Geometric properties of random walk loop soups

by

Quirin Thomas Simon Vogel

Supervised by Dr. Stefan Adams and Dr. Wei Wu

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Declarations

The work in this thesis was conducted by the author during the period June 2017 - June 2020 at the University of Warwick, in collaboration with Dr. Stefan Adams. Chapter 3 generalises some arguments from our work in [AV20] and Chapter 4 uses results from [LV19]. Where we make use of work not our own, or rework established arguments, we write (for instance): we follow [Geo88]". To the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise stated. This thesis has not been submitted for a degree at any other university.

Abstract

In this thesis the author examines geometric properties of (Poisson) loop soups generated from loop measures with varying weights. The framework incorporates the Markovian loop measure, see [LJ11], as well as the Bosonic loop measure, see [AV20]. The author characterises certain geometric features of the loop soup, such as its percolative properties and correlation structure.

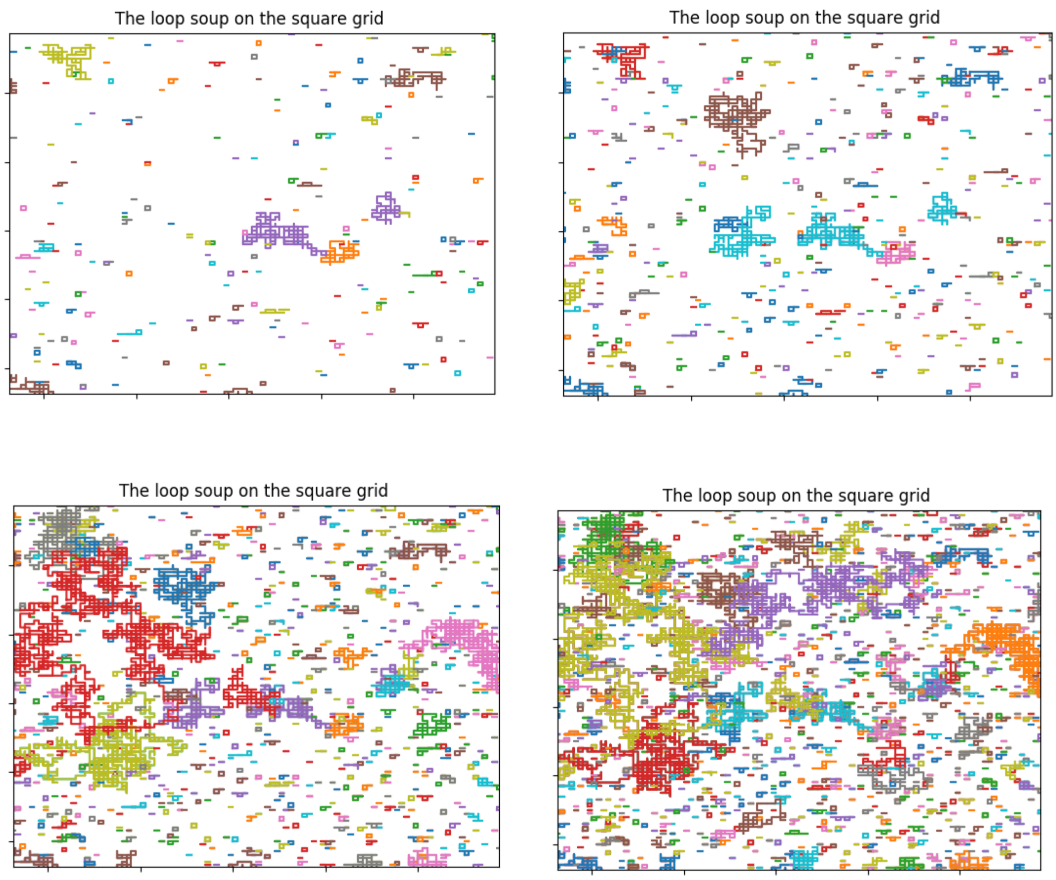

Figure 1: Different realisations of a two-dimensional random walk loop soup, with increasing intensity.

Chapter 1

Introduction

Statistical mechanics is a branch of physics which aims to make a connection between the macroscopic and the microscopic properties of a system. Often cited examples of macroscopic properties include temperature, magnetisation, and viscosity. The strength of molecular bonds or other interatomic forces are examples of microscopic properties of a system. As systems typically considered in statistical mechanics consist of a large number of interacting microscopic atoms (or agents), simplified probabilistic models are brought forward, with the hope that the qualitative behaviour is accurately rendered.

1.1 Loop models and loop soups

There are many models in statistical mechanics describing different physical systems. Our investigation is motivated by the fact that many have a (partial) representation in terms of a *loop model* or *loop soup*. We first clarify what we mean by a loop model or loop soup and then introduce some examples from the literature where a loop representation exists. As this is purely to motivate the study, the list will be far from exhaustive.

Given an at most countably infinite graph $\mathcal{G} = (V, E)$, we say that a *loop* ω is a function from $[0, t] \to V$ (for some $t > 0$) which is continuous from the right, has left limits, jumps across edges only and satisfies

$$\omega(0) = \omega(t)\,. \tag{1.1.1}$$

We say that t is the *length* of the loop ω. Let Γ be the space of all such loops (of any finite length). In our setting, we define a loop model to be a probability measure on $(\mathbb{N}_0)^\Gamma$. For $\sigma \in (\mathbb{N}_0)^\Gamma$, we interpret $\sigma_\omega = k \in \mathbb{N}$ as the loop ω being sampled k-times and $\sigma_\omega = 0$ as the loop being absent. The multiset, where loop ω is present σ_ω-times, is then referred to as the loop soup and denoted by $\mathcal{U}$. It is trivial to see that both viewpoints are equivalent: specifying the law of the random multiset $\mathcal{U}$ of Γ is equivalent to fixing the distribution of σ. A statistical mechanics model has a *loop representation* if certain features of the system can be computed in terms of a loop soup $\mathcal{U}$. Instead of giving a strict mathematical definition, we present an in-depth description of two cases.

1.1.1 Markovian loop soups and Gaussian fields

One of the most studied models in statistical mechanics and probability theory is the Gaussian (free) field: given a finite collection of vertices V and $Q \in \mathbb{R}^{V\times V}$ a positive definite, symmetric matrix over V, let $\mathcal{N}_Q$ be the Gaussian measure on $\mathbb{R}^V$[1] with covariance matrix Q^{-1}. That is, $\mathcal{N}_Q$ is a probability measure on $\mathbb{R}^V$ with density

$$\mathrm{d}\mathcal{N}_Q(\psi) = \frac{1}{Z}\exp\left(-\sum_{x,y\in V}\psi_x Q_{x,y}\psi_y\right)\mathrm{d}\psi, \qquad \text{for } \psi \in \mathbb{R}^V\,, \tag{1.1.2}$$

where Z is the normalising constant and $\mathrm{d}\psi$ is the Lebesgue measure on $\mathbb{R}^V$. In many cases it is possible to define a Gaussian measure on an infinite graph by taking weak limits of the above measures. For the purpose of this introduction, we restrict ourselves to finite graphs.

It is well-known that for a Gaussian field its covariance can be represented in terms of the local time of a continuous-time random walk, we refer the reader to [Fun05, Chapter 3] for a reference. Let $G(x,y) = \mathcal{N}_Q[\psi_x\psi_y]$ be the correlation function for $x, y \in V$. Enrich $\mathcal{G}$ with an additional symbol $\dagger$, often referred to as graveyard. Let $(X_s)_{s\geq 0}$ be the continuous-time random walk with semi-group $\left(\mathrm{e}^{sQ}\right)_{s\geq 0}$. For any $x \in V$, set the weight of going from x to $\dagger$

[1] We restrict ourselves to mean zero Gaussian measures.

to be $Q(x,x) - \sum_{y \neq x} Q(x,y)$. We have that

$$G(x,y) = \mathbb{E}_x \left[\int_0^{\tau} \mathbb{1}\{X_s = y\} \right] , \tag{1.1.3}$$

where † is an absorbing state and τ is the hitting time of †.
Building on the works of Symanzik (see e.g. [Sym68]), more powerful random walk representations (of Gaussian measures) have been found, such as the Dynkin Isomorphism (see [Dyn83] and [LJ08]), the Eisenbaum Isomorphism (see [Eis95]), the Ray-Knight theorems (see [Kni63] and [Ray63]) and the representations in [BFS82], to name a few. The above results are of the form that some functional of both the Gaussian field and the random walk is equal (in distribution) to a different functional applied to the Gaussian field alone. For an account of these representations together with their implications, we refer the reader to [Szn12] or [FFS13]. As we are primarily interested in representations of the Gaussian field in terms of loops, we do not examine the aforementioned results in greater detail.
In the works of Le Jan (see for example [LJ10]) a representation of the square of the Gaussian field in terms of the accrued local time of a loop soup is given. As the result is important to our work, we give a brief description: let Q be positive definite and symmetric, as above. Let $(X_s)_{s\geq 0}$ be the continuous-time random walk induced by Q, with measure $\mathbb{P}_x$. This means that the random walk starts at $x \in V$ and then evolves according to $(\mathrm{e}^{sQ})_{s\geq 0}$, as above. For $t > 0$, let $\mathbb{P}^t_{x,x}$ the measure $\mathbb{P}_x$ restricted (not conditioned) to the event $\{X_t = x\}$. Let M be the following measure on the space of loops Γ:

$$M = \sum_{x \in V} \int_0^{\infty} \frac{1}{t} \mathbb{P}^t_{x,x} \, \mathrm{d}t \, . \tag{1.1.4}$$

Let $\mathbb{P}_\lambda$ be the Poisson point process (PPP) on Γ with intensity measure λM for $\lambda > 0$. A sample from $\mathbb{P}_\lambda$ is a realisation of what we call the *Markovian loop soup.* We define the *occupation field* $\mathcal{L}$ as the combined sum of all the local times of the loops: let $L_x = L_x(\omega)$ be defined as $\int_0^t \mathbb{1}\{\omega_s = x\}\mathrm{d}s$, where

t is the length of the loop ω. Given a realisation of the loop soup $\mathcal{U}$, let

$$\mathcal{L}_x = \sum_{\omega \in \mathcal{U}} L_x(\omega)\,. \tag{1.1.5}$$

It then holds that:

Theorem 1.1.1. *[LJ10] The occupation field $(\mathcal{L})_{x\in V}$ under $\mathbb{P}_\lambda$ with $\lambda = 1/2$ has the same distribution as the square of the Gaussian field. This means in particular that for any continuous and bounded function $F\colon [0,\infty)^V \to \mathbb{R}$,*

$$\mathbb{E}_{1/2}\left[F(\mathcal{L})\right] = \mathcal{N}_Q\left[F(\psi^2)\right]\,, \tag{1.1.6}$$

where $(\psi^2)_x = (\psi_x)^2$ for all $x \in V$.

There exist several extensions of the above theorem: in [Lup16a,Cam15, LST19] the isomorphism is generalised to the whole field $(\psi_x)_{x\in V}$. The intuition is that one can first sample $(\psi_x^2)_{x\in V}$ and then sample the sign of ψ_x by an Ising type weight depending on $(\psi_x^2)_{x\in V}$. These results are restricted to the cases where Q is symmetric. For asymmetric random walks, one has to consider complex valued Gaussian measures and replace ψ_x^2 by $|\psi_x|^2$. This is done in [AV20]. In the same publication an isomorphism for the full complex-valued field is given. For a discussion for more general spaces, we refer the reader to [LJMR17].
The above results have the following consequence: a measure $\overline{\mathcal{N}}$ which has a density with respect to $\mathcal{N}_Q$ can be represented it in terms of a loop soup "with interaction". This is made precise in the following corollary.

Corollary 1.1.2. *Let $f\colon [0,\infty)^V \to \mathbb{R}$ be a continuous and bounded. Suppose that $\mathrm{d}\overline{\mathcal{N}} \propto f(\psi^2)\mathrm{d}\mathcal{N}_Q$. Let $\overline{\mathbb{E}}$ be the expectation with respect to the measure $\overline{\mathbb{P}}$ which satisfies $\mathrm{d}\overline{\mathbb{P}} \propto f(\mathcal{L})\mathrm{d}\mathbb{P}$. Then*

$$\overline{\mathbb{E}}[F(\mathcal{L})] = \overline{\mathcal{N}}[F(\psi^2)]\,. \tag{1.1.7}$$

An important example of such a field $\overline{\mathcal{N}}$ is the *Phi-4* model, see [FFS13]. To summarise, in this subsection we have seen that the Gaussian field can be represented as the occupation field of a random walk loop soup. A more general framework, which can be seen as generalisation of the above, is given in Theorem 3.3.1.

For more properties of the Markovian loop measure, we refer the reader to [LJ10, Szn12, Law18].

Recently, a number of new isomorphisms for non-Markovian random walks have been found, connecting their local time to spin systems in spherical or hyperbolic geometries. For more on that, we refer the reader to [BHS19].

1.1.2 Bosonic loop soups

Loop soups can also be used to describe a system of (non-)interacting Bosons. The following introduction into Bosonic particle systems is paraphrased from [AV20, Section 3]:

In quantum mechanics particles can either be Bosons or Fermions. Consider a system of (interacting) Bosons on some finite box $\Lambda \subset \mathbb{Z}^d$: a single particle can be described as a function in the one-particle Hilbert space $\mathcal{H}_\Lambda = \mathbb{R}^\Lambda$ (with the Euclidean inner product). The N-particle Hilbert space is given by the tensor product $\mathcal{H}_\Lambda^{\otimes N}$. The Hamilton operator $H_N \colon \mathcal{H}_\Lambda^{\otimes N} \to \mathcal{H}_\Lambda^{\otimes N}$ for N particles is

$$H_N = -\sum_{i=1}^{N} \Delta_i^{(\Lambda)} + \sum_{1\leq i<i\leq N} v(|x^i - x^j|)\,, \tag{1.1.8}$$

where $\Delta_i^{(\Lambda)}$ is the discrete Laplacian operator on Λ with Dirichlet boundary conditions[2] giving the kinetic energy for particle i. The distance $|x^i - x^j|$ between two points x^i, x^j is the usual Euclidean norm. Thus, the interaction depends only on the distance of particle i at $x^i \in \Lambda$ and particle j at $x^j \in \Lambda$ and the function v. We assume that the particle number is only known in expectation, and thus the thermodynamic equilibrium is given by the grand canonical ensemble. This means that we have to work with the Hilbert space

$$\mathcal{F} = \bigoplus_{N=0}^{\infty} \mathcal{H}_\Lambda^{\otimes N}\,, \tag{1.1.9}$$

[2]equivalently, the generator of the simple symmetric random walk killed upon entering the complement of Λ.

also called the *Fock space.*

States of identical and indistinguishable Bosons are described by symmetric functions: for N Bosons, their possible states are given by all symmetric functions in the tensor product $\mathcal{H}_\Lambda^{\otimes N}$. Here, symmetry refers to the exchangeability of arguments, i.e. if $f(x,y) = f(y,x)$ for all $x, y \in \Lambda$, we would say f is symmetric ($N = 2$). This symmetry is the unique distinguishing feature of Bosons. Note that we can project from $\mathcal{H}_\Lambda^{\otimes N}$ onto its subspace of symmetric function $\mathcal{H}_{\Lambda,+}^{\otimes N}$ by

$$f \mapsto \frac{1}{N!} \sum_{\sigma \in \mathfrak{S}_N} f \circ \sigma \,, \tag{1.1.10}$$

where $\mathfrak{S}_N$ is the symmetric group of N elements and $f \circ \sigma(x_1, \ldots, x_N)$ is given by $f(x_{\sigma(1)}, \ldots, x_{\sigma(N)})$. Write $\mathcal{F}_+$ for the Fock space of all symmetric functions. At thermodynamic equilibrium with inverse temperature β and chemical potential μ, the grand canonical partition function (which is the trace over the symmetrised Fock space) is given by

$$Z_{\Lambda,v}(\beta,\mu) = \mathrm{Tr}_{\mathcal{F}_+}(\mathrm{e}^{-\beta(H-\mu\mathsf{N})}), \tag{1.1.11}$$

where H is the quantised Hamilton operator having projection H_N on the subspace $\mathcal{H}_\Lambda^{\otimes N}$, N is the number operator in Λ taking the value N on the space $\mathcal{H}_\Lambda^{\otimes N}$, and $\mathrm{Tr}_{\mathcal{F}_+}$ is the trace operator on $\mathcal{F}_+$. Using the Feynman-Kac formula (see e.g. [Szn12]), one can derive the following representation of the grand canonical partition function

$$Z_{\Lambda,v}(\beta,\mu) = \sum_{N=0}^{\infty} \frac{\mathrm{e}^{\beta\mu N}}{N!} \sum_{\substack{x_i \in \Lambda \\ i=1,\ldots,N}} \sum_{\sigma \in \mathfrak{S}_N} \bigotimes_{i=1}^{N} \mathbb{P}^{\beta}_{x_i, x_{\sigma(i)}} \left[\mathrm{e}^{-\sum_{1\le i,j\le N} \int_0^\beta v(|X_t^i - X_t^j|)\,\mathrm{d}t} \right], \tag{1.1.12}$$

where $\mathfrak{S}_N$ is the set of all permutations of N elements, and the right-hand side can be interpreted as a system of N random walks $(X_t^i)_{t\ge 0}, i = 1, \ldots, N$ (see [AD08] for details). Following [Gin71] and [ACK11], one can employ cycle-expansion to simplify the above expression: define the *Bosonic loop measure* $M^B_{\Lambda,\mu,\beta}$ as

$$M^B_{\Lambda,\mu,\beta} = \sum_{x \in \Lambda} \sum_{j \ge 1} \frac{\mathrm{e}^{\beta\mu}}{j} \mathbb{P}^{\beta j}_{x,x} \,. \tag{1.1.13}$$

Using the definition of the Bosonic loop measure one obtains

$$Z_{\Lambda,v}(\beta,\mu) = \sum_{N=0}^{\infty} \frac{1}{N!} \bigotimes_{i=1}^{N} M^B_{\Lambda,\mu,\beta}(\mathrm{d}\omega^{(i)}) \big[\mathrm{e}^{-V(\omega^{(1)},\dots,\omega^{(N)})}\big], \tag{1.1.14}$$

where the interaction energy of N loops is the given by

$$\begin{aligned} V(\omega^{(1)},\dots,\omega^{(N)}) =& \frac{1}{2} \sum_{1\le i,j\le N} \sum_{k=0}^{[\ell(\omega^{(i)})-1]/\beta} \sum_{m=0}^{[\ell(\omega^{(j)})-1]/\beta} \\ & \mathbb{1}\{(i,k)\neq(j,m)\} \int_0^\beta v(|\omega^{(i)}(k\beta+t) - \omega^{(j)}(m\beta+t)|)\,\mathrm{d}t\,. \end{aligned} \tag{1.1.15}$$

Here we write $\ell(\omega^{(i)})$ for the length of the i-th loop. The derivation of the above representation of the partition function is non-trivial and is achieved through a series of combinatorial identities and the concatenation of paths (from x_i to $x_{\sigma(i)}$) of length β to form loops with lengths in $\beta\mathbb{N}$. We refer the reader to [Gin71] for the lengthy derivation. In [AV20], we show that the (quantum) correlation functions can also be represented in terms of the Bosonic loop soup. To summarise, we have defined a model of Bose particles and outlined how several of its characteristics, such as the partition function and the correlation functions, can be expressed in terms of a system of loops governed by the Bosonic loop measure (with an additional interaction term).
Previous work has been focused on the distribution of the loop lengths (cycle statistics), see [Lew86, Owe15, AD18]. In our work we are interested in more geometric properties of the Bose gas, such as connectivity properties and correlation functions, continuing the work from [AV20].

1.2 Loop percolation

Loop percolation generally refers to the connected components induced by a loop soup. Previous results are restricted to the Markovian loop soup, defined by the measure M from Equation (1.1.4). Assume that the underlying random walk is the simple symmetric random walk on $\mathbb{Z}^d$. A sample of the loop soup induces a bond-percolation model on $\mathbb{Z}^d$, where we declare a bond as *open* if there is at least one loop traversing through it. Let $\mathcal{C}_0$ be the set of all open bonds connected to the origin through other open bonds only. Note that in this formulation constant loops (i.e. loops which only visit one vertex) do not play any role. By considering the Poisson point process with intensity measure λM, we obtain a one-parameter percolation model (as $\lambda > 0$ varies). We explain here some of the past results in loop percolation, all of which are for the Markovian loop soup. After introducing the main references and results, we give a brief summary at the end of this section.

In [LJL13] percolation for the Markovian loop soup is introduced and then first results are given. The authors introduced an additional parameter $\kappa > 0$ which corresponds to the rate the random walk is killed. To be more precise, with probability $1/(1+\kappa)$ the random walk chooses one of its neighbouring sites uniformly for the next step and with probability $\kappa/(1+\kappa)$ it moves to the absorbing state †. The authors then showed the following: given any $\lambda > 0$, $\mathcal{C}_0$ is finite almost surely for κ sufficiently large. Conversely for any $\kappa \geq 0$, by making λ sufficiently large one has that the cluster of open bonds at the origin $\mathcal{C}_0$ is infinite with positive probability. For the first claim, they use a path counting argument, like it is done for Bernoulli percolation (see for example [Gri89, Chapter 1]). For the second statement, they use that loop percolation can be bounded from below by Bernoulli bond percolation.

In [Lem13], the same model is studied on the complete graph, with the killing-parameter κ proportional to the total number of vertices.

For $\mathbb{Z}^d$ with $d \geq 3$, a number of new results are given in [CS16]. All results in this paper are for $\kappa = 0$. The most important result is that $\mathcal{C}_0$ is finite almost surely for $\lambda > 0$ small enough. This implies (together with the results from [LJL13]) that the critical parameter λ_c (which is the smallest λ for which $\mathcal{C}_0$ being infinite with positive probability) is strictly between 0 and infinity. Another result is that for $d \geq 5$, they were able to show that there exists two constants $C_1, C_2 > 0$ such that the probability $\mathbb{P}\left(\mathcal{C}_0 \cap \boldsymbol{B}_n^c \neq \varnothing\right)$ can be sandwiched in the following way

$$C_1 n^{2-d} \leq \mathbb{P}_\lambda\left(\mathcal{C}_0 \cap \boldsymbol{B}_n^c \neq \varnothing\right) \leq C_2 n^{2-d}\,, \tag{1.2.1}$$

where $\boldsymbol{B}_n$ is the ball centred at 0 with radius n and one has to assume that $0 < \lambda < \lambda_r$ where λ_r is positive and bounded from above by λ_c. Note their other results include bounds on whether a point x is contained in $\mathcal{C}_0$, bounds on the tails of the size of $\mathcal{C}_0$ and more. Most of their results are limited to the case $\lambda < \lambda_r$.
The regime where $\lambda > \lambda_c$ (also called *supercritical phase*) is studied in [Cha17]. Here, the author gives heat kernel bounds for the random walk on the infinite connected component. Contrary to the behaviour of the subcritical phase, the behaviour of the loop soup for $\lambda > \lambda_c$ is similar to simpler percolation models.
In the important work [Lup16a], the author uses a novel coupling with the Gaussian free field to show that $\lambda_c \geq 1/2$ in the cases: for $d \geq 3$ and $\kappa = 0$, and for $G = \mathbb{Z} \times \mathbb{N}$ (the half space) with killing at the boundary $\mathbb{Z} \times \{0\}$. In [Lup16b], the author shows that for the latter case one has $\lambda_c = 1/2$. For this, previous results on conformal loop ensembles are used.
In [AS19], the authors study the vacant set, i.e. those bonds which have not been traversed by any loop. Decoupling inequalities for local functions on the vacant sets were proven.

To summarise: loop percolation has been studied for several years by now. While for several parameter regimes, such as $\lambda > \lambda_c$ and $\lambda < \lambda_r$, detailed estimates are available, there are open questions: Are the decay estimates from Equation (1.2.1) valid for all $\lambda < \lambda_c$? What does the structure of large clusters look like? Does the Bosonic loop soup percolate in a different way than the Markovian one? In this thesis, we provide (partial) answers to these questions.
In this section we have introduced loop percolation and given an overview over results from the literature. In the next section we will introduce sharpness and the recently developed framework of randomised algorithms.

1.3 Sharpness and random algorithms

It is common in percolation theory that, at first, certain decay estimates can only be proven for a parameter range $[0, \lambda_r)$ with $\lambda_r \leq \lambda_c$. This is also true for loop percolation, see Equation 1.2.1.
We compare this to the case of Bernoulli bond percolation $\mathbb{P}_p$ with parameter

$p \in [0,1]$: for many decades it was known (see [Gri89, Chapter 1]) that for every $p < \mathfrak{c}_2^{-1}$ (where $\mathfrak{c}_d$ is the connective constant of $\mathbb{Z}^d$) one has that

$$\mathbb{P}_p\left(\mathcal{C}_0 \cap \boldsymbol{B}_n^c \neq \varnothing\right) \leq \mathrm{e}^{-c_p n}\,, \tag{1.3.1}$$

for some $c_p > 0$. The question is whether this exponential decay continues to hold for every $p \in [0, p_c)$, where p_c is the critical parameter of Bernoulli bond percolation. An affirmative answer to that question is often referred to as *sharpness* (of the phase transition). In both [AB87] and [Men86], it is shown that for Bernoulli bond percolation sharpness holds[3]. While their proofs differ, in both references a system of differential inequalities is used together with an iteration scheme. In [DCT16], the authors utilise the relatively new OSSS inequality (named after O'Donnell, Saks, Schramm and Servedio, see [OSSS05]) to give a new and short proof of sharpness for Bernoulli percolation and the Ising model. The OSSS inequality can be seen as a generalisation of the Poincaré inequality in the sense that it gives an upper bound on the variance of functions. The strategy used in [DCT16] has the advantage of being flexible enough to be adaptable to various other settings: in [DCRT19b] sharpness for the random-cluster model is established, in [DCRT19a] for Voronoi percolation, in [BH19] for inhomogeneous percolation on quasi-transitive graphs, in [DCRT18] for Poisson-Boolean percolation, in [MV20] for Gaussian fields and in [DH18] for the Widom-Rowlinson model.
In this thesis we use the framework laid out in [DCT16] to show the validity of various decay bounds for loop percolation in the whole subcritical regime.

1.4 Main results and outline

In this section we briefly summarise the key results of this thesis.
The main novelty presented in this work is the development of a method which allows us to characterise various features and geometric properties of loop soups uniformly over a wide range of loop measures and the employment of that method. As giving the precise statements of the individual results needs

[3]This took almost 30 years to prove, exponential decay for parts of the subcritical regime was first shown in [BH57].

further notation, we only give rough characterisations and refer the reader to the respective chapters of the thesis for more details.

(Geometric) Properties of the loop soup	
Property	Chapter and Remarks
Occurrence of long loops through a point	Chapter 4.
Occurrence of long loops through an annuls with diverging radius	Chapter 4.
Derivation of the two-point function	Chapter 5.
Derivation of the cumulant function	Chapter 5.
Distribution of the occupation field	Chapter 3 and Chapter 5.
Occurance of vacant sets	Chapter 5.
Existence/Absence of infinite clusters	Chapter 6.
Decay of the one-arm connectivity	Chapter 6, strong decay assumption on the weights.
Occurrence of long loops in clusters	Chapter 6, strong decay assumption on the weights.
Equivalence of critical parameters/ Sharpness	Chapter 6, strong decay assumption on the weights.

We now give a brief description of the content of each chapter of the thesis. In Chapter 2 we fix notation and specify the class of admissible random walks. We also prove lemmas regarding hitting time estimates for the random walk. In Chapter 3 we define the loop measures and the induced loop soups used in this thesis. We use the following approach: instead of proving statements separately for different loop measures, we develop proofs which hold uniformly over a wide range of loop measures. The results for Bosonic and Markovian loop measure follow as special cases. This has the advantage that we no longer rely on the closed form expressions which exists for the Markovian loop measure (due to its connection to the Gaussian free field) only. We restrict ourselves mainly to weights decaying at a polynomial speed, as in that case the loop soup exhibits long-range correlations. In Chapter 3 we also generalise the work from [AV20], which illuminates the intricate relation between Bosonic and Markovian loop measures. We also show how the characterisation of the distribution of the occupation field is equivalent to solving a measure-valued differential equation.

In Chapter 4 we give various decay estimates for loop measures. This is done via representing quantities in terms of the *range* of the random walk bridge and then using concentration estimates for the range.

In Chapter 5 we characterise different (geometric) properties of the loop soup. We use the same strategy as employed in the previous chapter, to obtain novel results for a wide class of loop measures.

In Chapter 6 we study the behaviour of the connected component of the loop soup intersecting the origin. We employ the results from the previous chapters to show that (given certain decay assumptions on the weights) different critical parameters for loop percolation are equal. We use some standard tools from percolation theory, such as the FKG-inequality or Russo's formula, as well as the recently developed framework of randomised algorithms and the OSSS-inequality, see [OSSS05,DCRT18]. We also provide some finer estimates on the structure of the cluster in the subcritical phase.

In Chapter 7, we direct the reader's attention to potential uses of the techniques developed in this thesis and speculate how technical restrictions could be loosened. We give a number of conjectures we plan to verify in future studies.

In the Appendix we give several technical lemmas which we use throughout the text.

At the end of the thesis we provide an index which lists the symbols used throughout the text (not including the introduction) together with a number referencing the page where they are defined. Notation restricted to a small section of the thesis (such as a short proof) is not listed.

Chapter 2

Random walk path measures

In this chapter we introduce notation, the set-up, and prove various technical lemmas. We first define path spaces and then give the class of random walks used in this thesis. We then prove a coupling result with the Brownian bridge. In the last section of this chapter we establish several results on hitting times: computing the (sharp) asymptotics of hitting a single point and the boundary of a sphere.

2.1 Notation and set-up

We begin with a technical remark: in this work, we do not use the ":=" notation when it comes to defining new mathematical objects. Instead we use the "=" symbol. It will be clear from the context when "=" refers to the equality between two (predefined) mathematical objects and when "=" refers to a notational assignment. Furthermore, all equations in this thesis have been labelled to facilitate referencing.
We present a list of conventions used in this work.

I. *Constants*: usually denoted by C and may change value from line to line. Constants with sub/super-scripts are fixed and, unless stated otherwise, only depend on the underlying random walk and the dimension.

II. *Rounding*: given a real number $t \geq 0$, we define $\sum_{j=t} \dots$ as $\sum_{j=\lfloor t \rfloor} \dots$, where $t \mapsto \lfloor t \rfloor$ is the floor function. If we split a sum, we set

$$\sum_{j=1}^{t} \ldots + \sum_{j=t}^{\infty} \ldots \text{ to be equal to } \sum_{j=1}^{\lfloor t \rfloor} \ldots + \sum_{j=\lceil t \rceil}^{\infty} \ldots , \tag{2.1.1}$$

if $t \notin \mathbb{N}$, where $t \mapsto \lceil t \rceil$ is the ceiling function.

III. *Integration*: given a measure space $(\Omega, \mathcal{A}, \mathfrak{m})$ and a measurable function $f \colon \Omega \to \mathbb{C}$, we denote the integral of f with respect to $\mathfrak{m}$ by

$$\mathfrak{m}[f] = \int f \mathrm{d}\mathfrak{m} = \int_{\Omega} f(\omega) \mathrm{d}\mathfrak{m}(\omega) \, . \tag{2.1.2}$$

If $\mathfrak{n}$ is absolutely continuous with respect to $\mathfrak{m}$ and the Radon–Nikodym derivative is given by g, we then write $\mathrm{d}\mathfrak{n}(\omega) = g(\omega)\mathrm{d}\mathfrak{m}(\omega)$. If $\mathfrak{m}$ is the Lebesgue measure (on $\mathbb{R}^d$) and the integration variable is given by x, we simply write $\mathrm{d}x$ instead of $\mathrm{d}\mathfrak{m}(x)$.
The delta measure on a set/point A is denoted by δ_A. The indicator function on a set A is denoted by $\mathbb{1}_A$ or $\mathbb{1}\{A\}$.

IV. *Conditioning*: given an event A and a probability measure $\mathbb{P}$, we write $\mathbb{P}(B|A)$ for the conditional probability of B given A. This extends to events of measure 0, using regular conditional distributions, see [Kle13]. If f is a real-valued function, we sometimes write $\mathbb{E}[f, A]$ as a shorthand for $\mathbb{E}[f \mathbb{1}_A]$.

V. *Cardinalities:* given a countable set I, we denote its cardinality by $|I|$.

VI. *Asymptotic Equality:* given two real-valued sequences $(x_\varepsilon)_\varepsilon$ and $(y_\varepsilon)_\varepsilon$, depending on some sets of parameters ε, we write $x_\varepsilon \sim y_\varepsilon$ if there exist two positive constants C_1, C_2 such that

$$\forall \varepsilon \colon C_1 x_\varepsilon \leq y_\varepsilon \leq C_2 x_\varepsilon \, . \tag{2.1.3}$$

Unless stated otherwise C_1, C_2 only depend on the dimension of the space and the underlying random walk.

VII. *Derivatives*: for functions of multiple arguments, we use the notation $\partial_x f(\boldsymbol{x})$ for the derivative of f with respect to x (where x is a coordinate of $\boldsymbol{x}$). If a function g only depends on one argument, we write ∂g for the derivative.

VIII. *Gamma function*: the Gamma function is denoted by $\Gamma(s)$, $s>0$. The *upper incomplete Gamma function* is denoted by $\Gamma(s,x)=\int_x^\infty t^{s-1}\mathrm{e}^{-t}\mathrm{d}t$. The *lower incomplete Gamma function* is $\gamma(s,x)=\int_0^x t^{s-1}\mathrm{e}^{-t}\mathrm{d}t$. Note $\Gamma(s)=\Gamma(s,x)+\gamma(s,x)$.

IX. *Norms*: we denote the Euclidean norm on $\mathbb{R}^d$ by $|\cdot|$. When we refer to the p-norm (for $p\in[1,\infty]$), we write $|\cdot|_p$, where $|\cdot|_2=|\cdot|$. For the distance between a set and a point, write $\operatorname{dist}(x,A)=\inf_{y\in A}|x-y|$ and for two sets $\operatorname{dist}(B,A)=\inf_{x\in B}\operatorname{dist}(x,A)$. We see $\mathbb{Z}^d$ as a subset of $\mathbb{R}^d$ and thus the same notation is used on the lattice.

X. *Landau Symbols*: given two $\mathbb{R}^d$ valued functions f and g and a point y, we write $f=o(g)$ if for all $\varepsilon>0$ we have $|f(x)|\le\varepsilon|g(x)|$ in a neighbourhood of y. We write $f=\mathcal{O}(g)$ if $\limsup_{x\to y}|f(x)|/|g(x)|\le C$ for some $C>0$. We use the same notation for the limit $|x|\to+\infty$.

For spheres we use the following notation: for $x\in\mathbb{R}^d$ and $r>0$ we write

$$\boldsymbol{B}_r(x)=\{y\in\mathbb{R}^d\colon |x-y|\le r\}\,. \tag{2.1.4}$$

If $x=0$, we omit it from the notation, i.e. $\boldsymbol{B}_r(0)=\boldsymbol{B}_r$. If we are working on $\mathbb{Z}^d$, we use the same notation: in that case $\boldsymbol{B}_r(x)$ is understood as $\{y\in\mathbb{Z}^d\colon |x-y|\le r\}$. Note that care must be taken when considering the discrete ball in $\mathbb{Z}^d$: it is no longer rotationally invariant, see Figure 2.1.

2.1.1 Path spaces

For a metric space E (assumed to be separable and complete) with metric $\mathtt{d}$ (think of $\mathbb{Z}^d$ or $\mathbb{R}^d$ equipped with the Euclidean distance $|\cdot|$), we define various path spaces on which our stochastic processes live. We add an extra symbol to E, denoted by $\dagger$, and set $\mathtt{d}(x,\dagger)=\mathbb{1}\{x\neq\dagger\}$. For any $t\ge 0$ let

$$\mathcal{D}_t(E)=\{\omega\colon[0,t]\to E\cup\{\dagger\},\text{ with }\omega\text{ right continuous with left limits}\}. \tag{2.1.5}$$

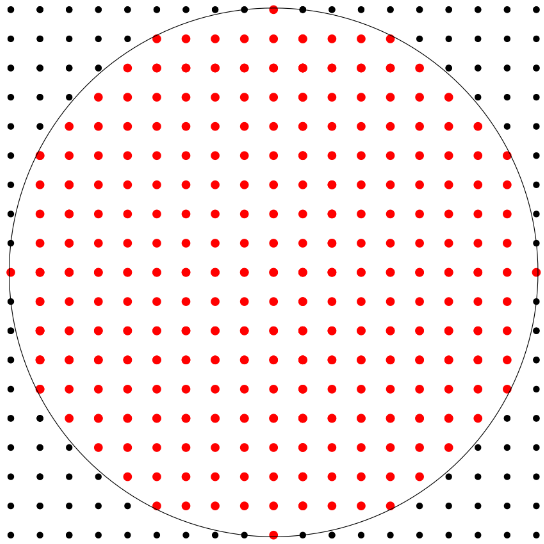

Figure 2.1: The points in the discrete ball $\boldsymbol{B}_{10} \subset \mathbb{Z}^2$ in red. Note the missing rotational invariance.

If the space E is apparent from the context, we omit it from the notation and simply write $\mathcal{D}_t$. The same applies to all subsequently defined spaces. Following [Bil68, Section 12], we define a metric $\mathtt{d}_t$ on $\mathcal{D}_t$ by first introducing a functional F. The functional F acts on non-decreasing functions g, satisfying $g(0) = t - g(t) = 0$, with

$$F(g) = \sup_{0<s_1<s_2<t} \left| \log \frac{\mathtt{d}\,(g(s_2), g(s_1))}{s_2 - s_1} \right| . \tag{2.1.6}$$

Thus, F takes values in $(0, +\infty]$. We define our metric

$$\mathtt{d}_t(\omega_1, \omega_2) = \inf_g \max\{F(g), \sup_{0\le s\le t} \mathtt{d}(\omega_1(s), \omega_2 \circ g(s))\}\,, \tag{2.1.7}$$

where the infimum is taken over those functions g for which we previously defined F. Denote furthermore

$$\mathcal{D} = \{\omega\colon [0,\infty) \to E \cup \{\dagger\},\ \text{ such that } \omega \text{ is right continuous with left limits}\}\,. \tag{2.1.8}$$

Define for $\omega_1, \omega_2 \in \mathcal{D}$,

$$\mathtt{d}_\infty(\omega_1,\omega_2) = \sum_{m=1}^{\infty} 2^{-m} \mathtt{d}_m(\omega_1,\omega_2)\,. \tag{2.1.9}$$

By [Bil68, Theorem 12.2 and 16.3] we have that $\mathcal{D}_t$ is separable and complete under $\mathtt{d}_t$. The same applies to $\mathcal{D}$ under $\mathtt{d}_\infty$. Denote the topology generated by $\mathtt{d}_t$ and $\mathtt{d}_\infty$ by $\tau(\mathcal{D}_t)$ and $\tau(\mathcal{D})$ respectively. These topologies are usually referred to as *Skorokhod* topologies. Let $\sigma(\mathcal{D}_t)$ and $\sigma(\mathcal{D})$ be the associated Borel sigma-algebras. We write $t-$ for the left limit and $t+$ for the limit from the right, $t \in \mathbb{R}$. Let

$$\Gamma_t = \{\omega \in \mathcal{D}_t \text{ such that } \omega(0) = \omega(t-) \text{ and } \omega(s) \neq \dagger,\ \forall s \in [0,t]\} \subset \mathcal{D}_t\,, \tag{2.1.10}$$

the space of *loops* of length t. We denote the subspace topology by $\tau(\Gamma_t)$ and the (sub) sigma-algebra by $\sigma(\Gamma_t)$. Define

$$\Gamma = \bigcup_{t>0} \Gamma_t\,. \tag{2.1.11}$$

For $\omega \in \Gamma$, define the length $l(\omega)$ as the unique $t > 0$ such that $\omega \in \Gamma_t$. Furthermore, we denote a loop's maximal diameter by

$$\|\omega\| = \sup_{0\le s,t \le l(\omega)} \mathtt{d}\left(\omega(t),\omega(s)\right)\,. \tag{2.1.12}$$

We can embed Γ into $\mathcal{D}$ by setting $\omega(t) = \dagger$ for $t > l(\omega)$. Write $x \in \omega$ if there exists $t \le l(\omega)$ such that $\omega(t) = x$. Henceforth one (unless stated otherwise) identifies Γ with its embedding into $\mathcal{D}$. Denote the topology and the sigma-algebra on Γ generated by this embedding by $\tau(\Gamma)$ and $\sigma(\Gamma)$.
We also define the family of coordinate projections $(X_t)_{X\ge 0}$ in the canonical way: $X_t(\omega) = \omega(t)$, for $\omega \in \mathcal{D}$. We also use the letters B and S instead of X, depending on the reference measure. This will be made clear in the next section.

2.1.2 Random walks on the lattice

In this section, we introduce the reader to the class of random walks used in this text. For this section, we consider the case $E = \mathbb{Z}^d$ (only in Chapter 3 we will have to consider $E \neq \mathbb{Z}^d$ or $E \neq \mathbb{R}^d$).
A generator matrix $q\colon \mathbb{Z}^d \cup \{\dagger\} \times \mathbb{Z}^d \cup \{\dagger\} \to \mathbb{R}$ induces a random walk. It has the following properties:

I. $q(x,y) \geq 0$ for all $x \neq y \in \mathbb{Z}^d \cup \{\dagger\}$.

II. For all $x \in \mathbb{Z}^d \cup \{\dagger\}$

$$\infty > \sum_{y \in \mathbb{Z}^d \cup \{\dagger\}} q(x,y) = -q(x,x)\,. \tag{2.1.13}$$

III. $\|q\|_\infty = \sup_{x \in \mathbb{Z}^d \cup \{\dagger\}} |q(x,x)| < \infty$.

Set $p = \|q\|_\infty^{-1} q + \mathsf{I}$, the (one-step) transition matrix. Henceforth assume that $q(x,x)$ is constant with respect to x. Apart from the space-time random walk to be defined in the next chapter, we always assume that $\|q\|_\infty = 1$. By [Kle13, Theorem 17.25], the matrix q uniquely defines a continuous-time Markov process whose coordinate projections we denote by X_t, $t \geq 0$. We refer to $(X_t)_t$ as *continuous-time random walk*. Its transition kernel is denoted by $\overline{p}_t(x,y)$ for $x, y \in \mathbb{Z}^d \cup \{\dagger\}$, $t \geq 0$, and satisfies

$$\partial_t \overline{p}_t(x,y)\Big|_{t=0} = q(x,y)\,. \tag{2.1.14}$$

Since we are going to think of $\dagger$ as a cemetery state, we require that $-q(\dagger, y) = \mathbb{1}\{y = \dagger\}$. The next assumption is key and therefore stated separately.

Assumption 2.1.1. *Suppose that there exist* $p^{(1)}\colon \mathbb{Z} \to [0,1]$ *such that*

I. Summability: $\sum_{x \in \mathbb{Z}} p^{(1)}(x) = 1$.

II. Symmetry: $p^{(1)}(x) = p^{(1)}(-x)$.

III. Interval-like support: let

$$I = \{x \in \mathbb{Z}\colon p^{(1)}(x) > 0\} = (\alpha - 1, -\alpha + 1) \cap \mathbb{Z}\,, \tag{2.1.15}$$

for some $\alpha \in \{-1, -2, \ldots\} \cup \{-\infty\}$.
Alternatively to the interval-like support, we may assume that $p^{(1)}(-1) = p^{(1)}(1) = 1/2$.

IV. *Square-exponential decay:*

$$p^{(1)}(x) = \mathcal{O}\left(\mathrm{e}^{-c|x|^2}\right), \tag{2.1.16}$$

for some $c > 0$ *as* $|x| \to \infty$.

V. *If the support of* $p^{(1)}$ *is non-compact, we require*

$$p^{(1)}(x) \geq p^{(1)}(x-1)p^{(1)}(x+1), \tag{2.1.17}$$

for any $x \in \mathbb{Z}$ *(this is equivalent to the distribution being strongly unimodal, see [DW19]). Furthermore, assume* $t \mapsto \sum_x p^{(1)}(x)\mathrm{e}^{tx}$ *is lower semi-continuous.*

Let $e_1, \ldots, e_d \in \mathbb{Z}^d$ *be the standard basis vectors in* $\mathbb{Z}^d$ *(over* $\mathbb{Z}$*) and denote* $(x)_i$ *the* i*-th coordinate of* $x \in \mathbb{Z}^d$ *(i.e. the projection of* x *onto the space spanned by* e_i*). We assume that for* $x \neq y \in \mathbb{Z}^d$,

$$q(x,y) = \frac{p^{(1)}\left((x-y)_i\right)}{d-(d-1)p^{(1)}(0)}\mathbb{1}\{(x-y)_i = 0 \text{ for all but at most one } i\}, \tag{2.1.18}$$

and $q(x,x) = 1$ *otherwise. In words, at each step the random walk chooses a direction* $i \in \{1, \ldots, d\}$ *uniformly at random and then moves in that direction distributed accordingly to* $p^{(1)}$*. For an illustration, see Figure 2.2.*

Remark 2.1.2. I. *Note that the above assumptions imply that the jump chain induced by* q *is aperiodic over* $\mathbb{Z}^d$ *(unless* $p^{(1)}(-1) = p^{(1)}(1) = 1/2$*). Indeed, the interval-like support ensures that* $p^{(1)}$ *is aperiodic and Equation 2.1.18 implies that this carries over to* $q(x,y)$.

II. *The "independence" assumption (i.e. Equation* (2.1.18)*) is only due to*

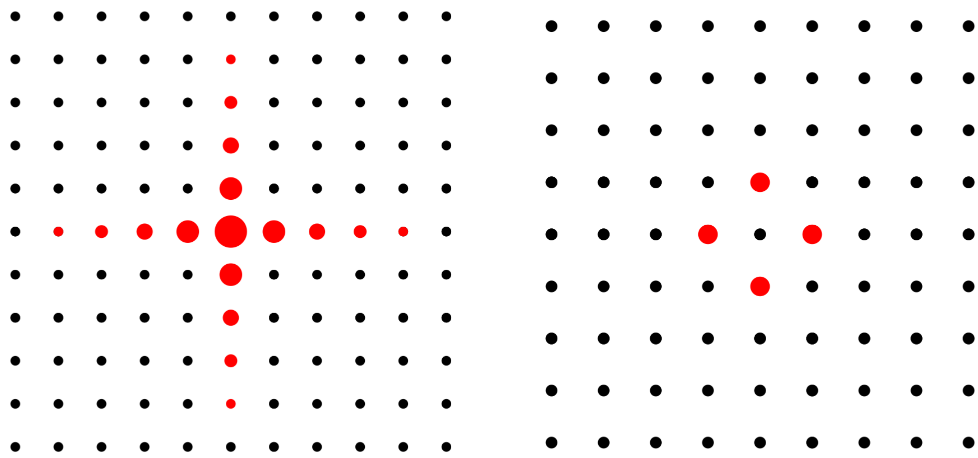

Figure 2.2: Two possibilities for the support of $q(x,y)$ (in red) on $\mathbb{Z}^2$.

the fact that the recently proved KMT coupling[1] *for the random walk bridge (see [DW19]) has not been generalised to higher dimensions yet. We expect the results in this work to hold for all random walks with finite exponential moments.*

The measure associated to $(X_t)_{t\geq 0}$ starting at $x \in \mathbb{Z}^d$ is denoted by $\overline{\mathbb{P}}_x$. The *jump-chain*[2] associated to $(X_t)_{t\geq 0}$ is denoted by $(S_n)_{n\in\mathbb{N}}$. The kernel of the jump-chain is denoted by $p_n(x,y)$ for $x,y \in \mathbb{Z}^d \cup \{\dagger\}$, $n \in \mathbb{N}$. We denote the measure governing the discrete jump-chain started in x by $\mathbb{P}_x$.
Let $t \in [0,\infty)$ and $j \in \mathbb{N}$. We define for $G \in \sigma(\mathcal{D})$ and $x,y \in \mathbb{Z}^d$ the bridge measures in continuous and discrete time

$$\overline{\mathbb{P}}^t_{x,y}(G) = \overline{\mathbb{P}}_x(G \cap \{X_t = y\}) \quad \text{and} \quad \mathbb{P}^j_{x,y}(G) = \mathbb{P}_x(G \cap \{S_j = y\})\,. \quad (2.1.19)$$

We furthermore denote the normalised version of the above measures as

$$\overline{\mathbb{B}}^t_{x,y}(G) = \overline{\mathbb{P}}_x(G | X_t = y) \quad \text{and} \quad \mathbb{B}^j_{x,y}(G) = \mathbb{P}_x(G | S_j = y)\,. \quad (2.1.20)$$

[1] A coupling which produces an error at scale $\log(n)$ over a time horizon n, named after Komlós–Major–Tusnády, see [KMT75].

[2] The discrete time random walk induced by p.

We also write $\Lambda \Subset \mathbb{Z}^d$ if $\Lambda \subset \mathbb{Z}^d$ and contains finitely many points.
For a set $A \subset \mathbb{Z}^d$, we define the *inner boundary*

$$\partial_i A = \{x \in A \colon \exists\, y \notin A \text{ with } |x-y| = 1\}\,. \tag{2.1.21}$$

If $A \subset \mathbb{R}^d$, we write ∂A for its boundary in the topological sense (with respect to any norm on $\mathbb{R}^d$).
We use the Brownian motion and its kernel. Denote a standard Brownian motion (in d dimensions, with the same covariance as the random walk) by $(B_t)_{t\geq 0}$ and write P_x for its distribution (started at $x \in \mathbb{R}^d$). The transition kernel of the Brownian motion is denoted by $\mathfrak{p}_t(x,y)$. As the kernel $\mathfrak{p}_t(x,y)$ is translation invariant, we occasionally write $\mathfrak{p}_t(x-y)$ for $\mathfrak{p}_t(y,x)$. For $r \geq 0$ we occasionally write $\mathfrak{p}_t(r)$ instead of $\mathfrak{p}_t(x_r)$, where x_r is any point in $\mathbb{R}^d$ satisfying $|x_r| = r$. The measure of the Brownian bridge transitioning from x to y in time $t \geq 0$ is denoted by $B^t_{x,y}$. We also use the unnormalised bridge measure: $P^t_{x,y} = \mathfrak{p}_t(x,y) B^t_{x,y}$.
As a rule of thumb, boldface notation refers to discrete objects whereas standard and fraktur typeface indicates continuous processes.
A word on densities: for continuous (on $\mathbb{R}^d$) processes (such as the Brownian motion/bridge) we denote densities by adding the letter d before the measure, i.e. $\mathrm{d}P_x(B_t = y)$ is understood as the unique function satisfying

$$P_x(B_t \in A) = \int_A \mathrm{d}P_x(B_t = y)\mathrm{d}y\,, \tag{2.1.22}$$

for every measurable $A \subset \mathbb{R}^d$.

2.2 Hitting time estimates

This section is devoted to estimating the distribution of certain hitting times of our random walk. These technical estimates will be of importance in later chapters, in particular Chapter 4 and Chapter 5. We first prove a coupling between the random walk bridge and the Brownian bridge. We then use this to show that the distribution of the hitting times for the random walk is close to those of the Brownian bridge. In this section we always assume that

$q(x, \dagger) = 0$.
The hitting time H_A of a set $A \subset \mathbb{Z}^d$ is defined in the following way

$$H_A = \inf\{k \geq 1 \colon S_k \in A\}\,. \tag{2.2.1}$$

For the continuous time random walk $(X_t)_t$ and the Brownian $(B_t)_t$, we define the hitting time analogously: replace $k \geq 1$ with $k > 0$ in the above equation. We use the superscript "B" when we refer to the hitting time of a set with respect to the Brownian motion, e.g. H_A^B (instead of H_A). If $A = \{x\}$, we write H_x instead of $H_{\{x\}}$. If $A = \boldsymbol{B}_m$, we use the following convention: if the random walk is started from $\boldsymbol{B}_m \setminus \partial_i \boldsymbol{B}_m$, we set H_m to be the first time we hit $\boldsymbol{B}_m^c$. If the random walk is started from any other point, we set H_m the first hitting time of $\boldsymbol{B}_m$. This means that if the random walk is started from inside the sphere, H_m is the first time it exits it. If the random walk is started from outside, H_m is the first time we hit the sphere. This simplifies notation in later chapters.
We begin by stating a coupling result, a consequence of the one-dimensional version established in [DW19].

Lemma 2.2.1. *For every $\alpha > 0$, there exists $c_\alpha > 0$ such that for $n \in \mathbb{N}$ (if the underlying random walk is the simple random walk, we need to assume n even) one can construct a coupling b^n between $\mathbb{B}^n_{0,0}$ (the random walk bridge of n steps) and $B^n_{0,0}$ (Brownian bridge of duration n) satisfying*

$$b^n\left(\sup_{0 \leq t \leq n} |S_t - B_t| \geq c_\alpha \log^2(n)\right) \leq \mathcal{O}\left(n^{-\alpha}\right)\,. \tag{2.2.2}$$

The same holds for the continuous-time random walk bridge.

Proof of Lemma 2.2.1. We prove the result for the discrete-time random walk, the continuous-time case follows analogously.
First the main ideas: let $M_n(i)$ be the number of times the random walk has chosen direction e_i. We begin by sampling $(M_n(i))_{i=1}^d$ first. We then couple each one-dimensional bridge of time-horizon $M_n(i)$ with a Brownian bridge of time-horizon $dM_n(i)$ (to adjust for covariance). We then use a large deviation-type bound to show that $dM_n(i) = n + \texttt{small}$. In the final step, we perform

a time change to get a Brownian bridge of time-horizon n and then use a continuity estimate to control the error.

