On the Orbit Closure Containment Problem and Slice Rank of Tensors

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Abstract

We consider the orbit closure containment problem, which, for a given vector and a group orbit, asks if the vector is contained in the closure of the group orbit. Recently, many algorithmic problems related to orbit closures have proved to be quite useful in giving polynomial time algorithms for special cases of the polynomial identity testing problem and several non-convex optimization problems. Answering a question posed by Wigderson, we show that the algorithmic problem corresponding to the orbit closure containment problem is NP-hard. We show this by establishing a computational equivalence between the solvability of homogeneous quadratic equations and a homogeneous version of the matrix completion problem, while showing that the latter is an instance of the orbit closure containment problem.

Secondly, we consider the notion of slice rank of tensors, which was recently introduced by Tao, and has subsequently been used for breakthroughs in several combinatorial problems like capsets, sunflower free sets, tri-colored sum-free sets, and progression-free sets. We show that the corresponding algorithmic problem, which can also be phrased as a problem about union of orbit closures, is also NP-hard, hence answering an open question by Bürgisser, Garg, Oliveira, Walter, and Wigderson. We show this by using a connection between the slice rank and the size of a minimum vertex cover of a hypergraph revealed by Tao and Sawin.

1 Introduction

1.1 Orbit containment and orbit closure containment problems

The problems related to group orbits have been ubiquitous in mathematics and computer science, both from the perspective of theory and practice. For a group G acting[1] on a vector space V, the orbit of a vector $v \in V$, denoted as Gv, is defined to be the set $\{gv \mid g \in G\}$. That is, the orbit Gv is the set of points that v gets mapped to on the action of G. The group problem that has received the widest attention in computer science is the orbit containment problem.

Problem 1.1. Orbit Containment: *For a group G acting on a vector space V, and given two elements $u, v \in V$ as inputs, decide if $u \in Gv$.*

Thus, it asks if a vector is in the orbit of another vector. This problem is quite general and captures many problems, for instance the graph isomorphism problem and the module isomorphism problem. We can see the graph isomorphism problem as an instance of the orbit containment problem as follows. Suppose we are given two graphs G_1 and G_2 on n vertices each, and we want to know if they are isomorphic to each other. This can be rephrased as whether the adjacency matrix of the graph G_1 is in the orbit of the adjacency matrix of the graph G_2, under the action of the permutation group S_n. Here S_n acts by permuting the rows and columns of the matrix, induced by the permutation of vertices of the graph corresponding to the matrix. Owing to its generality, the orbit containment problem has been very important from the point of view of both algorithm design as well as complexity theory for decades. While the graph isomorphism problem remains one of the central algorithmic problem in graph theory, the module isomorphism problem has been very crucial in cryptography [46, 10, 35, 53, 5, 56].

From the perspective of topology, it is more natural to consider *orbit closures* instead. For a group G acting on a vector space V, the orbit closure of $v \in V$, denoted as $\overline{Gv}$, is defined to be the smallest closed subset of V which contains Gv. In the standard Euclidean topology, this translates to $\overline{Gv}$ being the smallest superset of Gv which contains the limit points of all convergent sequences comprising of elements of Gv. In the Zariski topology, this translates to $\overline{Gv}$ being the smallest superset of Gv which contains all the common zeros of the set of polynomials that vanish on all the elements of Gv. In most of the cases of interest, in particular, when the underlying field is $\mathbb{C}$, the definitions of $\overline{Gv}$ obtained by considering the above two topologies,

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[1]When we say a group G acts on the ambient space S, we have a mapping $\cdot : G \times S \to S$ that satisfies the axioms $1 \cdot s = s$ and $(gh) \cdot s = g \cdot (h \cdot s)$ for all $s \in S$ and $g, h \in G$. Here gh is the group operation.

that is, the Euclidean (or analytic) closure and the Zariski (or algebraic) closure coincide[2]. Thus we can ask the following weakening of the orbit containment problem.

PROBLEM 1.2. ORBIT CLOSURE CONTAINMENT: *For a group G acting on a vector space V, and given two elements $u, v \in V$ as inputs, decide if $u \in \overline{Gv}$.*

This problem is quite general, too, and has appeared centrally in algorithmic and complexity theoretic problems related to algebra and combinatorial optimization, since it captures problems like border rank of tensors, the null cone problem, and the permanent versus determinant problem. As an example, to see the border rank problem as an orbit closure containment problem, let GL_n denote the group of all invertible $n \times n$ matrices. GL_n acts on $\mathbb{F}^n$ by the usual matrix-vector multiplication. $G_n := \mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$ acts on rank-one tensors $u \otimes v \otimes w$ by $(A, B, C) \cdot u \otimes v \otimes w = Au \otimes Bv \otimes Cw$ and on arbitrary tensors by linear continuation. The orbit of a tensor t under G_n is the set $G_n t := \{g \cdot t \mid g \in G_n\}$ and its orbit closure is the closure $\overline{G_n t}$ in the Zariski topology. It is well known that the set of all tensors of border rank $\leq r$ can be written with the help of an orbit closure [11], namely $\overline{G_r e_r}$ where e_r is the so-called unit tensor in $\mathbb{F}^{r \times r \times r}$: A tensor $t \in \mathbb{F}^{n \times n \times n}$ has border rank $\leq r$ iff $\tilde{t} \in \overline{G_r e_r}$, where $\tilde{t}$ is an embedding of t into the larger ambient space $\mathbb{F}^{r \times r \times r}$.

The null cone problem is a special case of the orbit closure intersection problem where vector u is always the 0 vector. That is, we ask the following:

PROBLEM 1.3. NULL CONE: *For a group G acting on a vector space V, and given $v \in V$ as input, decide if $0 \in \overline{Gv}$.*

For an example set up of the null cone problem, let us think of a tensor $t \in \mathbb{F}^{n \times n \times m}$ as a set of m matrices $A_1, \dots, A_m$ of size $n \times n$, stacked up on top of each other (also called slices). The group $\Gamma_n := \mathsf{SL}_n \times \mathsf{SL}_n$ acts on t by simultaneously multiplying each of the matrices from the left and the right. King [37] showed that the *noncommutative* rank of the matrix space given by $A_1, \dots, A_m$ is maximal iff $0 \in \overline{\Gamma_n t}$. (All such tensors t are said to lie in the *null cone*.) Garg et al. [24] show how to decide the null-cone problem in this setting in polynomial time, hence giving a deterministic noncommutative identity testing algorithm. Ivanyos, Qiao, and Subrahmanyam [36], based on work by Derksen and Makam [20], give a different algorithm for this problem, which works over arbitrary characteristic. (Unfortunately, we do not know whether something similar can be achieved in the commutative setting. More unfortunately, Makam and Wigderson proved recently that the commutative case cannot be written as a null-cone problem [43].)

