1,2,\dots,n\}} \| \underbrace{\underline{v}_i'B}_{\underline{\zeta}_i} \|_1 \tag{3.23}$$

or equivalently

$$\max_{\|B\|_\infty \le 1,\ i\in\{1,2,\dots,n\}} \min \{|\underline{v}_i'\underline{b}_1| + \dots + |\underline{v}_i'\underline{b}_m|\} \tag{3.24}$$

Set $\gamma = |\underline{\zeta}_i|_1$, then the constraints are defined as

$$\left.\begin{aligned} \gamma &\le |\underline{v}_1'\underline{b}_1| + \dots + |\underline{v}_1'\underline{b}_m| \\ \gamma &\le |\underline{v}_n'\underline{b}_1| + \dots + |\underline{v}_n'\underline{b}_m| \end{aligned}\right\} \tag{3.25}$$

Under the assumptions that all the eigenvectors are scaled to be placed in the same hyperplane, the absolute values can be omitted and the optimisation problem will be simplified to

$$\begin{aligned} &\max \gamma \\ \text{s.t.}\quad &\gamma \le \underline{v}_1'\underline{b}_1 + \dots + \underline{v}_1'\underline{b}_m \\ &\vdots \\ &\gamma \le \underline{v}_n'\underline{b}_1 + \dots + \underline{v}_n'\underline{b}_m \\ &b_{ij} \le 1, -b_{ij} \le 1,\ \ \forall i = 1,\dots,n, j = 1,\dots,m \end{aligned} \tag{3.26}$$

Similarly

$$\begin{aligned}
&\max \gamma \\
s.t.\ \gamma \leq &\begin{pmatrix} \underline{v}_1' & \underline{v}_1' & \dots & \underline{v}_1' \end{pmatrix} \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_m \end{pmatrix} \\
&\vdots \\
\gamma \leq &\begin{pmatrix} \underline{v}_n' & \underline{v}_n' & \dots & \underline{v}_n' \end{pmatrix} \begin{pmatrix} \underline{b}_1 \\ \underline{b}_2 \\ \vdots \\ \underline{b}_m \end{pmatrix} \\
&b_{ij} \leq 1, -b_{ij} \leq 1, \ \forall i = 1, \dots, n, j = 1, \dots, m
\end{aligned} \tag{3.27}$$

If the optimisation problem is defined in the standard form where the constraints are structured as block-diagonal matrices, then

$$\begin{aligned}
&\max \left(\begin{array}{ccc|c} \underline{0} & \dots & \underline{0} & 1 \end{array} \right) \left(\begin{array}{c} \underline{b}_1 \\ \vdots \\ \underline{b}_m \\ \hline \gamma \end{array} \right) \\
s.t. &\left(\begin{array}{ccc|c} -\underline{v}_1' & \dots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\underline{v}_1' & 1 \\ \hline -\underline{v}_i' & \dots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\underline{v}_i' & 1 \\ \hline -\underline{v}_n' & \dots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & -\underline{v}_n' & 1 \end{array} \right) \left(\begin{array}{c} \underline{b}_1 \\ \vdots \\ \underline{b}_m \\ \hline \gamma \end{array} \right) \leq \underline{0} \\
&b_{ij} \leq 1, -b_{ij} \leq 1, \ \forall i = 1, \dots, n, j = 1, \dots, m
\end{aligned} \tag{3.28}$$

The above approach finds the optimal solution which indicates the optimal placement of actuators. Again any optimal LP algorithms may be used.

If the infinity norm is chosen, then the optimisation problem can be formulated as follows:

$$\min_{\|B\|_\infty \leq 1,\ i\in\{1,2,\ldots,n\}} \| \underbrace{\underline{v}_i' B}_{\zeta_i} \| \Longleftrightarrow \tag{3.29}$$

$$\min_{\|B\|_\infty \leq 1} \max_{i\in\{1,2,\ldots,n\}} \{\underline{v}_i' b_1, \ldots, \underline{v}_i' b_m\} \tag{3.30}$$

Let t_i be an upper bound for every $i = 1, \ldots, n$. Then if γ is the minimum value, it is obvious that

$$\begin{aligned} \min t_1 : \underline{v}_1' \underline{b}_1 \leq t_1 \ldots \underline{v}_1' \underline{b}_m \leq t_1 \\ \vdots \\ \min t_n : \underline{v}_n' \underline{b}_1 \leq t_n \ldots \underline{v}_n' \underline{b}_m \leq t_n \end{aligned} \tag{3.31}$$

and at the same time

$$\begin{aligned} t_1 \geq \gamma \\ \vdots \\ t_n \geq \gamma \end{aligned} \tag{3.32}$$

The optimisation problem is defined as

$$\max \left(\begin{array}{ccc|ccc|c} \underline{0} & \dots & \underline{0} & 0 & \dots & 0 & 1 \end{array} \right) \left(\begin{array}{c} \underline{b}_1 \\ \vdots \\ \underline{b}_m \\ \hline t_1 \\ \vdots \\ t_n \\ \hline \gamma \end{array} \right)$$

$$s.t. \quad \left(\begin{array}{ccc|ccc|c} \underline{v}_1' & \dots & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \underline{v}_1' & 0 & \dots & -1 & 0 \\ \hline \underline{v}_i' & \dots & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \underline{v}_i' & 0 & \dots & -1 & 0 \\ \hline \underline{v}_n' & \dots & 0 & -1 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \underline{v}_n' & 0 & \dots & -1 & 0 \\ \hline 0 & \dots & 0 & -1 & \dots & 0 & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 & \dots & -1 & 1 \end{array} \right) \left(\begin{array}{c} \underline{b}_1 \\ \vdots \\ \underline{b}_m \\ \hline t_1 \\ \vdots \\ t_n \\ \hline \gamma \end{array} \right) \leq \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \hline 0 \end{array} \right) \tag{3.33}$$

$$b_{ij} \leq 1, -b_{ij} \leq 1, \ \ \forall i = 1, \dots, n, j = 1, \dots, m$$

Thus, in order to estimate the modal controllability measure, the problem can be solved in the following steps:

Step 1: Let $\underline{\zeta}_i$ be the controllability measure of the corresponding i-th mode, where $i = 1, \dots, n$. In order to estimate nearly-uncontrollable modes, the minimum $\underline{\zeta}_i$ has to be maximised

Step 2: For $i = 1, \dots, n$ identify the boundaries of $\underline{v}_i' \underline{b}_j$ for every eigenvector. Derive the upper bound for every $i = 1, \dots, n$

$$\max_{i=1,\dots,n} \{\underline{v}_i' \underline{b}_1, \dots, \underline{v}_i' \underline{b}_m\} = t_i, \tag{3.34}$$

leading to $\gamma \leq t_i, \forall i = 1, \dots, n$

Step 3: Solve a constrained optimisation problem, i.e. $\max \gamma$, subject to constraints

$$\begin{aligned} \gamma &\leq t_i, \ \ \forall i = 1, \ldots, n \\ \underline{v}_i' \underline{b}_j &\leq t_i, \ \ \forall i = 1, \ldots, n, j = 1, \ldots, m \end{aligned} \tag{3.35}$$

It is assumed for simplicity that $\|B\|_\infty \leq 1$

Step 4: Solve and repeat the procedure for all nearly uncontrollable pairs with the given tolerance ϵ, i.e. $\underline{v}_i' \underline{b}_j \geq \epsilon, \ \ \forall i = 1, \ldots, n, j = 1, \ldots, m$.

3.3 Summary

The modal measure criteria for the effective structure problem of sensor and actuator placement has been presented. The framework is defined as an optimisation problem that evaluates the optimal input (output) matrices with respect to the modal distance to uncontrollability (unobservability). It has been demonstrated that the optimal choice of the input/output structures of a given system can be derived based on the angle measure between the range of input (output) matrices and the system of eigenvectors (based on previous work of [15], [14], [13]). The procedure, proposed in the Chapter, aims to optimise modal measure of controllability (observability respectively) using two approaches: geometric intuition (for the SISO case) or by solving a Linear Programming optimisation problem (for both SISO and MIMO systems).

However, it has been noted that the degree of controllability (observability) does not remain invariant when feedback transformations are applied. In this case an alternative criteria for measuring invariant systems properties is required. The next Chapter introduces background results as well as an computational framework of distances from uncontrollability (unobservability) that are invariant under feedback transformations.

Chapter 4

Invariant Distance to Uncontrollability, Unobservability

The selection of sets of inputs and outputs in a system is a fundamental problem in Systems Design. This problem is independent from control system design, but the selection affects the resulting model and has a significant effect on the resulting structure and system properties, which determine the potential for control design [2]. The selection of inputs and outputs has been based so far on controllability, observability criteria for a given system (fixed input and output structure) [10], [47], [32]. The existing methods for selection do not take into account the fact that although controllability is invariant under state feedback and observability is invariant under output injection [80], [101], their corresponding degrees (measures of distance to uncontrollability, unobservability) are not invariant under compensation. Given that selecting input and output structures fixes the system and that different feedback schemes are used, it is essential to define selection procedures which are based on criteria that are invariant under feedback.

This Chapter establishes generalised framework for measuring the invariant system properties. The approach used in the definition of the distance measures to uncontrollability, unobservability is based on the use of system operators to characterise controllability (observability) which is invariant under state feedback (output injection). These operators are the Input-State Restriction pencil $R(s)$ (used to characterise controllability) and the State-Output Restriction pencils $Q(s)$ (used to characterise observability). The properties of matrix pencils are described in Section 4.1. It is demonstrated that by deploying notions of exterior algebra, reviewed in Section 4.2, the overall problem is reduced to the standard problem of the distance of a set of polynomials from non-coprimeness. In other words based on the interrelations of multivectors and the notion of GCD, presented in Section 4.3, the invariant criteria of controllability, observability can be defined as a distance problem that is evaluated in Section 4.4.

4.1 Invariant matrix pencils and their properties

The problem of selecting the best structure of sensors and actuators is a significant issue in Control Theory and Control Design as it affects structural invariance of a system $S(A, B, C, D)$ and its properties.

Consider a Linear Time Invariant system $S(A, B, C, D)$ described by

$$\dot{\underline{x}} = A\underline{x} + B\underline{u}, \quad \underline{y} = C\underline{x} + D\underline{u} \tag{4.1}$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{m\times n}$ and $D \in \mathbb{R}^{m\times p}$. The transfer function of the model defined by $G(s) = C(sI - A)^{-1}B + D \in \mathbb{R}^{m\times p}(s)$ can be represented by the right, left coprime matrix fraction descriptions (RCMFD, LCMFD) as

$$G(s) = N_r(s)D_r(s)^{-1} = D_l(s)^{-1}N_l(s) \tag{4.2}$$

with the coprime representations respectively

$$T_l(s) = [D_l(s) \;\; N_l(s)] \in \mathbb{R}^{m\times(m+p)}[s], \;\; T_r(s) = [D_r(s) \;\; N_r(s)]' \in \mathbb{R}^{(m+p)\times p}[s], \tag{4.3}$$

where $N_l(s), N_r(s) \in \mathbb{R}^{m\times p}[s]$ and $D_l(s) \in \mathbb{R}^{m\times m}[s], D_r(s) \in \mathbb{R}^{p\times p}[s]$. It has been shown in [102] that a rational matrix representation and the properties of rational vector spaces can be simply studied as matrix polynomials and corresponding polynomial vectors. The rank property of polynomial matrices is defined by the structure and degree of the resulting polynomials and can be expressed in terms of the GCD [86], [24], [33].

Matrix fraction description provides an alternative perspective of studying the crucial properties of LTI systems. Since the Rosenbrock's matrix pencil of a system $S(A, B, C, D)$ is a polynomial matrix denoted by

$$P(s) = \begin{pmatrix} sI - A & -B \\ -C & -D \end{pmatrix}, \tag{4.4}$$

then the properties of the system can be studied with respect to the transmission zeros of $P(s)$ defined precisely as those s, where the pencil is rank-deficient [102], [54], [58], [103]. The family of zero assignment problems has a common formulation providing an alternative approach for studying various system properties [104], [24], [37], [54]. The input and output decoupling zeros of the system are defined as zeros of the corresponding controllability and observability pencils, which is going to be discussed later in this chapter.

If a pair (A, B) is controllable, (A, C) is observable and $\text{rank}(B) = p, \text{rank}(C) = m$, then the controllability, observability pencils are defined as

$$C(s) = (sI - A, \ -B), \ K(s) = \begin{pmatrix} sI - A \\ -C \end{pmatrix} \tag{4.5}$$

It is well known that restricted matrix pencils hold important properties for measuring system properties that are summarised in the following theorem:

Theorem 4.1.1 ([89])**.** The system $S(A, B, C)$ is controllable if and only if the pencil $C(s)$ has no finite elementary divisors and it is observable if and only if the pencil $K(s)$ has no finite elementary divisors.

Proof. The proof can be found in [89]. □

Consider now a closed-loop system $\tilde{S}(A + BL, B, C, D)$ with applied state feedback $L \in \mathbb{R}^{p \times n}$. Then the system's controllability pencil becomes

$$\tilde{C}(s) = [sI - (A + BL), \ B]. \tag{4.6}$$

It is obvious that the feedback gain affects the eigenspace of the state matrix and the sensitivities of the eigenvalues in $\tilde{C}(s)$.

Let N be a left annihilator of B, where $NB = 0, \ N \in \mathbb{R}^{(n-p) \times n}$, $\text{rank}(N) = n - p$, and the $(p \times n)$ left inverse of B is denoted by $B^\dagger$ $(B^\dagger B = I_n)$. Further let M be a right annihilator of C such that is $CM = 0$, $M \in \mathbb{R}^{n \times (n-m)}$, $\text{rank}(M) = n - m$ and let $C^\dagger$ be a $(n \times m)$ right inverse of C $(CC^\dagger = I_m)$. Then we define matrices Z, W

$$Z = \begin{pmatrix} N \\ B^\dagger \end{pmatrix} \in \mathbb{R}^{n \times n}, |Z| \neq 0, \tag{4.7}$$

$$W = (M \ \ C^\dagger) \in \mathbb{R}^{n \times n}, |W| \neq 0 \tag{4.8}$$

Transforming, for example, the controllability pencil by multiplying by Z, the following equivalent description is obtained

$$\begin{pmatrix} N \\ B^\dagger \end{pmatrix} (sI - A, \ -B) \begin{pmatrix} \underline{x}(s) \\ \underline{u}(s) \end{pmatrix} = 0 \Longleftrightarrow \begin{cases} (sN - NA)\underline{x}(s) = 0 \\ \underline{u}(s) = B^\dagger (sI - A)\underline{x}(s) \end{cases} \tag{4.9}$$

A similar approach is carried out for the observability pencil leading to the new matrix pencils

$$R(s) = sN - NA, \tag{4.10}$$

$$Q(s) = sM - AM \tag{4.11}$$

known as the Restricted State-Input Pencil (RSIP) and Restricted State-Output Pencil (RSOP) respectively [26], [105], [54].

In order to define invariant measures of uncontrollability and unobservability it is essential to introduce some background results. The main results are based on the properties of compound matrices, decomposability property of multivectors as well as Grassmann representatives.

Definition 4.1.1 ([106])**.** Let $Q_{k,n}$ denote the set of lexicographically ordered, strictly increasing sequences of k integers from $\tilde{n} = \{1, 2, \ldots, n\}$. If $\{\underline{x_{i_1}}, \ldots, \underline{x_{i_k}}\}$ is a set of vectors of $\mathcal{V}, \omega = (i_1, \ldots, i_k) \in Q_{k,n}$, then $\underline{x_{i_1}} \wedge \cdots \wedge \underline{x_{i_k}} = \underline{x_\omega} \wedge$ denotes the exterior product, $\wedge^r\mathcal{V}$ denotes the r-th exterior power of $\mathcal{V}$. If $H \in \mathcal{F}^{m\times n}$ and $r \leq \min(m, n)$, then $C_r(H)$ denotes the r-th compound matrix of H.

Clearly, the compound matrix $C_r(H)$, where $r \leq \min(m, n)$ is a polynomial matrix representation obtained by calculating all minors of the entries of the given matrix H. A multilinear map that is characterised by the exterior product $\wedge^r\mathcal{V}$ holds important properties for the analysis of the mapping between affine and projective spaces. Such a property will be denoted later as decomposability of multivectors.

If $\underline{c}_i(s)^t$, denote the rows of $C(s) \in \mathbb{R}^{n\times(p+n)}[s]$, $\underline{k}_i(s), i \in \tilde{n}$ denote the columns of $K(s) \in \mathbb{R}^{(n+m)\times n}[s]$ and $C_r(X)$ denotes the r-th compound matrix of $X \in \mathbb{R}^{t\times w}[s], r \leq \min(t, w)$ [106] then the polynomial vectors may be defined as:

$$\begin{aligned} C_n(C(s)) &= \underline{c}_1(s)^t \wedge \cdots \wedge \underline{c}_n(s)^t = \underline{c}(s)^t \wedge^n \in \mathbb{R}^{1\times\rho}[s] \\ &= \tilde{\underline{c}}(s)^t, \;\; \rho = \binom{n+p}{n} \end{aligned} \tag{4.12}$$

$$\begin{aligned} C_n(K(s)) &= \underline{k}_1(s) \wedge \cdots \wedge \underline{k}_n(s) = \underline{k}(s) \wedge^n \in \mathbb{R}^{\rho'}[s] \\ &= \tilde{\underline{k}}(s), \;\; \rho' = \binom{n+m}{n} \end{aligned} \tag{4.13}$$

as the input decoupling zero and output decoupling zero polynomials of the system respectively.

Remark 4.1.1. The system is controllable, if the polynomials of $\tilde{\underline{c}}(s)$ are coprime and it is observable if the polynomials in $\tilde{\underline{k}}(s)$ are coprime. If the polynomials in $\tilde{\underline{c}}(s)$ and $\tilde{\underline{k}}(s)$ are coprime, then their distance from the corresponding GCD variety [107] defines the distances from uncontrollability, $d(A, B)$, unobservability, $d(A, C)$ of the $S(A, B), S(A, C)$ systems respectively.

Note that the distances $d(A, B), d(A, C)$ may vary when state feedback, respectively output injection, is applied. In fact, if $L \in \mathbb{R}^{p\times n}$ is a state feedback matrix, then the

controllability pencil of the closed loop system becomes

$$C'(s) = (sI - A - BL, \ -B) = (sI - A, \ -B) \begin{pmatrix} I_n & 0 \\ L & I_p \end{pmatrix} \tag{4.14}$$

and based on the equivalence property of the compound matrices the corresponding input decoupling polynomial is then defined by

$$\begin{aligned} \underline{\tilde{c}'}(s)^t &= C_n(C'(s)) = C_n([sI - A - BL, \ -B]) \\ &= C_n([sI - A, \ -B]) C_n \left(\begin{bmatrix} I_n & 0 \\ L & I_p \end{bmatrix} \right) = \underline{\tilde{c}}(s)^t T(L) \end{aligned} \tag{4.15}$$

where $T(L) \in \mathbb{R}^{\sigma \times \sigma}$.

Remark 4.1.2. Condition (4.15) clearly demonstrates that although $\underline{\tilde{c}'}(s)^t$ and $\underline{\tilde{c}}(s)^t$ have the same GCD, their distance from the GCD, such that can be characterised by the smallest singular value, is affected by the choice of L.

Similar results may be stated for unobservability and the effect of output injection on the distance from unobservability when $C_n(K(s)) = \underline{\tilde{k}}(s)$, is considered. The above raises the question of defining distance measures from uncontrollability, unobservability which are feedback invariant. To introduce the problem of invariant distance to uncontrollability some preliminary results of Exterior Algebra and properties of multilinear vectors should be addressed.

4.2 Exterior algebra and Multivectors: Background results

The problem of invariant distance to uncontrollability (observability) is studied via restriction matrix pencils and their projective spaces.

Projective geometry is often used to simplify the formulation of the vector spaces. The fundamental properties of the projective spaces [108], [109], [110] define the basis for studying the invariant controllability measures. Projective equivalence of an arbitrary controllability pencil described in terms of linear invariant subspaces is an elegant approach of measuring the distance to uncontrollability.

Definition 4.2.1 ([108])**.** Let $\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k$ and $\mathcal{U}$ be vector spaces over a field $\mathcal{F}$, where $0 \leq k \leq \infty$. Let the Cartesian product of the vector spaces be defined as $\mathcal{V}^k = \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_k$, where the corresponding k-tuples are of the form $(\underline{x_1}, \ldots, \underline{x_k})$. A function $\underline{f} : \mathcal{V}^k \longrightarrow \mathcal{U}$ is

called a multilinear map if it is linear in every vector variable, i.e.

$$\underline{f}(\underline{x}_1, \ldots, \underline{x}_i, c\underline{y} + d\underline{z}, \underline{x}_{i+1}, \ldots, \underline{x}_k) = \\ = c\underline{f}(\underline{x}_1, \ldots, \underline{x}_i, \underline{y}, \underline{x}_{i+1}, \ldots, \underline{x}_k) + d\underline{f}(\underline{x}_1, \ldots, \underline{x}_i, \underline{z}, \underline{x}_{i+1}, \ldots, \underline{x}_k) \quad (4.16)$$

where $c, d \in \mathcal{F}$ for all vectors $\underline{y}, \underline{z} \in \mathcal{V}$. The total set of multilinear maps is defined then as $\mathrm{Hom}_{\mathcal{F}}(\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_k; \mathcal{U})$, whereas the dual vector space is denoted by $\mathcal{V}^*$ and is defined as $\mathcal{V}^* = \mathrm{Hom}(\mathcal{V}, \mathcal{F})$.

Consider a vector space $\mathcal{V}$ with an integer $k \geq 2$, then the k-th exterior power of the vector space is defined by

$$\wedge^k : \wedge^k\mathcal{V} : (\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_k) \xrightarrow{\wedge^k} \underline{x}_1 \wedge \underline{x}_2 \wedge \cdots \wedge \underline{x}_k \quad (4.17)$$

The k-th exterior product $\wedge^k$ of the vector space has the following properties [108]:

Proposition 4.2.1 ([108])**.** (i) If the exterior product $\wedge^k\mathcal{V}$ defines a k-linear map $\wedge^k : \mathcal{V}^k \longrightarrow \wedge^k(\mathcal{V})$ that is skew symmetric, then $\wedge^k(\mathcal{V})$ is structured by $\wedge^k(\underline{x}_1, \underline{x}_2, \ldots, \underline{x}_m)$, for every $\underline{x}_i \in \mathcal{V}$;

(ii) There exists a linear map $\underline{f} : \wedge^k\mathcal{V} \longrightarrow \mathcal{U}$ if $\underline{f}$ is a skew symmetric k-linear map into the vector space $\mathcal{U}$.

If $\underline{e}_i,\ \ i = 1, \ldots, n$ is a basis of $\mathcal{V}$, than the exterior product of the basis vectors spans the vector space $\wedge^k\mathcal{V}$ as

$$\underline{e}_{i,1} \wedge \underline{e}_{i,2} \wedge \cdots \wedge \underline{e}_{i,k},\ \ 1 \leq i_1 \leq i_2 \leq \ldots i_p \leq n \quad (4.18)$$

Definition 4.2.2 ([108])**.** Let $\tau = \binom{v}{r}, r \leq v$ and let $\underline{k} \in \mathbb{R}^\tau$. The vector $\underline{k}$ is called decomposable, if there exists a set of vectors $\{\underline{h}_i, i = 1, \ldots, r, \underline{h}_i \in \mathbb{R}^v\}$ such that

$$\underline{h}_1 \wedge \cdots \wedge \underline{h}_r = \underline{h}\wedge^r = \underline{k} \quad (4.19)$$

The matrix $H = [\underline{h}_1, \ldots, \underline{h}_r] \in \mathbb{R}^{v \times r}$ defines a basis for the subspace $\mathcal{H} = \mathrm{span}\{\underline{h}_1, \ldots, \underline{h}_r\}$ which has length r and may be referred to as the parent space of the decomposable vector $\underline{k}$.

Let z an exterior product of k vectors in the vector space $\mathcal{V}$ be a decomposable element in $\wedge^k\mathcal{V}$ [108]:

$$\underline{z} = \underline{x_1} \wedge \underline{x_2} \wedge \cdots \wedge \underline{x_k} \quad (4.20)$$

Although $\wedge^k\mathcal{V}$ is generated by a set of decomposable vectors, not every vector in $\wedge^k\mathcal{V}$ is decomposable [39].

Theorem 4.2.1 ([108])**.** $0 \neq \underline{z} \in \wedge^k \mathcal{V}$ is decomposable if and only if there exists a linear independent set of vectors $\underline{u}_1, \underline{u}_2, \ldots, \underline{u}_k$ in the vector space $\mathcal{V}$, i.e. $\underline{u}_i \wedge \underline{z} = 0, \;\; i = 1, \ldots, k$.

Proof. For proof see [108]. □

If $\mathcal{U}$ is an n-dimensional vector space over the field $\mathcal{F}$, then we denote $\mathcal{G}(k,\mathcal{U})$ as Grassmannian that describes the set of k-dimensional subspaces $\mathcal{V}$ of $\mathcal{U}$ [36]. Let $\mathcal{B}_{\mathcal{U}} = \{u_i, i \in \underline{n}\}$ denote a basis of $\mathcal{U}$ and let $\mathcal{B}^k_{\mathcal{U}} = \{u_\omega \wedge : u_\omega = u_{i,1} \wedge \cdots \wedge u_{i,k}, \omega = (i_1, \ldots, i_k) \in Q_{k,n}\}$ be the basis of $\wedge^k \mathcal{U}$, then the exterior product z can be defined as $z = \sum_{\omega \in Q_{k,n}} a_\omega e^\wedge_\omega$, where a_ω are the coordinates of $\underline{z}$ [36], [108].

Theorem 4.2.2 ([108])**.** The exterior product $\underline{z}$, defined by

$$\underline{z} = \sum_{\omega \in Q_{k,n}} a_\omega e^\wedge_\omega \tag{4.21}$$

is decomposable if and only if there exists $A \in M_{k,n}(R)$, i.e. $a_\omega = \det A[1, \ldots, m|\omega]$, $\omega \in Q_{k,n}$.

Proof. Can be found in [108]. □

It has been shown in [108] that two decomposable vectors $\underline{z}, \underline{z}' \in \wedge^m \mathcal{U}$ are linearly dependent if and only if $\langle \underline{z} \rangle = \langle \underline{z}' \rangle = \mathcal{V}$, where $\mathcal{V} \in \mathcal{G}(k,\mathcal{U})$ and

$$\underline{z}' = \underline{v}'_1 \wedge \cdots \wedge \underline{v}'_k = c \cdot \underline{z} = c \cdot \underline{v}_1 \wedge \cdots \wedge \underline{v}_k, \;\; c \in \mathcal{F} - \{0\}. \tag{4.22}$$

For the nonzero k-dimensional subspace $\mathcal{V}$ of $\mathcal{U}$ any decomposable vector of the family $\{c \cdot \underline{z}, \underline{z} \in \wedge^k \mathcal{U}, c \in \mathcal{F} - \{0\}\}$ characterises the vector space $\mathcal{V}$ and is called the Grassmann Representative of $\mathcal{V}$ [108], [39], [36]. Grassmann representatives differ only by a scalar $c \in \mathcal{F}, c \neq 0$ and are denoted by $g(\mathcal{V})$ [108]. Coordinates $\{a_\omega, \omega \in Q_{k,n}\}$ of $g(\mathcal{V})$ are called Plücker coordinates of $\mathcal{V}$. The set of lexicographically ordered Plücker coordinates is ultimately induced by $\mathcal{V}$ to within $c \in \mathcal{F}$ [108], [36]. Since not every vector $\underline{z} \in \wedge^k \mathcal{U}$ is decomposable, it was shown in [108] and references therein that decomposable vectors should satisfy Quadratic Plücker Relations (QPR). The set of quadratics

$$\sum_{k=1}^{k+1} (-1)^m a_{i_1, \ldots i_{m-1}, j_m} a_{j_1, \ldots, j_{m-1}, j_{m+1}, \ldots, j_{k+1}} = 0, \tag{4.23}$$

where $1 \leq i_1 < \cdots < i_{k-1} \leq n, 1 \leq j_1 < j_2 < \cdots < j_{k+1} \leq n$, is called the Quadratic Plücker Relations [108] and defines the algebraic variety $\Omega(k,n)$. Such an $(n-k)k$-dimensional variety of the projective space $\mathbb{P}^{\sigma-1}, \sigma = \binom{m}{n}$ is known as the Grassmann variety and corresponding map, $\rho : \mathcal{G}(k,\mathcal{U}) \longrightarrow \mathbb{P}^{\sigma-1}$ is called Plücker Embedding [111].

Grassmann variety $\mathcal{G}(k,\mathcal{U})$ denotes the image of the map ρ in the projective space. Basic properties of $\Omega(k,n)$ specify the links between affine and projective spaces that are summarised below.

Theorem 4.2.3 ([36])**.** Let $\Omega(k,n)$ be the Grassmann variety of $\mathbb{P}^{\sigma-1}, \sigma = \binom{k}{n}$. Then the following properties hold:

1. $\Omega(k,n)$ is the irreducible algebraic variety of the dimension $k(n-k)$ and it contains decomposable vectors of $\mathbb{P}^{\sigma-1}$;
2. It is possible to evaluate coordinates of a generic point of the variety $\Omega(k,n)$ with respect to $k(n-k)$ independent indeterminates;
3. The projective equivalent of $\Omega(k,n)$ is $\Omega(n-k,n)$ and $\dim\Omega(n-k,n) = \dim\Omega(k,n) = k(n-k)$;
4. When $k=1$, $\Omega(1,n) = \mathbb{P}^{n-1}(\mathcal{F})$ with $\dim\Omega(1-n) = n-1$. Similarly, if $k = n-1$, then $\Omega(n-1,n) = \mathbb{P}^{n-1}(\mathcal{F})$. For such boundary cases the corresponding Plücker Embedding is bijective implying that every vector of $\wedge^1\mathcal{U}, \wedge^{n-1}\mathcal{U}$ is decomposable.

Proof. See [36] for more details. □

In [42] the authors introduced an alternative characterisation of the decomposability property based on Grassmann matrices. Let a structured matrix $\Phi_n^k(\underline{z}) \in \mathcal{F}$ be specified by (k,n) and the elements of the corresponding coordinates $\{a_\omega, \omega \in Q_{k,n}\}$ of the vector $\underline{z} \in \wedge^k\mathcal{U}$. $\Phi_n^k(\underline{z})$ is called the Grassmann matrix of $\underline{z}$.

Definition 4.2.3 ([42])**.** Consider a set of strictly increasing sequences $Q_{k,n}$, where $\{a_\omega, \omega \in Q_{k,n}\}$ are the coordinates of $\underline{z} \in \wedge^k\mathcal{U}$ with respect to the basis $\mathcal{B}_\mathcal{U}^k, k+1 \le n$. Let $\gamma = (i_1,\ldots,i_{k+1}) \in Q_{k+1,n}$, then the subset of $Q_{k,n}$ sequences with elements from γ is defined by $Q_{k,k+1}^\gamma$ as

$$Q_{k,k+1}^\gamma = \{a_{\gamma(\tilde{m})} = (i_1,\ldots,i_{m-1},i_{m+1},\ldots,i_{k+1})\}, \;\; m \in k+1 \tag{4.24}$$

Define the function ϕ by

$$\phi(i,\gamma) = \begin{cases} 0, & \text{if } i \notin \gamma \\ (-1)^{m-1}a_{\gamma(\tilde{m})}, & \text{if } i = i_m, \;\; m = 1,\ldots,k+1 \end{cases}$$

Proposition 4.2.2 ([42])**.** Consider an n-dimensional vector space $\mathcal{U}$ with the corresponding basis $\mathcal{B}_\mathcal{U} = \{\underline{u}_i, i \in n\}$. Let $\mathcal{B}_\mathcal{U}^k = \{\underline{u}_\omega\wedge, \omega \in Q_{k,n}\}$ be the basis of $\wedge^k(\mathcal{U})$, then if there exists a vector $\underline{v} = \sum_{i=1}^n c_i\underline{u}_i \in \mathcal{U}$, i.e. $\underline{v} \neq \underline{0}$ and the generator vector $\underline{z}$ is defined as

$$\underline{z} = \sum_{\omega\in Q_{k,n}} a_\omega \underline{u}_\omega\wedge \in \wedge^k\mathcal{U}, \;\; \underline{z} \neq \underline{0}, \tag{4.25}$$

then the necessary and sufficient condition for $\underline{v} \wedge \underline{z} = \underline{0}$ is

$$\sum_{i=1}^{n} \phi i, \gamma c_i = 0, \ \ \forall \gamma \in Q_{k+1,n}. \tag{4.26}$$

If the lexicographically ordered elements of $Q_{k,n}$ are defined by γ_t, where $t = \binom{n}{k+1}$, then (4.26) can be rearranged as follows

$$\underbrace{\begin{pmatrix} \phi^1_{\gamma_1} & \phi^2_{\gamma_1} & \dots & \phi^n_{\gamma_1} \\ \phi^1_{\gamma_2} & \phi^2_{\gamma_2} & \dots & \phi^n_{\gamma_2} \\ \vdots & \vdots & \dots & \vdots \\ \phi^1_{\gamma_t} & \phi^2_{\gamma_t} & \dots & \phi^n_{\gamma_t} \end{pmatrix}}_{\Phi^k_n(\underline{z})} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.27}$$

A structured matrix $\Phi^k_n(\underline{z})$ with zeroes allocated at the fixed positions is called the Grassmann matrix of the multivector $\underline{z}$.