We assume without loss of generality that the random walk has the identity

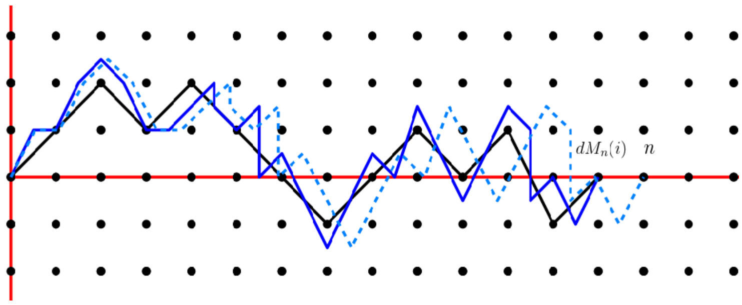

Figure 2.3: The coupling from Lemma 2.2.1: the random walk bridge in black, the continuous approximation in blue, with the time changed version in dashed style.

as covariance matrix. We begin by rewriting our random walk $(S_n)_n$ as

$$S_n = \sum_{i=1}^{d} e_i S^{(i)}_{M_n(i)}\,, \tag{2.2.3}$$

where the $S^{(i)}$'s are independent one-dimensional random walks (distributed with respect to $p^{(1)}$). Furthermore, $M_n \in \mathbb{N}^d$ (the coordinate process) is defined in the following way

$$M_n = \sum_{j=1}^{n} D_j\,, \tag{2.2.4}$$

where D_i are i.i.d. uniform on the standard basis $\{e_i\}_{i=1}^d$. For $C > 0$, let A_n be the event that the coordinate process is behaving atypically, i.e.

$$A_n = \{\exists i \in \{1,\dots,d\} \colon M_n(i) \notin [n/d - C\log(n), n/d + C\log(n)]\}\,. \tag{2.2.5}$$

A standard large deviation estimate shows that for any $\alpha > 0$, there is $C_1 > 0$ large enough, such that $\mathbb{P}(A_n) = o\left(n^{-\alpha}\right)$. Thus, we can now assume that we are on the event A_n^c.

Note that since the e_i's form a basis, we have that

$$S_n = 0 \iff \forall i \in \{1,\dots,d\} \colon S^{(i)}_{M_n(i)} = 0\,. \tag{2.2.6}$$

By [DW19, Theorem 1.2] (or [CD18, Theorem 8.1] in the case of the simple random walk), conditioned on $S_n = 0$, we can couple each of the $S^{(i)}_{M_t(i)}$'s with $B^{(i),M}_t$, a one-dimensional bridge of time-horizon $dM_n(i)$, such that on an event of mass at least $1 - o\left(n^{-\alpha}\right)$, the error is at most $c_\alpha \log M_n(i)$. Write

$$\beta_t = \sum_{i=1}^{d} e_i B^{(i),M}_t \,. \tag{2.2.7}$$

Note that by the scaling invariance of the Brownian motion we have that

$$B_t = \sum_{i=1}^{d} e_i \sqrt{\frac{n}{dM_n(i)}} B^{(i),M}_{t\frac{dM_n(i)}{n}} \,, \tag{2.2.8}$$

is distributed like a Brownian bridge on $[0, n]$. Since we conditioned to be on A_n^c, we have that

$$\left|\frac{n}{dM_n(i)} - 1\right| \le C_1 \frac{\log(n)}{n} \,. \tag{2.2.9}$$

From continuity estimates (see e.g. [MP10, Chapter 1]) it then follows that, outside a set of probability $o(n^{-\alpha})$, we have

$$\sup_{t=0,1,\ldots,n} |\beta_t - B_t| \le \log^2(n) \,. \tag{2.2.10}$$

Indeed, by the Markov inequality, we have that the probability that a centred Gaussian random variable with variance $\mathcal{O}(n^{-1}\log n)$ exceeds $\log^2(n)$ is bounded by $\mathcal{O}\left(\mathrm{e}^{-n\log^{3/2}(n)}\right)$. As there are at most $2C_1 dn \log(n)$ choices, the probability of the complement of the event in Equation (2.2.10) decays at an exponential scale.
Together with the triangle inequality the result follows. □

2.2.1 Hitting of a single point

The main result of is a technical estimate on the time it takes a random walk bridge to hit a distant point.

Lemma 2.2.2. *We have that for*

$$|x| = o\left((j-k)^{3/4}\right) \text{ and } |x| = o(k^{2/3}) \,, \tag{2.2.11}$$

that it holds for any $\varepsilon > 0$ and some constants $c_d > 0$,

$$\mathbb{P}^j_{x,x}(H_0 = k) = \begin{cases} c_1|x|k^{-1}\mathfrak{p}_k(x)\mathfrak{p}_{j-k}(x)\left(1+o(1)\right) + o(|x|^{-5+\varepsilon}) & d=1, \\ c_2\frac{\log(|x|)}{\log^2(k)}\mathfrak{p}_k(x)\mathfrak{p}_{j-k}(x)(1+o(1)) + k^{-1}\log^{-1}(k)o(1\wedge k^{1/2}|x|^{-1}) & d=2, \\ c_d\mathfrak{p}_k(x)\mathfrak{p}_{j-k}(x)(1+o(1)) & d\geq 3, \end{cases} \tag{2.2.12}$$

where in the case $d \geq 3$ we additionally require the existence of an $M > 1$ such that $0 \leq |x| \leq M\sqrt{k}$.
For the cumulative distribution function in one variable, we have for $M > 0$ fixed, $d \geq 3$ and $Mj \geq |x|^2$

$$\begin{aligned} \mathbb{P}^j_{x,x}\left(H_0 < j\right) =& \kappa_d j^{-d+1}\int_{1/M}^1 \mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)\mathrm{d}k\left(1+o(1)\right) \\ &+ \mathcal{O}\left(j^{-d+1}\int_0^{1/M}\mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)\mathrm{d}k\right), \end{aligned} \tag{2.2.13}$$

where $\kappa_d = \mathbb{P}_0(H_0 = \infty)$.
In the case $d = 2$, we have that for every $\rho \in (0,2)$,

$$\begin{aligned} \mathbb{P}^j_{x,x}\left(H_0 < j\right) =& \frac{4\pi\log|x|}{j}\int_{|x|^2/j\log(|x|^{2-\rho})}^1 \frac{\mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)}{\log^2(kj)}\mathrm{d}k\left(1+o(1)\right) \\ &+ \mathcal{O}\left(j^{-1}\int_0^{|x|^2/j\log(|x|^{2-\rho})}\mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)\mathrm{d}k\right). \end{aligned} \tag{2.2.14}$$

For $d \geq 2$, we also have the following bound

$$\mathbb{B}^j_{x,x}\left(H_0 < j\right) \leq C|x|^{2-d}\Gamma\left(d/2-1, 4|x|^2j^{-1}\right). \tag{2.2.15}$$

Proof of Lemma 2.2.2. Use the strong Markov property to write

$$\mathbb{P}^j_{x,x}\left(H_0 = k\right) = \mathbb{P}_x\left(H_0 = k\right)p_{j-k}(x). \tag{2.2.16}$$

The first part of the lemma follows immediately from [Uch11, Theorem 1.2, 1.4, 1.7]. Indeed, in that reference it is shown that

$$\begin{aligned}
&\mathbb{P}_x\left(H_0 = k\right) \\
&= c_d\left(\mathbb{1}\{d \geq 3\} + \frac{1 \vee \log|x|}{\log^2(k)}\mathbb{1}\{d = 2\} + |x|k^{-1}\mathbb{1}\{d = 1\}\right)\mathfrak{p}_k(x)\left(1 + \mathtt{E}\right),
\end{aligned} \tag{2.2.17}$$

where $\mathtt{E}$ are the Landau-symbols from Equation (2.2.12). Due to the assumptions we made on the decay of the tails of the random walk, we can employ [LL10, Theorem 2.3.11] to approximate $p_{j-k}(x)$ by $\mathfrak{p}_{j-k}(x)\left(1 + o(1)\right)$.
We now prove the second part of the lemma. Let us begin with $d \geq 3$. We expand

$$\mathbb{P}^j_{x,x}(H_0 < j) = \sum_{k=1}^{j} \mathbb{P}_x(H_0 = k)p_{j-k}(x)\,. \tag{2.2.18}$$

In the case that $k \geq |x|^2/M = j_0$ for some $M > 1$, we expand

$$\sum_{k=j_0}^{j} \mathbb{P}_x(H_0 = k)p_{j-k}(x) = \kappa_d\left(1 + \mathcal{O}(|x|^{2-d})\right)\sum_{k=j_0}^{j} \mathfrak{p}_k(x)p_{j-k}(x)\,, \tag{2.2.19}$$

by [Uch11, Theorem 1.7]. We can approximate the sum by an integral and thus

$$\begin{aligned}
\sum_{k=j_0}^{j}\mathfrak{p}_k(x)p_{j-k}(x) &= (1 + o(1))\int_{j_0}^{j} \mathfrak{p}_k(x)\mathfrak{p}_{j-k}(x)\mathrm{d}k \\
&= (1 + o(1))j^{-d+1}\int_{1/M}^{1} \mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)\mathrm{d}k\,.
\end{aligned} \tag{2.2.20}$$

Indeed, in the proof of Proposition 4.2.2, the approximation of a sum by an integral at cost of $(1 + o(1))$ is shown in a more general setting by computing second derivatives and using the approximation result [LL10, Lemma A.1.1]. Bounding $\mathbb{P}_x(H_0 = k) \leq p_k(x)$, we can estimate using Lemma 8.2.1 to bound the sum by an integral and a change of variables $k \mapsto kj$

$$\sum_{k=1}^{j_0} \mathbb{P}_x(H_0 = k) p_{j-k}(x) \leq C j^{-d+1} \int_0^{1/M} \mathfrak{p}_k \left(x/\sqrt{j}\right) \mathfrak{p}_{1-k} \left(x/\sqrt{j}\right) \mathrm{d}k \,. \tag{2.2.21}$$

Combining the two previous equations finishes the proof in the $d \geq 3$ case. In the case that $d = 2$, we have for $|x| \leq \sqrt{3k \log\log(k)} = p(k)$ that by [Uch11, Theorem 1.4],

$$\mathbb{P}_x(H_0 = k) = \frac{2\log(|x|)}{k \log^2(k)} \mathrm{e}^{-|x|^2/(2k)} \left(1 + \mathcal{O}\left(\frac{1}{\log^{1/4}(k)}\right)\right) . \tag{2.2.22}$$

Let $q(x)$ be the inverse function of $p(k)$. Plugging in the above then gives us

$$\sum_{k=q(x)}^{j} \mathbb{P}_x(H_0 = k) p_{j-k}(x) = (1 + \mathcal{O}\left(r(x)\right)) \sum_{k=q(x)}^{j} \frac{4\pi \log(|x|)}{\log^2(k)} \mathfrak{p}_k(k) \mathfrak{p}_{j-k}(x) \,, \tag{2.2.23}$$

where $r(x) = \log^{-1/4}(|x|)$. Note the bound $q(x) \leq 3|x|^2/8 \log\log(|x|)$. We can use [Uch11, Theorem 1.5] in conjunction with [Uch15, Theorem 2] to get approximations on $\mathbb{P}_x(H_0 = k)$ for $k \leq q(x)$: by the first theorem, there exists an explicit constant $r_0 > 0$, depending on the distribution of random walk, such that

$$\mathbb{P}_x(H_0 = k) = \mathrm{d}P_x(H^B_{r_0} = k) + \mathcal{O}\left(\frac{1}{|x^2| k \log(k)}\right) . \tag{2.2.24}$$

By the calculation done in [Uch15, Corollary 4], we have for $\rho > 0$ small enough and $|x|^2/(\log|x|^{2-\rho}) \leq k \leq q(x)$ that

$$\mathrm{d}P_x(H^B_{r_0} = k) = \frac{\pi}{\log(k/|x|)} \mathfrak{p}_k(x) \left(1 + \mathcal{O}\left(\frac{1}{\log|x|}\right)\right) . \tag{2.2.25}$$

Inspecting the error term in Equation (2.2.24) reveals that

$$\begin{aligned} \mathbb{P}_x(H_0 = k) &= \frac{\pi}{\log(k/|x|)} \mathfrak{p}_k(x) \left(1 + \mathcal{O}\left(\frac{1}{\log|x|}\right)\right) \\ &= \frac{4\pi \log|x|}{\log^2(k)} \mathfrak{p}_k(x) \left(1 + \mathcal{O}\left(\frac{1}{\log|x|}\right)\right) , \end{aligned} \tag{2.2.26}$$

for k's satisfying the above bounds. From there on can approximate the sum by an integral and proceed as in the case $d \geq 3$.
Bound: we now prove the last claim of this lemma. Let $d \geq 2$ and bound

$$\mathbb{P}^j_{x,x}(H_0 < j) \leq \sum_{k=1}^{j} \mathbb{P}^j_{x,x}(S_k = 0) \leq C\left(\mathrm{e}^{-c|x|^{2/3-\varepsilon}} + \int_0^j \mathfrak{p}_t(x)\mathfrak{p}_{1-t}(x)\mathrm{d}t\right), \tag{2.2.27}$$

where we use Lemma 8.2.1 in conjunction with Lemma 8.2.2 to approximate the sum by an integral and [LL10, Proposition 2.1.2] to bound the summand for small values of j. We apply a change of variables $t \mapsto jt$ to get

$$j^{d/2}\int_0^j \mathfrak{p}_t(x)\mathfrak{p}_{1-t}(x)\mathrm{d}t = j^{-d/2+1}\int_0^1 t^{-d/2}(1-t)^{-d/2}\mathrm{e}^{-\frac{|x|^2}{2j}\left(\frac{1}{t}+\frac{1}{1-t}\right)}\mathrm{d}t\,. \tag{2.2.28}$$

We bound the integral (using symmetry around the midpoint)

$$\int_0^1 t^{-d/2}(1-t)^{-d/2}\mathrm{e}^{-\frac{|x|^2}{4j}\left(\frac{1}{t}+\frac{1}{1-t}\right)}\mathrm{d}t \leq 2^{d/2+1}\int_0^{1/2} t^{-d/2}\mathrm{e}^{-\frac{|x|^2}{4tj}}\mathrm{d}t\,. \tag{2.2.29}$$

After performing a change of variables $t \mapsto t^{-1}|x|^2 j^{-1}$, we recognise the above as the incomplete Gamma function. Combining this with the previous estimates, we get that

$$\mathbb{B}^j_{x,x}(H_0 < j) \leq C\left(j^{d/2}\mathrm{e}^{-c|x|^{2/3-\varepsilon}} + |x|^{2-d}\Gamma\left(d/2-1, 4|x|^2 j^{-1}\right)\right), \tag{2.2.30}$$

from which the desired estimate follows. □

2.2.2 Hitting a sphere from inside

In this section we approximate the distribution of the time H_n it takes the random walk bridge, started at 0, to leave a ball of radius n. We use a classical result on the first hitting time of Bessel processes and the coupling from Lemma 2.2.1. As (for certain indices) Bessel processes have the same distribution as the Euclidean norm of Brownian motions, their appearance is natural here.
Before stating the next lemma, we recall the following fact from [LL10, Propo-

sition 2.4.5]: there exists a $C > 0$ such that for all $j > 0$

$$\mathbb{P}_0(H_n < j) + P_0\left(H_n^B < j\right) \leq C^{-1}\mathrm{e}^{-Cn^2/j}\,. \tag{2.2.31}$$

If one applies this for $j = n^2/c\log(n)$, the right-hand side of the equation above decays at polynomial speed, depending on c.

Lemma 2.2.3. *I. For $n^2 > j \geq n^{3/2}$ there exists $C > 0$,*

$$\mathbb{B}_{0,0}^j(H_n < j) \leq C^{-1}\mathrm{e}^{-Cn^2/j}\,. \tag{2.2.32}$$

II. For every $M > 0$, $T > 0$, $S \in (0,1)$, $n^{3-S} > j \geq \frac{n^2}{\log(n^M)}$ and $n \in \mathbb{N}$ large enough, the Brownian approximation reads as follows

$$\begin{aligned}\mathbb{B}_{0,0}^j(H_n < j) &= B_{0,0}^j(H_n^B < j)\left(1 + \mathcal{O}(n^{-S/2})\right) + \mathcal{O}\left(n^{-T}\right)\\ &= \int_0^j \frac{\mathfrak{p}_{j-t}(n)}{\mathfrak{p}_j(0)} \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu+1}\mathrm{e}^{-j_{\nu,k}^2 t/(2n^2)}}{z^2 2^{\nu}\Gamma(\nu+1)J_{\nu+1}(j_{\nu,k})}\mathrm{d}t\left(1 + \mathcal{O}(n^{-S/2})\right) + \mathcal{O}\left(n^{-T}\right),\end{aligned} \tag{2.2.33}$$

where $J_\nu(x)$ is the Bessel function (of the first kind) of $\nu-$th order with $j_{\nu,k}$ its strictly positive zeros, in increasing order (here, $\nu = d/2 - 1$). Note that summation and integration are not exchangeable here, this is shown in the proof.

III. Furthermore, for any $\varepsilon > 0$ and n large enough, we have

$$\inf_{j \geq \varepsilon n^2} \mathbb{B}_{0,0}^j(H_n < j) > 0\,. \tag{2.2.34}$$

Proof of Lemma 2.2.3.
Proof of I: by [LL10, Proposition 2.4.5], we can bound

$$\mathbb{P}_0(H_n < k) \leq \min\left\{c\mathrm{e}^{-r(n^2/k)},\, 1\right\}, \tag{2.2.35}$$

for some $r, c > 0$ and $k \in \mathbb{N}$. We expand using the local central limit theorem

$$\mathbb{B}^j_{0,0}(H_n < j) \sim j^{d/2} \sum_{k=n^{4/3}}^{j-n^{4/3}} \mathbb{P}_0(H_n = k) \underbrace{\mathfrak{p}_{j-k}(ne_1)}_{=\mathfrak{p}_{j-k}(n)}, \tag{2.2.36}$$

where e_1 is the unit vector into the first direction. We excluded k's in $\{1, \dots, n^{4/3}\}$ and $\{j - n^{4/3}, \dots, j\}$, as they contribute at most an exponential factor. We bound the above using integration by parts from Lemma 8.2.3

$$\begin{aligned} &j^{d/2} \sum_{k=n^{4/3}}^{j-n^{4/3}} \mathbb{P}_0(H_n = k)\mathfrak{p}_{j-k}(n) \\ &\quad \le j^{d/2} \sum_{k=n^{4/3}-1}^{j-n^{4/3}} \mathbb{P}_0(H_n \le k) \left[\mathfrak{p}_{j-k}(n) - \mathfrak{p}_{j-k-1}(n)\right] + E\,, \end{aligned} \tag{2.2.37}$$

with

$$E = j^{d/2} \left[\mathbb{P}_x(H_n \le n^{4/3} - 1)p_{j-n^{4/3}}(n) - \mathbb{P}_x(H_n \le j - n^{4/3})p_{n^{4/3}}(n)\right] . \tag{2.2.38}$$

Note that by the mean value theorem for $k \in \{n^{4/3}, \dots, j - n^{4/3}\}$, we can find a $C > 0$ such that

$$\left[\mathfrak{p}_{j-k}(n) - \mathfrak{p}_{j-k-1}(n)\right] \sim -C\partial_t \mathfrak{p}_t(n)\Big|_{t=j-k} . \tag{2.2.39}$$

For $k \in \{n^{4/3}, \dots, j - n^{4/3}\}$, we can find a $C, c > 0$ such that

$$\partial_t \mathfrak{p}_t(n)\Big|_{t=j-k} \le C \frac{\mathrm{e}^{-cn^2/(j-k)}}{(j-k)^{d/2+1}} . \tag{2.2.40}$$

Using Lemma 8.2.1 and 8.2.2 to bound the sum by an integral, we bound

$$\mathbb{B}^j_{0,0}(H_n < j) \le E + Cj^{d/2} \int_{n^{4/3}}^{j-n^{4/3}} \frac{\mathrm{e}^{-cn^2/(j-k)}}{(j-k)^{d/2+1}} \mathrm{e}^{-r(n^2/k)} \mathrm{d}k . \tag{2.2.41}$$

We simplify this further by changing variables $k \mapsto kj$ and altering the boundaries of integration

$$j^{d/2}\int_{n^{4/3}}^{j-n^{4/3}} \frac{\mathrm{e}^{-cn^2/(j-k)}}{(j-k)^{d/2+1}}\mathrm{e}^{-r(n^2/k)}\mathrm{d}k \le \int_0^1 \frac{\mathrm{e}^{-cn^2/j(1-k)}}{(1-k)^{d/2+1}}\mathrm{e}^{-r(n^2/jk)}\mathrm{d}k\,. \quad (2.2.42)$$

Expand for $\varepsilon > 0$

$$\int_0^1 \frac{\mathrm{e}^{-cn^2/j(1-k)}}{(1-k)^{d/2+1}}\mathrm{e}^{-r(n^2/jk)}\mathrm{d}k = \mathrm{e}^{-\varepsilon n^2/j}\int_0^1 \frac{\mathrm{e}^{-cn^2/j(1-k)+\varepsilon n^2/j}}{(1-k)^{d/2+1}}\mathrm{e}^{-r(n^2/jk)}\mathrm{d}k\,. \quad (2.2.43)$$

Now observe that for $\varepsilon > 0$ small enough

$$\sup_{j,n}\int_0^1 \frac{\exp\left(-\frac{n^2}{j}\left(\frac{c}{1-k}+\frac{r}{k}-\varepsilon\right)\right)}{(1-k)^{d/2+1}}\mathrm{d}k < \infty\,, \quad (2.2.44)$$

where the supremum is over all j, n's with $j^2 > n$. The boundary term E is of exponential order and can be absorbed into the main contribution. This concludes proof of the statement I.

Proof of II: for the second claim we use the coupling from Lemma 2.2.1 in conjunction with the explicit formula for the hitting time of Bessel processes. Several approximations will be necessary: since the coupling induces an error in space, we have to show that this error remains negligible for contributing loop of lengths j.

Let $\mathsf{P}^j_{0,0}$ be a coupling between $\mathbb{B}^j_{0,0}$ and $B^j_{0,0}$ such that

$$\mathsf{P}^j_{0,0}\Big(\max_{1\le i\le j}|S_i - B_i| \ge c_t \log^2(j)\Big) \le \frac{C}{j^t}\,, \quad (2.2.45)$$

with $c_t > 0$ increasing in $t > 0$. We rewrite

$$B^j_{0,0}(H^B_{n^+} < j) \le \mathbb{B}^j_{0,0}(H_n < j) + \mathcal{O}\left(j^{-t}\right) \le B^j_{0,0}(H^B_{n^-} < j)\,. \quad (2.2.46)$$

Next, we will show that in the above formula, n^- can be replaced by n at negligible cost. For this, we need to exclude certain atypical events.

Using the Markov property and the rotational invariance of the Brownian motion we get that

$$B^j_{0,0}(H^B_{n^-} < j) = (2\pi j)^d \int_0^j \mathrm{d}P_0(H^B_{n^-} = r)\mathfrak{p}_{j-r}(n^-)\mathrm{d}r\,. \tag{2.2.47}$$

The Brownian motion needs time of order $\sim n^2$ to reach the complement of $\boldsymbol{B}_n$ and thus, for some $d_1 > 0$ large enough, the integral over atypical times is of order

$$(2\pi j)^d \int_{[0,n_1]\cap[j-n_1,n_1]} \mathrm{d}P_0(H^B_{n^-} = r)\mathfrak{p}_{j-r}(n^-)\mathrm{d}r = \mathcal{O}\left(j^{-t}\right)\,, \tag{2.2.48}$$

where $n_1 = n^2/(d_1 \log(n))$.
By [HM13] we have that for $z, r > 0$ and $\nu = d/2 - 1$

$$\mathrm{d}P_0(H^B_z = r) = \sum_{k=1}^{\infty} \frac{j^{\nu+1}_{\nu,k}}{z^2 2^\nu \Gamma(\nu+1) J_{\nu+1}(j_{\nu,k})} \mathrm{e}^{-j^2_{\nu,k} r/(2z^2)}\,. \tag{2.2.49}$$

Note that we have that by [Zwi18, 6.15.12.1]

$$j_{\nu,k} = \pi k + (\nu/2 - 1/4)\pi - \frac{4\nu^2 - 1}{(8k + 4\nu - 2)\pi} + \mathcal{O}(k^{-2})\,, \tag{2.2.50}$$

and for $x > 0$

$$J_\nu(x) \sim \sqrt{2/(\pi x)}\left[\cos\left(x - \pi\nu/2 - \pi/4\right) + \mathcal{O}(x^{-1})\right]\,. \tag{2.2.51}$$

This implies that

$$J_{\nu+1}(j_{\nu,k}) = (-1)^{k+1}\sqrt{\frac{2}{\pi k}}\left(1 + \mathcal{O}(k^{-1})\right)\,. \tag{2.2.52}$$

For $r > n_1$, we can bound

$$\Big| \sum_{k=1}^{\infty} \frac{j^{\nu+1}_{\nu,k}}{n^2 2^\nu \Gamma(\nu+1) J_{\nu+1}(j_{\nu,k})} \mathrm{e}^{-j^2_{\nu,k} r/(2n^{-2})} \Big| \leq C(n) \sum_{k\geq 1} k^{3/2+\nu} \mathrm{e}^{-k^2/(d_1 \log(n))}\,, \tag{2.2.53}$$

where $C(n)$ depends on n but not r. Thus, by dominated convergence, we can rewrite

$$\int_{n_1}^{j-n_1} \mathrm{d}P_0(H^B_{n^-} = r)\mathfrak{p}_{j-r}(n^-)\mathrm{d}r = \sum_{k=1}^{\infty} \frac{j_{\nu,k}^{\nu+1}}{n^2 2^\nu \Gamma(\nu+1) J_{\nu+1}(j_{\nu,k})} \times \int_{n_1}^{j-n_1} \mathrm{e}^{-j_{\nu,k}^2 r/(2n^{-2})}\mathfrak{p}_{j-r}(n)\mathrm{d}r \left(1 + \mathcal{O}\left(\log(n)/n\right)\right) , \tag{2.2.54}$$

since $r \in [n_1, j-n_1]$. We used that, by the definition of the heat kernel, $\mathfrak{p}_{j-r}(n^-) = \mathfrak{p}_{j-r}(n)(1+\mathcal{O}(\log(n)/n))$ for $r \in [n_1, j-n_1]$.
Using the asymptotics for $j_{\nu,k}$ from Equation (2.2.50), we get that

$$\mathrm{e}^{-j_{\nu,k}^2 r/(2n^{-2})} = \mathrm{e}^{-j_{\nu,k}^2 t/(2n^2)}\left(1 + \mathcal{O}\left(\frac{k^2 r \log^4(n)}{n^3}\right)\right) . \tag{2.2.55}$$

We now show that large k's are negligible and thus the above $\mathcal{O}$-term is sufficiently small for contributing k's. First note that for $k \le n^{S/4}/\log^{1/2}(n) = k_n$ and $r \in [n_1, j-n_1]$, we have that

$$\mathcal{O}\left(\frac{k^2 r \log^4(n)}{n^3}\right) = \mathcal{O}\left(n^{-S/2}\right) , \tag{2.2.56}$$

and thus

$$\mathrm{e}^{-j_{\nu,k}^2 r/(2n^{-2})} = \mathrm{e}^{-j_{\nu,k}^2 r/(2n^2)}\left(1 + \mathcal{O}\left(n^{-S/2}\right)\right) . \tag{2.2.57}$$

Recall that $\Gamma(a,x)$ is the upper incomplete Gamma function (with index a and argument x) and that $\nu = d/2 - 1$. We can estimate the error term

$$\Big|\sum_{k\ge k_n}^{\infty} \frac{j_{\nu,k}^{\nu+1}}{n^2 2^\nu \Gamma(\nu+1) J_{\nu+1}(j_{\nu,k})}\mathrm{e}^{-j_{\nu,k}^2 r/(2n^{-2})}\Big| \le C \sum_{k\ge k_n}^{\infty} \frac{k^{\nu+1/2}}{n^2}\mathrm{e}^{-kr/n^2} \le C\int_{k_n}^{\infty} \mathrm{d}k \frac{k^{\nu+1/2}\mathrm{e}^{-kr/n^2}}{n^2} \le C\frac{n^{d+5}}{r^{d/2+3/2}}\Gamma\left(\frac{d+2}{2}, \frac{n^{S/2}}{\log(n)}\right) \le C\frac{\mathrm{e}^{-Cn^{S/4}}}{r^{d/2+3/2}} . \tag{2.2.58}$$

Integrating the above from n_1 to $j - n_1$ is of order $\mathcal{O}\left(\mathrm{e}^{-Cn^{S/8}}\right)$, which is negligible. This implies that we can neglect k's in Equation (2.2.54) with

$k > k_n$ and thus

$$\begin{aligned}\int_{n_1}^{j-n_1} \mathrm{d}P_0(H_{n^-} = r)\mathfrak{p}_{j-r}(n^-)\mathrm{d}r = \sum_{k=1}^{k_n} \frac{j_{\nu,k}^{\nu+1}}{n^2 2^\nu \Gamma(\nu+1) J_{\nu+1}(j_{\nu,k})} \\ \times \int_{n_1}^{j-n_1} \mathrm{e}^{-j_{\nu,k}^2 r/(2n^2)} \mathfrak{p}_{j-r}(n)\mathrm{d}r \left(1+\mathcal{O}(n^{-S/2})\right) + \mathcal{O}\left(j^{-t}\right) .\end{aligned} \tag{2.2.59}$$

since for $k \le k_n$, we have $\mathrm{e}^{-j_{\nu,k}^2 r/(2n^{-2})} = \mathrm{e}^{-j_{\nu,k}^2 r/(2n^2)}$ up to a multiplicative $\left(1+\mathcal{O}\left(n^{-S/2}\right)\right)$ term.

By arguments similar to the ones above, we can reintroduce terms with $k \ge k_n$ (this time with the correct n instead of n^- in the exponent) and adjust the areas of integration. This gives us

$$\int_{n_1}^{j-n_1} \sum_{k=1}^{k_n} \mathtt{Q} = \mathcal{O}\left(j^{-t}\right) + \int_{n_1}^{j-n_1} \sum_{k=1}^{\infty} \mathtt{Q} = \mathcal{O}\left(j^{-t}\right) + \int_0^j \sum_{k=1}^{\infty} \mathtt{Q}, \tag{2.2.60}$$

with

$$\mathtt{Q} = \frac{j_{\nu,k}^{\nu+1}}{n^2 2^\nu \Gamma(\nu+1) J_{\nu+1}(j_{\nu,k})} \mathrm{e}^{-j_{\nu,k}^2 r/(2n^2)} \mathfrak{p}_{j-r}(n)\mathrm{d}r \left(1+\mathcal{O}(n^{-S/2})\right) . \tag{2.2.61}$$

To summarise (as we could have carried out the above computations using n^+ instead of n^- with no changes), we have shown that

$$\mathbb{B}_{0,0}^j(H_n < j) = B_{0,0}^j(H_n^B < j)\left(1+\mathcal{O}(n^{-S/2})\right) + \mathcal{O}\left(j^{-t}\right) . \tag{2.2.62}$$

This, together with the expansion in Equation (2.2.49), implies the second claim.

Proof of III: we expand for some $\alpha > 0$ by the previous coupling argument

$$\mathbb{B}_{0,0}^j(H_n < j) \ge \mathtt{P}_{0,0}^j\left(H_{n+c_\alpha \log^2(j)}^B < j\right) + \mathcal{O}\left(j^{-\alpha}\right) . \tag{2.2.63}$$

We furthermore bound

$$B_{0,0}^j\left(H_{n+c_\alpha \log^2(j)}^B < j\right) \ge B_{0,0}^j\left(H_{2n}^B < \varepsilon n^2\right) \ge B_{0,0}^j\left(H_{2n,1}^B < \varepsilon n^2\right) , \tag{2.2.64}$$

where $H_{2n,1}^B$ is the hitting time of points $\{-2n, +2n\}$ for the first coordinate of a multidimensional Brownian bridge. From there on it is straightforward to see that above remains positive uniformly in j, n. Indeed, this is due to the scale invariance (map $j \mapsto j/n^2$) and the distribution of the maximum of the Brownian motion/bridge, see [MP10]. This concludes the proof. □

2.2.3 Hitting a sphere from outside

In this section we prove random walk analogues of known hitting time estimates for the Brownian motion. We begin by introducing the results for the continuum case. Let for $|x| \geq n$

$$q(x,t,n) = \partial_t P_x \left(H_n^B \leq t\right) , \tag{2.2.65}$$

the "density" (in $d \geq 3$ it does not integrate to 1) of the first hitting time of the centred ball with radius n. The main references are [Uch15], [Uch16] and [BMR13]. Firstly, note that by Brownian scaling we have that

$$q(x,t,n) = \frac{1}{n^2} q(x/n, t/n^2, 1) . \tag{2.2.66}$$

It is obvious that q is constant with respect to rotating its first argument and so we write $q(x,t,n)$ with $x \in [n, \infty)$.
In [BMR13] it is shown that for $d \geq 3$

$$q(x,t,1) \sim \frac{x-1}{x} \frac{\mathrm{e}^{(x-1)^2/(2t)}}{t^{3/2}} \frac{1}{t^{(d-3)/2} + x^{(d-3)/2}} , \tag{2.2.67}$$

and for $d = 2$

$$q(x,t,1) \sim \frac{x-1}{x} \frac{\mathrm{e}^{(x-1)^2/(2t)}}{t^{3/2}} \frac{(x+t)^{1/2}(1+\log x)}{(1+\log(1+t/x))(1+\log(t+x))} . \tag{2.2.68}$$

The following observation is useful.

Lemma 2.2.4. *Suppose n, n' with $n = n'(1+o(1))$. Suppose furthermore that $(x-n)^2 - (x-n')^2 = o(t)$. We then have that*

$$q(x,t,n) \sim q(x,t,n') . \tag{2.2.69}$$

Proof of Lemma 2.2.4. The lemma follows immediately after noticing that by the scaling relation and Equation (2.2.67)

$$q(x,t,n) \sim \frac{x-n}{t^{3/2}} \mathrm{e}^{-(x-n)^2/(2t)} \frac{1}{(t/n^2)^{(d-3)/2} + (x/n)^{(d-3)/2}} . \tag{2.2.70}$$

This concludes the proof. □

The above lemma is useful for the following reason: when we apply the coupling from Lemma 2.2.1, we have to shrink/enlarge a ball of radius n by a logarithmic factor in n. Lemma 2.2.4 shows that if x is sufficiently far away from the boundary of $\boldsymbol{B}_n$ and t large enough, this error is negligible.
However, it is not possible to infer the analogue of the density $q(x,t,n)$ for the random walk directly from the above and a coupling argument. Indeed, similar to [DW15], it is only possible to get bounds on the cumulative distribution function. This is the content of the next proposition. Note that we often write $\mathbb{P}_x$ and similar expressions for $x \in [0,\infty)$. This is shorthand for taking $y \in \mathbb{R}^d$ (or $\mathbb{Z}^d$) with $|y| = x(1+\mathcal{O}(1))$.

Proposition 2.2.5. *Let $d \geq 2$. Take $x,n,k > 0$. Suppose there exists an $M > 0$ such that $Mn \geq x$ and there exists $\delta > 0$ such that $k \geq \log^{5+\delta}(n)$. Furthermore, suppose that $k \leq n^L$ for some $L > 0$ and $(x-n)\log(n) = \mathcal{O}(k)$. Then*

$$\mathbb{P}_x\left(\infty > H_n \geq k\right) \sim \int_k^\infty q(x,t,n)\mathrm{d}t. \tag{2.2.71}$$

Proof of Proposition 2.2.5. We only prove the $d \geq 3$ case, the case $d = 2$ follows analogously.
The idea of the proof is as follows: first restrict the k's, as $\mathbb{P}_x\left(\infty > H_n \geq k\right)$ converges to a constant as $k \downarrow 0$. Given x very close to $\partial_i \boldsymbol{B}_n$, we first use standard estimates to let the random walk escape "a bit further" from $\partial \boldsymbol{B}_n$ and then use the coupling. We use that $q(x,n,t)$ has bounds which are slowly varying and then show that the errors from the coupling are negligible.
We begin by restricting the k's we need to consider. By [LL10, Proposition 6.4.2] we know that

$$\mathbb{P}_x\left(H_n < \infty\right) = c(x) = (n/x)^{d-2}\left(1 + \mathcal{O}(n^{-1})\right) . \tag{2.2.72}$$

Denote $m = n - x$. Choose $\varepsilon_1 > 0$ (depending increasingly on M) such that for $t \in \mathbb{N}$ we have that

$$\mathbb{P}_0\left(\max_{1\le i\le \varepsilon_1 t^2}|S_i| \ge t\right) \le c(x)/2\,. \tag{2.2.73}$$

For $k \le \varepsilon_1 m^2$, we bound

$$\mathbb{P}_x\left(\infty > H_n \ge k\right) = c(x) - \mathbb{P}_x\left(H_n < k\right) \ge c(x)/2\,. \tag{2.2.74}$$

Henceforth we assume $k \ge \varepsilon_1 m^2$. From now on, for $l > 0$, shorten $H_l = H_l \mathbb{1}\{H_l < \infty\}$, to simplify notation. We use the same shorthand notation for H_l^B. Let $n^{\pm} = n \pm c_\alpha \log^2(n)$.
By [BMR13, Theorem 3] and the rescaling relation (2.2.66) we have that

$$P_x\left(H^B_{n^\pm} \ge k\right) \sim \int_k^\infty \frac{n^\pm(x-n^\pm)}{xt^{3/2}} \frac{\mathrm{e}^{-\left(x-n^\pm\right)^2/(2t)}}{(t/(n^\pm)^2)^{(d-3)/2} + (x/n^\pm)^{(d-3)/2}}\mathrm{d}t\,. \tag{2.2.75}$$

We now assume that $x - n = m \ge \log(n)^{2+\delta/4}$. Note that we have $n^\pm = n(1+o(1))$, $x - n^\pm = m(1+o(1))$ and $m\log(n)/t = \mathcal{O}(1)$ for $t \ge k$. We thus get that

$$\begin{aligned} P_x\left(H^B_{n^\pm} \ge k\right) \sim P_x\left(H^B_{n} \ge k\right) &\sim \int_k^\infty \frac{nm}{xt^{3/2}} \frac{\mathrm{e}^{-(m)^2/(2t)}}{\left(\frac{t}{n^2}\right)^{(d-3)/2} + (x/n)^{(d-3)/2}}\mathrm{d}t \\ &\sim \frac{n}{x}\int_0^{m^2/k} \frac{\mathrm{e}^{-t/2}}{t^{1/2}} \frac{\mathrm{d}t}{\left(\frac{m^2}{n^2 t}\right)^{(d-3)/2} + \left(\frac{x}{n}\right)^{(d-3)/2}}\,. \end{aligned} \tag{2.2.76}$$

By [LL10, Theorem 7.1.1], for every $\alpha > 0$ we can choose $c_\alpha > 0$, such that there exists a coupling P_x between the random walk and the Brownian motion with

$$\mathsf{P}_x\left(\max_{1\le i\le k}|B_i - S_i| \ge \log(k)\right) \le c_\alpha k^{-\alpha}\,. \tag{2.2.77}$$

Note that by the coupling from above equation

$$\mathbb{P}_x\left(H^B_{n^+} \ge k\right) + \mathcal{O}\left(k^{-\alpha}\right) \le \mathbb{P}_x\left(H_n \ge k\right) \le \mathbb{P}_x\left(H^B_{n^-} \ge k\right) + \mathcal{O}\left(k^{-\alpha}\right)\,. \tag{2.2.78}$$

Making $\alpha > 0$ sufficiently large finishes the proof for the case $m \geq \log(n)^{2+\delta/4}$. We now treat the case $m = x - n = \mathcal{O}\left(\log(n)^{2+\delta/4}\right)$. Here we cannot employ the coupling directly, as we only know $|B_i - S_i| = \mathcal{O}\left(\log^2(n)\right)$ under the coupling and thus the error in Equation (2.2.75) may no longer be negligible. Therefore we let the random walk first "escape" a bit further from the sphere and then use the coupling: abbreviate $n(\delta) = n + \log^{2+\delta/3}(n)$ and decompose

$$\begin{aligned} \mathbb{P}_x\left(H_n \geq k\right) = &\mathbb{P}_x\left(H_n \geq k, H_n > H_{n(\delta)}\right) \\ &+ \sum_{l \geq 1} \sum_{z \in \partial_i \boldsymbol{B}_{n(\delta)}} \mathbb{P}_x\left(H_n < H_{n(\delta)} = l, X_l = z\right) \mathbb{P}_z\left(H_n \geq k - l\right) . \end{aligned} \tag{2.2.79}$$

We begin by bounding the first term. We claim that

$$\mathbb{P}_x\left(H_n \wedge H_{n(\delta)} \geq k/2\right) = \mathcal{O}\left(c^{k/\log^{4+(2\delta)/3}(n)}\right) \text{ for some } c \in (0,1)\,. \tag{2.2.80}$$

As $k \leq n^L$ and $k \geq \log^{5+\delta}(n)$

$$k/\log^{4+(2/3)\delta}(n) \geq \log^{1+\delta/3}(n) \geq c(L)\log^{1+\delta/3}(k) \geq C\log^{1+\delta/4}(n)\,. \tag{2.2.81}$$

This implies that the above term decays faster than any polynomial and is of order

$$\mathbb{P}_x\left(H_n \wedge H_{n(\delta)} \geq k/2\right) = \mathcal{O}\left(c^{\log^{1+\delta/4}(k)}\right)\,. \tag{2.2.82}$$

We now prove the claim, i.e. Equation (2.2.80). Let for $x \in \boldsymbol{B}_{n(\delta)} \setminus \boldsymbol{B}_n$ the box $C_n(x) \subset \mathbb{R}^d$ be defined as the smallest rotated $|\cdot|_\infty$ box[3] centred at x such that two faces of the box lie in $\left(\boldsymbol{B}_{n(\delta)} \setminus \boldsymbol{B}_n\right)^c$. To be more precise, $C_n(x)$ is $O\boldsymbol{B}_l^\infty(x)$ where $O \in \mathbb{R}^{d\times d}$ with $O^T = O$ and $|\det(O)| = 1$, for $l > 0$ the smallest side length such that two faces of $C_n(x)$ have zero intersection with the interior of $\boldsymbol{B}_{n(\delta)} \setminus \boldsymbol{B}_n$. See Figure 2.4 for an illustration.

Let $M(x)_{t_2}^{t_1}$ be the event that the random walk in the time interval $[t_1, t_2)$ first exits $C_n(x)$ on any but those two faces which lie outside $\boldsymbol{B}_{n(\delta)}$. One has that the length of the faces of $C_n(x) \sim \log^{2+\delta/3}(n)$. Using the Markov property on

[3] $\boldsymbol{B}_l^\infty(x) = \{y \in \mathbb{R}^d \colon |x - y|_\infty \leq l\}$

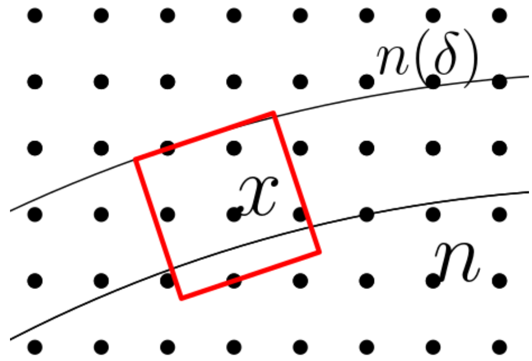

Figure 2.4: An illustration of $C_n(x)$

time scales of length $\log^{4+(2/3)\delta}(n)$

$$\begin{aligned}
&\mathbb{P}_x\left(H_n \wedge H_{n(\delta)} \geq k/2\right) \\
&\leq \left(\max_{x\in \boldsymbol{B}_{n(\delta)}\setminus \boldsymbol{B}_n} \max_{y\in C_n(x)} \mathbb{P}_y\left(M(y)^0_{\left\lfloor \log^{4+(2/3)\delta}(n)\right\rfloor}\right)\right)^{\left\lfloor k/\log^{4+(2/3)\delta}(n)\right\rfloor} \\
&= \mathcal{O}\left(c^{k/\log^{4+(2/3)\delta}(n)}\right) = \mathcal{O}\left(\exp\{-C\log^{1+\delta/4}(k)\}\right),
\end{aligned} \tag{2.2.83}$$

by the Donsker's invariance principle, see e.g. [MP10]. This proves the claim from Equation (2.2.80).
Set $m(\delta) = \log^{2+\delta/3}(n)$. By the results in [BMR13], we have

$$P_{n(\delta)}\left(H_n^B \geq k\right) \sim \int_k^\infty \frac{nm(\delta)}{n(\delta)t^{3/2}} \frac{\mathrm{e}^{-(m(\delta))^2/(2t)}}{\left(\frac{t}{n^2}\right)^{(d-3)/2} + (m(\delta)/n)^{(d-3)/2}}\mathrm{d}t\,, \tag{2.2.84}$$

where we recall that $n(\delta) = n + \log^{2+\delta/3}(n)$. Note that m^2/k remains bounded (see beginning of this proof). Thus, the above function satisfies that for any $l \in \{1, \ldots, k/2\}$ we have that

$$P_{n(\delta)}\left(H_n^B \geq k\right) \sim P_{n(\delta)}\left(H_n^B \geq k - l\right)\,. \tag{2.2.85}$$

Note that by a martingale (or harmonic function) argument (see [LL10, Proposition 6.3.5]), we have that

$$\mathbb{P}_x\left(H_n < H_{n(\delta)}\right) \sim \frac{n(\delta)^{2-d} - x^{2-d}}{n(\delta)^{2-d} - n^{2-d}} \sim \frac{m}{m(\delta)}\,, \tag{2.2.86}$$

due to the restrictions placed on x, i.e. $x - n = \mathcal{O}\left(\log^{2+\delta/4}(n)\right)$. We use this to expand

$$\begin{aligned}
\mathbb{P}_x(H_n \geq k) &= \mathcal{O}\left(c^{\log^{1+\delta/4}(k)}\right) \\
&\quad + \sum_{l=1}^{k/2} \sum_{z \in \partial_i \boldsymbol{B}_{n(\delta)}} \mathbb{P}_x\left(H_n < H_{n(\delta)} = l, X_l = z\right) \mathbb{P}_z\left(H_n \geq k - l\right) \\
&\sim \mathcal{O}\left(c^{\log^{2+\delta/4}(k)}\right) + \mathbb{P}_x\left(H_n < H_{n(\delta)}\right) P_{n(\delta)}\left(H_n^B \geq k\right) \\
&\sim \mathcal{O}\left(c^{\log^{2+\delta/4}(k)}\right) + P_x\left(H_n^B \geq k\right)\,.
\end{aligned} \tag{2.2.87}$$

Note that we used that Equation (2.2.86) cancels the factor of $m(\delta)$ in Equation (2.2.84). This concludes the proof. □

In the case $d = 1$ and the simple symmetric random walk, we can employ a different proof and get stronger results. The proof itself is a generalisation of [LL10, Proposition 5.1.2].

Lemma 2.2.6. *Let $d = 1$ with the random walk with $\mathbb{P}_x(S_1 = x+1) = \mathbb{P}_x(S_1 = x - 1) = p$ and $1 - 2p = \mathbb{P}_x(S_1 = x)$, $x \in \mathbb{Z}$ and $p \in (0, 1/2]$. We then have that for $x > n$*

$$\mathbb{P}_x(H_0 = k) \sim 2\sum_{l=0}^{x} p_{k-2}(l) - p_k(l)\,. \tag{2.2.88}$$

Proof of Lemma 2.2.6. Note that we can assume that $k \geq x$ and let us assume without loss of generality that x is even (the odd case follows analogously). Note that we have for $l \geq k$ that

$$\mathbb{P}_x(H_0 = k, S_l = y) = \mathbb{P}_x(H_0 = k, S_l = -y)\,. \tag{2.2.89}$$

From this we can infer that $\mathbb{P}_x(H_0 \leq k, S_l = y) = \mathbb{P}_x(H_0 \leq k, S_l = -y)$ and therefore

$$\mathbb{P}_x(H_n > k) = \sum_{y>0} \mathbb{P}_x(H_n > k, S_k = y) - \mathbb{P}_x(H_n > k, S_k = -y)$$
$$= \sum_{y>0} p_k(x,y) - p_k(x,-y) = p_k(0) + p_k(x) + 2\sum_{l=1}^{x-1} p_k(l)\,. \tag{2.2.90}$$

This concludes the proof. □

Chapter 3

Loop measures, soups and first properties

In this chapter we define different loop measures, the associated loop soups, and occupation fields. An important part of this chapter is the derivation of the Bosonic loop measure as a space-time limit. This part is based on and generalises the work from [AV20]. The last part of the chapter is devoted to isomorphism theorems: we show how one can compute the distribution of the accrued local time of all the loops by solving a measure-valued equation.

3.1 Loop measures

We begin by introducing the *Markovian loop measure*, following [LJ10, Section 3] and [AV20, Definition 1.1].

Definition 3.1.1. *For $G \in \sigma(\mathcal{D})$ and $\mu \leq 0$ (also called chemical potential)*

$$M_\mu[G] = M[G] = \sum_{x \in \mathbb{Z}^d} \int_0^\infty \frac{\mathrm{e}^{\mu t}}{t} \overline{\mathbb{P}}^t_{x,x}(G) \mathrm{d}t \,. \tag{3.1.1}$$

Remark 3.1.2. *The factor μ is non-standard and appears in [AV20]. An exponential decay in the above integral is usually achieved by introducing an exponential killing uniform on the vertices, see [Szn12]. The two approaches are equivalent.*

Another important measure on loops is the *Bosonic loop measure*. Following [AV20], we define.

Definition 3.1.3. *For $\mu \leq 0$ (chemical potential), $\beta \in (0, \infty)$ (also referred to as inverse temperature), we define the Bosonic loop measure*

$$M^B_{\mu,\beta}[G] = M^B[G] = \sum_{x \in \mathbb{Z}^d} \sum_{j \geq 1} \frac{e^{\beta\mu j}}{j} \overline{\mathbb{P}}^{\beta j}_{x,x}(G)\,, \tag{3.1.2}$$

where $G \in \sigma(\mathcal{D})$.

Remark 3.1.4. *The Bosonic loop measure has its origin in the physics community in the context of functional integration, where mainly its continuum analogue (replacing the random walk by a Brownian motion) is considered (see e.g. [BR03]). For random walks on graphs, first computations for $M^B_{\mu,\beta}$ are carried out in [Owe15]. These are restricted to finite graphs and follow from different methods compared to what we employ. In [AV20], various properties for $M^B_{\mu,\beta}$ are proven in the finite setting.*

We can unify the above definitions into a single framework. This will only be needed when talking about isomorphism theorems, as done in Section 3.3.

Definition 3.1.5. *Given a positive measure $\mathfrak{m}$ on $[0, \infty)$ we define the loop measure with weight $\mathfrak{m}$ as*

$$M^{\mathfrak{m}}[G] = \int_0^\infty \overline{\mathbb{P}}^t_{x,x}(G) \mathrm{d}\mathfrak{m}(t) = \mathfrak{m}\left[\overline{\mathbb{P}}^t_{x,x}(G)\right]\,, \tag{3.1.3}$$

where $G \in \sigma(\mathcal{D})$. We assume that for all $\varepsilon > 0$

$$\int_\varepsilon^\infty \frac{\mathrm{d}\mathfrak{m}(t)}{(1 \wedge t)^{d/2}} < \infty\,. \tag{3.1.4}$$

For the discrete-time random walk, we define the discrete time loop measure.

Definition 3.1.6. *Given a positive sequence $a = (a_j)_{j \in \mathbb{N}}$ and $G \in \sigma(\mathcal{D})$, we* *define the loop measure in discrete-time with weights* $(a_j)_j$

$$M^a[G] = \sum_{x\in\mathbb{Z}^d}\sum_{j\geq 0} a_j \mathbb{P}^j_{x,x}(G)\,. \tag{3.1.5}$$

Note that the underlying random walk for M^a *is a discrete-time random walk.*

We begin with a proposition relating the above defined measures.

Proposition 3.1.7. *For ease of notation we assume that* $q(x,\dagger) = 0$ *for all* $x \in \mathbb{Z}^d$.

I. *[AV20, Remark 2.2] Suppose that* $t \mapsto \mathrm{e}^{\mu t} t^{-1} \overline{\mathbb{P}}^t_{x,x}(G)$ *is Riemann-integrable and* $x \mapsto \sum_{j\geq 1} \frac{\mathrm{e}^{\beta\mu j}}{j} \overline{\mathbb{P}}^{\beta j}_{x,x}(G)$ *can be dominated by an integrable (with respect to the counting measure on* $\mathbb{Z}^d$*) positive function* $g(x) \geq 0$ *for all* $\beta > 0$ *small enough. We then have that*

$$\lim_{\beta\downarrow 0} M^B_{\mu,\beta}[G] = M_\mu[G]\,. \tag{3.1.6}$$

II. *Given* $\mu \leq 0$ *and*

$$a_j = \frac{1}{j}\left(\frac{1}{1-\mu}\right)^j\,. \tag{3.1.7}$$

For every G *that is in the sigma-algebra generated by the discrete jump chain* $(S_n)_{n\in\mathbb{N}}$*, we have that*

$$M^a[G] = M_\mu[G]\,. \tag{3.1.8}$$

III. *Given* $\mu \leq 0$

$$a_j = \frac{\beta^j}{j!}\texttt{PolyLog}_{1-j}\left(\mathrm{e}^{\beta(\mu-1)}\right)\,, \tag{3.1.9}$$

we have for every G *that is in the sigma-algebra generated by the discrete jump chain* $(S_n)_{n\in\mathbb{N}}$ *that*

$$M^a[G] = M^B_{\mu,\beta}[G]\,. \tag{3.1.10}$$

Here, the function $\texttt{PolyLog}_k(z)$ *is defined as*

$$\texttt{PolyLog}_k(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^k}\,. \tag{3.1.11}$$

IV. *We have for every* G *that is in the sigma-algebra generated by the discrete jump chain* $(S_n)_{n\in\mathbb{N}}$ *that for* $a_j = (j!)^{-1}\int_0^\infty \mathrm{e}^{-t}t^j \mathrm{d}\mathfrak{m}(t)$ *that*

$$M^a[G] = M^{\mathfrak{m}}[G]\,. \tag{3.1.12}$$

Remark 3.1.8. *This proposition allows us to interpret the Markovian loop measure as an infinite-temperature limit (i.e. the inverse temperature* $\beta \downarrow 0$*) of the Bosonic loop measure. In Section 3.2 we show how one can construct the Bosonic loop measure from the Markovian one.*
It furthermore shows that for events depending on the jump chain alone, the Bosonic and the Markovian loop measure can be represented by M^a*. Thus, when analysing connectivity properties of the loop measure in Chapter 4 and Chapter 6, we only use* M^a*.*

Proof of Proposition 3.1.7.