Orbit closure containment problems have played a central role in algebraic complexity theory in the recent years. On the algorithmic side, orbit closure containment has been crucial in several advancements in the fast matrix multiplication algorithms due to the notion of border rank of tensors, see e.g. [6]. On the complexity theoretic side, the famous permanent versus determinant problem can also be phrased as an orbit closure containment problem. This is the starting point of the geometric complexity program initiated by Mulmuley and Sohoni [44]. While all the above mentioned problems remain far from being completely understood, the interest towards studying algorithmic problems related to orbit closures has seen a rise in the past few years. Thanks to the sequence of works connecting several areas of mathematics, combinatorial optimization, and complexity theory, many special instances of the orbit closure containment problem, in particular the null cone problem, have proved to be useful in giving polynomial time algorithms for special cases of the polynomial identity testing problem and several non-convex optimization problems. See [13, 26, 23, 12, 14, 1, 25, 16, 24] for details.

As a result, Wigderson in his invited talk in CCC'17 posed the orbit containment problem, the orbit closure containment problem and the null cone problem to the community [58].

While there has been a lot progress recently towards the null cone problem and we have efficient algorithms in many setups, most of the instances of the orbit closure containment problem is not understood from the algorithmic perspective. In particular, we neither know the NP-hardness nor a polynomial time algorithm for the tensor border rank problem. This is in contrast to the tensor rank problem, where we know the NP-completeness for 30 years now. Similarly, we currently do not know whether it is hard to test whether a polynomial lies in the orbit closure of the determinant, which is an algebraic variant of the so-called minimum circuit size problem. The main challenge for proving hardness or getting an algorithm for the problems related to orbit closure is that it is difficult to get a hold on how the closure will behave.

1.2 Slice rank of tensors and relation to orbit closure containment The notion of slice rank was first used implicitly by Croot, Lev, and Pach in their application of the so-called polynomial method in their breakthrough work on progression-free sets, also known as capsets [19]. Later Tao [54] gave a symmetrized

[2] Unless stated otherwise, we assume the underline field to be $\mathbb{C}$ in this paper

formulation of this method and used slice rank explicitly. The term "slice rank", however, was first used by Blasiak et al. [9], who used the term for the notion that Tao introduced. They used this notion to extend the results on capsets and obtained some barrier results on the group-theoretic approach to the matrix multiplication, hence making slice rank quite important from the perspective of algorithm design. Tao and Sawin [55] explored the slice rank of tensors systematically. The methods based on slice rank have been very useful in advancement of several combinatorial problems like the sunflowers free sets, the tri-colored and multi-colored sum-free sets, the capsets and the progression-free problem, and multiplicative matching in nonabelian groups (see, for instance, [21, 45, 41, 48]). Finally, upper bounds on slice rank can be used to lower bound the matrix-multiplication exponent achievable by the so called universal method (which generalizes many known methods), and thus the computation of slice rank is interesting for analyzing the scope of the methods for finding fast matrix multiplication algorithms. See, for example, [2, Section 5].

We now describe the notion of slice rank and then the corresponding computational problem. For this, we consider the space $V_1 \otimes V_2 \otimes V_3$. It can also be written as $\bigotimes_{i=1}^3 V_i$, and is generated by the decomposable (also called rank-one) tensors $v_1 \otimes v_2 \otimes v_3$, where $v_i \in V_i$. The usual tensor rank is the minimum number of decomposable tensors that is needed to write a given tensor as a sum of decomposable tensors. The slice rank is defined in a similar manner, however, the basic building blocks are not decomposable tensors but tensors that can be decomposed into a matrix and a single vector. More formally, consider the smaller tensor products $\bigotimes_{1 \le i \le 3: i \ne j} V_i$ and the j-th tensor products $\otimes_j : V_j \times \bigotimes_{1 \le i \le 3: i \ne j} V_i \to \bigotimes_{i=1}^3 V_i$ with its natural definition. Now the rank one functions are the elements of the form $v_j \otimes_j v_{\hat{j}}$ for some $v_j \in V_j$ and $v_{\hat{j}} \in \bigotimes_{1 \le i \le 3: i \ne j} V_i$. The slice rank (or srk for short) of a tensor $T \in \bigotimes_{i=1}^3 V_i$ is the smallest nonnegative integer r such that T can be expressed as a linear combination of r rank one functions. For its comparison with other notions of rank of tensors, like subrank and multi-slice rank, see [17, Section 5]. For its relation to the analytic rank and the partition rank, see [42].

The algorithmic problem corresponding to slice rank problem is the following.

PROBLEM 1.4. SLICE RANK OF TENSORS: *Given* $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ *and a number* r*, decide if* $\mathrm{srk}(T) \le r$.

The notion of slice rank is closely related to the orbit closure containment problem. In particular, [9] established some interesting connections between the slice rank and the null cone problem. Bürgisser et al. [16] showed an equivalence between the *asymptotic* fullness of slice-rank and the null cone problem, and make some algorithmic progress towards it. In this work, we make the connection between slice rank and the orbit closure containment even more apparent, thanks to a formulation of slice rank by Tao and Sawin [55]. Bürgisser et al. [16, page 27] report that Sawin has an unpublished proof that *computing* the slice-rank of tensors of order three is NP-hard. However, they state that the decision version, that is, the above Problem 1.4 remains open, while expressing that it is plausible that this should be NP-hard as well.

1.3 Our contributions, relation to previous works, and proof methods In this work, we make progress on the above Problem 1.2 and Problem 1.4, that is, the orbit closure containment problem and the slice rank problem, by showing their NP-hardness, while we observe an upper bound for Problem 1.1, that is. the orbit containment problem.

1.3.1 Orbit closure containment problem Our first contribution is that we rule out the possibility of an efficient algorithm for the general case of the orbit closure containment problem under the assumption that $\mathsf{P} \neq \mathsf{NP}$, answering a question posed by Wigderson. We do so by showing that testing whether a 3-tensor t lies in the orbit closure of another 3-tensor t' under the group action $\mathsf{GL}_k \times \mathsf{GL}_m \times \mathsf{GL}_n$ is NP-hard.

THEOREM 1.1. *Given two tensors* t *and* t'*, deciding whether the orbit closure of* t *is contained in the orbit closure of* t' *(under the usual* $\mathsf{GL}_k \times \mathsf{GL}_m \times \mathsf{GL}_n$ *action) is* NP*-hard.*

We show this by defining a quantity called minrank (see Sections 2 and 4) and proving that deciding whether the minrank is bounded by some given bound b can be phrased as an orbit closure containment problem. We then show that it is an NP-hard question (see Section 2.1) by showing that it is polynomial time equivalent to the solvability of homogeneous quadratic equations. Since the solvability of homogeneous quadratic equations is NP-hard, we get that the orbit closure containment is NP-hard as well.