Example 4.2.1. Consider a simple example of the Grassmann matrix $\Phi^k_n(\underline{z})$, where $n = 4, k = 2$. Let the coordinates of the vector $\underline{z} \in \wedge^2\mathcal{U}, \dim\mathcal{U} = 4$ be as $\{a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34}\}$. Then matrix $\Phi^2_4(\underline{z})$ is structured as follows:

$$\Phi^2_4(\underline{z}) = \begin{pmatrix} a_{23} & -a_{13} & a_{12} & 0 \\ a_{24} & -a_{14} & 0 & a_{12} \\ a_{34} & 0 & -a_{14} & a_{13} \\ 0 & a_{34} & -a_{24} & a_{23} \end{pmatrix}$$

The Grassmann matrix $\Phi^k_n(\underline{z})$ holds important properties for the analysis of decomposability of $\underline{z}$. In [45] the results have been summarised in a form of a theorem.

Theorem 4.2.4 ([45])**.** Consider an n-dimensional vector space $\mathcal{U}$ over a field $\mathcal{F}$. Let $\mathcal{B}_\mathcal{U}$ be the basis of $\mathcal{U}$ and $\Phi^k_n(\underline{z})$ be the Grassmann matrix of z, such that $\underline{0} \neq \underline{z} \in \wedge^k\mathcal{U}$. Let the null space of the Grassmann matrix be denoted by $\mathcal{N}^k_n(\underline{z}) = \mathcal{N}_r(\Phi^k_n(\underline{z}))$. Then the following holds:

1. $\dim\mathcal{N}^k_n(\underline{z}) \leq k$ and the corresponding equality holds, if and only if $\underline{z}$ is decomposable and $\Phi^k_n(\underline{z})$ is canonical;

2. If $\dim\mathcal{N}^k_n = k$, then the solution of the vector space $\mathcal{V}, v_1 \wedge \dots \wedge v_k = \underline{z}$ is defined by the $\mathcal{N}^k_n(\underline{z})$.

Proof. The proof is presented in [45]. □

Theorem 4.2.4 provides alternative criteria for the analysis of multivectors and their decomposability properties. By combining Theorem 4.2.3 with the properties of Grassmann

matrices it is possible to define different decomposability conditions of $\underline{z}$ and corresponding forms of $\Phi_n^k(\underline{z})$.

Corollary 4.2.1 ([45])**.** If $\Phi_n^k(\underline{z})$ is the Grassmann Representative of $\underline{z} \in \wedge^k\mathcal{U}$ and $\underline{z} \neq 0$, then:

(i) For the case of $k = 1$ the corresponding vector $\underline{z}$ is always decomposable and $\Phi_n^1(\underline{z})$ is referred to as canonical. For $n \geq 3$ the rank of $\Phi_n^k(\underline{z})$ is equal to $\text{rank}\{\Phi_n^1(\underline{z})\} = n - 1$;

(ii) If $k = n - 1$, then again $\underline{z}$ is free corresponding to the canonical form of $\Phi_n^{n-1}(\underline{z})$, i.e. $\text{rank}\{\Phi_n^{n-1}(\underline{z})\} = 1$;

(iii) If k is not a boundary case, i.e. $k = n - p, k > 1$ and $p \geq 2$, then $\text{rank}\{\Phi_n^k(\underline{z})\} \geq n - k$ for all $\underline{z}$ and $\text{rank}\{\Phi_n^k(\underline{z})\} = n - k$ if and only if $\underline{z}$ is decomposable, hence $\Phi_n^k(\underline{z})$ is canonical.

Since QPRs are always satisfied for the special case of multivectors (parts (i) and (ii) in Corollary 4.2.1), it is possible to define some interesting links between the set of QPRs and rank test of the corresponding $\Phi_n^k(\underline{z})$ matrix.

Corollary 4.2.2 ([45])**.** If $\underline{z} \in \wedge^k\mathcal{U}$, i.e. $\underline{z} \neq 0$ for $n - k \geq 2, k > 1$, then $\Phi_n^k(\underline{z})$ is canonical iff $C_{n-k+1}(\Phi_n^k(\underline{z})) = 0$.

It is evident that decomposability of $\underline{z}$ can be evaluated from the corresponding Grassmann matrix. Moreover, the linearity of the maps, say from $\mathcal{V}$ to $\mathcal{U}$, can be characterised with respect to the compound matrices. Following this, some notations of the structured sequences and the properties related to decomposability of m-vectors should be introduced.

Definition 4.2.4. Let $Q_{k,n}$ be a set of $\binom{n}{k}$ strictly increasing sequences of k. The k-th compound matrix of $M \in \mathcal{F}^{m\times n}, 1 \leq k \leq \min\{m, n\}$ is denoted as a $\binom{m}{k} \times \binom{n}{k}$ matrix whose entries are defined in lexicographical order in α and β, such that $\det(M(\alpha, \beta)), \alpha \in Q_{k,n}, \beta \in Q_{k,n}$. The matrix is designated as $C_k(M)$.

It is clear that for a special case of the structure, that when $k = \binom{n}{m}$, a corresponding $\binom{n}{k}$-dimensional vector $C_k(M)$ is decomposable. Then for $M = \{\underline{k}_1, \ldots, \underline{k}_k\} \in \mathcal{F}^{n\times k}, 1 \leq k \leq n$ the compound representation is equivalent to

$$C_k(M) = \underline{k}_1 \wedge \underline{k}_2 \wedge \cdots \wedge \underline{k}_k, \tag{4.28}$$

where the entries of $C_k(M)$ are the Plücker coordinates.

Definition 4.2.5 ([42])**.** Let $\mathcal{V}_M$ be a rational vector space of a given matrix $M(s) = \{M(s) : M(s) \in \mathbb{R}^{p\times q}(s), q \leq p\}$, i.e. $\mathcal{V}_M = \text{colspan}(M(s))$. Then the Grassmann representative of $\mathcal{V}_M$, $g(\mathcal{V}_M)$, is a decomposable multivector of $\mathbb{R}^\sigma$, $\sigma = \binom{p}{q}$. If $\deg(g(\mathcal{V}_M)) = \delta$, corresponding

to Forney's dynamical order (see [112] for more details), then the Grassmann representative can be expressed as

$$g(\mathcal{V}_M) = P_\delta \underline{e}_\delta(s), \;\; \underline{e}_\delta(s) = (1, s, \ldots, s^\delta)^t, \tag{4.29}$$

where P_δ is the Plücker matrix of $\mathcal{V}_M$, such that $P_\delta \in \mathbb{R}^{\begin{pmatrix} p \\ q \end{pmatrix} \times (\delta+1)}$.

The Plücker matrix holds important properties of the decomposability characteristics. It has been shown in [42] that if the given multivector $\underline{z}(s) = P_\delta \underline{e}_\delta(s)$ is decomposable, then the polynomials of $P_\delta = (\underline{p}_0, \underline{p}_1, \ldots, \underline{p}_\delta)$ are also decomposable. Such a criteria leads to a result, defined in a form of a theorem.

Theorem 4.2.5 ([42])**.** The Plücker matrix P_δ of $g(\mathcal{V}_M)$ is a complete invariant of $\mathcal{V}_M$.

Proof. See [42]. □

To summarise, some fundamental properties and notations of Exterior Algebra have been presented. So far it has been highlighted that there exists a specific case of multivectors when decomposability property holds, namely $k = 1, k = n - 1$. Such a scenario will be designated as a linear problem later in the Chapter and is characterised by a one-to-one linear map. As mentioned in [108] the set of k-dimensional subspaces of a vector space $\mathcal{V}$, i.e. $\wedge^k\mathcal{V}$, can be characterised by the elements of the components that satisfy quadratic polynomial properties of the tensors. Then, for a general-case mapping ($k \neq 1, k \neq n - 1$) additional constraints have to be satisfied, referred to as Quadratic Plücker Relations to ensure decomposability of the generating multivector.

Since the decomposability property of the elements in the vector space $\wedge^k\mathcal{V}$ plays an important role in a variety of distance problems of polynomial matrices, the invariant distance problem to uncontrollability, unobservability can be separated into two subproblems: a linear (or special) case and a multilinear (also denoted as general) case; that is going to be described next.

4.3 Multivectors and their link to GCD

The study of the invariant distance from uncontrollability and unobservability uses early results for characterising controllability and observability based on the restriction pencils introduced above [101], [54], [26]. Note that

$$ZC(s) = \begin{pmatrix} N \\ B^\dagger \end{pmatrix} (sI - A, \;\; -B) = \begin{pmatrix} sN - NA & 0 \\ sB^\dagger - B^\dagger A & I_p \end{pmatrix} = C^*(s) \tag{4.30}$$

and

$$K(s)W = \begin{pmatrix} sI - A \\ C \end{pmatrix} (M \;\; C^\dagger) = \begin{pmatrix} sM - AM & sC^\dagger - AC^\dagger \\ 0 & I_m \end{pmatrix} = K^*(s) \tag{4.31}$$

Clearly, $C(s)$ and $C^*(s)$, as well as $K(s)$ and $K^*(s)$ are strict equivalent [103] and both $C(s)$ and $K(s)$ do not have zeros at infinity. Thus their finite zeros, if any, are given by their corresponding Smith forms defined by

$$C^*(s) = \begin{pmatrix} sN - NA & 0 \\ sB^\dagger - B^\dagger A & I_p \end{pmatrix} \triangleleft \begin{pmatrix} sN - NA & 0 \\ 0 & I_p \end{pmatrix} = C(s) \tag{4.32}$$

$$K^*(s) = \begin{pmatrix} sM - AM & sC^\dagger - AC^\dagger \\ 0 & I_m \end{pmatrix} \triangleleft \begin{pmatrix} sM - AM & 0 \\ 0 & I_m \end{pmatrix} = K(s) \tag{4.33}$$

where $\triangleleft$ denotes $\mathbb{R}[s]$-equivalence. The above lead to the following result [101], [54]:

Theorem 4.3.1. The system $S(A, B, C)$ is:

(i) Controllable if and only if the pencil $R(s)$ defined in (4.10) has no finite elementary divisors

(ii) Observable if and only if the pencil $Q(s)$ defined in (4.11) has no finite elementary divisors.

Proof. For the proof see [101], [54]. □

Corollary 4.3.1. The input restriction pencil $R(s) = sN - NA$ is invariant under state feedback and the output restriction pencil $Q(s) = sM - AM$ is invariant under output injection.

The above is rather obvious from the fact that if $L \in \mathbb{R}^{p\times n}$ is a state feedback matrix, then $R'(s) = sN - N(A + BL) = sN - NA = R(s)$. Similarly, if $F \in \mathbb{R}^{n\times m}$ is an output injection matrix, then $Q'(s) = sM - (A + FC)M = sM - AM = Q(s)$. The invariance of $R(s)$ and $Q(s)$ leads to the following definition.

Definition 4.3.1. (i) For the restriction pencil $R(s) = sN - NA$ the rows of $R(s)$ are defined by $\underline{r}_i(s)^t$, $i = 1, \ldots, n - p$. The polynomial defined as

$$C_{n-p}(R(s)) = \underline{r}_1(s)^t \wedge \cdots \wedge \underline{r}_{n-p}(s)^t = \underline{r}(s)^t \wedge^{n-p} \in \mathbb{R}^{1\times\sigma'}[s] = \underline{\tilde{r}}(s)^t, \;\; \sigma' = \binom{n}{n-p} \tag{4.34}$$

will be called the Invariant Controllability Polynomial (ICP) of the system.

(ii) For the restriction pencil $Q(s) = sM - AM$ the rows of $Q(s)$ correspond to $\underline{q}_i(s), i = 1, \ldots, n-m$. The polynomial defined as

$$C_{n-m}(Q(s)) = \underline{q}_1(s) \wedge \cdots \wedge \underline{q}_{n-m}(s) = \underline{q}(s) \wedge^{n-m} \in \mathbb{R}^{\sigma'}[s] = \tilde{\underline{q}}(s), \;\; \sigma' = \binom{n}{n-m} \tag{4.35}$$

will be called the Invariant Observability Polynomial (IOP) of the system.

The invariance under feedback of $R(s)$ and $Q(s)$ implies the invariance of $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)$ and these are used for the study of feedback invariant distances from uncontrollabity, respectively unobservability of the system. Note that $\deg\{\tilde{\underline{r}}(s)^t\} = n - p$ and that $\deg\{\tilde{\underline{q}}(s)\} = n - m$. Given that the GCD of $\tilde{\underline{r}}(s)^t$ provides a state feedback invariant characterisation of input decoupling zeros and $\tilde{\underline{q}}(s)$ provides an output injection invariant characterisation of output decoupling zeros, one is led to the following result:

Corollary 4.3.2. The distances of the set of polynomials of $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)$ from noncoprimeness define respectively the invariant distance from uncontrollability, unobservability.

Thus, such distance problems may be studied using the results on the "approximate" GCD of a set of polynomials [113], [114], [84], [92], [97] and express the distance of the corresponding sets of polynomials from their respective GCD variety [97]. The existing results on the approximate GCD assume that the set of polynomials is arbitrarily defined. However, this is not the case for the polynomials of $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)$ since these are defined as exterior products of rows, columns of matrix pencils and the above distance problems have to take into account these properties. Some basics on the exterior algebra which impact the subsequent analysis are summarised next.

The set of r-dimensional subspaces $\mathcal{H}$ of $\mathbb{R}^v$ is referred to as the r-Grassmaniann. It is denoted by $\mathcal{G}(r, \mathbb{R}^v)$ and the column space of H defines a basis for these subspaces. The mapping of each r-dimensional subspace $\mathcal{H}$ is $\underline{h}_1 \wedge \cdots \wedge \underline{h}_r = \underline{h} \wedge^r = \underline{k}$ (where $\underline{h}_i$ are the columns of H). A vector $\underline{k} \in \mathbb{R}^\tau$ defines a point in the projective space $\mathbb{P}^{\tau-1}(\mathbb{R})$; the points of $\mathbb{P}^{r-1}$ which satisfy (4.19) for some $H \in \mathbb{R}^{v \times r}$ are belong to the Grassmann variety $\Omega(r, v)$ of the projective space $\mathbb{P}^{\tau-1}(\mathbb{R})$ [115]. Let $\underline{k} \in \mathbb{R}^\sigma$ with coordinates $k_\omega, \omega = (i_1, \ldots, i_r) \in Q_{r,v}$.These are referred to as the Plücker coordinates of $\underline{k}$ and the mapping of $\mathcal{H}$ through $\wedge^r$ is known as the Plücker Embedding of the r-Grassmaniann $\mathcal{G}(r, \mathbb{R}^v)$ into the projective space $\mathbb{P}^{\tau-1}(R)$.

The variety $\Omega(r, v)$ is characterised by the following result [108]:

Theorem 4.3.2. 1. Let $\underline{k} \in \mathbb{R}^\tau, \tau = \binom{v}{r}$. A necessary and sufficient condition for matrix H to exist, where $H \in \mathbb{R}^{v \times r}, H = [\underline{h}_1, \ldots, \underline{h}_r] \in \mathbb{R}^{v \times r}$, such that

$$\underline{h} \wedge^r = \underline{h}_1 \wedge \cdots \wedge \underline{h}_r = \underline{k} = [\ldots, k_\omega, \ldots]^t \tag{4.36}$$

is that the coordinates k_ω satisfy the following quadratic relations

$$\sum_{k=1}^{r+1}(-1)^{r-1}k_{i_1,\ldots,i_{v-1},j_r^k j_1,\ldots,j_{r-1},j_{r+1},j_{v+1}}=0 \tag{4.37}$$

where $1 \leq i_1 < i_2 < \cdots < i_{v-1} \leq r$ and $1 \leq j_1 < j_2 < \cdots < j_{v+1} \leq r$. The quadratic relations in (4.37) denote the QPRs for the Grassmman variety.

2. If condition (4.37) is satisfied, there exists a uniquely defined space $\mathcal{H}$ that corresponds to $\underline{k} \in \mathbb{R}^\tau$ that satisfies equation (4.36).

Proof. The proof is given in [108] and hence it is omitted. □

The vectors $\underline{k}$ which satisfy (4.36) are called decomposable. The set of quadratics defined by (4.37) [108], [115], are refered to as Quadratic Plücker Relations and they define the Grassmann variety of $\mathbb{P}^{\tau-1}(\mathbb{R})$. Alternative conditions for decomposability are given in [44], in terms of the Grassmann matrix [44]; the latter criteria also provide the means for the reconstruction of the space $\mathcal{H}$ via its basis matrix H. The nature of the quadratics in QPR is explained in terms of an example.

Example 4.3.1. Let $v = 5, r = 3$ and let $(k_0, k_1, k_2, \ldots, k_8)$ be the coordinates of a vector defining a point in the projective space $\mathbb{P}^{\tau-1}(\mathbb{R}), \tau = 9$. The set of QPRs describing the Grassmann variety $\Omega(3,5)$ are given by

$$k_0k_5 - k_1k_4 + k_2k_3 = 0,\ \ k_0k_8 - k_1k_7 + k_2k_6 = 0,\ \ k_0k_9 - k_3k_7 + k_4k_6 = 0 \tag{4.38}$$

$$k_1k_9 - k_3k_8 + k_5k_6 = 0,\ \ k_2k_9 - k_4k_8 + k_5k_7 = 0 \tag{4.39}$$

It may be readily shown that the above set of equations is not minimal; in fact, the set in (4.39) may be obtained from the set in (4.38) and thus (4.38) is a minimal set of quadratics describing the Grassmann variety $\Omega(3,5)$.

4.4 Feedback invariant distance to uncontrollability

It has been observed that the multilinear nature of the problem leads to analysis of the exterior equation obtained from the corresponding matrix pencils. If the problem is linear and multivectors are decomposable, then the solution is found as a standard problem of Linear Algebra. However, if the problem is multilinear and QPRs have to be satisfied, then the problem is solvable only approximately, as the "approximate decomposable" polynomial combinant that satisfies QPRs and provides alternative criteria for the general case of the problem.

4.4.1 Special case of the invariant distance

As the main focus of the analysis is the decomposability of multivectors, consider first the following Lemma:

Lemma 4.4.1 ([108])**.** Let $\mathcal{U}$ be a vector space over a field $\mathcal{F}$ with $\dim\mathcal{U} = v$. Then any vector in $\wedge^{v-1}\mathcal{U}$ as well as $\wedge^{1}\mathcal{U}$ is decomposable.

Clearly in this case the Grassmann variety of $\mathbb{P}^{\tau-1}(\mathcal{F}), (\tau = \begin{pmatrix} v \\ r \end{pmatrix}, r = v-1 \text{ or } r = 1)$ coincides with the projective space $\mathbb{P}^{\tau-1}(\mathcal{F})$, or in other words there are no Quadratic Plücker Relations defining the variety. The above suggests that the study of the invariant distance to uncontrollability, unobservability is reduced to:

Problem 4.4.1. Linear Invariant Distance Problem (LIDP) is characterised by:

1. The invariant distance to uncontrollability is defined by the distance from the GCD variety of the polynomial $C_{n-p}(R(s)) = \underline{\tilde{r}}(s)^t$, where the vector $\underline{k} \in \mathbb{R}^p$ is the resulting polynomial vector or the generator of the polynomials, and it is free if $p = 1$ or $p = n-1$.

2. The invariant distance to unobservability is defined by the distance from the GCD variety of the polynomial $C_{n-m}(Q(s)) = \underline{\tilde{q}}(s)$, where the generator of the polynomials, the vector $\underline{k} \in \mathbb{R}^m, \underline{\tilde{q}}(s)$ is free if $m = 1$ or $m = n-1$.

The free vectors $\underline{\tilde{r}}(s)^t, \underline{\tilde{q}}(s)$ uniquely identify the projective space and, hence, the mapping is exact.

4.4.2 General case of the invariant distance

For the general case, where $p \neq 1, p \neq n-1$ and $m \neq 1, m \neq n-1$ implies that the polynomials $\underline{\tilde{r}}(s)^t$ and $\underline{\tilde{q}}(s)$ generated by vectors $\underline{k} \in \mathbb{R}^p, \underline{k} \in \mathbb{R}^m$ respectively are not any longer free, but they have to satisfy the corresponding QPRs. In this case the general problem of invariant distance to uncontrollability, unobservability becomes more complicated and is equivalent to the distance between the GCD variety and the corresponding Grassmann variety.

Problem 4.4.2. General Invariant Distance Problem (GIDP) can be defined as follows:

1. The case where $p \neq 1, p \neq n-1$ the general invariant distance to uncontrollability is defined as the distance from the GCD variety of the polynomial $C_{n-p}(R(s)) = \underline{\tilde{r}}(s)^t$, where the generator of the polynomials, the vector $\underline{k} \in \mathbb{R}^p, \underline{\tilde{r}}(s)^t$ also satisfies the corresponding QPRs.

2. The case where $m \neq 1, m \neq n-1$ the general invariant distance to unobservability is defined as the distance from the GCD variety of the polynomial $C_{n-m}(Q(s)) = \underline{\tilde{q}}(s)$, where the generator of the polynomials, the vector $\underline{k} \in \mathbb{R}^m, \underline{\tilde{q}}(s)$ also satisfies the corresponding QPRs.

The results in Theorem 4.2.4 provide alternative characterisations for the decomposability condition of multivectors. It has been shown that the rank of a Grassmann matrix can be used as a decomposability criterion. Hence, in order for the vectors $\underline{\tilde{r}}(s)^t, \underline{\tilde{q}}(s)$ to satisfy QPRs, an optimisation problem has to be solved.

Problem 4.4.3. Let $\underline{z} \in \mathbb{R}^k$ be a non-decomposable multivector. Find $\underline{\hat{z}} \in \mathbb{R}^k \in \Omega(r, v)$, such that the distance between the two multivectors, say $d(\underline{z}, \underline{\hat{z}})$, is minimised.

The solvability of the general case of the invariant distance leads to the notion of an "approximate" solution that is defined as the distance from the GCD variety of the corresponding controllability, observability polynomials, where the generator is the "best almost decomposable" multivector if the initial $\underline{\tilde{r}}(s)^t, \underline{\tilde{q}}(s)$ are not decomposable.

Reference [24] highlighted that the controllability property of the pair (A, B) ((A, C) for observability) can be measured with respect to the relative degree of corresponding Plücker matrices $P(A, B)$ ($P(A, C)$) or $P(N, NA)$ ($P(M, AM)$). This can be summarised in a form of the following theorem:

Theorem 4.4.1 ([24])**.** Let $P(A, B)$ be a Plücker controllability matrix (similarly for the observability $P(A, C)$). Then the smallest singular value of $P(A, B)$ ($P(A, C)$) provides a lower bound for the distance to the family of uncontrollable systems (unobservable, respectively). Equivalently, if $P(N, NA)$ ($P(M, AM)$) is the restricted Plücker controllability (observability) matrix, then the corresponding smallest singular value specifies a lower bound of the "feedback-invariant" distance from the family of uncontrollable (unobservable) systems.

The proof of Theorem 4.4.1 in [24] is based on the properties of polynomials and multivectors. Since the decomposability property is not always satisfied, then the smallest singular value defines the lower bound of the distance. Moreover, considering Lemma 4.4.1 for the boundary cases of the problem, when the multivectors are always decomposable, the minimal singular value becomes an exact distance that is referred to as LIDP.

Example 4.4.1. Consider a linear case of a controllability pencil $C(s)$ defined as

$$[sI - A, \ -B] = \left(\begin{array}{ccc|c} s & -1 & 0 & 0 \\ 0 & s & -1 & 0 \\ 0 & 0 & s & -1 \end{array}\right) = \begin{pmatrix} \underline{c}_1(s)^t \\ \underline{c}_2(s)^t \\ \underline{c}_3(s)^t \end{pmatrix}$$

Then the exterior product of the rows of $C(s)$ corresponds to the polynomial vector

$$\underline{c}(s)^t \wedge = \underline{c}_1(s)^t \wedge \underline{c}_2(s)^t \wedge \underline{c}_3(s)^t = [s^3 \ \ -s^2 \ \ s \ \ -1]$$

is equal to the result of the compound matrix.

According to Theorem 4.4.1 the distance to uncontrollability of the pair (A, B) is given by the smallest singular value of the corresponding Plücker matrix that is computed as

$$P(A,B)\underline{e}_3(s) = \underbrace{\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}}_{P(A,B)} \begin{pmatrix} s^3 \\ s^2 \\ s \\ 1 \end{pmatrix}$$

Since $P(A, B)$ has full rank and the smallest singular value of $P(A, B)$ is equal to 1, then the pair (A, B) is controllable.

Example 4.4.2. Consider now a Restricted Pencil derived from the previous example. The left annihilator of B is

$$N = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

Then

$$R(s) = sN - NA = \begin{pmatrix} 0 & s & 1 \\ -s & -1 & 0 \end{pmatrix} = \begin{pmatrix} \underline{r}_1(s)^t \\ \underline{r}_2(s)^t \end{pmatrix}$$

Deriving the exterior product of $R(s)$ by calculating the compound matrix, which results in

$$\underline{r}(s)^t \wedge = [s^2 \;\; s \;\; 1]$$

Then

$$P(N,NA)\underline{e}_2(s) = \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{P(N,NA)} \begin{pmatrix} s^2 \\ s \\ 1 \end{pmatrix}$$

corresponding to the controllable pair (N, NA), which is consistent with the fact that $P(N, NA)$ is nonsingular.

Projecting Theorem 4.4.1 on the special case $(p = 1, p = n - 1, m = 1, m = n - 1)$ of uncontrollability and unobservability it can be concluded that the Plüker Embedding is bijective implying that polynomials $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)$ are always decomposable. Then, in such a case the invariant distance to uncontrollability, unobservability is exactly characterised by the distance of the corresponding polynomials from the GCD variety.

4.4.3 "Approximate" decomposability of multivectors

The problem of "approximate" decomposability has found particular interest in various algebrogeometric problems in Control Theory and Mathematics [39], [36], [116], [117], [45]. In the study of DAP, for example, the multilinear nature under certain conditions can lead to a better solution of complex control problems.

For the problem of the invariant distance to uncontrollability, unobservability, the methodology is based on the polynomial vectors of the corresponding restricted pencils. Since the resulting polynomials are not necessarily decomposable the need of finding "approximate" solution appears. The framework for finding the "approximate decomposable" polynomial vectors has been studied in [45], [116], [117]. The problem of decomposability is equivalent to

$$\underline{u}_1 \wedge \underline{u}_2 \wedge \cdots \wedge \underline{v}_k = \underline{z}, \;\; \underline{u}_i \in \mathcal{V} \tag{4.40}$$

where the solvability of (4.40) is equal to the decomposability of $\underline{z} \in \wedge^k \mathcal{V}$. Clearly, the approximation of the solution is a difficult task that combines notions of multilinear algebra, projecting spaces as well as complex optimisation procedures. In [116], the authors demonstrated the approximation of multivectors from different perspectives: estimation of the skew symmetric tensors and estimation of the distance from the Grassmann variety.

The rank test of the Grassmann matrix in Theorem 4.2.1 can be used to examine the nontrivial cases of multivectors. Relaxation of the vector can be defined as an optimisation problem that has been studied in [117], [45], [116] and references therein. The "approximate" decomposability problem can be defined as follows:

Problem 4.4.4. Let $\underline{z}$ be a non-decomposable vector, it is required to find such $\underline{k}$ that minimises

$$\Phi(\underline{z}, \underline{k}) = \|\underline{z} - \underline{k}\|, \;\; \text{s.t.} \quad p(\underline{k}) = 0 \tag{4.41}$$

If $\underline{k}^*$ is the minimising solution of (4.41), then $\underline{k}^*$ is characterised by the embedded Grassmannian as

$$\|\underline{z}\| \cdot \sin(\theta) = \|\underline{z} - \underline{k}^*\| \tag{4.42}$$

where θ corresponds to the minimal angle between $\underline{z}$ and $\underline{k}^*$ [45].

The solution to the "approximate" decomposability problem can be obtained using standard optimisation techniques. Some crucial results of the optimisation procedure have been established in [117], [116], [118], [45]. In order to demonstrate the estimation framework, consider a simple numerical example from [45].

Example 4.4.3. Consider a vector $\underline{z} = (a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, a_{34})$ with the corresponding Grassmann matrix $\Phi_4^2(\underline{z})$ defined as in Example 4.2.1. The projective variety is characterised by the QPR as

$$a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = 0$$

where a_{ij} are the Plücker coordinates of $\underline{z}$. If the coordinates are defined as $(10, 2, 15, 3, 1, -20)$, the Grassmann matrix becomes

$$\Phi_4^2(\underline{z}) = \begin{pmatrix} 3 & -2 & 10 & 0 \\ 1 & -15 & 0 & 10 \\ -20 & 0 & -15 & 2 \\ 0 & -20 & -1 & 3 \end{pmatrix}$$

Based on the Corollary 4.2.1 $\Phi_4^2(\underline{z})$ is canonical if its rank is equal to 2. The singular values of $\Phi_4^2(\underline{z})$ are equal to $\sigma = \{27.1635, 27.1635, 5.6694, 5.6694\}$ corresponding to the numerical rank of 4 that does not satisfy the condition of QPRs, i.e. $p(\underline{z}) = -157 \neq 0$, which means that $\underline{z}$ is not decomposable.

In order to find the "approximate" decomposable $\underline{k}$ the optimisation problem in (4.41) has to be solved, and the optimal approximation is computed as

$$k^* = (5.83, 1.869, 16.489, 6.667, 0.582, -18.699)'$$

Remark 4.4.1. The rank test of the Grassmann matrix provides a simple numerical test for the decomposability criteria of multivectors. However, the problem of approximate decomposability of multivectors is clearly a non-standard problem for the general $\Phi_n^k(\underline{z})$, as the uniqueness of the solution is not guaranteed. Moreover, the general case of the restricted input (restricted output) pencils of the system appears to be a more complicated task due to the polynomial nature of the problem. This is going to be investigated in the future work.

Example 4.4.4. Consider a simple form of a system, where the left annihilator of B and the state matrix A are given as follows:

$$N = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & 1 \\ 3 & 1 & 0 & 2 \\ 0 & 3 & 4 & 0 \end{pmatrix}.$$

Then the Restricted Input Pencil is given by

$$R(s) = sN - NA = \begin{pmatrix} s-4 & -6 & s-7 & s-6 \\ -5 & s-5 & s-4 & s-3 \end{pmatrix}$$

The resulting $R(s)$ corresponds to the general case as $n = 4, p = 2$. In order to evaluate the distance to uncontrollability of the given pair (N, NA) the decomposability condition has to be checked. The resulting set of polynomials derived from taking all possible minors of

degree 2 is found as

$$C_2(R(s)) = \begin{pmatrix} s^2 - 9s - 10 \\ s^2 - 3s - 19 \\ s^2 - 2s - 18 \\ -s^2 + 6s - 11 \\ -s^2 + 5s - 12 \\ -3 \end{pmatrix} = \underline{\tilde{r}}(s)^t$$

It can be observed that $\text{rank}\{\Phi_4^2(\underline{z})\} = 2$ that corresponds to the canonical form of $\Phi_4^2(\underline{z})$ and decomposability of $\underline{z}$ respectively. Hence, the distance of the set of polynomials $\underline{\tilde{r}}(s)^t$ from the GCD variety evaluated in the projective space uniquely characterises the invariant distance to uncontrollability.