I. As this was a remark in [AV20], we give a proof here. Fix K large enough such that $M_\mu[G, l(\omega) < K] \geq M_\mu[G] - \varepsilon$. We then choose a sequence $\beta_n \downarrow 0$ and write the Riemann integral representation of $M_\mu[G, l(\omega) < K]$ in the following way

$$M_\mu[G, l(\omega) < K] = \lim_{n\to\infty} \sum_{x\in\mathbb{Z}^d} \sum_{j=1}^{\lfloor K/\beta_n \rfloor} \beta_n \frac{\mathrm{e}^{\beta_n j\mu}}{\beta_n j} \overline{\mathbb{P}}_{x,x}^{\beta_n j}(G)\,. \tag{3.1.13}$$

We use the dominated convergence theorem to switch limit and summation. This concludes the proof. □

II. By the definition of $\mathtt{d}_\infty$, we can write

$$G = \{S_0 = x_0,\, S_1 = x_1, \dots, S_k = x_0, S_{k+j} = \dagger,\, \forall j \in \mathbb{N}\}\,. \tag{3.1.14}$$

without loss of generality. The construction of $(X_t)_{t\geq 0}$ as done in [Kle13, Chapter 17] shows that we can rewrite $X_t = S_{N_t}$, where $(N_t)_{t\geq 0}$ is a Poisson process on the real line with intensity 1. Note that

$$\mathbb{P}(N_t = k) = \mathrm{e}^{-t}\frac{t^k}{k!}\,. \tag{3.1.15}$$

Thus,

$$\int_0^\infty \frac{\mathrm{e}^{\mu t}}{t}\overline{\mathbb{P}}^t_{x_0,x_0}(G)\mathrm{d}t = \sum_{k\geq 0}\int_0^\infty \frac{\mathrm{e}^{\mu t}}{t}\mathrm{e}^{-t}\frac{t^k}{k!}\mathrm{d}t\,\prod_{i=0}^{k-1} p(x_i, x_{i+1})\,, \tag{3.1.16}$$

where we identify $x_k = x_0$. The claim follows after a change of variables $t \mapsto t(\mu-1)^{-1}$ and using the integral representation of the factorial/Gamma function.

III. This follows analogously to the previous proof. Indeed, write $\overline{\mathbb{P}}^{\beta j}_{x,x} = \mathrm{e}^{-\beta j}\sum_{k\geq 0}(\beta j)^k(k!)^{-1}\mathbb{P}^k_{x,x}$. Exchanging the sum over the lengths j with the sum over the k's gives the result.

IV. This is similar to the above.

□

The Bosonic and the Markovian loop measure assign comparable weight to loops of the same length, this is shown in the next lemma.

Lemma 3.1.9. *For $\beta > 0$ and $\mu \leq 0$*

$$\frac{\beta^j}{j!}\mathtt{PolyLog}_{1-j}\left(\mathrm{e}^{\beta(\mu-1)}\right) = \frac{1}{j}\left(\frac{1}{1-\mu}\right)^j(1+o(1))\,. \tag{3.1.17}$$

As a consequence, for every $\beta > 0$ and for every $\mu \leq 0$ there exist constants C_1, $C_2 > 0$ such that for every event $G \in \sigma(\Gamma)$ that is generated by the jump chain we have

$$C_1 M_\mu[G] \leq M^B_{\mu,\beta}[G] \leq C_2 M_\mu[G]\,. \tag{3.1.18}$$

Proof of Lemma 3.1.9. This is a consequence of the limiting behaviour of the polylogarithm. By [Woo92], we have that

$$\mathtt{PolyLog}_s(\mathrm{e}^r) = \Gamma(1-s)\,[-r]^{s-1}(1+o(1))\,, \tag{3.1.19}$$

as $s \to -\infty$. Plugging in $s = 1 - j$ and $r = \beta(\mu - 1)$ gives us

$$\texttt{PolyLog}_{1-j}\left(\mathrm{e}^{\beta(\mu-1)}\right) = (j-1)!\,[\beta - \beta\mu]^{-j}\,(1 + o(1)) \;, \tag{3.1.20}$$

and thus the claim follows. □

3.1.1 Random walk soups and their occupation fields

In this section we introduce various loop soups, the notion of local times and occupations fields.

Definition 3.1.10. *For $\lambda > 0$ we introduce four different classes of Poisson point processes. For a general definition of Poisson point processes (PPP) on measurable spaces, see [Kal01, Chapter 12].*

I. We define $\mathbb{P}_\lambda^M$ as the PPP with intensity measure λM_μ.

II. We define $\mathbb{P}_\lambda^B$ as the PPP with intensity measure $\lambda M_{\mu,\beta}^B$.

III. We define $\mathbb{P}_\lambda^{\mathfrak{a}}$ as the PPP with intensity measure $\lambda M^{\mathfrak{a}}$.

IV. We define $\mathbb{P}_\lambda^{\mathfrak{m}}$ as the PPP with intensity measure $\lambda M^{\mathfrak{m}}$.

If we omit the superscript, it is either to be understood that we refer to all four types of PPPs simultaneously or that the superscript is clear from the context. A random measure sampled from $\mathbb{P}_\lambda$ is denoted by $\mathcal{U}$. We write $\mathcal{U}^\lambda$ when we want to emphasise the dependence on λ. Since $\|q\|_\infty < \infty$, we have that loops with infinitely many jumps on finite intervals have zero mass, thus our loop measures live on a Borel space and by [Kal01, Chapter 12] we can write

$$\mathcal{U} = \sum_{k \le \kappa} \delta_{\omega_k} \;, \tag{3.1.21}$$

with $\kappa \in \mathbb{N} \cup \{\infty\}$ and $\omega_k \in \Gamma$. The collection of $(\omega_k)_{k=1}^{\kappa}$ is often referred to as the loop soup. We use the (non-standard) notation $x \in \mathcal{U}$ if there exists ω in the support of $\mathcal{U}$ with $\{x\} \cap \omega[0, l(\omega)] \neq \varnothing$.

Remark 3.1.11. *Note that M_μ, M^B and $M^{\mathfrak{m}}$ are non-atomic and thus, almost surely, all ω_k's from the above representation are distinct. This means that the*

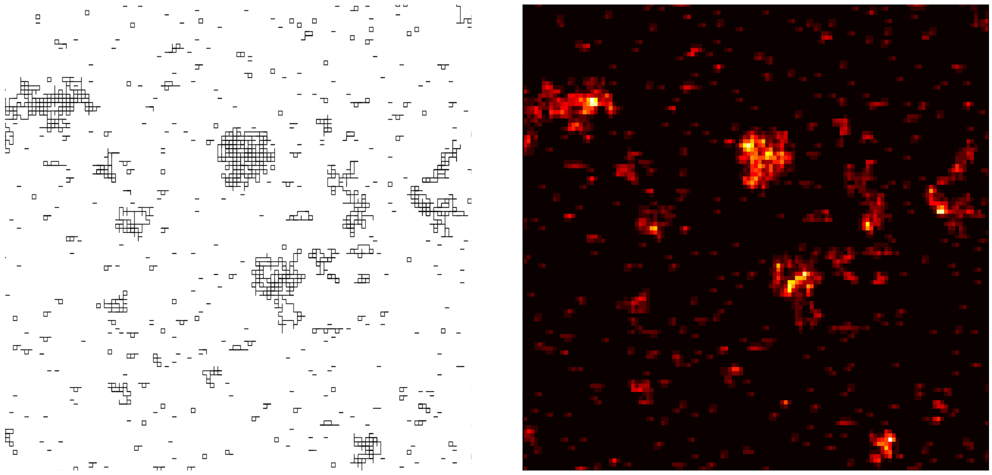

Figure 3.1: A sample from a simulation of the loop soup and its occupation field $\mathcal{L}$, by the author. Bright colours correspond to large values of the local time. Simulation obtained using Dirichlet boundary conditions on a larger square and unit intensity.

associated PPPs are simple. For $\mathbb{P}^a_\lambda$ *this is not the case, as* M^a *is a purely atomic measure.*

Another important concept is that of the *local time* and the *occupation field*.

Definition 3.1.12. *For* $\omega \in \Gamma$ *and* $x \in \mathbb{Z}^d$ *we define the local time as*

$$L_x = L_x(\omega) = \int_0^{l(\omega)} \mathbb{1}\{\omega(t) = x\} \mathrm{d}t \,, \tag{3.1.22}$$

where we recall that $l(\omega)$ *is the length of the loop.*
For $\mathcal{U} = \sum_{k \le \kappa} \delta_{\omega_k}$ *a sample from* $\mathbb{P}_\lambda$*, define the occupation field* $\mathcal{L}$ *as*

$$\mathcal{L}_x = \mathcal{L}_x(\mathcal{U}) = \mathcal{U}[L_x] = \int_0^{\infty} |\{\omega_k(t) = x,\, 1 \le k \le \kappa\}| \mathrm{d}t \,, \tag{3.1.23}$$

where in the last equality monotone convergence is applicable. We occasionally write L *and* $\mathcal{L}$ *instead of* $(L_x)_{x \in \mathbb{Z}^d}$ *or* $(\mathcal{L}_x)_{x \in \mathbb{Z}^d}$*.*

In Figure 3.1 we show a realisation of the loop soup together with a heat map of its occupation field.

3.2 The derivation of the Bosonic loop measure as a space-time limit

The goal of this section is to prove the converse of Proposition 3.1.7; this time constructing the Bosonic loop measure from the Markovian one. Partial success of that task was achieved in [Owe15, Theorem 3.12] and [Vog16, Theorem 3.3]. Our result is more general and shows a full convergence of the finite-dimensional distributions. This section is based on and generalises [AV20].

3.2.1 Space-time random walks

We begin with enlarging $\mathbb{Z}^d$ by taking the Cartesian product with a discrete torus: define for $N \in \mathbb{N}$

$$\mathbb{Z}_N^d = \mathbb{Z}^d \times \{0, \dots, N-1\} = \mathbb{Z}^d \times \mathbb{T}_N \,, \tag{3.2.1}$$

the *space-time torus*. Define $\Sigma \in \mathbb{R}^{\mathbb{T}_N \times \mathbb{T}_N}$ by setting

$$\Sigma(b_1, b_2) = \mathbb{1}\{b_2 = b_1 + 1\} \,. \tag{3.2.2}$$

In this definition, as well as throughout the whole section, we understand arithmetics on $\mathbb{T}_N$ always modulo N. For $\beta > 0$ and $(x_1, b_1) \neq (x_2, b_2)$ let

$$q_N\left((x_1, b_1), (x_2, b_2)\right) = \begin{cases} \beta^{-1} N \Sigma(b_1, b_2) & \text{if } x_1 = x_2,\ b_1 \neq b_2\,, \\ q(x_1, x_2) & \text{if } x_1 \neq x_2,\ b_1 = b_2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2.3}$$

Furthermore, set $q_N\left((x_1, b_1), (x_1, b_1)\right) = -\sum_{(x,b) \neq (x_1, b_1)} q_N\left((x_1, b_1), (x, b)\right)$. For an illustration of the space-time random walk see Figure 3.2.

Let $\pi \colon \Gamma(\mathbb{Z}_N^d) \to \Gamma(\mathbb{Z}^d)$ be the projection onto the coordinate in $\mathbb{Z}^d$, i.e.

$$\pi\left((\omega^{(1)}, \omega^{(2)})\right)(t) = \omega^{(1)}(t) \,. \tag{3.2.4}$$

Here we identify for $z \in \mathbb{Z}_N^d$ the coordinates $z = (z^{(1)}, z^{(2)})$ with $z^{(1)} \in \mathbb{Z}^d$ and

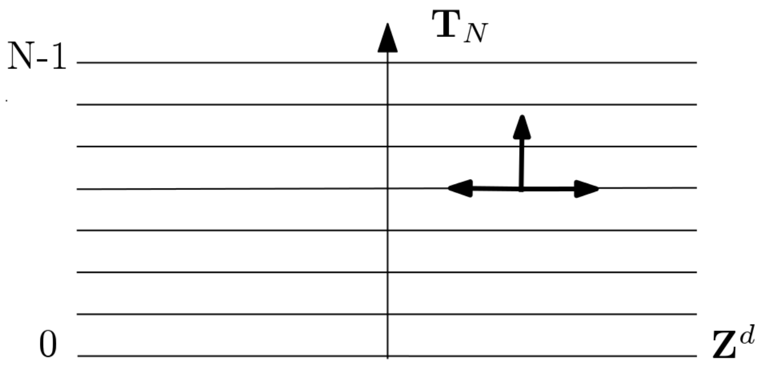

Figure 3.2: The space-time random walk can move freely on $\mathbb{Z}^d$, but on $\mathbb{T}_N$ it has to move upwards. Figure from [AV20].

$z^{(2)} \in \{0, \ldots, N-1\}$.

Definition 3.2.1. *Define the space-time loop measure M_N in the following way*

$$M_N[G] = \sum_{z \in \mathbb{Z}_N^d} \int_0^\infty \frac{\mathrm{e}^{\mu t}}{t} \overline{\mathbb{P}}_{z,z}^t(G) \mathrm{d}t \,, \tag{3.2.5}$$

for $G \in \sigma(\mathcal{D}(\mathbb{Z}_N^d))$ and the random walk induced by the generator q_N. It thus is the standard Markovian loop measure on the (enlarged) graph $\mathbb{Z}_N^d$.
For $G \in \sigma(\mathcal{D}(\mathbb{Z}^d))$, we define the projected loop measure

$$M_N^{\downarrow}[G] = \left(M_N \circ \pi^{-1}\right)[G] = \sum_{z \in \mathbb{Z}_N^d} \int_0^\infty \frac{\mathrm{e}^{\mu t}}{t} \overline{\mathbb{P}}_{z,z}^t\Big(\{(X_s)_{s \in [0,t]} : \pi(X) \in G\}\Big) \, \mathrm{d}t. \tag{3.2.6}$$

The associated local times and occupation fields are denoted by $L^N, L^{\downarrow}, \mathcal{L}^N, \mathcal{L}^{\downarrow}$ and the PPPs by $\mathbb{P}_\lambda^N$ and $\mathbb{P}_\lambda^{\downarrow}$, respectively.

We begin by analysing the distribution of $(\omega^{(1)}, \omega^{(2)})$ under $\overline{\mathbb{P}}_z$ for $z \in \mathbb{Z}_N^d$.

Lemma 3.2.2. *Under $\overline{\mathbb{P}}_z$ and $\overline{\mathbb{P}}_{z,z}^t$, we have that $\omega^{(1)}$ and $\omega^{(2)}$ are two independent stochastic processes with weight matrices q and Σ.*

Proof of Lemma 3.2.2. As the process is uniquely characterised by its transition kernel, it suffices to show that

$$\overline{p}_t\left((x_1, b_1), (x_2, b_2)\right) = \overline{p}_t^{\mathbb{Z}^d}(x_1, x_2) \overline{p}_t^{\Sigma}(b_1, b_2) \,, \tag{3.2.7}$$

where the kernels $\overline{p}^{\mathbb{Z}^d}$ and $\overline{p}^{\Sigma}$ are those generated by the respective projections. The superscripts will be omitted from now, as the kernel's arguments serve as an indicator for the underlying process. Recall $1 = \|q\|_\infty$ and expand

$$\begin{aligned} \mathrm{e}^{-t(1+\beta^{-1}N)}\overline{p}_t\left((x_1,b_1),(x_2,b_2)\right) &= \sum_{n=0}^{\infty} \frac{[t(1+\beta^{-1}N)]^n}{n!} p_n\left((x_1,b_1),(x_2,b_2)\right) \\ = \sum_{n=0}^{\infty} \frac{[t(1+\beta^{-1}N)]^n}{n!} \\ \times \sum_{k=0}^{n} \binom{n}{k} p_k(x_1,x_2) \left(\frac{1}{1+\beta^{-1}N}\right)^k p_{n-k}(b_1,b_2) \left(\frac{\beta^{-1}N}{1+\beta^{-1}N}\right)^{n-k} . \end{aligned} \tag{3.2.8}$$

In the last line we count how many times the space-time random walk will choose the torus coordinate.
Exchanging the two sums and expanding the binomial coefficient gives us

$$\sum_{k=0}^{\infty} p_k(x_1,x_2)\frac{1}{k!}\sum_{n=0}^{\infty} p_n(b_1,b_2)\left(\beta^{-1}N\right)^n \frac{1}{n!} = \mathrm{e}^{-t}\overline{p}_t(x_1,x_2)\mathrm{e}^{-t\beta^{-1}N}\overline{p}_t(b_1,b_2)\,. \tag{3.2.9}$$

This concludes the proof. □

3.2.2 Convergence of the finite-dimensional distributions

We begin by stating a set of necessary assumptions for this section.

Assumption 3.2.3. *Assume that $d \geq 3$ and*

$$\mu - \inf_{x\in\mathbb{Z}^d} p(x,\dagger) \leq 0\,. \tag{3.2.10}$$

Let $A \subset (0,\infty)$ be Lebesgue-measurable and assume

$$\beta\mathbb{N} \cap \partial A = \varnothing\,. \tag{3.2.11}$$

The main result of this section is the following theorem.

Theorem 3.2.4. *Let Assumption 3.2.3 hold. For $k \in \mathbb{N}$ and $0 < t_1 < \ldots < t_k < \infty$, with $A \subset (t_k, \infty)$, it holds that*

$$\lim_{N\to\infty} M_N^{\downarrow}[X_{t_1} = x_1, \ldots, X_{t_k} = x_k, l \in A] \\ = M_{\mu,\beta}^{B}[X_{t_1} = x_1, \ldots, X_{t_k} = x_k, l \in A]\,. \tag{3.2.12}$$

Remark 3.2.5. *This theorem is an extension of [AV20, Theorem 2.5]. Whilst the proof is similar, we remove the condition of confinement to a finite box. Before embarking on the proof, we briefly explain the necessity of Assumptions 3.2.3. The loop length l has the discrete support $\beta\mathbb{N}$ under M^B. By [Bil68, Theorem 13.1], in order to get a consistent notion of convergence on càdlàg spaces, one needs to exclude those times on which the path is discontinuous (except on a set of measure zero). If $\beta\mathbb{N} \cap \partial A = \varnothing$, we can ensure that all the coordinate projections are continuous almost surely. The conditions on μ and on q ensure that both sides have finite mass.*

Proof of Theorem 3.2.4. We begin with the case $k = 1$. Expanding the left-hand side of Equation (3.2.12), we get

$$\begin{aligned} &M_N^{\downarrow}[X_{t_1} = x_1,\, l \in A] \\ &\quad = \sum_{x\in\mathbb{Z}^d} \sum_{b,b_1\in\mathbb{T}_N} \int_A \frac{\mathrm{e}^{t\mu}}{t} \overline{p}_{t_1}\left((x,b),(x_1,b_1)\right) \overline{p}_{t-t_1}\left((x_1,b_1),(x,b)\right) \mathrm{d}t \\ &\quad = \sum_{b_1\in\mathbb{T}_N} \int_A \frac{\mathrm{e}^{t\mu}}{t} \overline{p}_t\left((x_1,b_1),(x_1,b_1)\right) \mathrm{d}t \\ &\quad = \sum_{b_1\in\mathbb{T}_N} \int_A \frac{\mathrm{e}^{t\mu}}{t} \overline{p}_t\left(x_1,x_1\right) \overline{p}_t\left(b_1,b_1\right) \mathrm{d}t = N \int_A \frac{\mathrm{e}^{t\mu}}{t} \overline{p}_t\left(x_1,x_1\right) \overline{p}_t\left(b_1,b_1\right) \mathrm{d}t\,, \end{aligned} \tag{3.2.13}$$

where in the last line, b_1 can be any element of $\mathbb{T}_N$. To go from the second to the third line, we first used monotone convergence (to exchange integration and summation) and then the Chapman-Kolmogorov equations. We used Lemma 3.2.2 to factorise the kernel $\overline{p}_t\left((x_1,b_1),(x_1,b_1)\right)$. In the last step we used that the process on $\mathbb{T}_N$ is translation invariant.

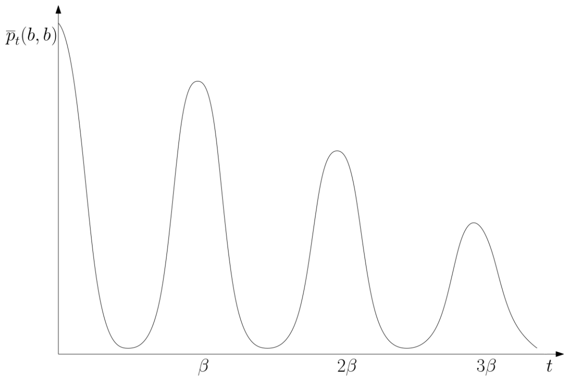

Figure 3.3: The graph of the transition kernel $\overline{p}_t\,(b_1, b_1)$. As N grows large, the peaks converge to a sum of (weighted) delta-measures.

We expand the kernel on the torus

$$\overline{p}_t\,(b_1, b_1) = \mathrm{e}^{-t\beta^{-1}N} \sum_{n=0}^{\infty} \frac{(t\beta^{-1}N)^n}{n!} p_n(b_1, b_1) = \mathrm{e}^{-t\beta^{-1}N} \sum_{j=0}^{\infty} \frac{(t\beta^{-1}N)^{jN}}{(jN)!}\,, \tag{3.2.14}$$

as the jump chain of the torus coordinate is deterministic. Thus, $p_n(b_1, b_1) \neq 0$ for $n \in N\mathbb{N}$ only. Recall that the density of a Gamma distributed variable X with parameters $(x, y) \in (0, \infty)^2$ is given by

$$\frac{y^x}{\Gamma(x)} t^{x-1} \mathrm{e}^{-yt} \mathrm{d}t\,, \tag{3.2.15}$$

where $\Gamma(x)$ denotes the Gamma function. Expectation with respect to X is denoted by $\mathbb{E}_{x,y}[X]$. Using monotone convergence and Equation (3.2.14), we can rewrite

$$M_N^{\downarrow}[X_{t_1} = x_1,\, l \in A] = \beta \sum_{j=0}^{\infty} \mathbb{E}_{jN+1,\beta^{-1}N} \left[\mathbb{1}_A(X) \overline{p}_X(x_1, x_1) \frac{\mathrm{e}^{\mu X}}{X} \right]. \tag{3.2.16}$$

Indeed, note that the density $N\beta^{-1}\overline{p}_t\,(b_1, b_1)$ is an infinite sum of Gamma densities with parameters $(jN+1, \beta^{-1}N)$. For a sketch of $\overline{p}_t\,(b_1, b_1)$, see Figure 3.3.

We can bound for any $t \geq 0$ and $x \in \mathbb{Z}^d$

$$\overline{p}_t(x,x) \le \exp\left(-t \inf_{x\in\mathbb{Z}^d} p(x,\dagger)\right) . \tag{3.2.17}$$

By [LL10, Theorem 2.5.6] we have that for any $\varepsilon > 0$ there exists $C > 0$ such that for all $t \ge \varepsilon$

$$\overline{p}_t(x,x) \le Ct^{-d/2} . \tag{3.2.18}$$

Since Assumption 3.2.3 holds true, at least one of the two above bounds converges to zero at speed $\mathcal{O}\left(t^{-d/2}\right)$. Thus

$$\mathbb{E}_{jN+1,\beta^{-1}N}\left[\mathbb{1}_A(X)\overline{p}_X(x_1,x_1)\frac{\mathrm{e}^{\mu X}}{X}\right] \le C\mathbb{E}_{jN+1,\beta^{-1}N}\left[\mathbb{1}_{[\varepsilon,\infty)}(X)X^{-d/2-1}\right] , \tag{3.2.19}$$

for some $\varepsilon > 0$ (as $\inf A > 0$). Using Lemma 8.3.1 to compute the moments the Gamma distribution, we can bound this by

$$\mathbb{E}_{jN+1,\beta^{-1}N}\left[\mathbb{1}_{[\varepsilon,\infty)}(X)X^{-d/2-1}\right] \le \left(\beta^{-1}N\right)^{d/2+1}\frac{\Gamma(jN-d/2)}{\Gamma(jN+1)} . \tag{3.2.20}$$

For $j \ge 1$ and N sufficiently large, we can expand the fraction of Gamma functions using Stirling's formula (see [LL10, Lemma A.1.4]) and bound

$$\frac{\Gamma(jN-d/2)}{\Gamma(jN+1)} \le \frac{C}{(jN+1)^{d/2+1}} . \tag{3.2.21}$$

We have shown that

$$\mathbb{E}_{jN+1,\beta^{-1}N}\left[\mathbb{1}_A(X)\overline{p}_X(x_1,x_1)\frac{\mathrm{e}^{\mu X}}{X}\right] \le Cj^{-d/2-1} \le Cj^{-3/2} , \tag{3.2.22}$$

and thus can exchange the limit as $N \to \infty$ with the sum over $j \in \mathbb{N}$.
Recall two basic properties of the Gamma function, which can be easily verified by hand: if X_i are i.i.d. Gamma distributed with parameters (x,y), then the sum $\sum_{i=1}^n X_i$ is Gamma distributed with parameters (nx,y). Furthermore, if X is Gamma distributed with parameters (x,y), then, for any $c > 0$, cX is Gamma distributed with parameters $(x,y/c)$. This implies that if X is Gamma

distributed with parameters $(jN+1, \beta^{-1}N)$

$$X \overset{d}{=} \frac{1}{N} \sum_{k=1}^{jN+1} X_i\,, \tag{3.2.23}$$

where each X_i is Gamma distributed with parameters $(1, \beta^{-1})$. Using that if X is Gamma distributed with parameters (x, y), its mean is given by x/y and the strong law of large numbers, we get that for $j > 0$

$$\lim_{N\to\infty} \mathbb{E}_{jN+1,\beta^{-1}N}\left[\mathbb{1}_A(X)\overline{p}_X(x_1,x_1)\frac{\mathrm{e}^{\mu X}}{X}\right] = \mathbb{1}_A(\beta j)\overline{p}_{\beta j}(x_1,x_1)\frac{\mathrm{e}^{\mu\beta j}}{\beta j}\,. \tag{3.2.24}$$

If X is Gamma distributed with parameters $(1, \beta^{-1}N)$, it converges to 0 almost surely and in L^p and thus (since $\inf(A) > 0$) we have that for $j = 0$

$$\lim_{N\to\infty} \mathbb{E}_{1,\beta^{-1}N}\left[\mathbb{1}_A(X)\overline{p}_X(x_1,x_1)\frac{\mathrm{e}^{\mu X}}{X}\right] = 0\,. \tag{3.2.25}$$

We can thus conclude that

$$\begin{aligned}\lim_{N\to\infty} M_N^{\downarrow}[X_{t_1} = x_1,\, l \in A] &= \beta \sum_{j=0}^{\infty} \lim_{N\to\infty} \mathbb{E}_{jN+1,\beta^{-1}N}\left[\mathbb{1}_A(X)\overline{p}_X(x_1,x_1)\frac{\mathrm{e}^{\mu X}}{X}\right] \\ &= \beta \sum_{j\in\beta^{-1}A} \overline{p}_{\beta j}(x_1,x_1)\frac{\mathrm{e}^{\mu j}}{j} = M^B[X_{t_1} = x_1,\, l \in A]\,.\end{aligned} \tag{3.2.26}$$

This finishes the proof for the case $k = 1$.
Let us now assume that $k \geq 2$. Rewrite

$$\begin{aligned}&M_N^{\downarrow}[X_{t_1} = x_1, \ldots, X_{t_k} = x_k, l \in A] \\ &\quad = \left(\prod_{i=1}^{k-1} \overline{p}_{t_{i+1}-t_i}(x_i, x_{i+1})\right) \int_A \overline{p}_{t-t_k+t_1}(x_k,x_1) N \overline{p}_t(b_1,b_1)\frac{\mathrm{e}^{t\mu}}{t}\mathrm{d}t\,,\end{aligned} \tag{3.2.27}$$

by the Chapman-Kolmogorov equations. One can use the same approximation procedure as employed in the case $k = 1$ to conclude the theorem. □

3.2.3 Convergence of local times and occupation fields

In this section we examine the convergence of the local time under $M_N^{\downarrow}$ and the occupation field under $\mathbb{P}_\lambda^{\downarrow}$. We want to show that (under $M_N^{\downarrow}$) the local time converges to the local time distributed with respect to M^B.

Assumption 3.2.6. *Assume that either*

I. The transition matrix satisfies

$$\mu - \inf_{x\in\mathbb{Z}^d} q(x,\dagger) < 0\,. \tag{3.2.28}$$

II. Or that $d \geq 3$ and $\mu = 0$.

The main result of this subsection is the following theorem.

Theorem 3.2.7. *Let $F\colon [0,\infty)^{\mathbb{Z}^d} \to \mathbb{R}$ be such that*

I. There exists $\Lambda_F \subset \mathbb{Z}^d$ bounded such that F is measurable with respect to the sigma algebra generated by the coordinates in Λ_F.

II. $F(0) = 0$ and that the right derivative at zero $\partial F_\cdot(0)$ exists for all coordinates in Λ_F. This means that for $x \in \mathbb{Z}^d, t > 0$, we abbreviate $\mathbb{R}^{\mathbb{Z}^d} \ni t_x$ for $t_x(y) = t\delta_x(y)$ and define $\partial F_x(0)$ by

$$F(t_x) = t\partial F_x(0) + o(t) \quad \text{as } t \downarrow 0\,. \tag{3.2.29}$$

III. It holds that

$$\sup_{(s_x)_{x\in\mathbb{Z}^d}} |F\left((s_x)_{x\in\mathbb{Z}^d}\right)| < \infty\,. \tag{3.2.30}$$

Given the above as well as Assumption 3.2.6, we have that

$$\lim_{N\to\infty} M_N^{\downarrow}[F(L^{\downarrow})] = \beta \sum_{x\in\mathbb{Z}^d} \partial_x F(0) + M^B[F(L)]\,. \tag{3.2.31}$$

Remark 3.2.8. *This theorem is an extension of [AV20, Theorem 2.7], where the case of random walks confined to a bounded set is considered. As we allow for the loop measure to be fully supported on $\mathbb{Z}^d$, more care needs to be taken. In fact, the largest part of the proof is to make sure that it takes the random*

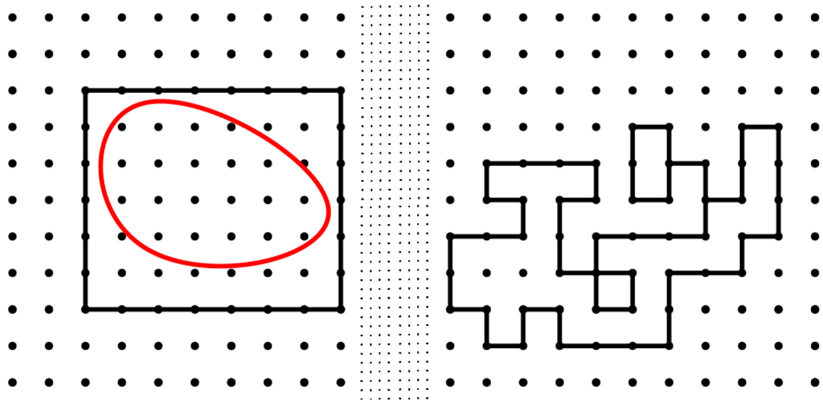

Figure 3.4: A loop (in black, on the right) started far away from the support of F (in red) are unlikely to reach $\boldsymbol{B}_m$.

walks sufficiently long to reach the support of F and we can thus interchange the limit as $N \to \infty$ together with the sum of $x \in \mathbb{Z}^d$.

Proof of Theorem 3.2.7. Without loss of generality, we may assume that $\Lambda_F \subset \boldsymbol{B}_m$ for some $m > 0$. Recall that $\boldsymbol{B}_m$ is the ball of radius m centred at zero. Abbreviate

$$L_x^{\downarrow} = \sum_{b_1 \in \mathbb{T}_N} L_{(x,b_1)} \,. \tag{3.2.32}$$

The idea of the proof is as follows: loops (of typical length) started far away from $\boldsymbol{B}_m$ are unlikely to reach the support of F. For an illustration see Figure 3.4. This will allow us to work with loops started in a finite neighbourhood around $\boldsymbol{B}_m$. We then use the convergence of the waiting times, similar to the proof of Theorem 3.2.4.

We begin by showing that loops started far away from $\boldsymbol{B}_m$ are negligible: expand, using the independence of the process on $\mathbb{Z}^d$ and on $\mathbb{T}_N$ and the translation invariance on $\mathbb{T}_N$,

$$M_N^{\downarrow}[|F(L^{\downarrow})|] = N \sum_{x \in \mathbb{Z}^d} \int_0^{\infty} \frac{\mathrm{e}^{t\mu}}{t} \overline{\mathbb{E}}^t_{(x,b_1),(x,b_1)}[|F(L^{\downarrow})|] \mathrm{d}t \,. \tag{3.2.33}$$

From now on we work with the assumption that $d \geq 3$ and $\mu = 0$. The alternative assumption (i.e. that $\mu - \inf_x q(x, \dagger) < 0$) induces an exponential decay (see Equation (3.2.17)) which is faster than the polynomial decay implied by $\mu = 0$.

We now estimate the integrand in the above equation: for $t \in [0,\varepsilon]$, we bound $\overline{p}_t(x,x) \le 1$. For $t > \varepsilon$, estimate $\overline{p}_t(x,x) \le Ct^{-d/2}$. Notice that $F\left((s_i)_i\right)$ is bounded uniformly and $F(0) = 0$ with F differentiable at zero. We can thus bound

$$\begin{aligned}\int_0^\infty \frac{1}{t}\overline{\mathbb{E}}^t_{(x,b_1),(x,b_1)}[|F(L^\downarrow)|]\mathrm{d}t &\le \sup_{x\in \boldsymbol{B}_M} |\partial_x F(0)| \int_0^\infty \frac{C}{t\vee\varepsilon}\overline{\mathbb{P}}^t_{(x,b_1),(x,b_1)}(H_m < t)\mathrm{d}t \\ &\le C\int_0^\infty \frac{1}{t^{d/2+1}\vee\varepsilon}\overline{\mathbb{B}}^t_{x,x}\left(H_m < t\right)\overline{p}_t(b_1,b_1)\mathrm{d}t\,,\end{aligned} \tag{3.2.34}$$

where the constant $C > 0$ depends on F.

Take $x \in \mathbb{Z}^d \setminus \boldsymbol{B}_m$ and define $x_m = |x| - \mathrm{dist}(x,\boldsymbol{B}_m)$. Using the union bound and Lemma 2.2.2

$$\overline{\mathbb{B}}^t_{x,x}\left(H_m < t\right) \le \sum_{y\in\boldsymbol{B}_m} \overline{\mathbb{B}}^t_{x,x}\left(H_y < t\right) \le Cm^d|x_m|^{2-d}\mathrm{e}^{-\frac{|x_m|^2}{4t}}\,. \tag{3.2.35}$$

Use the expansion in terms of Gamma functions from Theorem 3.2.4 and the above bound to write

$$\begin{aligned}N\int_0^\infty \frac{1}{t}\overline{\mathbb{E}}^t_{(x,b_1),(x,b_1)}[|F(L^\downarrow)|]\mathrm{d}t &\le CN\int_0^\infty \frac{|x_m|^{2-d}\mathrm{e}^{-\frac{|x_m|^2}{4t}}}{t^{d/2+1}\vee\varepsilon}\overline{p}_t(b_1,b_1)\mathrm{d}t \\ &\le C\sum_{j=0}^\infty \mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{|x_m|^{2-d}\mathrm{e}^{-\frac{|x_m|^2}{4t}}}{\left(t^{d/2+1}\vee\varepsilon\right)}\right]\,,\end{aligned} \tag{3.2.36}$$

where the Gamma distributed random variable is denoted by t and the constant C depends on m.

The strategy for the next part of the proof is the following: we want to show that as we move the base point x of the loop further away from $\boldsymbol{B}_m$ (i.e. $|x_m| \to \infty$), the sum above becomes a negligible contribution to Equation (3.2.33).

For this purpose fix $K > 2m > 0$ and use the Lemmas 8.2.1, 8.2.2 to bound a sum by an integral:

$$\begin{aligned}
&\sum_{x\in\mathbb{Z}^d\colon |x|>K}\sum_{j=0}^{\infty}\mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{|x_m|^{2-d}\mathrm{e}^{-\frac{|x_m|^2}{4t}}}{(t^{d/2+1}\vee\varepsilon)}\right]\\
&\qquad=\sum_{j=0}^{\infty}\mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{1}{(t^{d/2+1}\vee\varepsilon)}\sum_{x\in\mathbb{Z}^d\colon |x|>K}|x_m|^{2-d}\mathrm{e}^{-\frac{|x_m|^2}{4t}}\right]\\
&\qquad\le C\sum_{j=0}^{\infty}\mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{1}{(t^{d/2+1}\vee\varepsilon)}\int_K^{\infty}r^{d-1}r^{2-d}\mathrm{e}^{-\frac{r^2}{4t}}\,\mathrm{d}r\right]\\
&\qquad\le C\sum_{j=0}^{\infty}\mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{K}{(t^{d/2}\vee\varepsilon)}\mathrm{e}^{-\frac{K^2}{4t}}\right].
\end{aligned}\tag{3.2.37}$$

We begin with treating the case $j=0$ in the above sum. By comparing densities, we note that a Gamma distributed random variable with parameters $(1,\beta^{-1}N)$ has the same distribution as an exponentially distributed random variable with parameter $\beta^{-1}N$. Thus

$$\mathbb{E}_{1,\beta^{-1}N}\left[\frac{\mathrm{e}^{-\frac{K^2}{4t}}}{(t^{d/2}\vee\varepsilon)}\right]\le C\mathrm{e}^{-K^{1/3}}+C\int_{K^{2/3}}^{\infty}\left(\beta^{-1}N\right)^{-1}\mathrm{e}^{-tN\beta^{-1}}\mathrm{d}t\le C\mathrm{e}^{-K^{1/3}}.\tag{3.2.38}$$

Therefore, we can choose $K_1\in\mathbb{N}$ such that the K-times the above is smaller than $\delta/2>0$ for all $N\in\mathbb{N}$, $K>K_1$ and $\delta>0$ arbitrary but fixed.
For a fixed $K>0$, split the remaining sum from Equation (3.2.37)

$$\sum_{j=1}^{\infty}\mathbb{E}_{jN+1,\beta^{-1}N}[\ldots]=\sum_{j=1}^{K^{2/3}}\mathbb{E}_{jN+1,\beta^{-1}N}[\ldots]+\sum_{j=K^{2/3}}^{\infty}\mathbb{E}_{jN+1,\beta^{-1}N}[\ldots].\tag{3.2.39}$$

Begin with $j\le K^{2/3}$ and split the expectation into the regime where $t<K^{3/4}$ and its complement:

$$\mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{1}{(t^{d/2}\vee\varepsilon)}\mathrm{e}^{-\frac{K^2}{4t}}\right]\le C\mathrm{e}^{-K^{1/4}}+\mathbb{P}_{jN+1,\beta^{-1}N}\left(t\ge K^{3/4}\right).\tag{3.2.40}$$

Recall the fact that the mean of a Gamma $(jN+1,\beta^{-1}N)$ distributed random variable is given by $\beta(j+1/N)$.
Recall the large deviation inequality $P(Y\ge y)\le\exp\left(-\Lambda(y)\right)$, for Y a real-

valued random variable, $y > \mathbb{E}[Y]$ and Λ the associated large deviation rate function. In Lemma 8.3.2, we show that for a Gamma random variable (with parameter $(jN+1, \beta^{-1}N)$), the rate function is given by

$$\Lambda(y) = \begin{cases} \beta^{-1}Ny + (jN+1)\left(\log\left((jN+1)\right) - 1 + \log\left(\beta N y\right)\right) & \text{if } y > 0\,, \\ +\infty & \text{otherwise.} \end{cases} \tag{3.2.41}$$

Thus, using that j is bounded by $K^{2/3}$, we get

$$\mathbb{P}_{jN+1,\beta^{-1}N}\left(t \geq K^{3/4}\right) \leq \mathbb{P}_{jN+1,\beta^{-1}N}\left(t \geq CjK^{1/12}\right) \leq \exp\left(-CK^{1/12}\right)\,. \tag{3.2.42}$$

Therefore,

$$\sum_{j=0}^{K^{2/3}} \mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{K}{(t^{d/2} \vee \varepsilon)} \mathrm{e}^{-\frac{K^2}{4t}}\right] \leq CK^{5/3} \exp\left(-C^{-1}K^{1/12}\right)\,, \tag{3.2.43}$$

and so we can choose K_2 large enough such that the above is smaller than $\delta/4$ for all $K > K_2$ and $N \in \mathbb{N}$.

For $j \geq K^{2/3}$ we bound using Lemma 8.3.1 for the moments of the Gamma distribution

$$\mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{K}{(t^{d/2} \vee \varepsilon)} \mathrm{e}^{-\frac{K^2}{4t}}\right] \leq C\mathbb{E}_{jN+1,\beta^{-1}N}\left[t^{-d/2}\right] \leq C\left(\frac{\beta^{-1}N}{jN - d/2}\right)^{d/2}\,. \tag{3.2.44}$$

Thus, we can bound

$$\sum_{j=K^{2/3}}^{\infty} \mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{1}{(t^{d/2} \vee \varepsilon)} \mathrm{e}^{-\frac{K^2}{4t}}\right] \leq C \sum_{j=K^{2/3}}^{\infty} j^{-d/2}\,. \tag{3.2.45}$$

As the above sum is convergent, we can choose K_3 such that for all $N \in \mathbb{N}$ and $K > K_3$ we have

$$\sum_{j=K^{2/3}}^{\infty} \mathbb{E}_{jN+1,\beta^{-1}N}\left[\frac{1}{(t^{d/2} \vee \varepsilon)} \mathrm{e}^{-\frac{K^2}{4t}}\right] \leq \delta/4\,. \tag{3.2.46}$$

By collecting the previous estimates, we conclude that for $\delta > 0$ there exists

$K_0 \in \mathbb{N}$ such that for all $N \in$ and all $K > K_0$ we have that

$$N \sum_{x\in\mathbb{Z}^d\colon |x|>K} \int_0^\infty \frac{1}{t} \overline{\mathbb{E}}^t_{(x,b_1),(x,b_1)}[|F(L^\downarrow)|]\mathrm{d}t \le \delta\,. \tag{3.2.47}$$

Thus, we can exchange the limit of $N \to \infty$ with the sum over all $x \in \mathbb{Z}^d$ in Equation (3.2.33). By the independence of the processes on $\mathbb{Z}^d$ and on $\mathbb{T}_N$ established in Lemma 3.2.2 we can write

$$\overline{\mathbb{E}}^t_{(x,b_1),(x,b_1)}[F(L^\downarrow)] = \mathbb{E}^t_{x,x}[F(L)]\overline{p}_t(b_1,b_1)\,. \tag{3.2.48}$$

Thus, as in the proof of Theorem 3.2.4,

$$N\int_0^\infty \frac{\mathrm{e}^{t\mu}}{t}\overline{\mathbb{E}}^t_{(x,b_1),(x,b_1)}[F(L^\downarrow)]\mathrm{d}t = \beta\sum_{j=0}^\infty \mathbb{E}_{jN+1,\beta^{-1}}\left[\frac{\mathrm{e}^{t\mu}}{t}\mathbb{E}^t_{x,x}[F(L)]\right]\,. \tag{3.2.49}$$

For $j \ge 1$ we use the convergence (similar to Equation (3.2.23)) to

$$\lim_{N\to\infty} \mathbb{E}_{jN+1,\beta^{-1}}\left[\frac{1}{t}\overline{\mathbb{E}}^t_{x,x}[F(L)]\right] = \frac{1}{\beta j}\overline{\mathbb{E}}^{\beta j}_{x,x}[F(L)]\,. \tag{3.2.50}$$

For $j = 0$ expand

$$F(t_x) = t\partial_x F(0) + o(t)\,. \tag{3.2.51}$$

Write

$$\overline{\mathbb{E}}^t_{x,x}[F(L)] = F(t_x)\overline{\mathbb{P}}^t_{x,x}\,(\text{RW does not jump}) + \overline{\mathbb{E}}^t_{x,x}[F(L)\mathbb{1}\{\text{RW does jump}\}]. \tag{3.2.52}$$

By Equation (2.1.14), we have that

$$\overline{\mathbb{P}}^t_{x,x}\,(\text{RW does not jump}) = 1 - \mathcal{O}(t^2) \text{ and } \overline{\mathbb{P}}^t_{x,x}\,(\text{RW does jump}) = \mathcal{O}(t^2)\,. \tag{3.2.53}$$

As F is a bounded function and $\mathcal{O}(t^2)$ is stronger than $o(t)$ we can expand

$$\overline{\mathbb{E}}^t_{x,x}[F(L)] = t\partial_x F(0) + o(t)\,. \tag{3.2.54}$$

Thus

$$\mathbb{E}_{1,\beta^{-1}N}\left[\frac{1}{t}\overline{\mathbb{E}}^{t}_{x,x}[F(L)]\right] = \mathbb{E}_{1,\beta^{-1}N}\left[\partial_x F(0) + o(1)\right] = \partial_x F(0) + o(1)\,. \quad (3.2.55)$$

In the last equality in the above Equation, we use that if t is distributed with respect to a Gamma distribution with parameters $(1, \beta^{-1}N)$, then $t \to 0$ almost surely as $N \to \infty$. This concludes the proof. □

As a corollary, we deduce the convergence of the occupation field in a suitable topology.

Corollary 3.2.9. *In the topology of local convergence (for a definition see Definition 8.4.1) it holds that $\mathcal{L}^{\downarrow} - \beta$ converges to $\mathcal{L}$ distributed with respect to $\mathbb{P}^{B}_{\lambda}$, given $\lambda > 0$ and Assumption 3.2.6. Here, β denotes the constant field: $\beta_x = \beta$.*

Proof of Corollary 3.2.9. By Proposition 8.4.2, it suffices to show the convergence of $f^{\Lambda}_{m}(\mathcal{L})$, where $(f^{\Lambda}_{m})_m$ is a separating class for coordinates with values $[0,\infty)^{\Lambda}$ and $\Lambda \Subset \mathbb{Z}^d$. By [Kle13, Theorem 15.6], we have that

$$\left\{ f\colon f(\varphi) = \exp\left(-\sum_{x\in\Lambda} r_x \varphi_x\right),\, r_x \geq 0 \right\}, \quad (3.2.56)$$

where Λ ranges over all finite subsets of $\mathbb{Z}^d$, is such a class of functions. By [Kal01, Lemma 12.2], we have that

$$\mathbb{E}^{\downarrow}_{\lambda}\left[\mathrm{e}^{-\sum_{x\in\Lambda} r_x \mathcal{L}^{\downarrow}_x}\right] = \exp\left(-\lambda M^{\downarrow}_{N}\left[1 - \mathrm{e}^{-\sum_{x\in\Lambda} r_x L^{\downarrow}_x}\right]\right)\,. \quad (3.2.57)$$

Note that as $\sup_{x\in\Lambda}\{|L^{\downarrow}_x|\} \to 0$ we have that

$$1 - \exp\left(-\sum_{x\in\Lambda} r_x L^{\downarrow}_x\right) = \sum_{x\in\Lambda} r_x L^{\downarrow}_x + o\left(\sup_{x}\{|L^{\downarrow}_x|\}\right)\,. \quad (3.2.58)$$

Thus, by applying Theorem 3.2.7, we have that

$$\lim_{N\to\infty} \mathbb{E}^{\downarrow}_{\lambda}\left[\mathrm{e}^{-\sum_{x\in\Lambda} r_x \mathcal{L}^{\downarrow}_x}\right] = \exp\left(-\lambda\beta\sum_{x\in\Lambda} r_x - \lambda M^{B}\left[1 - \mathrm{e}^{-\sum_{x\in\Lambda} r_x L_x}\right]\right)\,. \quad (3.2.59)$$

This concludes the proof. □

3.3 Isomorphism theorems

This section provides results regarding the distribution of the occupation field of the loop soup. We only treat isomorphism in finite volume as infinite volume versions of the fields may not exist. Extending our results to the whole of $\mathbb{Z}^d$ can be done using similar arguments to the proof of Proposition 3.2.7.
We restrict the random walk to some finite, connected subset Λ of $\mathbb{Z}^d$. Define

$$q_\Lambda(x,y) = \begin{cases} q(x,y) & \text{if } x,y \in \Lambda \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.1}$$

This induces a random walk with Dirichlet boundary conditions. Let $Q \in \mathbb{R}^{\Lambda\times\Lambda}$ be the matrix with q_Λ as entries. Enumerate the real eigenvalues $(q_y)_{y\in\Lambda}$ of Q and write $a_y = -q_y$. For $(v_x)_{x\in\Lambda}$ with $v_x \geq 0$, define $V \in \mathbb{R}^{\Lambda\times\Lambda}$ the matrix with $(v_x)_x$ on the diagonal and zero everywhere else. Write p_y for the eigenvalues of $Q - V$ and set $b_y = -p_y$. For a weight measure $\mathfrak{m}$, denote $\overline{\mathfrak{m}}$ the measure defined by its Radon–Nikodym derivative

$$\mathrm{d}\overline{\mathfrak{m}}(t) = t\mathrm{d}\mathfrak{m}(t)\,. \tag{3.3.2}$$

For example, in the case of the Markovian loop measure, we have that $\mathrm{d}\mathfrak{m}(t) = t^{-1}\mathrm{d}t$ and thus $\overline{\mathfrak{m}}$ is the Lebesgue measure. For the Bosonic loop measure, we have that $\overline{\mathfrak{m}}$ is a weighted counting measure on $\beta\mathbb{N}$. We restrict ourselves to $\mathfrak{m}$ being a positive measure and refer to [AV20, Equation 4.37] for a construction of PPP for signed measures.
For a measure $\mathfrak{m}$ on $[0,\infty)$, we define its Laplace transform $\mathfrak{L}_\mathfrak{m} = \mathfrak{L}(\mathfrak{m})$ as follows

$$\mathfrak{L}_\mathfrak{m}(x) = \mathfrak{L}(\mathfrak{m},x) = \int_0^\infty \mathrm{e}^{-xt}\mathrm{d}\mathfrak{m}(t), \ \text{ with } x > 0\,. \tag{3.3.3}$$

If $\mathfrak{m}$ has density f with respect to the Lebesgue measure, we write $\mathfrak{L}_f$. Denote the inverse Laplace transform by $\mathfrak{L}^{-1}$.
We are now in the position to state our isomorphism theorem. It gives the distribution of the occupation field for loop measures with general weight by

computing the Laplace transform.

Theorem 3.3.1. *Let $h\colon (-\infty, 0] \to \mathbb{R}$ be analytically extendable to the half plane $\{z \in \mathbb{C}\colon \Re(z) \leq 0\}$. Fix $\lambda > 0$. Suppose that*

$$\overline{\mathfrak{m}} = -\mathfrak{L}^{-1}\left(\frac{\partial h}{h}\right), \tag{3.3.4}$$

satisfies the conditions of Definition 3.1.5. Furthermore, assume that for any $\varepsilon > 0$ the integral $\int_\varepsilon^x \mathfrak{L}_{\overline{\mathfrak{m}}}(s)\mathrm{d}s$ exists for all $x > \varepsilon$. Then:

I. There exists a measure Σ on $[0,\infty)^\Lambda$ such that its Laplace transform is given by

$$\Sigma\left[\mathrm{e}^{-\langle v,\psi\rangle}\right] = \left(\frac{\det h(Q)}{\det h(Q-V)}\right)^\lambda . \tag{3.3.5}$$

II. Under $\mathbb{P}_\lambda^{\mathfrak{m}}$, we have that the occupation field $\mathcal{L}$ is distributed like ψ under Σ, i.e. for any bounded test function $u\colon \mathbb{R}^d \to \mathbb{R}$, we have that

$$\mathbb{E}_\lambda^{\mathfrak{m}}\left[u(\mathcal{L})\right] = \Sigma\left[u(\psi)\right] . \tag{3.3.6}$$

Proof of Theorem 3.3.1. Since the Laplace transform uniquely characterises a measure, the theorem follows upon showing that

$$\mathbb{E}_\lambda^{\mathfrak{m}}\left[\mathrm{e}^{-\langle v,\mathcal{L}\rangle}\right] = \left(\frac{\det h(Q)}{\det h(Q-V)}\right)^\lambda . \tag{3.3.7}$$

By the Campbell formula for $\mathbb{P}_\lambda^{\mathfrak{m}}$

$$\mathbb{E}_\lambda^{\mathfrak{m}}\left[\mathrm{e}^{-\langle v,\mathcal{L}\rangle}\right] = \exp\left(-\lambda M^{\mathfrak{m}}\left[1-\mathrm{e}^{-\langle v,L\rangle}\right]\right) . \tag{3.3.8}$$

Note that due to the Dirichlet boundary conditions, the eigenvalues of Q are contained in $(-\infty,0)$. Choose $\varepsilon > 0$ such that $-\varepsilon$ is larger than the largest eigenvalue of Q. By Weyl's inequality (see e.g. [HJ12, Theorem 4.3.1]), we have that the eigenvalues of $Q-V$ are also contained in $(-\infty,-\varepsilon)$. We recall that $(a_y)_y$ are the eigenvalues of Q with their sign flipped and $(b_y)_y$ for $Q-V$.