This is in contrast to the recent results on the null cone problem, for which polynomial time algorithms have been discovered for several group actions. Since the null cone has equivalent characterizations via invariant theory, we have more tools there. On the other hand, for the orbit closure containment problem corresponding to a group action, no such characterization is available, and we need to understand the corresponding orbit closure better. Unfortunately, in most of the interesting settings, our understanding of the closure of the set is in quite limited. For instance, we do not understand the tensor border rank well, neither do we understand the closures of algebraic complexity classes. Thus, the main challenge is to find a set up where one has a good control over the orbit closure. In this work, we find one such set up.

The initial inspiration of the set up that we find is the NP-hardness of the completion rank and the border completion rank [8]. Let us briefly look at those notions.

We can phrase the *matrix completion problem* as a problem on tensors or on tuples of matrices. Many variants of matrix completion problem has been studied in the literature. In its most general form, we are given a tuple of $n \times n$ matrices $(A_1, A_2, \ldots, A_m)$. We can view $(A_1, A_2, \ldots, A_m)$ as a tensor in $\mathbb{F}^{n \times n \times m}$ with slices $A_1, \ldots, A_m$ of size $n \times n$, stacked up on top of each other. Then the matrix completion problem can be phrased as follows:

Problem 1.5. Matrix Completion: *Given a tensor t as a tuple of $n \times n$-matrices $(A_1, A_2, \ldots, A_m)$, and a number r, decide if there exist $\lambda_2, \ldots, \lambda_m \in K$ such that* $\mathrm{rk}(A_1 + \lambda_2 A_2 + \cdots + \lambda_m A_m) \le r$.

Here rk denotes the usual matrix rank. The minimum achievable value of r above is called the *completion rank* of t. Matrix completion has many applications, for instance, in machine learning and network coding, we here just refer to [47, 32, 31], which contain relevant hardness results. When we consider minimization, the problem is NP-hard, even when the resulting matrix has rank 3 [47]. When we consider maximization, then the problem is NP-hard over finite fields [32]. Over large enough fields, there is a simple randomized polynomial time algorithm that simply works by plugging in random elements from a large enough set. The correctness of this algorithm follows from the well-known Schwartz-Zippel lemma.

In [8], it is shown that given t and a bound r, deciding whether the completion rank of t is bounded by r is NP-hard. Furthermore—and this is the interesting case here—even testing whether t is in the closure of the set of all tensors of completion rank $\le r$ is NP-hard. The smallest r such that this is the case, is called the *border completion rank*. It is shown in [8] that given t and a bound r, deciding whether the border completion rank of t is bounded by r is NP-hard. Thus, completion rank is one of the rare examples where we understand the border well. Thus the hope was to exploit this understanding.

However, the above result could not help us simply because the border completion rank problem cannot be phrased as an interesting orbit closure problem. We overcome this challenge by defining a homogeneous version of matrix completion problem, which we call as the *minrank* problem, where, in contrast to the completion rank, we allow any nontrivial linear combination of the slices.

Problem 1.6. Minrank: *Given $A_1, \ldots, A_k$ of the same size $m \times n$ and a number r, decide whether there exists a nonzero linear combination $x_1 A_1 + \cdots + x_k A_k$ with rank at most r. The smallest r for which the answer is YES is called the* minrank *of $A_1, \ldots, A_k$.*

Here again, instead of thinking of a tuple of matrices, we can also view $A_1, \ldots, A_k$ as a tensor in $\mathbb{F}^{k \times m \times n}$ with $A_1, \ldots, A_k$ being its slices. We will use both views in this paper. We show that the obtained homogeneous version of the problem can indeed be phrased as an orbit closure containment problem. For this, we first show that the set of matrix tuples (or tensor) with minrank at most r is a Zariski closed set by showing that the set can be viewed as a projective variety. Next, in order to show that we can phrase the minrank problem as an orbit closure problem, we give an explicit tensor $T_{k,n,r}$ such that every tensor (or matrix tuple) with slice rank at most r lies inside the orbit closure of this tensor $T_{k,n,r}$. We now elaborate on the above.

For a tensor $T \in \mathbb{F}^{k \times m \times n}$ given as $e_1 \otimes A_1 + \cdots + e_k \otimes A_k$ (indicating that $A_1, A_2, \ldots, A_k$ correspond to different slices of T) and a linear form $x \in (\mathbb{F}^k)^*$, we define the *contraction* Tx by $Tx := x(e_1)A_1 + \cdots + x(e_k)A_k$, where $x(e_i)$ denotes the i-th coordinate of x. That is, we form a linear combination of the slices. If we take the set of all (T, x) with $\mathrm{rk}(Tx) \le r$ and $x \ne 0$ and project on the first component, we get all tensors of minrank at most r. Since the set of all such (T, x) is invariant under scaling of T or x by nonzero factors, it also defines a *projective* variety, and the projection on the first component is a projective variety, too (see Section 4 for more details). So we are in the nice situation where the set of all tensors of minrank at most r is Zariski closed (Theorem 4.1). Thus we do not need an additional border complexity measure, i. e., minrank and border minrank coincide. This is different to the situation with completion rank and border completion rank or tensor rank and border rank. We denote the corresponding variety of all tensors $T \in U \otimes V \otimes W$ of minrank at most r by $\mathcal{M}_{U \otimes V \otimes W, r}$ or just $\mathcal{M}_r$ when the tensor space is clear from the context.

Next, we want to write the minrank varieties $\mathcal{M}_{U \otimes V \otimes W, r}$ as orbit closures. Note that we can always embed a tensor $T \in U \otimes V \otimes W$ into a larger ambient space $U \otimes L \otimes L$, where V and W are subspaces of L, by filling the new entries with zeros. (This process is called *padding*.) We then show (Corollary 4.1), that $\mathcal{M}_{U \otimes V \otimes W, r}$ is the $\mathsf{GL}(U) \times \mathsf{GL}(L) \times \mathsf{GL}(L)$-orbit closure of the tensor

$$T_{k,n,r} = e_1 \otimes \left(\sum_{j=1}^{r} e_{1j} \otimes e_{1j}\right) + \sum_{i=2}^{k} e_i \otimes \left(\sum_{j=1}^{n} e_{ij} \otimes e_{ij}\right)$$

intersected with the ambient space $U \otimes V \otimes W$ (here

$k = \dim U$, $n = \dim L$). This means that we can reduce the question whether a tensor has minrank at most r to the question whether it is contained in the orbit closure of $T_{k,n,r}$.

Now, one might hope that the proof of hardness of border completion rank in [8] can be adapted to the homogeneous setting. However, unfortunately, this NP-hardness proof breaks down in the homogeneous setting since the hard instance in this proof critically used the fact that we are in the affine setting, since A_1 was a matrix that had rank linear in the input size whereas all other matrices had the same, constant rank. Thus the hardness proofs do not work in the homogeneous setting, since all instances created in the proofs trivially have the same minrank. Thus, we need to do something completely different. We solve the problem by showing the equivalence of solvability of homogeneous quadratic equations and the minrank problem, hence establishing the NP-hardness of the minrank problem. In fact, it turns out that even deciding whether the minrank is ≤ 1 is already NP-hard. Thus, we get that the orbit closure containment problem in NP-hard as well. See section 2.1 for details.