Example 4.4.5. Let now $\underline{\tilde{r}}(s)^t$ be not free with the set of polynomials given by

$$C_2(R(s)) = \begin{pmatrix} s^2 - 9s - 10 \\ 5s \\ 4s^2 + s - 19 \\ -s^2 + 5s \\ -3s^2 + 9s - 12 \\ 2s^2 - 3s \end{pmatrix}$$

Assume that the numerical value of the multivector is obtained at $s = 1$, then $\underline{z} = (-18, 1, -14, 4, -6, -1)'$, for which $\text{rank}\{\Phi_4^2(\underline{z})\} = 4 > 2$ is not canonical. The problem leads to finding the best decomposable polynomial $\underline{k}$, by solving

$$\begin{aligned} &\min_{\underline{k}} \|\underline{z} - \underline{k}\|, \\ &\text{s.t.} \quad k_1k_6 - k_2k_5 + k_3k_4 = 0 \end{aligned} \tag{4.43}$$

If λ corresponds to the Lagrange multiplier, then the Lagrangian of the function is defined as:

$$L(\underline{k}, \lambda) = \frac{1}{2}\left[(z_1 - k_1)^2 + (z_2 - k_2)^2 + \cdots + (z_6 - k_6)^2\right] - \lambda(k_1k_6 - k_2k_5 + k_3k_4)$$

that leads to the following optimality conditions:

$$\frac{\partial L}{\partial k_1} = (k_1 - z_1) - \lambda k_6 = 0, \quad \frac{\partial L}{\partial k_2} = (k_2 - z_2) + \lambda k_5 = 0, \ldots, \frac{\partial L}{\partial k_6} = (k_6 - z_6) - \lambda x_1 = 0,$$
$$\frac{\partial L}{\partial \lambda} = -k_1k_6 + k_2k_5 - k_3k_4 = 0.$$

Solving the above set of equations the optimal value of $\underline{k}^*$ is computed as

$$\underline{k}^* = (-18.2832, 1.6167, -13.7373, 2.6252, -6.1618, -2.8297)'$$

It can be observed that the elements of $\underline{k}^*$ are very close to the initial vector $\underline{z}$ and hence corresponds to the "almost" decomposable $\underline{z}$.

If s takes an arbitrary value, then the optimisation of (4.43) becomes a more challenging problem. However, for such a simple example the approximate decomposable multivector can be computed using constraint optimisation techniques and it is achieved at

$$\begin{aligned}\underline{k}(s) = (&7.6394s^2 - 2.3606s - 3.3606,\ \ 5.6693s,\ \ 4.0883s^2 + 1.0883s - 18.9117,\\ &-1.1689s^2 + 4.3011s,\ \ -3.1923s^2 + 8.8077s - 12.1923,\ \ 2.0976s^2 - 2.9024s)\end{aligned}$$

that is relatively close to the initial $\underline{z}$. However, in general the optimisation of the polynomial coefficients of the multivector is a non-generic problem and it should be investigated in more detail.

So far, the criteria of the invariant distance to uncontrollability, unobservability, leads to the analysis of a set of polynomials, generated as $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)$, respectively, that uniquely characterise the given system. This corresponds to a distance problem of the resulting polynomials from the GCD variety. It should be noted that the calculation of the non-trivial GCD is a difficult problem. Before stating the main results of the work the powerful notion of the Sylvester Resultant matrix is introduced next.

4.5 Sylvester Resultant and characterisation of the distance

Invariant distance problems that are reviewed as the distance of a set of polynomials from the GCD variety can be studied from the perspective of the Sylvester Resultant matrix. This is a powerful tool for the analysis of the "coprimeness" and "almost coprimeness" of polynomials. This will be described for the simple, unconstrained case where there are no QPRs involved in the analysis and then the results will be extended to the general case where the set of QPRs is non-trivial. In the following it will be referred to as the invariant distance to uncontrollability that is equivalent to the distance of the polynomials $C_{n-p}(R(s)) = \tilde{\underline{r}}(s)^t$ from the GCD variety, while the results for the invariant distance to unobservability based on $\tilde{\underline{q}}(s)$ follow by duality.

It is known [119] that necessary and sufficient conditions for the polynomials to have a common root are defined by the singularity of the corresponding Sylvester matrix. Consider a set of polynomials $\mathcal{P}_{h+1,n}$:

$$\begin{aligned}\mathcal{P}_{h+1,n} = \{&a(s), b_i(s) \in \mathbb{R}[s],\ \ i = 1, \ldots, h;\\ &n = deg\{a(s)\},\ \ t = deg\{b_i(s)\},\ \ i = 1, 2, \ldots, h,\ \ n \geq t\}\end{aligned} \tag{4.44}$$

The notation is occasionally simplified to $\mathcal{P}$ if the integers n and h can be inferred from the context. Let $\phi(s)$ be the GCD of the polynomial set $\mathcal{P}_{h+1,n}$. If the polynomials are coprime, i.e. $\phi(s) = 1$, the polynomial set is denoted as $\mathcal{P}^0_{h+1,n}$. The Sylvester matrix background is considered next [119], [120], [94], [92],[85], [121]:

Definition 4.5.1. Consider the set of monic polynomials $\mathcal{P}_{h+1,n} = \{a(s), b_i(s) \in \mathbb{R}[s],\ i = 1, \ldots, h\}$, where $a(s)$ and $b_i(s)$, $\forall i = 1, \ldots, h$, are given as:

$$\begin{aligned} a(s) &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0, \\ b_i(s) &= s^t + \beta_{t-1,i}s^{t-1} + \cdots + \beta_{0,i},\ \ i = 1, \ldots, h \end{aligned} \tag{4.45}$$

Assuming that with no loss of generality, the polynomials are monic, i.e. $\alpha_n = \beta_{t,i} = 1$, $i = 1, 2, \ldots, h$.

(i) Let $S_0 \in \mathbb{R}^{t\times(n+t)}$ be the Sylvester Resultant associated with $a(s)$:

$$S_0 = \begin{pmatrix} 1 & \alpha_{n-1} & \alpha_{n-2} & \ldots & \ldots & \alpha_0 & 0 & \ldots & 0 \\ 0 & 1 & \alpha_{n-1} & \ldots & \ldots & \alpha_1 & \alpha_0 & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \ldots & \alpha_{t-1} & \ldots & \ldots & \alpha_0 \end{pmatrix} \tag{4.46}$$

(ii) Corresponding to $b_i(s)$ for each $i = 1, \ldots, h$ the resultant matrices $S_i \in \mathbb{R}^{n\times(n+t)}$ are structured as:

$$S_i = \begin{pmatrix} 1 & \beta_{i,t-1} & \beta_{i,t-2} & \ldots & \ldots & \beta_{i,0} & 0 & \ldots & 0 \\ 0 & 1 & \beta_{i,t-1} & \ldots & \ldots & \beta_{i,1} & \beta_{i,0} & \ldots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \ldots & 0 & 1 & \ldots & \beta_{i,n-1} & \ldots & \ldots & \beta_{i,0} \end{pmatrix} \tag{4.47}$$

(iii) The Generalized Resultant for the set $\mathcal{P}_{h+1,n}$ is defined as:

$$S_{\mathcal{P}} = \begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_h \end{pmatrix} \in \mathbb{R}^{(t+hn)\times(n+t)} \tag{4.48}$$

The Sylvester resultant matrix holds important properties for GCD computations. Moreover, the concept of "approximate" GCD and the distance of the polynomials to the GCD variety can be analysed with respect to the singular values of the corresponding Sylvester resultant. In previous research papers [94], [93], [91] "approximate coprimeness" is studied

alongside the Sylvester matrix representation leading to some important notions that are considered next.

Theorem 4.5.1. [92]: Let $S_{\mathcal{P}}$ be a Sylvester resultant matrix of the set of polynomials $\mathcal{P}_{h+1,n}$

$$S_{\mathcal{P}} = \begin{pmatrix} 1 & \alpha_{n-1} & \alpha_{n-2} & \dots & \dots & \alpha_0 & 0 & \dots & 0 \\ 0 & 1 & \alpha_{n-1} & \dots & \dots & \alpha_1 & \alpha_0 & \dots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & \alpha_{t-1} & \dots & \dots & \alpha_0 \\ 1 & \beta_{1,t-1} & \beta_{1,t-2} & \dots & \dots & \beta_{1,0} & 0 & \dots & 0 \\ 0 & 1 & \beta_{1,t-1} & \dots & \dots & \beta_{1,1} & \beta_{1,0} & \dots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & \beta_{1,n-1} & \dots & \dots & \beta_{1,0} \\ \vdots & & \vdots & \vdots & & \vdots & & & \vdots \\ 1 & \beta_{h,t-1} & \beta_{h,t-2} & \dots & \dots & \beta_{h,0} & 0 & \dots & 0 \\ 0 & 1 & \beta_{h,t-1} & \dots & \dots & \beta_{h,1} & \beta_{h,0} & \dots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 1 & \dots & \beta_{h,n-1} & \dots & \dots & \beta_{h,0} \end{pmatrix}. \tag{4.49}$$

If $\phi(s)$ denotes the GCD of the corresponding polynomial set $\mathcal{P}_{h+1,n} = \{a(s), b_i(s), \forall i = 1, \dots, h\}$, then the following properties hold true:

1. For the polynomial set to be coprime it is necessary and sufficient that the Sylvester Resultant has full rank, i.e. $\text{rank}(S_{\mathcal{P}}) = n + t$;

2. If $\mathcal{P}_{h+1,n}$ has non-trivial GCD ($\deg(\phi(s)) \geq 1$), then

$$\text{rank}(S_{\mathcal{P}}) = n + t - deg\{\phi(s)\}; \tag{4.50}$$

3. Since the GCD of $S_{\mathcal{P}}$ stays invariant under elementary row operations, then by reducing it to the row-echelon form the last non-vanishing row will provide the coefficients of $\phi(s)$.

The proof of these properties can be found in [92].

Remark 4.5.1. The degree of the "approximate" GCD can be determined from the values of the smaller singular values of the Sylvester resultant matrix. For example, the matrix pencil technique [84], [85] defines the degree of the approximate GCD with respect to a specified tolerance level *tol*, i.e. as the number of singular values that are less than or equal to *tol*. This approach is refined in this thesis by replacing "singular values" by "structured singular values" which take explicitly into account the structure of the resultant matrix.

The Sylvester Resultant result stated above is central in establishing a number of important computational procedures for the GCD of many polynomials [114]. Clearly, the set of polynomials is not coprime if $S_{\mathcal{P}}$ is rank-deficient, which occurs when the polynomial vector $\underline{\tilde{r}}(s)^t$ is on the GCD variety defined as:

Definition 4.5.2. For a polynomial vector $\underline{\tilde{r}}(s)^t$ the GCD variety is defined by the set of maximal order minors of $S_{\mathcal{P}}$ which are equal to zero. This variety is defined by a polynomial in α_j, β_j parameters defined by the coefficients of the polynomials of $\underline{\tilde{r}}(s)^t$ specified in (4.45).

An immediate Corollary of the above result is:

Corollary 4.5.1. The distance of the polynomial vector $\underline{\tilde{r}}(s)^t$ ($\underline{\tilde{q}}(s)$) from the GCD variety is defined by the distance of the corresponding $S_{\mathcal{P}}$ Sylvester matrix from the set of rank-deficient matrices.

Clearly, the Sylvester Resultant and the properties of the GCD are central in the analysis of the polynomial matrices that characterise system properties. If the Restricted Input pencil is defined by the corresponding polynomial vector $\underline{\tilde{r}}(s)^t$ as in (4.34), then the distance of the corresponding Sylvester matrix from rank-deficiency identifies the distance of $\underline{\tilde{r}}(s)^t$ from the GCD variety, and equivalently the distance to uncontrollability. Similarly, the distance to uncontrollability can be measured whether the polynomial vector $\underline{\tilde{q}}(s)$ is on the GCD variety. Then, an obvious Corollary of the above result is:

Combining characteristics of the restriction matrix pencils $R(s) = sN - NA$ and $Q(s) = sM - MA$ respectively and the notion of the distance to the GCD variety of a polynomial set, it is possible to specify invariant measures to uncontrollability and unobservability.

Corollary 4.5.2. Let $\mathcal{P}$ be a set of polynomials, obtained from $C_{n-p}(R(s)) = \underline{\tilde{r}}(s)^t$. If $p = 1, p = n - 1$, then the smallest singular value of a corresponding Sylvester Resultant $S_{\mathcal{P}}$ denotes the invariant distance to uncontrollability. If $p \neq 1, p \neq n - 1$ and there exists such a generator of the polynomials $\underline{k} \in \mathbb{R}^p$ that satisfies the QPRs, then the smallest singular value of a respective $S_{\mathcal{P}}$ defines a lower bound on the distance.

Similar results can be defined for the distance to unobservability that follows by duality.

Corollary 4.5.3. Let $\mathcal{P}$ be a set of polynomials, obtained from $C_{n-m}(Q(s)) = \underline{\tilde{q}}(s)$. For $m = 1, m = n - 1$ the smallest singular value of $S_{\mathcal{P}}$ defines the invariant distance to unobservability. In the case of $m \neq 1, m \neq n-1$ the Sylvester matrix $S_{\mathcal{P}}$ is structured from a corresponding generator of the polynomials $\underline{k} \in \mathbb{R}^m$ that satisfy the QPRs; then the smallest singular value of $S_{\mathcal{P}}$ is a lower bound of the distance to unobservability.

It is clear that constrained polynomials lead to a bound on the feedback invariant distance to uncontrollability or unobservability. It can be observed that for an arbitrary system, the

problem of finding invariant measures of system properties can be narrowed to the analysis of the lower bound, where it is necessary to satisfy the constraints of the Grassmann variety. Such observations form a basis of future work along with ways of finding a tighter bound of the distance to uncontrollability and unobservability for the design problem.

4.6 Summary

Overall, an alternative approach for measuring fundamental properties of a system, namely controllability and observability that are invariant under state feedback and output injection respectively has been presented. The framework uses concepts of restricted input and output pencils, denoted respectively as $R(s)$ and $Q(s)$, in order to derive criteria for evaluating the distance to uncontrollability, unobservability.

By studying properties of the invariant polynomials of a system, $\tilde{\underline{r}}(s)^t, \tilde{\underline{q}}(s)$, it is shown that the problem is equivalent to the distance of a set of polynomials, say $\mathcal{P}$, to the GCD variety that is reduced to two special cases: LIDP, where polynomials are decomposable and the QPRs are satisfied, and GIDP that requires additional optimisation of the polynomials in order to satisfy the QPRs. Then the invariant distance to uncontrollability (unobservability by duality) is characterised by the smallest singular value of the corresponding Sylvester matrix $S_{\mathcal{P}}$.

Chapter 5

Methods for Solving the Distance Problems

The importance of defining the notion of an "almost common factor" for a set of polynomials has been highlighted in [39], [97], [24]. This is based on the relaxation of the conditions involved in exact GCD computations. Along with the classical computational methods alternative approaches for calculating the distance of a set of monic coprime polynomials to the set of polynomials sharing a common root and, more generally to the set of polynomials with a GCD of fixed degree k, $k \geq 1$ are proposed.

One of the methods is inspired from a factorisation result of a set of polynomials with exact GCD. Specifically, it can be shown that a set of polynomials has GCD of degree k if and only if the generalised resultant of the set, $S_{\mathcal{P}}$, can be written as a product of an augmented resultant matrix $[0_k \mid S_{\mathcal{P}^*}]$ corresponding to a set of polynomials of reduced degrees and a lower triangular Toeplitz matrix $\hat{\Phi}_\lambda$ defined from the coefficients of the GCD polynomial, i.e. $S_{\mathcal{P}} = [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda$. If the polynomial set has GCD of degree less than k the factorisation is not exact and gives rise to an error matrix $E = S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda$. Here, the aim is to minimise the Frobenious norm of E with respect to the free parameters contained in $S_{\mathcal{P}^*}$ and $\hat{\Phi}_\lambda$. The minimisation gives rise to a nonlinear least squares-problem which is computationally demanding. Hence an alternating least-squares projection algorithm for minimising $\|E\|_F$ involving a series of linear projections is proposed.

Another technique is based on singular values to define and solve approximate GCD problems. Consider, for simplicity, the case of two polynomials. These are coprime if and only if the corresponding Sylvester Resultant matrix, $S_{\mathcal{P}}$ say, is nonsingular. Thus, it is reasonable to associate proximity to singularity with the smallest singular value of $S_{\mathcal{P}}$. This follows from the fact that the smallest singular value of a matrix A is equal to the minimum norm perturbation Δ such that $A + \Delta$ is singular (here the norm of Δ, $\|\Delta\|$, indicates the spectral norm, i.e. the largest singular value). Unfortunately, as a measure of proximity

to singularity, the smallest singular value of $S_{\mathcal{P}}$ can be conservative. The reason is that perturbations in $S_{\mathcal{P}}$ arise from the perturbations in the polynomials' coefficients which enter $S_{\mathcal{P}}$ in a highly structured way.

Some limitations of the use of the singular values of $S_{\mathcal{P}}$ as indicators of the existence of an approximate GCD of a specific degree (for the case of two polynomials) have been highlighted in [95], [96]. In these references the notion of the *generalised structured singular value* (μ-value) was introduced to quantify the distance of the corresponding Sylvester resultant matrix from the set of matrices with nullity of at least k. This corresponds to the distance from the set of polynomials with a GCD of degree of at least k. In the case where $k = 1$ this notion reduces to the well-known *structured singular value* which is a well-researched tool in the area of robust control.

The recent paper [96] generalised the results of [95] to the general case of an arbitrary number of polynomials $h \geq 2$. In this case the procedure is based on the calculations of the distance of the Sylvester resultant matrix to a matrix of reduced rank under appropriate structured perturbations; this corresponds to the computation of the closest (minimum distance) nontrivial GCD, i.e. a GCD with degree of at least one.

It can be demonstrated that in the general case involving an arbitrary number of polynomials the problem is still equivalent to the calculation of a structured singular value, however in this case the solution is significantly more intricate compared to the case when $h = 2$ (the results have been presented in [96]). Part of the technical difficulties arise due to the fact that the derived equivalent structured approximation problem no longer involves square matrices and also that the structured approximation sets need to be redefined during the solution procedure. Moreover, the notion of generalised structured singular value can be extended to general structured rank-approximation problems of non-square matrices, for which approximate GCD problems of arbitrary degree can be formulated in a natural way (although their solution is more complicated). This topic is discussed next.

The structure of the Chapter is as follows: Section 5.1 summarises some fundamental results and methodologies for the GCD and "approximate" GCD calculations. Section 5.2 presents a factorisation result of Sylvester resultants when the polynomial set has an exact GCD and presents an alternating least-squares projection algorithm for calculating the "best" approximate GCD (in a least-squares sense) when the polynomial set is coprime. Section 5.3 shows that the problem of calculating the closest common root of a set of polynomials subject to minimal magnitude perturbations in the polynomials' coefficients is equivalent to the calculation of a structured singular value of a matrix. It is also shown that the calculation of the optimal approximate GCD of degree $k > 1$ is associated with the solution of a generalised structured singular value problem. Computational techniques for solving structured singular value problems of the type arising in this work are described in Section 5.4. Finally, the main conclusions of the results and suggestions for further research are included in Section 5.5.

5.1 Computation of GCD and "approximate" GCD: Background results

The study of the Greatest Common Divisors of a set of polynomials has received considerable interest in recent years. The notion of GCD of polynomial sets has several applications in Control Theory (e.g. determinantal assignment problems, distance to controllability or observability), Robust Control (stability of systems subject to structured perturbations), Linear Systems, Numerical Analysis and other engineering fields. Many systems properties depend on the notion of zeros. Computation of the GCD is a non-generic problem: generically any set of polynomials is coprime and hence "coprimeness" needs to be reinterpreted from a binary notion (with a yes or no answer) to one involving a continuously-valued function, a notion which is more suitable for finite-precision calculations. Thus, we can define a set of polynomials to be "approximately coprime" if its distance from the nearest non-trivial common divisor (different from one) is "sufficiently large" in an appropriate sense. The concept of "almost zero" was first introduced in [113]. Subsequently, this has led to the introduction of the notion of "approximate GCD" [114], [84], [93], [95], [91], [122]. The overall philosophy in developing this new notion is based on the relaxation of the conditions defining the exact GCD. The "approximate GCD" is a polynomial which in some sense is closest to the non-coprime set of polynomials.

The problem of finding the GCD of many polynomials has been a subject of interest for a long time in mathematics, numerical analysis and control theory. The origins go back to Euclid's algorithms for two polynomials and more recently to the work based on generalised resultants [123], [119], [120], [121]. Euclid's algorithm has provided an algebraic approach for two polynomials and its iterative use provides an extension for many polynomials. For many polynomials a matrix method based on the invariance property of GCD under row equivalence and shifting was introduced in [124]. An alternative method of reducing GCD computation to matrix pencils was introduced in [84]. Several numerical methods for GCD computation based on the relaxation of exact methods have also been developed. Numerical techniques developed for GCD computations were presented in [94], [92], [125], [84], [85], [91], [126], [114], [122], [127], [128].

The importance of defining the notion of an "almost common factor" for a set of polynomials has been highlighted in [39], [97], [24]. This is based on the relaxation of the conditions involved in exact GCD computations. Alongside the classical computational framework of GCD various strategies have been considered: The invariance property of GCD leads to the numerical method of ERES methodologies [94], [114]; [122] considers a variety of approaches using the Euclidian algorithm; [36], [39] rely on a matrix pencil technique, while [125] applies an augmented optimisation methodology based on factorisation techniques. The majority of the proposed methods in the literature analyse characteristics of the set of polynomials based

on the Sylvester matrix representation and Generalised Resultants [120], [92], [95], [39], [97], as these formulate the GCD problem in linear-algebraic terms which can be implemented via reliable algorithms. The ERES methodology [114] differs from the Resultant Approaches since it is based on the invariance property of GCD under Gaussian transformations and shifting [124]. The current work proposes two general methods for calculating the distance of a set of co-prime polynomials to the set of polynomials sharing a common root and, more generally to the set of polynomials with a GCD of fixed degree k, $k \geq 1$. From this viewpoint the work provides an alternative characterisation of the notion of "almost zero" [113] that establishes computational frameworks for the distance problems.

The Generalised Resultant matrix characterises fully the properties of the GCD of the polynomial set. Further, the k-th smallest singular value of this matrix can be used to define a proximity measure to the nearest GCD of degree k for the perturbed polynomial set. Such an approach is used in the matrix pencil methodologies [84], [85], where the number of singular values falling below a specified tolerance indicates the approximate GCD degree.

A considerable amount of research work has also been devoted to the problem of "approximate co-primness" and "almost zeros" of a set of polynomials [94], [93], [39], [113], [85], [91], [97], [24], [126], [122]. In the context of the problem presented here, a nearly singular Sylvester resultant matrix identifies the existence of an almost common root in the polynomial set, while the smallest singular value may be regarded as a measure of distance to non-coprimeness [94], [93], [91]. Note, however, that this measure may be crude. This can be explained as follows:

Consider a coprime set of polynomials with Generalised Sylvester matrix $S_{\mathcal{P}} \in \mathbb{R}^{(t+hn)\times(n+t)}$, so that $S_{\mathcal{P}}$ has full (column) rank. Consider the distance-to-rank-reduction problem:

$$\gamma = \min\{\|\Delta\|,\ \Delta \in \mathbb{R}^{(t+hn)\times(n+t)},\ \text{rank}(S_{\mathcal{P}} + \Delta) \leq n + t - 1\}$$

where $\|\cdot\|$ is the spectral norm. From the Eckart–Young–Mirsky theorem $\gamma = \sigma_{n+t-1}(S_{\mathcal{P}})$. Note, however, that any perturbations in the polynomial coefficients enter $S_{\mathcal{P}}$ in a highly structured way and hence γ underestimates the distance from non-coprimeness, which can be obtained by solving a structured singular value problem. The same argument applies if $\sigma_{n+t-k}(S_{\mathcal{P}})$ is used as an estimate of the distance of the polynomial set from the set of polynomials with GCD of degree greater than or equal to k. Again this proximity measure can be obtained from the solution of a generalised structured singular value problem. Note however, that (generalised) structured singular values are difficult to calculate for problems of high dimensionality (and even more difficult to optimise) and thus this approach may not be practical.

In [97] techniques for calculating the "best approximate GCD" for a polynomial set were introduced based on the factorisation of the generalised Sylvester matrix which forms the basis of the algorithms proposed in the next section.

Definition 5.1.1. [97]: Consider $\lambda(s) = \lambda_k s^k + \cdots + \lambda_1 s + \lambda_0$ to be the GCD of a polynomial set $\mathcal{P}_{h+1,n}$ and define the Toeplitz matrix $\hat{\Phi}_\lambda \in \mathcal{T}_n$ corresponding to $\lambda(s)$:

$$\hat{\Phi}_\lambda = \begin{pmatrix} \lambda_0 & 0 & 0 & \dots & \dots & \dots & 0 \\ \lambda_1 & \lambda_0 & 0 & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & & \vdots \\ \lambda_k & & \ddots & \ddots & \ddots & & \vdots \\ 0 & \lambda_k & & \lambda_1 & \lambda_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \lambda_k & \dots & \lambda_1 & \lambda_0 \end{pmatrix}. \tag{5.1}$$

Note that $\hat{\Phi}_\lambda$ is invertible if and only if $\lambda_0 \neq 0$ in which case its inverse, denoted as Φ_λ, is also a Toeplitz matrix, with structure:

$$\Phi_\lambda = \begin{pmatrix} y_0 & 0 & 0 & \dots & \dots & 0 \\ y_1 & y_0 & 0 & & & \vdots \\ y_2 & y_1 & y_0 & \ddots & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ y_{n-2} & y_{n-3} & \cdots & \cdots & y_0 & 0 \\ y_{n-1} & y_{n-2} & \cdots & \cdots & y_1 & y_0 \end{pmatrix}, \tag{5.2}$$

where the elements y_i satisfy

$$y_0 = \frac{1}{\lambda_0}, \;\; y_1 = \frac{\lambda_1}{\lambda_0} y_0, \dots, y_i = -\frac{1}{\lambda_0} \sum_{i=1}^{\min\{j,k\}} \lambda_i y_{j-i}, \;\; j = 2, \dots, n-1. \tag{5.3}$$

Theorem 5.1.1. [97]: Consider the Sylvester resultant matrix $S_\mathcal{P}$ of the polynomial set $\mathcal{P}_{h+1,n}$. Let $\lambda(s) = \lambda_k s^k + \cdots + \lambda_1 s + \lambda_0$ be the greatest common divisor of degree k of the set. Then $S_\mathcal{P}$ can be factored as

$$S_\mathcal{P} = S^{(k)}_{\mathcal{P}^*} \hat{\Phi}_\lambda, \;\; S^{(k)}_{\mathcal{P}^*} = \begin{pmatrix} 0_{t,k} & S^{(k)}_0 \\ 0_{n,k} & S^{(k)}_1 \\ \vdots & \vdots \\ 0_{n,k} & S^{(k)}_h \end{pmatrix} = [0_{(t+hn),k} \mid \mathcal{S}_{p^*}]$$

where $\mathcal{S}_p^*$ is the reduced set of polynomials $\mathcal{P}_{h+1,n-k}^*$ obtained by dividing the polynomials of the set $\mathcal{P}_{h+1,n}$ by $\lambda(s)$,

$$S_0^{(k)} = \begin{pmatrix} \alpha_{n-k}^{(k)} & \alpha_{n-k-1}^{(k)} & \dots & \alpha_1^{(k)} & \alpha_0^{(k)} & 0 & \dots & 0 \\ 0 & \alpha_{n-k}^{(k)} & \dots & \dots & \alpha_1^{(k)} & \alpha_0^{(k)} & \dots & 0 \\ \vdots & \ddots & & \ddots & & \ddots & & \vdots \\ 0 & \dots & 0 & \alpha_{n-k}^{(k)} & \dots & \dots & \alpha_1^{(k)} & a_0^{(k)} \end{pmatrix} \in \mathbb{R}^{t\times(n+t-k)} \tag{5.4}$$

and

$$S_i^{(k)} = \begin{pmatrix} \beta_{i,t-k}^{(k)} & \beta_{i,t-k-1}^{(k)} & \dots & \beta_{i,1}^{(k)} & \beta_{i,0}^{(k)} & 0 & \dots & 0 \\ 0 & \beta_{i,t-k}^{(k)} & \dots & \dots & \beta_{i,1}^{(k)} & \beta_{i,0}^{(k)} & \dots & 0 \\ \vdots & \ddots & & \ddots & & \ddots & & \vdots \\ 0 & \dots & 0 & \beta_{i,t-k}^{(k)} & \dots & \dots & \beta_{i,1}^{(k)} & \beta_{i,0}^{(k)} \end{pmatrix} \in \mathbb{R}^{n\times(n+t-k)} \tag{5.5}$$

for each $i = 1, \dots, h$. Further $\hat{\Phi}_\lambda$ is the lower-triangular Toeplitz matrix defined in equation (5.1).

Equivalently, if $\lambda(s)$ does not have a zero root we can write

$$S_{\mathcal{P}^*}^{(k)} = S_{\mathcal{P}}\Phi_\lambda = [0_{(t+hn),k} \mid S_{\mathcal{P}^*}]$$

where $\Phi_\lambda \in \mathbb{R}^{(n+t)\times(n+t)}$ is the lower-triangular Toeplitz matrix defined as in equations (5.2) and (5.3).

Proof. The proof follows from a slight modification of a parallel result in [97]. □

The factorisation of the generalised resultant presented above is exact if and only if the polynomial set $\mathcal{P}_{h+1,n}$ has an (exact) GCD. In case the GCD is only approximate, the necessary and sufficient conditions for Toeplitz matrix representation stated in Theorem 5.1.1 imply that there is a non-zero residual error matrix:

$$E = S_{\mathcal{P}} - S_{\mathcal{P}^*}^{(k)}\hat{\Phi}_\lambda = S_{\mathcal{P}} - [0_k \mid \mathcal{S}_{\mathcal{P}^*}]\hat{\Phi}_\lambda \tag{5.6}$$

for every selection of parameters defined as:

$$\{\alpha_j^{(k)}\}_{j=0,1,\dots,n-k-1}, \;\; \{\beta_{i,j}^{(k)}\}_{i=1,2,\dots,h}^{j=0,1,\dots,t-k-1} \text{ and } \{\lambda_j\}_{j=0,1,\dots k-1} \tag{5.7}$$

The number of parameters has been reduced by setting $\lambda_k = 1$, $\alpha_{n-k}^{(k)} = 1$ and $\beta_{i,t-k}^{(k)} = 1$ for all $i = 1, 2, \dots, h$. This assumes that all polynomials are monic, effectively restoring uniqueness to the factorisation.

The next section introduces an algorithm that aims to identify the "best" approximate GCD by minimising the Frobenious norm of the residual matrix E with respect to the set of free parameters in (5.7). In particular, the aim is to minimise the function:

$$f(\alpha_0^{(k)}, \dots \alpha_{n-k-1}^{(k)}, \beta_{1,0}^{(k)}, \dots \beta_{h,t-k-1}^{(k)}, \lambda_0, \dots \lambda_{k-1}) := \|S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda\|_F^2 \tag{5.8}$$

with respect to the parameters defined in equation (5.7). The optimal parameters define the "best" approximate GCD and the corresponding coefficients of the reduced-degree polynomials, while the minimum value of f quantifies the strength of the approximation (i.e. the lower the minimum value of f achieved, the better the approximation). This is a nonlinear least squares-problem which is computationally demanding for high-dimensional problems. Note, however, that since both $S_{\mathcal{P}^*}^{(k)}$ and $\hat{\Phi}$ are Toeplitz representations of polynomials, f has continuous derivatives of any order. Here, a least-squares alternating projection algorithm is proposed for minimising the residual error involving a sequence of linear projections which are easy to calculate.

Nonlinear least-squares problems have been of interest for many years [129], [130], [131], [132], [133], [134], [135], [136], [137]. Most methods in the literature are based on the Taylor series expansion or gradient-based techniques. Alternatively the nonlinear least-squares problem can be decomposed into two nested sub-problems for which variable projection algorithms can be applied, [132], [133], [138].

5.2 An "alternating" least-squares algorithm for calculating the "approximate" GCD

Expanding on the ideas of [97] a least-squares "alternating projection" algorithm is proposed for solving the optimisation problem. This solves iteratively two linear least squares approximation sub-problems, thus avoiding non-linear programming, including the calculation of steplengths, Jacobian or Hessian matrices. Certain advantages of the proposed method are illustrated via numerical examples included in the last part of the section.

5.2.1 Convergence analysis: Separable problems

Various algorithms proposed for solving nonlinear least-squares problems rely on Gauss-Newton or Levenberg-Marquardt strategies with various modifications. Convergence is typically analysed using information based on the derivatives of the nonlinear function.