Expand, using that the trace is the sum of the eigenvalues,

$$\begin{aligned}
M^{\mathfrak{m}}\left[1-\mathrm{e}^{-\langle v,L\rangle}\right] &= \sum_{x\in\Lambda}\int_0^\infty \left(\mathrm{e}^{tQ}(x,x)-\mathrm{e}^{t(Q-V)}(x,x)\right)\mathrm{d}\mathfrak{m}(t)\\
&= \int_0^\infty \operatorname{Tr}\left(\mathrm{e}^{Qt}-\mathrm{e}^{(Q-V)t}\right)\mathrm{d}\mathfrak{m}(t)\\
&= \sum_{y\in\Lambda}\int_0^\infty \left(\mathrm{e}^{-a_y t}-\mathrm{e}^{-tb_y}\right)\mathrm{d}\mathfrak{m}(t)\\
&= \sum_{y\in\Lambda}\int_0^\infty \left(\mathrm{e}^{-a_y t}-\mathrm{e}^{-t\varepsilon}\right)\mathrm{d}\mathfrak{m}(t)-\int_0^\infty \left(\mathrm{e}^{-b_y t}-\mathrm{e}^{-t\varepsilon}\right)\mathrm{d}\mathfrak{m}(t)\,.
\end{aligned} \tag{3.3.9}$$

Fix $y\in\Lambda$ and observe that $g(a_y)=\int_0^\infty\left(\mathrm{e}^{-a_y t}-\mathrm{e}^{-t\varepsilon}\right)\mathrm{d}\mathfrak{m}(t)$ satisfies the following ODE

$$\begin{cases}\partial g(a_y) & = -\mathfrak{L}_{\overline{\mathfrak{m}}}(a_y)\,,\\ g(\varepsilon) & = 0\,.\end{cases} \tag{3.3.10}$$

If $g(x)=\log(h(x))$, this implies that

$$h(x)=\exp\left(-\int_\varepsilon^x \mathfrak{L}_{\overline{\mathfrak{m}}}(s)\mathrm{d}s\right)\,. \tag{3.3.11}$$

Thus

$$\begin{aligned}
\exp\left(-\lambda M^{\mathfrak{m}}\left[1-\mathrm{e}^{-\langle v,L\rangle}\right]\right) &= \exp\left(-\lambda\sum_{x\in\Lambda} g(a_y)-g(b_y)\right)\\
&= \exp\left(-\lambda\sum_{y\in\Lambda}\log\left(\frac{h(b_y)}{h(a_y)}\right)\right)=\prod_{y\in\Lambda}\left(\frac{h(a_y)}{h(b_y)}\right)^\lambda\\
&= \left(\frac{\det h(Q)}{\det h(Q-V)}\right)^\lambda\,.
\end{aligned} \tag{3.3.12}$$

In the last line we use that h can be written as a power series, and thus the eigenvalues of $h(Q)$ are the images of h applied to the eigenvalues of Q.

On the other hand, since

$$\log(h(x)) = -\int_{\varepsilon}^{x} \mathfrak{L}_{\overline{\mathfrak{m}}}(s)\mathrm{d}s\,, \tag{3.3.13}$$

we have that

$$\frac{\partial h(x)}{h(x)} = -\mathfrak{L}_{\overline{\mathfrak{m}}}(x)\,, \tag{3.3.14}$$

and thus we get the condition on $\mathfrak{m}$ stated in Equation (3.3.4) is satisfied. This concludes the proof. □

In the next remark we collect some examples of loop weights.

Remark 3.3.2. *I. The above theorem is a straight-forward extension of the Le Jan isomorphism, as presented in [LJ11, LJ10]. Indeed, choose $\lambda = 1/2$ and $h(x) = x$, we solve*

$$\overline{\mathfrak{m}} = -\mathfrak{L}^{-1}\left(\frac{\partial h}{h}\right) = -\mathfrak{L}^{-1}\left(\frac{1}{\mathrm{Id}}\right) = \textit{Lebesgue measure}\,, \tag{3.3.15}$$

and thus $\mathrm{d}\mathfrak{m}(t) = t^{-1}\mathrm{d}t$. The resulting measure Σ is the distribution of the square of the Gaussian free field with covariance Q.

II. For the Bosonic field (the occupation field under $\mathbb{P}^{B}_{\lambda}$) introduced in [AV20], we choose $h(x) = \beta^{-1}(1 + \mathrm{e}^{\beta(x+\mu)})$ for some $\beta > 0$. It is easy to see that in this case

$$-\,\mathfrak{L}^{-1}\left(\frac{\partial h}{h}\right) = \sum_{j\geq 1} \mathrm{e}^{\beta\mu}\delta_{\beta j} = \overline{\mathfrak{m}}\,. \tag{3.3.16}$$

This implies that $\mathfrak{m} = \sum_j \delta_{\beta j}\mathrm{e}^{\beta\mu j}/j$ and thus we recover the Bosonic loop measure. See also [AV20, Lemma 4.2].

III. Choosing the positive measure $\mathfrak{m}$ via

$$\mathrm{d}\mathfrak{m}(t) = \frac{1}{t}\left(\sum_{n=1}^{\infty} \mathrm{e}^{-(2n-1)^2\pi^2 t^2/4}\right)\mathrm{d}t\,, \tag{3.3.17}$$

results in

$$\mathfrak{L}_{\overline{\mathfrak{m}}}(s) = \frac{\sinh\left(\sqrt{s}\right)}{\sqrt{s}\cosh\left(\sqrt{s}\right)} \qquad \textit{and thus} \qquad h(x) = \cosh^{-2}\left(\sqrt{s}\right)\,. \tag{3.3.18}$$

The above is shown in [LSL12, Equation 32.150]. Thus,

$$\mathbb{E}^{\mathfrak{m}}_{\lambda}\left[\mathrm{e}^{-\langle v,\mathcal{L}\rangle}\right]=\left(\frac{\det\cosh\left(\sqrt{-Q}\right)}{\det\cosh\left(\sqrt{V-Q}\right)}\right)^{-2\lambda}. \tag{3.3.19}$$

Remark 3.3.3. *In [AV20] it is shown that one can also define the PPP for signed measures. Extending the above theorem to signed measures* $\mathfrak{m}$ *gives these two additional examples:*

I. *For the Fermionic loop measure (introduced in [BR03, Theorem 6.3.14])* $M^F_{\beta,\mu}=\sum_x\sum_{j\geq 1}\frac{(-1)^j\mathrm{e}^{\beta\mu j}}{j}\mathbb{P}^{(\beta\mu)}_{x,x}$*, we can do the same calculation as we did for the Bosonic loop measure, where* $1+\mathrm{e}^{\beta(x+\mu)}$ *is replaced by* $1-\mathrm{e}^{\beta(x+\mu)}$.

II. *Suppose we choose*

$$\mathrm{d}\mathfrak{m}(t)=\frac{\cos(at)}{t}\mathrm{d}t\,, \tag{3.3.20}$$

for some $a>0$*. Note that the resulting loop measure is no longer a positive measure. By [LSL12, Equation 32.33]*

$$\mathfrak{L}_{\overline{\mathfrak{m}}}(s)=\frac{s}{s^2+1}\qquad\text{and thus}\qquad h(x)=\sqrt{x^2+a^2}\,. \tag{3.3.21}$$

This implies

$$\mathbb{E}^{\mathfrak{m}}_{\lambda}\left[\mathrm{e}^{-\langle v,\mathcal{L}\rangle}\right]=\left(\frac{\det Q^2+a^2\mathrm{Id}}{\det(Q-V)^2+a^2\mathrm{Id}}\right)^{\lambda/2}. \tag{3.3.22}$$

However, as in this work we restrict ourselves to positive measures, we do not prove the above.

Chapter 4

Connectivity results for loop measures

In this chapter we prove various estimates for connectivity events with respect to the loop measure. As connectivity features of the loop soup solely depend on the jump chain $(S_n)_n$, we work with general discrete-time loop measure $M = \sum_x \sum_j a_j \mathbb{P}^j_{x,x}$. This is justified by Proposition 3.1.7.
In the first part of the chapter, we prove concentration inequalities for the range of random walk bridges. We then use those to prove a sharp estimate for the mass of all the loops connecting the origin to the complement of large spheres. The last part of the chapter is devoted to a technical estimate, which will be useful later on. The whole chapter treats the case where $q(x, \dagger) = 0$ for all $x \in \mathbb{Z}^d$.

4.1 Introduction and preliminary results

We define the range of the random walk as follows: let $\mathcal{R}_j$ be the number of vertices visited up to time j, i.e.

$$\mathcal{R}_j = |\{x \in \mathbb{Z}^d \colon \exists k \in \{1, \dots, j\} \text{ such that } S_k = x\}| . \tag{4.1.1}$$

Recall that Assumption 2.1.1 (defining the class of admissible random walks) still holds. Let (for any $\delta > 0$) and $j \in \mathbb{N}$

$$r_j = \begin{cases} \sqrt{\pi j/2} + \mathcal{O}\left(j^{1/4}\right) & \text{if } d = 1\,, \\ \pi j/\log j + \mathcal{O}\left(j \log\log(j)/\log^2(j)\right) & \text{if } d = 2\,, \\ \kappa_3 j + \mathcal{O}\left(j^{1/2}/\log^{\delta}(j)\right) & \text{if } d = 3\,, \\ \kappa_4 j - 8\kappa_4^2 \log(j)/\pi^2 + \mathcal{O}(1) & \text{if } d = 4\,, \\ \kappa_d j + \mathcal{O}(1) & \text{if } d \geq 5\,. \end{cases} \tag{4.1.2}$$

It is shown in [Ham06, Theorem 2.2] that the above is the expected range of a random walk bridge, i.e.

$$r_j = \mathbb{B}^j_{0,0}\left[\mathcal{R}_j\right] = \frac{1}{p_j(0)}\mathbb{E}^j_{0,0}\left[\mathcal{R}_j\right] = \frac{1}{\mathfrak{p}_j(0)}\mathbb{E}^j_{0,0}\left[\mathcal{R}_j\right]\left(1 + \mathcal{O}\left(\frac{1}{j}\right)\right)\,. \tag{4.1.3}$$

In the next lemma we give some bounds on the probability that the range $\mathcal{R}_j$ deviates from r_j (on the scale of r_j). Combining a number of fairly recent results, the proof is short except in the case $d = 2$. There, one needs to introduce an additional argument.

Lemma 4.1.1. *Let $d \geq 3$. For every $\varepsilon > 0$, there exists $\alpha > 0$ and $c > 0$ such that*

$$\mathbb{B}^j_{0,0}\left(|\mathcal{R}_j - r_j| \geq \varepsilon r_j\right) \leq \mathcal{O}\left(\mathrm{e}^{-cj^{\alpha}}\right)\,. \tag{4.1.4}$$

For $d = 2$ and for every $\varepsilon > 0$, we have that

$$\mathbb{B}^j_{0,0}\left(|\mathcal{R}_j - r_j| \geq \varepsilon r_j\right) = \frac{1}{\varepsilon^2}\mathcal{O}\left(\frac{\log\left(\log(j)\right)}{\log(j)}\right)\,. \tag{4.1.5}$$

Additionally, for $d = 2$ and for $\varepsilon > 0$ fixed

$$\mathbb{B}^j_{0,0}\left[\mathcal{R}_j,\, |\mathcal{R}_j - r_j| \geq \varepsilon r_j\right] = \mathcal{O}\left(\frac{j\log^2(\log(j))}{\log^3(j)}\right)\,. \tag{4.1.6}$$

Remark 4.1.2. *This lemma is crucial for the following reason: the strong concentration of the range $\left(\mathcal{R}_j\right)_j$ onto the deterministic sequence $(r_j)_j$ allows us to use $(r_j)_j$ instead of $\left(\mathcal{R}_j\right)_j$.*

Proof of Lemma 4.1.1. We begin with the case $d \geq 3$. The result is implied by various large deviation type upper and lower bounds: in [HK01, Theorem 1] and [Ham06, Theorem 2.3], it is shown that

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}_x \left(\mathcal{R}_j \geq bj\right) = \lim_{n\to\infty} \frac{1}{n} \log \mathbb{B}^j_{x,x} \left(\mathcal{R}_j \geq bj\right) = -I^{(1)}(b)\,, \tag{4.1.7}$$

where $I^{(1)}(b) > 0$ if $b > \kappa_d$ (where $\kappa_d = \mathbb{P}_0(H_0 = \infty)$ is the escape probability for the random walk). In [Phe11, Theorem 1.2.10], it is shown that for the simple random walk

$$\lim_{n\to\infty} \frac{1}{n^{(d-2)/d}} \log \mathbb{P}_x \left(\mathcal{R}_j \leq bj\right) = -I^{(2)}(b)\,, \tag{4.1.8}$$

where $I^{(2)}(b) > 0$ if $b < \kappa_d$. In [LV19], it is proved that the result also holds for random walks with finite moment generating function. Thus, the result is applicable to our setting. Combining the above bounds finishes the proof for the case $d \geq 3$, as the exponential decay from the large deviation type bounds dominates the polynomial decay from the bridge condition:

$$\begin{aligned}\mathbb{P}^j_{0,0} \left(|\mathcal{R}_j - r_j| \geq \varepsilon r_j\right) &\leq \mathbb{P}_x \left(\mathcal{R}_j \leq (1-\varepsilon) r_j\right) + \mathbb{P}_x \left(\mathcal{R}_j \geq (1+\varepsilon) j\right) \\ &\leq \mathcal{O}\left(\mathrm{e}^{-\mathcal{O}\left(n^{1-2/d}\right)}\right).\end{aligned} \tag{4.1.9}$$

In the case $d = 2$, the first result follows from Chebyshev's inequality. Indeed, note that by [Ham06, Theorem 2.3], we have that

$$\mathbb{B}^j_{0,0} \left[\left(\mathcal{R}_j - \mathbb{B}^j_{0,0}[\mathcal{R}_j]\right)^2\right] = \mathcal{O}\left(\frac{j^2 \log\log(j)}{\log^3(j)}\right). \tag{4.1.10}$$

Thus, by Chebyshev's inequality and noting that $r_j = \mathbb{B}^j_{0,0}[\mathcal{R}_j]$, we get that for every $\varepsilon > 0$

$$\mathbb{B}^j_{0,0} \left(|\mathcal{R}_j - r_j| \geq \varepsilon r_j\right) = \frac{1}{\varepsilon^2} \mathcal{O}\left(\frac{\log\left(\log(j)\right)}{\log(j)}\right). \tag{4.1.11}$$

To verify the second claim made for $d = 2$, we introduce a new argument: partition the event $\{|\mathcal{R} - r_j| \geq \varepsilon r_j\}$ into sub-events by dividing the interval $[0, \varepsilon r_j]$ into shorter scales. We use Chebyshev's inequality on each scale and then sum the resulting error.

Fix a positive increasing sequence $b_j = o(\log\log(j))$ with $\lim_j b_j = \infty$. Let A_k be the event

$$A_k = \left\{ |\mathcal{R}_j/r_j - 1| \in \left[\varepsilon + k\log(j)b_j^{-1}, \varepsilon + (k+1)\log(j)b_j^{-1}\right] \right\}. \qquad (4.1.12)$$

for $k \geq 0$. We then estimate

$$\begin{aligned}
&\mathbb{B}^j_{0,0}\left[\mathcal{R}_j,\, |\mathcal{R}_j - r_j| \geq \varepsilon r_j\right] \\
&\quad \leq \sum_{k=0}^{b_j-1}\left((1+\varepsilon)r_j + (k+1)\frac{r_j\log(j)}{b_j}\right)\mathbb{B}^j_{0,0}(A_k) \\
&\quad \leq C\sum_{k=0}^{b_j-1}\left((1+\varepsilon)r_j + (k+1)\frac{r_j\log(j)}{b_j}\right)\frac{1}{\left(\varepsilon + k\log(j)b_j^{-1}\right)^2}\left(\frac{\log\log(j)}{\log(j)}\right) \\
&\quad \leq C\sum_{k=0}^{b_j-1}\frac{jb_j\log(\log(j))}{(k+1)\log^3(j)} \leq C\frac{j\log\log(j)b_j\log(b_j)}{\log^3(j)},
\end{aligned} \qquad (4.1.13)$$

where Chebyshev's inequality gives an estimate on the probability of A_k. Using the assumptions on $(b_j)_j$ concludes the lemma.

□

In the following sections, we use the concentration inequalities from the previous lemma and our new approach to prove results for the connectivity. Before continuing, we offer a guiding principle: for the random walk/bridge (or Brownian motion) to traverse a distance proportional to $\sim n$, we need time $\sim n^2$. Usually it will be of interest to know the behaviour for times large than n^{2-}, where one should think of n^{2-} as slightly smaller than n^2 (say up to a logarithmic scale). Characterising the behaviour for times in between n^{2-} and n^2 is usually the most challenging part of our proofs.

4.2 Sharp connectivity estimates for connecting 0 to B_n

In this section we prove a sharp loop estimate. The result is, to our best knowledge, new even in the case of the Markovian loop measure.

We begin by stating a class of sequences $(a_j)_j$ such that the next proposition holds. Concrete examples are given in the second part of the proposition.

Assumption 4.2.1. *Let $d \geq 3$. Assume that:*

- *$(a_j)_j$ a sequence with values in $[0, \infty)$.*
- *$a_j \geq Cj^\nu$ for some $C > 0$ and $\nu > -\infty$. Also, $a_j = \mathcal{O}\left(j^{d/2-2}\right)$.*
- *Furthermore, for $S \in (2, 3)$:*

$$\sum_{j \geq n^S} \frac{a_j j}{j^{d/2}} = o\left(\sum_{j \geq n^2} \frac{a_j r_j}{j^{d/2}}\right) . \tag{4.2.1}$$

For $d = 2$, assume that the above holds with two additional conditions: fix $\varepsilon > 0$ and let n_1 be

$$n_1 = \frac{n^2}{\log\left(\log^{1+\varepsilon}(n)\right)} . \tag{4.2.2}$$

Assume

$$\sum_{j=n_1}^{n^S} a_j \frac{\log^2(\log(n))}{\log^3(n)} = o\left(\sum_{j=n_1}^{n^S} a_j r_j \mathfrak{p}_j(0) B^j_{0,0}\left(H^B_n < j\right)\right) , \tag{4.2.3}$$

and

$$\sum_{j=1}^{n_1} a_j \mathrm{e}^{-cn^2/j} = o\left(\sum_{j=n_1}^{n^S} a_j r_j \mathfrak{p}_j(0) B^j_{0,0}\left(H^B_n < j\right)\right) , \tag{4.2.4}$$

for some $c > 0$ sufficiently small.

The above assumptions are often quick to verify in practice as we will see in the proof of the next proposition.
For a loop measure M, denote $M[A \overset{\omega}{\longleftrightarrow} B]$ the mass of all loops which intersect both A and B, with $A, B \subset \mathbb{Z}^d$.

Proposition 4.2.2. *If Assumption 4.2.1 holds, then:*

I. For $d \geq 2$

$$M^a[0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}^c_n] = (1 + o(1)) \sum_{j=1}^{\infty} a_j r_j \mathfrak{p}_j(0) B^j_{0,0}\left(H^B_n < j\right) . \tag{4.2.5}$$

II. Fix $\nu < d/2-2$. If $d \geq 3$ and $a_j = j^\nu(1+o(1))$ or $a_j = j^\nu \log(j)(1+o(1))$ for $d = 2$, we get the decay

$$M^a[0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c] = \kappa_d G_{d,\nu} n^{4-d+2\nu} (1 + o(1)) \ , \tag{4.2.6}$$

where we recall that κ_d is the escape probability of the random walk (and we set $\kappa_2 = \pi$, as the escape probability is zero for $d = 2$) and the explicit constant $G_{d,\nu}$ is given in the Equation (4.2.35), as an integral over Bessel function.

Remark 4.2.3. *I. The condition $\nu < d/2 - 2$ is needed so that $M^a[0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c]$ converges to zero.*

II. For the simple random walk, $d \geq 3$ and the case $a_j = j^{-1}$, only an upper and lower bound for $M^a[0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c]$ has been known before, see [CS16, Lemma 2.7]. The proof in [CS16] is different and only covers the sequence $a_j = 1/j$. For more on that, see the remark after Theorem 4.3.1.

III. Many results in this thesis follow the same pattern: while our method allows for results for a very general class of sequences $(a_j)_j$, closed form expressions are only available in special cases. We use the sequence $a_j = j^\nu$ to generate closed form expressions.

Proof of Proposition 4.2.2. We begin the proof by counting paths and then estimating stochastic quantities. Expand

$$\begin{aligned} M^a[0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c] &= \sum_{x\in\mathbb{Z}^d}\sum_{j\geq 0} a_j \mathbb{P}^j_{x,x}\left(H_n < j,\, H_0 < j\right) \\ &= \sum_{x\in\mathbb{Z}^d}\sum_{j\geq 0} a_j \mathbb{P}^j_{0,0}\left(H_n < j,\, H_x < j\right) \\ &= \sum_{j\geq 0} a_j \mathbb{E}^j_{0,0}\left[\mathbb{1}\{H_n < j\}, \sum_{x\in\mathbb{Z}^d} \mathbb{1}\{\, H_x < j\}\right] \\ &= \sum_{j\geq 0} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j \mathbb{1}\{H_n < j\}\right] . \end{aligned} \tag{4.2.7}$$

The second equality is due to the time-homogeneity of the random walk. Monotone convergence implies the third equality above. For an illustration, see Figure 4.1.

We begin by proving the first statement of Proposition 4.2.2.

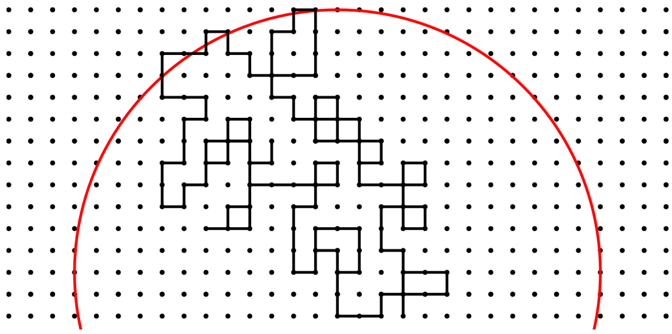

Figure 4.1: The set of possible starting points of a loop is equal to the points visited by it.

First Statement: our strategy is as follows:

I. Firstly, restrict to $d \geq 3$.

II. Show that loop lengths far below n^2 can be neglected.

III. Show that loop lengths bigger than n^S can be ignored, for $S \in (2, 3)$.

IV. For the remaining loop lengths, use concentration inequalities for the range and the hitting time estimates from Chapter 2.

V. Repeat the strategy for the case $d = 2$, with different concentration bounds.

We begin with the case $d \geq 3$. Define $n_1 = n^2/c_1 \log(n)$ for some $c_1 > 0$, to be adjusted later. We have

$$\begin{aligned} \sum_{j=1}^{n_1} a_j \mathbb{E}_{0,0}^j \left[\mathcal{R}_j \mathbb{1}\{H_n < j\}\right] &\leq \sum_{j=1}^{n_1} a_j j^{1-d/2} j \mathbb{B}_{0,0}^j \left(H_n < j\right) \\ &\leq C \sum_{j=1}^{n_1} a_j j^{1-d/2} \mathrm{e}^{-Cn^2/j} \leq C \sum_{j=1}^{n_1} \mathrm{e}^{-Cn^2/j} = \mathcal{O}\left(n^{-f(c_1)}\right) , \end{aligned} \tag{4.2.8}$$

for some $f(c_1) \to \infty$ as $c_1 \to \infty$. This is because $\mathcal{R}_j \leq j$, the polynomial growth of a_j, and the bound on $\mathbb{B}_{0,0}^j \left(H_n < j\right)$ from Lemma 2.2.3.

Furthermore, note that by Lemma 2.2.3 for $j \geq n^2$

$$\mathbb{B}_{0,0}^{j}\left(H_n < j\right) \geq C\,, \tag{4.2.9}$$

for some $C > 0$. We can thus obtain the lower bound

$$\sum_{j=n^2}^{2n^2} a_j \mathbb{E}_{0,0}^{j}\left[\mathcal{R}_j, H_n < j\right] \geq C \sum_{j=n^2}^{2n^2} \frac{a_j}{j^{d/2}} \mathbb{B}_{0,0}^{j}(H_n < j) \geq C \sum_{j=n^2}^{2n^2} j^{\nu - d/2} \geq C n^{2+2\nu-d}, \tag{4.2.10}$$

due to the assumption $a_j \geq C j^{\nu}$ and the bound $\mathcal{R}_j \geq 1$.
Comparing the two previous equations, we see that by making c_1 sufficiently large, the sum over $j \leq n_1$ is of lower order than the sum of $j \in \{n_1, \ldots, n^S\}$.
To show that the sum over $j \geq n^S$ with $S \in (2,3)$ is negligible is easier: indeed, this is the third part of Assumption 4.2.1, Equation (4.2.1).
To finish the proof of the first statement, we need to show that

$$\sum_{j=n_1}^{n^S} a_j \mathbb{E}_{0,0}^{j}\left[\mathcal{R}_j \mathbb{1}\{H_n < j\}\right] = (1 + o(1)) \sum_{j=n_1}^{n^S} a_j r_j \mathfrak{p}_j(0) B_{0,0}^{j}\left(H_n^B < j\right)\,. \tag{4.2.11}$$

Fix $\varepsilon > 0$. We use the concentration inequality from Lemma 4.1.1 to bound for some $\alpha > 0$ and all $j > n_1$

$$\mathbb{E}_{0,0}^{j}\left[|\mathcal{R}_j - r_j| > \varepsilon j\right] = \mathcal{O}\left(\mathrm{e}^{-n^{\alpha}}\right)\,. \tag{4.2.12}$$

By Equation (4.2.10), this error lives on a negligible scale:

$$\sum_{j \geq n_1} a_j \mathcal{O}\left(\mathrm{e}^{-n^{\alpha}}\right) = o\left(n^{2+2\nu-d}\right)\,. \tag{4.2.13}$$

Thus, we have

$$\sum_{j=n_1}^{n^S} a_j \mathbb{E}_{0,0}^{j}\left[\mathcal{R}_j \mathbb{1}\{H_n < j\}\right] \leq (1+\varepsilon) \sum_{j=n_1}^{n^S} a_j r_j p_j(0) \mathbb{B}_{0,0}^{j}\left(H_n < j\right)\,, \tag{4.2.14}$$

and the corresponding lower bound

$$\sum_{j=n_1}^{n^S} a_j \mathbb{E}_{0,0}^j \left[\mathcal{R}_j \mathbb{1}\{H_n < j\}\right] \geq (1-\varepsilon) \sum_{j=n_1}^{n^S} a_j r_j p_j(0) \mathbb{B}_{0,0}^j \left(H_n < j\right) , \quad (4.2.15)$$

for $\varepsilon > 0$ small enough.
For any $T > 0$ fixed, we can apply Lemma 2.2.3 to approximate the random walk bridge by the Brownian bridge:

$$\mathbb{B}_{0,0}^j \left(H_n < j\right) = B_{0,0}^j \left(H_n^B < j\right) (1 + o(1)) + \mathcal{O}\left(n^{-T}\right) . \quad (4.2.16)$$

By making $T > 0$ sufficiently large and taking the limit $\varepsilon \downarrow 0$, we arrive at

$$\sum_{j=n_1}^{n^S} a_j \mathbb{E}_{0,0}^j \left[\mathcal{R}_j \mathbb{1}\{H_n < j\}\right] = (1+o(1)) \sum_{j=n_1}^{n^S} a_j r_j \mathfrak{p}_j(0) B_{0,0}^j \left(H_n^B < j\right) . \quad (4.2.17)$$

The sum over $j \leq n_1$ and $n \geq n^S$ is negligible as seen above. This shows the first statement for the case $d \geq 3$.
For $d = 2$, the reasoning is the same: by Assumption 4.2.1, the sum over $j \leq n_1$ and $j \geq n^S$ is negligible. Lemma 4.1.1 gives us the scale of the error term:

$$\mathbb{B}_{0,0}^j \left[\mathcal{R}_j, H_n < j\right] \leq \mathcal{O}\left(\frac{\log^2(\log(j))}{\log^3(j)}\right) + (1+\varepsilon) r_j \mathbb{B}_{0,0}^j \left(H_n < j\right) . \quad (4.2.18)$$

By the additional assumption made for $d = 2$ (i.e. Equation (4.2.3)), the $\mathcal{O}$-term is negligible in the limit. From there on, one proceeds analogously to the case $d \geq 3$.

This finishes the proof of the first statement.

Second statement: the proof of the second statement consists of two steps:

I. Show that $(a_j)_j$ with $a_j = (j^\nu \mathbb{1}\{d \geq 3\} + j^\nu \log(j) \mathbb{1}\{d = 2\})$ satisfies Assumption 4.2.1.

II. We then compute the expression given by the first statement of this proposition (Equation (4.2.5)) by approximating the sum by an integral.

We have to be careful in the second step, as $B_{0,0}^j(H_n^B < j)$ itself is expressed as an infinite sum and (as we will see later) the order of summation is not exchangeable. We begin by showing that the sequence $(a_j)_j$ satisfies Assumption 4.2.1.

Step I: the first requirements (polynomial growth and bounded decay speed) of Assumption 4.2.1 are trivially satisfied. Note that for the sum over long loop lengths

$$\begin{aligned}\sum_{j\geq n^S} a_j j r_j j^{-d/2} \geq C\sum_{j\geq n^S} j^{\nu+1-d/2} \geq C n^{(S-2)(\nu+2-d/2)} \sum_{j\geq n^2} j^{\nu+1-d/2}\\ = o(1)\sum_{j\geq n^2} a_r r_j j^{-d/2}\,.\end{aligned} \tag{4.2.19}$$

This shows that $(a_j)_j$ satisfies the assumptions for the case $d \geq 3$.
For the case $d = 2$, recall that $n_1 = n^2/c_1 \log(n)$. Note that for the sum over small loop lengths

$$\begin{aligned}\sum_{j=1}^{n_1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_n < j\right] &\leq C\log(n_1)\sum_{j=1}^{n_1} j^\nu \mathbb{B}^j_{0,0}\left(H_n < j\right)\\ &\leq C\log(n_1)\sum_{j=1}^{n_1} j^\nu \mathrm{e}^{-cn^2/j} \leq C\log(n_1) n^{2\nu+2}\Gamma\left(-\nu-1, \frac{cn^2}{n_1}\right)\\ &= \mathcal{O}\left(\frac{n^{2\nu+2}}{\log^c(n)}\right)\,,\end{aligned} \tag{4.2.20}$$

for some $c > 0$. The above holds as $a_j \sim j^\nu \log(j)$, Lemma 8.2.1 lets us approximate the sum by an integral, and Lemma 2.2.3 gives a bound on $\mathbb{B}^j_{0,0}\left(H_n < j\right)$.
To verify the last remaining condition for the case $d = 2$, note that the sum over the error term

$$\begin{aligned}\sum_{j\geq n_1} a_j \frac{\log^2(\log(n))}{\log^3(n)} &\leq C\sum_{j\geq n_1} j^\nu \frac{\log^3(\log(j))}{\log^2(j)}\\ &\leq C\frac{\log^3(\log(n))}{\log^2(n)}\frac{n^{2\nu+2}}{\log^{\nu+1}\left(\log^{1+\varepsilon}(n)\right)} = o\left(\frac{n^{2\nu+2}}{\log(n)}\right)\,.\end{aligned} \tag{4.2.21}$$

This finishes the proof that the sequence $(a_j)_j$ satisfies Assumption 4.2.1 and thus the first statement of Proposition 4.2.2 holds.

Step II: we now calculate the infinite sum $\sum_j a_j r_j \mathfrak{p}_j(0) B^j_{0,0}\left(H^B_n < j\right)$. We do this by successively removing areas of summation/integration, similar to Chapter 2. We begin by excluding the event that the Brownian bridge "quickly" hits the boundary of $\boldsymbol{B}_n$: define $n_2 = c_2 n^2/\log(n)$ and choose $c_2 > 0$ small enough such that for the hitting time H_n^B of the Brownian motion

$$\sum_{j=n_1}^{n^{5/2}} j^{-d/2+1+\nu} B_{0,0}^j \left(n_2 > H_n^B\right) = o\left(n^{4-d+2\nu}\right) . \tag{4.2.22}$$

Such a choice is possible by noting

$$\{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \colon |x_i| \leq n\} \subset \boldsymbol{B}_n \subset \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \colon |x_1| \leq n\} , \tag{4.2.23}$$

and using the independence of the Brownian motion coordinates.
Our goal is to evaluate

$$\sum_{j=n_1}^{n^{5/2}} j^{-d/2+1+\nu} B_{0,0}^j \left(H_n^B < j,\, n_2 < H_n^B\right) . \tag{4.2.24}$$

We recall the explicit formula for $\mathrm{d}B_{0,0}^j\left(H_n^B = t\right)$ introduced in Lemma 2.2.3

$$\mathrm{d}B_{0,0}^j \left(H_n^B = t\right) = \sum_{k=1}^{\infty} \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 t/(2n^2)}}{z^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathfrak{p}_{j-t}(n) . \tag{4.2.25}$$

where $\mu = d/2 - 1$. As this formula involves an infinite sum, we begin by showing that most terms do not contribute to the asymptotics.
Recall the expansion

$$J_{\mu+1}(j_{\mu,k}) = (-1)^{k+1} \sqrt{\frac{2}{\pi k}} \left(1 + \mathcal{O}(k^{-1})\right) . \tag{4.2.26}$$

Fix $T > 0$. We now show that for T large enough, those terms with $k > T$ in Equation (4.2.25) are negligible. Bounding the sum by an integral using Lemma 8.2.1 leads to

$$\sum_{k \geq T} k^{d/2+1/2} \mathrm{e}^{-k^2 t/(2n^2)} \leq C \int_T^\infty k^{d/2+1/2} \mathrm{e}^{-k^2 t/(2n^2)} \mathrm{d}k . \tag{4.2.27}$$

Note that

$$\int_T^\infty k^{d/2+1/2} \mathrm{e}^{-k^2 t/(2n^2)} \mathrm{d}k = \left(\frac{(2n)^2}{t}\right)^{d/2+3/2} \int_{\frac{T^2}{(2n)^2}}^\infty k^{d/4-1/4} \mathrm{e}^{-k} \mathrm{d}k \,. \quad (4.2.28)$$

Recalling the definition of the upper incomplete Gamma function implies that Equation (4.2.27) is proportional to $\left(\frac{n^2}{t}\right)^{d/2+3/2} \Gamma\left(\frac{d+3}{4}, \frac{T^2 t}{n^2}\right)$. Choosing $T = \log^{1+\delta_1}(n)$ for any $\delta_1 > 0$ fixed, gives the bound for $t > n_2$

$$\Gamma\left(\frac{d+3}{4}, \frac{T^2 t}{n^2}\right) \leq \Gamma\left(\frac{d+3}{4}, \mathcal{O}\left(\log^{1+\delta_1/2}(n)\right)\right) = \mathcal{O}\left(\mathrm{e}^{-\log^{1+\delta'}(n)}\right), \quad (4.2.29)$$

by the asymptotics of the Gamma function from Lemma 8.3.3, for some $\delta' > 0$. As the above decays fast than any polynomial in n, we have that

$$\sum_{j=n_1}^{n^{5/2}} j^{1+\nu} \int_{n_2}^{j} \mathfrak{p}_{j-t}(n) \sum_{k=T}^{\infty} \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 t/(2n^2)}}{n^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}t = o\left(n^{4-d+2\nu}\right). \quad (4.2.30)$$

We now calculate the main contributing factor

$$\begin{aligned} &\sum_{j=n_1}^{n^{5/2}} j^{1+\nu} \int_{n_2}^{j} \mathfrak{p}_{j-t}(n) \sum_{k=1}^{T} \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 t/(2n^2)}}{n^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}t \\ &= \sum_{k=1}^{T} \sum_{j=n_1}^{n^{5/2}} \int_{n_2/j}^{1} \mathfrak{p}_{j(1-t)}(n) j^{2+\nu} \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 tj/(2n^2)}}{n^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}t \,, \end{aligned} \quad (4.2.31)$$

by the change of variables $t \mapsto jt$. In order to eliminate the scale n, we would like to approximate the sum over j with an integral. Indeed, this would allow us to perform a change of variables $j \mapsto n^2 j$. To approximate the sum by an integral, we use [LL10, Lemma A.1.1]. This lemma states that if the second derivative decays sufficiently fast, we can replace the sum by an integral at the cost of a $(1 + o(1))$ factor. Let us calculate the second derivative (take now

$j \in \mathbb{R}$)

$$\partial_j^2 \left(j^{2+\nu} \mathfrak{p}_{j(1-t)}(n) \mathrm{e}^{-j_{\mu,k}^2 tj/(2n^2)} \right) = j^{-d/2+2+\nu} \mathfrak{p}_{j(1-t)}(n) \mathrm{e}^{-j_{\mu,k}^2 tj/(2n^2)} \times \left(c^2 + \frac{2ac}{j} + \frac{2}{j^2}\left(a^2+bc\right) - \frac{2}{j^3}\left(ab+2b\right) + \frac{b^2}{j^4} \right), \tag{4.2.32}$$

for $a = 2+\nu$, $b = n^2/2(1-t)$, $c = j_{\mu,k}^2 t/(2n^2)$. Using the above bounds on j and k, we can bound the above by

$$Cj^{2+\nu} \mathfrak{p}_{j(1-t)}(n) \mathrm{e}^{-j_{\mu,k}^2 tj/(2n^2)} \frac{\log^4(n)}{n^3} \left(1 + \frac{1}{(1-t)^2} \right). \tag{4.2.33}$$

By [LL10, Lemma A.1.1]

$$\begin{aligned} &\int_0^1 \sum_{k=1}^T \sum_{j=n_1}^{n^{5/2}} \mathfrak{p}_{j(1-t)}(n) j^{2+\nu} \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 tj/(2n^2)}}{n^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}t = \\ &\int_0^1 \sum_{k=1}^T \int_{n_1}^{n^{5/2}} \mathfrak{p}_{j(1-t)}(n) j^{2+\nu} \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 tj/(2n^2)}}{n^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}j \, \mathrm{d}t (1+o(1)) . \end{aligned} \tag{4.2.34}$$

By a similar approximation, we can rewrite the above (omitting the $o\left(n^{4-d+2\nu}\right)$ factor to aid legibility)

$$\begin{aligned} &\int_0^1 \sum_{k=1}^T \int_{n_1}^{n^{5/2}} \mathfrak{p}_{j(1-t)}(n) j^{2+\nu} \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 tj/(2n^2)}}{n^2 2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}j \, \mathrm{d}t \\ &= n^{4+2\nu-d} \int_0^1 \int_0^\infty \mathfrak{p}_{(1-t)j}(1) j^{2+\nu} \sum_{k=1}^\infty \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 tj/2}}{2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}j \, \mathrm{d}t \\ &= G_{d,\nu} n^{4+2\nu-d} , \end{aligned} \tag{4.2.35}$$

with

$$G_{d,\nu} = \int_0^1 \int_0^\infty \mathfrak{p}_{(1-t)j}(1) j^{2+\nu} \sum_{k=1}^\infty \frac{j_{\mu,k}^{\mu+1} \mathrm{e}^{-j_{\mu,k}^2 tj/2}}{2^\mu \Gamma(\mu+1) J_{\mu+1}(j_{\mu,k})} \mathrm{d}j \, \mathrm{d}t . \tag{4.2.36}$$

To summarise, we have shown that

$$M^a[0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c] n^{d-4-2\nu} = G_{d,\nu}\kappa_d(1+o(1)) \,. \tag{4.2.37}$$

This finishes the proof of the second statement of Proposition 4.2.2. □

We now state the result for the connectivity associated to the Bosonic loop measure separately.

Corollary 4.2.4. *For $\beta > 0$, $\mu < 0$, and $d \geq 3$, it holds that $M^B_{\mu,\beta}[0 \longleftrightarrow \boldsymbol{B}_n^c]$ decays exponentially fast, with speed increasing as $\mu \downarrow -\infty$.*
If $\mu = 0$ and $d \geq 3$, we have that

$$M^B_{\mu,\beta}[0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c] = \kappa_d G_{d,-1} n^{2-d}(1+o(1)) \,, \tag{4.2.38}$$

where the $o(1)$ term depends on β.

As Lemma 3.1.9 implies that the Bosonic loop measure with $\mu = 0$ gives weight $j^{-1}(1+o(1))$ to a loop of length j, the proof is immediate. Notice the transition from exponential decay for non-zero chemical potential ($\mu < 0$) to algebraic decay for $\mu = 0$.

4.3 Connecting large annuli

The next theorem gives upper and lower bounds on the mass of connecting two spheres of diverging radius.

Theorem 4.3.1. *Let the underlying random walk have bounded support. Let $\nu < -1/2$ and $d \geq 3$. We then have that for every $\gamma_0 > 1$, there exists a $C = C(\gamma_0) > 1$ such that for all $\gamma > \gamma_0$, for n large enough and $a_j \sim j^\nu$*

$$C^{-1} n^{2\nu+2} \gamma^{3-d+2\nu} \leq M^a[\boldsymbol{B}_n \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_{\gamma n}^c] \leq C n^{2\nu+2} \gamma^{\nu'} \,, \tag{4.3.1}$$

where $\nu' = \max\{2\nu+1, -4\}$.

Remark 4.3.2. *I. Contrary to Proposition 4.2.2, we only give this theorem for the case $a_j \sim j^\nu$, as other sequences do not not yield closed* *form bounds. For more general sequences $(a_j)_j$, we summarise the (more lengthy) bounds in the Appendix, see Proposition 8.1.1.*

II. *For $d \geq 3$ and $a_j = j^{-1}$, a stronger version of the above theorem is established in [CS16, Lemma 2.7]: it gives bounds on $M[K \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_R]$ with $K \subset \boldsymbol{B}_n$ and $R > \gamma n$. The proof exploits the fact that $a_j = j^{-1}$ in an elegant way: if $\dot{M}$ is the push-forward measure of M under the equivalence class of forgetting the base point of the loop, we have that for a loop $\dot{\omega}$ (of length n) that*

$$\dot{M}[\dot{\omega}] = \frac{n}{m(\dot{\omega})} M[\omega] \,. \tag{4.3.2}$$

Here, ω is an arbitrary representative of the equivalence class $\dot{\omega}$ and $m(\dot{\omega})$ is the loop's multiplicity. Noting that $M[\omega]$ has a factor of n^{-1}, one can rearrange the sum over loops intersecting K (for any $K \subset \boldsymbol{B}_n$) and $\boldsymbol{B}_R^c$ in a way which aides estimation, as the sum over lengths can be interchanged with a sum over multiplicities. Our proof works in a different way, we estimate the contribution of each length directly.

III. *The restriction for $\nu < -1/2$ is technical. Indeed, note that for $\nu > -1$, we have that $M^a[\boldsymbol{B}_n \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_{\gamma n}^c]$ diverges to $+\infty$ as $n \to \infty$. This makes the associated loop soups less interesting to study as long loops cover the whole space.*

Proof of Theorem 4.3.1. As in the proof of Proposition 4.2.2, we begin with a combinatorial argument. Expand

$$\begin{aligned}
M^a[\boldsymbol{B}_n \longleftrightarrow \boldsymbol{B}_{\gamma n}^c] &= \sum_{x \in \mathbb{Z}^d} \sum_{j \geq 1} a_j \mathbb{P}_{x,x}^j \left(H_n < j,\, H_{\gamma n} < j\right) \\
&= \sum_{x \in \mathbb{Z}^d} \sum_{y \in \boldsymbol{B}_n} \sum_{j \geq 1} a_j \mathbb{P}_{x,x}^j \left(H_n < j,\, S_{H_n} = y,\, H_{\gamma n} < j\right) \\
&= \sum_{x \in \mathbb{Z}^d} \sum_{y \in \boldsymbol{B}_n} \sum_{j \geq 1} a_j \mathbb{P}_{y,y}^j \left(H_x < j,\, H_{\gamma n} < j,\, H_n(y) \geq H_x\right) \\
&= \sum_{y \in \boldsymbol{B}_n} \sum_{j \geq 1} a_j \mathbb{E}_{y,y}^j \left[\mathbb{1}\{H_{\gamma n} < j\} \sum_{x \in \mathbb{Z}^d} \mathbb{1}\{H_n(y) \geq H_x,\, H_x < j\}\right] \\
&= \sum_{y \in \boldsymbol{B}_n} \sum_{j \geq 1} a_j \mathbb{E}_{y,y}^j \left[\mathbb{1}\{H_{\gamma n} < j\} \mathcal{R}_{H_n(y)}\right] ,
\end{aligned} \tag{4.3.3}$$

where $H_n(y)$ is the first time that the random walk hits the set $\boldsymbol{B}_n \setminus \{y\}$. If under $\mathbb{P}^j_{y,y}$ the random walk bridge does not hit $\boldsymbol{B}_n \setminus \{y\}$, we set $H_n(y) = j$. All steps apart from the third equality are fairly standard, see also the proof of Proposition 4.2.2. For the third equality, we use the Markov property to

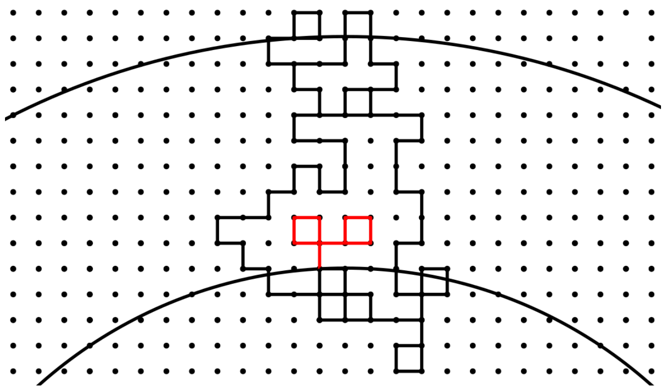

Figure 4.2: A loop intersecting both $\boldsymbol{B}_n$ and $\boldsymbol{B}^c_{\gamma n}$. The points visited up to $H_n(y)$ are coloured red.

start the random walk at y and time-reverse it. As y is the first point at which we hit $\boldsymbol{B}_n$, the time-reversed walk has to hit x before it hits $\boldsymbol{B}_n \setminus \{y\}$. For an illustration, see Figure 4.2.

Note that since the random walk has finite support, the above sum is non-zero only for $\mathcal{O}\left(n^{d-1}\right)$ many y's, as we need to move outside of $\boldsymbol{B}_n$ with the first step of the random walk. Thus, we assume without loss of generality that $p^{(1)}$, the jump distribution in each coordinate, is supported on $\{-1, 0, 1\}$.

Let us restrict the loop lengths j we need to consider. Fix $\varepsilon > 0$ small, let $\boldsymbol{\gamma} = \gamma - 1$ and expand

$$\sum_{j=1}^{(\boldsymbol{\gamma} n)^{2-\varepsilon}} a_j \mathbb{E}^j_{y,y}\left[\mathcal{R}_{H_n}, H_{\gamma n} < j\right] \leq \sum_{j=1}^{(\boldsymbol{\gamma} n)^{2-\varepsilon}} a_j j^{-d/2+1} \mathbb{B}^j_{0,0}\left(H_{\boldsymbol{\gamma} n} < j\right)$$
$$\leq \sum_{j=1}^{(\boldsymbol{\gamma} n)^{2-\varepsilon}} a_j j^{-d/2+1} \mathrm{e}^{-C(\boldsymbol{\gamma} n)^2/j} \leq \mathrm{e}^{-C(\boldsymbol{\gamma} n)^{\varepsilon}/2} \sum_{j=1}^{\infty} a_j j^{-d/2+1} = \mathcal{O}\left(\mathrm{e}^{-C(\boldsymbol{\gamma} n)^{\varepsilon}/2}\right), \tag{4.3.4}$$

for some $C > 0$. We use Lemma 2.2.3 to bound $\mathbb{B}^j_{0,0}\left(H_{\boldsymbol{\gamma} n} < j\right)$ and the fact that $(a_j)_j$ grows at most polynomially.

As the above lives on a smaller scale (exponentially decreasing) than our result (of polynomial order), henceforth assume that $j \geq (\boldsymbol{\gamma} n)^{2-\varepsilon} = n_\gamma^{2-}$ (where we suppress the dependence on $\varepsilon > 0$ in that notation).
In the spirit of the proof of Proposition 4.2.2 we want to replace the range $(\mathcal{R}_j)_j$ evaluated at the stopping time $H_n(y)$, with the stopping time itself. The intuition is that by Lemma 4.1.1, the range is approximately linear for large enough arguments. We begin by cutting off small values of $H_n(y)$: for $\delta \in (0, 1/2)$ we bound

$$\mathbb{E}_{y,y}^j \left[\mathbb{1}\{H_{\gamma n} < j\} \mathcal{R}_{H_n(y)},\, H_n(y) < j^\delta \right] \leq \mathbb{E}_{y,y}^j \left[\mathcal{R}_{j^\delta} \right] = \mathcal{O}\left(j^{-d/2+\delta} \right) , \quad (4.3.5)$$

as $\mathcal{R}_{j^\delta}$ can be bounded from above by j^δ. We impose the following constraint on the values of (ε, δ):

$$2\delta - \varepsilon \left(-d/2 + \delta + \nu + 1 \right) < 1 \,. \quad (4.3.6)$$

Indeed, with this we have that

$$\sum_{y \in \partial_i \boldsymbol{B}_n} \sum_{j \geq n_\gamma^{2-}} a_j \mathbb{E}_{y,y}^j \left[\mathcal{R}_{H_n(y)}, H_n(y) < j^\delta \right] \leq C n^{d-1} \sum_{j \geq n_\gamma^{2-}} j^{-d/2+\delta+\nu} = o\left(n^{2\nu+2} \right) . \quad (4.3.7)$$

We now turn our attention to values of $H_n(y)$ larger than j^δ. For $H_n(y)$ large, we know that by Lemma 4.1.1 that $\mathcal{R}_{H_n(y)} \approx H_n(y)$. We make this rigorous now: for any $t > 0$ and two constants $c_t, C_t > 0$, depending on t, we do a case distinction whether $\mathcal{R}_k$ is close to its mean or not:

$$\begin{aligned}
&\mathbb{E}_{y,y}^j \left[\mathcal{R}_{H_n(y)}, H_n(y) \geq j^\delta, H_{\gamma n} < j \right] \\
&\quad = \mathbb{E}_{y,y}^j \Big[\mathcal{R}_{H_n(y)} \Big(\mathbb{1}\{ \forall k \geq j^\delta : |\mathcal{R}_k - r_k| \leq c_t r_k \} \\
&\qquad\qquad + \mathbb{1}\{ \exists k \geq j^\delta : |\mathcal{R}_k - r_k| \geq c_t r_k \} \Big), H_n(y) \geq j^\delta, H_{\gamma n} < j \Big] \\
&\quad \leq C_t \mathbb{E}_{y,y}^j \left[H_{\gamma n} < j,\, H_n(y) \right] \\
&\qquad + \mathcal{O}\left(\mathrm{e}^{-c_t^{-1} j^{\delta'}} \right) \sum_{k=j^\delta}^{j} \mathbb{E}_{y,y}^j \left[H_{\gamma n} < j,\, \mathcal{R}_k,\, H_n(y) = k \big| |\mathcal{R}_k - r_k| \geq c_t r_k \right] \\
&\quad \leq C_t \mathbb{E}_{y,y}^j \left[H_{\gamma n} < j,\, H_n(y) \right] + \mathcal{O}\left(j^{-\delta t - \mathrm{d}/2 + 1} \right) ,
\end{aligned} \quad (4.3.8)$$

for some $\delta' > 0$. This follows after using the union bound on k and Lemma 4.1.1 to estimate the probability that the range deviates by c_t from the mean. Choose $t > 0$ sufficiently large (if necessary, adjust $\varepsilon > 0$) such that

$$2t\delta + \varepsilon(-d/2 + 2 - \delta t + \nu) > 2\,. \tag{4.3.9}$$

Indeed, if the above holds, we have that

$$\begin{aligned}\sum_{y\in\partial_i \boldsymbol{B}_n}\sum_{j\geq n_\gamma^{2-}} a_j \mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, \mathcal{R}_{H_n(y)}\right] \leq C &\sum_{y\in\partial_i \boldsymbol{B}_n}\sum_{j\geq n_\gamma^{2-}} a_j \mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n(y)\right] \\ &+ o\left(n^{2\nu+2}\right)\,,\end{aligned} \tag{4.3.10}$$

by a computation analogous to Equation (4.3.7). Repeating the argument which give the bound above, results in

$$\begin{aligned}\sum_{y\in\partial_i \boldsymbol{B}_n}\sum_{j\geq n_\gamma^{2-}} a_j \mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, \mathcal{R}_{H_n(y)}\right] \geq C &\sum_{y\in\partial_i \boldsymbol{B}_n}\sum_{j\geq n_\gamma^{2-}} a_j \mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n(y)\right] \\ &+ o\left(n^{2\nu+2}\right)\,.\end{aligned} \tag{4.3.11}$$

Having reduced the initial problem to an analysis of $\mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n(y)\right]$, we now prove upper and lower bounds for $\mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n(y)\right]$. The proof of the lower bound is shorter and uses the FKG inequality. The justification of the upper bound is lengthier and involves a series of approximations. We begin with the lower bound.