When the underlying field is the set of real numbers, and we are taking the Euclidean closure, then we can say something more about the orbit closure containment problem. In Section 3.1, we show the equivalence of the orbit closure containment problem and the existential theory over reals (see [49]) in this case.

Theorem 1.2. *The (Euclidean) orbit closure containment problem over the reals is polynomial-time equivalent to the existential theory over the reals.*

For the *tensor rank problem*, such an equivalence with the existential theory over reals was recently established by Shitov [52].

1.3.2 Orbit Containment Problem

We also show an upper bound for the algorithmic problem of the orbit containment problem in Section 7 by reducing it to the solvability of polynomial equations. Since the solvability of polynomial equations is known to be in the complexity class AM[3], assuming the generalized Riemann hypothesis (GRH), by a result of Koiran [38], we deduce that over the field of complex numbers, the orbit containment problem can be shown to be in the complexity class AM under the same assumption.

[3]AM refers to the complexity class containing the set of decision problems decidable in polynomial time by an Arthur-Merlin protocol with 2 messages. It is contained in the complexity class Π_2P, and is hence contained in the second level of polynomial hierarchy. See [3] for details.

Theorem 1.3. *Over $\mathbb{C}$,* orbit containment problem $\in$ AM*, assuming the generalized Riemann hypothesis.*

1.3.3 Slice rank problem

Our second main result is the progress towards understanding Problem 1.4, the algorithm corresponding to slice rank problem. We rule out an efficient algorithm under the assumption that P $\neq$ NP by showing that the problem is NP-hard under polynomial time many-one reductions. (see Section 5).

Theorem 1.4. *Given a 3-tensor T and a positive integer r, determining if the slice rank of T is at most is r, is* NP*-hard.*

For this, we use a connection of the slice rank to the size of a minimum vertex cover of a hypergraph by Tao and Sawin [55]. They showed that for every 3-uniform, 3-partite hypergraph H, one can associate a tensor T_H, and if the edge set of the hypergraph forms an antichain, then the slice rank of the associated tensor T_H equals the size of the minimum vertex cover of the hypergraph H. To our best knowledge, the complexity of the decision version of the slice rank problem for order-three tensors has been open so far. Prahladh Harsha, Aditya Potukuchi, and Srikanth Srinivasan kindly sent us an unpublished manuscript, in which they prove that the order-four case is NP-hard. However, this one more tensor leg gives an additional degree of freedom, which easily allows to establish the antichain condition. Bürgisser et al. [16, page 27] report that Sawin has an unpublished proof that *computing* the slice-rank of tensors of order three is NP-hard. However, they also state that the decision version is open.

We show the NP-hardness of the slice rank problem for order-three tensors by showing that the 3-uniform, 3-partite hypergraph minimum vertex cover problem where the edge set forms an antichain is NP-hard. The corresponding hypergraph minimum vertex cover problem without the antichain restriction is known to be NP-hard [27] by reduction from the usual 3-SAT problem. However, their reduction does not work if one wants to adapt it to the antichain restriction. We use a reduction from a restricted SAT-variant, the bounded-occurrence mixed SAT (bom-SAT) problem, in which there are 3-clauses and 2-clauses, and every variable occurs exactly thrice, once in a 3-clause and twice in 2-clauses. Because of the antichain restriction, our labelling of the gadget becomes very delicate and needs to be handled very carefully in the reduction (see Lemma 5.3).

Next, we phrase the slice rank problem in terms of orbit closures. More specifically, we show that testing whether a tensor $T \in \mathbb{F}^{n\times n\times n}$ has $\mathrm{srk}(T) \leq r$ is equivalent to testing if the tensor T is contained in a polynomially large union of orbit closures. Let (r_1, r_2, r_3) be such that $r_1 + r_2 + r_3 = r$. We first embed T in a larger subspace $U' \otimes V' \otimes W' \cong \mathbb{F}^{s_1} \otimes \mathbb{F}^{s_2} \otimes \mathbb{F}^{s_3}$ (this is called padding), where $s_1 = r_1 + nr_2 + nr_3$, $s_2 = nr_1 + r_2 + nr_3$ and $s_3 = nr_1 + nr_2 + r_3$, and define

$$S_{n,r_1,r_2,r_3} = \sum_{i=1}^{r_1}\sum_{j=1}^{n} e_i^1 \otimes e_{ij}^1 \otimes e_{ij}^1 + \sum_{i=1}^{r_2}\sum_{j=1}^{n} e_{ij}^2 \otimes e_i^2 \otimes e_{ij}^2 + \sum_{i=1}^{r_3}\sum_{j=1}^{n} e_{ij}^3 \otimes e_{ij}^3 \otimes e_i^3.$$

Intuitively, in the sum above, we have r_1 rank-one elements of the form $v_1 \otimes_1 v_{\hat{1}}$ with $v_1 \in V_1$ and $v_{\hat{1}} \in \bigotimes_{1 \le i \le 3: i \ne 1} V_i$, r_2 elements of the form $v_2 \otimes_2 v_{\hat{2}}$, and r_3 elements of the form $v_3 \otimes_3 v_{\hat{3}}$. Now $\operatorname{srk}(T) \le r$ becomes equivalent to testing whether T is in the orbit closure of S_{n,r_1,r_2,r_3} for some (r_1, r_2, r_3) with $r_1 + r_2 + r_3 = r$. Thus we show that the slice rank variety $\mathcal{SV}_{\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n, r}$ is the union of orbit closure of S_{n,r_1,r_2,r_3} over all (r_1, r_2, r_3) with $r_1 + r_2 + r_3 = r$, intersected with the ambient space $\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$. It is worth noting that S_{n,r_1,r_2,r_3} is very similar to $T_{k,n,r}$ defined for the minrank (see Sections 4 and 6 for details). Note that Tao showed that the set of all T with $\operatorname{srk}(T) \le r$ is closed, so, similar to minrank, there is no need to define a notion of border slice rank either (see [55, Corollary 2]).

1.4 Organization of the paper We give the algorithmic hardness of the minrank problem in Section 2. In Section 4, we show that the minrank problem is an instance of the orbit closure containment problem. Combining the above two, we conclude that the NP-hardness of the orbit closure containment problem in Corollary 3.1. We discuss the case when the underlying field is $\mathbb{R}$ in Section 3.1, showing an equivalence between the orbit closure containment problem and the existential theory over reals. We discuss the complexity of the slice rank problem in Section 5 showing that it is NP-hard using Lemma 5.2 and Lemma 5.3. In Section 6, we phrase the slice rank problem as a union of orbit closures. We point out that the sections where we show algorithmic hardness of our problems can be read independently of the sections where we phrase the problems as orbit closure problems. We finally close with an algorithmic upper bound in the case of orbit containment problem in Section 7.