In [133], [132], [138], [139] authors study nonlinear least-squares problems in which the unknown parameters can be separated into two sets. It was argued that the specialised methods developed in this case guarantee convergence in a fewer number of iterations compared to traditional gradient-based techniques. Moreover, a separable approach tends to

be more reliable in practice, when standard nonlinear least-squares optimisation fails [138], [139].

The separable approach addressed in [132] reduces the problem to two nested optimisations, each over a different set of parameters. The outer minimisation problem is linear over one parameter set and therefore reduces to a projection.

Thus, the overall problem reduced to a nonlinear minimisation involving a projection operation parametrised over the the second parameter set which can be tackled via nonlinear programming techniques. It is shown that this method is applicable to the problem posed here and guarantees convergence under certain conditions. Next this approach is considered in more detail, both in general terms and also specifically in the context of the problem.

First, consider the function

$$f(\theta, \lambda) = \|y - \Psi(\theta)\lambda\|^2 \tag{5.9}$$

where $\theta \in \mathbb{R}^p$, $\lambda \in \mathbb{R}^q$ are two parameter vectors and $y \in \mathbb{R}^n$. Assume also that $\Psi(\theta)$ has constant rank for all vectors θ inside an open set Ω. Assume now that $\theta \in \Omega$ is fixed and consider the minimisation:

$$\min_{\lambda \in \mathbb{R}^p} \|y - \Psi(\theta)\lambda\|$$

The optimal solution is given by projecting onto the Range of $\Psi(\theta)$, i.e. $\hat{\lambda}(\theta) = \Psi^+(\theta)y$ corresponding to a minimum value

$$r_2(\theta) = \|y - \Psi(\theta)\Psi^+(\theta)y\|^2 = \|P^{\perp}_{\Psi(\theta)}y\|$$

where $P^{\perp}_{\Psi(\theta)}$ is the projection operator onto the orthogonal complement of the range of $\Psi(\theta)$. Thus, to minimise $f(\theta, \lambda)$ simultaneously over θ and λ we need to solve:

$$\min_{\theta \in \Omega} r_2(\theta) = \min_{\theta \in \Omega} \|P^{\perp}_{\Psi(\theta)}y\|$$

which is a nonlinear least-squares problem. The following Theorem is the main result of [132]:

Theorem 5.2.1. Let Ω be an open set in $\mathbb{R}^p$ and assume that $\Psi(\theta)$ has constant rank $\text{rank}[\Psi(\theta)] = r \leq \min(n, q)$ for all $\theta \in \Omega$. Then

(i) If $\hat{\theta}$ is a critical point (or global minimiser in Ω) of $r_2(\theta)$ and $\hat{\lambda} = \Psi^+(\hat{\theta})y$, then $(\hat{\theta}, \hat{\lambda})$ is a critical point of $f(\theta, \lambda)$ (or global minimiser for θ in Ω) and $f(\hat{\theta}, \hat{\lambda}) = r_2\hat{(}\theta)$.

(ii) If $(\hat{\theta}, \hat{\lambda})$ is a global minimiser of $f(\theta, \lambda)$ for $\theta \in \Omega$, then $\hat{\theta}$ is a global minimiser for $r_2(\theta)$ in Ω and $r_2(\hat{\theta}) = f(\hat{\theta}, \hat{\lambda})$. Furthermore if there is a unique $\hat{\lambda}$ among the minimising pairs of $f(\theta, \lambda)$, then $\hat{\lambda}$ must satisfy $\hat{\lambda} = \Psi^+(\hat{\theta})y$.

Proof. See [132] for more details. □

Consider now the specific function

$$f(\theta,\lambda) = \|S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda\|_F^2 \tag{5.10}$$

with parameter vectors:

$$\theta = (\alpha_0^{(k)}, \dots \alpha_{n-k-1}^{(k)}, \beta_{1,0}^{(k)}, \dots \beta_{h,t-k-1}^{(k)})' \text{ and } \lambda = (\lambda_0, \dots \lambda_{k-1})'$$

defined in the previous section. Note that the problem is separable since the θ_i's enter (linearly) matrix $S_{\mathcal{P}^*}$, while the λ_i's enter (linearly) matrix $\hat{\Phi}_\lambda$. To bring $f(\theta,\lambda)$ in the form of (5.9), the vec$(\cdot)$ operation is applied to (5.10). Then:

$$f(\theta,\lambda) = \|S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda\|_F^2 = \|\text{vec}(S_{\mathcal{P}}) - (I \otimes [0_k \mid S_{\mathcal{P}^*}])\text{vec}(\hat{\Phi}_\lambda)\|^2.$$

Define a matrix Q such that $\lambda = Q\text{vec}(\hat{\Phi}_\lambda)$. Then

$$f(\theta,\lambda) = \|\text{vec}(S_{\mathcal{P}}) - (I \otimes [0_k \mid S_{\mathcal{P}^*}])Q\text{vec}(\hat{\Phi}_\lambda)\|^2.$$

Identifying $y := \text{vec}(S_{\mathcal{P}})$ and $\Psi(\theta) = (I \otimes [0_k \mid S_{\mathcal{P}^*}])Q$ shows that $f(\theta,\lambda)$ has the same form as (5.9). Note that due to the bilinearity of the problem this can also be achieved by interchanging the two parameter vectors. Writing:

$$f(\theta,\lambda) = \|S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda\|_F^2 = \|\text{vec}(S_{\mathcal{P}}) - (\hat{\Phi}_\lambda \otimes I)\text{vec}([0_k \mid S_{\mathcal{P}^*}])\|^2$$

and introducing a matrix Q so that

$$\text{vec}([0_k \mid S_{\mathcal{P}^*}]) = \hat{Q}\theta$$

then $f(\theta,\lambda)$ can again be written in the form of (5.9), i.e. as $f(\theta,\lambda) = \|y - \Upsilon(\lambda)\theta\|^2$ in which $y = \text{vec}(S_{\mathcal{P}})$ and $\Upsilon(\lambda) = (\hat{\Phi}_\lambda \otimes I)\hat{Q}$. The next section will make use of this property to define a least-squares alternating projection algorithm for solving the problem.

5.2.2 Algorithm for the estimation of "best approximate GCD"

Let $\mathcal{P}_{h+1,n}$ be a set of monic polynomials with dominant degrees (n,t), $t \leq n$ and let $S_{\mathcal{P}}$ be the generalised Sylvester resultant of the set. Assume that there exists an "approximate GCD" of degree k defined by the monic polynomial $\lambda(s) = s^k + \lambda_{k-1}s^{k-1} + \cdots + \lambda_1 s + \lambda_0$, $k \leq t$. The Toeplitz representation of the GCD is given in equation (5.1) with $\lambda_k = 1$. To estimate the optimal coefficients of the GCD polynomial and the corresponding strength of approximation the following steps are followed:

Step 1: Define $E = S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda$. The objective is to determine the approximate GCD of degree k such that the function $f = \|E\|_F^2$ is minimised. Write E in the form:

$$E = S_{\mathcal{P}} - \left(\tilde{N}_0 + \sum_{j=0}^{n-k-1} \alpha_j^{(k)} N_j + \sum_{i=1}^{h} \sum_{j=0}^{t-k-1} \beta_{ij}^{(k)} N_{ij}\right)\left(\tilde{M}_0 + \sum_{i=0}^{k-1} \lambda_i M_i\right)$$

where $\tilde{N}_0, N_j, N_{ij} \in \mathbb{R}^{(t+hn)\times(n+t)}$ and $\tilde{M}_0, M_i \in \mathbb{R}^{(n+t)\times(n+t)}$. The algorithm proceeds by solving sequentially two subproblems iteratively; these are described in the following two steps (in Step 2 and 3):

Step 2: Assume that parameters $(\lambda_0, \lambda_1, \ldots, \lambda_{k-1})$ are fixed and set

$$M = \tilde{M}_0 + \sum_{i=0}^{k-1} \lambda_i M_i = \hat{\Phi}_\lambda$$

Then:

$$\begin{aligned} E &= S_{\mathcal{P}} - \left(\tilde{N}_0 + \sum_{j=0}^{n-k-1} \alpha_j^{(k)} N_j + \sum_{i=1}^{h} \sum_{j=0}^{t-k-1} \beta_{ij}^{(k)} N_{ij}\right) M \\ &= \tilde{Y}_0 - \sum_{j=0}^{n-k-1} \alpha_j^{(k)} Y_j - \sum_{i=1}^{h} \sum_{j=0}^{t-k-1} \beta_{ij}^{(k)} Y_{ij} \end{aligned}$$

where set $\tilde{Y}_0 = S_{\mathcal{P}} - \tilde{N}_0 M$, $Y_j = N_j M$ and $Y_{ij} = N_{ij} M$ for $i = 1, 2, \ldots, h$ and $j = 0, 1, \ldots, t-k-1$. The first least-squares subproblem is defined as the minimisation of the function:

$$f_1(\alpha_0^{(k)}, \ldots, \beta_{h,t-k-1}^{(k)}) = f(\alpha_0^{(k)}, \ldots, \alpha_{n-k-1}^{(k)}, \beta_{1,0}^{(k)}, \ldots, \beta_{h,t-k-1}^{(k)}, \lambda_0, \ldots, \lambda_{k-1}) \quad (5.11)$$

This is equivalent to the minimisation of:

$$\left\| \tilde{Y}_0 - \sum_{j=0}^{n-k-1} \alpha_j^{(k)} Y_j - \sum_{i=1}^{h} \sum_{j=0}^{t-k-1} \beta_{ij}^{(k)} Y_{ij} \right\|_F^2 = c_1 - 2y'\theta + \theta'\Gamma\theta \quad (5.12)$$

Here $c_1 = \|\tilde{Y}_0\|_F^2$,

$$\theta' = \begin{pmatrix} \alpha_0^{(k)} & \ldots & \alpha_{n-k-1}^{(k)} & \beta_{1,0}^{(k)} & \ldots & \beta_{h,t-k-1}^{(k)} \end{pmatrix} \in \mathbb{R}^{n-k+h(t-k)}$$

and

$$y' = \begin{pmatrix} y_0' & y_1' & \ldots & y_h' \end{pmatrix} \in \mathbb{R}^{n-k+h(t-k)}$$

where

$$y_0' = \begin{pmatrix} \text{trace}(\tilde{Y}_0 Y_0') & \dots & \text{trace}(\tilde{Y}_0 Y_{n-k-1}') \end{pmatrix} \in \mathbb{R}^{n-k}$$

and

$$y_i' = \begin{pmatrix} \text{trace}(\tilde{Y}_0 Y_{i,0}') & \dots & \text{trace}(\tilde{Y}_0 Y_{i,t-k-1}') \end{pmatrix} \in \mathbb{R}^{t-k}$$

for $i = 1, 2, \dots, h$. Also

$$\Gamma = \Gamma' = \begin{pmatrix} \Gamma^{0,0} & \Gamma^{0,1} & \dots & \Gamma^{0,h} \\ \Gamma^{1,0} & \Gamma^{1,1} & \dots & \Gamma^{1,h} \\ \vdots & \vdots & \ddots & \vdots \\ \Gamma^{h,0} & \Gamma^{h,1} & \dots & \Gamma^{h,h} \end{pmatrix} \in \mathbb{R}^{(n-k-h(t-k))\times(n-k-h(t-k))}$$

where

$$\begin{aligned} \Gamma_{i,j}^{0,0} &= \text{trace}(Y_{i-1} Y_{j-1}'), \;\; i = 1, 2, \dots, n-k, \; j = 1, 2, \dots, n-k \\ \Gamma_{i,j}^{0,m} &= \text{trace}(Y_{i-1} Y_{m,j-1}'), \;\; m = 1, 2, \dots, h, \; i = 1, 2, \dots, n-k, \; j = 1, 2, \dots, t-k \\ \Gamma_{i,j}^{\rho,m} &= \text{trace}(Y_{\rho,i-1} Y_{m,j-1}'), \;\; m, \rho = 1, 2, \dots, h, \; i, j = 1, 2, \dots, t-k \end{aligned}$$

Since Γ is positive definite (see Lemma 5.2.1 below), the (unique) minimiser of f_1 is given by $\hat{\theta} = \Gamma^{-1} y$.

Step 3: Here assume that parameters:

$$\{\alpha_j^{(k)}\}_{j=0,1,\dots,n-k-1}, \;\; \{\beta_{i,j}^{(k)}\}_{i=1,2,\dots,h}^{j=0,1,\dots,t-k-1}$$

are fixed and define

$$N = [0_k \mid S_{\mathcal{P}}^*] = \tilde{N}_0 + \sum_{j=0}^{n-k-1} \alpha_j^{(k)} N_j + \sum_{i=1}^{h} \sum_{j=0}^{t-k-1} \beta_{ij}^{(k)} N_{ij}$$

Then:

$$E = S_{\mathcal{P}} - (N\tilde{M}_0 + \sum_{i=0}^{k-1} \lambda_i N M_i) = \tilde{X}_0 - \sum_{i=0}^{k-1} \lambda_i X_i$$

where we have set:

$$\tilde{X}_0 = S_{\mathcal{P}} - N\tilde{M}_0 \;\text{ and }\; X_i = NM_i, \;\; i = 0, 1, \dots, k-1$$

The second least-squares subproblem involves the minimisation of the function:

$$f_2(\lambda_0, \dots, \lambda_{k-1}) = f(\alpha_0^{(k)}, \dots, \alpha_{n-k-1}^{(k)}, \beta_{1,0}^{(k)}, \dots, \beta_{h,t-k-1}^{(k)}, \lambda_0, \dots, \lambda_{k-1}) \qquad (5.13)$$

This is equivalent to the minimisation of:

$$\left\| \tilde{X}_0 - \sum_{i=0}^{k-1} \lambda_i X_i \right\|_F^2 = c - 2\sum_{i=0}^{k-1} \eta_i \lambda_i + \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \sigma_{ij}\lambda_i\lambda_j = c_2 - 2\eta'\lambda + \lambda'\Sigma\lambda \qquad (5.14)$$

where we have defined:

$$c_2 = \|\tilde{X}_0\|_F^2, \;\; \lambda' = \begin{pmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{k-1} \end{pmatrix} \in \mathbb{R}^k$$

and also

$$\eta' = \begin{pmatrix} \eta_0 & \eta_1 & \cdots & \eta_{k-1} \end{pmatrix} \in \mathbb{R}^k, \;\; \eta_i = \text{trace}(\tilde{X}_0 X_i')$$

for $i = 0, 1, \ldots, k-1$ and

$$\Sigma = \Sigma' \in \mathbb{R}^{k\times k} \;\text{ where }\; \Sigma_{ij} = \text{trace}(X_i X_j')$$

for $i = 0, 1, \ldots, k-1$ and $j = 0, 1, \ldots, k-1$. It is shown (Lemma 5.2.2 below) that $\Sigma = \Sigma' > 0$ and hence the (unique) minimiser of f_2 is given by

$$\hat{\lambda} = \begin{pmatrix} \hat{\lambda}_0 & \hat{\lambda}_1 & \cdots & \hat{\lambda}_{k-1} \end{pmatrix}' = \Sigma^{-1}\eta$$

Step 4: Starting from an initial vector:

$$\hat{\lambda}' = \begin{pmatrix} \hat{\lambda}_0 & \hat{\lambda}_1 & \cdots & \hat{\lambda}_{k-1} \end{pmatrix}$$

whose entries are the initial estimates of the approximate GCD iterate between steps 2 and 3 of the algorithm until numerical convergence is attained, i.e. the difference between two consecutive values of f falls below a pre-specified tolerance.

Lemma 5.2.1. $\Gamma = \Gamma' > 0$.

Proof. For $A, B \in \mathbb{R}^{(t+hn)\times(n+t)}$ define the inner product $(A, B) = \text{trace}(AB')$ and note that $(A, A) = \|A\|_F^2$. Then

$$\Gamma = \Gamma' = \left(\begin{array}{c|c|c} \Gamma^{0,0} & \cdots & \Gamma^{0,h} \\ \hline \vdots & \ddots & \vdots \\ \hline \Gamma^{h,0} & \cdots & \Gamma^{h,h} \end{array}\right) \in \mathbb{R}^{(n-k-h(t-k))\times(n-k-h(t-k))}$$

where

$$\Gamma^{0,0} = \begin{pmatrix} (Y_0, Y_0) & \dots & (Y_0, Y_{n-k-1}) \\ \vdots & & \vdots \\ (Y_{n-k-1}, Y_0) & \dots & (Y_{n-k-1}, Y_{n-k-1}) \end{pmatrix}$$

$$\Gamma^{0,h} = \begin{pmatrix} (Y_0, Y_{h,0}) & \dots & (Y_0, Y_{h,t-k-1}) \\ \vdots & & \vdots \\ (Y_{n-k-1}, Y_{h,0}) & \dots & (Y_{n-k-1}, Y_{h,t-k-1}) \end{pmatrix} = (\Gamma^{h,0})'$$

$$\Gamma^{h,h} = \begin{pmatrix} (Y_{h,0}, Y_{h,0}) & \dots & (Y_{h,0}, Y_{h,t-k-1}) \\ \vdots & & \vdots \\ (Y_{h,t-k-1}, Y_{h,0}) & \dots & (Y_{h,t-k-1}, Y_{h,t-k-1}) \end{pmatrix}$$

and similarly for the remaining blocks. Thus Γ is the Gramian matrix of the matrix set

$$\Pi_Y = \{Y_0, \dots, Y_{n-k-1}, Y_{1,0}, \dots Y_{1,t-k-1}, \dots, Y_{h,0}, \dots Y_{h,t-k-1}\}$$

and hence $\Gamma > 0$ if and only if the matrices of this set are linearly independent. If $\lambda_0 \neq 0$ then M is nonsingular and hence Π_Y is linearly independent if and only if the set

$$\Pi_N = \{N_0, \dots, N_{n-k-1}, N_{1,0}, \dots N_{1,t-k-1}, \dots, N_{h,0}, \dots N_{h,t-k-1}\}$$

is linearly independent. Form a linear combination of the matrices in set Π_N and set it equal to zero, i.e.

$$\sum_{j=0}^{n-k-1} \alpha_j^{(k)} N_j + \sum_{i=1}^{h} \sum_{j=0}^{t-k-1} \beta_{ij}^{(k)} N_{ij} = 0 \tag{5.15}$$

Set also $\hat{S}_{\mathcal{P}^*}$ equal to $S_{\mathcal{P}^*}$ but with all leading 1's (i.e. the leading coefficients $\alpha_{n-k}^{(k)}$ and $\beta_{i,t-k}^{(k)}$ which have been absorbed into $\tilde{N}_0$) replaced by 0's. Note that the term in the LHS of equation (5.15) is equal to $[0_k \mid \hat{S}_{\mathcal{P}^*}]$. Thus,

$$\alpha_0^{(k)} = \dots = \alpha_{n-k-1}^{(k)} = \beta_{1,0}^{(k)} = \dots \beta_{h,t-k-1}^{(k)} = 0 \tag{5.16}$$

and hence Π_N is linearly independent. Next suppose that $\lambda_0 = \lambda_1 = \dots = \lambda_\mu = 0$ and $\lambda_{\mu+1} \neq 0$, $\mu \leq k-1$. Form a linear combination of matrices

$$\sum_{j=0}^{n-k-1} \alpha_j^{(k)} Y_j + \sum_{i=1}^{h} \sum_{j=0}^{t-k-1} \beta_{ij}^{(k)} Y_{ij} = 0 \tag{5.17}$$

This implies that $\hat{S}_{\mathcal{P}^*} M = 0$ where

$$\hat{S}_{\mathcal{P}^*} = \left(\begin{array}{cccccccccc}
0_{k+1} & \alpha^{(k)}_{n-k-1} & \dots & \alpha^{(k)}_{1} & \alpha^{(k)}_{0} & 0 & 0 & \dots & 0 \\
0_{k+1} & 0 & \alpha^{(k)}_{n-k-1} & \dots & \alpha^{(k)}_{1} & \alpha^{(k)}_{0} & 0 & \dots & 0 \\
\vdots & \vdots & & \ddots & & \ddots & \ddots & & \vdots \\
0_{k+1} & 0 & \dots & 0 & \alpha^{(k)}_{n-k-1} & \dots & \alpha^{(k)}_{1} & \alpha^{(k)}_{0} & 0 \\
\hline
0_{k+1} & \beta^{(k)}_{1,t-k-1} & \dots & \beta^{(k)}_{1,1} & \beta^{(k)}_{1,0} & 0 & 0 & \dots & 0 \\
0_{k+1} & 0 & \beta^{(k)}_{1,t-k-1} & \dots & \beta^{(k)}_{1,1} & \beta^{(k)}_{1,0} & 0 & \dots & 0 \\
\vdots & \vdots & & \ddots & & \ddots & \ddots & & \vdots \\
0_{k+1} & 0 & \dots & 0 & \beta^{(k)}_{1,t-k-1} & \dots & \beta^{(k)}_{1,1} & \beta^{(k)}_{1,0} & 0 \\
\hline
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\hline
0_{k+1} & \beta^{(k)}_{h,t-k-1} & \dots & \beta^{(k)}_{h,1} & \beta^{(k)}_{h,0} & 0 & 0 & \dots & 0 \\
0_{k+1} & 0 & \beta^{(k)}_{h,t-k-1} & \dots & \beta^{(k)}_{h,1} & \beta^{(k)}_{h,0} & 0 & \dots & 0 \\
\vdots & \vdots & & \ddots & & \ddots & \ddots & & \vdots \\
0_{k+1} & 0 & \dots & 0 & \beta^{(k)}_{h,t-k-1} & \dots & \beta^{(k)}_{h,1} & \beta^{(k)}_{h,0} & 0
\end{array}\right)$$

and

$$M = \left(\begin{array}{cccccccc}
0 & 0 & 0 & \dots & \dots & \dots & \dots & 0 \\
\vdots & \ddots & & & & & & \vdots \\
\lambda_{\mu+1} & & 0 & & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & & & & \vdots \\
1 & & \lambda_{\mu+1} & \ddots & \ddots & & & \vdots \\
0 & 1 & & \lambda_{\mu+1} & 0 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & & \ddots & \ddots & \ddots & 0 \\
0 & \dots & 0 & 1 & \dots & \lambda_{\mu+1} & 0 & 0
\end{array}\right).$$

Considering the first row of the product $\hat{S}_{\mathcal{P}^*} M = 0$, $\alpha^{(k)}_j = 0$, $j = 0, 1, \dots, n-k-1$ is obtained. Similarly, by considering the $t + (i-1)n + 1$-th row, $i = 1, 2, \dots, h$ this results in $\beta^{(k)}_{i,j} = 0$, $i = 1, 2, \dots, h$, $j = 0, 1, \dots, t-k-1$. Hence, equation (5.17) again implies (5.16) and hence the matrix set Π_Y is linearly independent. □

Example 5.2.1. Consider the case of two polynomials

$$\alpha(s) = s^3 + \alpha_2 s^2 + \alpha_1 s + \alpha_0 \text{ and } \beta(s) = s^2 + \beta_1 s + \beta_0$$

of degree $n = 3$ and $t = 2$ respectively. An approximate GCD $\phi(s) = s + \lambda_0$ of degree $k = 1$ is required. In this case:

$$\mathcal{S}_P = \begin{pmatrix} 1 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_2 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 & 0 \\ 0 & 0 & 1 & \beta_1 & \beta_0 \end{pmatrix}, \quad [0 \mid \mathcal{S}_{P^*}] = \left(\begin{array}{c|cccc} 0 & 1 & \alpha_1^{(1)} & \alpha_0^{(1)} & 0 \\ 0 & 0 & 1 & \alpha_1^{(1)} & \alpha_0^{(1)} \\ 0 & 1 & \beta_0^{(1)} & 0 & 0 \\ 0 & 0 & 1 & \beta_0^{(1)} & 0 \\ 0 & 0 & 0 & 1 & \beta_0^{(1)} \end{array}\right)$$

and

$$M = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 & 0 \\ 1 & \lambda_0 & 0 & 0 & 0 \\ 0 & 1 & \lambda_0 & 0 & 0 \\ 0 & 0 & 1 & \lambda_0 & 0 \\ 0 & 0 & 0 & 1 & \lambda_0 \end{pmatrix}$$

Hence:

$$N_0 = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_1 = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad N_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

are linearly independent. Further

$$Y_0 = \begin{pmatrix} 0 & 0 & 1 & \lambda_0 & 0 \\ 0 & 0 & 0 & 1 & \lambda_0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 1 & \lambda_0 & 0 & 0 \\ 0 & 0 & 1 & \lambda_0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Y_{10} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & \lambda_0 & 0 & 0 \\ 0 & 0 & 1 & \lambda_0 & 0 \\ 0 & 0 & 0 & 1 & \lambda_0 \end{pmatrix}$$

are also linearly independent regardless on whether $\lambda_0 = 0$ or $\lambda_0 \neq 0$.

Lemma 5.2.2. $\Sigma = \Sigma' > 0$.

Proof. Again note that $\Sigma_{i,j} = (X_i, X_j)$, $i = 0, 1, \ldots, k-1$, $j = 0, 1, \ldots, k-1$, and so Σ is the Gram matrix of the matrix set:

$$\Pi_X = \{X_0, X_1, \ldots, X_{k-1}\} = \{NM_0, NM_1, \ldots, NM_{k-1}\}$$

Thus $Y = Y' \geq 0$. Further, $Y > 0$ if and only if the matrix set Π_X is linearly independent. Form a linear combination of the matrices in Π_X and set it to zero, i.e.

$$\sum_{i=0}^{k-1} \lambda_i N M_i = N \sum_{i=0}^{k-1} \lambda_i M_i = 0$$

and note that $\sum_{i=0}^{k-1} \lambda_i M_i$ is the matrix $\hat{\Phi}_\lambda$ in equation (5.1) with all entries λ_k replaced by zero, $\tilde{\Phi}_\lambda$ say. Hence

$$[0_k \mid S_i^{(k)}]\tilde{\Phi}_\lambda = 0, \quad i = 0, 1, 2, \ldots, h$$

For $i = 0$ the above equation implies that:

$$\begin{pmatrix} 1 & 0 & \ldots & \ldots & 0 \\ \alpha_{n-k-1}^{(k)} & 1 & \ddots & & \vdots \\ \alpha_{n-k-2}^{(k)} & \alpha_{n-k-1}^{(k)} & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \alpha_{n-2k+1}^{(k)} & \ldots & \alpha_{n-k-2}^{(k)} & \alpha_{n-k-1}^{(k)} & 1 \end{pmatrix} \begin{pmatrix} \lambda_{k-1} \\ \lambda_{k-2} \\ \vdots \\ \lambda_1 \\ \lambda_0 \end{pmatrix} = 0$$

where it is assumed that $n \geq 2k - 1$. The situation is similar to the case where $n < 2k - 1$, except that some entries below the main diagonal in the coefficient matrix of the above matrix equation are zero. In either case the equation implies that $\lambda_{k-1} = \lambda_{k-2} = \ldots = \lambda_0 = 0$ and hence Π_X is linearly independent. □

Remark 5.2.1. Define the polynomial coefficient vectors:

$$\alpha = \begin{pmatrix} 1 & \alpha_{n-1} & \ldots & \alpha_0 \end{pmatrix}, \quad \alpha^{(k)} = \begin{pmatrix} 1 & \alpha_{n-k-1}^{(k)} & \ldots & \alpha_0^{(k)} \end{pmatrix}$$
$$\beta_i = \begin{pmatrix} 1 & \beta_{i,t-1} & \ldots & \beta_{i,0} \end{pmatrix}, \quad \beta_i^{(k)} = \begin{pmatrix} 1 & \beta_{i,n-t-1}^{(k)} & \ldots & \beta_{i,0}^{(k)} \end{pmatrix}$$

where $i = 1, 2, \ldots, h$. Let also

$$\lambda^{(k)} = \begin{pmatrix} 1 & \lambda_{k-1} & \ldots & \lambda_0 \end{pmatrix}$$

Then

$$\|S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda\|_F^2 = t\|\alpha - \alpha^{(k)} * \lambda^{(k)}\|^2 + n\sum_{i=1}^{h} \|\beta_i - \beta_i^{(k)} * \lambda^{(k)}\|^2 \tag{5.18}$$

where $*$ denotes the discrete convolution of two vectors (i.e. the vector of coefficients resulting from the multiplication of the corresponding two polynomials). Note that the two terms in the above expression are weighted by the polynomial degrees t and n, respectively.

These factors can be removed, if required, by eliminating the appropriate rows of the matrices $S_{\mathcal{P}}$ and $[0_r \mid S_{\mathcal{P}^*}]$. This does not reduce significantly the complexity of the algorithm presented above.

5.2.3 Rate of convergence of alternating projection algorithm

In this section the asymptotic rate of convergence of the alternating projection algorithm is estimated. To simplify notation define:

$$N_{n+(i-1)t-ik+j} = N_{i,j}, \;\; 1 \le i \le h, \; 0 \le j \le t-k-1$$

and $\theta_i = \alpha_{i-1}^{(k)}$ for $i = 1, 2, \ldots, n-k-1$, $\theta_{n+(i-1)t-ik+j} = \beta_{i,j}^{(k)}$, $1 \le i \le h$, $0 \le j \le t-k-1$. Then

$$E = S_{\mathcal{P}} - \left(\tilde{N}_0 + \sum_{i=0}^{p} \theta_i N_i\right)\left(\tilde{M}_0 + \sum_{i=0}^{q} \lambda_i M_i\right)$$

where $p = n-k-1$ and $q = k-1$. Suppose that the alternating projection algorithm converges to the optimal solution with parameters $\{\theta_i^o\}_{i=0}^p$, $\{\lambda_j^o\}_{j=0}^q$ and consider small perturbations $\{\delta\theta_i\}_{i=0}^p$, $\{\delta\lambda_j\}_{j=0}^q$ around the optimal set of parameters. Then,

$$E = S_{\mathcal{P}} - \left(\tilde{N}_0 + \sum_{i=0}^{r} (\theta_i^o + \delta\theta_i) N_i\right)\left(\tilde{M}_0 + \sum_{i=0}^{q} (\lambda_i^o + \delta_i^o) M_i\right)$$

This can be written as:

$$E = Y - \sum_{i=0}^{p} \delta\theta_i N_i (\tilde{M}_0 + \sum_{j=0}^{q} \lambda_j^0 M_j) - \sum_{i=0}^{q} \delta\lambda_i (\tilde{N}_0 + \sum_{j=0}^{p} \theta_j^0 N_j) M_i - \sum_{i-0}^{p} \sum_{j=0}^{q} \delta\theta_i \delta\lambda_j N_i M_j$$

where

$$Y = S_{\mathcal{P}} - \tilde{N}_0 \tilde{M}_0 - \sum_{i=0}^{p} \theta_i^o N_i \tilde{M}_0 - \sum_{i=0}^{q} \lambda_i^o \tilde{N}_0 M_i - \sum_{i-0}^{p} \sum_{j=0}^{q} \delta\theta_i^o \delta\lambda_j^o N_i M_j$$

Vectorising:

$$\begin{aligned} \mathrm{vec}(E) = \mathrm{vec}(Y) &- \sum_{i=0}^{p} \delta\theta_i \mathrm{vec}[N_i (\tilde{M}_0 + \sum_{j=0}^{q} \lambda_j^0 M_j)] - \sum_{i=0}^{q} \delta\lambda_i \mathrm{vec}[(\tilde{N}_0 + \sum_{j=0}^{p} \theta_j^0 N_j) M_i] \\ &- \sum_{i=0}^{p} \sum_{j=0}^{q} \delta\theta_i \delta\lambda_j \mathrm{vec}[N_i M_j] \end{aligned}$$

Thus asymptotically as $\delta\theta_i \to 0$ and $\delta\lambda_i \to 0$, $\|E\|_F^2 = \|\mathrm{vec}(E)\|^2$ gets arbitrarily close to

$$\gamma = \|y - S_1 \delta\theta - S_2 \delta\lambda\|^2 \tag{5.19}$$

where

$$y = \text{vec}(Y), \;\; (\delta\theta)' = \begin{pmatrix} \delta\theta_1 & \delta\theta_2 & \dots & \delta\theta_p \end{pmatrix}, \quad (\delta\lambda)' = \begin{pmatrix} \delta\lambda_1 & \delta\lambda_2 & \dots & \delta\lambda_q \end{pmatrix}$$

and

$$S_1 = \begin{pmatrix} \xi_1 & \xi_2 & \dots & \xi_p \end{pmatrix}, \;\; \xi_i = \text{vec}[N_i(\tilde{M}_0 + \sum_{j=0}^{q} \lambda_j^0 M_j)], \;\; i = 1, 2, \dots, p$$

$$S_2 = \begin{pmatrix} \psi_1 & \psi_2 & \dots & \psi_q \end{pmatrix}, \;\; \psi_i = \text{vec}[(\tilde{N}_0 + \sum_{j=0}^{p} \theta_j^0 N_j)M_i], \;\; i = 1, 2, \dots, q$$

Note that geometrically the norm in equation (5.19) corresponds to the distance of y from the subspace $\mathcal{R}(S_1)+\mathcal{R}(S_2)$, where $\mathcal{R}(S)$ is the range (image) of subspace S. The following result can now be used which guarantees convergence and establishes the asymptotic convergence rate of the algorithm:

Theorem 5.2.2. (von Neumann) Let $\mathbb{X}$ be a Hilbert space and $\mathbb{U}$, $\mathbb{V}$, two closed subspaces of $\mathbb{X}$. Let also $\mathbb{W} = \overline{\mathbb{U} + \mathbb{V}}$ where $\overline{(\cdot)}$ denotes set closure. For a given $x \in \mathbb{X}$ define the sequence:

$$x_n = [(I - P_v)(I - P_u)]^n x, \; n \in \mathbb{N}$$

where P_v and P_u denote the orthogonal projection operators onto $\mathbb{V}$ and $\mathbb{U}$, respectively. Then the sequence $\{x_n\}$, $n \in \mathbb{N}$ converges to $x - w$ where w is the best approximation to x from $\mathbb{W}$. Further, convergence is geometric with a convergence rate:

$$\beta = \text{incl}(\mathbb{U}^\perp, \mathbb{V}^\perp)$$

where $\text{incl}(\cdot, \cdot)$ denotes the inclination between two subspaces, i.e.

$$\text{incl}(\mathbb{U}^\perp, \mathbb{V}^\perp) = \sup\{|(z, w)| : z \in \mathbb{U}^\perp, \; w \in \mathbb{V}^\perp, \; \|z\| = \|w\| = 1\}$$

and where $\mathbb{U}^\perp$ and $\mathbb{V}^\perp$ denote the orthogonal complements of $\mathbb{U}$ and $\mathbb{V}$, respectively.