Lower Bound: we firstly bound $H_n(y) \geq H_n$. Indeed, hitting $\boldsymbol{B}_n \setminus \{y\}$ takes longer than hitting $\boldsymbol{B}_n$.

The main idea of the lower bound is the following: if y is the north pole[1], we can bound the hitting time H_n by the hitting time of the half space which consists of those points whose first coordinate is less than n, see Figure 4.3. We then use the FKG inequality to separate the events $\{H_{\gamma n} < j\}$ and $\{H_n = k\}$.

We begin by symmetrising the problem so that we can assume that y is equal to the north pole. The strategy for the symmetrisation is to approximate the

[1]The point $(n, 0, \ldots, 0) \in \mathbb{R}^d$.

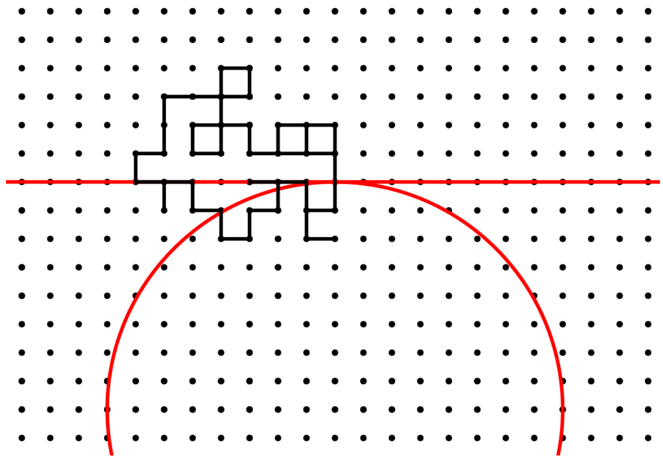

Figure 4.3: The random walk started from the north pole hits the horizontal red line before hitting the sphere.

random walk by a Brownian motion and then use the rotational invariance of the Brownian motion. As we only need a lower bound, henceforth discard $j \in \{n_\gamma^{2-}, \ldots, (\boldsymbol{\gamma} n)^2\}$.

For a point $x \in \partial_i \boldsymbol{B}_n$, let T_x be the approximate tangent (for a graphical representation, see Figure 4.4) through x defined as follows: let $x^* \in \mathbb{R}^d$ be the unique point of absolute value n which lies on the line connecting x with the origin and satisfies $|x - x^*| \leq d$. Let T_x^* be the tangent through x^* on the surface of the ball of radius n (this time in $\mathbb{R}^d$). Then

$$T_x = \{y \in \mathbb{R}^d \colon y + (x^* - x) \in T_x^*\} . \tag{4.3.12}$$

Let $\bar{\mathbf{H}}_x$ be the half space which contains the origin and has T_x as its boundary. Let τ_x be the hitting time of that $\bar{\mathbf{H}}_x \cap \mathbb{Z}^d$. We bound

$$\mathbb{E}_{x,x}^j \left[H_{\gamma n} < j,\, H_n\right] \geq \mathbb{E}_{x,x}^j \left[H_{\gamma n} < j,\, \tau_x\right] \geq \frac{j}{2} \mathbb{P}_{x,x}^j \left(H_{\gamma n} < j/4,\, \tau_x > j/2\right) . \tag{4.3.13}$$

By Lemma 2.2.1, there exists a coupling $\mathsf{P}_{x,x}^j$ between $\mathbb{B}_{x,x}^j$ and $B_{x,x}^j$ such that

$$\mathsf{P}_{x,x}^j \Big(\max_{1 \leq i \leq j} |S_i - B_i| \geq c_\alpha \log^2(j) \Big) \leq \frac{C}{j^\alpha} , \tag{4.3.14}$$

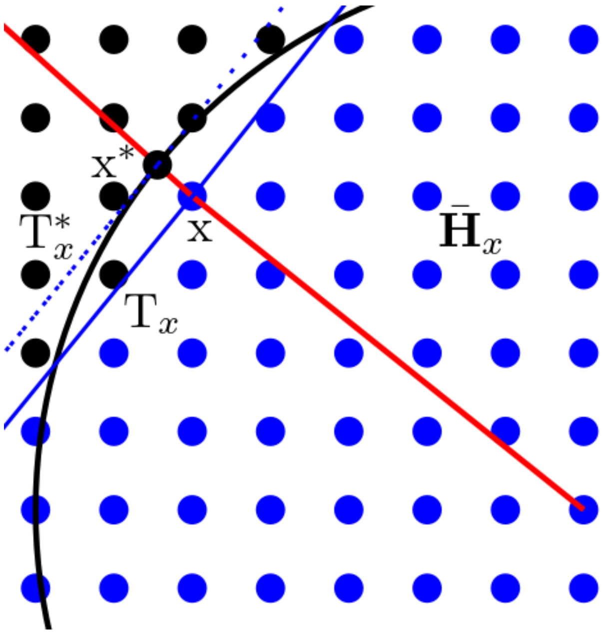

Figure 4.4: The half-space $\bar{\mathbf{H}}_x$, in blue. τ_x is the first time we hit any of the blue-coloured dots.

with $c_\alpha > 0$. Choose $\alpha > 0$ large enough such that

$$n^{d-1} \sum_{j \geq (\gamma n)^2} \frac{C j^{1+\nu}}{j^{\alpha+d/2}} = o\left(n^{2\nu+2}\right) . \tag{4.3.15}$$

For some $C_1 > 0$, let $\tau^B_{n^-}$ be the hitting time of the half space

$$\overline{\mathbf{H}}_{n^-} = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d \colon x_1 \leq n - C_1 \log^2(n)\} . \tag{4.3.16}$$

By adjusting the constant $C_1 > 0$, we have that by the coupling and the rotational invariance of the Brownian motion

$$\begin{aligned}
&\sum_{y \in \partial_i \boldsymbol{B}_n} \sum_{j \geq (\gamma n)^2} j^\nu \mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n\right] \\
&\geq C \sum_{y \in \partial_i \boldsymbol{B}_n} \sum_{j \geq (\gamma n)^2} j^{-d/2+1+\nu} \mathbb{B}^j_{y,y}\left(H_{\gamma n} < j/4,\, \tau_y > j/2\right) \\
&\geq C \sum_{y \in \partial_i \boldsymbol{B}_n} \sum_{j \geq (\gamma n)^2} j^{-d/2+1+\nu} B^j_{\bar{n},\bar{n}}\left(H^B_{2\gamma n} < j/4,\, \tau^B_{n^-} > j/2\right) + o\left(n^{2\nu+2}\right) .
\end{aligned} \tag{4.3.17}$$

Here $\bar{n} = (n, 0, \ldots, 0) \in \mathbb{R}^d$ is the north pole. Denote $B_t^{(i)}$ the i-th coordinate of the Brownian motion for $i = 1, \ldots, d$. Bounding the ball $\boldsymbol{B}_{\gamma n}$ by the box with larger radius $\boldsymbol{B}^{\infty}_{2\gamma n}$, we reduce the our expression to a one-dimensional problem:

$$\begin{aligned}
&B^j_{\bar{n},\bar{n}}\left(H^B_{2\gamma n} < j/4,\, \tau^B_{n^-} > j/2\right)\\
&\quad \geq B^j_{\bar{n},\bar{n}}\left(\forall i \in \{1,\ldots,d\}\, \exists 0 < t_i < j/4\colon |B^{(i)}_{t_i}| = 2\boldsymbol{\gamma} n,\, \tau^B_{n^-} > j/2\right)\\
&\quad \geq C B^{j,(1)}_{n,n}\left(\exists 0 < t_i < j/4\colon |B^{(1)}_{t_i}| = 2\boldsymbol{\gamma} n,\, \tau^B_{n^-} > j/2\right),
\end{aligned} \tag{4.3.18}$$

where $B^{j,(1)}_{n,n}$ is a one-dimensional Brownian bridge of length j. Note $j \geq (\boldsymbol{\gamma} n)^2$ and thus the probability of hitting the complement of a box with length $4\gamma n$ remains bounded away from zero uniformly in n and j. This allows us to discard the other coordinates in the equation above. Use the coupling from Lemma 2.2.1 once more (adjusting the constant C_1 in Equation (4.3.16)),

$$\begin{aligned}
&\sum_{y\in\partial_i \boldsymbol{B}_n} \sum_{j\geq(\gamma n)^2} j^{\nu} \mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n\right]\\
&\geq C \sum_{y\in\partial_i \boldsymbol{B}_n} \sum_{j\geq(\boldsymbol{\gamma} n)^2} j^{1-d/2+\nu} B^{j,(1)}_{n,n}\left(\exists 0 < t_i < j/4\colon |B^{(1)}_{t_i}| = 2\boldsymbol{\gamma} n,\, \tau^B_{n^-} > j/2\right)\\
&\geq C \sum_{y\in\partial_i \boldsymbol{B}_n} \sum_{j\geq(\boldsymbol{\gamma} n)^2} j^{1-d/2+\nu} \mathbb{B}^{j,(1)}_{\bar{n},\bar{n}}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} > j/2\right) + o\left(n^{2\nu+2}\right).
\end{aligned} \tag{4.3.19}$$

Here, $\mathbb{B}^{j,(1)}_{n,n}$ is a one-dimensional random walk bridge of length j.

Decompose the event $\{H_{4\gamma n} < j/4,\, \tau_{n^-} > j/2\}$ by conditioning on the value of τ_{n^-}: let $\mathbb{P}_{\bar{n}}^{j,(1)}$ be a one-dimensional random walk and

$$
\begin{aligned}
&\mathbb{B}_{\bar{n},\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} > j/2\right) \\
&\geq \sum_{k=j/2}^{j} \mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} = k\right) p_j^{-1}(0) p_{j-k}(n^-, n) \\
&\geq \sum_{k=j/2}^{2j/3} \mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} = k\right) p_j^{-1}(0) p_{j-k}(-C_1^2 \log(n)) \\
&\geq C \sum_{k=j/2}^{2j/3} \left(\frac{j}{j-k}\right)^{1/2} \mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} = k\right) \exp\left(-C \log^4(n)/(j-k)\right) \\
&\geq C \sum_{k=j/2}^{2j/3} \left(\frac{j}{j-k}\right)^{1/2} \mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} = k\right) .
\end{aligned}
\tag{4.3.20}
$$

Using discrete integration by parts from Lemma 8.2.3 and using that $(j-k)^{-1/2} - (j-k-1)^{-1/2}$ is bounded above and from below by $\mathcal{O}\left((j-k)^{-3/2}\right)$, we rewrite the above as

$$
C \sum_{k=j/2}^{2j/3} \frac{j^{1/2}}{(j-k)^{3/2}} \mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} \geq k\right) + E\,, \tag{4.3.21}
$$

where

$$
E \geq C \mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} > j/2\right) . \tag{4.3.22}
$$

We want to apply the FKG inequality to separate the two events $\{H_{4\gamma n} < j/4\}$ and $\{\tau_{n^-} \geq k\}$. For this purpose, firstly bound

$$
\mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} \geq k\right) \geq \mathbb{P}_{\bar{n}}^{j,(1)}\left(H_{4\gamma n}^{+} < j/4,\, \tau_{n^-} \geq k\right) , \tag{4.3.23}
$$

where $H_{4\gamma n}^{+}$ is the first time the one-dimensional random walk enters $\{4\boldsymbol{\gamma} n, 4\boldsymbol{\gamma} n+1, \dots\}$.

We now set up the FKG-inequality: set $\Omega = \mathbb{Z}^j$, $\mathfrak{n} = \left(p^{(1)}\right)^{\otimes j}$, where recall that $p^{(1)}$ is the law induced by S_1 on $\mathbb{Z}$. We have the following partial order on Ω: take $a, b \in \Omega$ and say $a \leq b$ if for all $i \in \{1, \dots, j\}$ we have $a_i \leq b_i$ (in

$\mathbb{Z}$), see Figure 4.5 for an illustration. Note that the two events $\{H_{4\gamma n} < j/4\}$ and $\{\tau_{n^-} \geq k\}$ are both non-decreasing. Applying the FKG-inequality (see for

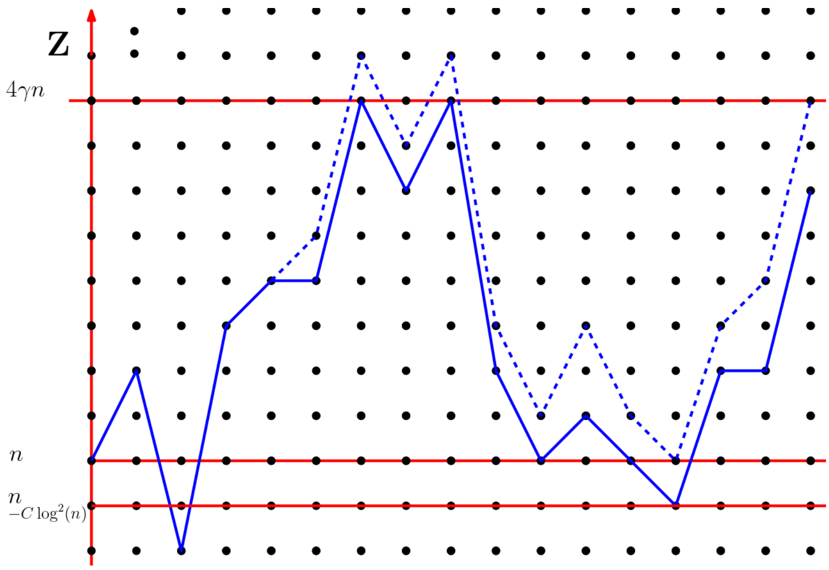

Figure 4.5: Two ordered configurations, with the larger one being represented in the dashed style

example [FV17, Theorem 3.50]) to $\mathfrak{n}$, we get that

$$\mathbb{P}^{j,(1)}_{\bar{n}}\left(H^+_{4\gamma n} < j/4,\, \tau_{n^-} \geq k\right) \geq \mathbb{P}^{j,(1)}_{\bar{n}}\left(H^+_{4\gamma n} < j/4\right)\mathbb{P}^{j,(1)}_{\bar{n}}\left(\tau_{n^-} \geq k\right). \quad (4.3.24)$$

Using Lemma 2.2.3 for the random walk, we then bound

$$\mathbb{P}^{j,(1)}_{\bar{n}}\left(H^+_{4\gamma n} < j/4\right)\mathbb{P}^{j,(1)}_{\bar{n}}\left(\tau_{n^-} \geq k\right) \geq C\mathbb{P}^{j,(1)}_{\bar{n}}\left(\tau_{n^-} \geq k\right), \quad (4.3.25)$$

where the constant C does not depend on γ. Indeed, as $j \geq (\gamma n)^2$, the probability of $\{H^+_{4\gamma n} < j/4\}$ stays bounded away from zero. We then use Lemma 2.2.6 to estimate

$$\mathbb{P}^{j,(1)}_{\bar{n}}\left(\tau_{n^-} \geq k\right) \geq Ck^{-1/2}. \quad (4.3.26)$$

Plugging the above into Equation (4.3.21), we get that for $j \geq (\gamma n)^2$

$$j\mathbb{B}^{j,(1)}_{\bar{n},\bar{n}}\left(H_{4\gamma n} < j/4,\, \tau_{n^-} > j/2\right) \geq Cj^{1/2}. \quad (4.3.27)$$

Thus, by Equation (4.3.13)

$$\mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n\right] \geq C j^{1/2-d/2}\,. \tag{4.3.28}$$

This then leads to the estimate

$$\begin{aligned}\sum_{y\in\partial_i \boldsymbol{B}_n}\sum_{j\geq(\boldsymbol{\gamma}n)^2} \mathbb{E}^j_{y,y}\left[H_{\gamma n} < j,\, H_n\right] &\geq C n^{d-1}\sum_{j\geq(\boldsymbol{\gamma}n)^2} j^{-d/2+1/2+\nu} \\ &\geq C n^{d-1}\int_{(\boldsymbol{\gamma}n)^2}^{\infty} j^{-d/2+1/2+\nu}\mathrm{d}j \geq C\left(n\sqrt{\boldsymbol{\gamma}}\right)^{2\nu+2}\boldsymbol{\gamma}^{1-d}\,,\end{aligned} \tag{4.3.29}$$

where Lemma 8.2.1 implies that one can approximate the sum by an integral. This concludes the proof of the lower bound.

Upper Bound: we begin by recalling the definition of $n^{2-}_{\gamma} = (\boldsymbol{\gamma}n)^{2-\varepsilon}$ where $\boldsymbol{\gamma} = \gamma - 1$ and $\varepsilon > 0$ small, subject to some constraints.

The proof is organised as follows: for $j \geq (\boldsymbol{\gamma}n)^2$ we bound $\mathbb{E}^j_{y,y}[H_n(y), H_{\gamma n} < j] \leq \mathbb{E}^j_{y,y}[H_n(y)]$. This is justified by the fact that the event $\{H_{\gamma n} < j\}$ has constant mass for such j. The expected value of $H_n(y)$ is then analysed using results from [Uch16], [BMR13] and [DW19]. The analysis of $H_n(y)$ under a bridge measure is more complex compared to not fixing the endpoint, as we need to know where the bridge hits $\boldsymbol{B}_n$. Finally, we need to consider the case $j \in \{n^{2-}_{\gamma}, \ldots, (\boldsymbol{\gamma}n)^2\}$; however, in that regime we can no longer discard the event $\{H_{\gamma n} < j\}$. From now, shorten $H_n(y)$ as H_n.

Before embarking on the proof, we offer the following heuristics (ignoring the condition $\{H_{\gamma n} < j\}$ for now) for the upper bound: rewrite $\mathbb{P}^j_{y,y}\left(H_n = k\right)$ in the following way by conditioning on the site at which the random walk hits the sphere:

$$\mathbb{P}^j_{y,y}\left(H_n = k\right) = \mathbb{P}_y\left(H_n = k\right)\sum_{z\in\partial_i \boldsymbol{B}_n}\mathbb{P}_y\left(S_k = z | H_n = k\right) p_{j-k}(z,y)\,. \tag{4.3.30}$$

We then expect that $\mathbb{P}_y\left(H_n = k\right) \sim q(y,k,n) \sim k^{-3/2}$ (using the notation from Chapter 2). Furthermore, for $k \geq n^2$ it is reasonable to expect that the hitting distribution on $\partial_i \boldsymbol{B}_n$ is uniform, i.e. $\mathbb{P}_y\left(S_k = z | H_n = k\right) \sim n^{1-d}$, as the random walk should have "mixed" on the scale of $\boldsymbol{B}_n$ by that time. For k's smaller than n^2, we can bound $p_{j-k}(z,y)$ by $C(j-k)^{-d/2}$ and thus obtain the bounds

$$\mathbb{P}^j_{y,y}\left(H_n=k\right)\leq C\begin{cases}k^{-3/2}\frac{1}{n^{d-1}}\sum_{z\in\partial_i\boldsymbol{B}_n}p_{j-k}(z,y) & \text{for } k\geq n^2\,,\\ k^{-3/2}(j-k)^{-d/2} & \text{otherwise}\,.\end{cases}\tag{4.3.31}$$

The difficulty is making the above intuition rigorous: only bounds on the cumulative distribution function of $\mathbb{P}_y(H_n=k)$ are available and the "mixing" result regarding the first hitting location is only available for the Brownian motion. To overcome this, we employ coupling arguments similar to those used in Chapter 2 and integration by parts. Finally, for $j\leq(\gamma n)^2$, we need to refine the above bounds, as the bound for $k<n^2$ is too rough in that case. We now give the various steps into which we have subdivided the proof of the upper bound:

I. Step 1: bounding small values of H_n.

II. Step 2: various expansion of H_n for $j\geq\gamma n^2$.

III. Step 3: estimating $\sum_{j\geq\gamma n^2}\sum_{y\in\boldsymbol{B}_n}\mathbb{E}^j_{y,y}\left[H_n\right]$.

IV. Step 4a and Step 4b: estimating $H_n\mathbb{1}\{H_{\gamma n}<j\}$ for $j\leq\gamma n^2$. Step 4b is further split into two parts, treating the cases H_n small and large.

Step 1: in this step we show that we can neglect values of H_n smaller than $j^{1/5}$. Expand

$$\mathbb{E}^j_{y,y}[H_n]=\sum_{k=0}^{j}k\mathbb{P}^j_{y,y}(H_n=k)=\sum_{k=1}^{j^{1/5}}k\mathbb{P}^j_{y,y}(H_n=k)+\sum_{k=j^{1/5}}^{j}k\mathbb{P}^j_{y,y}(H_n=k)\,.\tag{4.3.32}$$

Using the bound $p_j(x)=\mathcal{O}(j^{-d/2})$ for any $x\in\mathbb{Z}^d$, the first sum can be bounded

$$\sum_{k=1}^{j^{1/5}}k\mathbb{P}^j_{y,y}(H_n=k)\leq\sum_{k=1}^{j^{1/5}}kp_j(y,y)\leq Cj^{-d/2+0.4}\,.\tag{4.3.33}$$

This implies for $\varepsilon > 0$ small enough

$$\sum_{j\geq n_\gamma^{2-}} \sum_{y\in\partial_i \boldsymbol{B}_n} a_j \mathbb{E}^j_{y,y}[H_n \mathbb{1}\{k \leq j^{1/5}\}] \leq C n^{d-1} \sum_{j\geq n_\gamma^{2-}} j^{\nu+0.4-d/2} = o\left(n^{2\nu+2}\right) . \tag{4.3.34}$$

Thus, henceforth assume that $k \geq j^{1/5}$.

Step 2: Assume $j \geq (\boldsymbol{\gamma} n)^2$ and bound $\mathbb{E}^j_{y,y}[H_n, H_{\gamma n} < j] \leq \mathbb{E}^j_{y,y}[H_n]$.

The next step consists of getting good estimates for the cumulative distribution function of H_n. For that we employ the following strategy: since we are interested in values of $H_n > j^{1/5}$ (see previous step), we know that with overwhelming probability, the random walk first hits a shell with radius slightly larger n before hitting $\boldsymbol{B}_n$. Once the random walk has hit that shell, we employ the coupling with the Brownian bridge. We need this step, because otherwise the error from the coupling would be non-negligible. We then use the bounds on H^B_n from the literature to estimate the expectation of H_n.

Rewrite using integration by parts

$$\sum_{k=j^{1/5}}^{j} k\mathbb{P}^j_{y,y}(H_n = k) = \sum_{k=j^{1/5}}^{j} \mathbb{P}^j_{y,y}(H_n \geq k) . \tag{4.3.35}$$

Note that by the same argument used in the previous step, we can neglect k's with $j - k < j^{1/5}$.

Set $n^{\pm} = n \pm \log^{2+\delta}(n)$ for some $\delta > 0$ and expand

$$\begin{aligned}\mathbb{P}^j_{y,y}(H_n \geq k) = &\sum_{l\geq 1} \sum_{z\in\partial_i \boldsymbol{B}_{n^+}} \mathbb{P}_y(H_{n^+} = l > H_n, S_l = z)\mathbb{P}^{j-l}_{z,y}(H_n \geq k - l) \\ &+ \mathbb{P}^j_{y,y}(H_n \geq k,\, H_{n^+} > H_n) .\end{aligned} \tag{4.3.36}$$

By the argument made in the proof of Proposition 2.2.5, we have that for $F > 0$ large enough that $l < \log^F(k)$ apart from a negligible set. By a similar argument as stated in that proof

$$\sum_{k=j^{1/5}}^{j-j^{1/5}} \mathbb{P}^j_{y,y}(H_n \geq k) \leq C \sup_{\substack{l\leq \log^F(k) \\ z\in R_y}} \sum_{k=j^{1/5}}^{j-j^{1/5}} \mathbb{P}^{j-l}_{z,y}(H_n \geq k - l) , \tag{4.3.37}$$

where $R_y = \{x \in \mathbb{Z}^d \colon x \in \partial_i \boldsymbol{B}_{n^+} \text{ and } |x-y| \leq \log^{2+2\delta}(n)\}$ are those points in the boundary of $\boldsymbol{B}_{n^+}$ which have small distance to y. For an illustration of R_z, see Figure 4.6.

Using the coupling from Lemma 2.2.1, we bound

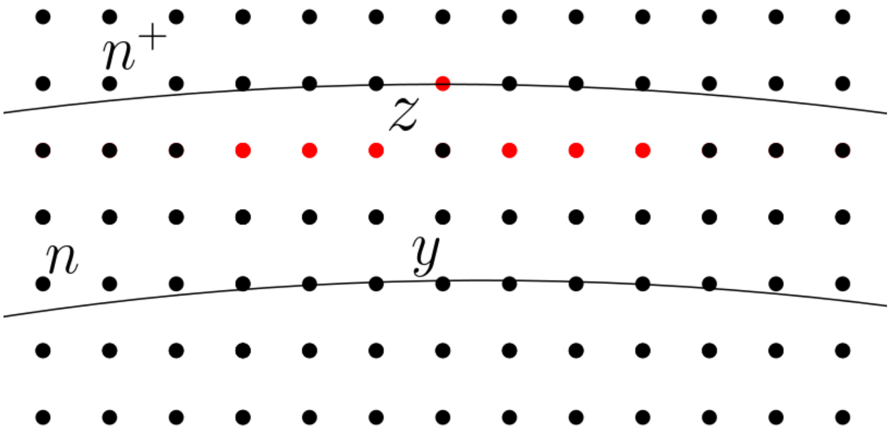

Figure 4.6: The points z in R_y are coloured red.

$$\mathbb{P}_{z,y}^{j-l}(H_n \geq k-l) \leq C P_{z,y}^{j-l}(H_{n^-}^B \geq k-l)\,. \tag{4.3.38}$$

We will bound $P_{z,y}^j(H_{n^-} \geq k-l)$ uniformly in $0 \leq l \leq \log^F(k)$. The bound will not depend on z. Denote this bound by $C(k,y)$. We then express Equation (4.3.36) as follows

$$\begin{aligned} \mathbb{P}_{y,y}^{j-l}(H_n \geq k) &\leq C P_{z,y}^{j-l}(H_{n^-}^B \geq k-l)\mathbb{P}_y(H_{n^+} < H_n) \\ &\leq C(k,y)\mathbb{P}_y(H_{n^+} < H_n) \sim \frac{C(k,y)}{\log^{2+\delta}(n)}\,, \end{aligned} \tag{4.3.39}$$

where martingale argument from proof of Proposition 2.2.5 gives the estimate on $\mathbb{P}_y(H_{n^+} < H_n)$. We now show that $C(k,y) \sim k^{-1/2}\log^{2+\delta}(n)$.

We proceed as follows: to bound $\mathrm{d}P_{z,y}^{j-l}(H_{n^-}^B \geq k-l)$, we firstly we bound the density $\mathrm{d}P_{z,y}^{j-l}(H_{n^-} = k)$. By the Markov property we have

$$\mathrm{d}P_{z,y}^{j-l}(H_{n^-}^B = k) = \int_{\partial \boldsymbol{B}_{n^-}} \mathrm{d}P_z(H_{n^-}^B = k)\mathrm{d}P_z(B_k = z | H_{n^-}^B = k)\mathfrak{p}_{j-l-k}(z,y)\mathrm{d}z. \tag{4.3.40}$$

Indeed, the above is simply a conditioning on the location at which we hit

$\partial \boldsymbol{B}_{n^-}$. Note that by the restrictions placed on l and k, we have that $\mathfrak{p}_{j-l-k}(z,y) \sim \mathfrak{p}_{j-k}(z,y)$.

Using the result from [Uch16] on $\mathrm{d}P_z(B_k = z | H_{n^-} = k)$, we bound the above expression in the following way: fix $\beta_o > 0$ (to be adjusted later) and denote $\mathcal{U}_{\partial \boldsymbol{B}_{n^-}}$ the uniform (Haar with respect to rotation) measure on $\partial \boldsymbol{B}_{n^-}$. We then have two bounds

$$\mathrm{d}P^j_{z,y}(H^B_{n^-} = k) \leq \begin{cases} \mathrm{d}P^j_z(H^B_{n^-} = k)\mathfrak{p}_{j-k}(y^-, y) & \text{if } k \leq \beta_o n^2\,, \\ C\mathrm{d}P_z(H^B_{n^-} = k)\mathcal{U}_{\partial \boldsymbol{B}_{n^-}}\left[\mathfrak{p}_{j-k}(Z,y)\right] & \text{otherwise}\,, \end{cases} \tag{4.3.41}$$

where y^- is the projection (through the origin) of y onto $\boldsymbol{B}_{n^-}$ and Z is distributed with respect to $\mathcal{U}_{\partial \boldsymbol{B}_{n^-}}$. Indeed, for the first bound in the equation above, a quick estimate using Equation (4.3.40) gives

$$\mathrm{d}P^j_{z,y}(H^B_{n^-}= k) \leq \mathrm{d}P_z(H^B_{n^-}= k) \sup_{z \in \partial \boldsymbol{B}_{n^-}} \mathfrak{p}_{j-k}(z,y) \leq C\mathrm{d}P_z(H^B_{n^-}= k)\mathfrak{p}_{j-k}(y^-, y)\,, \tag{4.3.42}$$

by the definition of the heat kernel. To verify the second bound in Equation (4.3.41), recall that [Uch16, Theorem 2.2] states that for $k \geq \beta_o n^2$ the hitting location of the Brownian motion is uniform (up to a multiplicative constant in the density) on $\boldsymbol{B}_{n^-}$. This fact gives the second bound in Equation (4.3.41). We furthermore bound (using that $k \geq (\gamma n)^{1/5}$) and Lemma 2.2.4

$$\mathrm{d}P_z(H^B_{n^-} = k) = q(z,k,n) = q(n + \log^{2+\delta}(n), k, n) \leq C\frac{\log^{2+\delta}(n)}{k^{3/2}}\,. \tag{4.3.43}$$

This implies

$$\mathrm{d}P^j_{z,y}(H^B_{n^-} = k) \leq \begin{cases} \frac{\log^{2+\delta}(n)}{k^{3/2}}\mathfrak{p}_{j-k}(y^-, y) & \text{if } k \leq \beta_o n^2\,, \\ C\frac{\log^{2+\delta}(n)}{k^{3/2}}\mathcal{U}_{\partial \boldsymbol{B}_{n^-}}\left[\mathfrak{p}_{j-k}(Z,y)\right] & \text{otherwise}\,. \end{cases} \tag{4.3.44}$$

We now compute $\mathcal{U}_{\partial \boldsymbol{B}_{n^-}}\left[\mathfrak{p}_{j-k}(Z,y)\right]$. Note that we can expand

$$n^{d-1}\mathcal{U}_{\partial \boldsymbol{B}_{n^-}}\left[\mathfrak{p}_{j-k}(Z,y)\right] \leq \frac{C}{(j-k)^{d/2}} \int_0^{2n^-} r^{d-2} \mathrm{e}^{-C(r^2 + c_\alpha^2 \log(n)^4)/2(j-k)} \mathrm{d}r\,. \tag{4.3.45}$$

Indeed, note that since $Z \in \partial \boldsymbol{B}_{n^-}$, we have that $\mathfrak{p}_{j-k}(Z,y)$ only depends on

$|y-Z|$. We can use the inner product to rewrite this as

$$|y-Z|^2 = 2\,(1-\cos(\theta))\,(n^2 + c_\alpha \log(n)) + c_\alpha^2 \log^4(n)\,, \tag{4.3.46}$$

where θ is the angle between Z and y. Approximating $(1-\cos(\theta)) \sim \theta^2$ and incorporating the fact that $\partial \boldsymbol{B}_{n^-}$ is a $(d-1)$-dimensional submanifold (hence the factor r^{d-2}) gives us the bound Equation (4.3.45). Note that $\log^4(n) = o(j-k)$. We change variables,

$$\begin{aligned}\frac{1}{(j-k)^{d/2}} \int_0^{2n^-} & r^{d-2} \mathrm{e}^{-C(r^2+c_\alpha^2 \log(n)^4)/2(j-k)} \mathrm{d}r \\ &= \frac{\mathcal{O}(1)}{(j-k)^{1/2}} \int_0^{C(n^-)^2/(j-k)} \mathrm{e}^{-r} r^{d/2-3/2} \mathrm{d}r\,.\end{aligned} \tag{4.3.47}$$

We recognise that the integral above as the incomplete Gamma function. Thus

$$\mathcal{U}_{\partial \boldsymbol{B}_{n^-}}\left[\mathfrak{p}_{j-k}(Z,y)\right] \le \frac{1}{n^{d-1}(j-k)^{1/2}} \gamma\left(d/2-1/2, C\frac{(n^-)^2}{j-k}\right)\,. \tag{4.3.48}$$

Recall the asymptotics from Lemma 8.3.3

$$\gamma(s,x) \sim x^s\,, \tag{4.3.49}$$

as $x \downarrow 0$. The above bounds imply that when we plug Equation (4.3.38) into Equation (4.3.37), we can bound the sum by an integral using Lemma 8.2.1. Furthermore, the function $l \mapsto P^j_{z,y}(H^B_{n^-} \ge k-l)$ satisfies for $l \le \log^F(k)$

$$P^j_{z,y}(H^B_{n^-} \ge k-l) \sim P^j_{z,y}(H^B_{n^-} \ge k/2)\,, \tag{4.3.50}$$

and

$$\sum_{k \ge j^{1/5}} P^j_{z,y}(H^B_{n^-} \ge k/2) \le C \int_{j^{1/5}}^{j} P^j_{z,y}(H^B_{n^-} \ge k) \mathrm{d}k \le C \int_{j^{1/5}}^{j} k P^j_{z,y}(H^B_{n^-} = k) \mathrm{d}k. \tag{4.3.51}$$

Thus, by Equation (4.3.39), we have

$$\sum_{k \ge j^{1/5}} \mathbb{P}^j_{y,y}(H_n \ge k-l) \le \int_{j^{1/5}}^{j} \frac{k}{\log^{2+\delta}(n)} P^j_{z,y}(H^B_{n^-} = k) \mathrm{d}k\,. \tag{4.3.52}$$

Fix $\beta_1 > 0$ with $\boldsymbol{\gamma} > \beta_1 > \beta_o$ and split the integration into three regions

$$\int_{j^{1/5}}^{j} \frac{kP_{z,y}^j(H_{n^-}^B = k)}{\log^{2+\delta}(n)}\mathrm{d}k = \int_{j^{1/5}}^{\beta_o n^2} \ldots \mathrm{d}k + \int_{\beta_o n^2}^{j-\beta_1 n^2} \ldots \mathrm{d}k + \int_{j-\beta_1 n^2}^{j} \ldots \mathrm{d}k\,. \tag{4.3.53}$$

We begin with the integral from $j^{1/5}$ to $\beta_o n^2$, where we employ the upper bound from Equation (4.3.41) and Equation (4.3.43), so that we get

$$\begin{aligned}
&\mathbb{E}_{y,y}^j\left[H_n, H_n \in \{j^{1,5}, \ldots, \beta_o n^2\}\right] \\
&\quad\le \frac{C}{\log^{2+\delta}(n)} \int_{j^{1/5}}^{\beta_o n^2} kP_{z,y}^j(H_{n^-}^B = k)\mathrm{d}k \le C\int_{j^{1/5}}^{\beta_o n^2} k\frac{\mathfrak{p}_{j-k}(y^-, y)}{k^{3/2}}\mathrm{d}k \\
&\quad\le \int_0^{\beta_o n^2} \frac{C}{\sqrt{k}(j-k)^{d/2}}\mathrm{d}k \le \frac{C}{(j-\beta_o n^2)^{d/2}}\int_0^{\beta_o n^2}\frac{1}{\sqrt{k}}\mathrm{d}k \le \frac{C(\beta_o)^{1/2}n}{(j-\beta_o n^2)^{d/2}}\,,
\end{aligned} \tag{4.3.54}$$

as $j \ge (n\boldsymbol{\gamma})^2$ and

$$\beta_o < \beta_1 < \boldsymbol{\gamma} = (\gamma - 1)\,. \tag{4.3.55}$$

Next we integrate from $\beta_o n^2$ to $j - \beta_1 n^2$. Due to this restriction, one has that

$$\beta_1 n^2 \le j - k \le j - \beta_o n^2\,. \tag{4.3.56}$$

We apply the second bound from Equation (4.3.41) together with Equation (4.3.43) and Equation (4.3.48) to bound

$$\begin{aligned}
&\mathbb{E}_{y,y}^j\left[H_n, H_n \in \{\beta_o n^2, \ldots, j - \beta_1 n^2\}\right] \\
&\quad\le \frac{1}{\log^{2+\delta}(n)}\int_{\beta_o n^2}^{j-\beta_1 n^2} kP_{z,y}^j(H_{n^-}^B = k)\mathrm{d}k \\
&\quad\le \frac{C}{n^{d-1}}\int_{\beta_o n^2}^{j-\beta_1 n^2} \frac{1}{\sqrt{k(j-k)}}\gamma\left(d/2 - 1/2, \frac{n^2}{j-k}\right)\mathrm{d}k \\
&\quad\le C\int_{\beta_o n^2}^{j-\beta_1 n^2}\frac{1}{\sqrt{k(j-k)^d}}\mathrm{d}k \\
&\quad\le Cj^{1/2-d/2}\int_{\beta_o n^2/j}^{1-\beta_1 n^2/j}\frac{1}{\sqrt{k(1-k)^d}}\mathrm{d}k \le Cj^{1/2-d/2}\,.
\end{aligned} \tag{4.3.57}$$

The above uses the bounds in Equation (4.3.56) on $j - k$ which allows us to

apply the asymptotics of the Gamma function given in (4.3.49). Observe that in the last equality, the bounds on β_o and β_1 make sure that the boundaries of integration stay away from 1 uniformly in n, j. This implies that the integral does not blow up.
For the last integral (k from $j-\beta_1 n^2$ to j) we use the fact that $\gamma(d/2-1/2, x) \leq \Gamma(d/2-1/2) < \infty$ and Equation (4.3.46)

$$\begin{aligned}&\mathbb{E}^j_{y,y}\left[H_n, H_n \in \{j-\beta_1 n^2, \dots, j\}\right]\\ &\quad \frac{1}{\log^{2+\delta}(n)} \int_{j-\beta_1 n^2}^{j} k P^j_{z,y}(H^B_{n^-} = k) \mathrm{d}k \leq \frac{C}{n^{d-1}} \int_{j-\beta_1 n^2}^{j} \frac{1}{k^{1/2}(j-k)^{1/2}} \mathrm{d}k\\ &\quad \frac{C}{n^{d-1}(j-\beta_1 n^2)^{1/2}} \int_0^{\beta_1 n^2} \frac{1}{k^{1/2}} \mathrm{d}k \leq \frac{C\sqrt{\beta_1}}{n^{d-2}\sqrt{j-\beta_1 n^2}}\,.\end{aligned} \tag{4.3.58}$$

This concludes the second step. In the next step we sum the above estimates.
Step 3: in this step we bound the estimates from the previous step and calculate the asymptotics for j with $j \geq (\boldsymbol{\gamma} n)^2$. We use Equation (4.3.39) together with the various estimates made for the integral over $k P^j_{z,y}(H^B_{n^-} = k)$. We begin with the part of $\mathbb{P}^j_{y,y}(H_n = k)$ for $k \in \{j^{1/5}, \dots, \beta_o n^2\}$. By Equation (4.3.54), it holds that

$$\begin{aligned}&\sum_{j \geq (\boldsymbol{\gamma} n)^2} \sum_{y \in \partial_i \boldsymbol{B}_n} j^\nu \mathbb{E}^j_{y,y}\left[H_n, H_n \in \{j^{1,5}, \dots, \beta_o n^2\}\right]\\ &\quad \leq \sum_{j \geq (\boldsymbol{\gamma} n)^2} \sum_{y \in \partial_i \boldsymbol{B}_n} \frac{Cn}{(j-\beta_o n^2)^{d/2}} j^\nu \sim n^d \int_{(\boldsymbol{\gamma} n)^2}^{\infty} \frac{j^\nu}{j^{d/2}} \frac{1}{(1-\beta_o n^2/j)^{d/2}} \mathrm{d}j\\ &\quad \sim C(\boldsymbol{\gamma} n)^{2\nu+2} \boldsymbol{\gamma}^{-d}\,.\end{aligned} \tag{4.3.59}$$

We furthermore have that by Equation (4.3.57)

$$\begin{aligned}&\sum_{j \geq (\boldsymbol{\gamma} n)^2} \sum_{y \in \partial_i \boldsymbol{B}_n} j^\nu \mathbb{E}^j_{y,y}\left[H_n, H_n \in \{\beta_o n^2, \dots, j-\beta_1 n^2\}\right]\\ &\quad \leq C \sum_{j \geq (\boldsymbol{\gamma} n)^2} \sum_{y \in \partial_i \boldsymbol{B}_n} j^{1/2-d/2+\nu} \sim C(\boldsymbol{\gamma} n)^{2\nu+2} \boldsymbol{\gamma}^{1-d}\,.\end{aligned} \tag{4.3.60}$$

For the third summation, notice that Equation (4.3.58) implies that

$$\begin{aligned}\sum_{j\geq(\gamma n)^2}\sum_{y\in\partial_i \boldsymbol{B}_n} j^\nu \mathbb{E}^j_{y,y}\left[H_n, H_n\in\{j-\beta_1 n^2,\dots,j\}\right] \\ \leq \sum_{j\geq(\gamma n)^2}\sum_{y\in\partial_i \boldsymbol{B}_n} \frac{j^\nu}{n^{d-2}\sqrt{j-\beta_1 n^2}} \sim C n^{2\nu+2}\boldsymbol{\gamma}^{2\nu+1}\,,\end{aligned} \tag{4.3.61}$$

where the assumption that $\nu < -1/2$ allows for a computation of the above sum.

To summarise, we have shown that

$$\sum_{j\geq(\gamma n)^2}\sum_{y\in\partial_i \boldsymbol{B}_n} \mathbb{E}^j_{y,y}[H_n \mathbb{1}\{H_{\gamma_n}<j\}] \leq \sum_{j\geq(\gamma n)^2}\sum_{y\in\partial_i \boldsymbol{B}_n} \mathbb{E}^j_{y,y}[H_n] \leq C n^{2\nu+2}\boldsymbol{\gamma}^{\nu'}\,, \tag{4.3.62}$$

where $\nu' = \max\{2\nu+1,\, 2\nu+2-d\}$.

It remains to analyse the sum over $n_\gamma^{2-} \leq j \leq (\boldsymbol{\gamma} n)^2$ which is done in the next step.

Step 4: in order to analyse the expectation of $H_n\mathbb{1}\{H_{\gamma n}<j\}$, we split the associated density into two parts by distinguishing whether $\boldsymbol{B}_{\gamma n}$ is hit before $\boldsymbol{B}_n$ or not:

$$\mathbb{P}^j_{y,y}\left(H_n = k, H_{\gamma n}< j\right) = \mathbb{P}^j_{y,y}\left(H_n = k, k< H_{\gamma n}< j\right) + \mathbb{P}^j_{y,y}\left(H_n = k, H_{\gamma n}< k\right)\,. \tag{4.3.63}$$

In Step 4a, we analyse the first summand, in Step 4b the second.

Step 4a: we use the Markov property to decompose the first summand into

$$\begin{aligned}\mathbb{P}^j_{y,y}\left(H_n = k, k< H_{\gamma n}< j\right) = \mathbb{P}_y\left(H_n = k, H_{\gamma n} > k\right) \\ \times \sum_{z\in\partial_i \boldsymbol{B}_n} \mathbb{P}_z\left(1< H_{\gamma n}< j-k, S_{j-k}=y\right)\mathbb{P}_y\left(S_k = z | H_n = k\right)\,.\end{aligned} \tag{4.3.64}$$

Indeed, as $H_n < H_{\gamma n}$, the random walk has to hit $\partial_i \boldsymbol{B}_n$ at point z after k steps, it then hits $\partial_i \boldsymbol{B}_{\gamma n}$ before returning to y.

As reasoned previously, since we have to cover a distance of $(\boldsymbol{\gamma} n)^2$ in $j-k$ steps, we can neglect $j-k \leq (\boldsymbol{\gamma} n)^2 \log^{-1}((\boldsymbol{\gamma} n)^M)$, for some $M > 1$ large enough. By a reasoning analogous to proof of Lemma 2.2.3, we have that for some $t > 0$ fixed

$$\mathbb{P}_z\left(1< H_{\gamma_n} < j-k, S_{j-k}=y\right) \le \mathrm{e}^{-t(\gamma n)^2/(j-k)}(j-k)^{-d/2}\,. \qquad (4.3.65)$$

Indeed,

$$\begin{aligned}\mathbb{P}_z\left(H_{\gamma_n} < j-k, S_{j-k}=y\right) &= \sum_{b\in\partial \boldsymbol{B}_{\gamma n}} \sum_{r=1}^{j-k} \mathbb{P}_z\left(H_{\gamma n}=r\right) \\ &\qquad\times \mathbb{P}_z\left(S_r=b|H_{\gamma n}=r\right) p_r(b,y) \\ &\le C\sum_{r=1}^{j-k} \mathbb{P}_z\left(H_{\gamma n}=r\right) p_{j-k}(y^+,y) \le \mathbb{P}_z\left(H_{\gamma n}\le j-k\right)(j-k)^{-d/2}\,,\end{aligned} \qquad (4.3.66)$$

where y^+ is some point in the $\mathcal{O}(1)$ neighbourhood around the intersection of the line connecting the origin and y with $\partial\boldsymbol{B}_{\gamma n}$. The heat-kernel approximation of the random walk kernel for the contributing $r > 0$ large enough was used above.

Plug the above estimate (i.e. Equation (4.3.65)) into Equation (4.3.64) to bound

$$\begin{aligned}&\sum_{z\in\partial_i \boldsymbol{B}_n} \mathbb{P}_z\left(1< H_{\gamma_n} < j-k, S_{j-k}=y\right)\mathbb{P}_y\left(S_k=z|H_n=k\right) \\ &\le C\mathrm{e}^{-t(\gamma n)^2/(j-k)}(j-k)^{-d/2}\,.\end{aligned} \qquad (4.3.67)$$

Bounding $\mathbb{P}_y\left(H_n=k, H_{\gamma_n}>k\right) \le \mathbb{P}_y\left(H_n=k\right)$ and performing a discrete integration by parts as described in Lemma 8.2.3, we bound

$$\begin{aligned}\mathbb{E}^j_{y,y}\left[H_n\mathbb{1}\{H_{\gamma n}<j\}, H_{\gamma_n}>H_n\right] &\le C\sum_{k=(\gamma n)^2\log^{-1}((\gamma n)^M)}^{j} \\ &\mathbb{P}_y\left(H_n\ge k\right)\left(\mathrm{e}^{-t(\gamma n)^2/(j-k)}(j-k)^{-d/2} - \mathrm{e}^{-t(\gamma n)^2/(j-k-1)}(j-k-1)^{-d/2}\right)\,.\end{aligned} \qquad (4.3.68)$$

Using Proposition 2.2.5 to bound $\mathbb{P}_y\left(H_n\ge k\right) \le Ck^{-1/2}$ and Lemma 8.2.1 to approximate the integral by a sum, we rewrite the above as

$$\mathbb{E}_{y,y}^{j}\left[H_n \mathbb{1}\{H_{\gamma n} < j\}, H_{\gamma_n} > H_n\right] \leq C \int_1^j k^{-1/2} \mathrm{e}^{-t(\gamma n)^2/(j-k)} (j-k)^{-d/2} \mathrm{d}k\,. \tag{4.3.69}$$

By changing variables $k \mapsto jk$, we can estimate

$$\int_1^j k^{-1/2} \mathrm{e}^{-t(\gamma n)^2/(j-k)} (j-k)^{-d/2} \mathrm{d}k \leq C(n,j) j^{1/2-d/2} \mathrm{e}^{-t(\gamma n)^2/j}\,, \tag{4.3.70}$$

where

$$C(n,j) \leq \int_{1/j}^1 k^{-1/2} (1-k)^{-d/2} \mathrm{e}^{-t(\gamma n)^2[1/(1-k)-1]/j} \mathrm{d}k\,. \tag{4.3.71}$$

However, as γ is bounded away from 1 and we assumed that $j \leq (\boldsymbol{\gamma} n)^2$, we have that

$$\sup_{j,n} C(n,j) < \infty\,, \tag{4.3.72}$$

where the supremum is taken over all n, j satisfying $j \leq (\boldsymbol{\gamma} n)^2$. This allows us to estimate

$$\begin{aligned}
&\sum_{j=n_\gamma^{2-}}^{(\boldsymbol{\gamma} n)^2} \sum_{y \in \partial_i \boldsymbol{B}_n} a_j \mathbb{E}_{y,y}^{j}\left[H_n \mathbb{1}\{H_{\gamma n} < j\}, H_{\gamma_n} > H_n\right] \\
&\qquad \leq C \sum_{j=n_\gamma^{2-}}^{(\boldsymbol{\gamma} n)^2} a_j n^{d-1} j^{1/2-d/2} \mathrm{e}^{-t(\gamma n)^2/j} \\
&\qquad \leq C (\boldsymbol{\gamma} n)^{2+2\nu} \boldsymbol{\gamma}^{1-d} \int_0^t \mathrm{e}^{-j} j^{-5/2+d/2-\nu} \mathrm{d}j \sim (\boldsymbol{\gamma} n)^{2+2\nu} \boldsymbol{\gamma}^{1-d}\,,
\end{aligned} \tag{4.3.73}$$

where Lemma 8.2.1 allows for an approximation of the sum by an integral and a change of variables $j \mapsto t(\boldsymbol{\gamma} n)^2/j$ was used. The above is of the right order and thus we have finished the case $H_{\gamma n} > H_n$.

Step 4b: it remains to estimate

$$\mathbb{E}_{y,y}^{j}\left[H_n \mathbb{1}\{H_{\gamma n} < j\}, H_{\gamma_n} < H_n\right]\,. \tag{4.3.74}$$

For $C_1 > 0$ abbreviate $n_1 = C_1(\boldsymbol{\gamma} n)^2 \log^{-1}(\boldsymbol{\gamma} n)$ and bound

$$\sum_{k\le n_1} k\mathbb{P}^j_{y,y}\left(H_n = k, H_{\gamma_n} < H_n\right) \le n_1 \mathbb{P}_0(H_{\boldsymbol{\gamma} n} \le n_1) \le C n_1 (\boldsymbol{\gamma} n)^{-t/C_1}\,, \quad (4.3.75)$$

for some $t > 0$ by [LL10, Proposition 2.4.5]. By making C_1 sufficiently small, we conclude that it suffices to estimate

$$\mathbb{E}^j_{y,y}\left[H_n \mathbb{1}\{H_{\gamma n} < j\}, H_{\gamma_n} < H_n,\, H_n \ge n_1\right]. \quad (4.3.76)$$

Two different estimates have to be made for the case $H_n = k \le j/2$ and $H_n = k \ge j/2$.