2 Complexity of the minrank problem

We consider the following problem: given a tuple of matrices $A_1, \dots, A_k$ of the same size $m \times n$ and a number r, does there exist a nonzero linear combination $x_1 A_1 + \dots + x_k A_k$ with rank at most r? This is a homogeneous variant of the generalized matrix completion problem considered in [8], where instead of a linear combination we have an affine expression $A_0 + x_1 A_1 + \dots + x_k A_k$. A restricted variant of this problem was first considered in [15], where it is proven that the problem is NP-hard. The related problem of low rank matrix completion is widely studied in optimization.

Clearly, the answer depends on the field from which we take the coefficients of the linear combination. For example, the pair of matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

does not have any nontrivial linear combinations of rank 1 over $\mathbb{R}$, but over $\mathbb{C}$ we have $\operatorname{rk}(A_1 + iA_2) = 1$. We will mostly work over algebraically closed fields such as $\mathbb{C}$, but many results are also true over other fields.

Let $\mathbb{F}$ be a field. Instead of talking about matrices $A_1, \dots, A_k \in \mathbb{F}^{m \times n}$, we can also phrase the homogeneous minrank problem in terms of a linear subspace $\langle A_1, \dots, A_k \rangle$, a matrix of linear forms $A \colon \mathbb{F}^k \to \mathbb{F}^{m \times n}$ where $A(x) = \sum_{i=1}^k x_i A_i$ or a tensor $T \in \mathbb{F}^k \otimes \mathbb{F}^m \otimes \mathbb{F}^n$ such that $T = \sum e_i \otimes A_i$. We will mainly use the tensor language.

Recall the definition of minrank.

Definition 2.1. *Let U, V, W be finite-dimensional vector spaces over some field $\mathbb{F}$. The minrank of a tensor $T \in U \otimes V \otimes W$ is the minimal number r such that there exists a nonzero $x \in U^*$ with* $\operatorname{rk}(Tx) = r$.

Let S be a finite or countable subset of $\mathbb{F}$.

Problem 2.1. (HMinRank$_{S,\mathbb{F}}$) *Given a tensor $T \in \mathbb{F}^{k \times m \times n}$ with all components in S and a number r, decide if the minrank of T is at most r.*

In section 2.1 we will prove that this problem is NP-hard. Moreover, it is hard even when r is fixed to one.

Problem 2.2. (HMinRank1$_{S,\mathbb{F}}$) *Given a tensor $T \in \mathbb{F}^{k \times m \times n}$ with all components in S, decide if the minrank of T is at most 1.*

2.1 Equivalence of minrank and solvability of quadratic equations In this section we prove NP-hardness of HMinRank by reducing it to the following problem:

Problem 2.3. (HQuad$_{S,\mathbb{F}}$) *Given a set of quadratic forms with coefficients from S, represented by lists of coefficients, determine if it has a nonzero common zero over $\mathbb{F}$.*

To implement the reduction, we need to perform linear algebra computations with elements of the field.

DEFINITION 2.2. *An* effective field *is a finite or countable field* $\mathbb{F}$ *with a binary encoding of elements of* $\mathbb{F}$ *such that the following operations can be performed in time polynomial in the length of the encoding of arguments:*

- *multiplication and addition of two elements over* $\mathbb{F}$*,*
- *multiplication of an arbitrary number of matrices over* $\mathbb{F}$*,*
- *equality comparison of two elements of* $\mathbb{F}$*,*
- *division of two elements of* $\mathbb{F}$ *(if the denominator is zero, the algorithm should fail).*

Furthermore, we want that polynomial identity testing is in BPP*, that is, there is a* BPP*-machine that given an algebraic circuit computing a polynomial over* $\mathbb{F}$*, decides whether this polynomial is identically zero.*

In our paper, we usually deal with polynomials over uncountable fields like $\mathbb{C}$. In the algebraic complexity setting, this is no problem. However, when we want to compute with Turing machines, we have to restrict ourselves to appropriate subfields. This is modelled by effective fields. In particular, $\mathbb{Q}$ is effective and the natural effective subfield of $\mathbb{R}$ and $\mathbb{Q} + i\mathbb{Q}$ is a natural choice for $\mathbb{C}$. Finite fields are effective, when we drop the last condition about identity testing, which we only need in the second part of this section.

Efficient multiplication of several matrices implies that products and linear combinations of elements can also be computed in polynomial time. It also allows for various polynomial-time linear algebra procedures. In particular, we are interested in the following:

THEOREM 2.1. *For an effective field K there is a polynomial time algorithm which, given a matrix A over K, computes a basis of* $\ker A$.

Proof. Determinants of matrices over an effective field are computable in polynomial time, because determinant can be represented as an iterated matrix multiplication of polynomial size (see e. g. [34]). This allows computing the inverse of a nonsingular matrix. Also, we can find one of the maximal nonzero minors of a given nonzero matrix, by starting from any nonzero entry and trying to enlarge the minor by checking all rows and columns at each step. We can then compute the basis of the kernel by basic linear algebra. □

Hillar and Lim [33, Thm. 2.6] proved that HQUAD is NP-hard over the fields $\mathbb{R}$ and $\mathbb{C}$. Their proof also works for any field of characteristic different from 3 containing cubic roots of unity. The NP-hardness for arbitrary fields was proven by Grenet, Koiran and Portier in [28]. We give another proof for arbitrary fields based on the idea of Hillar and Lim. Compared to [28], we describe a general construction for all fields instead of treating characteristic 2 as a special case, and only use coefficients from $\{-1, 0, 1\}$.

THEOREM 2.2. $\mathrm{HQUAD}_{\{0,1,-1\},\mathbb{F}}$ *is* NP*-hard for any field* $\mathbb{F}$*.*

Proof. We reduce from graph 3-colorability.

Given a graph $G = (V, E)$, we will construct a system of quadratic homogeneous equation, solutions of which correspond to colorings of the graph. The set of variables consists of two variables x_v and y_v for each vertex $v \in V$ and one additional variable z. Consider a system of homogeneous quadratic equations which contains for each vertex v the three equations

$$x_v y_v = 0$$
$$x_v^2 - x_v z = 0$$
$$y_v^2 - y_v z = 0$$

and for each edge $(v, w) \in E$ the equation

$$x_v^2 + y_v^2 + x_w^2 + y_w^2 - x_v y_w - x_w y_w - z^2 = 0$$

If $z = 0$, then from vertex equations we deduce $x_v = y_v = 0$ for all $v \in V$. Therefore, a nontrivial solution must have nonzero z. We can scale it so that $z = 1$. When $z = 1$, the vertex equations give $(x_v, y_v) \in \{(0,0), (0,1), (1,0)\}$. Restricted to these values, the left-hand side of the edge equation has the following values:

v \ w	$(0,0)$	$(0,1)$	$(1,0)$
$(0,0)$	-1	0	0
$(0,1)$	0	1	0
$(1,0)$	0	0	1

That is, the edge equation forces the tuples (x_v, y_v) and (x_w, y_w) to be different. Thus, nontrivial solutions with $z = 1$ are in one-to-one correspondence with colorings of the graph G into three colors, given by the three possible solutions of the vertex equations. □

THEOREM 2.3. *Let $\mathbb{F}$ be a field and K be an effective subfield of $\mathbb{F}$. Then* $\mathrm{HMINRANK1}_{K,\mathbb{F}}$ *is polynomial-time equivalent to* $\mathrm{HQUAD}_{K,\mathbb{F}}$.