Proof. The proof can be found in [140]. □

Using the previous analysis in this section and Theorem 5.2.2 the asymptotic convergence rate of the algorithm can now be established. This can be easily obtained from the singular value decomposition of matrices S_1 and S_2. Also note that since $\mathcal{R}(S_1)$ and $\mathcal{R}(S_2)$ are finite-dimensional they are closed and so is their sum $\mathcal{R}(S_1) + \mathcal{R}(S_2)$.

Theorem 5.2.3. Let U_2 and $\tilde{U}_2$ be matrices whose columns are orthonormal and span the range of $S_1^\perp$ and $S_2^\perp$, respectively. Then the asymptotic convergence rate of the alternating

projection algorithm is

$$\beta = \mathrm{incl}(\mathcal{R}(S_1^{\perp}), \mathcal{R}(S_2^{\perp})) = \sigma_1(U_2' \tilde{U}_2)$$

Proof. Let $\{u_i\}_{i=1}^{r_1}$ and $\{\tilde{u}_i\}_{i=1}^{r_2}$ be orthonormal bases of $\mathcal{R}(S_1^{\perp})$ and $\mathcal{R}(S_2^{\perp})$, respectively, and set $U_2 = [u_1, u_2, \ldots, u_{r_1}]$ and $\tilde{U}_2 = [\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_{r_1}]$. Then arbitrary unit-norm vectors in $\mathcal{R}(S_1^{\perp})$ and $\mathcal{R}(S_2^{\perp})$ can be expressed as $z = \sum_{i=1}^{r_1} x_i u_i$ and $w = \sum_{i=1}^{r_2} y_i \tilde{u}_i$, respectively. Thus, setting $x = [x_1, x_2, \ldots, x_{r_1}]$ and $y = [y_1, y_2, \ldots, y_{r_2}]$,

$$\begin{aligned}\mathrm{incl}(\mathcal{R}(S_1^{\perp}), \mathcal{R}(S_2^{\perp})) &= \max\{|(z,w)| : z \in \mathcal{R}(S_1^{\perp}), w \in \mathcal{R}(S_2^{\perp}), \|z\| = \|w\| = 1\} \\ &= \max\{|x' U_2' \tilde{U}_2 y| : \|x\| = \|y\| = 1\} = \sigma_1(U_2' \tilde{U}_2)\end{aligned}$$

as required. □

5.2.4 Numerical examples

The advantage of the algorithm is illustrated with several numerical examples.

Example 5.2.2. Consider the polynomials

$$\begin{cases}\alpha(s) &= s^3 + 6s^2 + 11s + 6 = (s+1)(s+2)(s+3) \\ \beta(s) &= s^2 + 5.1s + 4.4 = (s+1.1)(s+4)\end{cases}$$

It is necessary to find an approximate GCD of degree 1. Thus:

$$S_{\mathcal{P}} = \begin{pmatrix} 1 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_2 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 & 0 \\ 0 & 0 & 1 & \beta_1 & \beta_0 \end{pmatrix} = \begin{pmatrix} 1 & 6 & 11 & 6 & 0 \\ 0 & 1 & 6 & 11 & 8 \\ 1 & 5.1 & 4.4 & 0 & 0 \\ 0 & 1 & 5.1 & 4.4 & 0 \\ 0 & 0 & 1 & 5.1 & 4.4 \end{pmatrix}$$

and

$$[0_k \mid S_{\mathcal{P}^*}] = \begin{pmatrix} 0 & 1 & \gamma_1 & \gamma_0 & 0 \\ 0 & 0 & 1 & \gamma_1 & \gamma_0 \\ 0 & 1 & \delta_0 & 0 & 0 \\ 0 & 0 & 1 & \delta_0 & 0 \\ 0 & 0 & 0 & 1 & \delta_0 \end{pmatrix}, \quad \hat{\Phi}_\lambda = \begin{pmatrix} \lambda_0 & 0 & 0 & 0 & 0 \\ 1 & \lambda_0 & 0 & 0 & 0 \\ 0 & 1 & \lambda_0 & 0 & 0 \\ 0 & 0 & 1 & \lambda_0 & 0 \\ 0 & 0 & 0 & 1 & \lambda_0 \end{pmatrix}$$

Here it is necessary to minimise:

$$\begin{aligned} f(\gamma_0, \gamma_1, \delta_0, \lambda_0) &= \|S_{\mathcal{P}} - [0_k \mid S_{\mathcal{P}^*}]\hat{\Phi}_\lambda\|_F^2 \\ &= 2(\gamma_0 + \gamma_1\lambda_0 - 11)^2 + 3(\delta_0 + \lambda_0 - 5.1)^2 + 2(\gamma_1 + \lambda_0 - 6)^2 \\ &+ 3(\delta_0\lambda_0 - 4.4)^2 + 2(\gamma_0\lambda_0 - 6)^2 \end{aligned}$$

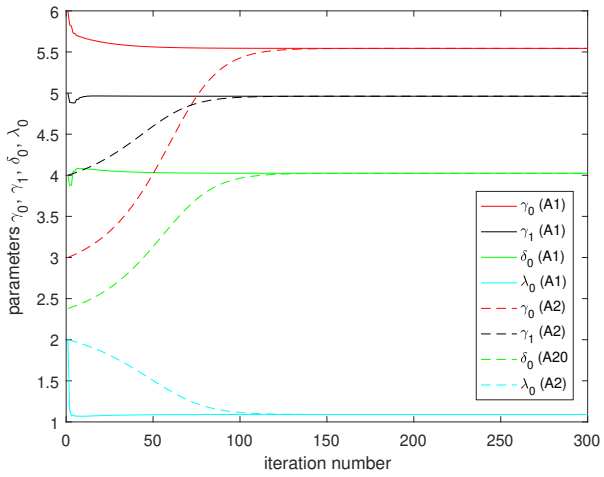

Fig. 5.1 Parameter Convergence Algorithm 1 (dotted) and 2 (solid)

over parameters $(\gamma_0, \gamma_1, \delta_0, \lambda_0)$. An alternating projection algorithm (denoted as Algorithm 1 in the example) has been applied to this problem, initialised at $\lambda_0 = 2$. The results are compared with those obtained via a steepest descent algorithm (later denoted as Algorithm 2, see next) and can be seen in Figure 5.1.

For comparison, a nonlinear steepest-descent algorithm has been applied to the optimisation of f. The gradient and Hessian matrix of f are given as

$$\nabla f = \begin{pmatrix} 4\gamma_0 + 4\lambda_1\lambda_0 + 4\lambda_0(\gamma_0\lambda_0 - 6) - 44 \\ 4\gamma_1 + 4\lambda_0 + 4\lambda_0(\gamma_0 + \gamma_1\lambda_0 - 11) - 24 \\ 6\delta_0 + 6\lambda_0 + 6\lambda_0(\delta_0\lambda_0 - 4.4) - 30.6 \\ f_{\lambda_0} \end{pmatrix}$$

where

$$f_{\lambda_0} = 6\delta_0 + 4\gamma_1 + 10\lambda_0 + 6\delta_0(\delta_0\lambda_0 - 4.4) + 4\gamma_0(\gamma_0\lambda_0 - 6) + 4\gamma_1(\gamma_0 + \gamma_1\lambda_0 - 11) - 54.6$$

and

$$H = H' = \begin{pmatrix} 4\lambda_0 + 4 & 4\lambda_0 & 0 & 4\gamma_1 + 8\gamma_0\lambda_0 - 24 \\ * & 4\lambda_0^2 + 4 & 0 & 4\gamma_0 + 8\gamma_1\lambda_0 - 40 \\ * & * & 6\lambda_0^2 + 6 & 12\delta_0\lambda_0 - 20.4 \\ * & * & * & 6\delta_0^2 + 4\gamma_0^2 + 4\gamma_1^2 + 10 \end{pmatrix}$$

respectively. The algorithm was initialised at the point

$$x_0 = (\gamma_0^0, \gamma_1^0, \delta_0^0, \lambda_0^0)' = (6\ 5\ 4\ 2)'$$

and generated a sequence of points according to the iteration:

$$x_{k+1} = x_k - \mu_k \nabla f(x_k), \;\; \mu_k = \frac{\|\nabla f(x_k)\|^2}{(\nabla f(x_k))' H(x_k) \nabla f(x_k)}, \;\; k = 0, 1, 2, \ldots$$

Parameter μ_k is an estimate of the optimal step-length along the negative gradient direction using Taylor's series expansion around the point x_k (up to and including quadratic terms) assuming that $H(x_k) \geq 0$. This assumption was tested a-posteriori and found to be true, except for the first two iterations (for which the Hessian had one negative eigenvalue), but this did not cause any problems in this case. An alternative choice (according to Armijo's method) would be to determine the optimal step-length by calculating numerically the first local minimum of f along the descent direction, but this proved to be very expensive computationally.

Both algorithms converge to the value $f_0 = 0.014756367409376$ and hence the strength of approximation is 0.121475789395978. The parameter estimates of the GCD and the remainder polynomials are summarised in the following table:

Parameter	Algorithm 1	Algorithm 2
λ_0	1.090161226364660	1.090161226364671
γ_0	5.543683184332871	5.543683184332813
γ_1	4.961546466747018	4.961546466747016
δ_0	4.024100224189135	4.024100224189107

Table 5.1 Parameters at convergence (Example 5.2.2)

Note that on termination of Algorithm 2 (steepest descent) the norm of the gradient vector is equal to $5.100467000228771 \cdot 10^{-14}$ and the Hessian matrix is positive-definite (eigenvalues $(0.7358, 5.6438, 13.1258, 339.6905)$) which indicates that the algorithm has converged to a local minimum of f. Reproduction of the results from many different initial conditions indicates that the solution is in fact a global optimum. The convergence rate of the alternating least-squares as the cosine of the angle between the complement projections is equal to 0.8944 that characterises the slower convergence of the algorithm.

The steepest-descent algorithm converges to the optimal solution in fewer iterations compared to the alternating least-squares projection algorithm. In this example the steepest-descent algorithm needed 52 iterations to converge to within $\pm 1\%$ of the optimal solution whereas the alternating least-squares projection algorithm converged in 126 iterations. However, each iteration of the steepest-descent algorithm is computationally more expensive (14.87 ms CPU time per iteration on average for the steepest descent algorithm versus 2.62

ms CPU time per iteration on average for the alternating least-squares projection algorithm). If a numerical optimisation method was used to calculate the optimal step-length along each descent direction, the difference in computational expense would be considerably larger. Further, the performance of the alternating projection algorithm seems to be less sensitive to inaccurate initial conditions (and requires initial conditions only on the λ_k parameters). However, the main advantage of the alternating projection algorithm is its numerical robustness. Note that each iteration step consists of the solution of two standard linear least-squares problems for which very efficient numerical algorithms exist. By contrast, experimentation with the steepest-descent algorithm showed that some erratic behaviour can arise for some problems, which seems to be associated with Hessian matrices of mixed inertia (as shown in Lemma 5.2.1 and Lemma 5.2.2 the Hessian matrices arising in the two linear least squares problems are always positive-definite).

Example 5.2.3. Consider now three polynomials:

$$\begin{aligned} a(s) =& s^5 - 0.9s^4 - 15.2s^3 + 23.7s^2 + 17.8s - 26.4 \\ b(s) =& s^4 + 1.5s^3 - 49s^2 + 46.5s + 180 \\ c(s) =& s^4 - 1.2s^3 - 39.4s^2 + 58.8s + 216 \end{aligned}$$

In order to illustrate the advantage and robustness of the algorithm with general nonlinear least-squares methodologies, the proposed alternating-least squares algorithm (Algorithm 1), standard steepest descent (Algorithm 2) and nonlinear least-squares Levenberg-Marquardt algorithm (Algorithm 3) are applied in order to compare the performance of each. Starting from the same initial conditions ($\lambda_0 = 1$) the estimated parameters are evaluated and presented in Table 5.2.

Parameter (1.0e+02 *)	Algorithm 1	Algorithm 2	Algorithm 3
a_3	-0.075479785502581	-0.075479774084458	-0.019944048953005
a_2	0.002866727696739	0.002866689509170	-0.129391731512693
a_1	0.213927327799795	0.213927385868718	0.378729306122778
a_0	-0.164012757747432	-0.164012788876443	-0.239976828676686
b_2	-0.043920475311190	-0.043920403980712	-0.030853848990869
b_1	-0.391806867259189	-0.391807029951176	-0.342579521931123
b_0	1.096178193936529	1.096178369685109	1.197436017782389
c_2	0.006479438100252	0.006479519009375	0.000800956132876
c_1	-0.425890527145751	-0.425890705212694	-0.487571675212330
c_0	1.303845714182917	1.303845917713361	1.203483647324654
λ_0	0.016506716625806	0.016506714035634	0.016086238800988

Table 5.2 Parameters at convergence (Example 5.2.3)

Algorithm 1 and 2 converge to the solution with the given tolerance of 10^{-8}, however for the "alternating projection" approach the solution is obtained after 323 iterations, while it takes almost ten times longer for the steeps descent to converge to within $\pm 1\%$ of the optimum. However, the interesting observation is that Algorithm 3 terminates after 173 iterations and fails to converge to the optimal value. This methodology is based on the built-in MATLAB procedure for the calculation of nonlinear least-squares problems. It demonstrates that the alternating linearisation for the larger number of nonlinear parameters converges to the solution in fewer iterations (in comparison to steepest descent) and acts more reliably under practical implementation (as opposed to Levenberg-Marquardt).

5.3 Structured singular value approach for the distance problems

Structured singular values [141], [142] are a powerful tool for the analysis and synthesis of robust control systems. They can be used to model uncertainty in system dynamics arising from multiple sources, e.g. parametric uncertainty in the coefficients of the differential or difference equations, unstructured norm-bounded uncertainty arising due to unmodelled high-frequency dynamics, or combinations of these two types. Structured singular values can be employed to establish non-conservative conditions for robust-stability and robust-performance analysis and in combination with $\mathcal{H}_\infty$ optimal control they can provide a systematic framework for robust control system design [141], [143], [144].

The computation of the structured singular value (μ) of a matrix is an NP-hard problem. Convex relaxation methods are normally employed to calculate upper bounds of μ; techniques for reducing the duality gap have also been developed. In the present work, structured uncertainty arises from the perturbations in the coefficients of the polynomials entering the generalised resultant matrix and will be represented as a diagonal matrix of repeated scalar perturbations. The definition of the appropriate uncertainty structure requires a sequence of preliminary transformations which are introduced later in the section.

The definition of the structured singular value of a matrix is given next. Note that the underlying perturbation structure is not the most general possible but is adequate for the purposes of this work:

Definition 5.3.1. [141] Let $M \in \mathbb{R}^{n\times n}$ and consider the structured set of uncertainties as

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}) : \delta_i \in \mathbb{R}, i = 1, \ldots, s\} \subseteq \mathbb{R}^{n\times n}, \tag{5.20}$$

where r_i are positive integers corresponding to the block-structure of $\mathbf{\Delta}$, i.e. $\sum_{i=1}^{s} r_i = n$. If there exists $\Delta \in \mathbf{\Delta}$, such that $\det(I_n - M\Delta) = 0$, then the structured singular value of M is:

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\min\{\|\Delta\| : \Delta \in \mathbf{\Delta},\ \det(I_n - M\Delta) = 0\}} \tag{5.21}$$

If for all $\Delta \in \mathbf{\Delta}$, $\det(I_n - M\Delta) \neq 0$ then $\mu_{\mathbf{\Delta}}(M) = 0$.

The first problem considered is as follows:

Problem 5.3.1. (Distance to non-coprimeness). Consider the set $\mathcal{P}^0_{h+1,n}$ of coprime polynomials

$$\begin{aligned} a_0(s) &= s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_0, \\ b_{0,i}(s) &= s^t + \beta_{t-1,i}s^{t-1} + \cdots + \beta_{0,i},\ \ i = 1, \ldots, h \end{aligned} \tag{5.22}$$

with corresponding Sylvester resultant $S_{\mathcal{P}_0}$ and define the set of perturbed polynomials $\mathcal{P}_{h+1,n}$:

$$\begin{aligned} a(s) &= s^n + (\alpha_{n-1} + \delta_{n-1})s^{n-1} + \cdots + (\alpha_0 + \delta_0), \\ b_i(s) &= s^t + (\beta_{t-1,i} + \epsilon_{t-1,i})s^{t-1} + \cdots + (\beta_{0,i} + \epsilon_{0,i}),\ \ i = 1, \ldots, h \end{aligned} \tag{5.23}$$

and the corresponding Sylvester resultant $S_{\mathcal{P}}$. Then, what is the minimal absolute value perturbation in the coefficients of the nominal polynomials $\mathcal{P}^0_{h+1,n}$ so that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have a common root? Formally define:

$$\gamma = \max\{|\delta_0|, \ldots, |\delta_{n-1}|, |\epsilon_{0,1}|, \ldots, |\epsilon_{t-1,1}|, \ldots, |\epsilon_{0,h}|, \ldots, |\epsilon_{t-1,h}|\}$$

It is required to minimise γ so that the perturbed polynomials (5.23) have a common root. The Sylvester resultant of the perturbed polynomials is introduced as $S_{\mathcal{P}} = S_{\mathcal{P}_0} + E$, where:

$$E = \begin{pmatrix} 0 & \delta_{n-1} & \delta_{n-2} & \dots & \dots & \delta_0 & 0 & \dots & 0 \\ 0 & 0 & \delta_{n-1} & \dots & \dots & \delta_1 & \delta_0 & \dots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \delta_{t-1} & \dots & \dots & \delta_0 \\ 0 & \epsilon_{1,t-1} & \epsilon_{1,t-2} & \dots & \dots & \epsilon_{1,0} & 0 & \dots & 0 \\ 0 & 0 & \epsilon_{1,t-1} & \dots & \dots & \epsilon_{1,1} & \epsilon_{1,0} & \dots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \epsilon_{1,n-1} & \dots & \dots & \epsilon_{1,0} \\ \vdots & & \vdots & \vdots & & \vdots & & & \vdots \\ 0 & \epsilon_{h,t-1} & \epsilon_{h,t-2} & \dots & \dots & \epsilon_{h,0} & 0 & \dots & 0 \\ 0 & 0 & \epsilon_{h,t-1} & \dots & \dots & \epsilon_{h,1} & \epsilon_{h,0} & \dots & 0 \\ \vdots & \ddots & \ddots & & & \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & 0 & \dots & \epsilon_{h,n-1} & \dots & \dots & \epsilon_{h,0} \end{pmatrix}. \tag{5.24}$$

Then, in view of Theorem 4.5.1 the problem can be formulated as:

$$\inf\{\gamma : \text{null}(S_{\mathcal{P}}) \geq 1\}.$$

Remark 5.3.1. For compatibility with the previous notation and the framework of [84], [97] the h polynomials $b_i(s), i = 1, \dots, h$, are considered to be of the same degree t. Note, however, that all results can be easily amended to the general case (i.e. any set of $h+1$ polynomials $\{a(s), b_i(s)\}, i = 1, 2, \dots, h$, of arbitrary degrees $\deg\{a(s), b_i(s)\} \geq 1, i = 1, 2, \dots, h$) with only minor modifications.

Problem 5.3.1 can be generalised as follows:

Problem 5.3.2. (Approximate GCD of degree k) Let all variables be defined as in Problem 5.3.1 above. Here it is required to minimise γ so that the perturbed polynomials in (5.23) have a GCD $\phi(s)$ with $\deg(\phi(s)) \geq k$. Equivalently the problem can be formulated as: $\inf\{\gamma : \text{null}(S_{\mathcal{P}}) \geq k\}$.

In the remaining part of this section it is shown that Problem 5.3.1 is equivalent to the computation of a structured singular value with respect to a diagonal set of repeated perturbations, while Problem 5.3.2 involves the computation of a "generalised structured singular value", defined over a similar diagonal set. These generalise the results in [95] to the case of multiple polynomials and rely on the following Lemma:

Lemma 5.3.1. Let $A \in \mathbb{R}^{n\times m}$, $n \geq m$ and define:

$$M = \begin{pmatrix} I_n & A \\ A' & 0_m \end{pmatrix} \in \mathbb{R}^{(n+m)\times(n+m)} \tag{5.25}$$

Then $\text{null}(A) = \text{null}(M)$. In particular A has full column rank if and only if matrix M is nonsingular.

Proof. Let $\text{rank}(A) = r$ and $A = U\Sigma V'$ be the singular value decomposition of A with $UU' = U'U = I_n$, $VV' = V'V = I_m$ and

$$\Sigma = \begin{pmatrix} \Sigma_r & 0 \\ 0 & 0 \end{pmatrix}, \;\; \Sigma_r = \text{diag}(\Sigma_r) \in \mathbb{R}^{r\times r}, \;\; \det(\Sigma_r) \neq 0$$

Then:

$$\begin{pmatrix} U' & 0 \\ 0 & V' \end{pmatrix} M \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} I_r & 0 & \Sigma_r & 0 \\ 0 & I_{n-r} & 0 & 0 \\ \Sigma_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and hence:

$$\text{rank}(M) = n - r + \text{Rank}\begin{pmatrix} I_r & \Sigma_r \\ \Sigma_r & 0 \end{pmatrix} = n + r$$

Hence from the rank-nullity theorem $\text{null}(M) = n + m - (n + r) = m - r = \text{null}(A)$ as required. The equivalence between the conditions: (i) A is full column rank, and (ii) M is nonsingular follows in the special case $r = m$. □

Considering Lemma 5.3.1 and the properties of the Sylvester matrix listed in Theorem 4.5.1, Problem 5.3.1 can be reformulated as follows:

Theorem 5.3.1. Let $S_{\mathcal{P}_0}$ be the Sylvester Resultant matrix of the set of the $h+1$ coprime polynomials $\mathcal{P}^0_{h+1,n} = \{a_0(s), b_{0,i}(s), \;\; \forall i = 1, \ldots, h\}$. Introduce perturbations $\{\delta_i\}_{i=0,1,\ldots,n-1}$ and $\{\epsilon_{i,j}\}_{i=1,2,\ldots,h}^{j=0,1,\ldots,t-1}$ in the polynomials' coefficients and define γ as in Problem 5.3.1. Then the minimum-magnitude perturbation γ^* in the coefficients of $\mathcal{P}^0_{h+1,n}$ such that the perturbed polynomials in $\mathcal{P}_{h+1,n}$ have a common root is given by:

$$\gamma^* = \min\{\gamma : \text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}\}$$

where $\mathbf{\Delta}$ is the structured set:

$$\mathbf{\Delta} = \{\text{diag}(\Delta_1, \Delta_2)\} \subseteq \mathbb{R}^{nt(h+1)\times nt(h+1)} \tag{5.26}$$

where:

$$\Delta_1 = \operatorname{diag}(\delta_{n-1} I_t, \delta_{n-2} I_t, \ldots, \delta_0 I_t) \in \mathbb{R}^{nt \times nt} \tag{5.27}$$

and

$$\Delta_2 = \operatorname{diag}(\epsilon_{1,t-1} I_n, \ldots, \epsilon_{1,0} I_n, \ldots, \epsilon_{h,t-1} I_n, \ldots, \epsilon_{h,0} I_n) \in \mathbb{R}^{nth \times nth} \tag{5.28}$$

in which $\delta_i \in \mathbb{R}, \epsilon_{i,j} \in \mathbb{R}$. Matrix $\Theta \in \mathbb{R}^{(t+nh) \times nt(h+1)}$ is defined as:

$$\Theta = \operatorname{diag}(1'_n \otimes I_t, 1'_t \otimes I_n, \ldots, 1'_t \otimes I_n)$$

which can be also written in expanded form as:

$$\Theta = \left(\begin{array}{ccc|ccc|c|ccc} I_t & \ldots & I_t & O_{t,n} & \ldots & O_{t,n} & \ldots & O_{t,n} & \ldots & O_{t,n} \\ \hline O_{n,t} & \ldots & O_{n,t} & I_n & \ldots & I_n & \ldots & O_{n,n} & \ldots & O_{n,n} \\ \hline \vdots & & \vdots & \vdots & & \vdots & \ddots & \vdots & & \vdots \\ \hline O_{n,t} & \ldots & O_{n,t} & O_{n,n} & \ldots & O_{n,n} & \ldots & I_n & \ldots & I_n \end{array}\right) \tag{5.29}$$

Matrix Z is defined as:

$$Z = \begin{pmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_2 \end{pmatrix} = \begin{pmatrix} Z_1 \\ 1_h \otimes Z_2 \end{pmatrix} \in \mathbb{R}^{nt(h+1) \times (n+t)}$$

where

$$Z_1 = \begin{pmatrix} Z_{t,n}^0 \\ Z_{t,n}^1 \\ \vdots \\ Z_{t,n}^{n-1} \end{pmatrix} \in \mathbb{R}^{nt \times (n+t)} \text{ and } Z_2 = \begin{pmatrix} Z_{n,t}^0 \\ Z_{n,t}^1 \\ \vdots \\ Z_{n,t}^{t-1} \end{pmatrix} \in \mathbb{R}^{nt \times (n+t)}$$

in which

$$Z_{t,n}^k = \left(\begin{array}{c|c|c} O_{t,k+1} & I_t & O_{t,n-k-1} \end{array} \right) \in \mathbb{R}^{t \times (n+t)}, \quad k = 0, 1, \ldots, n-1$$

and

$$Z_{n,t}^k = \left(\begin{array}{c|c|c} O_{n,k+1} & I_n & O_{n,t-k-1} \end{array} \right) \in \mathbb{R}^{n \times (n+t)}, \quad k = 0, 1, \ldots, t-1$$

Proof. According to Theorem 4.5.1 polynomials $\mathcal{P}_{h+1,n}$ have a common root if and only if $S_{\mathcal{P}} = S_{\mathcal{P}_0} + E$ loses rank. By "pulling out the uncertainty" it is straightforward to verify that $S_{\mathcal{P}_0} + E = S_{\mathcal{P}_0} + \Theta \Delta Z$ where $\Delta \in \mathbf{\Delta}$. This follows by writing the Sylvester perturbation

matrix in (5.24) as:

$$E = \begin{pmatrix} \sum_{\lambda=0}^{n-1} \delta_\lambda Z_{t,n}^{n-\lambda-1} \\ \sum_{\lambda=0}^{t-1} \epsilon_{1,\lambda} Z_{n,t}^{t-\lambda-1} \\ \vdots \\ \sum_{\lambda=0}^{t-1} \epsilon_{h,\lambda} Z_{n,t}^{t-\lambda-1} \end{pmatrix} = \begin{pmatrix} \delta_{n-1} Z_{t,n}^0 + \delta_{n-2} Z_{t,n}^1 + \ldots + \delta_0 Z_{t,n}^{n-1} \\ \epsilon_{1,t-1} Z_{n,t}^0 + \epsilon_{1,t-2} Z_{n,t}^1 + \ldots + \epsilon_{1,0} Z_{n,t}^{t-1} \\ \vdots \\ \epsilon_{h,t-1} Z_{n,t}^0 + \epsilon_{h,t-2} Z_{n,t}^1 + \ldots + \epsilon_{h,0} Z_{n,t}^{t-1} \end{pmatrix}$$

which reveals the displacement structure of the Sylvester resultant matrix. Further note that $\gamma = \|\Delta\|$ and hence:

$$\gamma^* = \inf\{\|\Delta\| : \text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}\} \tag{5.30}$$

Consider the perturbations: $\delta_0 = -\alpha_0$, $\epsilon_{i,0} = -\beta_{i,0}$, $i = 1, 2, \ldots, h$. Then all perturbed polynomials $\mathcal{P}_{h+1,n}$ have a common root at the origin and hence $\gamma^* \leq \hat{\gamma} := \max\{|\alpha_0|, |\beta_{1,0}|, \ldots, |\beta_{h,0}|\}$. Hence the constraint set in (5.30) can be restricted to the compact set:

$$\{\Delta : \text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}, \|\Delta\| \leq \hat{\gamma}\}$$

Since the function $\Delta \to \|\Delta\|$ is continuous the infimum in (5.30) is attained. □

The following result shows that the solution of Problem 5.3.1 is equivalent to the calculation of a structured singular value.

Theorem 5.3.2. Let all variables be defined as in Theorem 5.3.1. Then the minimum-magnitude perturbation γ^* in the coefficients of $\mathcal{P}_{h+1,n}^0$ such that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have a common root is $\gamma^* = \mu_{\tilde{\mathbf{\Delta}}}^{-1}(\tilde{M})$ where

$$\tilde{M} = -P' \begin{pmatrix} \Theta' S_{\mathcal{P}_0} (S'_{\mathcal{P}_0} S_{\mathcal{P}_0})^{-1} Z' & \Theta'(I - S_{\mathcal{P}_0}(S'_{\mathcal{P}_0} S_{\mathcal{P}_0})^{-1} S'_{\mathcal{P}_0})\Theta \\ -Z(S'_{\mathcal{P}_0} S_{\mathcal{P}_0})^{-1} Z' & Z(S'_{\mathcal{P}_0} S_{\mathcal{P}_0})^{-1} S'_{\mathcal{P}_0} \Theta \end{pmatrix} P$$

and P is a permutation matrix such that $\text{diag}(\mathbf{\Delta}, \mathbf{\Delta}) = P\tilde{\mathbf{\Delta}}P'$ in which

$$\mathbf{\Delta} = \{\text{diag}(\delta_{n-1} I_t, \ldots, \delta_0 I_t, \epsilon_{1,t-1} I_n, \ldots, \epsilon_{1,0} I_n, \ldots, \epsilon_{h,t-1} I_n, \ldots, \epsilon_{h,0} I_n)\} \tag{5.31}$$

and

$$\tilde{\mathbf{\Delta}} = \{\text{diag}(\delta_{n-1} I_{2t}, \ldots, \delta_0 I_{2t}, \epsilon_{1,t-1} I_{2n}, \ldots, \epsilon_{1,0} I_{2n}, \ldots, \epsilon_{h,t-1} I_{2n}, \ldots, \epsilon_{h,0} I_{2n})\} \tag{5.32}$$

Proof. According to Theorem 5.3.1

$$\gamma^* = \min\{\|\Delta\| : \text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1, \Delta \in \mathbf{\Delta}\} \tag{5.33}$$

From Lemma 5.3.1:

$$\text{null}(S_{\mathcal{P}_0} + \Theta\Delta Z) \geq 1 \quad \Leftrightarrow \quad \det \begin{pmatrix} I & S_{\mathcal{P}_0} + \Theta\Delta Z \\ S'_{\mathcal{P}_0} + Z'\Delta\Theta & 0 \end{pmatrix} = 0$$

which is also equivalent to condition:

$$\det \left\{ \begin{pmatrix} I & S_{\mathcal{P}_0} \\ S'_{\mathcal{P}_0} & 0 \end{pmatrix} + \begin{pmatrix} \Theta & 0 \\ 0 & Z' \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} 0 & Z \\ \Theta' & 0 \end{pmatrix} \right\} = 0 \tag{5.34}$$

Since the set of polynomials $\mathcal{P}_{h+1,n}$ are assumed coprime the Sylvester resultant $S_{\mathcal{P}_0}$ has full column rank and hence the first matrix in equation (5.34) is nonsingular (see Lemma 5.3.1). Next let $\Delta \in \mathbf{\Delta}$ and introduce permutation P so that $\text{diag}(\Delta, \Delta) = P\tilde{\Delta}P'$, $\tilde{\Delta} \in \tilde{\mathbf{\Delta}}$ and note that:

$$\Delta \in \mathbf{\Delta} \quad \Leftrightarrow \quad \tilde{\Delta} \in \tilde{\mathbf{\Delta}}$$

Thus condition (5.34) is equivalent to: $\det(I + \tilde{M}\tilde{\Delta}) = 0$ where

$$\tilde{M} = P' \begin{pmatrix} \Theta' & 0 \\ 0 & Z \end{pmatrix} \begin{pmatrix} I & S_{\mathcal{P}_0} \\ S'_{\mathcal{P}_0} & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & \Theta \\ Z' & 0 \end{pmatrix} P$$

Thus from equation (5.33) it follows that:

$$\gamma^* = \min\{\|\tilde{\Delta}\| : \det(I + \tilde{M}\tilde{\Delta}) = 0, \tilde{\Delta} \in \tilde{\mathbf{\Delta}}\} = \mu^{-1}_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \tag{5.35}$$

as required. The form of $\tilde{M}$ given in the statement of the Theorem follows on noting that:

$$\begin{pmatrix} I & S_{\mathcal{P}_0} \\ S'_{\mathcal{P}_0} & 0 \end{pmatrix}^{-1} = \begin{pmatrix} I - S_{\mathcal{P}_0}(S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1}S'_{\mathcal{P}_0} & S_{\mathcal{P}_0}(S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1} \\ (S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1}S'_{\mathcal{P}_0} & -(S'_{\mathcal{P}_0}S_{\mathcal{P}_0})^{-1} \end{pmatrix}$$

after some algebra. □

Remark 5.3.2. The proposed method can be easily generalised to the case, where the nominal coefficients of the polynomials are complex and the perturbations are real. First, Lemma 5.3.1 can be generalised to the complex case by replacing M' by M^* in equation (5.25). This leads to the computation of the real structured singular value of a complex matrix (relative to a real diagonal structure with repeated elements). This can be calculated by separating real and imaginary parts. Generalised results are presented in Appendix A.