Step 4b, Part I: the case $k \le j/2$: we use integration by parts to rewrite

$$\sum_{k=n_1}^{j/2} k\mathbb{P}^j_{y,y}\left(H_n = k, H_{\gamma_n} < H_n\right) = \sum_{k=n_1}^{j/2} \mathbb{P}^j_{y,y}\left(H_n \ge k, H_{\gamma_n} < H_n\right). \quad (4.3.77)$$

The random walk bridge now has to hit first $\partial_i \boldsymbol{B}_{\gamma n}$ before hitting the shell $\partial_i \boldsymbol{B}_n$. We estimate again, by conditioning on the point at which the random walk hits $\partial_i \boldsymbol{B}_{\gamma n}$,

$$\begin{aligned}\mathbb{P}^j_{y,y}\left(H_n \ge k, H_{\gamma_n} < H_n\right) = \sum_{z\in\partial_i \boldsymbol{B}_{\gamma n}} \sum_{l=0}^{j} &\mathbb{P}_y\left(H_{\gamma_n} = l,\, H_n > H_{\gamma n}\right)\\ &\times \mathbb{P}_z\left(H_n \ge k-l, S_{j-l} = y\right) \mathbb{P}_y\left(S_l = z | H_{\gamma_n} = l\right).\end{aligned} \quad (4.3.78)$$

By the same reasoning as above it suffices to consider the sum over $n_1 \le l \le k - n_1$. We expand again

$$\begin{aligned}&\mathbb{P}_z\left(H_n \ge k-l, S_{j-l} = y\right)\\ &= \sum_{t=k-l}^{j-l} \mathbb{P}_z\left(H_n = t\right) \sum_{w\in\partial_i \boldsymbol{B}_n} \mathbb{P}_z\left(S_t = w | H_n = t\right) p_{j-l-t}(w,y)\\ &\le C \sum_{t=k-l}^{j-l} \mathbb{P}_z\left(H_n = t\right)(j-l-t)^{-d/2}\,,\end{aligned} \quad (4.3.79)$$

by the same reasoning as employed in Equation (4.3.67). We integrate by parts to bound the above

$$\sum_{t=k-l+1}^{j-l-1} \mathbb{P}_z\left(H_n \geq t\right)\left[(j-l-t)^{-d/2}-(j-l-t-1)^{-d/2}\right]+E\,, \tag{4.3.80}$$

where

$$E=\mathbb{P}_z\left(H_n \geq k-l\right)(j-k)^{-d/2}\,. \tag{4.3.81}$$

By Proposition 2.2.5, we have that

$$\mathbb{P}_z\left(H_n \geq t\right) \sim \int_t^{\infty} q(n+1,s,n)\mathrm{d}s \sim \int_t^{\infty} \frac{n\boldsymbol{\gamma}\mathrm{e}^{-(\gamma n)^2/2t}}{s^{3/2}}\mathrm{d}s \sim P_z\left(H_n^B \geq t\right)\,. \tag{4.3.82}$$

Using discrete integration by parts, we can rewrite

$$\begin{aligned}&\sum_{t=k-l+1}^{j-l-1} \mathbb{P}_z\left(H_n \geq t\right)\left[(j-l-t)^{-d/2}-(j-l-t-1)^{-d/2}\right]+E\\ &\leq C\sum_{t=k-l+1}^{j-l-1} q(n+1,t,n)(j-l-t)^{-d/2}\,.\end{aligned} \tag{4.3.83}$$

Approximating the sum by an integral using Lemma 8.2.1

$$\mathbb{P}_z\left(H_n \geq k-l, S_{j-l}=y\right) \leq C\int_{k-l}^{j-l} \frac{n\boldsymbol{\gamma}\mathrm{e}^{-(\gamma n)^2/2t}}{\gamma t^{3/2}}(j-l-t)^{-d/2}\,\mathrm{d}t\,. \tag{4.3.84}$$

Shorten the above as $C(k,l,n)$ and note that it does not depend on z anymore. For $f\colon \mathbb{N}\to\mathbb{R}$, denote $\nabla_l f = f(l)-f(l+1)$ and plug the above result into

Equation (4.3.78) to get that

$$\begin{aligned}
&\sum_{k=n_1}^{j/2} \mathbb{P}^{j}_{y,y}\left(H_n \geq k, H_{\gamma n} < H_n\right) \\
&\quad \leq C \sum_{k=n_1}^{j/2} \sum_{l=n_1}^{k-n_1} \mathbb{P}_y\left(H_{\gamma_n} = l,\ H_n > H_{\gamma n}\right) C(k,l,n) \\
&\quad \leq C \sum_{k=n_1}^{j/2} k \sum_{l=n_1}^{k-n_1} \mathbb{P}_y\left(H_{\gamma_n} = l,\ H_n > H_{\gamma n}\right) \left[C(k,l,n) - C(k+1,l,n)\right] \\
&\quad \leq C \sum_{k=n_1}^{j/2} k \sum_{l=n_1}^{k-n_1} \mathbb{P}_y\left(H_{\gamma_n} = l,\ H_n > H_{\gamma n}\right) \frac{n\boldsymbol{\gamma}\mathrm{e}^{-(\gamma n)^2/2(k-l)}}{\gamma(k-l)^{3/2}} (j-k)^{-d/2} \\
&\quad \leq C \sum_{k=n_1}^{j/2} k \sum_{l=n_1}^{k-n_1} \mathbb{P}_y\left(H_{\gamma_n} \leq l,\ H_n > H_{\gamma n}\right) \nabla_l \left[\frac{n\boldsymbol{\gamma}\mathrm{e}^{-(\gamma n)^2/2(k-l)}}{\gamma(k-l)^{3/2}} (j-k)^{-d/2}\right],
\end{aligned} \tag{4.3.85}$$

using discrete integration by parts (and neglecting the boundary terms on account of the reasoning behind Equation (4.3.75)). To estimate the above, we now need to bound $\mathbb{P}_y\left(H_{\gamma_n} \leq l,\ H_n > H_{\gamma n}\right)$. By the Markov property and the bound [LL10, Proposition 2.4.5], we have that for some $\beta > 0$

$$\mathbb{P}_y\left(H_{\gamma_n} \leq l,\ H_n > H_{\gamma n}\right) \leq \mathbb{P}_y\left(H_n > H_{\gamma n}\right) \mathbb{P}_z\left(H_{\gamma_n} \leq l\right) \leq \frac{C\mathrm{e}^{-\beta(\gamma n)^2/l}}{\boldsymbol{\gamma} n}, \tag{4.3.86}$$

where the martingale argument from the proof of Proposition 2.2.5 gives the bound on $\mathbb{P}_y\left(H_n > H_{\gamma n}\right)$. Inserting the above into Equation (4.3.85) implies that

$$\sum_{k=n_1}^{j/2}\mathbb{P}^j_{y,y}\left(H_n\geq k, H_{\gamma n}<H_n\right)$$
$$\leq C\sum_{k=n_1}^{j/2}k\sum_{l=n_1}^{k-n_1}\frac{\mathrm{e}^{-\beta(\gamma n)^2/l}}{\gamma n}\nabla_l\left[\frac{(n\gamma)\mathrm{e}^{-(\gamma n)^2/2(k-l)}}{\gamma(k-l)^{3/2}}\left(j-k\right)^{-d/2}\right]$$
$$\leq C\sum_{k=n_1}^{j/2}k\sum_{l=n_1}^{k-n_1}\frac{\beta(\gamma n)^2\mathrm{e}^{-\beta(\gamma n)^2/l}(n\gamma)\mathrm{e}^{-(\gamma n)^2/2(k-l)}}{(\gamma n)l^2\gamma(k-l)^{3/2}\left(j-k\right)^{d/2}}$$
$$\leq C\int_1^{j/2}\int_{n_1}^{k-n_1}\frac{\beta(\gamma n)^2\mathrm{e}^{-(\gamma n)^2/2(k-l)}\mathrm{e}^{-\beta(\gamma n)^2/l}}{l^2\gamma(k-l)^{3/2}\left(j-k\right)^{d/2}}\,k\,\mathrm{d}l\,\mathrm{d}k$$
$$\leq C(\gamma n)^2j^{-d/2-1/2}\int_{1/j}^{1/2}\int_{n_1/j}^{k-n_1/j}\frac{\mathrm{e}^{-(\gamma n)^2/2j(k-l)}\mathrm{e}^{-\beta(\gamma n)^2/(jl)}}{l^2\gamma(k-l)^{3/2}\left(1-k\right)^{d/2}}\,k\,\mathrm{d}l\,\mathrm{d}k$$
$$\leq C(\gamma n)^2j^{-d/2-1/2}\mathrm{e}^{-C(\gamma n)^2/j}\,, \tag{4.3.87}$$

using integration by parts. The factors of $j^{-2}(k-l)^{-3/2}$ are neutralised by the exponential. Furthermore observe that $(1-k)^{-d/2}\leq 2^{d/2}$, due to the assumption that $k\leq j/2$. Integrating the above from n_γ^{2-} up to $(\gamma n)^2$ with respect to j gives the upper bound for $k\leq j/2$.

Step 4b, Part II: it remains to analyse the case $k\in\{j/2,\dots j\}$. The problem is that for such k's, the previously applied bound

$$\mathbb{E}_z\left[p_l(S_k,x)|H_n=k\right]\leq\max_{y\in\partial_i\boldsymbol{B}_n}p_l(y,x)\,, \tag{4.3.88}$$

is not good enough, as l is small with non-negligible mass. This would lead to a blow-up in the penultimate line of Equation (4.3.87). However, contrary to the case that $j\geq(\gamma n)^2$, we cannot assume uniformity of the hitting time anymore. We begin by bounding (as done previously)

$$\mathbb{P}_z(H_n\geq k-l,\,S_{j-l}=y)\leq P^{j-l}_{z,y}(H^B_{n^-}\geq k-l)+\mathcal{O}\left(\left(\frac{1}{j-l}\right)^{\alpha}\right)\,, \tag{4.3.89}$$

with $n^- = n - c_\alpha \log^2(n)$, $\alpha > 0$ to be determined later. We expand

$$P^{j-l}_{z,y}(H^B_{n^-} \geq k-l) = C(z,j,l) \\ + \int_{\partial \boldsymbol{B}_{n^-}} \mathrm{d}z \int_{k-l}^{j-l} \mathrm{d}t \, \mathrm{d}P_z\left(H^B_{n^-} = t\right) \mathrm{d}P_z\left(B_t = z | H^B_{n^-} = t\right) \mathfrak{p}_{j-t-l}(z,y) \,. \tag{4.3.90}$$

Here, we use

$$C(z,j,l) = P^{j-l}_{z,y}(H^B_{n^-} \geq j-l) \,. \tag{4.3.91}$$

Let us firstly neglect the $\mathcal{O}\left(\left(\frac{1}{j-l}\right)^\alpha\right)$ term and expand

$$\sum_{k=j/2}^{j} \mathbb{P}^j_{y,y}\left(H_n \geq k, H_{\gamma n} < H_n\right) \leq C \sum_{k=j/2}^{j} k \sum_{l=n_1}^{k-n_1} \mathbb{P}_y\left(H_{\gamma_n} = l,\, H_n > H_{\gamma n}\right) \\ \times \sum_{z \in \partial_i \boldsymbol{B}_{\gamma n}} \left[P^{j-l}_{z,y}(H^B_{n^-} \geq k-l) - P^{j-l}_{z,y}(H^B_{n^-} \geq k+1-l)\right] \\ \times \mathbb{P}_y\left(S_l = z | H_{\gamma_n} = l,\, H_n > H_{\gamma n}\right) \,. \tag{4.3.92}$$

We apply the same strategy as before: if $k \mapsto \mathrm{d}P^{j-l}_{z,y}(H^B_{n^-} = k-l)$ varies sufficiently slowly with k, we have that

$$\left(P^{j-l}_{z,y}(H^B_{n^-} \geq k-l) - P^{j-l}_{z,y}(H^B_{n^-} \geq k+1-l)\right) \sim \mathrm{d}P^{j-l}_{z,y}(H^B_{n^-} = k-l) \,. \tag{4.3.93}$$

To show that, we use that

$$\mathrm{d}P^{j-l}_{z,y}(H^B_{n^-} = k-l) = \int_{\partial B_{n-}} \mathrm{d}P_z(B_{k-l} = x, H^B_{n^-} = k-l)\mathfrak{p}_{j-k}(x,y) \,. \tag{4.3.94}$$

We now estimate the above quantity. Let, for $z \in \partial \boldsymbol{B}_{\gamma n}$, $r = |z - \underline{n}|$ and $t \in [0,\infty)$, h^*_n denote a function which can be bounded in the following way

$$h^*_n(z,t) \leq \begin{cases} Cq(z,t,n) & \text{if } t \geq nr \,, \\ C\frac{rn^{d-1}}{t^{1+d/2}} \mathrm{e}^{-r^2/(2t)} & \text{otherwise.} \end{cases} \tag{4.3.95}$$

By [Uch11, Lemma 4.5], we can choose h^*_n such that one has that for any

$a > 0, t > 0, \xi \in \partial \boldsymbol{B}_a$

$$\frac{P_z\left[B(H_a^B) \in \mathrm{d}\xi,\, H_a^B \in \mathrm{d}t\right]}{\mathcal{U}_{\partial \boldsymbol{B}_a}(\mathrm{d}\xi)\mathrm{d}t} = h_a^*(z \cdot \xi / a, t)\,, \tag{4.3.96}$$

where $\mathcal{U}_{\partial \boldsymbol{B}_a}$ is the uniform measure on $\partial \boldsymbol{B}_a$. We apply it for $a = n^-$ and $t = k - l$. In the following, set $y = \underline{n}$, due to the rotational invariance of the Brownian motion. We do a case distinction on whether $k-l$ is smaller or large than $\boldsymbol{\gamma} n^2$.

Case 1: we can apply Equation (4.3.95) to find for $r = |z \cdot \xi / n^- - \underline{n^-}|$ we have that in the case $t = k - l \leq \boldsymbol{\gamma} n^2$

$$\mathrm{d}P_{z,y}^{j-l}(H_{n^-}^B = k - l) \leq \frac{C}{n^{d-1}} \int_{\partial B_{n^-}} \frac{r n^{d-1}}{t^{1+d/2}} \mathrm{e}^{-r^2/(2t)} \mathfrak{p}_{j-k}(\xi, y) \mathrm{d}\xi\,. \tag{4.3.97}$$

We use that

$$r = n\sqrt{\gamma^2 + 1 - 2\gamma\cos(\theta) + 2\frac{c_\alpha \log(n)}{n}(\cos(\theta) - 1) + \frac{c_\alpha^2 \log^2(n)}{n^2}}\,, \tag{4.3.98}$$

where θ is not the angle between $\underline{n}$ and $z \cdot \xi$. Note that for $m \in \mathbb{Z}$ we have $\theta = \theta_\xi + \theta_z = \theta_\xi + \theta_z + 2m\pi$, where θ_ξ is the angle between ξ and $\underline{n}$ and θ_z is the angle between z and $\underline{n}$. We can apply the angular identities to bound the integrand

$$\begin{aligned}
&\frac{r n^{d-1}}{t^{1+d/2}} \mathrm{e}^{-r^2/(2t)} \mathfrak{p}_{j-k}(\xi, y) \\
&\leq C \frac{\sqrt{\gamma^2 + 1 - 2\gamma\cos(\theta_z + \theta_\xi)}\, n^d}{t^{1+d/2}(j-k)^{d/2}} \mathrm{e}^{-Cn^2\left[\left(\gamma^2 + 1 - 2\cos(\theta_z + \theta_\xi)\right)/t + \left(1 - \cos(\theta_\xi)\right)^2/(j-k)\right]} \\
&\leq \frac{C \boldsymbol{\gamma} n^d}{t^{1+d/2}(j-k)^{d/2}} \mathrm{e}^{-Cn^2\left[\boldsymbol{\gamma}^2/t + \theta_\xi^2/(j-k)\right]}\,.
\end{aligned} \tag{4.3.99}$$

Use a change of variables and approximate $|\sin(\theta)|^{d-2} \sim |\theta|^{d-2}$ to get

$$
\begin{aligned}
\mathrm{d}P_{z,y}^{j-l}(H_{n^-}^B = k-l) \leq C\frac{n\boldsymbol{\gamma}\mathrm{e}^{-Cn^2\boldsymbol{\gamma}^2/t}}{t^{1+d/2}(j-k)^{d/2}}\int_0^n \theta^{d-2}\mathrm{e}^{-C\theta^2/(j-k)}\mathrm{d}\theta \\
\leq C\frac{n\boldsymbol{\gamma}\gamma\left(\frac{d-1}{2},\frac{n^2}{(j-k)}\right)\mathrm{e}^{-Cn^2\boldsymbol{\gamma}^2/t}}{t^{1+d/2}(j-k)^{1/2}}\,.
\end{aligned}
\tag{4.3.100}
$$

Case 2: consider case $\boldsymbol{\gamma}n^2 \leq t = k-l \leq C(\boldsymbol{\gamma}n)^2$. In that case, we have that for the rescaled parameters $t' = t/n^2$ and $z' = z/n$ that

$$
\frac{z'}{t'} \in \left[\frac{1}{\gamma\max\{1,C\}},1\right]\,. \tag{4.3.101}
$$

Thus, we can apply [Uch11, Theorem 2.2] (which gives uniformity of hitting location) to conclude

$$
\mathrm{d}P_{z,y}^{j-l}(B_{k-l} = \xi | H_{n^-}^B = k-l) \leq \frac{C}{n^{d-1}}\mathfrak{p}_{j-k}(\xi,y)\mathrm{d}\xi\,. \tag{4.3.102}
$$

Similarly to before, this allows us to bound

$$
\begin{aligned}
\mathrm{d}P_{z,y}^{j-l}(H_{n^-}^B = k-l) \leq \frac{Cn^{-d+1}\gamma\left(\frac{d-1}{2},\frac{n^2}{(j-k)}\right)}{\sqrt{j-k}}q_n(z,k-l) \\
\leq \frac{C\boldsymbol{\gamma}n^{2-d}\gamma\left(\frac{d-1}{2},\frac{n^2}{(j-k)}\right)}{\gamma\sqrt{j-k}\,t^{3/2}}\frac{\mathrm{e}^{-(\gamma n)^2/(2t)}}{2\gamma^{(d-3)/2}}\,.
\end{aligned}
\tag{4.3.103}
$$

Using Equations (4.3.100) and (4.3.103), we can see that by choosing $\alpha > 0$ (the polynomial scale from of the error from the coupling) sufficiently large, we can absorb the error term into a fixed universal constant. Use the previous results to expand

$$
\begin{aligned}
&\sum_{k=j/2}^{j} \mathbb{P}^{j}_{y,y}\left(H_n \geq k,\ H_{\gamma n} < H_n\right) \\
&\qquad \leq C \sum_{k=j/2}^{j} \sum_{l=n_1}^{k-n_1} k \mathbb{P}_y \left(H_{\gamma n} = l > H_n\right) \mathrm{d}P^{j-l}_{z,y}(H_{n^-} = k-l) \qquad (4.3.104)\\
&\qquad \leq C \sum_{k=j/2}^{j} \sum_{l=n_1}^{k-n_1} k \frac{\beta(\boldsymbol{\gamma} n)^2 \mathrm{e}^{-\beta(\boldsymbol{\gamma} n)^2/l}}{(\boldsymbol{\gamma} n) l^2} \mathrm{d}P^{j-l}_{z,y}(H_{n^-} = k-l)\,.
\end{aligned}
$$

Let us firstly treat the case that $j \in [n_{\gamma}^{2-}, \boldsymbol{\gamma} n^2]$. In that case, we bound the sum

$$
\begin{aligned}
&\sum_{j=n_{\gamma}^{2-}}^{\boldsymbol{\gamma} n^2} \sum_{y \in \partial_i \boldsymbol{B}_n} j^{\nu} \mathbb{E}^{j}_{y,y}\left[H_{\gamma n} > H_n > j/2\right] \\
&\leq C \sum_{j=n_{\gamma}^{2-}}^{\boldsymbol{\gamma} n^2} j^{\nu} \sum_{k=j/2}^{j} \sum_{l=n_1}^{k-n_1} k \frac{n^{d-1}(\boldsymbol{\gamma} n)^2 \mathrm{e}^{-C(\boldsymbol{\gamma} n)^2/l}}{(\boldsymbol{\gamma} n) l^2} \frac{n \boldsymbol{\gamma} \gamma \left(\frac{d-1}{2}, \frac{n^2}{(j-k)}\right) \mathrm{e}^{-C(\boldsymbol{\gamma} n)^2/(j-l)}}{(j-l)^{1+d/2}(j-k)^{1/2}} \\
&\leq C \sum_{j=n_{\gamma}^{2-}}^{\boldsymbol{\gamma} n^2} j^{\nu} \sum_{k=j/2}^{j} \sum_{l=n_1}^{k-n_1} \frac{k n^{d+1} \boldsymbol{\gamma}^2 \mathrm{e}^{-C(\boldsymbol{\gamma} n)^2/l} \gamma \left(\frac{d-1}{2}, \frac{n^2}{(j-k)}\right) \mathrm{e}^{-C(\boldsymbol{\gamma} n)^2/(j-l)}}{l^2 (j-l)^{1+d/2}(j-k)^{1/2}} \\
&\leq C n^{2\nu+2} \boldsymbol{\gamma}^{-4} \int_0^{\gamma^{-1}} \mathrm{d}j \int_{1/2}^{1} \mathrm{d}k \int_0^{k} \mathrm{d}l \\
&\qquad\qquad \times \frac{j^{-\nu+d/2-3/2} k \gamma \left(\frac{d-1}{2}, \frac{j}{(1-k)}\right) \mathrm{e}^{-Cj\left(l^{-1}+(1-l)^{-1}\right)}}{l^2 (1-l)^{1+d/2}(l-k)^{1/2}}\,,
\end{aligned}
\qquad (4.3.105)
$$

where we can bound the integral by a universal constant, not depending on γ. Now assume that $j \in [n^2\boldsymbol{\gamma}, (n\boldsymbol{\gamma})^2]$ (and thus implicitly that $\boldsymbol{\gamma} \geq 1$). As we have to different bounds for $\mathrm{d}P^{j-l}_{z,y}$, we bound that term by the sum of the two (previously) proven bounds. For the ease of reading, we suppress this in the next two equations and simply treat each term separately.

Let us firstly employ the bound from Equation (4.3.103), using that the lower incomplete Gamma function can be bounded by a finite constant in that case

$$\begin{aligned}
&\sum_{j=\gamma n^2}^{(\gamma n)^2} \sum_{y\in\partial_i \boldsymbol{B}_n} j^\nu \mathbb{E}^j_{y,y}\left[H_{\gamma n} > H_n > j - n^2/2\right] \\
&\quad\le C \int_{\gamma n^2}^{(\gamma n)^2} \mathrm{d}j \int_{j/2}^{j} \mathrm{d}k \int_0^k \mathrm{d}l \frac{k j^\nu n^{d-1+2-d+1} \mathrm{e}^{-Cn^2(l^{-1}+(j-l)^{-1})}}{l^2 (j-k)^{1/2}(j-l)^{3/2} \boldsymbol{\gamma}^{(d-5)/2}} \\
&\quad\le C \frac{n^{2+2\nu}}{\boldsymbol{\gamma}^{(d-4\nu-5)/2}}\,,
\end{aligned} \tag{4.3.106}$$

which can be seen after the following changes of variables: $l \mapsto kl$, then $k \mapsto jk$ and finally $j \mapsto n^2 j$. If we use the bound from Equation (4.3.100), we get that by the same change of variables

$$\begin{aligned}
&\sum_{j=\gamma n^2}^{(\gamma n)^2} \sum_{y\in\partial_i \boldsymbol{B}_n} j^\nu \mathbb{E}^j_{y,y}\left[H_{\gamma n} > H_n > j - n^2/2\right] \\
&\quad\le C \int_{\gamma n^2}^{(\gamma n)^2} \mathrm{d}j \int_{j/2}^{j} \mathrm{d}k \int_0^k \mathrm{d}l \frac{k j^\nu n^{d-1+2} \mathrm{e}^{-Cn^2(l^{-1}+(j-l)^{-1})}}{l^2 (j-k)^{1/2}(j-l)^{1+d/2} \boldsymbol{\gamma}^{(d-5)/2}} \\
&\quad\le C n^{2+2\nu} \boldsymbol{\gamma}^{3-d+2\nu}\,.
\end{aligned} \tag{4.3.107}$$

Collecting the bounds from Step 3, Step 4a and Step 4b finishes the proof. □

A bound like the one we proved in the theorem above is essential to renormalisation arguments (see [CS16, DCRT18]). We apply it in Chapter 6 to make statements about the connected component of the loop soup.

Chapter 5

Properties of the occupation field

In this chapter we characterise important features of the occupation field. We begin by giving the scaling limit of the two-point function and then generalise the result to correlation functions of arbitrary order. We also give scaling limits for the moments of the occupation field. We then analyse the divergence in two dimensions before giving scaling results on the probability of observing large vacant sets.

Similar to the previous chapter, we restrict ourselves to the discrete-time loop measure $M^a = \sum_x \sum_j a_j \mathbb{P}^j_{x,x}$ and $q(x, \dagger) = 0$. Continuous-time results follow analogously.

Throughout the whole chapter, we observe the advantages of the method (rewriting events in terms of the range) developed in the previous chapter: not only do our proofs work for the Bosonic and the Markovian loop measure alike but we also obtain quantitative estimates and precise scaling limits for many different expressions.

5.1 The two-point function

In the next proposition, we prove sharp asymptotics for the two-point correlation function ψ_2, where

$$\psi_2(x, y) = \mathbb{P}_\lambda (x \in \mathcal{U}, y \in \mathcal{U}) - \mathbb{P}_\lambda (x \in \mathcal{U}) \mathbb{P}_\lambda (y \in \mathcal{U}) \ . \tag{5.1.1}$$

Recall that $\mathbb{P}^a_\lambda$ is the PPP with intensity measure λM^a. As M^a is fixed in the entire chapter, we omit the superscript ^{a} from the notation.

Proposition 5.1.1. *We have that the two-point function $\psi_2(x,y)$ is given by*

$$\begin{aligned}\psi_2(x,y) &= \mathbb{P}_\lambda\left(x\in\mathcal{U}, y\in\mathcal{U}\right) - \mathbb{P}_\lambda\left(x\in\mathcal{U}\right)\mathbb{P}_\lambda\left(y\in\mathcal{U}\right)\\ &= \mathrm{e}^{-2\lambda\sum_{j\geq 1} a_j \mathbb{E}^j_{0,0}[\mathcal{R}_j]}\left[\exp\left(\lambda\sum_{j\geq 1} a_j\mathbb{E}^j_{0,0}[\mathcal{R}_n, H_x<j]\right)-1\right]\,.\end{aligned} \tag{5.1.2}$$

For $d\geq 3$, $\lambda>0$, $\nu<d-3$ and $a_j = j^\nu(1+o(1))$, we have that as $|x-y|\to\infty$

$$\psi_2(x,y) = \lambda\kappa_d^2 K_{d,\nu}|y-x|^{6+2\nu-2d}\mathrm{e}^{-2\lambda K^o_{d,\nu}}\left(1+o(1)\right)\,, \tag{5.1.3}$$

with

$$K_{d,\nu} = \int_0^\infty j^{2-d+\nu}\int_0^1 \mathfrak{p}_k(j^{-1/2})\mathfrak{p}_{1-k}(j^{-1/2})\mathrm{d}k\,\mathrm{d}j\,, \tag{5.1.4}$$

and

$$K^o_{d,\nu} = \sum_{j\geq 1} a_j\mathbb{E}^j_{0,0}[\mathcal{R}_j]\,. \tag{5.1.5}$$

If $d=2$ and $a_j = j^\nu\log(j)(1+o(1))$ we have that for $\nu<-1$

$$\begin{aligned}\psi_2(x,y) &= \pi^2\mathrm{e}^{-2\lambda K^o_{2,\nu}}\lambda|x-y|^{2\nu+2}\log(|x-y|)\\ &\quad\times\int_0^\infty j^\nu\int_0^1 \frac{\mathfrak{p}_k\left(1/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(1/\sqrt{j}\right)}{\log^2(kj|x-y|^2)}\mathrm{d}k\,\mathrm{d}j\left(1+o(1)\right)\,.\end{aligned} \tag{5.1.6}$$

An estimation of the integral reveals that the leading-order of the above expression is given by $|x-y|^{2\nu+2}\log^{-1}(|x-y|)$.

Remark 5.1.2. *I. Higher-order correlations are given in Proposition 5.2.1.*

II. The main difficulty in the proof consists in controlling $\mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x<j\right]$ uniformly over lengths j. With the analysis provided in the proof below, we can get an asymptotic expression for a wide range of different sequences $(a_j)_j$. We summarise this in a separate statement, see Proposition 5.2.3.

III. It is interesting to note that for $d\geq 3$ the expression in Equation (5.1.2) *gives the correct asymptotical behaviour (up to a multiplicative constant) by employing the crude bounds $\mathcal{R}_n \leq n$ and $\mathbb{P}_0(H_x = k) \leq p_k(x)$.*

IV. *In [LJ11], closed form expressions for the correlation functions are also given. From those, one can compute the asymptotics more directly (also compare [CS16, Lemma 2.5]). However, our approach also works for loop measures where no closed from expressions exists, and thus we follow our approach.*

Proof of Proposition 5.1.1. Note that due to translation invariance

$$\psi_2(x, y) = \mathbb{P}_\lambda\left(x \notin \mathcal{U}, y \notin \mathcal{U}\right) - \mathbb{P}_\lambda\left(x \notin \mathcal{U}\right)^2 . \tag{5.1.7}$$

Let us assume without loss of generality that $y = 0$. We have that by the properties of the PPP that

$$\mathbb{P}_\lambda\left(x \notin \mathcal{U}, 0 \notin \mathcal{U}\right) = \exp\left(-\lambda M^a\left[\{0, x\} \cap \omega \neq \varnothing\right]\right) . \tag{5.1.8}$$

We can rewrite

$$\begin{aligned} M^a\left[\{0, x\} \subset \omega\right] &= \sum_{y \in \mathbb{Z}^d} \sum_{j \geq 1} a_j \mathbb{P}^j_{y,y}\left(H_x < j,\, H_0 < j\right) \\ &= \sum_{y \in \mathbb{Z}^d} \sum_{j \geq 1} a_j \mathbb{P}^j_{0,0}\left(H_x < j,\, H_y < j\right) \\ &= \sum_{j \geq 1} a_j \mathbb{E}^j_{0,0}\left[\mathbb{1}\{H_x < j\} \sum_{y \in \mathbb{Z}^d} H_y < j\right] \\ &= \sum_{j \geq 1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x < j\right] . \end{aligned} \tag{5.1.9}$$

Similarly, one finds

$$M^a\left[0 \in \omega\right] = \sum_{j \geq 1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j\right] = K^o_{d,\nu} . \tag{5.1.10}$$

The proof of the first statement (i.e. Equation (5.1.2)) of Proposition 5.1.1 now follows from applying the inclusion-exclusion principle to Equation (5.1.8): $M^a\left[\{0, x\} \cap \omega \neq \varnothing\right] = 2M^a\left[0 \in \omega\right] - M^a\left[\{0, x\} \subset \omega\right]$.

We now prove the asymptotic expression of the two-point function, beginning with the case $d \geq 3$. To accomplish this, we need to analyse

$$\sum_{j\geq 1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x < j\right] . \tag{5.1.11}$$

Note that, similarly to the proof of Proposition 4.2.2, the sum over $j \leq |x|^{5/3}$ is negligible. Our strategy is as follows: we first show that we can neglect loops of short length. For loops of typical length, we then use the precise asymptotic results on the first hitting time of single points from Lemma 2.2.2.
Fix $\varepsilon > 0$. For $j \geq |x|^{5/3}$ one has $\mathcal{R}_j \leq (1+\varepsilon)\kappa_d j$ outside a set of negligible probability, similar to the proof of Proposition 4.2.2. Thus, it holds

$$\sum_{j\geq |x|^{5/3}} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x < j\right] \leq \kappa_d(1+\varepsilon) \sum_{j\geq |x|^{5/3}} a_j j \mathbb{P}^j_{0,0}\left(H_x < j\right) . \tag{5.1.12}$$

Fix $M > 1$. Lemma 2.2.2 gives us the bound

$$\mathbb{B}^j_{0,0}\left(H_x < j\right) \leq C j^{d/2} \mathbb{P}^j_{0,0}\left(H_x < j\right) \leq C|x|^{2-d}\Gamma(d/2-1, |x|^2 j^{-1}) . \tag{5.1.13}$$

Thus,

$$\begin{aligned}
&\sum_{j=|x|^{5/3}}^{|x|^2/M} a_j j \mathbb{P}^j_{0,0}\left(H_x < j\right) \leq \frac{C}{|x|^{d-2}} \sum_{j=|x|^{5/3}}^{|x|^2/M} j^{1-d/2+\nu}\Gamma\left(\frac{d-2}{2}, \frac{|x|^2}{j}\right) \\
&\leq \frac{C}{|x|^2} \sum_{j=|x|^{5/3}}^{|x|^2/M} j^{3-d+\nu} \mathrm{e}^{-|x|^2/j} \leq \frac{C}{|x|^2} \int_0^{|x|^2/M} j^{3-d+\nu} \mathrm{e}^{|x|^2/j} \mathrm{d}j \\
&\leq C|x|^{6+2\nu-2d}\Gamma(d+\nu, M) ,
\end{aligned} \tag{5.1.14}$$

using Lemma 8.3.3 for the asymptotics of the incomplete Gamma function. By Lemma 2.2.2, we have that for $j \geq |x|^2/M$

$$\begin{aligned}
\mathbb{P}^j_{0,0}\left(H_x < j\right) =& \kappa_d j^{-d+1}\left(1+o(1)\right) \int_{1/M}^1 \mathfrak{p}_k\left(x/\sqrt{j}\right) \mathfrak{p}_{1-k}\left(x/\sqrt{j}\right) \mathrm{d}k \\
&+ \mathcal{O}\left(j^{-d+1} \int_0^{1/M} \mathfrak{p}_k\left(x/\sqrt{j}\right) \mathfrak{p}_{1-k}\left(x/\sqrt{j}\right) \mathrm{d}k\right) .
\end{aligned} \tag{5.1.15}$$

Using similar approximation arguments to those employed in the proof of Proposition 4.2.2 and the change of variables $j \mapsto j|x|^2$, we write

$$\begin{aligned}
&\sum_{j \geq |x|^2/M} j^{2-d+\nu} \int_{1/M}^{1} \mathfrak{p}_k(xj^{-1/2})\mathfrak{p}_{1-k}(xj^{-1/2})\mathrm{d}k \\
&\qquad = \int_{|x|^2/M}^{\infty} j^{2-d+\nu} \int_{1/M}^{1} \mathfrak{p}_k(xj^{-1/2})\mathfrak{p}_{1-k}(xj^{-1/2})\mathrm{d}k\mathrm{d}j\,(1+o(1)) \\
&\qquad = |x|^{6+2\nu-2d} \int_{1/M}^{\infty} j^{2-d+\nu} \int_{1/M}^{1} \mathfrak{p}_k(j^{-1/2})\mathfrak{p}_{1-k}(j^{-1/2})\mathrm{d}k\,\mathrm{d}j\,(1+o(1))\ .
\end{aligned} \tag{5.1.16}$$

To summarise, we have shown that

$$\begin{aligned}
&\sum_{j \geq |x|^{5/3}} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x < j\right] \\
&\qquad \leq (1+\varepsilon)\kappa_d^2 |x|^{6+2\nu-2d} \int_{1/M}^{\infty} j^{2-d+\nu} \int_{1/M}^{1} \mathfrak{p}_k(j^{-1/2})\mathfrak{p}_{1-k}(j^{-1/2})\mathrm{d}k\,\mathrm{d}j \\
&\qquad\quad + \mathcal{O}\left(|x|^{6+2\nu-2d}\Gamma(d+\nu, M)\right)\ .
\end{aligned} \tag{5.1.17}$$

Note that we can get the analogous lower bound, replacing $(1+\varepsilon)$ by $(1-\varepsilon)$. Taking $M \to \infty$ and $\varepsilon \downarrow 0$ finishes the proof for $d \geq 3$.

We now treat the case $d = 2$. The first part is analogous to the proof of Proposition 4.2.2: fix $\delta > 0$ and

$$x_1 = \frac{|x|^2}{\log\left(\log^{\delta}(|x|)\right)}\ . \tag{5.1.18}$$

We then expand

$$\sum_{j \geq 1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x < j\right] = \mathcal{O}\left(\frac{|x|^{2\nu+2}}{\log^{c\delta/2}(|x|)}\right) + \sum_{j \geq x_1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x < j\right]\ , \tag{5.1.19}$$

for some $c > 0$. Choose $\delta > 0$ such that $\delta c/2 > 1$. Fix $\varepsilon > 0$ and partition the remaining sum

$$\sum_{j\geq x_1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_j, H_x < j\right] \leq (1+\varepsilon)\sum_{j\geq x_1} a_j r_j \mathbb{P}^j_{0,0}\left(H_x < j\right) + \sum_{j\geq x_1} j^\nu p_j(0)\log(j)\mathcal{O}\left(\mathbb{B}^j_{0,0}\left[\mathcal{R}_j,\, |\mathcal{R}_j - r_j| \geq \varepsilon r_j\right]\right)\,. \tag{5.1.20}$$

Using Lemma 4.1.1 to bound $\mathbb{B}^j_{0,0}\left[\mathcal{R}_j,\, |\mathcal{R}_j - r_j| \geq \varepsilon r_j\right]$ by $\mathcal{O}\left(j\frac{\log^3(\log(j))}{\log^2(j)}\right)$, we have

$$\sum_{j\geq x_1} j^{\nu-1}\log(j)\mathcal{O}\left(\mathbb{B}^j_{0,0}\left[\mathcal{R}_j,\, |\mathcal{R}_j - r_j| \geq \varepsilon r_j\right]\right) \leq C\sum_{j\geq x_1} j^\nu \frac{\log^3(\log(j))}{\log^2(j)} \leq C\frac{\log^3(\log(|x|))}{\log^2(|x|)}\frac{|x|^{2\nu+2}}{\log^{\nu+1}\left(\log^\delta(|x|)\right)} = o\left(\frac{|x|^{2\nu+2}}{\log^{3/2}(|x|)}\right)\,. \tag{5.1.21}$$

We now analyse

$$\sum_{j\geq x_1} a_j r_j \mathbb{P}^j_{0,0}\left(H_x < j\right)\,. \tag{5.1.22}$$

We get two contributions to $\mathbb{P}^j_{0,0}\left(H_x < j\right)$ from Lemma 2.2.2, a leading-order term and an error term. Let us begin with the error term:

$$\mathcal{O}\left(j^{-1}\int_0^{|x|^2/j\log(|x|^{2-\rho})} \mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)\mathrm{d}k\right)\,. \tag{5.1.23}$$

Recall that $a_j \sim j^\nu \log(j)$ and that $r_j \sim j/\log(j)$. We estimate

$$\begin{aligned}
&\sum_{j\geq x_1} j^{\nu+1}\mathcal{O}\left(j^{-1}\int_0^{|x|^2/j\log(|x|^{2-\rho})} \mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)\mathrm{d}k\right)\\
&\leq C\int_{x_1}^\infty j^\nu \int_0^{|x|^2/j\log(|x|^{2-\rho})} \mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)\mathrm{d}k\mathrm{d}j\\
&\leq C|x|^{2+2\nu}\int_{x_1/|x|^2}^\infty j^\nu \int_0^{1/j\log(|x|^{2-\rho})} \mathfrak{p}_k\left(1/\sqrt{j}\right)\mathrm{d}k\mathrm{d}j\\
&\leq C|x|^{2+2\nu}\int_{x_1/|x|^2}^\infty j^\nu \Gamma(0,\log(|x|^{2-\rho})\mathrm{d}j \leq C|x|^{\rho+2\nu}\log^{-(\nu+1)}(\log^\varepsilon(|x|))\,.
\end{aligned} \tag{5.1.24}$$

We recall that $\rho > 0$ can be chosen arbitrarily small and we have used the

asymptotics of the incomplete Gamma function from Lemma 8.3.3.
Having estimated the error term, we turn our attention to the main contribution to $\mathbb{P}^j_{0,0}(H_x < j)$, calculated in Lemma 2.2.2:

$$\frac{4\pi \log|x|}{j}\int_{|x|^2/j\log(|x|^{2-\rho})}^{1}\frac{\mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)}{\log^2(kj)}\mathrm{d}k\,. \tag{5.1.25}$$

Estimate

$$\begin{aligned}&\sum_{j\geq x_1} j^{\nu}\log(|x|)\int_{|x|^2/j\log(|x|^{2-\rho})}^{1}\frac{\mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)}{\log^2(kj)}\mathrm{d}k\\ =&\int_{x_1}^{\infty} j^{\nu}\log|x|\int_{|x|^2/j\log(|x|^{2-\rho})}^{1}\frac{\mathfrak{p}_k\left(x/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(x/\sqrt{j}\right)}{\log^2(kj)}\mathrm{d}k(1+o(1))\\ =&|x|^{2+2\nu}\log|x|\int_{x_1/|x|^2}^{\infty} j^{\nu}\int_{1/j\log(|x|^{2-\rho})}^{1}\frac{\mathfrak{p}_k\left(1/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(1/\sqrt{j}\right)}{\log^2(kj|x|^2)}\mathrm{d}k(1+o(1))\\ =&|x|^{2+2\nu}\log|x|\int_{0}^{\infty} j^{\nu}\int_{0}^{1}\frac{\mathfrak{p}_k\left(1/\sqrt{j}\right)\mathfrak{p}_{1-k}\left(1/\sqrt{j}\right)}{\log^2(kj|x|^2)}\mathrm{d}k(1+o(1))\,,\end{aligned} \tag{5.1.26}$$

where similar arguments as in the proof of Proposition 4.2.2 imply that the sum over j can be approximated by an integral. A quick computation reveals that for $\rho > 0$ small enough, the estimate obtained in Equation (5.1.24) is $o(1)$ of the above. By letting $\varepsilon > 0$ tend to zero, we show the right upper bound. The lower bound is then established analogously. This finishes the proof. □

We give the analogue for the Bosonic loop measure.

Corollary 5.1.3. *If $\mu < 0$ and $\beta > 0$ we have that for the Bosonic Loop soup*

$$\mathbb{P}^B_{\lambda}\left(x\in\mathcal{U},\, y\in\mathcal{U}\right) = \psi^B_2(x,y) \leq \mathcal{O}\left(\mathrm{e}^{-f(\mu)|x-y|}\right)\,, \tag{5.1.27}$$

where $f(\mu)$ increases as $\mu \downarrow -\infty$. This implies that correlations decay exponentially fast.
If $d \geq 3$ and $\mu = 0$ we have that for fixed $\beta > 0$

$$\psi^B_2(x,y) = \lambda\kappa_d^2 K_{d,-1}|x-y|^{4-2d}\mathrm{e}^{-2\lambda K^o_{d,\beta}}(1+o(1))\,, \tag{5.1.28}$$

where $K^o_{d,\beta} = \sum_{j\geq 1} j^{-1}\mathbb{E}^{\beta j}_{0,0}\left[\mathcal{R}_{\beta j}\right]$ and the error term depends on β.

Note that as $\beta \uparrow \infty$, we have that

$$K^o_{d,\beta} = (1+o(1))\,\kappa_d \beta \sum_{j=1}^{\infty} \frac{(2\pi)^{-d/2}}{(\beta j)^{d/2}} = \frac{\kappa_d}{4\pi^2}(2\beta\pi)^{1-d/2}\,(1+o(1))\;. \tag{5.1.29}$$

Transitions from an exponential decay to a power law decay are often referred to as BKT transitions, see [KT73, FS81].

5.2 Higher-order correlations

In order to deal with higher-order correlations of the loop soup, we need additional notation. We abbreviate $\{1,\dots,n\}$ by $[n]$ in this section. Let $\mathfrak{P}_n$ be the set of all partitions of $[n]$ into non-empty subsets. For $I \in \mathfrak{P}_n$ a partition, we set $|I|$ to be the number of (disjoint) blocks $(I_i)_{i=1}^{|I|}$ in I.
The higher-order analogue of the covariance $\psi_2(x,y)$ is the cumulant. Let $n \geq 2$ and $x_1,\dots,x_n \in \mathbb{Z}^d$ and define the cumulant ψ_n

$$\psi_n(x_1,\dots,x_n) = \sum_{I\in\mathfrak{P}_n} (-1)^{|I|-1}(|I|-1)! \prod_{I_i\in I} \mathbb{P}_\lambda\left(x_j \in \mathcal{U},\, \forall j \in I_i\right)\;. \tag{5.2.1}$$

For $I \subset [n]$, abbreviate

$$M[I] = M\left[\{x_j \colon j \in I\} \subset \omega\right]\;. \tag{5.2.2}$$

For $J \subset [n]$ shorten the inclusion-exclusion formula without singletons

$$A_1(J) = \sum_{i\geq 2}(-1)^{i-1} \sum_{\substack{I\subset J\\ |I|=i}} M\left[I\right]\;. \tag{5.2.3}$$

Let $\mathfrak{S}_n$ be the set of permutations of n points. Define $\mathfrak{S}^1_n$ be the set of permutations which map 1 onto 1 and only have two cycles, i.e.

$$\mathfrak{S}^1_n = \{\sigma \in \mathfrak{S}_n \colon \sigma(1) = 1 \text{ and } \sigma \text{ has two cycles }\}\,. \tag{5.2.4}$$

Define for $a, b \in \mathbb{R}^d$ and $t, j > 0$, the rescaled kernel

$$\mathfrak{p}_t^j(a,b) = \mathfrak{p}_t\left(\frac{a}{\sqrt{j}}, \frac{b}{\sqrt{j}}\right) . \tag{5.2.5}$$

Let for $y_1, \ldots, y_n \in \mathbb{R}^d$ and $d \geq 3$

$$\varphi\left(y_1, \ldots, y_n\right) = \int_0^\infty j^{\nu n - dn/2} \int_{\Delta_{n-1}} \mathfrak{p}^j_{1-\sum_i t_i}(y_n, y_1) \prod_{i=1}^{n-1} \mathfrak{p}^j_{t_i}\left(y_i, y_{i+1}\right) \mathrm{d}(t_i)_i \, \mathrm{d}j \, , \tag{5.2.6}$$

where $\Delta_k = \{t \in [0,\infty)^k \colon \sum_{i=1}^k t_i \leq 1\}$ for $k \in \mathbb{N}$. For $d = 2$ and $N > 0$, set

$$\begin{aligned}&\varphi\left(y_1, \ldots, y_n\right)\\ &\quad= \int_0^\infty j^{\nu n - dn/2} \int_{\Delta_{n-1}} \mathfrak{p}^j_{1-\sum_i t_i}(y_n, y_1) \prod_{i=1}^{n-1} \frac{\mathfrak{p}^j_{t_i}\left(y_i, y_{i+1}\right)}{\log^2(t_i j N^2 |y_i - y_{i+1}|^2)} \mathrm{d}(t_i)_i \, \mathrm{d}j \, .\end{aligned} \tag{5.2.7}$$

For $y_1, \ldots, y_n \in \mathbb{R}^d$ and $d \geq 2$ let

$$\Upsilon\left(y_1, \ldots, y_n\right) = \sum_{\sigma \in \mathfrak{S}_n^1} \varphi\left(y_{\sigma(1)}, \ldots, y_{\sigma(n)}\right) . \tag{5.2.8}$$

We now state the main result of this section.

Proposition 5.2.1. *Let $n \geq 3$ and $x_1, \ldots, x_n \in \mathbb{Z}^d$. We then have that for $I = \{i_0, \ldots, i_k\}$*

$$M[I] = \sum_{j \geq 1} a_j \mathbb{E}^j_{x_{i_0}, x_{i_0}}\left[\mathcal{R}_j, H_{x_{i_1}} < j, \ldots, H_{x_{i_k}} < j\right] , \tag{5.2.9}$$

and

$$\psi_n(x_1, \ldots, x_n) = \mathrm{e}^{-\lambda n M[1]}\left(\sum_{I \in \mathfrak{P}_n} (-1)^{|I|-1}(|I|-1)! \mathrm{e}^{-\sum_{I_i \in I} A_1(I_i)}\right) . \tag{5.2.10}$$

Fix $C_1, C_2 > 0$. Consider distinct $y_1, \ldots, y_n \in \mathbb{R}^d$ satisfying

$$0 < C_1 < \sup_{i \neq j} |y_i - y_j| < C_2 \inf_{i \neq j} |y_i - y_j| < C_1^{-1} . \tag{5.2.11}$$

Pick $x_i \in \mathbb{Z}^d$ such that $x_i = N(y_i + o(1))$. Suppose that $a_j = j^{-\nu}(1+o(1))$ with $\nu \le -1$ and $d \ge 3$. We then have that (uniformly in y)

$$\psi_n(x_1, \dots, x_n) = A_{d,n} \Upsilon(y_1, \dots, y_n) N^{2+2\nu+n(2-d)}(1+o(1)) \,, \tag{5.2.12}$$

with

$$A_{d,n} = \lambda(-1)^n \kappa_d^n \mathrm{e}^{-\lambda n M[1]} \,. \tag{5.2.13}$$

If $d = 2$ and $a_j = j^\nu \log(j)(1+o(1))$ with $\nu < -1$, we then have that

$$\psi_n(x_1, \dots, x_n) = A_{2,n} \Upsilon(y_1, \dots, y_n) N^{2+2\nu} \log^n(N)(1+o(1)) \,, \tag{5.2.14}$$

with

$$A_{2,n} = \lambda(-1)^n \pi^n \mathrm{e}^{-\lambda n M[1]} \,. \tag{5.2.15}$$

Remark 5.2.2. *Similar to the case $n = 2$, we need to analyse*

$$\mathbb{E}^j_{x_1,x_1}[\mathcal{R}_j, H_{x_2} < j, \dots, H_{x_n} < j] \,. \tag{5.2.16}$$

However, for $n \ge 3$ a different reasoning must be used. Several combinatorial estimates are needed to prove a cancellation of lower order terms in the expansion of the exponential in Equation (5.2.10). Similar to previously, the analysis allows for a wide range of sequences $(a_j)_j$, which we state separately in Proposition 5.2.3.

Proof of Proposition 5.2.1. To show that

$$M[I] = \sum_{j \ge 1} a_j \mathbb{E}^j_{x_{i_0},x_{i_0}} \left[\mathcal{R}_j, H_{x_{i_1}} < j, \dots, H_{x_{i_j}} < j\right] \,, \tag{5.2.17}$$

one uses a similar reasoning to the case $|I| = 2$, i.e. Proposition 5.1.1.
The remaining proof is split into three steps. We first obtain an abstract representation of the cumulant ψ_n in terms of products of $M[I]$'s (for $I \subset [n]$) and then devise the precise asymptotics for $M[I]$. In the final step we use the assumption $\nu \le -1$ and combine the results from the two previous steps.

Step 1: combinatorial identities

We now prove Equation (5.2.10). Note that since the cumulant is invariant under adding constants, we have that

$$\psi_n(x_1,\ldots,x_n) = \sum_{I\in\mathfrak{P}_n}(-1)^{|I|-1}(|I|-1)!\prod_{i=1}^{|I|}\mathbb{P}_\lambda\left(x_j\notin\mathcal{U},\,\forall j\in I_i\right)\,, \quad (5.2.18)$$

where we work with an arbitrary ordering on the I_i's in $\mathfrak{P}_n$. Note that by the fundamental properties of the PPP

$$\mathbb{P}_\lambda\left(x_j\notin\mathcal{U},\,\forall j\in I_i\right) = \exp\left(-\lambda M\left[\exists j\in I_i\colon x_j\in\omega\right]\right)\,. \quad (5.2.19)$$

By the inclusion-exclusion formula and the translation invariance of M, we have that

$$M\left[\exists j\in I_i\colon x_j\in\omega\right] = \sum_{r\geq 1}(-1)^{r-1}\sum_{\substack{I\subset I_i\\|I|=r}}M[I_r]\,. \quad (5.2.20)$$

and thus

$$\mathbb{P}_\lambda\left(x_j\notin\mathcal{U},\,\forall j\in I_i\right) = \exp\left(-\lambda|I_i|M[1]-\lambda A_1(I_i)\right)\,, \quad (5.2.21)$$

where we recall that $A_1(I_i)$ is the inclusion-exclusion formula without the singletons, defined in Equation (5.2.3).
Inserting Equation (5.2.21) into Equation (5.2.18) gives the abstract representation stated in Equation (5.2.10).
Now expand the exponential and use the alternating combinatorial factor of $(-1)^{|I|-1}(|I|-1)!$ to cancel a large proportion of the $M[J]$'s, for $J\subset[n]$. We begin by setting up some new notation. For an illustration see Figure 5.1.
For $(J_i)_{i=1}^k$ subsets of $[n]$, we abbreviate

$$M[J,k] = \left(\prod_{i=1}^{k}(-1)^{|J|-1}\lambda M[J_i]\right)\,. \quad (5.2.22)$$

Given a collection $(J_i)_{i=1}^k$, we introduce $(\mathfrak{J}_j)_{j=1}^T$ where $\mathfrak{J}_i$ are the connected (in this context we say J_i connected to J_r if $J_i\cap J_r\neq\varnothing$) components of $\cup_i J_i$ and T is the number of disjoint connected components. Let $\Theta\colon\{1,\ldots,n\}\to\{1,\ldots,T\}$ with $\Theta(i)=j$ whenever $J_i\subset\mathfrak{J}_j$. Given a partition $I=\{I_1,\ldots,I_r\}$ and $J\subset[n]$, we write $J\prec I$ if there exists an $i\in[r]$ such that $J\subset I_i$, i.e. J is fully contained in one of the blocks of the partition. Expanding the

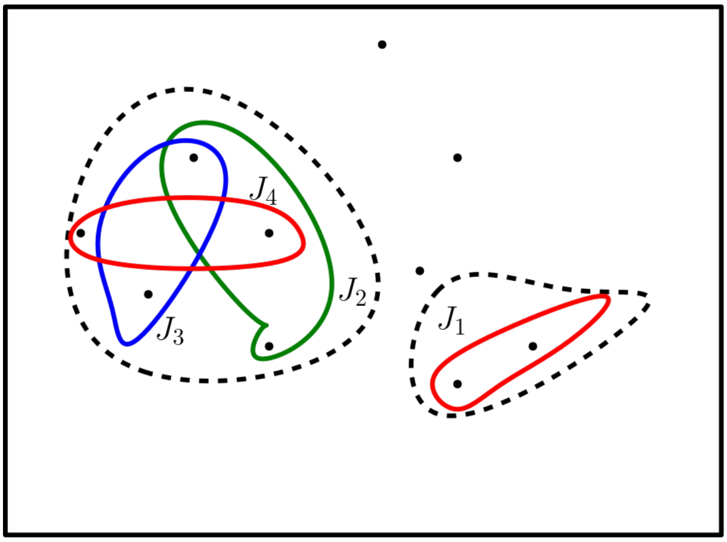

Figure 5.1: The coloured lines represent subsets J_i, the dashed lines are the boundaries of the $\mathfrak{J}_{\Theta(i)}$'s. Here $k = 4$, $T = 2$ and $m = 3$, as there are 4 J_i's, 2 $\mathfrak{J}_{\Theta(i)}$'s and 3 points are not contained in any of the J_i's.

exponential, we get that

$$\begin{aligned}
&\sum_{I\in\mathfrak{P}_n}(-1)^{|I|-1}(|I|-1)!\mathrm{e}^{-\lambda\sum_{I_i\in I}A_1(I_i)}\\
&=\sum_{I\in\mathfrak{P}_n}(-1)^{|I|-1}(|I|-1)!\sum_{k=0}^{\infty}\frac{1}{k!}\left(-\lambda\sum_{I_i\in I}A_1(I_i)\right)^k\\
&=\sum_{I\in\mathfrak{P}_n}\sum_{k=0}^{\infty}\frac{(-1)^{k+|I|-1}(|I|-1)!}{k!}\left(\sum_{i=1}^{|I|}\sum_{J\subset[n]}(-1)^{|J|-1}\lambda M[J]\mathbb{1}\{J\subset I_i\}\right)^k\\
&=\sum_{I\in\mathfrak{P}_n}\sum_{k=0}^{\infty}\frac{(-1)^{k+|I|-1}(|I|-1)!}{k!}\left(\sum_{J\subset[n]}(-1)^{|J|-1}\lambda M[J]\mathbb{1}\{J\prec I\}\right)^k\\
&=\sum_{I\in\mathfrak{P}_n}\sum_{k=0}^{\infty}\frac{(-1)^{k+|I|-1}(|I|-1)!}{k!}\sum_{J_1,\dots,J_k\subset[n]}M[J,k]\mathbb{1}\{J_i\prec I,\,\forall i\in[k]\}\,,
\end{aligned}\tag{5.2.23}$$

since $J \prec I$ being true is equivalent to the sum of the indicator functions $\mathbb{1}\{J \subset I_i\}$ being one. All sets J_i are understood to have at least two elements.