Proof. To reduce from HMINRANK1 to HQUAD, note that the condition $\mathrm{rk}(Tx) \le 1$ can be expressed by homogeneous quadratic equations on x, namely, vanishing of 2×2 minors of the matrix of linear forms Tx.

Now we describe the reduction from HQuad to HMinRank1. Let k be a number of given quadratic forms and n be the number of variables. Each quadratic form $q(x) = \sum_{1\le i\le j\le n} a_{ij}x_ix_j$ on $\mathbb{F}^n$ corresponds to a linear form $Q(X) = \sum_{1\le i\le j\le n} a_{ij}x_{ij}$ on the space $\mathsf{Sym}^2\mathbb{F}^n \subset \mathbb{F}^n\otimes\mathbb{F}^n$ of symmetric matrices, and a vector x is a zero of q if and only if $x\otimes x$ is a zero of Q. Therefore, a set of k linear forms on $\mathbb{F}^n$ corresponds to a linear map $L: \mathsf{Sym}^2\mathbb{F}^n \to \mathbb{F}^k$ given by a matrix consisting from the coefficients of quadratic forms, and x is a common zero if and only if $x \otimes x$ is contained in $\ker L$. Since all the coefficients lie in K, the map L is an extension of a linear map $\mathsf{Sym}^2 K^n \to K^k$, and its kernel has a basis consisting of vectors in $\mathsf{Sym}^2 K^n$, which, by Theorem 2.1, can be computed in polynomial time. Let $A_1, \dots, A_m$ be such basis and $T = \sum_{i=1}^m e_i \otimes A_i \in K^m \otimes K^n \otimes K^n$. Nontrivial common zeros $x \in \mathbb{F}^n$ of the original set of quadratic forms corresponds to rank 1 symmetric matrices $x \otimes x$ which can be presented as a nontrivial linear combination $\sum_{i=1}^m y_iA_i$ with $y_i \in \mathbb{F}$ or, equivalently, as a contraction Ty with nonzero $y \in \mathbb{F}^m$. This is the resulting instance of HMinRank1 problem. □

Corollary 2.1. *Let $\mathbb{F}$ be a field and K be an effective subfield of $\mathbb{F}$. Then* $\text{HMinRank1}_{K,\mathbb{F}}$ *is* NP*-hard.*

The HMinRank problem is also hard in other regimes.

Theorem 2.4. *Let $\mathbb{F}$ be a field of characteristic 0 and K be an effective subfield of $\mathbb{F}$. Then* $\text{HMinRank}_{K,\mathbb{F}}$ *is* NP*-hard for $n \times (2n+1) \times (2n+1)$ tensors and $r = n+1$.*

Proof. The proof is based on a similar theorem for finite fields is sketched in [18, §3.3], which uses the NP-completeness of the minimum distance problem for linear codes proved in [57].

We reduce from a variant of the Subset Sum problem: given a set of $2n$ integers, and a number S, determine if a subset of these integers sum up to S. NP-completeness of this variant is noted in [22, SP13].

From the input $\{a_1, \dots, a_{2n}\}$ of the Subset Sum problem, construct a $(n+1) \times (2n+1)$ matrix

$$A = \begin{bmatrix} 1 & 1 & \dots & 1 & 0 \\ a_1 & a_2 & \dots & a_{2n} & 0 \\ a_1^2 & a_2^2 & \dots & a_{2n}^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_1^{n-2} & a_2^{n-2} & \dots & a_{2n}^{n-2} & 0 \\ a_1^{n-1} & a_2^{n-1} & \dots & a_{2n}^{n-1} & 1 \\ a_1^n & a_2^n & \dots & a_{2n}^n & S \end{bmatrix}$$

From the properties of Vandermonde determinants we see that any $(n+1) \times (n+1)$ minor is nonzero if it does not contain the last column. If a minor does contain the last column and columns $i_1, \dots, i_n$, it vanishes if and only if $S = a_{i_1} + \dots + a_{i_n}$ [57, Lem. 1].

Thus, the matrix A has rank $n+1$. Moreover, it has $n+1$ linearly dependent columns if and only if the original Subset Sum problem has a solution.

Let $b_1, \dots, b_n$ be a basis of $\ker A$. Since subsets of k linearly dependent columns corresponds to vectors in $\ker A$ which have at most k nonzero coordinates, the original problem has a solution if and only if there is a nonzero linear combination of b_i with at most $n+1$ nonzero coordinates.

Let B_i be a $(2n+1) \times (2n+1)$ matrix constructed from b_i by placing its coordinates on the diagonal. The rank of a linear combination of B_i is equal to the number of nonzero coordinates in the corresponding linear combination of vectors b_i. Thus, the answer to the HMinRank problem for the $n \times (2n+1) \times (2n+1)$ tensor $\sum_{i=1}^n e_i \otimes B_i$ and $r = n+1$ determines the answer to the original problem. □

3 Complexity of the orbit closure containment problem

Thus, Theorem 2.3 proves that the homogeneous minrank problem is computationally equivalent to the solvability of homogeneous quadratic equations. This, combined with the fact that the minrank problem can be phrased as an orbit closure containment problem (Section 4), proves that the orbit closure containment problem ("$w \in \overline{Gv}$") is at least as hard as the solvability of homogeneous quadratic equations. In particular, the orbit closure containment problem is NP-hard.

Corollary 3.1. (Theorem 1.1 restated) *Given two tensors t and t', deciding whether the orbit closure of t is contained in the orbit closure of t' (under the usual $\mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$ action) is* NP*-hard.*

3.1 Orbit closure containment and existential theory over reals Over the reals, we can say even more, when closure means Euclidean closure. Let ETR denote the problem of the existential theory over the reals, ETR is the set of true sentences of the form $\exists x_1, \dots, x_n : \phi(x_1, \dots, x_n)$, where ϕ is a quantifier-free Boolean formula over the signature 0, 1, + ∗, <, = interpreted in the intended way over the real numbers. w being in the orbit closure can be expressed by

$$\forall \epsilon > 0 \, \exists g \in G : \det(g) \neq 0 \wedge \|w - gv\|_2^2 < \epsilon.$$

Except for the first quantifier, this is a statement in ETR. By the results of Grigoriev and Vorobjov [29], see also [4, Thm 3.15], this universal quantifier can be removed and ϵ can be replaced by a double exponentially small constant, which can be expressed in ETR.