In order to illustrate the structured singular value method for many polynomials a simple numerical example is presented below.

Example 5.3.1. Consider the set of coprime polynomials $\mathcal{P}_{3,2}$:

$$\begin{cases} p_1(s) = s^2 + s - 2 \\ p_2(s) = s^2 + 6.0020s - 6.9860 \\ p_3(s) = s^2 + 4s - 5 \end{cases}$$

The corresponding resultant matrix is:

$$S_{\mathcal{P}_0} = \begin{pmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & 1 & -2 \\ 1 & 6.0020 & -6.9860 & 0 \\ 0 & 1 & 6.0020 & -6.9860 \\ 1 & 4 & -5 & 0 \\ 0 & 1 & 4 & -5 \end{pmatrix}$$

with singular values $\sigma(S_{\mathcal{P}_0}) = \{13.6359, 8.9945, 0.9044, 0.0067\}$. Since the smallest singular value is almost zero the numerical rank of $S_{\mathcal{P}_0}$ is 3, indicating an "approximate GCD" of degree one.

As $S_{\mathcal{P}_0} \in \mathbb{R}^{6\times 4}$ the generalised approach of Theorem 5.3.1 can be applied, leading to an augmented matrix:

$$M = \left(\begin{array}{cccccc|cccc} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 6.0020 & -6.9860 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 6.0020 & -6.9860 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 4 & -5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 4 & -5 \\ \hline 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 6.0020 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & -6.9860 & 6.0020 & -5 & 4 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & -6.9860 & 0 & -5 & 0 & 0 & 0 & 0 \end{array}\right)$$

Structured singular value calculations were performed with MATLAB's μ-analysis and synthesis toolbox [145]. The minimum distance to singularity (i.e. the minimum norm perturbation in the coefficients of the three polynomials such that the perturbed polynomials have a common root) is obtained (with an accuracy of four decimal places) as $\gamma^* = 0.0035$. This is the exact μ-value as the lower and upper bounds, obtained via MATLAB, coincide

(within the specified tolerance):

$$\begin{cases} p_1 = s^2 + 1.0017s - 1.9983 \\ p_2 = s^2 + 5.9986s - 6.9894 \\ p_3 = s^2 + 4.0034s - 4.9965 \end{cases}$$

which have GCD $\phi(s) = s - 0.9989$.

The last part of this section is focused on the solution to Problem 5.3.2 which involves the calculation of the numerical GCD (of arbitrary degree) of a set of polynomials $\mathcal{P}^0_{h+1,n}$ (assumed coprime without loss of generality). One possible approach is to develop an approximate solution to this problem in the form of an iterative algorithm. This extracts sequentially approximate common factors $\phi_i(s)$, by calculating the corresponding structured singular value $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ and a corresponding minimum-norm singularising matrix perturbation by repeated application of Theorem 5.3.1. After the extraction of each factor, the quotient $a_{i+1}(s) = a_i(s)/\phi_i(s)$ and $b_{i+1,j}(s) = b_{i,j}(s)/\phi_i(s)$ are calculated, ignoring possible (small) remainder terms of the divisions. The procedure is initialised by setting $a_0(s) = a(s)$, $b_{0,j}(s) = b_j(s)$, $j = 1, 2, \ldots, h$, and iterates by constructing at each step of the algorithm the reduced-dimension Sylvester matrix corresponding to the polynomials $(a_{i+1}(s), \{b_{i+1,j}(s)\}_{j=1,2,\ldots,h})$; this is followed by calculating the new structured singular value $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ and a corresponding $\tilde{\Delta}_0 \in \tilde{\mathbf{\Delta}}$, which in turn leads to the extraction of the new approximate factor $\phi_{i+1}(s)$. The whole process is repeated until a tolerance condition is met, at which stage the approximate GCD $\phi(s)$ can be constructed by accumulating the extracted common factors $\phi_i(s)$. Special care is needed to ensure that any complex roots in $\phi(s)$ appear in conjugate pairs.

Compared to this procedure a more elegant (and exact) approach is to extract the approximate GCD of the polynomial set $\mathcal{P}^0_{h+1,n}$ by solving a single optimisation problem. This involves the following generalisation of the notion of the structured singular value of a matrix:

Definition 5.3.2. Let $M \in \mathbb{R}^{n\times n}$ and define the "structured" set:

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_{r_1},\ \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}) \ :\ \delta_i \in \mathbb{R},\ i = 1, 2, \ldots, s\} \tag{5.36}$$

where the r_i are positive integers such that $\sum_{i=1}^{s} r_i = n$. (Note that $\mathbf{\Delta}$ is a subspace of $\mathbb{R}^{n\times n}$). The *generalised structured singular value* of M relative to "structure" $\mathbf{\Delta}$ and for a non-negative integer k is defined as:

$$\hat{\mu}_{\mathbf{\Delta},k}(M) = \frac{1}{\min\{\|\Delta\| \ :\ \Delta \in \mathbf{\Delta},\ \mathrm{null}(I_n - M\Delta) > k\}} \tag{5.37}$$

unless there does not exist a $\Delta \in \mathbf{\Delta}$ such that $\text{null}(I_n - M\Delta) > k$, in which case $\hat{\mu}_{\mathbf{\Delta},k}(M) = 0$.

It follows immediately from the definition that $\hat{\mu}_{\mathbf{\Delta},0}(M) = \mu_{\mathbf{\Delta}}(M)$ and that $\hat{\mu}_{\mathbf{\Delta},k}(M) \geq \hat{\mu}_{\mathbf{\Delta},k+1}(M)$ for each integer $k \geq 0$. Further if for some integer k, $\hat{\mu}_{\mathbf{\Delta},k}(M) > 0$ and $\hat{\mu}_{\mathbf{\Delta},k+1}(M) = 0$, then any Δ_0 that minimises the denominator in (5.37) has $\text{null}(I_n - M\Delta) = k + 1$. The following Theorem can now be stated and proved:

Theorem 5.3.3. Consider Problem 5.3.2 and let all variables be defined as in Theorem 5.3.1. Then the minimum-magnitude perturbation γ^* in the coefficients of the polynomial set $\mathcal{P}^0_{h+1,n}$ such that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have a GCD of degree at least k $(1 \leq k \leq t)$ is $\gamma^* = \hat{\mu}^{-1}_{\tilde{\mathbf{\Delta}},k-1}(\tilde{M})$ where $\tilde{M}$ and $\tilde{\mathbf{\Delta}}$ are as defined in Theorem 5.3.2.

Proof. This is a simple generalisation of the proof of Theorem 5.3.2 based on the general conditions given in (5.25). Note that the inverse of $\hat{\mu}_{\tilde{\mathbf{\Delta}},k-1}(\tilde{M})$, $k = 1, 2, \ldots, t$ always exists since t is the minimum degree of all polynomials in set $\mathcal{P}^0_{h+1,n}$ and there are always perturbations in the coefficients of $\mathcal{P}^0_{h+1,n}$ such that the perturbed polynomials $\mathcal{P}_{h+1,n}$ have at least k common roots. For example the perturbations:

$$\delta_j = -\alpha_j, \;\; \epsilon_{i,j} = -\beta_{i,j}, \;\; i = 1, 2, \ldots, h, \;\; j = 0, 1, \ldots, k-1$$
$$\delta_j = 0, \;\; \epsilon_{i,j} = 0, \;\; i = 1, 2, \ldots, h, \;\; j \geq k$$

result in a perturbed set of polynomials $\mathcal{P}_{h+1,n}$ which have at least k common roots at the origin and hence

$$\hat{\mu}(\tilde{M}) \geq \frac{1}{\max\{|\alpha_j|, |\beta_{i,j}|\}^{j=0,1,\ldots,k-1}_{i=1,2,\ldots,h}} > 0$$

since the polynomials $\mathcal{P}^0_{h+1,n}$ have been assumed to be coprime. □

Theorem 5.3.3 suggests that the GCD of the polynomial set $\mathcal{P}^0_{h+1,n}$ can be obtained by calculating successively $\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M})$ for $k = 0, 1, \ldots, t-1$. The procedure terminates when either $k = t - 1$ is reached, or when the inverse of the generalised structured singular value falls below a pre-specified tolerance level.

The calculation of $\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M})$ is a nonconvex optimisation problem. An upper bound can be obtained as:

$$\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M}) \leq \inf_{D \in \mathbf{D}} \sigma_{k+1}(D\tilde{M}D^{-1}) \tag{5.38}$$

where $\mathbf{D}$ is the set of all positive definite matrices which commute with $\tilde{\mathbf{\Delta}}$ [95], [142]. The efficient calculation of $\hat{\mu}_{\tilde{\mathbf{\Delta}},k}(\tilde{M})$ is a challenging problem for which (it is belived) no solution is currently available. In the following section the literature for calculating the (standard) structured singular value is reviewed and an algorithm for estimating the upper bound of the generalised structured singular value given in equation (5.38) is proposed.

5.4 Computational algorithms for μ and generalised μ problems

The computation of the structured singular value of a matrix M is an NP-hard problem [146]. Thus, given any algorithm to compute $\mu(M)$, there will be "worst-case problems" for which the algorithm will fail to find the answer in polynomial-time. Although the structured singular value can be obtained by maximising the spectral radius of a matrix over a set of scaling matrices, the objective function is non-concave and the corresponding algorithms rarely converge to the optimal solution. In practice, upper bounds are often sought by applying the so-called D-iteration procedure, which solves a convex minimisation problem and is equivalent to a Linear Matrix Inequality (LMI). It can be shown that for certain simple perturbation structures the gap between μ and its convex upper bound is zero. The same is also true for certain other problems of special structure (rank-1 matrices [147], reciprocal matrices [148], etc). These problem classes, however, are typically too small for most practical applications. In general, the gap between μ and its convex upper bound can be arbitrarily large (but grows no faster than linearly in the number of uncertainty blocks) [146].

A systematic investigation of the gap between μ and its convex upper bound was presented in [149]. It was shown that the gap can be breached by solving an eigenvalue problem, provided a sufficiently tight bound can be obtained for an auxiliary reduced-rank μ-problem, defined from the optimal scaling matrices of the D-iteration procedure. In many cases the complexity of the auxiliary problem is significantly reduced and breaching the convex upper bound is feasible. Several other optimisation methods for the general μ problem, or varieties of its specialised versions have been reported in the literature [145], [150], [151], [141], [142] as well as a more recent one [152].

In the remaining of the section a method for minimising the bound in (5.38) using gradient descent algorithms is outlined. Note that the problem is in general nonconvex, so convergence to the global optimum cannot be guaranteed. However, the approach can prove useful in practice if a good starting point for the descent algorithm is available.

Consider the minimisation problem: $\inf_{D\in\mathbf{D}} \sigma_{k+1}(DMD^{-1})$ in which $M \in \mathbb{R}^{n\times n}$ and

$$\mathbf{D} = \{\mathrm{diag}(D_1, D_2, \ldots, D_s) : D_i \in \mathbb{R}^{r_i\times r_i},\ D_i = D_i' > 0\},\ \sum_{i=1}^{s} r_i = n$$

Note that matrices in $\mathbf{D}$ commute with matrices in the underlying perturbation set $\mathbf{\Delta}$ defined in (5.20). Let

$$d \in \mathbb{R}^q,\ q = \frac{1}{2}\sum_{i=1}^{s} r_i(r_i+1)$$

be the vector of the (non-repeated) variables of $D \in \mathbf{D}$. Further assume that Ω is an open subset of $\mathbb{R}^q$ and let $A(d) = DMD^{-1} : \Omega \to \mathbb{R}^{n\times n}$ be a real matrix function of $d \in \Omega$. Let

$\Sigma(d) = \text{diag}(\sigma_1(d), \sigma_2(d), \ldots, \sigma_n(d)), \sigma_1(d) \geq \sigma_2(d) \geq \cdots \geq \sigma_n(d) \geq 0$ be the singular values of $A(d)$. Assume that $\sigma_{k+1}(d)$ is non-repeated for every $d \in \Omega$. This assumption is made for simplicity and ensures the differentiability of $\sigma_{k+1}(d)$ in Ω; if it fails at some $d \in \Omega$ the gradient of $\sigma_{k+1}(d)$ is not defined at that point and the descent-direction algorithm may need to be modified using subgradient techniques [153], [154].

Define the matrix:

$$H(d) = \begin{pmatrix} 0 & A(d) \\ A'(d) & 0 \end{pmatrix} \in \mathbb{R}^{2n \times 2n} \tag{5.39}$$

The eigenvalues of $H(d)$ (arranged in non-increasing order) are related to the singular values of $A(d)$ as follows:

$$\sigma_1(d) \geq \cdots \geq \sigma_n(d) \geq 0 \geq -\sigma_n(d) \geq \cdots \geq -\sigma_1(d), \ \ \forall d \in \Omega \tag{5.40}$$

Thus the sensitivity of the singular values of $A(d)$ can be inferred from the sensitivity of the eigenvalues of $H(d)$ [154].

Next fix $d_0 \in \Omega$ and obtain the spectral decomposition of $H(d_0)$:

$$H(d_0) = W \begin{pmatrix} \Sigma(d_0) & 0 \\ 0 & -\Sigma(d_0) \end{pmatrix} W' \tag{5.41}$$

in which the eigenvector matrix $W \in \mathbb{R}^{2n \times 2n}$ is orthogonal. Denote by W_{k+1} the $(k+1)$-th column of W ($(k+1)$-th eigenvector of $H(d_0)$). Then, for each vector $x \in \mathbb{R}^q$, $\|x\| = 1$ the directional derivative of $\sigma_{k+1}(DMD^{-1})$ at d_0 along direction x is given as:

$$\langle \nabla \sigma_{k+1}(d_0), x \rangle = W'_{k+1} \left\{ \sum_{i=1}^{q} x_i \begin{pmatrix} 0 & \frac{\partial (DMD^{-1})}{\partial d_i}(d_0) \\ \frac{\partial (DMD^{-1})}{\partial d_i}(d_0) & 0 \end{pmatrix} \right\} W_{k+1} \tag{5.42}$$

where x_i is the i-th component of x. Equivalently, if W_{k+1} is partitioned as:

$$W_{k+1} = \begin{pmatrix} U_{k+1} \\ V_{k+1} \end{pmatrix}, \ U_{k+1} \in \mathbb{R}^n, \ \ V_{k+1} \in \mathbb{R}^n \tag{5.43}$$

then:

$$\langle \nabla \sigma_{k+1}(d_0), x \rangle = 2U'_{k+1} \left(\sum_{i=1}^{q} x_i \frac{\partial (DMD^{-1})}{\partial d_i}(d_0) \right) V_{k+1} \tag{5.44}$$

Note that from the chain rule:

$$\frac{\partial (DMD^{-1})}{\partial d_i} = \frac{\partial D}{\partial d_i} MD^{-1} - DMD^{-1} \frac{\partial D}{\partial d_i} D^{-1}, \ \ i = 1, 2 \ldots, q \tag{5.45}$$

The following steepest-descent algorithm can now be applied for solving the optimization problem: $\inf_{D\in\mathbf{D}} \sigma_{k+1}(DMD^{-1})$:

Algorithm:

Initialise: $j = 0$, $D_j = I_n$ and $d_j = \overline{\text{vec}}(D_j) \in \mathbb{R}^r$ where $r = \frac{n(n+1)}{2}$ and $\overline{\text{vec}}(D_j)$ denotes the vectorisation operation of stacking the columns of D_j into a vector. Set tolerance parameters $\epsilon_1 > 0$ and $\epsilon_2 > 0$.

Step 1: Define $A(d_j) = D_j M D_j^{-1}$, $H(d_j)$ as in equation (5.39) and perform the spectral decomposition (5.41) to obtain $\Sigma(d_j)$ and $W(d_j)$. Set $W_{k+1}(d_j)$ as the $(k+1)$-th column of $W(d_j)$ and decompose it as in equation (5.43) to obtain the two Schmidt vectors $U_{k+1}(d_j)$ and $V_{k+1}(d_j)$.

Step 2: Using equation (5.44) calculate $\nabla\sigma_{k+1}(d_j)$ by setting

$$(\nabla\sigma_{k+1}(d_j))_i = \langle \nabla\sigma_{k+1}(d_j), e_i\rangle, \;\; i = 1, 2, \ldots, q$$

where e_i is the i-th column of I_q.

Step 3: If $\|\nabla\sigma_{k+1}(d_j)\| \leq \epsilon_1$ stop and exit.

Step 4: Set

$$\Psi_j = \overline{\text{vec}}^{-1}(\nabla\sigma_{k+1}(d_j)), \;\; \Phi_j(t) = D_j - t\Psi_j, \;\; t \geq 0$$

and define the function:

$$f_j : I_j \to \mathbb{R}_+, f_j(t) = \sigma_{k+1}\left(\Phi_j(t) M \Phi_j^{-1}(t)\right)$$

whose domain $I_j \subseteq \mathbb{R}_{0+}$ is defined as follows: Solve the generalised (symmetric) eigenvalue problem $\det(D_j - t\Psi_j) = 0$ and let $\lambda(D_j, \Psi_j)$ be the set of eigenvalues. If $\lambda(D_j, \Psi_j) \subseteq \mathbb{R}_{0-}$ set $I_j = \mathbb{R}_{0+}$, otherwise set $I_j = [0, \gamma_j)$ where γ_j is the smallest positive eigenvalue.

Step 5: Find the optimal step length t_j^*, $t > 0$, and the optimal scaling matrix D_{j+1} such that

$$t_j^* \in \text{argmin}\{f_j(t) : t \in I_j\}, \;\; D_{j+1} = D_j - t_j^*\Psi_j$$

Step 6: If $\|D_{j+1} - D_j\| \leq \epsilon_2$ stop and exit.

Step 7: Set $d_{j+1} = \overline{\text{vec}}(D_{j+1})$.

Step 8: Set $D_j \leftarrow D_{j+1}$, $d_j \leftarrow d_{j+1}$, $j \leftarrow j+1$ and go to step 1.

Remark 5.4.1. (1) It is stressed again that the steepest descent method implemented in the algorithm above guarantees convergence to a local minimum only. Since the objective function is not in general convex this may not correspond to the globally optimal solution.

(2) Restricting the optimisation in step 5 to the interval $[0, \gamma_j)$ ensures that the optimal solution of the scaling matrix remains always within the positive-definite cone so that f_j remains bounded.

(3) Ideally the optimal steplength at every iteration has to be selected as the global minimum of f_j in I_j, which is typically estimated numerically. If this is too computationally expensive, approximate methods can be used. In this programme implementation the optimal steplength is estimated by gridding the interval $[0, \eta\gamma_j)$ where η is a fixed parameter such that $0 < \eta < 1$ (typically, take $\eta = 0.95$).

The following examples illustrate the proposed algorithm:

Example 5.4.1. Consider two coprime polynomials

$$a_0(s) = s^2 + \alpha_1 s + \alpha_0, \;\; b_0(s) = s^2 + \epsilon_1 s + \epsilon_0$$

and the corresponding Sylvester resultant matrix:

$$S_0 = \begin{pmatrix} 1 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 \end{pmatrix},$$

in which $\alpha_1 = -1.3026$, $\alpha_0 = -0.4218$, $\beta_1 = -1.0026$ and $\beta_0 = -0.3218$. The singular value set of S_0 is $\sigma(S_0) = \{2.5323, 1.8778, 0.1667, 0.0140\}$ indicating a numerical rank of 2 (or 3, depending on the required tolerance), hence identifying the approximate GCD degree of the corresponding polynomials as two (or one, again depending on the tolerance level). In this example the exact analytical solution can be obtained for $\hat{\mu}_{\mathbf{\Delta},1}(M)$, where

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_2, \delta_0 I_2, \epsilon_1 I_2, \epsilon_0 I_2)\} \subseteq \mathbb{R}^{8\times 8}$$

and $M = -ZS_0^{-1}\Theta$, where Z, Θ are as specified in Theorem 5.3.1. Since the two perturbed (monic) polynomials

$$a(s) = s^2 + (\alpha_1 + \delta_1)s + (\alpha_0 + \delta_0), \;\; b(s) = s^2 + (\beta_1 + \epsilon_1)s + (\beta_0 + \epsilon_0)$$

have two common roots if and only if they are *identical*, then:

$$\hat{\mu}^{-1}_{\mathbf{\Delta},1} = \max\left\{\frac{|\alpha_1 - \beta_1|}{2}, \frac{|\alpha_0 - \beta_0|}{2}\right\}$$

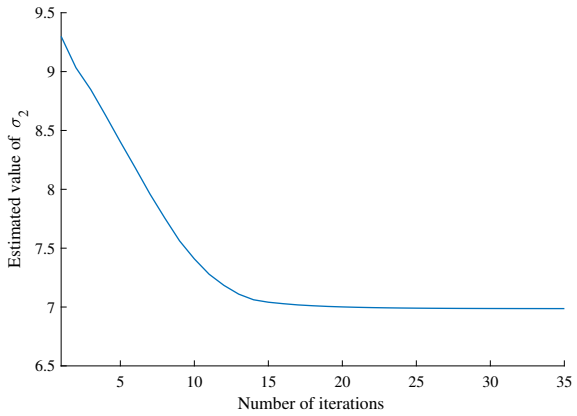

Fig. 5.2 Steepest-descent method for $\hat{\mu}_{\mathbf{\Delta},1}(M)$ upper bound (Example 5.4.1)

For the selected numerical values of the coefficients $\hat{\mu}^{-1}_{\mathbf{\Delta},1} = 0.15$. The upper bound of $\hat{\mu}_{\mathbf{\Delta},1}$ given in (5.38) was calculated via the steepest descent algorithm implemented in MATLAB. The local minimum was achieved at the 35-th iteration as illustrated in figure 5.2. The value of $\sigma_2(DMD^{-1})$ obtained at convergence results in the bound:

$$\hat{\mu}_{\mathbf{\Delta},1}(M) \leq 6.9867$$

This says that no structured perturbation $\Delta \in \mathbf{\Delta}$ with norm $\|\Delta\| \leq 6.9867^{-1} = 0.1431$ can give rise to a pair of perturbed polynomials with two common roots, which is consistent with the exact value of $\hat{\mu}^{-1}_{\mathbf{\Delta},1} = 0.15$ obtained above.

Next the proposed algorithm is applied to bound the distance of the Sylvester matrix to singularity (which corresponds to at least one common root for the pair of perturbed polynomials), given by

$$\hat{\mu}^{-1}_{\mathbf{\Delta},0}(M) = \mu^{-1}_{\mathbf{\Delta}}(M)$$

In this case the steepest-descent algorithm produced the bound:

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\gamma^*} \leq 119.1796$$

which is in fact the exact value of $\mu_{\mathbf{\Delta}}(M)$ (up to the 9-th decimal point) as calculated by MATLAB's μ-Control toolbox.

Example 5.4.2. Consider now a case of three polynomials,

$$\begin{cases} a(s) = s^3 - 6s^2 + 11s - 6 = (s-1)(s-2)(s-3) \\ b(s) = s^2 - 3s + 2.09 = (s-1.1)(s-1.9) \\ c(s) = s^2 - 2.9s + 1.68 = (s-0.8)(s-2.1) \end{cases}$$

that have an approximate common divisor of degree 2. As it is numerically hard to obtain an exact solution $\hat{\mu}_{\mathbf{\Delta},1}(M)$ can be evaluated iteratively as specified in Theorem 5.3.3. Such an approach does not guarantee the exact solution, but can be used as an approximation.

At first, with the Sylvester resultant

$$S_0 = \begin{pmatrix} 1 & \alpha_2 & \alpha_1 & \alpha_0 & 0 \\ 0 & 1 & \alpha_2 & \alpha_1 & \alpha_0 \\ 1 & \beta_1 & \beta_0 & 0 & 0 \\ 0 & 1 & \beta_1 & \beta_0 & 0 \\ 0 & 0 & 1 & \beta_1 & \beta_0 \\ 1 & \theta_1 & \theta_0 & 0 & 0 \\ 0 & 1 & \theta_1 & \theta_0 & 0 \\ 0 & 0 & 1 & \theta_1 & \theta_0 \end{pmatrix} = \begin{pmatrix} 1 & -6 & 11 & -6 & 0 \\ 0 & 1 & -6 & 11 & -6 \\ 1 & -3 & 2.09 & 0 & 0 \\ 0 & 1 & -3 & 2.09 & 0 \\ 0 & 0 & 1 & -3 & 2.09 \\ 1 & -2.9 & 1.68 & 0 & 0 \\ 0 & 1 & -2.9 & 1.68 & 0 \\ 0 & 0 & 1 & -2.9 & 1.68 \end{pmatrix}$$

an augmented matrix M is structured based on Theorem 5.3.1 proceeding with calculating the minimum norm perturbations, corresponding to a bound of $\mu^{(1)}_{\mathbf{\Delta}}(M) = 29.3311$. From the perturbed polynomials evaluated at the first iteration one can form a reduced Sylvester matrix, S_1, by factoring out the nearest common root from the derived polynomials. Then, the structure of M, based on the reduced Sylvester matrix is used for the computation of the structured singular values at the second iteration corresponding to a bound on $\mu^{(2)}_{\mathbf{\Delta}}(M) = 11.7145$.

Merging the results achieved an estimate of $\hat{\mu}_{\mathbf{\Delta},1}(M) = 4.8538$. This corresponds to the maximal absolute value of the perturbations in the coefficients of the original polynomials. The final generalised structured singular value estimate obtained as a result of this factorisation scheme is suboptimal in general. However, for the purpose of the analysis it can be compared with the upper bound obtained via the proposed steepest-descent algorithm which also gives an approximation of $\hat{\mu}_{\mathbf{\Delta},1}(M)$. In this case this is obtained at the 8-th iteration as

$$\hat{\mu}_{\mathbf{\Delta},1}(M) \leq 8.7760.$$

Example 5.4.3. Assume that polynomials from Example 5.4.2 are modified as follows

$$\begin{cases} a(s) = (s-1)(s-2)(s-3) \\ b(s) = (s-1.1+\epsilon)(s-1.9-\delta) \\ c(s) = (s-0.8-2\epsilon)(s-2.1+\delta) \end{cases}$$

where δ and ϵ are given perturbations, i.e. $(\delta, \epsilon) \in \{0.02, 0.04, 0.06, 0.08\}^2$. Applying the steepest-descent algorithm for all the combinations of δ and ϵ, the results of $\hat{\mu}_{\mathbf{\Delta},1}(M)$ upper bound are obtained as presented in Table 5.3. It can be observed that within the specified tolerance the smaller the gap between the common roots in the given polynomials, the greater is the value of the bound.

Parameters	$\delta_1 = 0.02$	$\delta_2 = 0.04$	$\delta_3 = 0.06$	$\delta_4 = 0.08$
$\epsilon_1 = 0.02$	11.0057	9.8397	8.8328	7.9497
$\epsilon_2 = 0.04$	17.1778	14.7157	12.6656	10.9659
$\epsilon_3 = 0.06$	33.2325	27.9176	22.1343	17.6500
$\epsilon_4 = 0.08$	34.0382	53.1205	67.4566	44.388

Table 5.3 $\hat{\mu}_{\mathbf{\Delta},1}(M)$ upper bound for the different values of δ and ϵ

5.5 Summary

In this Chapter two novel methods for calculating the distance of a set of coprime polynomials to the set of polynomials with a common root are presented. The problem is motivated by the distance to uncontrollability (unobservability) in algebraic control theory, which also has several important applications in Numerical Analysis, Robust Control, Linear Systems and other Engineering fields.

The approximate factorisation of the Sylvester matrix motivated the definition of an alternating least-squares projection algorithm that minimises the Frobenious norm of the error matrix of the factorisation. It is shown that a general nonlinear problem can be divided into two linear sub-problems and solved iteratively. Such an approach avoids significant complexity in numerical calculations of the "best" approximate GCD of a coprime set of polynomials. Moreover, the advantage of the method is demonstrated by simple numerical examples and compared with a non-linear steepest-descent algorithm as well as the standard nonlinear least squares Levenberg-Marquardt type of optimisation.

The structured singular value approach seeks to identify the minimum-magnitude perturbations in the coefficients of the polynomials so that the perturbed polynomials have a common root. It is demonstrated that the problem is equivalent to the calculation of a

structured singular value of a matrix which is also extensively studied in Robust Control. The method is generalised and applied to the calculation of the approximate GCD of an arbitrary set of polynomials. This leads naturally to the concept of the "generalised structured singular value" which involves the solution of a structured approximation problem with rank constraints. Although in this case an upper bound can be obtained via an optimal pair of positive-definite scaling matrices which commute with the uncertainty structure, the resulting optimisation problem is non-convex and convergence to the global optimum cannot be guaranteed. A steepest descent algorithm is proposed as a possible approach for tackling the problem which is shown to perform well for problems of small complexity. However further work is required to assess its numerical properties and its applicability to problems of higher complexity.

Significance of the structured singular value approach can be demonstrated on a wide variety of distance problems. Based on the obtained results the next Chapter introduces special case of distance problems that appear in Robust Control Theory, namely implicit systems.

Chapter 6

Implicit Case as a Special Case for the Distance Problems

It has been demonstrated that distance problems of matrix pencils have wide applications in Control Theory. Moreover, as mentioned in [19] and [66], general perturbation structures, i.e. $(A + E, B + F)$, can be constrained to a special structure, for example, perturbations may appear only in the A matrix as $(A + E, B)$. The structured singular approach, developed in Chapter 5 might be beneficial for such a special case, where the B matrix acts as as an additional constraint for the system description. Moreover, an overconstraint system description is useful in the analysis of implicit systems, where an extension of the μ-value methodology can provide further insights to the notion of Robust stability.

The input-output framework has dominated the study of the Systems and Control Theory paradigm. Building on early work of Rosenberg [104], the behavioural framework was introduced in [155] as an alternative. In this framework, the system is described by the collection of its trajectories rather than input-output relations. Implicit systems play an important role in this context. By incorporating constraints in the model description, implicit systems provide additional versatility relative to standard input-output models. Areas where implicit systems have proved useful include the analysis and synthesis of complex Interconnected Systems, Systems Identification and Robust Control [156], [141].