Let m be the number of points in $[n]$ not contained in any of the J_i's, i.e.

$$m = |[n] \setminus \cup_j \mathfrak{J}_j| \,. \tag{5.2.24}$$

Recall that $S(n,k)$ are the Stirling numbers of second kind, i.e. the number of ways to partition $[n]$ into k non-empty subsets. Fix $k \in \mathbb{N}$ and expand

$$\begin{aligned}
&\sum_{J_1,\ldots,J_k \subset [n]} M[J,k] \sum_{r=1}^{m+T} (-1)^{r-1}(r-1)! \sum_{\substack{I \in \mathfrak{P}_n \\ |I|=r}} \mathbb{1}\{J_i \prec I, \, \forall i \in [k]\} \\
&= \sum_{J_1,\ldots,J_k \subset [n]} M[J,k] \sum_{r=1}^{m+T} (-1)^{r-1}(r-1)! \sum_{\substack{I \in \mathfrak{P}_n \\ |I|=r}} \mathbb{1}\{\mathfrak{J}_{\Theta(i)} \prec I, \, \forall i \in [k]\} \qquad (5.2.25) \\
&= \sum_{J_1,\ldots,J_k \subset [n]} M[J,k] \sum_{r=1}^{m+T} (-1)^{r-1}(r-1)! S(m+T, r) \,.
\end{aligned}$$

Indeed, "collapse " each $\{\mathfrak{J}_{\Theta(i)}\}_{i=1}^{T}$ onto a single point. This gives a total of $m+T$ points. Since we partition those into r subsets, this gives $S(m+T,r)$, compare also Figure 5.1.
Note that by [AS65, p.825]

$$\sum_{r=1}^{m+T} (-1)^{r-1}(r-1)! S(m+T, r) = \delta_1(T+m) \,. \tag{5.2.26}$$

Write $\mathfrak{P}_c^k(n)$ for all the subsets $J_1, \ldots, J_k$ such that $m+T=1$. By the above cancellation, we can expand

$$\psi_n(x_1, \ldots, x_n) = \mathrm{e}^{-n\lambda M[1]} \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \sum_{J_1,\ldots,J_k \in \mathfrak{P}_c^k(n)} M[J,k] \,. \tag{5.2.27}$$

This concludes the expansion.
Step 2: analysis of M[I]
We can assume without loss of generality that $I = [n]$. We restrict ourselves to the case $d \geq 3$, as the case $d = 2$ follows by using reasoning from the cases $d = 2, n = 2$ and $d \geq 3, n \geq 3$.
We use the following approach: suppose that the random loop (started at x_1)

hits the points $x_2, \ldots, x_n$. Then partition that event by specifying the order in which the points are hit. This leads to a sum over $\mathfrak{S}_n^1$. We then show that the probability of going from a point x_i to a point x_j without hitting any other x_l (for $i \neq j \neq l$) is dominated by going from x_i to x_j (due to restrictions placed on $(y_i)_i$).

We begin by excluding a certain class of loop lengths, similar to previous proofs. Set $H_i = H_{x_i}$ and

$$N_1 = \frac{N^2}{C \log N}\,, \tag{5.2.28}$$

for some $C > 0$ sufficiently large such that

$$\sum_{j=1}^{N_1} a_j j \mathbb{P}^j_{x_1,x_1}\left(\bigcup_{i=2}^{n}\{H_i < j\}\right) = o\left(N^{2+2\nu+n(2-d)}\right)\,. \tag{5.2.29}$$

The existence of such a $C > 0$ follows from the same reasoning which is used in the proof of Proposition 4.2.2 or Theorem 4.3.1.

Let us assume that $j \geq N_1$. Similar to previous proofs, we approximate

$$\mathbb{E}^j_{x_1,x_1}\left[\mathcal{R}_j, H_2 < j, \ldots, H_n < j\right] = r_j \mathbb{P}^j_{x_1,x_1}\left(\bigcup_{i=2}^{n}\{H_i < j\}\right)(1 + o(1))\,, \tag{5.2.30}$$

where we recall that r_j is the expected range of the random walk bridge of length j. We then need to estimate

$$\sum_{j \geq N_1} a_j r_j \mathbb{P}^j_{x_1,x_1}\left(\bigcup_{i=2}^{n}\{H_i < j\}\right)\,. \tag{5.2.31}$$

We do a case distinction by summing over all the different orders in which the points $(x_i)_i$ can be hit: expand the different permutations of $x_2, \ldots, x_n$

$$\mathbb{P}^j_{x_1,x_1}\left(\bigcup_{i=2}^{n}\{H_i < j\}\right) = \sum_{\sigma \in \mathfrak{S}_n^1} \mathbb{P}^j_{x_1,x_1}\left(H_{\sigma_i} < H_{\sigma_{i+1}}, \forall i = 2, \ldots, n-1\right)\,. \tag{5.2.32}$$

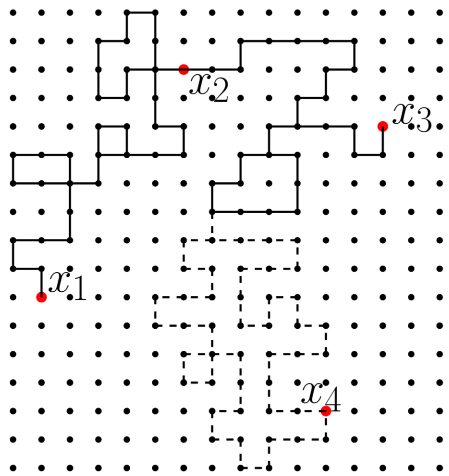

Figure 5.2: The solid line is a random walk starting from x_1, then hitting x_2 and then x_3. The dashed line represents a realisation of the event E_2: the point x_4 has been hit before x_3.

We use the (strong) Markov property and recursively decouple

$$\begin{aligned}
&\mathbb{P}^j_{x_1,x_1}\left(H_{\sigma_i} < H_{\sigma_{i+1}},\, \forall i = 2,\ldots,n-1\right) \\
&= \sum_{\substack{0\le t_1,\ldots,t_{n-1}\le j \\ t_1+\ldots+t_{n-1}\le j}} \mathbb{P}^j_{x_1,x_1}\left(t_{i-1} = H_{\sigma_i} < H_{\sigma_{i+1}} = t_i,\, \forall i = 2,\ldots,n-1\right) \\
&= \sum_{\substack{0\le t_1,\ldots,t_{n-1}\le j \\ t_1+\ldots+t_{n-1}\le j}} p_{j-\sum_i t_i}(x_{\sigma_n}, x_1) \prod_{i=1}^{n-1} \mathbb{P}_{x_{\sigma(i)}}\left(H_{\sigma(i+1)} = t_i > H_{\sigma(j)},\, \forall j > i+1\right).
\end{aligned} \tag{5.2.33}$$

Let E_i be the event that, for $r > i+1$, we hit a point $x_{\sigma(r)}$ before hitting $x_{\sigma(i+1)}$, i.e.

$$E_i = \left\{H_{\sigma(i+1)} = t_i \text{ and } \exists\, r \in \{i+2,\ldots,n\}\, \exists t \in \{0,\ldots,t_i\} \colon H_{\sigma(r)} = t\right\}. \tag{5.2.34}$$

For an illustration of the event E_i in the case $\sigma = \mathrm{Id}$, see Figure 5.2.

We can then subtract

$$\mathbb{P}_{x_{\sigma(i)}}\left(H_{\sigma(i+1)} = t_i > H_{\sigma(j)},\, \forall j > i+1\right) = \mathbb{P}_{x_{\sigma(i)}}\left(H_{\sigma(i+1)} = t_i\right) - \mathbb{P}_{x_{\sigma(i)}}(E_i)\,. \tag{5.2.35}$$

Thus, we get the expansion

$$\prod_{i=1}^{n-1}\mathbb{P}_{x_{\sigma(i)}}\left(H_{\sigma(i+1)}=t_i>H_{\sigma(j)},\,\forall j>i+1\right)=\prod_{i=1}^{n-1}\mathbb{P}_{x_{\sigma(i)}}\left(H_{\sigma(i+1)}=t_i\right)$$
$$-\sum_{\substack{I\subset\{1,\dots,n-1\}\\ I\neq\varnothing}}(-1)^{|I|}\left(\prod_{j\in I}\mathbb{P}_{x_{\sigma(j)}}(E_j)\right)\left(\prod_{j\notin I}\mathbb{P}_{x_{\sigma(j)}}(H_{\sigma(j+1)}=t_j)\right). \tag{5.2.36}$$

The intuition is as follows: the probability of E_i is of lower order than the one of $\{H_{\sigma(i+1)}=t_i\}$. This is because E_i requires hitting an additional one of the x_j's. Thus, only the first term on the right-hand side in Equation (5.2.36) remains in the limit. We make this rigorous below.
Let us evaluate $\mathbb{P}_{x_{\sigma(j)}}(E_j)$. Note that by the same reasoning applied above, we use $t_i\geq N_1$. This allows us to approximate the random walk transition density. We bound

$$\begin{aligned}\mathbb{P}_{x_{\sigma(j)}}(E_j)&\leq\max_{r\in\{i+2,\dots,n\}}\sum_{t=0}^{t_i}p_t(x_{\sigma(j)},x_{\sigma(r)})p_{t_j-t}(x_{\sigma(j)},x_{\sigma(j+1)})\\ &\leq Ct_j^{-d+1}\max_{r\in\{i+2,\dots,n\}}\int_0^1 t^{-d/2}(1-t)^{-d/2}\\ &\qquad\times\exp\left(-\frac{1}{2t_i}\left(\frac{|x_{\sigma(j)}-x_{\sigma(r)}|^2}{t}+\frac{|x_{\sigma(r)}-x_{\sigma(j+1)}|^2}{1-t}\right)\right)\mathrm{d}t\\ &=\mathtt{err}(t_i)\,,\end{aligned} \tag{5.2.37}$$

by the change of variables $t \mapsto t_j t$. Note that

$$\begin{aligned}
&\sum_{\substack{N_1 \le t_1, \ldots, t_{n-1} \le j \\ t_1 + \ldots + t_{n-1} \le j - N_1}} p_{j - \sum_i t_i}(x_{\sigma_n}, x_1) \mathbb{P}_{x_{\sigma(1)}}(E_1) \left(\prod_{j \neq 1} \mathbb{P}_{x_{\sigma(j)}}(H_{\sigma(j+1)} = t_j) \right) \\
&\le C \int_{t_1 + \ldots + t_{n-1} < j} \left(\prod_{j \neq 1} \mathfrak{p}_{t_j}(x_{\sigma(j)}, x_{\sigma(j+1)}) \right) \mathfrak{p}_{j - \sum_i t_i}(x_{\sigma_n}, x_1) \mathtt{err}(t_1) \mathrm{d}(t_i)_i \\
&\le C j^{(d-1)-(n-1)d/2} \int_{\Delta_{n-1}} \left(\prod_{j \neq 1} \mathfrak{p}_{t_j} \left(x_{\sigma(j)} j^{-1/2}, x_{\sigma(j+1)} j^{-1/2} \right) \right) \mathfrak{p}_{1 - 1 \sum_i t_i}(x_{\sigma_n}, x_1) \\
&\qquad \times \mathtt{err}(t_1 j) \, \mathrm{d}(t_i)_i \\
&= \mathtt{Er}(1, j) \,.
\end{aligned} \tag{5.2.38}$$

This leads to the following asymptotics

$$\begin{aligned}
&\mathbb{P}^j_{x_1, x_1} \left(H_{\sigma(2)} < \ldots < H_{\sigma(n)} \right) \\
&= \sum_{\substack{N_1 \le t_1, \ldots, t_{n-1} \le j \\ t_1 + \ldots + t_{n-1} \le j - N_1}} p_{j - \sum_i t_i}(x_{\sigma_n}, x_1) \prod_{i=1}^{n-1} \mathbb{P}_{x_{\sigma(i)}} \left(H_{\sigma(i+1)} = t_i \right) + \mathcal{O} \left(\mathtt{Er}(1, j) \right) .
\end{aligned} \tag{5.2.39}$$

Let $M > 1$ be large. For $t_i \ge N^2/M$, we have that by [Uch11, Theorem 1.7]

$$\mathbb{P}_{x_{\sigma(i)}} \left(H_{\sigma(i+1)} = t_i \right) = \kappa_d \mathfrak{p}_{t_i}(x_{\sigma(i)}, x_{\sigma(i+1)}) \left(1 + \mathcal{O}_M \left(N^{-d} \right) \right) . \tag{5.2.40}$$

We then have that, approximating the sum by an integral,

$$
\begin{aligned}
&\sum_{\substack{t_1+\ldots+t_{n-1}<j \\ |t_i|\geq N^2/M}} p_{j-\sum_i t_i}(x_{\sigma_n}, x_1) \prod_{i=1}^{n-1} \mathbb{P}_{x_{\sigma(i)}}\left(H_{\sigma(i+1)} = t_i\right) \\
&= \sum_{\substack{t_1+\ldots+t_{n-1}<j \\ |t_i|\geq N^2/M}} \kappa_d^{n-1} p_{j-\sum_i t_i}(x_{\sigma_n}, x_1) \prod_{i=1}^{n-1} \mathfrak{p}_{t_i}(x_{\sigma(i)}, x_{\sigma(i+1)}) \left(1 + \mathcal{O}\left(N^{-d}\right)\right) \\
&= \kappa_d^{n-1} \int_{\substack{t_1+\ldots+t_{n-1}<j \\ |t_i|\geq N^2/M}} \mathfrak{p}_{j-\sum_i t_i}(x_{\sigma_n}, x_1) \prod_{i=1}^{n-1} \mathfrak{p}_{t_i}(x_{\sigma(i)}, x_{\sigma(i+1)}) \mathrm{d}(t_i)_i \left(1 + o(1)\right) \\
&= \kappa_d^{n-1} j^{(n-1)-nd/2} \\
&\quad \times \int_{\substack{\Delta_{n-1} \\ t_i > 1/M}} \mathfrak{p}^j_{1-\sum_i t_i}(y_{\sigma_n} N, y_1 N) \left(\prod_{i=1}^{n-1} \mathfrak{p}^j_{t_i}\left(y_{\sigma(i)} N, y_{\sigma(i+1)} N\right)\right) \mathrm{d}(t_i)_i \left(1 + o(1)\right).
\end{aligned}
\tag{5.2.41}
$$

We also estimate (we assume $t_i \geq N_1$ in the first two lines) the error term

$$
\begin{aligned}
&\sum_{\substack{t_1+\ldots+t_{n-1}<j \\ |t_i|\leq N^2/M}} p_{j-\sum_i t_i}(x_{\sigma_n}, x_1) \prod_{i=1}^{n-1} \mathbb{P}_{x_{\sigma(i)}}\left(H_{\sigma(i+1)} = t_i\right) \\
&\leq C \sum_{\substack{t_1+\ldots+t_{n-1}<j \\ |t_i|\leq N^2/M}} p_{j-\sum_i t_i}(x_{\sigma_n}, x_1) \prod_{i=1}^{n-1} \mathfrak{p}_{t_i}(x_{\sigma(i)}, x_{\sigma(i+1)}) \\
&\leq C j^{(n-1)-nd/2} \int_{\substack{\Delta_{n-1} \\ t_i < 1/M}} \mathfrak{p}^j_{1-\sum_i t_i}(y_{\sigma_n} N, y_1 N) \left(\prod_{i=1}^{n-1} \mathfrak{p}^j_{t_i}\left(y_{\sigma(i)} N, y_{\sigma(i+1)} N\right)\right) \mathrm{d}(t_i)_i.
\end{aligned}
\tag{5.2.42}
$$

We have (ignoring the $(1+o(1))$ term to aid legibility) that

$$\begin{aligned}
&\sum_{j\geq N_1} a_j r_j j^{n-1-dn/2}\kappa_d^{n-1}\int_{\substack{\Delta_{n-1}\\ t_i>1/M}} \mathfrak{p}^j_{1-\sum_i t_i}(y_{\sigma_n}N, y_1 N)\\
&\qquad\qquad\qquad\qquad \times\prod_{i=1}^{n-1}\mathfrak{p}^j_{t_i}\left(y_{\sigma(i)}N, y_{\sigma(i+1)}N\right)\mathrm{d}(t_i)_i\\
&=\int_{N_1}^{\infty}\mathrm{d}j\, a_j r_j j^{n-1-dn/2}\kappa_d^{n-1}\int_{\substack{\Delta_{n-1}\\ t_i>1/M}} \mathfrak{p}^j_{1-\sum_i t_i}(y_{\sigma_n}N, y_1 N)\\
&\qquad\qquad\qquad\qquad \times\prod_{i=1}^{n-1}\mathfrak{p}^j_{t_i}\left(y_{\sigma(i)}N, y_{\sigma(i+1)}N\right)\mathrm{d}(t_i)_i\\
&=\kappa_d^n N^{2+2\nu+n(2-d)}\int_0^{\infty} j^{\nu n-dn/2}\int_{\substack{\Delta_{n-1}\\ t_i>1/M}} \mathfrak{p}^j_{1-\sum_i t_i}(y_{\sigma_n}, y_1)\\
&\qquad\qquad\qquad\qquad\qquad \times\prod_{i=1}^{n-1}\mathfrak{p}^j_{t_i}\left(y_{\sigma(i)}, y_{\sigma(i+1)}\right)\mathrm{d}(t_i)_i\,\mathrm{d}j\,.
\end{aligned} \tag{5.2.43}$$

The result follows by observing that the (error) term containing $\mathcal{O}\left(\mathtt{Er}(1,j)\right)$ is of lower order and by letting $M\to\infty$. To summarise, we have that

$$M\left[\{x_1,\dots,x_n\}\in\omega\right]=\kappa_d^n N^{2+2\nu+n(2-d)}\sum_{\sigma\in\mathfrak{S}_n^1}\varphi\left(y_{\sigma(1)},\dots,y_{\sigma(n)}\right)(1+o(1))\,. \tag{5.2.44}$$

Step 3: conclusion

Fix $k\in\mathbb{N}$ and, for $i\in[k]$, choose sets $J_i\subset[n]$. Write $|J|=|J_1|+\ldots+|J_k|$ and fix $d\geq 3$. We then have by the second step

$$M[J,k]=\left(\prod_{i=1}^{k}(-1)^{|J|-1}\lambda M[J_i]\right)\sim\lambda^k(-1)^{|J|-k}N^{2k(1+\nu)+(2-d)|J|}\,. \tag{5.2.45}$$

We examine for which k and $(J_i)_i$ the exponent is maximised, under the condition that $J_1,\dots,J_k\in\mathfrak{P}_c^k(n)$, see Equation (5.2.27). $J_1,\dots,J_k\in\mathfrak{P}_c^k(n)$ implies that $|J|\geq n$. For $k=1$, this implies that $J_1=[n]$. As for $k\geq 2$, a collection $J_1,\dots,J_k\in\mathfrak{P}_c^k(n)$ has to have non-zero intersection, we can conclude $|J|>n$ for $k>1$. Thus, $J_1=[n]$ and $k=1$ maximise the exponent in

Equation (5.2.45). This implies that

$$\sum_{k=1}^{\infty}\frac{(-1)^k}{k!}\sum_{J_1,\dots,J_k\in\mathfrak{P}_c^k(n)} M[J,k]=(-\kappa_d\lambda)^n N^{2+2\nu+n(2-d)}\Upsilon\left(y_1,\dots,y_n\right)(1+o(1))\,. \tag{5.2.46}$$

This finishes the proof in the case $d\geq 3$. For $d=2$, the same reasoning applies. □

Similar to previous results, we can give a more general version of the above result. As the proof is similar, we omit it.

Proposition 5.2.3. *I. Suppose $(a_j)_j$ satisfies $a_j\geq Cj^{-\nu_0}$ for some $\nu_0>-\infty$, $C>0$ and the x_i's are as in Proposition 5.2.1. We then have that*

$$\begin{aligned}M[x_i\in\omega,\,\forall i\in\{1,\dots,n\}]&=(1+o(1))\kappa^{n-1}\sum_{j=1}^{\infty}\sum_{\sigma\in\mathfrak{S}_n^1} a_j r_j j^{n-1-dn/2}\\&\times\int_{\Delta_{n-1}}\mathfrak{p}_{1-\sum_i t_i}(y_{\sigma_n}N,y_1N)\left(\prod_{i=1}^{n-1}\mathfrak{p}_{t_i}\left(y_{\sigma(i)}N,y_{\sigma(i+1)}N\right)\right)\mathrm{d}(t_i)_i\,,\end{aligned} \tag{5.2.47}$$

where in the case $d=2$ we need to add $\log^{-2}(t_ij)$ to each factor in the product. In the case that $d=2$ we furthermore need that

$$\sum_{j\geq n_1}\frac{|a_j|\log^3(\log(j))}{\log^3(j)}=o\left(\sum_{j\geq n_1}|a_j|r_j\right)\,, \tag{5.2.48}$$

for $n_1=N^2/\log\left(\log^{\varepsilon}(N)\right)$ with $\varepsilon>0$.

II. Suppose that for any sets $(I_k)_k, J\subset[n]$ we have that

$$\prod_k M[I_k]=o(M[J])\ \text{if}\ |\cup_k I_k|>|J|\,. \tag{5.2.49}$$

We then have that

$$\psi_n(x_1,\dots,x_n)=\lambda M[\{1,\dots,n\}](-1)^n\mathrm{e}^{-\lambda nM[1]}(1+o(1))\,. \tag{5.2.50}$$

Note that the cumulant uniquely determines the distribution. For the Bosonic loop measure, analogous statements to the above proposition hold.

5.3 The occupation field

Similar to Proposition 5.2.1, we can completely characterise the distribution of the occupation field in the limit. Due to the similarity in the proof, we only write out the parts where the proofs differ. Similarly to the correlation functions, the distribution of the moments of the occupation field have been studied in [LJ11] for the Markovian loop soup.

Proposition 5.3.1. *Let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ as in Proposition 5.2.1. Let $a_j = j^\nu (1 + o(1))$ for $d \geq 3$ and $a_j = j^\nu \log(j) (1 + o(1))$ for $d = 2$. We then have that joint cumulant of the family $(\mathcal{L}_{x_i})_{i=1}^n$ is given by*

$$\kappa_d N^{4+2\nu+n(2-d)-d} \bar{\varphi}(y_1, \ldots, y_n)(1 + o(1)) \,, \tag{5.3.1}$$

where

$$\bar{\varphi}(y_1, \ldots, y_n) = \int_0^\infty j^{\nu+1+n-d(n+1)/2} \int_{[0,1]^n} \mathrm{d}t \prod_{i=0}^n \mathfrak{p}^j_{t_{\sigma(i)}} \left(y_{\sigma(i)}, y_{\sigma(i+1)}\right) \,, \tag{5.3.2}$$

where for $t \in [0,1]^n$ we set σ such that $t_{\sigma(1)} \leq \ldots \leq t_{\sigma(n)}$. Furthermore, we set $\sigma(0) = \sigma(n+1) = 1$.

Proof of Proposition 5.3.1. First, note that

$$M\left[\prod_{i=1}^n L_{x_i}\right] = \sum_{j\geq 1} a_j \mathbb{E}^j_{x_1,x_1}\left[\mathcal{R}_j \prod_{i=1}^n L_{x_i}\right] \,, \tag{5.3.3}$$

using by now well-known arguments. The computation of the asymptotics follows the same strategy as in the proof of Proposition 5.3.1 from there on. Note that for $I \subset [n]$

$$\mathbb{E}_\lambda\left[\prod_{i\in I} \mathcal{L}_{x_i}\right] = \frac{\partial^{|I|}}{\prod_{i\in I}\partial v_i} \mathbb{E}_\lambda\left[\mathrm{e}^{-\langle v,\mathcal{L}\rangle}\right]\Big|_{v=0} . \tag{5.3.4}$$

We can use Campbell's formula to compute $\mathbb{E}_\lambda\left[\mathrm{e}^{-\langle v,\mathcal{L}\rangle}\right]$ in terms of $M\left[1 - \mathrm{e}^{-\langle v,L\rangle}\right]$. A multivariate version of Faà di Bruno's formula (e.g. [Har06]) then shows that the joint cumulant of the family $(\mathcal{L}_{x_i})_{i=1}^n$ is given by

$$M\left[\prod_{i=1}^{n} L_{x_i}\right]. \tag{5.3.5}$$

Together with the asymptotics this implies the main result and thus finishes the sketch of the proof. □

Remark 5.3.2. *A version of the above result for more general sequences $(a_j)_j$, similar to Proposition 5.2.3, can be given. As the conditions should be clear by now, we leave it to the reader.*

The above result could serve as an important tool for any cluster expansion for a loop soup with interaction.

5.4 Divergence in two dimensions

By Lemma 6.1.1, we have that for $d = 2$ and $a_j = 1/j$, every vertex in $\mathbb{Z}^2$ is covered by at least one loop. In this section we explore the speed of this occupation by approximating the loop measure with $a_j = 1/j$.
We restrict the lengths of the loops: for $T > 0$, let

$$M^T = \sum_{x\in\mathbb{Z}^2}\sum_{j=1}^{T}\frac{1}{j}\mathbb{P}_{x,x}^j . \tag{5.4.1}$$

Let $\mathbb{P}_\lambda^T$ be the PPP process with intensity measure λM^T. We then have the following limiting behaviour.

Proposition 5.4.1. *We have that*

$$\lim_{T\to\infty}\frac{1}{\log\log(T)}\log\mathbb{P}_\lambda^T\left(0\notin\mathcal{U}\right) = -\frac{\lambda}{2}. \tag{5.4.2}$$

This shows that the divergence occurs at a very slow speed.

Proof of Proposition 5.4.1. We use the fundamental property of the PPP to write

$$\mathbb{P}^T_\lambda\left(0\notin\mathcal{U}\right)=\exp\left(-\lambda M^T\left[0\in\omega\right]\right)=\exp\left(-\lambda\sum_{j=1}^{T}\frac{1}{j}\mathbb{E}^j_{0,0}[\mathcal{R}_j]\right). \tag{5.4.3}$$

By [Ham06, Theorem 2.2] we have that the logarithm of the above is equal to

$$-\lambda\sum_{j=2}^{T}\frac{1}{2j\log(j)}\left(1+\mathcal{O}\left(1/\log\log(j)\right)\right)+\mathcal{O}(1). \tag{5.4.4}$$

Indeed, $\mathbb{B}^j_{0,0}[\mathcal{R}_j]=\pi j/\log(j)\left(1+\mathcal{O}\left(1/\log\log(j)\right)\right)$ and thus the above follows by approximating $p_j(0)$ by $\mathfrak{p}_j(0)$. Note that for $f(j)=1/(j\log(j))$ one has that

$$|\partial^2_j f(j)|=\mathcal{O}\left(\frac{1}{j^3\log(j)}\right), \tag{5.4.5}$$

and thus by [LL10, Lemma A.1.1] one has that

$$\sum_{j=2}^{T}\frac{1}{2j\log(j)}\left(1+\mathcal{O}\left(1/\log\log(j)\right)\right)=\frac{1}{2}\int_2^T\frac{1}{j\log(j)}\mathrm{d}j+C+\mathcal{O}\left(1/T^2\right), \tag{5.4.6}$$

where the constant C is uniformly bounded in T. By computing the integral, we get that

$$\log\mathbb{P}^T_\lambda\left(0\notin\mathcal{U}\right)=-\frac{\lambda}{2}\log\log(T)+\mathcal{O}(1). \tag{5.4.7}$$

This concludes the proof. □

We can also analyse the divergence of the expectation of the occupation field. Due to the similarity of the proof, we have chosen to omit it.

Proposition 5.4.2. *We have that asymptotically*

$$\lim_{T\to\infty}\frac{1}{\log(T)}\mathbb{E}^T_\lambda\left[\mathcal{L}_0\right]=\frac{\lambda}{2\pi}. \tag{5.4.8}$$

5.5 Vacant sets

It is increasingly unlikely to observe a large unoccupied region of the space. We derive a precise limit for ℓ^∞-boxes, as their symmetry corresponds to the random walk. We make this precise later.

Proposition 5.5.1. *Let* $\boldsymbol{B}_n^\infty = \{x \in \mathbb{Z}^d \colon |x|_\infty \leq n\}$*. Furthermore, assume that* $a_j j^{-d/2} r_j$ *is summable (see beginning of Chapter 4 for a definition of* r_j*). We then have the following limit*

$$\lim_{n\to\infty} \frac{1}{(2n+1)^d} \log \mathbb{P}_\lambda \left(\boldsymbol{B}_n^\infty \cap \mathcal{C} = \varnothing\right) = -\lambda \sum_{j\geq 1} a_j p_j(0) = -\lambda C_d^\infty \,. \tag{5.5.1}$$

Furthermore, we can even get the next order term

$$\lim_{n\to\infty} \frac{\left(\log \mathbb{P}_\lambda \left(\boldsymbol{B}_n^\infty \cap \mathcal{U} = \varnothing\right) + \lambda(2n+1)^d C_d^\infty\right)}{\lambda(2n+1)^{d-1}} = -2d \sum_{j\geq 1} a_j \mathbb{E}_{0,0}^j \left[\mathcal{R}_{\tilde{H}}\right] = -D_d^\infty, \tag{5.5.2}$$

where $\tilde{H}$ *is the first time the random walk hits the half space* $\{x \in \mathbb{Z}^d \colon x^{(1)} \leq 0\} \setminus \{0\}$.

Proof of Proposition 5.5.1. Note that by the fundamental properties of the PPP we have that

$$\log \mathbb{P}_\lambda \left(\boldsymbol{B}_n^\infty \notin \mathcal{U}\right) = -\lambda M \left[\omega \cap \boldsymbol{B}_n^\infty \neq \varnothing\right] \,. \tag{5.5.3}$$

Let H_n^∞ be the hitting time of $\boldsymbol{B}_n^\infty$. We expand

$$\begin{aligned} M\left[\omega \cap \boldsymbol{B}_n^\infty \neq \varnothing\right] &= \sum_{x\in\mathbb{Z}^d} \sum_{j\geq 1} a_j \mathbb{E}_{x,x}^j \left[H_n^\infty < j\right] \\ &= \sum_{x\in\mathbb{Z}^d\setminus \boldsymbol{B}_n^\infty} \sum_{j\geq 1} a_j \mathbb{E}_{x,x}^j \left[H_n^\infty < j\right] + \sum_{x\in \boldsymbol{B}_n^\infty} \sum_{j\geq 1} a_j p_j(0) \\ &= \sum_{x\in\partial \boldsymbol{B}_n^\infty} \sum_{j\geq 1} a_j \mathbb{E}_{x,x}^j \left[\mathcal{R}_{H_n^\infty(x)}\right] + (2n+1)^d \sum_{j\geq 1} a_j p_j(0) \,, \end{aligned} \tag{5.5.4}$$

where $H_n^\infty(x)$ is the first time of hitting $\boldsymbol{B}_n^\infty \setminus \{x\}$. We can bound the first term

$$\sum_{x\in\partial \boldsymbol{B}_n^\infty} \sum_{j\geq 1} a_j \mathbb{E}_{x,x}^j \left[\mathcal{R}_{H_n^\infty(x)}\right] \leq \sum_{x\in\partial \boldsymbol{B}_n^\infty} \sum_{j\geq 1} a_j \mathbb{E}_{x,x}^j \left[\mathcal{R}_j\right] \leq C n^{d-1} \sum_{j\geq 1} a_j j^{-d/2} r_j \,. \tag{5.5.5}$$

Thus, dividing by $(2n+1)^d$ and taking the limit as $n \to \infty$ shows the first part of Proposition 5.5.1.

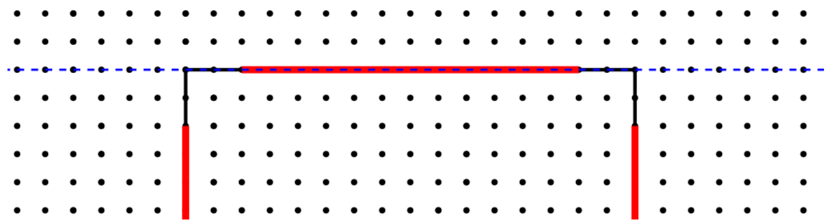

Figure 5.3: The (black) square is $\partial \boldsymbol{B}_n^\infty$, the red subset is $\partial \boldsymbol{B}_{n,\varepsilon_1}^\infty$, with $\varepsilon_1 = 2$. Hitting the blue line corresponds to $\tilde{H}(x)$, for x in the horizontal part of $\partial \boldsymbol{B}_{n,\varepsilon_1}^\infty$.

We now prove the second statement, the characterisation of the second order term. Fix $\varepsilon > 0$ and choose $M > 0$ large enough such that

$$\frac{1}{n^{d-1}} \sum_{x \in \partial \boldsymbol{B}_n^\infty} \sum_{j \geq M_1} a_j \mathbb{E}_{x,x}^j \left[\mathcal{R}_{H_n^\infty(x)} \right] \leq \varepsilon/4 \,, \tag{5.5.6}$$

for n sufficiently large. Let $1 > \varepsilon_1 > 0$ and define $\partial \boldsymbol{B}_{n,\varepsilon_1}^\infty$ as follows

$$\partial \boldsymbol{B}_{n,\varepsilon_1}^\infty = \{ x \in \partial \boldsymbol{B}_n^\infty \colon \forall i \in \{1, \ldots, d\} \forall j \in \{-1, +1\} \colon |x - jne_i|_\infty > \varepsilon_1 n \} \,, \tag{5.5.7}$$

where we recall that $(e_i)_{i=1}^d$ are the standard basis vectors in $\mathbb{Z}^d$. For an illustration of $\partial \boldsymbol{B}_{n,\varepsilon_1}^\infty$ in two dimensions, see Figure 5.3. Choose $\varepsilon_1 > 0$ sufficiently small such that

$$\frac{1}{n^{d-1}} \sum_{x \in \partial \boldsymbol{B}_n^\infty \setminus \partial \boldsymbol{B}_{n,\varepsilon_1}^\infty} \sum_{j=1}^{M_1} a_j \mathbb{E}_{x,x}^j \left[\mathcal{R}_{H_n^\infty(x)} \right] \leq \varepsilon/4 \,. \tag{5.5.8}$$

This is possible as $|\partial \boldsymbol{B}_{n,\varepsilon_1}^\infty| \sim \varepsilon_1 n^{d-1}$. For $x \in \partial \boldsymbol{B}_{n,\varepsilon_1}^\infty$ let

$$\tilde{H}(x) = \inf\{ n \geq 1 \colon |S_n^{(i)}| \leq n \} \,, \tag{5.5.9}$$

where $S_n^{(i)}$ is the i-th coordinate of S_n and i is the unique coordinate such that $|x^{(i)}| = n$. Note that for $x \in \partial \boldsymbol{B}_{n,\varepsilon_1}^\infty$ we have that bound

$$\mathbb{P}_{x,x}^j \left(\tilde{H}(x) \neq H_n^\infty(x) \right) \leq \mathbb{P}_{x,x}^j \left(\sup_{0 \leq k \leq n} |x - S_k| \geq \varepsilon_1 n \right) \,. \tag{5.5.10}$$

As the above goes exponentially fast to zero uniformly in $x \in \partial \boldsymbol{B}_{n,\varepsilon_1}^\infty$, we can write

$$\frac{1}{n^{d-1}} \sum_{x \in \partial \boldsymbol{B}^\infty_{n,\varepsilon_1}} \sum_{j=1}^{M_1} a_j \mathbb{E}^j_{x,x}\left[\mathcal{R}_{H^\infty_n(x)}\right] = \frac{1+o(1)}{n^{d-1}} \sum_{x \in \partial \boldsymbol{B}^\infty_{n,\varepsilon_1}} \sum_{j=1}^{M_1} a_j \mathbb{E}^j_{x,x}\left[\mathcal{R}_{\tilde{H}(x)}\right] . \tag{5.5.11}$$

Choose n sufficiently large such that the $o(1)$ term is smaller than $\varepsilon/4$. By symmetry, we have that

$$\mathbb{E}^j_{x,x}\left[\mathcal{R}_{\tilde{H}(x)}\right] = \mathbb{E}^j_{0,0}\left[\mathcal{R}_{\tilde{H}(0)}\right] , \tag{5.5.12}$$

and thus

$$\sum_{x \in \partial \boldsymbol{B}^\infty_{n,\varepsilon_1}} \sum_{j=1}^{M_1} a_j \mathbb{E}_{x,x}\left[\mathcal{R}_{\tilde{H}(x)}\right] = 2d(2n(1-\varepsilon)+1)^{d-1} \sum_{j=1}^{M_1} a_j \mathbb{E}_{0,0}\left[\mathcal{R}_{\tilde{H}(0)}\right] . \tag{5.5.13}$$

By further increasing M_1 and using the triangle inequality, we have shown that

$$\left| \sum_{x \in \partial \boldsymbol{B}^\infty_n} \sum_{j \geq 1} a_j \mathbb{E}^j_{x,x}\left[\mathcal{R}_{H^\infty_n(x)}\right] - 2d(2n+1)^{d-1} \sum_{j=1}^{M_1} a_j \mathbb{E}^j_{0,0}\left[\mathcal{R}_{\tilde{H}(0)}\right] \right| \leq \varepsilon n^{d-1} . \tag{5.5.14}$$

This concludes the proof. □

Remark 5.5.2. *The volume order limit in Proposition 5.5.1 holds true more generally: given a sequence of connected sets $(A_n)_n$ which is strictly increasing and satisfies $|\partial A_n| = o(|A_n|)$, we have*

$$\lim_{n\to\infty} \frac{1}{|A_n|} \log \mathbb{P}_\lambda(A_n \cap \mathcal{U} = \varnothing) = -\lambda C^\infty_d . \tag{5.5.15}$$

For the second order term, we need some knowledge of the "scaling limit" of the geometry of ∂A_n. In the case of $\boldsymbol{B}^\infty_n$, it scales to the half space and we can thus compute the second order limit.

Chapter 6

Loop percolation

In this chapter we study the connected component of $\mathcal{U}$ which intersects the origin, denoted by $\mathcal{C}_0$. As in all percolation models, there are different parameters:

$$\begin{aligned}
\lambda_c &= \inf\{\lambda > 0\colon \mathbb{P}_\lambda\left(|\mathcal{C}_0| = \infty\right) > 0\}\,, \\
\lambda_r &= \inf\{\lambda > 0\colon \limsup_{n\to\infty} \mathbb{P}_\lambda\left(\boldsymbol{B}_n \text{ connected to } \boldsymbol{B}_{2n}^c\right) = 1\}\,, \\
\lambda_\# &= \inf\{\lambda > 0\colon \mathbb{E}_\lambda\left[\#\mathcal{C}_0\right] = \infty\}\,.
\end{aligned} \tag{6.0.1}$$

It is obvious that $\lambda_r \leq \lambda_c$ and $\lambda_\# \leq \lambda_c$. In [CS16], it is shown that $\lambda_\# \leq \lambda_r$ for $d \geq 5$ and $a_j = 1/j$ [1].

We firstly introduce loop percolation rigorously and recall some results from the literature before applying them to our setting. We then use the estimates obtained in Chapter 4 to prove equality of critical parameters for $(a_j)_j$ decaying sufficiently fast. Important will be the framework of the OSSS inequality, which is applied in [DCRT18] to show $\lambda_c = \lambda_r$ for the Poisson-Boolean and other models. In the last section of the chapter, we prove some finer estimates on the structure of $\mathcal{C}_0$, some of which were predicted in [CS16].

We restrict ourselves to random walks such that the increments in each dimension are supported on $\{-1, 0, 1\}$ in this chapter.

Remark 6.0.1. *A brief comment regarding decay assumptions: while in the previous chapters, assumptions on the decay of a_j were generous, in this chap-*

[1] Their definition of λ_r is slightly different: take λ_k the smallest λ such that $\limsup_{n\to\infty} \mathbb{P}_\lambda\left(\boldsymbol{B}_n \text{ connected to } \boldsymbol{B}_{kn}^c\right) = 1$ and set $\lambda_r = \sup_{k>1} \lambda_k$.

ter we often assume that a_j decays much faster than j^{-1}. It is common in long-range percolation models (see e.g. [DCRT18]) that some restrictions on the correlation decay is imposed. One can interpret sequences a_j with $\sum_{j\geq 1} a_j j < \infty$ as introducing an additional (slow) killing to the Markovian loop measure: set V_a the measure on $\mathbb{N}$ with $V_a(X = j) = a_j j$. We can then rewrite

$$M_a = \sum_{j\geq 1} a_j \mathbb{P}^j_{x,x} = V_a(\mathbb{N}) \int \frac{\mathrm{d}V_a}{V_a(\mathbb{N})}(j)\, \frac{\mathbb{P}^j_{x,x}}{j}\,. \tag{6.0.2}$$

6.1 Introduction and preliminary results

Given a random point measure $\mathcal{U} = \sum_{k\leq\kappa} \delta_{\omega_k}$, we define $\mathcal{C} \subset \mathbb{Z}^d \times \mathbb{Z}^d$ to be the subset of bonds in $\mathbb{Z}^d$ which are open in the following way: given (discrete-time) loops $(\omega_k)_k$ with $\omega_k = (\omega_k(0), \omega_k(1), \ldots, \omega_k(n_k) = \omega_k(0))$ (with $\omega_k(i) \in \mathbb{Z}^d$), set

$$\mathcal{C} = \bigcup_{k\leq\kappa} \bigcup_{l=1}^{n_k} \{\omega_k(l-1), \omega_k(l)\}\,. \tag{6.1.1}$$

Note that bonds are not directed in this setting. If the bond $b = \{b^1, b^2\} \in \mathcal{C}$, we say that $b = \{b^1, b^2\}$ is *open*. For $x \in \mathbb{Z}^d$, we often say that $x \in \mathcal{C}$ (or equivalently, x open) when we mean that $\{x, y\} \in \mathcal{C}$ for some $y \in \mathbb{Z}^d$.

Let λ_c be the smallest $\lambda \geq 0$ such that for all $\lambda > \lambda_c$ there exists an unbounded connected component of $\mathcal{C}$ almost surely. Note that $\lambda_c < \infty$ as the random walk loop soup is bounded from below by the Bernoulli bond percolation. This argument is made for $a_j = j^{-1}$ in [LJL13] and applies for $(a_j)_j$ positive.

For $x, y \in \mathbb{Z}^d$ we say that x is connected to y if there exists a sequence of open bounds $b_1, \ldots, b_n$ such that

I. $x \in b_1$.

II. $y \in b_n$.

III. For all $i \in \{1, \ldots, n-1\}$ we have $b_i \cap b_{i+1} \neq \varnothing$.

For $A, B \subset \mathbb{Z}^d$ we say that A is connected to B (denoted by $A \longleftrightarrow B$) if there is $x \in A$ connected to $y \in B$. If one of the sets consists of a singleton, we write $x \longleftrightarrow B$ instead of $\{x\} \longleftrightarrow B$. If there exists a single loop connecting

A and B, we write $A \stackrel{\omega}{\longleftrightarrow} B$.

The next lemma gives us a minimal condition on the weights $(a_j)_j$ such that the percolation problem is not trivial. We recall that $\mathbb{P}^a_\lambda$ denotes the loop soup with intensity measure λM^a.

Lemma 6.1.1. *For any $\lambda > 0$ we have that*

$$\sum_{j\geq 0} a_j r_j j^{-d/2} = \infty \Longrightarrow \mathbb{P}^a_\lambda (0 \in \mathcal{C}) = 1\,. \tag{6.1.2}$$

Due to the translation invariance of the loop measure, this implies that every edge is covered by at least one loop.

Proof of Lemma 6.1.1. We expand

$$\begin{aligned} M^a[0\in\omega] &= \sum_{x\in\mathbb{Z}^d}\sum_{j\geq 0} a_j \mathbb{P}^j_{x,x}(H_0<j) = \sum_{x\in\mathbb{Z}^d}\sum_{j\geq 0} a_j \mathbb{P}^j_{0,0}(H_x<j) \\ &= \sum_{j\geq 0} a_j \mathbb{E}^j_{0,0}\left[\sum_{x\in\mathbb{Z}^d} \mathbb{1}\{H_x<j\}\right] = \sum_{j\geq 0} a_j \mathbb{E}^j_{0,0}[\mathcal{R}_j] \geq C\sum_{j\geq 1} a_j r_j j^{-d/2}\,, \end{aligned} \tag{6.1.3}$$

by the time-homogeneity of the random walk, monotone convergence and finally [Ham06, Theorem 2.2] (to evaluate the expectation of the random walk bridge). Note that by the fundamental properties of the PPP and by a limiting argument

$$\mathbb{P}^a_\lambda (0 \notin \mathcal{C}) = \exp\left(-\lambda M^a[0\in\omega]\right)\,. \tag{6.1.4}$$

This concludes the proof of the first statement. By an inclusion-exclusion argument, it is straightforward to see that every edge is covered by at least one loop. This concludes the proof. □

Remark 6.1.2. *This lemma is a slight generalisation of [CS16, Proposition 3.4], where the case $a_j = a^j$ with $a > 1$ is treated. The above lemma can also be applied more generally: replacing r_j with j, the lemma is valid for any random walk, as $\mathcal{R}_j \leq j$ holds true always.*

We incorporate the result of the above lemma into an assumption.

Assumption 6.1.3. *Henceforth assume that*

$$\sum_{j\geq 0} a_j r_j j^{-d/2} < \infty\,,$$

so that the induced percolation process is not trivial.

Next, we state a proposition establishing some basic properties of the connected component. As its proof is essentially the same as the one given in [CS16], we chose to omit it.

Proposition 6.1.4. *The loop soup is ergodic under lattice shifts. It has at most one unique infinite cluster.*

6.2 Decay estimates of the loop soup

In this section we prove decay estimates for the entire loop soup. We work in the regime that $a_j \leq Cj^{-1}$ for the entire section. We recall that

$$\lambda_r = \inf\{\lambda > 0\colon \liminf_{n\to\infty} \mathbb{P}_\lambda\left(\partial \boldsymbol{B}_n \longleftrightarrow \partial \boldsymbol{B}_{2n}\right) = 1\}\,. \tag{6.2.1}$$

Note that $0 \leq \lambda_r \leq \lambda_c$. The next proposition follows immediately form [CS16, Lemma 4.1].

Proposition 6.2.1. ***[CS16]*** *For $d \geq 3$ and $a_j = \mathcal{O}(1)j^{-1}$ we have that $\lambda_r > 0$.*
Furthermore, for $\lambda < \lambda_r$, we have that the connectivity $\mathbb{P}_\lambda\left(0 \longleftrightarrow \boldsymbol{B}_n^c\right)$ is bounded from above by $C(\lambda)n^{-c(\lambda)}$ for some $C(\lambda), c(\lambda) > 0$ both depending on $\lambda > 0$.

Indeed, in aforementioned reference the special case $a_j = j^{-1}$ is examined. However, for $a_j = \mathcal{O}(1)j^{-1}$, we can bound the associated loop soup from above by the special case.
The next proposition uses a proof strategy laid out in [CS16, Section 5]. Let $\lambda_\# > 0$ be the largest $\lambda > 0$ such that $\mathbb{E}_\lambda[|\mathcal{C}_0|] < \infty$. We then have that:

Proposition 6.2.2. *Let $\lambda < \lambda_{\#}$. Let $a_j \sim j^{\nu}$ ($\nu \leq -1$) if $d \geq 3$ and $a_j \sim j^{\nu} \log j$ ($\nu < -1$, $d = 2$). We then have that*

$$\mathbb{P}_{\lambda}\left(0 \longleftrightarrow \boldsymbol{B}_n^c\right) \sim n^{4+2\nu-d}\,. \tag{6.2.2}$$

Proof of Proposition 6.2.2. The lower bound follows from the Proposition 4.2.2. The upper bound is analogous the proof of Proposition [CS16, Proposition 5.2] where the only ingredient needed is an estimate of the probability of having a single loop connecting zero to the boundary of a ball with diverging radius. This we compute in Proposition 4.2.2.

By [CS16, Proposition 5.1], we know that for $d \geq 5$ that $\lambda_{\#} > 0$.

Remark 6.2.3. *In [DCRT18], it is shown that for Poisson-Boolean percolation*[2] *$\mathbb{Pb}$ one has*

$$\lim_{n\to\infty} \frac{\mathbb{Pb}_{\lambda}\left(0 \longleftrightarrow \boldsymbol{B}_n^c\right)}{\mathbb{Pb}_{\lambda}\left(0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c\right)} = 1\,, \tag{6.2.3}$$

where $\mathbb{Pb}_{\lambda}\left(0 \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_n^c\right)$ is the probability of connecting 0 to the complement of $\boldsymbol{B}_n^c$ through a single ball. Proposition 6.2.2 might seduce one into thinking that such a statement for loop percolation could be true as well. However, this is not the case: connecting the origin to $\boldsymbol{B}_n^c$ through a single loop is of the same order than having a loop of diameter $\mathcal{O}(1)$ which intersects both the origin as well as different loop, which intersects $\partial \boldsymbol{B}_n$ and the first loop, but not the origin. This reasoning is true for $d \geq 3$ due to the transience of the random walk. For $d = 2$ the above reasoning no longer applies, and it remains an open question whether for sufficiently fast decaying weights the above equation holds true for loop percolation in two dimensions. As Theorem 4.3.1 has only been shown for $d \geq 3$, an important tool used in [DCRT18] is not available and so we do not explore this question further. See Chapter 7 for further remarks.

[2]In Poisson-Boolean percolation one studies the PPP with intensity measures $\nu \times \lambda\mathsf{Lebesgue}$ on $(0,\infty) \times \mathbb{R}^d$, with ν a probability measure and intensity $\lambda > 0$. A sample (r, x) is interpreted as a sphere with radius r, centred at x. The overlapping of the spheres induces clusters in $\mathbb{R}^d$.

6.2.1 Equivalence of two critical parameters for small ν

In this section we prove that for $\nu < -1$ sufficiently small, we have the equivalence of the two critical parameter λ_r and $\lambda_\#$. Given Theorem 4.3.1, the proof is short and classical.

Theorem 6.2.4. *Given $a_j \sim j^\nu$, $s > 0$, $\nu < -1$, $d \geq 3$ and $2 + 2\nu < -sd - 1$, we have that for $\lambda < \lambda_r$ that*

$$\mathbb{E}_\lambda \left[|\mathcal{C}_0|^s \right] < \infty \,. \tag{6.2.4}$$

In particular, if $\nu < -d - 1$ (i.e. $s \geq 1$), we have that $\lambda_r = \lambda_\#$.

Proof of Theorem 6.2.4. We follow [GT19], where the case of the Poisson-Boolean model is considered.
We begin by noting that

$$\mathbb{P}_\lambda \left(\boldsymbol{B}_n \longleftrightarrow \boldsymbol{B}_{4n}^c \right) \leq \mathbb{P}_\lambda \left(\boldsymbol{B}_n \longleftrightarrow \boldsymbol{B}_{2n}^c \right)^2 + \mathbb{P}_\lambda \left(\boldsymbol{B}_{2n} \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_{4n}^c \right) \,. \tag{6.2.5}$$

Indeed, if there does not exist a loop connecting $\partial_i \boldsymbol{B}_{2n}$ to $\boldsymbol{B}_{4n}^c$, then $\boldsymbol{B}_n$ is connected to $\boldsymbol{B}_{2n}^c$ through loops contained in $\boldsymbol{B}_{3n-1}$ and the same for $\partial \boldsymbol{B}_{3n}$ to $\boldsymbol{B}_{4n}$. Note that by Theorem 4.3.1

$$\mathbb{P}_\lambda \left(\boldsymbol{B}_{2n} \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_{4n}^c \right) \leq \lambda M \left[\boldsymbol{B}_{2n} \overset{\omega}{\longleftrightarrow} \boldsymbol{B}_{4n}^c \right] \leq C \lambda n^{2\nu+2} \,. \tag{6.2.6}$$

Let $\pi(n) = \mathbb{P}_\lambda \left(\boldsymbol{B}_n \longleftrightarrow \boldsymbol{B}_{2n}^c \right)$. By covering $\boldsymbol{B}_n$ with smaller balls, we can find K such that for all n we have

$$\mathbb{P}_\lambda \left(\boldsymbol{B}_n \longleftrightarrow \boldsymbol{B}_{3n/2}^c \right) \leq K \mathbb{P}_\lambda \left(\boldsymbol{B}_{n/4} \longleftrightarrow \boldsymbol{B}_n^c \right) \,. \tag{6.2.7}$$

Choose n_0 sufficiently large that $4K\pi(n) < 1/2$ for all $n \geq n_0/4$. This is possible as $\lambda < \lambda_r$. Then for all $n > n_0$

$$\begin{aligned}\int_{n_0}^{n} m^{s-1}\pi(m)\mathrm{d}m &\le K\int_{n_0}^{n} m^{s-1}\pi(m/4)^2\mathrm{d}m + \lambda C\int_{n_0}^{n} m^{s-1}m^{2\nu+2}\mathrm{d}m \\ &\le \frac{1}{2}\int_{n_0}^{n} m^{s-1}\pi(m/4)\mathrm{d}m + \lambda C\int_{n_0}^{\infty} m^{s-1}m^{2\nu+2}\mathrm{d}m \\ &\le \frac{1}{2}\int_{n_0/4}^{n/4} m^{s-1}\pi(m/4)\mathrm{d}m + \lambda C\,.\end{aligned} \tag{6.2.8}$$

Rearranging, we get that

$$\int_{n_0}^{n} m^{s-1}\pi(m)\mathrm{d}m \le \frac{1}{2}\int_{n_0/4}^{n_0} m^{s-1}\pi(m/4)\mathrm{d}m + \lambda C \tag{6.2.9}$$

As the right-hand side of the above equation no longer depends on n, we can let n go to infinity and obtain

$$\int_{0}^{\infty} m^{s-1}\pi(m)\mathrm{d}m < \infty\,. \tag{6.2.10}$$

As $\{|\mathcal{C}_0| \ge 2n\} \subset \{\boldsymbol{B}_{n^{1/d}} \longleftrightarrow \boldsymbol{B}^c_{2n^{1/d}}\}$, we have proven the claim. □

6.3 The OSSS inequality and sharpness

In this section we prove sharpness for loop percolation, i.e. that $\lambda_r = \lambda_c$, given $(a_j)_j$ decays sufficiently fast. We use the strategy laid out in [DCT16, DCRT18].