On the other hand, we can also reduce ETR to orbit (Euclidean) closure containment over the reals. By results of Schaefer [49, Cor. 3.10], Hilbert's Homogeneous Nullstellensatz H_2N over the reals is equivalent for ETR. In Schaefer's construction all equations have degree two except for one, which has degree four. However, it is easy to see that the degree of this equation can be reduced to two, too, see [50]. Therefore, from our reduction in Theorem 2.3, it follows that orbit closure containment over the reals is computationally equivalent to ETR.

THEOREM 3.1. (THEOREM 1.2 RESTATED) *The (Euclidean) orbit closure containment problem over the reals (with coefficients computable by polynomial-size circuits) is polynomial-time equivalent to the existential theory over the reals* ETR.

Since the minrank problem can be phrased as an orbit closure containment problem when we have the action of $\mathsf{GL}_n \times \mathsf{GL}_n \times \mathsf{GL}_n$ on 3-tensors (as shown in Section 4), the above equivalence between orbit closure containment problem and ETR still holds if one restricts the orbit closure containment problem to the tensor action.

4 Minrank as an orbit closure containment problem

In this section, we show that over algebraically closed fields, the answer to the homogeneous minrank problem is determined by membership in a certain orbit closure.

We first show that the set of tensors with minrank at most r is Zariski closed.

THEOREM 4.1. *Let U, V, W be vector spaces over an algebraically closed field $\mathbb{F}$. The set of all tensors $T \in U \otimes V \otimes W$ with minrank at most r is Zariski closed.*

Proof. Define an affine variety

$$\mathcal{X}_{U\otimes V\otimes W,r} = \{(T,x) \in (U \otimes V \otimes W) \times U^* \mid \operatorname{rk}(Tx) \le r\}.$$

Since the condition $\operatorname{rk}(Tx) \le r$ is scale-invariant with respect to both T and x, we can define the corresponding projective variety

$$\begin{aligned}\mathbb{P}\mathcal{X}_{U\otimes V\otimes W,r} &= \\ &\{([T],[x]) \in \mathbb{P}(U \otimes V \otimes W) \times \mathbb{P}U^* \mid \operatorname{rk}(Tx) \le r\} \\ &\qquad \subset \mathbb{P}(U \otimes V \otimes W) \times \mathbb{P}U^*.\end{aligned}$$

Let $\pi: \mathbb{P}(U \otimes V \otimes W) \times \mathbb{P}U^* \to \mathbb{P}(U \otimes V \otimes W)$ be the projection onto the first component of the product. Consider the image of $\mathbb{P}\mathcal{X}_{U\otimes V\otimes W,r}$ under π:

$$\pi\mathbb{P}\mathcal{X}_{U\otimes V\otimes W,r} = \{[T] \in \mathbb{P}(U\otimes V\otimes W) \mid \exists x \neq 0 : \operatorname{rk}(Tx) \le r\}.$$

As an image of a projective variety, it is a closed subvariety of $\mathbb{P}(U \otimes V \otimes W)$ (see e. g. [51, Thm. 1.10]). The affine cone over this subvariety is therefore also closed. This affine cone is exactly the set of tensors of minrank at most r. ∎

DEFINITION 4.1. *We call the projective variety*

$$\mathbb{P}\mathcal{M}_{U\otimes V\otimes W,r} = \{[T] \in \mathbb{P}(U\otimes V\otimes W) \mid \exists x \neq 0 : \operatorname{rk}(Tx) \le r\}$$

the projective minrank variety*, and the corresponding affine cone*

$$\mathcal{M}_{U\otimes V\otimes W,r} = \{T \in U \otimes V \otimes W \mid \exists x \neq 0 : \operatorname{rk}(Tx) \le r\}$$

the affine minrank variety*, or just the* minrank variety*. We omit the index $U \otimes V \otimes W$ if it is clear from context.*

Some simple properties of minrank varieties follow directly from the definition:

LEMMA 4.1. *Let V' and W' be subspaces of V and W respectively. Then*

$$\mathcal{M}_{U\otimes V'\otimes W',r} = \mathcal{M}_{U\otimes V\otimes W,r} \cap (U \otimes V' \otimes W').$$

Proof. Trivial. A tensor lies in $\mathcal{M}_{U\otimes V'\otimes W',r}$ iff it is an element of the space $U \otimes V' \otimes W'$ and has minrank at most r, i. e., lies in $\mathcal{M}_{U\otimes V\otimes W,r}$. ∎

LEMMA 4.2. *Let $\dim U = k$, $\dim V = n$ and $\dim W > s = n(k-1) + r$. Then*

$$\mathcal{M}_{U\otimes V\otimes W,r} = \bigcup_{\substack{W'\subset W \\ \dim W'=s}} \mathcal{M}_{U\otimes V\otimes W',r}.$$

Proof. Let T be a tensor in $\mathcal{M}_{U\otimes V\otimes W,r}$ and x_1 be a nonzero vector in U^* such that $\operatorname{rk}(Tx_1) \le r$. Choose $x_2, \dots, x_k$ such that $\{x_i\}$ is a basis of U^* and set $A_i = Tx_i \in V \otimes W$. Since $\operatorname{rk} A_1 \le r$, there exists a subspace $W_1 \subset W$ of dimension at most r such that $A_1 \in V \otimes W_1$. Analogously, for $i > 1$ we have $A_i \in V \otimes W_i$ for some subspace $W_i \subset W$ of dimension at most n. The sum W' of all W_i is a subspace of dimension at most s. We extend it to dimension s in arbitrary way if needed. The tensor T lies in $U \otimes V \otimes W'$ and, therefore, in $\mathcal{M}_{U\otimes V\otimes W',r}$. ∎

LEMMA 4.3. *The variety $\mathcal{M}_{U\otimes V\otimes W,r}$ is invariant under the standard action of $\mathsf{GL}(U) \times \mathsf{GL}(V) \times \mathsf{GL}(W)$ on $U \otimes V \otimes W$.*

Proof. Straightforward. If $\operatorname{rk}(Tx) \le r$, then $(F \otimes G \otimes H)T \cdot (Fx) = (G \otimes H)(Tx)$ also has rank at most r (here Fx denotes the dual action of $GL(U)$ on U^*). ∎

4.1 Minrank varieties and orbit closures The minrank varieties are related to orbit closures of some tensors. Let $L = (\mathbb{F}^n)^{\oplus(k-1)} \oplus \mathbb{F}^r$ be a vector space of dimension $s = n(k-1)+r$ decomposed into k summands of dimension n each, except the first one, which is of dimension r. Let L_i be the i-th summand and denote the standard basis of L_i by e_{ij}, $1 \le j \le \dim L_i$. Let $U = \mathbb{F}^k$ be a k-dimensional space with a standard basis e_i. Define the tensor $T_{k,n,r} \in U \otimes L \otimes L$ as

$$T_{k,n,r} = e_1 \otimes (\sum_{j=1}^{r} e_{1j} \otimes e_{1j}) + \sum_{i=2}^{k} e_i \otimes (\sum_{j=1}^{n} e_{ij} \otimes e_{ij}),$$

that is, the first slice of $T_{k,n,r}$ consists of an $r \times r$ identity matrix at the top-left corner, with zero everywhere else. Whereas, for $i > 1$, the i-th slice of $T_{k,n,r}$ is a block diagonal matrix, whose only nonzero block is the i-th block, which is an identity matrix of size $n \times n$.