In the area of Robust Control, implicit analysis is useful in the formulation of uncertainty models of various types, e.g. LTI, time-varying, polytopic, delay, sector-nonlinearity, etc. All these descriptions can be unified via Integral Quadratic Constraints (IQC) resulting in constrained system descriptions [157]. The main idea of IQC's is to replace uncertain operators which are difficult to model by integral quadratic constraints on the input-output pair which applies for all uncertain conditions. This results in robust stability and performance conditions which can be checked via semi-definite programming.

Considerable work has been done recently in formulating and solving distance problems involving uncertain systems [158]. Some examples of constrained H_∞ and H_2 problems are presented in [141], [156], [159] providing systematic methods for the control structure selection. A standard frequency-domain description of implicit LTI uncertain systems can be expressed in the form:

$$\begin{pmatrix} I + \Delta M \\ N \end{pmatrix} z = 0, \;\; M \in \mathbb{C}^{n\times n}, N \in \mathbb{C}^{p\times n}, \Delta \in \mathbf{\Delta} \subset \mathbb{C}^{n\times n} \tag{6.1}$$

where $\mathbf{\Delta}$ is a structured uncertainty set. Stability is studied with respect to the nullity of the matrix in (6.1) for all Δ in the bounded set

$$\mathbf{B}_{\mathbf{\Delta}} = \{\Delta \in \mathbf{\Delta} : \|\Delta\| \leq 1\} \tag{6.2}$$

When $\mathbf{\Delta}$ has a diagonal structure this is a generalisation of the structured singular value (μ) problem.

The structured singular value (μ) is a fundamental analysis and synthesis tool of Robust Control theory with several applications in model validation and the characterisation of robust stability and performance of dynamic systems, [142], [141], [144], [143], [160], [161], [157]. The structured singular value of a matrix $M \in \mathbb{R}^{n\times n}$ (or $M \in \mathbb{C}^{n\times n}$) is defined as:

$$\mu_{\mathbf{\Delta}}(M) = \frac{1}{\min\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \det(I_n + M\Delta) = 0\}} \tag{6.3}$$

where $\mathbf{\Delta}$ is a set of structured perturbations. Although more general structures can be defined (combining real/complex and scalar/matrix perturbations), here $\mathbf{\Delta}$ is restricted to the set:

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}) : \delta_i \in \mathbb{R}, i = 1, 2, \ldots, s\}$$

If for any $\Delta \in \mathbf{\Delta}$ $\det(I_n + M\Delta) \neq 0$, then $\mu_{\mathbf{\Delta}}(M) = 0$.

In the Linear Time Invariant case robust stability properties of an implicit system defined by the pair (M, N) can be analysed using a Scaled Linear Fractional Transformation (SLFT). This leads to a generalised μ-problem involving additional constraints in the form of equation (6.1). Since exact calculations are numerically hard, it is more appropriate to develop optimisation procedures for lower and upper bounds. Calculation of an upper bound can be achieved via a convex relaxation technique using Linear Matrix Inequalities, similar to the non-negative scaling approach used for the standard problem [162]. Unfortunately, however, in contrast to the standard version of the problem, there is not sufficient numerical experience to assess the effectiveness of this method, i.e. the proximity of the upper bound to the exact generalised μ value.

This Chapter focuses on a special class of generalised μ problems involving repeated real scalar parameters. It is shown that in this case the problem is equivalent to a standard μ problem which is well studied and for which several numerical algorithms are applicable. The proposed results are based on a matrix dilation technique and the redefinition of the uncertainty structure of the transformed problem.

6.1 Analysis of Implicit systems

In the behavioural paradigm, dynamic systems are specified by the family of time trajectories they generate with no prior need to establish a mapping between inputs and outputs [163], [164]. The notion of implicit systems allows for a broader analysis of robust control problems within this framework. System uncertainties can be formulated in different forms, i.e. constant or dynamic, where the latter may be linear time invariant, linear time variant or of the nonlinear type. The basic model of an implicit system is $G\omega = 0$, where G is a linear operator and ω denotes a vector signal. Robust l_2 stability theory is characterised by the parametrisation of the linear operator $G(\Delta)$, where Δ is a structured set of uncertainties, assumed for simplicity to be norm bounded as defined in equation (6.2).

Robust stability of an implicit system can be evaluated via a Linear Fractional Transformation (LFT) of a nominal system and an uncertainty structure [157]. Its canonical form is given by equation (6.1) in which $M \in \mathbb{C}^{n\times n}$, $N \in \mathbb{C}^{m\times n}$ are given matrices that characterise the system and $\mathbf{\Delta}$ is a set of perturbations which in general combines different types of structured uncertainty. This canonical representation is derived from a state-space description in discrete time [163], [157] that leads to stability analysis under performance constraints. In the LTI case (6.1) typically represents the uncertain implicit system at a single frequency. The implicit system (M, N) is said to be robustly stable if:

$$\text{null} \begin{pmatrix} I + \Delta M \\ N \end{pmatrix} = 0, \ \ \forall \Delta \in \mathbf{B}_{\mathbf{\Delta}} \tag{6.4}$$

where $\mathbf{\Delta}$ is a set of norm-bounded structured matrices. Equivalently, (M, N) is robustly stable if the pencil in equation (6.4) is left invertible for all $\Delta \in \mathbf{B}_{\mathbf{\Delta}}$.

The generalised structured singular value for implicit systems is defined as follows (see [159], [158], [157] and [163] for more details). Note that the set $\mathbf{\Delta}$ in the definition is restricted to the set of real repeated scalar parameters, which corresponds to the structure considered in this work. Although more general diagonal structures can be defined (combining real/complex and scalar/matrix perturbations) these will not be considered here.

Definition 6.1.1 ([157])**.** Let M and N be real (or complex) matrices of dimension $n \times n$ and $m \times n$, respectively, with $m \leq n$ Then, the generalised structured singular value $\hat{\mu}_{\Delta}(M, N)$

of the implicit system is defined as:

$$\hat{\mu}_{\Delta}(M,N) = \left(\min\left\{\|\Delta\| : \Delta \in \mathbf{\Delta}, \text{null}\begin{pmatrix} I + \Delta M \\ N \end{pmatrix} \neq 0\right\}\right)^{-1} \tag{6.5}$$

where

$$\mathbf{\Delta} = \{\text{diag}(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \ldots, \delta_s I_{r_s}) : \delta_i \in \mathbb{R}, i = 1, 2, \ldots, , s\} \tag{6.6}$$

in which $\sum_{i=1}^{s} r_i = n$. If for all $\Delta \in \mathbf{\Delta}$ the nullity of the pencil is zero, then $\hat{\mu}_{\Delta}(M,N) = 0$.

The computational difficulties of the standard μ-problem are amplified in the general case. Thus, it is more promising to rely on the lower and upper bound calculations in this case as well. The lower bound can be obtained by maximising the modulus of a generalised eigenvalue of a pair of matrices over a bounded structured set. This procedure inherits the problems of algorithms corresponding to the standard μ-problem by exhibiting multiple local maxima [142]. Fundamental work on upper-bound estimation of the generalised μ-value has been reported in [163], [157], [158] and [159]. Reference [159] uses the right annihilator of N, $N_{\perp}$, to transform the nullity constraint:

$$\text{null}\begin{pmatrix} I + \Delta M \\ N \end{pmatrix} \neq 0 \Longleftrightarrow \text{null}(N_{\perp} + \Delta M N_{\perp}) \neq 0 \tag{6.7}$$

and then applies convex relaxations to calculate the upper bound of $\hat{\mu}(M,N)$ based on a scaling technique. Consider a set $\mathbf{X}$ of positive scaling matrices X that commute with the structure of $\mathbf{\Delta}$, i.e.

$$X = \text{diag}(X_1, X_2, \ldots, X_s), X_i = X_i' > 0, X_i \in \mathbb{R}^{r_i \times r_i} \tag{6.8}$$

Then an upper bound of $\mu_{\Delta}(M,N)$ is obtained as:

$$\hat{\mu}_{\mathbf{\Delta}}(M,N) = \inf\{\beta > 0 : \exists X \in \mathbf{X} : M'XM - \beta^2 X - N'N < 0\} \tag{6.9}$$

Unfortunately the class of μ-simple structures for which this convex bound coincides with the actual value of $\hat{\mu}_{\mathbf{\Delta}}(M,N)$ is rather restricted as shown in the following result:

Theorem 6.1.1 ([159])**.** In the implicit case the following are simple-μ structures:

(i) $\{\Delta = \delta I : \delta \in \mathbb{C}\}$.

(ii) Full real blocks: $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_F) : \Delta_i \in \mathbb{R}^{n_i \times n_i}, F \leq 2$ for M, N real.

(iii) Full complex blocks: $\Delta = \text{diag}(\Delta_1, \ldots, \Delta_F) : \Delta_i \in \mathbb{C}^{n_i \times n_i}, F \leq 3$

The following section develops an alternative methodology for calculating $\hat{\mu}_{\mathbf{\Delta}}(M,N)$. This is based on a matrix dilation technique and the transformation of the uncertainty structure $\mathbf{\Delta}$. It is shown that the transformed problem is equivalent to a standard μ-problem. This is potentially significant since, in contrast to $\hat{\mu}_{\mathbf{\Delta}}(M,N)$ the computational load with μ calculations is extensive.

6.2 Generalised μ-value of implicit systems

Let $M \in \mathbb{R}^{n\times n}$ and $N \in \mathbb{R}^{m\times n}$ with $m \leq n$. The $\hat{\mu}_{\mathbf{\Delta}}(M,N)$ problem defined in equation (6.5) is considered here. The uncertainty structure $\mathbf{\Delta}$ throughout the rest of the Chapter is assumed to be as defined in equation (6.6).

The following Lemma is standard but is included here for ease of reference.

Lemma 6.2.1. Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n\times m}$ with $n \geq m$. Then $\text{null}(I_n+BA) = \text{null}(I_m+AB)$.

Proof. First note that

$$\begin{pmatrix} I_n & B \\ 0_{m,n} & I_m \end{pmatrix}\begin{pmatrix} 0_{n,n} & 0_{n,m} \\ A & AB \end{pmatrix}\begin{pmatrix} I_n & -B \\ 0_{m,n} & I_m \end{pmatrix} = \begin{pmatrix} BA & 0_{n,m} \\ A & 0_{m,m} \end{pmatrix}$$

Thus the two matrices:

$$\begin{pmatrix} 0_{n,n} & 0_{n,m} \\ A & AB \end{pmatrix} \text{ and } \begin{pmatrix} BA & 0_{n,m} \\ A & 0_{m,m} \end{pmatrix}$$

are similar and hence:

$$\lambda^n \det(\lambda I_m - AB) = \lambda^m \det(I_n - BA) \Leftrightarrow \lambda^{n-m}\phi_{AB}(\lambda) = \phi_{BA}(\lambda)$$

and hence

$$\phi_{I_m+AB}(\lambda) = \lambda^q \prod_{i\in Z}(\lambda - \lambda_i(AB) - 1),\ Z = \{i : \lambda_i(AB) \neq -1\}$$

and

$$\phi_{I_n+BA}(\lambda) = \lambda^q(\lambda-1)^{n-m}\prod_{i\in Z}(\lambda - \lambda_i(AB) - 1)$$

Hence $\text{null}(I_m + AB) = \text{null}(I_n + BA) = q$, the algebraic multiplicity of the eigenvalue $\lambda = -1$ of AB (or BA). □

Theorem 6.2.1. In previously defined notation,

$$\hat{\mu}_{\mathbf{\Delta}}(N,M) = \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}), \tag{6.10}$$

where

$$\tilde{M} = P' \begin{pmatrix} (I + N'N)^{-1}M' & N'N(I + N'N)^{-1} \\ -M(I + N'N)^{-1}M' & M(I + N'N)^{-1} \end{pmatrix} P \tag{6.11}$$

in which:

$$\mathbf{\Delta} = \{\text{diag}(\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}) \ : \ \delta_i \in \mathbb{R}, \ i = 1, 2, \ldots, s\} \subseteq \mathbb{R}^{n \times n} \tag{6.12}$$

$$\tilde{\mathbf{\Delta}} = \{\text{diag}(\delta_1 I_{2r_1}, \ldots, \delta_s I_{2r_s}) \ : \ \delta_i \in \mathbb{R}, \ i = 1, 2, \ldots, s\} \subseteq \mathbb{R}^{2n \times 2n} \tag{6.13}$$

and P is a permutation matrix such that $\text{diag}(\mathbf{\Delta}, \mathbf{\Delta}) = P\tilde{\mathbf{\Delta}}P'$.

Proof. From Lemma 5.3.1 (section 5.3) it follows that

$$\nu = \text{null} \begin{pmatrix} I + \Delta M \\ N \end{pmatrix} = \text{null} \begin{pmatrix} I_n & 0 & I_n + \Delta M \\ 0 & I_m & N \\ I_n + M'\Delta & N' & 0 \end{pmatrix}$$

which is equivalent to:

$$\nu = \text{null} \left\{ \underbrace{\begin{pmatrix} I_n & 0 & I_n \\ 0 & I_m & N \\ I_n & N' & 0 \end{pmatrix}}_{\Phi} + \begin{pmatrix} 0 & I_n \\ 0 & 0 \\ M' & 0 \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & M \end{pmatrix} \right\} \tag{6.14}$$

Let $\Delta \in \mathbf{\Delta}$ and introduce a permutation matrix P such that $\text{diag}(\Delta, \Delta) = P\text{diag}(\tilde{\Delta})P'$, $\tilde{\Delta} \in \tilde{\mathbf{\Delta}}$. (This is just a "reshufling" of $\text{diag}(\Delta, \Delta)$). Let Φ be as defined in equation (6.14). From Lemma 5.3.1, Φ is non-singular. Then, using Lemma 6.2.1,

$$\nu = \text{null} \left\{ I_{2n} + P' \begin{pmatrix} I_n & 0 & 0 \\ 0 & 0 & M \end{pmatrix} \Phi^{-1} \begin{pmatrix} 0 & I_n \\ 0 & 0 \\ M' & 0 \end{pmatrix} P\tilde{\Delta} \right\}$$

On noting (after some algebra) that

$$\Phi^{-1} = \begin{pmatrix} N'N(I + N'N)^{-1} & -(I + N'N)^{-1}N' & (I + N'N)^{-1} \\ -N(I + N'N)^{-1} & I - N(I + N'N)^{-1}N' & N(I + N'N)^{-1} \\ (I + N'N)^{-1} & (I + N'N)^{-1}N' & -(I + N'N)^{-1} \end{pmatrix}$$

it is concluded that:

$$\nu = \text{null} \begin{pmatrix} I_n + \Delta M \\ N \end{pmatrix} = \text{null}(I_{2n} + \tilde{M}\tilde{\Delta})$$

where

$$\tilde{M} = P' \begin{pmatrix} (I + N'N)^{-1}M' & N'N(I + N'N)^{-1} \\ -M(I + N'N)^{-1}M' & M(I + N'N)^{-1} \end{pmatrix} P$$

and $\tilde{\Delta} \in \tilde{\mathbf{\Delta}}$. Furthermore, $\Delta \in \mathbf{\Delta} \iff \tilde{\Delta} \in \tilde{\mathbf{\Delta}}$ and $\|\Delta\| = \|\tilde{\Delta}\|$. Since

$$\left\{ \text{null} \begin{pmatrix} I_n + \Delta M \\ N \end{pmatrix} \neq 0, \forall \Delta \in \mathbf{\Delta} \right\} \iff \left\{ \det(I_{2n} + \tilde{\Delta}\tilde{M}) \neq 0, \forall \tilde{\Delta} \in \tilde{\mathbf{\Delta}} \right\}$$

it is concluded that:

$$\hat{\mu}_{\mathbf{\Delta}}(N, M) = 0 \iff \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}) = 0.$$

Similarly, if $\hat{\mu}_{\Delta}(N, M) \neq 0$ then

$$\begin{aligned} \hat{\mu}^{-1}_{\mathbf{\Delta}}(N, M) &= \min \left\{ \|\Delta\| : \ \Delta \in \mathbf{\Delta}, \ \text{null} \begin{pmatrix} I_n + \Delta M \\ N \end{pmatrix} \neq 0 \right\} \\ &= \min \left\{ \|\tilde{\Delta}\| : \ \tilde{\Delta} \in \tilde{\mathbf{\Delta}}, \ \det(I_{2n} + \tilde{\Delta}\tilde{M}) = 0 \right\} \\ &= \mu^{-1}_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \end{aligned}$$

and hence $\hat{\mu}_{\mathbf{\Delta}}(N, M) = \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ as required. □

Theorem 6.2.1 shows that the standard μ-problem and the generalised-μ problem are essentially equivalent. In particular, all theoretical results and numerical algorithms developed for the solution of the former can also be applied directly to the latter. This is significant since μ is a well studied problem, with an extensive range of theoretical and numerical work devoted to its calculation.

The simplicity of the method can be illustrated by means of numerical examples.

Example 6.2.1. Consider the matrix pair:

$$M = \begin{pmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{pmatrix}, \ N = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0.5 & 2 \end{pmatrix}$$

The singular values of $[M' \ N']$ are $\sigma(M, N) = \{11.7798, \ 3.9297, \ 0.2075\}$. Let Δ be a set of diagonal scalar perturbations:

$$\mathbf{\Delta} = \{\text{diag}(\delta_1, \delta_2, \delta_3) : \delta_i \in \mathbb{R}, i = 1, 2, 3\}$$

Let $\tilde{M}$ be defined as in equation (6.11). Lower and upper bounds of the structured singular value of $\tilde{M}$ were obtained using MATLAB's μ-Toolbox [145] as:

$$3.000000006862770 \leq \mu_{\tilde{\Delta}}(\tilde{M}) \leq 3.000000009072207$$

which indicates a value $\mu = 3$ (within an accuracy of 10^{-8}). This indicates a distance of $\gamma^* = \frac{1}{3}$ from the set of reduced-rank matrices. Indeed it can be easily verified if $\Delta = \frac{1}{3}I_3$, say, the rank of $[I_3 - M'\Delta \mid N']$ is equal to 2.

Example 6.2.2. Consider now another example, where M and N are defined as:

$$M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad N = (1 \quad -1)$$

Let $\mathbf{\Delta}$ be a set of diagonal scalar perturbations:

$$\mathbf{\Delta} = \{\mathrm{diag}(\delta_1, \delta_2) : \delta_i \in \mathbb{R}, i = 1, 2\}$$

Then for $\Delta \in \mathbf{\Delta}$:

$$\begin{pmatrix} I_2 + \Delta M \\ N \end{pmatrix} = \begin{pmatrix} 1+\delta_1 & \delta_1 \\ 0 & 1+\delta_2 \\ 1 & -1 \end{pmatrix}$$

loses rank if and only if there exists $(x, y)' \in \mathbb{R}^2, (x, y) \neq 0$ such that

$$\begin{pmatrix} 1+\delta_1 & \delta_1 \\ 0 & 1+\delta_2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$$

which implies that $x = y \neq 0, \delta_1 = -\frac{1}{2}$ and $\delta_2 = -1$. Thus $\hat{\mu}_{\mathbf{\Delta}}(N, M) = 1$. Next write

$$\left(\begin{array}{cc|c} I_2 & 0_{2,1} & I_2 + \Delta M \\ 0_{1,2} & 1 & N \\ \hline I_2 + M'\Delta & N' & 0 \end{array}\right) = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1+\delta_1 & \delta_1 \\ 0 & 1 & 0 & 0 & 1+\delta_2 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1+\delta_1 & 0 & 1 & 0 & 0 \\ \delta_1 & 1+\delta_2 & -1 & 0 & 0 \end{array}\right)$$

which can be decomposed as

$$\left(\begin{array}{cc|c} I_2 & 0_{2,1} & I_2 + \Delta M \\ 0_{1,2} & 1 & N \\ \hline I_2 + M'\Delta & N' & 0 \end{array}\right) = \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ \hline 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{array}\right) + \left(\begin{array}{ccc|cc} 0 & 0 & 0 & \delta_1 & \delta_1 \\ 0 & 0 & 0 & 0 & \delta_2 \\ 0 & 0 & 0 & 0 & 0 \\ \hline \delta_1 & 0 & 0 & 0 & 0 \\ \delta_1 & \delta_2 & 0 & 0 & 0 \end{array}\right)$$

Then Theorem 6.2.1 says that $\hat{\mu}_{\mathbf{\Delta}}(N, M) = \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ where

$$\tilde{M} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\tilde{\mathbf{\Delta}} = \{\text{diag}(\delta_1, \delta_1, \delta_2, \delta_2); \delta_1, \delta_2 \in \mathbb{R}\}$$

The singular values of $\tilde{M}$ are:

$$\{3.29623546,\ \ 0.78314066,\ \ 0.69807278,\ \ 0.18497798\}$$

Setting $\Delta = \text{diag}(-\frac{1}{2}, -\frac{1}{2}, -1, -1)$ verifies that $\det(I_4 + \tilde{M}\Delta) = 0$ so that $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \geq 1$. Using Matlab's *mussv* function gives the convex upper bound of $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \leq 1.000599$. Thus,

$$1 \leq \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \leq 1.000599 \tag{6.15}$$

Using the symbolic Matlab toolbox the solutions of equation $\det(I_4 + \tilde{M}\Delta) = 0$ are obtained as

$$\delta_2 = -1 \pm i\frac{2\delta_1 + 1}{\sqrt{\delta_1^2 + 2\delta_1 + 2}}$$

and hence the only real solutions are $\delta_1 = -\frac{1}{2}$ and $\delta_2 = -1$ as expected. Thus, $\mu_{\tilde{\mathbf{\Delta}}}(\tilde{M})$ coincides with the lower bound given in equation (6.15) and is equal to $\mu_{\mathbf{\Delta}}(N, M)$.

Remark 6.2.1. The proposed analysis of overconstrained systems, motivated by the implicit uncertainty structure, can be applied to general controllability (observability) pencils.

6.3 Summary

Motivated by the structured singular value approach, developed in Chapter 5, an extension of the computational methodology that is defined as a special case of the distance problems that also appeared in the analysis of implicit systems is presented. Since the general controllability (observability) problem is specified as the distance of $[sI - A,\ \ B]$ to rank deficiency, then it can be also reviewed as an overconstrained problem, where perturbations in the pair (A, B) are restricted as $(A + \Delta, B)$. However, such a description requires optimisation over all $s \in \mathbb{C}$, which is a difficult task and will be studied in future work. However, the advantage of the

proposed generalised approach has found application in the analysis of the implicit systems presented in this Chapter.

Thus, an alternative framework for calculating the structured singular value of implicit systems in the case of real scalar repeated perturbations is proposed. By dilating the problem it is shown that it is equivalent to a standard structured singular value calculation for which extensive numerical algorithms are available. The proposed approach applies to implicit systems described by both real and complex data; analysis of the latter is presented in Appendix B. Extensions to more general classes of uncertainty are currently under investigation.

A wide variety of control problems can be studied using the proposed strategy. Structured constrained μ analysis seems to show promising results and continues to be developed. Computation of the implicit uncertainties with various block structures of Δ is also of particular interest. Moreover, robust performance criteria defined with respect to the structured bounded perturbations along with the computational framework can be performed as an optimisation procedure and might establish a link with the generalised methodology of the μ-bounds with imposed constraints.

Chapter 7

Conclusion

7.1 Conclusions and future work

The current work develops and presents an alternative framework for evaluating the distance to uncontrollability, unobservability and specifies methodologies for the solution of distance problems that appear in the area of Control Design and Robust Stability analysis.

The contributions of the thesis can be summarised as follows:

(i) Chapter 3 was devoted to the problem of modal measurement of controllability, observability and studies the problem of actuator/sensor location based on angles between the eigenvectors of the state matrix and the input/output vectors (columns of the input matrix or rows of the output matrix, respectively). The problem of modal controllability (observability) appears in a variety of Engineering disciplines, however the already existing approaches found in the literature do not consider evaluation of the optimal structure of sensors and actuators in order to improve systems behaviour and increase controllability and observability properties. Thus, the framework developed in the present work demonstrates that the solution for the best structure selection can be defined as the problem optimising the direction of the input, output vectors of nearly uncontrollable, unobservable modes respectively can lead to a better IO structure. It has been demonstrated that for a simple SISO system the solution can be found with respect to the geometric interpretation, while for the higher order dimensional systems, as well as the MIMO systems it is appropriate to evaluate a linear programming problem that can be used in order to obtain the desired systems measures.

(ii) The problem of invariant measures of uncontrollability and unobservability has been highlighted in Chapter 4. As the sensitivity of systems properties, studied in the work, changes with respect to the feedback transformations it is required to evaluate an alternative measure that would remain invariant under the state feedback and output injection. Hence, Chapter 4 suggests to study the invariant distance to uncontrollability,

unobservability in the projective space using the notions of Exterior Algebra that leads to a novel alternative approach of invariant systems properties. It is demonstrated that the problem falls into two categories, the boundary case (LIDP), where the distance is exact and Quadratic Plücker Relations are always satisfied, and the general case (GIDP), when the quadratic conditions have to be imposed as constraints in order to ensure decomposability of the resulting multivectors. If the resulting vector is not free, it is necessary to implement an additional optimisation procedure in order to find the "approximate" decomposable multivector. For this problem a wide range of techniques and procedures have been developed in the literature [117], [116], [118], [45]. However, existing methodologies are beneficial for the numerical polynomial multivectors, while in the case of multivectors derived from the Restricted Matrix Pencils the problem becomes more complicated. Minimisation of the distance between two multilinear polynomials with imposed constraints is a NP-hard problem and require additional research that will be fully addressed in future work. Moreover, there is still an open research problem of evaluating the "best" approximate decomposable polynomial, where the gap is minimised. This is definitely a complex problem and also provides perspectives for future research.

Nevertheless, the main result of Chapter 4, stated in the corollaries, evaluates the invariant distance measure as the distance of the Sylvester matrix $S_{\mathcal{P}}$, structured from the corresponding generator of polynomials $\underline{k} \in \mathbb{R}^p, \underline{k} \in \mathbb{R}^m$, from rank deficiency. Presented framework of the invariant distance measure is a novel approach in Control Theory. Such a statement gives rise to a new problem of calculating the best "approximate" GCD of a set of polynomials has been addressed in details in Chapter 5.

(iii) Computational procedures of greatest common divisors and, in particular, "approximate" common roots is a complicated mathematical problem due to the non-generic nature and it founds applications in a variety of Engineering disciplines. The problem of "approximate" GCD was extensively studied in Chapter 5 resulting into two independent solutions, in particular the "alternating" projection algorithm and the structured singular value approach. The former method avoids significant complexity in the "approximate" GCD calculations by separating the nonlinear problem into two linear sub-problems that are solved iteratively. It has been demonstrated that the method seems to be robust with smooth convergence to the optimal solution in comparison to the standard nonlinear least squares approaches. In addition, "alternating" projection algorithm in principle can be extended to a robust least squares methodology. The future work leads to applying proposed algorithm for the minimisation problem, where residual error matrix is defined as $E = S_{\mathcal{P}}\Phi_\lambda - [0_k \mid S_{\mathcal{P}^*}]$ and Φ_λ is a Toeplitz matrix representation, corresponding to the inverse of $\hat{\Phi}_\lambda$.

The latter method, presented in Chapter 5 for the GCD problem based on the μ-value is a powerful tool in the Robust Control area. The method is generalised and applied to the calculation of the "approximate" GCD of an arbitrary set of polynomials that leads to the notion of "generalised structured singular value". It has been highlighted that for the arbitrary number of polynomials some technical difficulties arise that require additional matrix transformations in order to bring the problem to the standard μ-value form that has already established computational frameworks. The simplicity of the structured singular value approach provided motivation to explore potential applications of the proposed solution. Thus, a constraint structured uncertainty problem was introduced and analysed.

(iv) The approach in Chapter 6 extended the proposed structured singular value solution to the special case of over-constrained systems. Such a description is used for robust stability analysis of implicit systems. The present work suggests to carry out matrix reshuffling prior the computation that develops solution of the generalised-μ problem that is equivalent to the standard structured singular value one.

Moreover, the proposed generalised μ-value computational procedure has potential application for the general problem of uncontrollability that can be stated in the general form as

$$d(A,B) = \min\{\|\Delta A, \Delta B\| : \ \ (A+\Delta A, B+\Delta B) \text{ is uncontrollable}\} \tag{7.1}$$

or equivalently

$$d(A,B) = \min_{s\in\mathbb{C}} \sigma_n([A - sI, \ \ B])$$

It gives rise to an alternative approach of studying the invariant distance of uncontrollability from the perspectives of a structured uncertainty set $\mathbf{\Delta}$, such that $d(A,B) = \min\{\|\Delta\| : \ \ (A+\Delta A, B) \text{ is uncontrollable}\}$, details of which will be presented in future work.

(v) Some problems that lead to μ-value calculations can be NP-hard. The possibility of exploring structured distance to uncontrollability, proposed by [66] is of the particular interest. Traditionally, the norm of uncertainties in the entries of A and B denotes the measure of how much the system is perturbed before it becomes uncontrollable. In the problem analysed by Karow the distance is evaluated with respect to a structured perturbation of the form

$$d_{\Delta}^{F,G}(A,B) = \inf\{\|\Delta\| : (A+E, B+F) \text{ is uncontrollable}, \ \ [E,F] = F\Delta G, \Delta \in \mathbf{\Delta}\} \tag{7.2}$$

However, the notion of the structured distance with respect to $s \in \mathbb{C}$ is computationally difficult. This analysis opens new research directions related to the solution of the structured distance problem, or at least the estimation of tight upper bounds.

The theoretical results as well as computational procedures developed in the thesis provide a structured framework for evaluating novel measures to uncontrollability, unobservability and can be used as pre-requirements for better control structure selection. It can be observed that for an arbitrary system the problem of finding invariant measures of system properties can be narrowed down to the analysis of the lower bound, where the constraints of the Grassmann variety have to be satisfied. Such observations form a basis for future work and possible ways of finding a tighter bound of the distance to uncontrollability and unobservability for the design problem.

Finally, the proposed structured singular value approach for distance problems is not a "typical" solution and can be used as a creative mathematical approach for a variety of open Robust Control, Systems Design and other mathematical problems.

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Appendix A

Mixed μ problem

A.1 Real μ case

In robust control and stability analysis norm-bounded uncertainties are more naturally real than complex. Thus the importance of real structured singular value arises.

The mixed μ problem follows the similar approach considering complex matrix and purely real perturbations. In order to introduce it let $M \in \mathbb{C}^{n\times m},\ n \geq m$ be complex where $M = X + iY$, where $X, Y \in \mathbb{R}^{n\times m}$ are real and complex parts respectively. The real μ case, say $\mu_{\mathbb{R}}(M)$, is defined as

$$\mu_{\mathbb{R}}(M) = \inf\{\|\Delta\|_{\mathbb{R}} : \Delta \in \mathbf{\Delta}, \det(I - \Delta M) = 0\}^{-1} \tag{A.1}$$

It is possible to separate complex and real parts of M, i.e. $M^{\mathbb{R}} = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}$, leading to the alternative interpretation of the μ-value [142] as follows:

$$\det(I - \Delta M) = 0 \Leftrightarrow \det(I - \Delta M)^{\mathbb{R}} = \det\left(I_{2n} - \begin{bmatrix} \Delta & 0 \\ 0 & \Delta \end{bmatrix}\begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}\right) = 0. \tag{A.2}$$

where $\mathbf{\Delta}$ is assumed to be a block diagonal repeated real scalar perturbations $\|\mathrm{diag}(\Delta, \Delta)\|_{\mathbf{\Delta}} = \|\Delta\|$.

Definition A.1.1. For $A \in \mathbb{R}^{m\times n}$ consider $\ker(A) = \{x \in \mathbb{R}^n,\ A\underline{x} = 0\}$, then $\mathrm{null}(A) = \dim\{\ker(A)\}$ having $\mathrm{rank}(A) + \mathrm{null}(A) = n$ according to the rank-nullity theorem .

Lemma A.1.1. Let $M = X + iY \in \mathbb{C}^{n\times m}, X \in \mathbb{R}^{n\times m}, Y \in \mathbb{R}^{n\times m}$ and $n \geq m$. Then $\mathrm{rank}(M) < m$ if and only if

$$\mathrm{rank}\left\{\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\right\} < 2m \tag{A.3}$$

Proof. Let $\text{rank}(M) < m$, then $\exists\ \underline{\alpha} + i\underline{\beta} \in \mathbb{C}^m,\ \ \underline{\alpha} + i\underline{\beta} \neq \underline{0},\ \ \alpha \in \mathbb{R}^m,\ \ \beta \in \mathbb{R}^m$, such that

$$(X + iY)(\underline{\alpha} + i\underline{\beta}) = 0 \iff \left.\begin{array}{c} X\underline{\alpha} - Y\underline{\beta} = 0 \\ Y\underline{\alpha} + X\underline{\beta} = 0 \end{array}\right\} \iff$$

$$\iff \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \begin{pmatrix} \underline{\alpha} \\ \underline{\beta} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and result follows since $\underline{\alpha} + i\underline{\beta} \neq 0$ if and only if $(\underline{\alpha}', \underline{\beta}') \neq \underline{0}$. □

Considering Lemma A.1.1 and the augmented structure of a given matrix $M \in \mathbb{C}^{n\times m}$ some useful properties are highlighted below.