We begin by explaining the framework of the OSSS inequality, as proved in [OSSS05] and used in [DCRT18]. Let I be a finite index set, $\Omega = \bigtimes_{i\in I}\Omega_i$ the product space over some probability spaces (Ω_i, π_i) and $\pi = \bigotimes_{i\in I}\pi_i$. Take $f\colon \Omega \to \{0,1\}$ (think of $f = \mathbb{1}\{0 \longleftrightarrow \boldsymbol{B}^c_n\}$). An *algorithm* `T` takes a point in the sample space $\omega \in \Omega$ and checks the value of each of its coordinates, one after the other. It stops as soon as the value of f does not change with the remaining coordinates. For example, if we need to check whether $0 \longleftrightarrow \boldsymbol{B}^c_n$, we may stop as soon as we have found a lattice path connecting 0 to $\boldsymbol{B}^c_n$. Note that it is not necessary to check all coordinates in $\boldsymbol{B}_n$.

Given an algorithm `T` and a product space (Ω, π), we define two important functions

I. The *revealment*: it quantifies how likely it is for the algorithm to visit Ω_i for $i \in I$. It is henceforth denoted by $\delta_i(\mathtt{T})$ and is defined as

$$\delta_i(\mathtt{T}) = \pi \left[\mathtt{T} \text{ reveals the value in } \Omega_i\right] . \tag{6.3.1}$$

It will turn out that it is desirable to have a uniformly low revealment.

II. The *influence*. It quantifies how important a coordinate is to the outcome of f. It is defined as

$$\mathtt{Inf}_i(f) = \pi \otimes \overline{\pi_i} \left[f(\omega) \neq f\left(\overline{\omega}\right)\right] , \tag{6.3.2}$$

where $\overline{\pi_i}$ is an independent copy of π_i and $\overline{\omega}$ is the tuple where we take ω and re-sample the i-th coordinate with respect to $\overline{\pi_i}$.

The OSSS inequality can then be seen as a generalisation of the Poincaré inequality for product spaces.

Theorem 6.3.1. ***[OSSS05]*** *Given the above set-up, we have that*

$$\pi[f] - \pi[f]^2 = \mathrm{var} f \leq \sum_{i \in I} \delta_i(\mathtt{T}) \mathtt{Inf}_i(f) . \tag{6.3.3}$$

The central idea in [DCT16] is to combine the OSSS-inequality with two other tools from percolation theory to obtain a short proof of sharpness. These are a *Russo type formula* (see Lemma 6.3.3) and a *differential inequality*. We begin by stating the latter one:

Lemma 6.3.2. ***Differential Inequality, [DCRT19b, Lemma 3.1]*** *Given a converging sequence of differentiable functions $f_n \colon [0,1] \to [0,1]$ satisfying*

$$\partial f_n \geq \frac{n}{\sum_{k=0}^{n-1} f_k} f_n , \tag{6.3.4}$$

for all $n \geq 1$. Then, there exists $\beta_1 \in [0,1]$ such that

I. For any $\beta < \beta_1$, there exists $c_\beta > 0$ such that $f_n(\beta) = \mathcal{O}\left(\mathrm{e}^{-c_\beta n}\right)$, as $n \to \infty$.

II. There exists $C > 0$ such that for any $\beta > \beta_1$, $\lim_n f_n(\beta) \geq C(\beta - \beta_1)$.

Next, we present a proof of the Russo's formula adapted to our setting.

Lemma 6.3.3. ***Russo*** *Let $A = \{0 \longleftrightarrow \partial B_n\}$, with $n > 0$. We then have that*

$$\partial_\lambda \mathbb{P}_\lambda(A) = \sum_{\omega \in \Gamma} M[\{\omega\}] \mathbb{P}_\lambda\left(\omega \textit{ pivotal for } A\right) . \tag{6.3.5}$$

Furthermore, the above formula holds for any increasing event A satisfying

$$M[\{\omega \textit{ could be pivotal for } A\}] < \infty .$$

Proof of Lemma 6.3.3. Let $\Gamma_n = \{\omega \colon \omega \cap \boldsymbol{B}_n \neq \varnothing\}$. We write the loop soup $\mathcal{U}^{\lambda+h} = \mathcal{U}^\lambda \cup \mathcal{U}^h$ using the superposition of Poisson point processes. Let P denote this coupling between $\mathbb{P}_{\lambda+h}$ and $\mathbb{P}_\lambda$ and expand

$$\begin{aligned} \mathbb{P}_{\lambda+h}(A) - \mathbb{P}_\lambda(A) &= \mathsf{P}\left(A \in \mathcal{U}^\lambda \cup \mathcal{U}^h,\, A \notin \mathcal{U}^\lambda\right) = \\ &= \sum_{\omega \in \Gamma_r} \mathsf{P}\left(A \in \mathcal{U}^\lambda \cup \{\omega\},\, A \notin \mathcal{U}^\lambda,\, \omega = \mathcal{U}^h \cap \Gamma_n\right) \\ &\quad + \mathsf{P}\left(A \in \mathcal{U}^\lambda \cup \mathcal{U}^h,\, A \notin \mathcal{U}^\lambda, |\mathcal{U}^h \cap \Gamma_n| \geq 2\right) . \end{aligned} \tag{6.3.6}$$

A quick calculation in the spirit of Lemma (6.1.1) reveals that $M[\Gamma_n] < \infty$ as long as the percolation process is non-trivial. As a consequence, the second term in the above equation is of order $\mathcal{O}(h^2)$ and thus negligible.
We expand further, using the independence of $\mathcal{U}^\lambda$ and $\mathcal{U}^h$,

$$\begin{aligned} &\mathsf{P}\left(A \in \mathcal{U}^\lambda \cup \{\omega\},\, A \notin \mathcal{U}^\lambda,\, \omega = \mathcal{U}^h \cap \Gamma_n\right) \\ &\quad = \mathbb{P}_\lambda\left(\omega \text{ pivotal for } A\right) \mathrm{e}^{-hM^a[\{\omega\}]} hM[\{\omega\}] . \end{aligned} \tag{6.3.7}$$

Dividing by h and taking the limit $h \to 0$ yields

$$\begin{aligned} \partial_\lambda \mathbb{P}_\lambda(A) &= \sum_{\omega \in \Gamma_n} M[\{\omega\}] \mathbb{P}_\lambda\left(\omega \text{ pivotal for } A\right) \\ &= \sum_{\omega \in \Gamma} M[\{\omega\}] \mathbb{P}_\lambda\left(\omega \text{ pivotal for } A\right) . \end{aligned} \tag{6.3.8}$$

In the last equality we used that for ω to be pivotal, it has to hold that $\omega \in \Gamma_n$. □

We recall the renormalization parameter

$$\lambda_r = \inf\{\lambda \geq 0\colon \lim_{n\to\infty} \mathbb{P}_\lambda\left(\partial \boldsymbol{B}_n \longleftrightarrow \boldsymbol{B}_{2n}\right) = 1\} \leq \lambda_c\,. \tag{6.3.9}$$

6.3.1 Estimating influence

In this subsection we quantify the influence of re-sampling a coordinate. We begin with a lower bound on connectivity. The following lemma is given in [DCRT18] for the Poisson-Boolean percolation in $\mathbb{R}^d$. The proof is similar, we adapt it here. We say that $A \overset{Z}{\longleftrightarrow} B$ if A is connected to B through loops which are contained inside Z.

Lemma 6.3.4. *Let $d \geq 2$. We then have for every $\lambda > \lambda_r$ and $x \in \partial \boldsymbol{B}_n$*

$$\mathbb{P}_\lambda\left(0 \overset{\boldsymbol{B}_n}{\longleftrightarrow} x\right) \geq \frac{C}{n^{2d-2}}\,. \tag{6.3.10}$$

Proof of Lemma 6.3.4. We begin by noting that

$$\mathbb{P}_\lambda\left(\partial \boldsymbol{B}_n \longleftrightarrow \partial \boldsymbol{B}_{2n}\right) \leq C n^{d-1} \mathbb{P}_\lambda\left(0 \longleftrightarrow \boldsymbol{B}_n\right)\,, \tag{6.3.11}$$

by the union bound. Since we have that $\lambda > \lambda_r$, we get that for some $C_o > 0$ we have that

$$\mathbb{P}_\lambda\left(0 \longleftrightarrow \boldsymbol{B}_n\right) \geq \frac{C_o}{n^{d-1}}\,. \tag{6.3.12}$$

Define $Y = \partial_i \boldsymbol{B}_n$ and the finite set $Z = \boldsymbol{B}_n \setminus Y$. Note that if 0 is connected to $\partial \boldsymbol{B}_n$ then either we have that for one $z \in Z$ that z is connected to 0 in $\boldsymbol{B}_n$ or that there exists a $y \in Y$ such that $A(y)$ occurs with

$$A(y) = \{0 \overset{\boldsymbol{B}_n}{\longleftrightarrow} y\} \cap \{\exists \omega \text{ intersecting both } y \text{ and } \partial \boldsymbol{B}_n\}\,. \tag{6.3.13}$$

By independence and the estimates from Proposition 4.2.2, we have for $\bar{n} = n - |y|$

$$\mathbb{P}_\lambda(A(y)) \leq c \bar{n}^{4+2\nu-d} \mathbb{P}_\lambda(0 \overset{\boldsymbol{B}_n}{\longleftrightarrow} y)\,, \tag{6.3.14}$$

by the independence of the PPP. Note that by the union bound we have

$$\sum_{z\in Z} \mathbb{P}_\lambda\left(0 \overset{\boldsymbol{B}_n}{\longleftrightarrow} z\right) + \sum_{y \in Y} \mathbb{P}_\lambda\left(A(y)\right) \geq \mathbb{P}_\lambda\left(0 \longleftrightarrow \boldsymbol{B}_n\right) \geq \frac{C_o}{n^{d-1}}\,. \tag{6.3.15}$$

Note that we have by the FKG inequality that

$$\begin{aligned}\mathbb{P}_\lambda\left(0 \xleftrightarrow{\boldsymbol{B}_n} x\right) &\geq \mathbb{P}_\lambda\left(0 \xleftrightarrow{\boldsymbol{B}_n} y\right)\mathbb{P}_\lambda\left(y \xleftrightarrow{\boldsymbol{B}_{\bar{n}}(y)} x\right) \\ &\geq C\bar{n}^{d-4-2\nu}\mathbb{P}_\lambda(A(y))\mathbb{P}_\lambda\left(y \xleftrightarrow{\boldsymbol{B}_{\bar{n}}(y)} x\right)\end{aligned} \tag{6.3.16}$$

If we assume that $\mathbb{P}_\lambda\left(0 \xleftrightarrow{\boldsymbol{B}_n} x\right) \geq Cn^{2-2d}$, one can use induction, as Equation (6.3.16) reduces the question from $x \in \partial\boldsymbol{B}_n$ to $x \in \partial\boldsymbol{B}_{n-r}$. One readily checks that assuming $a_j \sim j^\nu$ and $2\nu \leq -d-3$ ensures the success of the inductive step. Indeed, combining the above equations we have

$$U(n) \geq \frac{C}{C_1 n^{2d-2} + C_2\sum_{y\in Y}\frac{n^{d-1}}{U(n-|y|)(n-|y|)^{d-4-2\nu}}}\,, \tag{6.3.17}$$

where $U(n) = \sup_{x\in\partial\boldsymbol{B}_n}\mathbb{P}_\lambda\left(0 \xleftrightarrow{\boldsymbol{B}_n} x\right)$. The condition imposed on ν ensures that the sum over y (in the above equation) does not grow faster than n^{2d-2}. If $2\nu \geq -d-3$, note that the loop process can be written as a sum of two loop soups, one with weight j^{-d-3} and the other with weight $a_j - j^{-d-3}$. Since the event in question is increasing and we are seeking a lower bound, the sum of the two processes fulfils the inequality in question. This concludes the proof. □

Write $\mathcal{C}_0$ for the points connected to 0 and $\mathcal{C}_n$ for those connected to $\partial\boldsymbol{B}_n$. Fix $n, m \geq 1$ and $x \in \mathbb{Z}^d$ and define the event

$$\mathcal{P}_x(m) = \{\mathcal{C}_0 \cap \boldsymbol{B}_m(x) \neq \varnothing\} \cap \{\boldsymbol{B}_m(x) \longleftrightarrow \boldsymbol{B}_n^c\} \cap \{0 \longleftrightarrow \boldsymbol{B}_n^c\}^c\,. \tag{6.3.18}$$

The following lemma is proven in [DCRT18] for the Poisson-Boolean case and can be adapt easily to the setting of loop percolation.

Lemma 6.3.5. ***[DCRT18]*** *For some constant $C > 0$ we have that*

$$\mathbb{P}_\lambda\left(\mathcal{P}_x(m) \text{ and } \operatorname{dist}(\mathcal{C}_0 \cap \boldsymbol{B}_{3m}(x), \mathcal{C}_n) < 2\right) \geq \frac{C}{m^{3d-2}}\mathbb{P}_\lambda\left(\mathcal{P}_x(m)\right)\,. \tag{6.3.19}$$

For an increasing event A, we define the random variable $\texttt{Piv}_{x,A}$ in the following way

$$\texttt{Piv}_{x,A}(\mathcal{U}) = \mathbb{1}\{\mathcal{U} \notin A\}\sum_{\omega\colon\,\omega(0)=x}\mathbb{1}\{\mathcal{U}\cup\omega \in A\}M[\omega]\,. \tag{6.3.20}$$

The next lemma is an important result and is a consequence of Lemma 6.3.5. It is also given in [DCRT18] for the Poisson-Boolean case. We give its proof for the sake of completeness. Let $M_{||}$ be the distribution of $\omega \mapsto \|\omega\|$ under M. We recall that $\|\omega\|$ is the maximal distance between any two points in the loop.

Lemma 6.3.6. ***[DCRT18]*** *We have for some $C > 0$ that for every $m, n \geq 1$ and every $\lambda > \lambda_r$*

$$\sum_{x \in \mathbb{Z}^d} \boldsymbol{\mathtt{Inf}}_{(x,m)}(f_n) \leq C m^{4d-2} M_{||}[m-1, m] \sum_{x \in \mathbb{Z}^d} \mathbb{E}_\lambda \left[\boldsymbol{\mathtt{Piv}}_{x,A} \right] . \tag{6.3.21}$$

Proof of Lemma 6.3.6. We firstly note that

$$\mathtt{Inf}_{(x,m)}(f_n) \leq \lambda M_{||}[m-1, m] \mathbb{P}_\lambda \left(\mathcal{P}_x(m) \right) , \tag{6.3.22}$$

as $\mathcal{P}_x(m)$ has to occur and we need to have at least one loop connecting x to $\boldsymbol{B}_m(x)$. Note that if $\operatorname{dist}(\mathcal{C}_0 \cap \boldsymbol{B}_{3m}(x), \mathcal{C}_n) < 2$, we have to have at least one y with $|y - x| \leq 4m$ such that $\mathbb{P}_y(1)$ occurs. By the union bound together with Lemma 6.3.5, this implies that

$$\mathbb{P}_\lambda \left(\mathcal{P}_x(m) \right) \leq C m^{4d-2} \mathbb{P}_\lambda \left(\mathcal{P}_y(1) \right) , \tag{6.3.23}$$

for such y's. As we have $\mathbb{P}_\lambda \left(\mathcal{P}_y(1) \right) \leq C \mathbb{E}_\lambda \left[\mathtt{Piv}_{y,A} \right]$, this finishes the proof. $\square$

6.3.2 Proving sharpness

The main result of this section is the following theorem.

Theorem 6.3.7. *Consider loop percolation induced by the loop measure with weight sequence $(a_j)_j$. Supposed that $a_j \leq C j^\nu$, for some $C > 0$.*
We have that for $d \geq 3$ and $\nu < -2d - 1/2$ that

$$\lambda_r = \inf\{\lambda > 0\colon \liminf_{n \to \infty} \mathbb{P}_\lambda \left(\partial \boldsymbol{B}_n \longleftrightarrow \partial \boldsymbol{B}_{2n} \right) = 1\} = \lambda_c \,. \tag{6.3.24}$$

This implies (together with Theorem 6.2.4) that under the above conditions $\lambda_c = \lambda_\# = \lambda_r$. It also implies that the estimates in Proposition 6.2.1 and Proposition 6.2.2 hold for the entire subcritical regime $\lambda \leq \lambda_c$.

Proof of Theorem 6.3.7. At first, we restrict our probability space. Let $\alpha = -4d(2 - d/2 + \nu)$ and $L > 2n^{\alpha} > 0$ and set $A = \{0 \longleftrightarrow \boldsymbol{B}_n^c\}$ and $f = \mathbb{1}\{A\}$. Define the space of restricted coordinates I_L in the following way

$$S_L = \{\omega \in \Gamma \colon |\omega(0)| + \|\omega\| \leq L\}\,. \tag{6.3.25}$$

Let $g = \Gamma \setminus S_L$ and let $\mathcal{U}_g$ be the PPP restricted to loops in g. Denote π_g the law of $\mathcal{U}_g$ and let, for (x, m), $\pi_{x,m}$ be the law of $\mathbb{P}_\lambda$ restricted to loops with $\omega(0) = x$ and $\|\omega\| \in [m-1, m)$. Denote the space of such loops by $\Gamma_{x,m}$ and $\mathcal{U}_{x,m}$ the restricted PPP. Let I_L be those (x, m) such that $\Gamma_{x,m} \subset S_L$. Write then $\Omega = (g, \pi_g) \times \bigtimes_{(x,m)}(\Gamma_{x,m}, \pi_{x,m})$. Write $I = \{g\} \cup I_L$.
In order to apply the OSSS-inequality, we need to choose an appropriate algorithm $\mathtt{T}_L$. Fix an arbitrary ordering of I. Set $i_0 = g$ and reveal $\mathcal{U}_g$. Suppose that $\{i_0, \ldots, i_{t-1}\} \subset I$ have been revealed, and denote $\mathcal{C}_t^s$ the connected components formed by $\cup_{l=0}^{t-1}\mathcal{U}_{i_l}$ intersecting $\partial \boldsymbol{B}_s$. The algorithm $\mathtt{T}_L$ then takes one of the two following steps:

I. If there exists $(x, m) \in I \setminus \{i_0, \ldots, i_{t-1}\}$ with the distance between x and $\mathcal{C}_t^s$ being less than m, reveal the first (x, m) in the ordering which fulfils that. Set $i_t = (x, m)$.

II. Halt the algorithm if such (x, m) does not exist.

For an illustration of a configuration explored after the above algorithm terminates, see Figure 6.1 . By Theorem 6.3.1 we then have that

$$\mathbb{P}_\lambda(A) - \mathbb{P}_\lambda(A)^2 = \theta_n(1 - \theta_n) \leq 2 \sum_{i \in I} \delta_i(\mathtt{T}_L)\mathtt{Inf}_i(f)\,, \tag{6.3.26}$$

where $\theta_n = \mathbb{P}_\lambda(A)$. We begin by bounding the influence of the coordinate g. From the choice of $L > 0$, it follows that

$$\begin{aligned}\mathtt{Inf}_g(f) &\leq \mathbb{P}_\lambda\left(\exists \omega \colon \|\omega\| > L \text{ and } \omega \cap \boldsymbol{B}_n \neq \varnothing\right)\\ &\leq C \sum_{|x| \geq L} \sum_{j \geq 1} a_j \mathbb{E}_{x,x}^j\left[H_n < j\right] \leq C n^{d-1} \sum_{j \geq L^{3/2}} a_j j^{1+d/2} = o(1)\,,\end{aligned} \tag{6.3.27}$$

due to the conditions placed on $(a_j)_j$ and L and using similar reasoning to the

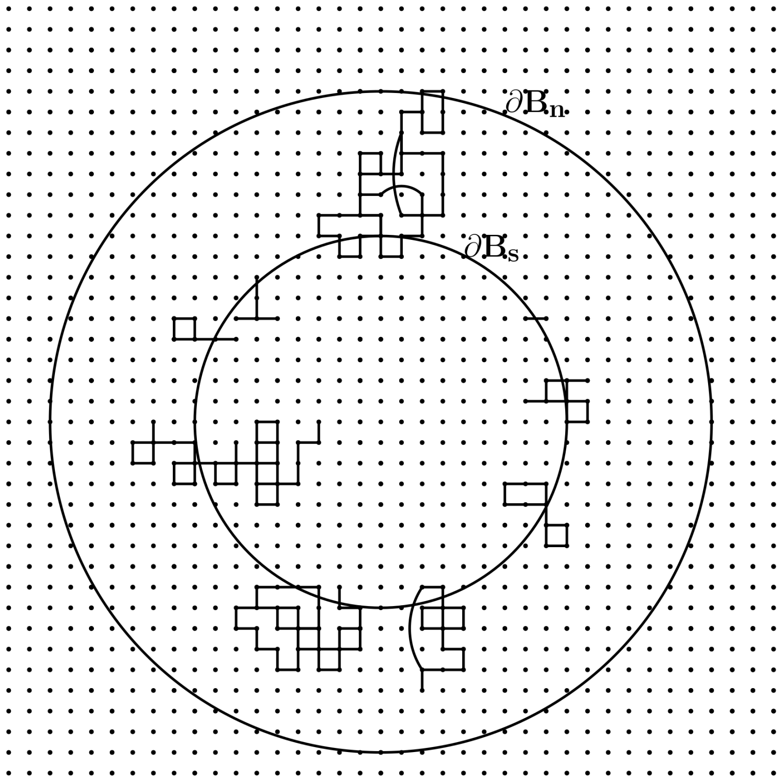

Figure 6.1: The loops connected to $\partial \boldsymbol{B}_s$ have been revealed by $\mathtt{T}_L$. In this instance the origin is not connected to $\boldsymbol{B}_n^c$

proof of Proposition 4.2.2. We can also bound

$$\delta_i(\mathtt{T}_L) \leq \mathbb{P}_\lambda \left(\boldsymbol{B}_m(x) \longleftrightarrow \boldsymbol{B}_n^c \right) . \tag{6.3.28}$$

By letting $L \to \infty$, we thus obtain

$$\theta_n(1-\theta_n) \leq 2 \sum_{\substack{x \in \mathbb{Z}^d \\ m \geq 1}} \mathbb{P}_\lambda \left(\boldsymbol{B}_m(x) \longleftrightarrow \boldsymbol{B}_n^c \right) \mathtt{Inf}_{(x,m)}(f) \,. \tag{6.3.29}$$

We then estimate

$$\sum_{s=1}^{n-1} \mathbb{P}_\lambda \left(\boldsymbol{B}_m(x) \longleftrightarrow \boldsymbol{B}_n^c \right) \leq C m^{d-1} \sum_{s=1}^{n-1} \theta_s \,, \tag{6.3.30}$$

by the union bound. Using the above equation, we obtain

$$\frac{n\theta_n(1-\theta_n)}{\sum_{s=1}^{n-1}\theta_s} \le 2C \sum_{\substack{x\in\mathbb{Z}^d\\ m\ge 1}} m^{d-1}\mathtt{Inf}_{(x,m)}(f)\,. \tag{6.3.31}$$

Assume that $\lambda > \lambda_r$. We use Lemma 6.3.6 and bound

$$\sum_{\substack{x\in\mathbb{Z}^d\\ m\ge 1}} m^{d-1}\mathtt{Inf}_{(x,m)}(f) \le C \sum_{\substack{x\in\mathbb{Z}^d\\ m\ge 1}} m^{5d-3} M_{||}[m-1,m]\mathbb{E}_\lambda[\mathtt{Piv}_{x,A}]\,. \tag{6.3.32}$$

Applying Proposition 4.2.2, integration by parts and the assumption on ν, we get that

$$\sum_{m\ge 1} m^{5d-3} M_{||}[m-1,m] \le \sum_{m\ge 1} m^{5d-4+4-d+2\nu} = C < \infty\,. \tag{6.3.33}$$

Plugging that into the above equation, we get that

$$\frac{n\theta_n(1-\theta_n)}{\sum_{s=1}^{n-1}\theta_s} \le C \sum_{x\in\mathbb{Z}^d} \mathbb{E}_\lambda[\mathtt{Piv}_{x,A}] \le C\partial\theta_n\,, \tag{6.3.34}$$

where in the last step we use Lemma 6.3.3.
Fix $\lambda_o > \lambda_r$. By Lemma 6.3.2 there exists a $\beta_1 \ge \lambda_o$ such that for $\lambda < \beta_1$, we have exponential decay of connectivity and for $\lambda > \beta_1$ we have a positive bound from below. This implies that $\beta_1 = \lambda_c$. On the other hand, since the decay of connectivity is exponentially fast, this implies that $\beta_1 \ge \lambda_r$ and thus $\beta_1 = \lambda_r = \lambda_c$. □

Remark 6.3.8. *Like the results in [DCRT18], our results put some moment conditions on the decay of the connectivity and do not cover all non-trivial weights. We conjecture that by refining the estimation of the influence and using different algorithms, one can allow for a wider range of sequences* $(a_j)_j$.

6.4 Finer properties of the subcritical phase

We now turn our attention to the structure of the loop soup for $\lambda > 0$ small.

The following bound is predicted in [CS16] and we give a proof here:

Lemma 6.4.1. *Let $\lambda < \lambda_{\#}$, $d \geq 3$ and $a_j \sim j^{\nu}$. We then have that*

$$\mathbb{P}_\lambda \left(0 \longleftrightarrow \boldsymbol{B}_n \text{ through loops of diameter at most } m\right) = \mathcal{O}\left(\mathrm{e}^{-cn/m}\right) . \quad (6.4.1)$$

Proof of Lemma 6.4.1. We show that

$$\mathbb{P}_\lambda \left(0 \longleftrightarrow \boldsymbol{B}_n \text{ through loops of diameter at most } m\right) = \mathcal{O}\left(\mathrm{e}^{-c\lfloor n/m \rfloor}\right) , \quad (6.4.2)$$

and prove the result via induction over $\lfloor n/m \rfloor$. Note that this result implies the lemma. Let $n = mk + r$ with $k, n, m \in \mathbb{N}$ and $0 \leq r < m$. Let A_m be the event that loops of diameter at most m are used to facilitate the connectivity. Then

$$\begin{aligned} &\mathbb{P}_\lambda \left(0 \longleftrightarrow \boldsymbol{B}_{km},\, A_m\right) \\ &\leq \mathbb{P}_\lambda \left(0 \overset{\boldsymbol{B}_{2k}}{\longleftrightarrow} \boldsymbol{B}_m,\, A_m,\, \exists x \in \partial \boldsymbol{B}_{2k} \cap \mathcal{C}_0 \colon x \overset{\mathcal{U}\setminus \boldsymbol{B}_{2k}}{\longleftrightarrow} \partial \boldsymbol{B}_{kn}\right) \\ &\leq \mathbb{P}_\lambda \left(0 \longleftrightarrow \boldsymbol{B}_m,\, A_m\right) \mathbb{E}_\lambda \left[\mathcal{C}_0\right] \mathbb{P}_\lambda \left(0 \longleftrightarrow \boldsymbol{B}_{(k-2)m},\, A_m\right) , \end{aligned} \quad (6.4.3)$$

where we use the independence of the loops which are contained in $\boldsymbol{B}_{2k}$ and those which are not. To go from the penultimate to the last line, we conditioned on $x \in \mathcal{C}_0$. For $n = m$, we have that

$$\mathbb{P}_\lambda \left(0 \longleftrightarrow \boldsymbol{B}_m,\, A_m\right) = \mathcal{O}\left(m^{2-d}\right) , \quad (6.4.4)$$

by Proposition 6.2.2. This finishes the proof of the lemma. □

Note that in the above proof, Equation (6.4.4) implies that $\mathcal{O}\left(\mathrm{e}^{-cn/m}\right)$ can be replaced by the stronger $\mathcal{O}\left(\mathrm{e}^{-c(n/m)\log m}\right)$.
The next bound had also been predicted in [CS16] for the Markovian case. We give a general proof here.

Proposition 6.4.2. *Given that $\lambda < \lambda_{\#}$, $d \geq 3$ and $a_j \sim j^{\nu}$. We then have that*

$$\mathbb{P}_\lambda(\mathcal{C}_0 \text{ contains at least two loops of diameter bigger than } m) = \mathcal{O}\left(m^{10-4\nu-2d}\right). \quad (6.4.5)$$

Proof of Proposition 6.4.2. Let A_m the event that $\mathcal{C}_0$ contains at least two loops of diameter bigger than m. Let $\mathcal{C}_0^{<m}$ be the open cluster formed by loops of length less than m. We have two possibilities: either there exists two or more loops contained in $\mathcal{C}_0^{<m}$ or the $\mathcal{C}_0^{<m}$ intersects only one loop of length greater than m. Define $\mathcal{C}_+^m$ the sub-cluster of $\mathcal{C}_0$ formed in the following way: in the first scenario described above, take the cluster formed by loops of length less than m and ω_0, where ω_0 is the first loop of diameter bigger than m (in some arbitrary ordering on Γ). In the second scenario, take the cluster formed by loops of diameter less than m together with ω_0, where ω_0 is the unique loop intersecting $\mathcal{C}_0^{<m}$. We show $\mathbb{E}_\lambda\left[\mathcal{C}_+^m\right] \leq \mathbb{E}_\lambda[\mathcal{C}_0]\mathbb{E}_\lambda\left[\mathcal{R}(\omega_0)\right]$. The intuition

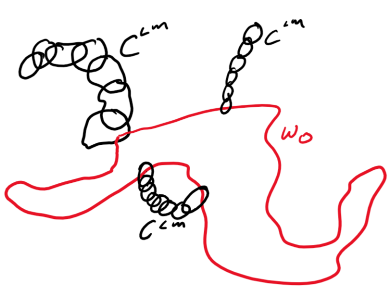

Figure 6.2: The long loop in red, together with 3 clusters of small loops attached to it.

is as follows: if ω_0 contains j points, the maximum size of the cluster $\mathcal{C}_+^m$ is bounded by j-times the size of the cluster $\mathcal{C}_0^{<m}$ as we can attach at most one "version" of $\mathcal{C}_0^{<m}$ to each point in ω_0, see Figure 6.2. Expand

$$\begin{aligned}\mathbb{E}_\lambda\left[\mathcal{C}_+^m\right] &= \sum_{j\geq m}\sum_{x\in\mathbb{Z}^d}\mathbb{E}_\lambda\left[\mathcal{C}_+^m,\, \mathcal{R}(\omega_0)=j,\, \omega_0(0)=x\right]\\ &\leq \sum_{j\geq m}\sum_{x\in\mathbb{Z}^d}\mathbb{E}_\lambda\left[j\mathcal{C}_0^{<m},\, \mathcal{R}(\omega_0)=j,\, \omega_0(0)=x\right] \qquad (6.4.6)\\ &\leq \mathbb{E}_\lambda[\mathcal{C}_0^{<m}]\mathbb{E}_\lambda\left[\mathcal{R}(\omega_0)\right]\,,\end{aligned}$$

where we use that the loops which form $\mathcal{C}_0^{<m}$ are independent from loops with diameter greater than m.

Note that by the same reasoning used in the proof of Proposition 4.2.2

$$\mathbb{E}_\lambda\left[\mathcal{R}(\omega_0)\right] = \sum_{j\geq 1} a_j \mathbb{E}_{0,0}^j\left[\mathcal{R}_j^2\right] \sim m^{6-d+2\nu}\,. \qquad (6.4.7)$$

By the same reasoning to above, we can bound

$$\mathbb{P}_\lambda\left(A_m\right) \leq \mathbb{E}_\lambda\left[\mathcal{C}_+^m\right] \mathbb{P}_\lambda\left(\exists \omega_1 \colon 0 \in \omega_1 \text{ and } \omega_1 \cap \boldsymbol{B}_m^c \neq \varnothing\right) = \mathcal{O}\left(m^{10-4\nu-2d}\right), \tag{6.4.8}$$

where we bound $\mathbb{P}_\lambda\left(\exists \omega_1 \colon 0 \in \omega_1 \text{ and } \omega_1 \cap \boldsymbol{B}_m^c \neq \varnothing\right)$ by $\mathcal{O}\left(m^{4+2\nu-d}\right)$ using Proposition 4.2.2. This concludes the proof. □

Chapter 7

Outlook

In this chapter we briefly comment on continuations of the results proved in this thesis.

7.1 Sharpness

The sharpness result in Theorem 6.3.7 is limited by the decay assumption $a_j \leq Cj^{-2d-1/2}$ for some $C > 0$. We believe that this is a purely technical assumption and that one can show in general that $\lambda_r = \lambda_c$ using our method and thus answering a question posed in [CS16] for the Markovian case. The route to such a result will probably use different algorithms for the OSSS inequality and refined intersection estimates for random loops. Indeed, most proofs for loop percolation (so far) usually do not involve classical random walk intersection estimates (compiled in the classic reference [Law13]).
We compare loop percolation to the Poisson-Boolean model: in the Poisson-Boolean model each realisation consists of a collection of spheres. This means that the "base" element (a sphere) has volume of the same order as the space. In loop percolation this is not the case: a loop of diameter n consists of only $n^{2+o(1)}$ points (with overwhelming probability under $\mathbb{P}_\lambda$). Since we "know" (strictly speaking only for $\lambda < \lambda_\#$) that large clusters $\mathcal{C}_0$ typically have one large loop, we can argue $\mathbb{P}\left(|\mathcal{C}_0| = n^{2+o(1)}|0 \longleftrightarrow \boldsymbol{B}_n^c\right) = 1 - o(1)$. If we want to use a renormalisation approach, we need to factor in that if we want to connect the loops contained in $\boldsymbol{B}_{2n}$ (denoted by $\mathcal{C}^{2n}$) with a single loop to $\boldsymbol{B}_{4n}^c$, it is very unlikely that this loop intersects $\mathcal{C}^{2n}$. Using similar computations as done

in Chapter 4, we can find bounds such as

$$\mathbb{P}_{o,x}^{j_1,j_2}(\text{int}) \leq C \int_{\mathbb{R}^d} |y|^{2-d}|y-x|^{2-d}\Gamma\left(\frac{d-2}{2}, |y|^2/j_1\right)\Gamma\left(\frac{d-2}{2}, |y-x|^2/j_2\right)\mathrm{d}y, \tag{7.1.1}$$

where $\mathbb{P}_{o,x}^{j_1,j_2}(\text{int})$ is the probability that two independent random walk bridges of length j_1 and j_2 intersect (where the first loop is started at the origin and the second one at $x \in \mathbb{Z}^d$). One can then use the properties of the Poisson process to estimate intersection probabilities of the loop soup. We believe that the above strategy is key to reducing the moment conditions from Chapter 6.

7.2 One-arm domination in two dimensions

For $d = 2$ and a_j sufficiently fast decaying we make the following conjecture: for any $\lambda < \lambda_c$, we have that

$$\lim_{n\to\infty} \frac{\lambda M[0 \stackrel{\omega}{\longleftrightarrow} \boldsymbol{B}_N^c]}{\mathbb{P}_\lambda(0 \longleftrightarrow \boldsymbol{B}_N^c)} = 1\,. \tag{7.2.1}$$

This kind of result is known as *one-arm* domination. It is proven in [DCRT18] for the Poisson-Boolean case for $d \geq 2$. In loop percolation, it can only hold for $d \leq 2$ as the random walk is transient for $d \geq 3$. Indeed, transience implies that we could connect 0 to e_1 (the point $(1, 0\ldots, 0)$) and e_1 to the boundary of $\boldsymbol{B}_n$ through a loop which avoids the origin and have a comparable cost to connecting 0 to $\boldsymbol{B}_n^c$ through a single loop.
To prove one-arm domination result in the planar case, it would be vital to have a stronger version of Theorem 4.3.1 for $d = 2$. The large deviation bounds for range process from [BCR09,LV19] together with the explicit bounds from [BMR13] can be used to prove Theorem 4.3.1 for $d = 2$ for $\nu < -1$ sufficiently negative. Indeed, our method does not need to be adapted for that. To prove a one-arm domination result, we could strengthen the result by showing

$$M[\boldsymbol{B}_N \stackrel{\omega}{\longleftrightarrow} \boldsymbol{B}_{\gamma N}^c] = f(\gamma, N)N^{2\nu-2}\left(1 + o(1)\right)\,, \tag{7.2.2}$$

with $f(\gamma, N)$ converging as $N \to \infty$ uniformly in γ. Indeed, this would allow us to utilise the strategy from the proof of [DCRT18, Theorem 2] to show one arm domination. This seems out of reach, as the upper and lower bounds in Theorem 4.3.1 differ substantially. The aim would therefore be to exploit the recurrence of the planar random walk to show that

$$M[\boldsymbol{B}_N \stackrel{\omega}{\longleftrightarrow} \boldsymbol{B}^c_{\gamma N},\, 0 \notin \omega] \sim N^{2\nu-2} f(\gamma)\,, \qquad (7.2.3)$$

where $f(\gamma)$ goes to zero as $N \to \infty$. Equipped with such an estimate, the proof of one-arm domination would follow rather quickly.

7.3 The disordered loop soup

Given the results on the correlation function in Chapter 5, we could study the *loop soup with disorder*. We briefly explain the setting of disordered models, following the notes [CSZ16]. Given a statistical mechanics model with law $\mathbb{P}^r_N$ on some domain Ω_N, governing the behaviour of a family of spins $(\sigma_x)_{x\in\Omega_N}$ with $\sigma_x \in \{0,1\}$. We assume that $\Omega_N = (N^{-1}\mathbb{Z})^d$ for a continuum domain $\Omega \subset \mathbb{R}^d$ as $N \to \infty$. We model disorder by a family of i.i.d. centred random variables $(\omega_x)_{x\in\Omega_N}$.Given two parameters, $\beta > 0$ and $h \in \mathbb{R}$, we define the *disordered model* $\mathbb{P}^\omega_{\beta,h}$ through its Radon–Nikodym derivative

$$\mathrm{d}\mathbb{P}^\omega_{\beta,h} = \frac{\exp\left(-\sum_{x\in\Omega_N} (\beta\omega_x + h)\,\sigma_x\right)}{Z^\omega_{N,\beta,h}} \mathrm{d}\mathbb{P}^r_N(\sigma)\,. \qquad (7.3.1)$$

Some examples of disordered models are the disordered pinning model (see e.g. [DGLT09]), the directed polymer model (see e.g. [CSY04]) and the random field Ising model (see e.g. [CSZ17]). It is important to know whether the model is disorder relevant or not, i.e. does an arbitrarily small amount of disorder change the statistical properties of the model. Harris in [Har74] proposed the following criterion: let $\gamma > 0$ be the correlation length (i.e. the correlation functions of order k of $\mathbb{P}^r_N$ scale like $N^{-\gamma k}$). Then the model is disorder irrelevant if $\gamma < d/2$ and disorder relevant for $\gamma > d/2$. In [CSZ17], the authors give a different viewpoint on disorder relevance: does there exist $\beta_N, h_N \downarrow 0$ such that the limit of $Z^\omega_{N,\beta,h}$ converges to a non-constant random variable? If the answer is yes, then the model is disorder irrelevant. If any scaling of $\beta_N, h_N \downarrow$ leads to a trivial limit, the model is disorder relevant. One key

advantage of that method is that the existence of the scaling limit of the correlation functions (pointwise and in L^2) suffice (together with some uniform, in N, bounds for k large). See [CSZ16, Assumption 1.1] for a precise statement. One can study the disordered loop soup model by making the identification $\sigma_x = \mathbb{1}\{x \in \mathcal{U}\}$. At least for $d \geq 3$ and $a_j = j^{-\nu}(1 + o(1))$ (thus including the Bosonic and Markovian case) the required bounds follow immediately from Proposition 5.2.1. As the framework in [CSZ16] assumes the finiteness of Ω, one could study the disordered loop soup on the continuum torus first (making small adjustments in Proposition 5.2.1), before extending the disorder to the whole space. As the correlation length is $d - 2$ (compare Equation (5.2.12)), Harris criterion would predict that $d < 4$ is disorder irrelevant and $d > 4$ is disorder relevant. This shows that $d = 4$ is the critical dimension. Only small gaps need to be filled for computing the scaling limit of $Z^{\omega}_{N,\beta,h}$ for the loop soup and we will close them in a forthcoming publication.
Note that besides a criterion to classify order/disorder relevance, the existence of the continuum limit allows for statements on the free energy of the system. This can be used to make statements regarding localisation/delocalisation transitions. Using Proposition 5.3.1 on the asymptotic behaviour of the occupation field, we could extend the study to disorder on non-compact state spaces.

Chapter 8

Appendix

8.1 Bounds for connecting annuli

The next proposition summarises the bounds from Theorem 4.3.1. Given some sequence $(a_j)_j$, one can use these bounds to compute asymptotics of connectivity.

Proposition 8.1.1. *Let $\boldsymbol{\gamma}_0 = \gamma_0 - 1$. Let $c_1 > 0$ arbitrary but fixed. Let $n_1 = n^2/c_1 \log(n)$. The following bounds hold for all $j \geq n_1$, $\boldsymbol{\gamma} > \boldsymbol{\gamma}_0$:*
Lower bound*: we have that for some $C > 0$*

$$\mathbb{E}^j_{y,y}\left[\mathbb{1}\{H_{\gamma n} < j\}\,\mathcal{R}_{H_n(y)}\right] \geq C j^{1/2} \mathrm{e}^{-C(\boldsymbol{\gamma} n)^2/j} \mathfrak{p}_j(0)\,. \tag{8.1.1}$$

Upper bounds*: fix $0 < \beta_0 < \beta_1 < \boldsymbol{\gamma}$. We then have that*

$$\mathbb{E}^j_{y,y}\left[\mathbb{1}\{H_{\gamma n} < j\}\,\mathcal{R}_{H_n(y)}\right] \leq C \begin{cases} \left(\frac{n}{(j-\beta_0 n^2)^{d/2}} + j^{1/2-d/2} + \frac{1}{n^{d-2}(j-\beta_1 n^2)^{1/2}}\right) & , \\ \left(j^{1/2} + (\boldsymbol{\gamma} n)^2 j^{-1/2}\right) \mathrm{e}^{-C(\boldsymbol{\gamma} n)^2/j} + \sum_{k=j/2}^{j} \sum_{l=n_1}^{(k-n_1)\wedge n_1} b(n,j,k,l) & , \end{cases} \tag{8.1.2}$$

where the first bound is for $j \geq (\boldsymbol{\gamma} n)^2$ and the second for $j \in [n_1, (\boldsymbol{\gamma} n)^2]$.

Furthermore, $b(n,j,k,l) =$

$$\frac{(\gamma n)^2 k\,\gamma\left((d-1)/2, n^2/(j-k)\right)}{e^{C(\gamma n)^2/l} e^{C(\gamma n)^2/(k-l)} l^2}\left(\frac{n^{2-d}}{\gamma\sqrt{j-k}(k-l)^{3/2}} + \frac{(\gamma n)}{(k-l)^{1+d/2}\sqrt{j-k}}\right). \tag{8.1.3}$$

The above proposition follows by collecting the bounds from the proof of Theorem 4.3.1.

8.2 Sum and integral techniques

In this section we collect various ways of approximating integrals by finite sums.

Lemma 8.2.1. *Given* $f\colon [a,b] \to (0,\infty)$*,* $a,b \in \mathbb{N} \cup \{\pm\infty\}$ *measurable and*

$$\sup_{k\in[a,b]\cap\mathbb{N}} \sup_{r\in[0,1]} \frac{f(k+r)}{f(k)} < \infty \text{ and } \inf_{k\in[a,b]\cap\mathbb{N}} \inf_{r\in[0,1]} \frac{f(k+r)}{f(k)} > 0\,, \tag{8.2.1}$$

then there exists $C > 1$ *with*

$$C^{-1}\sum_{k=a}^{b-1} f(k) \le \int_a^b f(t)\mathrm{d}t \le C\sum_{k=a}^{b-1} f(k)\,. \tag{8.2.2}$$

Proof of Lemma 8.2.1. We bound

$$\int_a^b f(t)\mathrm{d}t \le \sum_{k=a}^{b-1} f(k)\int_k^{k+1}\frac{f(t)}{f(k)}\mathrm{d}t \le \sum_{k=a}^{b-1} f(k)\sup_{r\in[0,1]}\frac{f(k+r)}{f(k)}\,, \tag{8.2.3}$$

from which the result follows by taking the supremum over all $k \in [a,b]$. The lower bound works analogously.

Lemma 8.2.2. *We have that for* $f(k) = \mathfrak{p}_k(x)$

$$\sup_{k\in[a,b]\cap\mathbb{N}} \sup_{|x|\le k} \sup_{r\in[0,1]} \frac{f(k+r)}{f(k)} < \infty \text{ and } \inf_{k\in[a,b]\cap\mathbb{N}} \inf_{|x|\le k} \inf_{r\in[0,1]} \frac{f(k+r)}{f(k)} > 0\,, \tag{8.2.4}$$

Furthermore

$$\sup_{x\in\mathbb{Z}^d}\sup_{k\geq|x|}\sup_{r\in[0,1]^d}\frac{\mathfrak{p}_k(x+r)}{\mathfrak{p}_k(x)}<\infty \textit{ and } \inf_{x\in\mathbb{Z}^d}\inf_{k\geq|x|}\inf_{r\in[0,1]^d}\frac{\mathfrak{p}_k(x+r)}{\mathfrak{p}_k(x)}>0\,, \quad (8.2.5)$$

Proof of Lemma 8.2.2. Note that

$$\frac{\mathfrak{p}_{k+r}(x)}{\mathfrak{p}_k(x)}=\left(\frac{k}{k+r}\right)^{d/2}\exp\left(-\frac{|x|^2}{2k(k+r)}\right)\,. \quad (8.2.6)$$

Due to the restrictions placed on $|x|$, we can conclude the statement. The second part of the theorem follows analogously by expanding $|x+r|^2=|x|^2+2\langle x,r\rangle+|r|^2$ and using the Cauchy-Schwarz inequality.

Lemma 8.2.3. *For* $a,b\in\mathbb{Z}$ *and* $f,g\colon[a-1,b]\to\mathbb{C}$ *we have that*

$$\sum_{k=a}^{b}f(k)g(k)=\sum_{k=a}^{b-1}F(k)\left[g(k)-g(k+1)\right]+F(a-1)g(a)-F(b)g(b)\,, \quad (8.2.7)$$

where $F(k)=\sum_{l\leq k}f(l)$

Proof of Lemma 8.2.3. We have that

$$\begin{aligned}\sum_{k=a}^{b}f(k)g(k)&=\sum_{k=a}^{b}\left[F(k)-F(k-1)\right]g(k)\\&=\sum_{k=a}^{b}F(k)g(k)-\sum_{k=a-1}^{b-1}F(k)g(k+1)\\&=\sum_{k=a}^{b-1}F(k)\left[g(k)-g(k+1)\right]+F(a-1)g(a)-F(b)g(b)\,.\end{aligned} \quad (8.2.8)$$

8.3 Properties of the Gamma function and Gamma distribution

Let $\mathbb{E}_{\alpha,\beta}$ be the expectation with respect to a Gamma distributed variable with parameter (α,β).

Lemma 8.3.1. *If $\gamma \in \mathbb{R}$ with $\alpha > \gamma$, we have that*

$$\mathbb{E}_{\alpha,\beta}\left[X^{-\gamma}\right] = \frac{\beta^{\gamma}\Gamma(\alpha-\gamma)}{\Gamma(\alpha)}\,. \tag{8.3.1}$$

Lemma 8.3.2. *The moment generating function of a Gamma distributed random variable is given by*

$$\varphi(r) = \mathbb{E}_{\alpha,\beta}\left[\mathrm{e}^{rX}\right] = \left(1-\frac{r}{\beta}\right)^{-\alpha}\mathbb{1}\{r<\beta\} + \infty\mathbb{1}\{r \geq \beta\}\,. \tag{8.3.2}$$

Furthermore, its large deviation rate function satisfies

$$\Lambda(x) = \sup_{r\in\mathbb{R}}\{xr - \log\varphi(r)\} = \begin{cases} \beta x + \alpha\left(\log(\alpha) - 1 - \log(x\beta)\right) & \textit{if } x > 0\,, \\ +\infty & \textit{if } x \leq 0\,. \end{cases} \tag{8.3.3}$$

Proof of Lemma 8.3.2. The first part of the lemma is standard and follows easily from observing that $\varphi(r) = C(r)\mathbb{E}_{\alpha,\beta-r}[1]$ and solving for $C(r)$.
For $r < \beta$, we differentiate $xr - \log\varphi(r)$ to obtain that (given $r < \beta$ and $x > 0$)

$$xr - \log\varphi(r) = \begin{cases} \text{decreasing} & \text{if } r > \beta - \alpha/x\,, \\ \text{increasing} & \text{if } r < \beta - \alpha/x \end{cases} \tag{8.3.4}$$

whereas for $x \leq 0$ it is strictly decreasing. This implies that for $x \leq 0$ that $\Lambda(x) = \infty$, as xr dominates the log term. For $x > 0$, we have that $\beta - \alpha/x < \beta$ and thus we attain a maximum at $\beta - \alpha/x$. Plugging that value back into the definition of $\Lambda(x)$, we obtain the result.

We also include the following asymptotics of the incomplete Gamma function. As they are easy to derive, we omit the proof.

Lemma 8.3.3. *We have*

$$\lim_{x\to\infty}\frac{\Gamma(s,x)}{x^{s-1}\mathrm{e}^{-x}} = 1 \qquad \textit{and} \qquad \lim_{x\downarrow 0}\frac{\gamma(s,x)}{x^{s}} = 1\,. \tag{8.3.5}$$

8.4 The topology of local convergence

This section relies heavily on [Geo88, Definition 4.2]. Let $(E, \tau(E))$ be a locally compact Polish space (e.g. $\mathbb{R}^m$, $\mathbb{Z}^n$ or the real half-line).

Definition 8.4.1. *Denote $\varphi \in E^{\mathbb{Z}^d} = \Omega$ a field with values in E. Let $p_x\colon E^{\mathbb{Z}^d} \to E$ be the projection which maps φ to $\varphi_x \in E$. Let $\mathcal{F} = \mathcal{E}^{\mathbb{Z}^d}$ be the product sigma algebra on $E^{\mathbb{Z}^d}$. For a finite subset $\Lambda \subset \mathbb{Z}^d$ let $\mathcal{F}_\Lambda$ be the sigma algebra generated by the maps $(p_x)_{x\in\Lambda}$. Let $\mathcal{F}^0$ be the sigma algebra of cylindrical events defined by*

$$\mathcal{F}^0 = \bigcup_{\Lambda\subset\mathbb{Z}^d\colon|\Lambda|<\infty} \mathcal{F}_\Lambda\,. \tag{8.4.1}$$

The topology of local convergence is then the coarsest topology such that the map $\nu \mapsto \nu(A)$ is measurable for all $\nu \in \mathcal{M}_1(\Omega, \mathcal{F})$ and $A \in \mathcal{F}^0$.

We need the following results about the topology of local convergence. We call a function local if it is measurable with respect to $\mathcal{F}_\Lambda$ for some $\Lambda \subset \mathbb{Z}^d$ finite. We call it quasilocal, if it can be approximated by a sequence of local functions in the infinity norm.

Proposition 8.4.2. *I. $\mathcal{M}_1(\Omega, \mathcal{F})$ equipped with the topology of local convergence is Hausdorff.*

II. $\nu_n \to \nu$ in the topology of local convergence if and only if $\nu_n(f) \to \nu(f)$ for all f quasilocal.

III. Let $\{f_m^\Lambda,\, n \in \mathbb{N},\, f_m^\Lambda - \mathcal{F}_\Lambda$ measurable $\}$ be a collection of separating classes for $(E^\Lambda, \mathcal{E}^\Lambda)$. Then $\nu_n \to \nu$ in the topology of local convergence if and only if $\nu_n(f_m^\Lambda) \to \nu(f_m^\Lambda)$ for all m, Λ.

Proof of Proposition 8.4.2. The first two statements follow directly from [Geo88, Remark 4.3]. For the third statement, choose $\Lambda \Subset \mathbb{Z}^d$ and f^Λ-$\mathcal{F}_\Lambda$ measurable such that $|f - f^\Lambda| \le \varepsilon/3$ for some $\varepsilon > 0$. We can write

$$|\nu_n(f) - \nu(f)| \le 2\varepsilon/3 + |\nu_n(f^\Lambda) - \nu(f^\Lambda)|\,. \tag{8.4.2}$$

But as $\nu_n(f_m^\Lambda) \to \nu(f_m^\Lambda)$ and $(f_m^\Lambda)_m$ a separating class for $\mathcal{F}_\Lambda$, we have that for n large enough $|\nu_n(f^\Lambda) - \nu(f^\Lambda)| < \varepsilon/3$. This concludes the proof.

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