Lemma A.1.2. Let $A \in \mathbb{R}^{n\times m}, B \in \mathbb{R}^{m\times n}$. Then

$$\text{null}(I_n + AB) = \text{null}(I_m + BA). \tag{A.4}$$

Proof. Since the non-zero eigenvalues of (AB) and BA are identical $\text{null}(I_n + AB) = \text{null}(I_m + BA)$, the number of eigenvalues of AB (or equivalently BA) is equal to -1. □

Let $M \in \mathbb{C}^{n\times m},\ \ n \geq m,\ \ M = X + iY,\ \ X, Y \in \mathbb{R}^{n\times m}$ and say that there exists such a set

$$\Delta = \{\Delta = \text{diag}(\delta_1 I_{r_1}, \ldots, \delta_s I_{r_s}\} \tag{A.5}$$

where $\sum_{i=1}^{s} r_i = n$. Then Problem 5.3.1 (section 5.3) is reformulated as a real structured singular value problem that seeks to find such a Δ that

$$\mu_{\Delta}^{\mathbb{R}}(M) = \inf\{\|\Delta\| : \Delta \in \mathbf{\Delta},\ \ \det(I - \Delta M) = 0\}^{-1} \tag{A.6}$$

unless $\det(I - \Delta M) \neq 0,\ \ \forall \Delta \in \mathbf{\Delta}$ in which case $\mu_{\Delta}^{\mathbb{R}}(M) = 0$.

Lemma A.1.3. Let $M = X + iY \in \mathbb{C}^{n\times m}, X, Y \in \mathbb{R}^{n\times m}$ and $n \geq m$. Then $\det(I - \Delta M) = 0$ if and only if

$$\det\left\{I_{2n} - \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\right\} = 0. \tag{A.7}$$

Proof. Assume that $\det(I - \Delta M) = 0$. Then $\forall \underline{\xi} = \underline{\alpha} + i\underline{\beta},\ \alpha, \beta \in \mathbb{R}^m,\ \underline{\xi} \neq \underline{0}$ the following holds

$$\begin{aligned}
\Delta M\underline{\xi} = \underline{\theta} &\iff \Delta(X+iY)(\underline{\alpha}+i\underline{\beta}) = \underline{\alpha}+i\underline{\beta},\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \neq 0 \\
&\iff \left.\begin{array}{c}\Delta X\underline{\alpha} - \Delta Y\underline{\beta} = \underline{\alpha} \\ \Delta Y\underline{\alpha} + \Delta X\underline{\beta} = \underline{\beta}\end{array}\right\},\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \neq 0 \\
&\iff \left.\begin{array}{c}(I-\Delta X)\underline{\alpha} + \Delta Y\underline{\beta} = 0 \\ -\Delta Y\underline{\alpha} + (I-\Delta X)\underline{\beta} = 0\end{array}\right\},\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \neq 0 \\
&\iff \begin{pmatrix} I-\Delta X & \Delta Y \\ -\Delta Y & I-\Delta X\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \underline{0},\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \neq 0 \\
&\iff \left\{\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix} - \begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix} X & -Y\\ Y & X\end{pmatrix}\right\}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \underline{0},\ \begin{pmatrix}\alpha\\ \beta\end{pmatrix} \neq 0 \\
&\iff \det\left\{I_{2n} - \begin{pmatrix}\Delta & 0\\ 0 & \Delta\end{pmatrix}\begin{pmatrix} X & -Y\\ Y & X\end{pmatrix}\right\} = 0.
\end{aligned}$$

And this completes the proof. □

Theorem A.1.1. Let $M = X + iY \in \mathbb{C}^{n\times m}$, where $X, Y \in \mathbb{R}^{n\times m}$ and $\operatorname{rank}(M) = m$. Let also $\Theta \in \mathbb{R}^{n\times k},\ Z \in \mathbb{R}^{k\times m}$ and $\Delta \in \mathbf{\Delta} \cap \mathbb{R}^{k\times k}$ and define

$$\gamma = \inf{}^{-1}\left\{\|\Delta\| :\ \Delta \in \mathbf{\Delta} \cap \mathbb{R}^{k\times k},\ \operatorname{rank}(M + \Theta\Delta Z) < m\right\} \tag{A.8}$$

unless $\operatorname{rank}(M + \Theta\Delta Z) = m$ for all $\Delta \in \mathbf{\Delta} \cap \mathbb{R}^{k\times k}$, in which case $\gamma = 0$. Then

$$\gamma = \mu_{\tilde{\Delta}\cap\mathbb{R}^{4k\times 4k}}(\tilde{M}), \tag{A.9}$$

where $\tilde{\Delta} = \{\operatorname{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s}) : \delta_i \in \mathbb{R}\}$ and $\tilde{M} = P\Theta E^{-1}\Gamma P'$, where

$$E = \begin{pmatrix} I & 0 & X & -Y\\ 0 & I & Y & X\\ X' & Y' & 0 & 0\\ -Y' & X' & 0 & 0\end{pmatrix},\ \Gamma = \begin{pmatrix} 0 & 0 & \Theta & 0\\ 0 & 0 & 0 & \Theta\\ Z' & 0 & 0 & 0\\ 0 & Z' & 0 & 0\end{pmatrix},$$

$$\Theta = \begin{pmatrix} \Theta' & 0 & 0 & 0\\ 0 & \Theta' & 0 & 0\\ 0 & 0 & Z & 0\\ 0 & 0 & 0 & Z\end{pmatrix}$$

and P is the permutation matrix, such that

$$P(I_4 \otimes \Delta)P' = \tilde{\Delta} = \mathrm{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s}). \tag{A.10}$$

Proof. Write $M + \Theta\Delta Z = (X + \Theta\Delta Z) + iY$, where $X + \Theta\Delta Z, Y \in \mathbb{R}^{n\times m}$. Then

$$\mathrm{rank}(M + \Theta\Delta Z) < m \iff \mathrm{rank}\begin{pmatrix} X + \Theta\Delta Z & -Y \\ Y & X + \Theta\Delta Z \end{pmatrix} < 2m$$

Separating real and complex parts respectively we get

$$\det\begin{pmatrix} I & 0 & X + \Theta\Delta Z & -Y \\ 0 & I & Y & X + \Theta\Delta Z \\ X' + Z'\Delta\Theta' & Y' & 0 & 0 \\ -Y' & X' + Z'\Delta\Theta' & 0 & 0 \end{pmatrix} = 0$$

$$\iff \det\left\{\begin{pmatrix} I & 0 & X & -Y \\ 0 & I & Y & X \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \Theta\Delta Z & 0 \\ 0 & 0 & 0 & \Theta\Delta Z \\ Z'\Delta\Theta' & 0 & 0 & 0 \\ 0 & Z'\Delta\Theta' & 0 & 0 \end{pmatrix}\right\} = 0$$

$$\iff \det\left\{\underbrace{\begin{pmatrix} I & 0 & X & -Y \\ 0 & I & Y & X \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{pmatrix}}_{E} + \right.$$

$$\left. + \underbrace{\begin{pmatrix} 0 & 0 & \Theta & 0 \\ 0 & 0 & 0 & \Theta \\ Z' & 0 & 0 & 0 \\ 0 & Z' & 0 & 0 \end{pmatrix}}_{\Gamma} \begin{pmatrix} \Delta & 0 & 0 & 0 \\ 0 & \Delta & 0 & 0 \\ 0 & 0 & \Delta & 0 \\ 0 & 0 & 0 & \Delta \end{pmatrix} \underbrace{\begin{pmatrix} \Theta' & 0 & 0 & 0 \\ 0 & \Theta' & 0 & 0 \\ 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \end{pmatrix}}_{\Theta}\right\} = 0$$

Introduce the permutation matrix P such that

$$P(I_4 \otimes \Delta)P' = \tilde{\Delta} = \mathrm{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s}).$$

Then

$$\begin{aligned}
&\det(E+\Gamma(I_4\otimes\Delta)\Theta)=0 \iff\\
&\det(E+\Gamma P'\tilde{\Delta}P\Theta)=0 \iff\\
&\det(I+E^{-1}\Gamma P'\tilde{\Delta}P\Theta)=0 \iff\\
&\det(I+\underbrace{P\Theta E^{-1}\Gamma P'}_{\tilde{M}}\tilde{\Delta})=0 \iff\\
&\det(I+\tilde{M}\tilde{\Delta})=0.
\end{aligned}$$

Equivalently $\|\Delta\|=\|\tilde{\Delta}\|$, where $\Delta\in\mathbf{\Delta}\cap\mathbb{R}^{k\times k}$ and $\tilde{\Delta}\in\tilde{\mathbf{\Delta}}\cap\mathbb{R}^{4k\times 4k}$ respectively. Then it is possible to define two cases:

(i) If $\text{rank}(M+\Theta\Delta Z)=m,\ \Delta\in\mathbf{\Delta}\cap\mathbb{R}^{k\times k}$, then $\det(I+\tilde{M}\tilde{\Delta})\neq 0,\ \forall\tilde{\Delta}\in\tilde{\mathbf{\Delta}}\cap\mathbb{R}^{4k\times 4k}$. In which case

$$\gamma=\mu_{\tilde{\Delta}\cap\mathbb{R}^{4k\times 4k}}(\tilde{M})=0.$$

(ii) If $\text{rank}(M+\Theta\Delta Z)<m$, then

$$\begin{aligned}
\gamma=&\inf{}^{-1}\left\{\|\Delta\|:\ \Delta\in\mathbf{\Delta}\cap\mathbb{R}^{k\times k},\ \text{rank}(M+\Theta\Delta Z)<m\right\}\\
=&\inf\left\{\|\tilde{\Delta}\|:\ \tilde{\Delta}\in\tilde{\mathbf{\Delta}}\cap\mathbb{R}^{4k\times 4k},\ \det(I+\tilde{M}\tilde{\Delta})=0\right\}\\
=&\mu_{\tilde{\Delta}\cap\mathbb{R}^{4k\times 4k}}(\tilde{M}).
\end{aligned}$$

Note that

$$E=\begin{pmatrix} I & \tilde{M}\\ \tilde{M}' & 0\end{pmatrix},\text{where }\ \tilde{M}=\begin{pmatrix} X & -Y\\ Y & X\end{pmatrix}$$

has full column rank equal to $2m$. Hence,

$$\begin{pmatrix} I & \tilde{M}\\ \tilde{M}' & 0\end{pmatrix}=\begin{pmatrix} I-\tilde{M}(\tilde{M}'\tilde{M})^{-1}\tilde{M}' & \tilde{M}(\tilde{M}'\tilde{M})^{-1}\\ (\tilde{M}'\tilde{M})^{-1}\tilde{M}' & -(\tilde{M}'\tilde{M})^{-1}\end{pmatrix},$$

where

$$\tilde{M}'\tilde{M}=\begin{pmatrix} X' & Y'\\ -Y' & X'\end{pmatrix}\begin{pmatrix} X & -Y\\ Y & X\end{pmatrix}=\begin{pmatrix} X'X+Y'Y & -X'Y+Y'X\\ -Y'X+X'Y & X'X+Y'Y\end{pmatrix}$$

is invertible. Moreover, note that $\tilde{M}'\tilde{M}=\left(\begin{array}{c|c} A & B\\ \hline B' & A\end{array}\right)$, where $A=A'>0$ and $B+B'=0$ (skew-symmetric). □

A.2 General μ problem

Apparently the case of complex data and complex perturbations is considerably more complicated when the augmented structure of M is taken into account. Let $M \in \mathbb{C}^{n\times m}, n \geq m$, then the matrix representation with respect to (5.25) is

$$A = \begin{pmatrix} I_n & M \\ M^* & 0_m \end{pmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}, \tag{A.11}$$

where M^* is a complex conjugate transpose. Then, it is required to find $\tilde{\Delta} \in \mathbf{\Delta} \cap \mathbb{C}^{k\times k}$ to be a block diagonal matrix $\tilde{\Delta} = \mathrm{diag}(\Delta, \Delta^*)$ such that $\det(I - \tilde{\Delta}\tilde{M}) = 0$. For the real and mixed cases of $\mu_{\tilde{\Delta}}$ it is obvious that $\Delta = \Delta' \cap \mathbb{R}$, while for the complex case there are different blocks of structured perturbations $\Delta \neq \Delta^*$. At the moment the case of $\mu^{\mathbb{C}}_{\Delta}(M), M \in \mathbb{C}^{n\times m}$ of the generalised problem is not solved exactly. However, it can be shown that alternatively $\mu^{\mathbb{C}}_{\Delta}$ can be transferred to analysis of μ problem.

Lemma A.2.1. Let $M = X + iY \in \mathbb{C}^{n\times m}, X, Y \in \mathbb{R}^{n\times n}$ with $n \geq m$. If there exists $\Delta = \Delta_R + i\Delta_I$, then $\det(I - \Delta M) = 0$ if and only if the following holds

$$\det\left\{ I_{2n} - \begin{pmatrix} \Delta_R & -\Delta_I \\ \Delta_I & \Delta_R \end{pmatrix} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \right\} = 0. \tag{A.12}$$

Proof. The proof of (A.2.1) follows the same path as argued in (A.1.3). Assume now that $\det(I - \Delta M) = 0$. Then $\forall \underline{\xi} = \underline{\alpha} + i\underline{\beta},\ \alpha, \beta \in \mathbb{R}^m,\ \underline{\xi} \neq \underline{0}$ the following holds

$$\begin{aligned}
\Delta M\underline{\xi} = \underline{\theta} &\iff (\Delta_R + i\Delta_I)(X + iY)(\underline{\alpha} + i\underline{\beta}) = \underline{\alpha} + i\underline{\beta},\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0 \\
&\iff \left.\begin{matrix} (\Delta_R X - \Delta_I Y)\underline{\alpha} - (\Delta_R Y + \Delta_I X)\underline{\beta} = \underline{\alpha} \\ (\Delta_R Y + \Delta_I X)\underline{\alpha} + (\Delta_R X - \Delta_I Y)\underline{\beta} = \underline{\beta} \end{matrix}\right\},\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0 \\
&\iff \left.\begin{matrix} (I - \Delta_R X + \Delta_I Y)\underline{\alpha} + (\Delta_R Y + \Delta_I X)\underline{\beta} = 0 \\ -(\Delta_R Y + \Delta_I X)\underline{\alpha} + (I - \Delta_R X + \Delta_I Y)\underline{\beta} = 0 \end{matrix}\right\},\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0 \\
&\iff \begin{pmatrix} I - \Delta_R X + \Delta_I Y & (\Delta_R Y + \Delta_I X) \\ -(\Delta_R Y + \Delta_I X) & I - \Delta_R X + \Delta_I Y \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \underline{0},\ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0 \\
&\iff \det\left\{ I_{2n} - \begin{pmatrix} \Delta_R & -\Delta_I \\ \Delta_I & \Delta_R \end{pmatrix} \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix} \right\} = 0.
\end{aligned}$$

□

Theorem A.2.1. Let $M = X + iY \in \mathbb{C}^{n\times m}$, where $X, Y \in \mathbb{R}^{n\times m}$ and $\operatorname{rank}(M) = m$. Let also $\Theta \in \mathbb{R}^{n\times k}$, $Z \in \mathbb{R}^{k\times m}$ and $\Delta \in \boldsymbol{\Delta} \cap \mathbb{R}^{2k\times 2k}$ and define

$$\gamma = \inf^{-1}\left\{\|\Delta\|_F : \ \Delta \in \boldsymbol{\Delta} \cap \mathbb{R}^{2k\times 2k}, \ \operatorname{rank}(M + \Theta\Delta Z) < m\right\} \tag{A.13}$$

unless $\operatorname{rank}(M + \Theta\Delta Z) = m$ for all $\Delta \in \boldsymbol{\Delta} \cap \mathbb{R}^{2k\times 2k}$, in which case $\gamma = 0$. Then

$$\gamma = \mu_{\tilde{\Delta}\cap\mathbb{R}^{8k\times 8k}}(\tilde{M}), \tag{A.14}$$

where $\tilde{\Delta} = \{\operatorname{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s}, \gamma_1 I_{4r_1}, \ldots, \gamma_s I_{4r_s}) : \delta_i, \gamma_i \in \mathbb{R}\}$ and $\tilde{M} = P\Theta E^{-1}\Gamma P'$, where

$$E = \begin{pmatrix} I & 0 & X & -Y \\ 0 & I & Y & X \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{pmatrix}, \ \Gamma = \left(\begin{array}{cccc|cccc} \Theta & 0 & 0 & 0 & \Theta & 0 & 0 & 0 \\ 0 & \Theta & 0 & 0 & 0 & \Theta & 0 & 0 \\ 0 & 0 & Z' & 0 & 0 & 0 & Z' & 0 \\ 0 & 0 & 0 & Z' & 0 & 0 & 0 & Z' \end{array}\right),$$

$$\Theta = \left(\begin{array}{cc|cc} 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \\ \Theta' & 0 & 0 & 0 \\ 0 & \Theta' & 0 & 0 \\ \hline 0 & 0 & 0 & -Z \\ 0 & 0 & Z & 0 \\ 0 & \Theta' & 0 & 0 \\ -\Theta' & 0 & 0 & 0 \end{array}\right)$$

and P is the permutation matrix, such that

$$\begin{aligned} P(I_8 \otimes \Delta)P' = \tilde{\Delta} &= \operatorname{diag}(I_4 \otimes \Delta_R, I_4 \otimes \Delta_I) \\ &= \operatorname{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s}, \gamma_1 I_{4r_1}, \ldots, \gamma_s I_{4r_s}) : \delta_i, \gamma_i \in \mathbb{R}). \end{aligned} \tag{A.15}$$

Proof. Let perturbations be applied for the complex and real parts respectively $M + \Theta\Delta Z = (X + \Theta\Delta_R Z) + i(Y + \Theta\Delta_I Z)$, where $X + \Theta\Delta_R Z, Y + \Theta\Delta_I Z) \in \mathbb{R}^{n\times m}$. Then

$$\operatorname{rank}(M + \Theta\Delta Z) < m \iff \operatorname{rank}\begin{pmatrix} X + \Theta\Delta_R Z & -Y - \Theta\Delta_I Z \\ Y + \Theta\Delta_I Z & X + \Theta\Delta_R Z \end{pmatrix} < 2m$$

Separating real and complex parts respectively we get

$$\det\left\{\begin{pmatrix} I & 0 & X+\Theta\Delta_R Z & -Y-\Theta\Delta_I Z \\ 0 & I & Y+\Theta\Delta_I Z & X+\Theta\Delta_R Z \\ X'+Z'\Delta_R\Theta' & Y'+Z'\Delta_I\Theta' & 0 & 0 \\ -Y'-Z'\Delta_I\Theta' & X'+Z'\Delta_R\Theta' & 0 & 0 \end{pmatrix}\right\} = 0$$

$$\iff \det\left\{\begin{pmatrix} I & 0 & X & -Y \\ 0 & I & Y & X \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \Theta\Delta_R Z & -\Theta\Delta_I Z \\ 0 & 0 & \Theta\Delta_I Z & \Theta\Delta_R Z \\ Z'\Delta_R\Theta' & Z'\Delta_I\Theta' & 0 & 0 \\ -Z'\Delta_I\Theta' & Z'\Delta_R\Theta' & 0 & 0 \end{pmatrix}\right\} = 0$$

$$\iff \det\left\{\underbrace{\begin{pmatrix} I & 0 & X & -Y \\ 0 & I & Y & X \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{pmatrix}}_{E} +\right.$$

$$\left. + \underbrace{\left(\begin{array}{cccc|cccc} \Theta & 0 & 0 & 0 & \Theta & 0 & 0 & 0 \\ 0 & \Theta & 0 & 0 & 0 & \Theta & 0 & 0 \\ 0 & 0 & Z' & 0 & 0 & 0 & Z' & 0 \\ 0 & 0 & 0 & Z' & 0 & 0 & 0 & Z' \end{array}\right)}_{\Gamma} \begin{pmatrix} I_4\otimes\Delta_R & 0 \\ 0 & I_4\otimes\Delta_I \end{pmatrix} \underbrace{\left(\begin{array}{cc|cc} 0 & 0 & Z & 0 \\ 0 & 0 & 0 & Z \\ \Theta' & 0 & 0 & 0 \\ 0 & \Theta' & 0 & 0 \\ \hline 0 & 0 & 0 & -Z \\ 0 & 0 & Z & 0 \\ 0 & \Theta' & 0 & 0 \\ -\Theta' & 0 & 0 & 0 \end{array}\right)}_{\Theta}\right\} = 0$$

Then the proof is identical to that of in Theorem A.1.1 (section *A*.1), where the above equation is equivalent to $\det(I+\tilde{M}\tilde{\Delta}) = 0$. □

Remark A.2.1. The proposed alternative methods for complex matrices deal with the higher dimensional $\mathbb{R}^{2k\times 2k}$ or $\mathbb{R}^{4k\times 4k}$ augmented structures that increase complexity of the computations. The problem of finding the distance to the nearest common root of many polynomials can be analysed with respect to the estimated bound of the structured singular value under the assumption that the perturbation structure is negligible. For the case of non-repeated largest singular values a fast upper bound can be easily evaluated by solving a convex optimisation problem [165]. Geometric equivalence of the problem corresponds to calculation of the extreme point of zonotope that will be presented in future work.

A.3 Numerical Examples

In this section the obtained results are demonstrated by means of simple numerical examples. Calculations are carried out in MATLAB using the existing *mu.m* function in order to compute the structured singular values of the given matrices. To avoid tremendous structures of the augmented matrix and compatible structured matrices Θ and Z, Sylvester matrices will only be specified and the computational results, leaving the rest for the reader to observe and compute from the Theorems provided in the main part of the thesis.

Example A.3.1. (Complex-Real) Let the given set of coprime polynomials be complex with the Sylvester matrix given below

$$S_R = \begin{pmatrix} 1.0000+0.0000i & 0.0172-0.7530i & 0.2312+0.0300i & 0.0000+0.0000i \\ 0.0000+0.0000i & 1.0000+0.0000i & 0.0172-0.7530i & 0.2312+0.0300i \\ 1.0000+0.0000i & 0.2338+0.4280i & 0.9850-0.0673i & 0.0000+0.0000i \\ 0.0000+0.0000i & 1.0000+0.0000i & 0.2338+0.4280i & 0.9850-0.0673i \\ 1.0000+0.0000i & 0.7996+0.4574i & 0.3689-0.1840i & 0.0000+0.0000i \\ 0.0000+0.0000i & 1.0000+0.0000i & 0.7996+0.4574i & 0.3689-0.1840i \end{pmatrix}$$

and restrict perturbations of Δ to be real. Then the problem of finding the nearest common root is equivalent to the real structured singular value problem as demonstrated in Theorem A.1.1 (section $A.1$). Pulling out the structure for the μ-analysis the structured perturbations are computed as:

$$\gamma^* = (-0.0232, -0.3311, 0.0420, -0.3309, -0.3309, 0.3153)$$

It can be evaluated that the perturbed set of polynomials have a common root at $s = -0.0617 + 0.6147i$ that is obtained by applying structured perturbations, i.e. $\|\Delta\| = 0.3311$

Example A.3.2. (Complex-Complex) Consider a set of complex polynomials with the corresponding non-singular Sylvester matrix

$$S_C = \begin{pmatrix} 1.0000+0.0000i & 0.1017-0.5718i & 0.9954-0.2500i & 0.0000+0.0000i \\ 0.0000+0.0000i & 1.0000+0.0000i & 0.1017-0.5718i & 0.9954-0.2500i \\ 1.0000+0.0000i & 0.0620-1.3380i & 0.2982+0.0303i & 0.0000+0.0000i \\ 0.0000+0.0000i & 1.0000+0.0000i & 0.0620-1.3380i & 0.2982+0.0303i \\ 1.0000+0.0000i & 0.7614-0.7006i & 0.6311-1.6305i & 0.0000+0.0000i \\ 0.0000+0.0000i & 1.0000+0.0000i & 0.7614-0.7006i & 0.6311-1.6305i \end{pmatrix}$$

Allow now perturbations to be complex that is equivalent to the general structured singular value problem. Separating real and imaginary parts of S_C with respect to Theorem A.2.1

(section $A.2$) the set of structured perturbations is obtained

$$\gamma_R^* = (-0.0547, 0.0977, -0.1207, -0.1207, 0.1207, 0.1207)$$

for real part of the perturbations and

$$\gamma_C^* = (-0.0831, -0.0775, 0.0348, -0.1207, -0.1207, 0.1207)$$

for complex part respectively. Then the minimal magnitude complex perturbations in the coefficients of the polynomials are defined with respect to the Frobenius norm as $\|\Delta\|_F = 1.1333$.

Appendix B

Generalised implicit μ problem

Consider now $M \in \mathbb{C}^{n\times n}$ and $\Delta \in \mathbb{R}^{n\times n}$. Then we have the following result:

Lemma B.0.1. Let $M = X + iY \in \mathbb{C}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$ and $\Delta \in \mathbb{R}^{n\times n}$. Then $\det(I + \Delta M) = 0$ if and only if

$$\det\left\{I_{2n} + \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\right\} = 0. \tag{B.1}$$

Proof. Assume that $\det(I + \Delta M) = 0$. Then there exists $\xi = \alpha + i\beta \in \mathbb{C}^n$, $\alpha, \beta \in \mathbb{R}^n$, $\xi \neq 0$ such that

$$\Delta M\xi = -\xi \iff \Delta(X + iY)(\alpha + i\beta) = -\alpha - i\beta, \ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0$$

This is further equivalent to:

$$\begin{aligned} \Delta M\xi = -\xi &\iff \left.\begin{matrix} \Delta X\alpha - \Delta Y\beta = -\alpha \\ \Delta Y\alpha + \Delta X\beta = -\beta \end{matrix}\right\}, \ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0 \\ &\iff \begin{pmatrix} I + \Delta X & -\Delta Y \\ \Delta Y & I + \Delta X \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = 0, \ \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \neq 0 \\ &\iff \det\left\{I_{2n} + \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\right\} = 0. \end{aligned}$$

This completes the proof. □

The following Lemma is a direct consequence of Lemma B.0.1:

Lemma B.0.2. Let $M = X + iY \in \mathbb{C}^{n\times n}$, $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $N = A + iB \in \mathbb{C}^{m\times n}$, $A \in \mathbb{R}^{m\times n}$, $Y \in \mathbb{R}^{m\times n}$ with $m \leq n$ and $\Delta \in \mathbb{R}^{n\times n}$. Then:

$$\text{null}\begin{pmatrix} I_n + \Delta M \\ N \end{pmatrix} \neq 0 \iff \text{null}\begin{pmatrix} I_n + \Delta X & -\Delta Y \\ \Delta Y & I_n + \Delta X \\ A & -B \\ B & A \end{pmatrix} \neq 0 \tag{B.2}$$

Proof. Note that:

$$\text{null}\begin{pmatrix} I_n + \Delta M \\ N \end{pmatrix} \neq 0 \iff \exists x \neq 0, x \in \mathcal{N}(I + \Delta M) \cap \mathcal{N}(N)$$

From Lemma B.0.1 this is equivalent to the existence of a vector $\xi \neq 0$ such that:

$$\left(I_{2n} + (I_2 \otimes \Delta)\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}\right)\xi = 0 \text{ and } \begin{pmatrix} A & -B \\ B & A \end{pmatrix}\xi = 0$$

which is also equivalent to the condition stated in the RHS of equation (B.2). □

The next Theorem generalised Theorem 6.2.1 to the case of complex M and N matrices. The solution involves only real data. This is achieved at the expense of dilating the uncertainty structure.

Theorem B.0.1. Let $M = X + iY \in \mathbb{C}^{n\times n}$, $N = A + iB \in \mathbb{C}^{m\times n}$, where $X \in \mathbb{R}^{n\times n}$, $Y \in \mathbb{R}^{n\times n}$, $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{m\times n}$ and $m \leq n$. Then,

$$\hat{\mu}_{\mathbf{\Delta}}(M, N) = \mu_{\tilde{\mathbf{\Delta}}}(\tilde{M}) \tag{B.3}$$

where

$$\tilde{M} = P'\begin{pmatrix} (I_{2m} + \tilde{N}'\tilde{N})^{-1}\Omega & \tilde{N}'\tilde{N}(I_{2m} + \tilde{N}'\tilde{N})^{-1} \\ -\Omega(I_{2m} + \tilde{N}'\tilde{N})^{-1}\Omega' & \Omega(I_{2m} + \tilde{N}'\tilde{N})^{-1} \end{pmatrix} P, \tag{B.4}$$

$\mathbf{\Delta}$ is defined in equation (6.6), $\tilde{\mathbf{\Delta}} = \{\text{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s}) : \delta_i \in \mathbb{R}\}$, P is a permutation matrix such that $P(I_4 \otimes \mathbf{\Delta})P' = \tilde{\mathbf{\Delta}}$,

$$\tilde{N} = \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \text{ and } \tilde{\Omega} = \begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

Proof. Applying Lemma B.0.2 gives:

$$\nu = \text{null} \begin{pmatrix} I_n + \Delta M \\ N \end{pmatrix} \neq 0 \iff \text{null} \begin{pmatrix} I_n + \Delta X & -\Delta Y \\ \Delta Y & I_n + \Delta X \\ A & -B \\ B & A \end{pmatrix} \neq 0$$

and hence from Lemma 5.3.1, $\nu \neq 0$ if and only if, assuming that $\Delta \in \mathbf{\Delta}$:

$$\nu = \text{null} \begin{pmatrix} I_n & 0 & 0 & 0 & I_n + \Delta X & -\Delta Y \\ 0 & I_n & 0 & 0 & \Delta Y & I_n + \Delta X \\ 0 & 0 & I_m & 0 & A & -B \\ 0 & 0 & 0 & I_m & B & A \\ I_n + X'\Delta & Y'\Delta & A' & B' & 0 & 0 \\ -Y'\Delta & I_n + X'\Delta & -B' & A' & 0 & 0 \end{pmatrix} \neq 0$$

This is equivalent to:

$$\text{null} \left\{ \Phi + \begin{pmatrix} 0 & 0 & 0 & 0 & \Delta X & -\Delta Y \\ 0 & 0 & 0 & 0 & \Delta Y & \Delta X \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ X'\Delta & Y'\Delta & 0 & 0 & 0 & 0 \\ -Y'\Delta & X'\Delta & 0 & 0 & 0 & 0 \end{pmatrix} \right\} \neq 0 \tag{B.5}$$

where we have defined:

$$\Phi = \begin{pmatrix} I_n & 0 & 0 & 0 & I_n & 0 \\ 0 & I_n & 0 & 0 & 0 & I_n \\ 0 & 0 & I_m & 0 & A & -B \\ 0 & 0 & 0 & I_m & B & A \\ I_n & 0 & A' & B' & 0 & 0 \\ 0 & I_n & -B' & A' & 0 & 0 \end{pmatrix}$$

Note that from Lemma 5.3.1 matrix Φ is nonsingular. The second matrix in equation (B.5) can be factored as:

$$\begin{pmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{pmatrix} (I_4 \otimes \Delta) \begin{pmatrix} I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & -Y \\ 0 & 0 & 0 & 0 & Y & X \end{pmatrix}$$

Next define permutation matrix P such that

$$P(I_4 \otimes \Delta)P' = \tilde{\Delta} = \text{diag}(\delta_1 I_{4r_1}, \ldots, \delta_s I_{4r_s})$$

(this is just a reshuffling of the diagonal entries of $I_4 \otimes \Delta$). Using Lemma (6.2.1) then shows that $\nu \neq 0$ if and only if $\text{null}(I_{4n} + \tilde{M}\tilde{\Delta}) \neq 0$ where

$$\tilde{M} = P' \begin{pmatrix} I_n & 0 & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & X & -Y \\ 0 & 0 & 0 & 0 & Y & X \end{pmatrix} \Phi^{-1} \begin{pmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ X' & Y' & 0 & 0 \\ -Y' & X' & 0 & 0 \end{pmatrix} P$$

Using the definitions of μ and $\hat{\mu}$ establishes (B.3). The expression of $\tilde{M}$ given in equation (B.4) follows after several intricate but straightforward calculations. □