The group $\mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s$ acts in a usual way on $U \otimes L \otimes L$. The minrank variety $\mathcal{M}_r$ can be defined using the orbit closure of $T_{k,n,r}$:

Theorem 4.2. *Let V be an n-dimensional subspace of L. Then*

$$\mathcal{M}_{U\otimes V\otimes L,r} = \overline{(\mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s)T_{k,n,r}} \cap (U \otimes V \otimes L).$$

Proof. We have $T_{k,n,r} \in \mathcal{M}_{U\otimes L\otimes L,r}$. Since the minrank variety is invariant, the entire orbit $(\mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s)T_{k,n,r}$ lies in it. Since the minrank variety is Zariski closed, it also contains the orbit closure. By Lemma 4.1 we have $\overline{(\mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s)T_{k,n,r}} \cap (U \otimes V \otimes L) \subset \mathcal{M}_{U\otimes V\otimes L,r}$.

Conversely, let $T \in \mathcal{M}_{U\otimes V\otimes L,r}$. We can write T as $\sum_{i=1}^{k} u_i \otimes A_i$ where $\{u_i\}$ is some basis of U and A_1 is a slice with $\operatorname{rk}(A_1) \le r$.

Since $\operatorname{rk}(A_1) \le r$, it can be presented as $(P_1 \otimes Q_1)(\sum_{j=1}^{r} e_{1j} \otimes e_{1j})$ where $P_1 : L_1 \to V$ and $Q_1 : L_1 \to L$ are some linear maps. Analogously, for $i > 1$ we have $\operatorname{rk}(A_i) \le \dim V = n$ and $A_i = (P_i \otimes Q_i)(\sum_{j=1}^{n} e_{ij} \otimes e_{ij})$ for some $P_i : L_i \to V$ and $Q_i : L_i \to L$. Let $P : L \to V$ and $Q : L \to L$ be the linear maps which are equal to P_i and Q_i respectively when restricted to L_i. Let $R : U \to U$ be the map sending each e_i to the corresponding u_i. Then $T = (R \otimes P \otimes Q)T_{k,n,r}$. The closure of $\mathsf{GL}(L)$ consists of all linear endomorphisms of L and thus contains P and Q. Therefore, T lies in the closure $\overline{(\mathsf{GL}_k \times \mathsf{GL}_s \times \mathsf{GL}_s)T_{k,n,r}}$. ◻

Corollary 4.1. *Let $\dim U = k$ and $\dim V = n$. Suppose V and W are subspaces of a vector space L of dimension $s = (k-1)n + r$. Then* $\mathcal{M}_{U\otimes V\otimes W,r} =$

$$\overline{(\mathsf{GL}(U) \times \mathsf{GL}(L) \times \mathsf{GL}(L))T_{k,n,r}} \cap (U \otimes V \otimes W).$$

5 Complexity of the slice rank problem

In this section, we show that the problem of testing if a given 3-tensor has slice rank at most r is NP-hard.

Theorem 5.1. (Theorem 1.4 restated) *Given a 3-tensor T and a positive integer r, determining if the slice rank of T is at most is r, is NP-hard.*

We prove this by showing that a variant of hypergraph vertex cover testing is NP-hard. Tao and Sawin [55] showed the equivalence of the slice rank problem to this variant of hypergraph vertex cover testing. For stating this equivalence precisely, we now set up some notations.

We fix a field $\mathbb{F}$. Given a 3-uniform, 3-partite hypergraph H with 3 partitions U, V and W with $|U| = n_1$, $|V| = n_2$, and $|W| = n_3$, $n_i \in \mathbb{N}$, $i \in [3]$, and edge set $E \subseteq U \times V \times W$, we can define a 3-tensor $T_H(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3})$ corresponding to H in the following way, where $\mathbf{x_i}$ is a tuple of $[n_i]$ variables:

$$T_H(\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}) = \sum_{(u_{i_1}, v_{i_2}, w_{i_3}) \in E} x_{1,i_1} \cdot x_{1,i_2} \cdot x_{3,i_3}$$

We label the nodes in U, V and W from the set of integers. For two hyperedges $e_1 := (u_{a_1}, v_{b_1}, w_{c_1})$ and $e_2 := (u_{a_2}, v_{b_2}, w_{c_2})$, we say that $e_1 \le e_2$ iff $(a_1 \le a_2) \wedge (b_1 \le b_2) \wedge (c_1 \le c_2)$. If neither $e_1 \le e_2$ nor $e_2 \le e_1$ holds, we say that e_1 and e_2 are incomparable. In E, if every pair of hyperedges is incomparable to each other, we say that E is an antichain.

Tao and Sawin (see [55, Proposition 4]) showed the following.

Lemma 5.1. *If the hyperedge set E is an antichain, then the slice rank of T_H is the same as the size of the minimum vertex cover of the hypergraph H.*

Thus, in order to show that computing the slice rank of 3-tensors is NP-hard, we show that the hypergraph minimum vertex cover problem for a 3-partite, 3-uniform graph, where the edge set is an antichain, is NP-hard.

Our reduction is inspired by [27] where they show the NP-hardness of the hypergraph vertex cover problem for 3-uniform 3-partite graphs. Their reduction involved reducing 3-SAT to this problem. Here we need to show the hardness under the extra condition that the hyperedge set of the graph is an antichain. This makes the reduction far more involved, and we also change the hard problem that we reduce to our problem.

The NP-hard problem that we use for our reduction is a bounded occurrence mixed SAT problem (bom-SAT), where we have 3-clauses and 2-clauses, such that every variable appears exactly thrice, once in a 3-clause, while the other two occurrences are in 2-clauses (note that the number of variables, $n = 3t$, for some t, where t is the number of 3-clauses).

Remark 5.1. *It is easy to see that the above mentioned bom-SAT is* NP*-hard. For this, start with any 3-SAT instance. Now assume that a variable* Z *appears* m *times. Introduce* m *copies* $Z_1, \ldots, Z_m$ *of* X. *Replace every occurrence of* Z *by one* Z_i. *We do this for all the variables. Now every variables appears only once. However, we have to ensure consistency, that is,* $Z_1, \ldots Z_m$ *should have the same value. So we add the 2-clauses:* $(Z_1 \vee \neg Z_2) \wedge (Z_2 \vee \neg Z_3) \wedge \cdots \wedge (Z_m \vee \neg Z_1)$. *These 2-clauses can only be satisfied if we set all the* Z_i*'s to* 0 *or all the* Z_i*'s to* 1. *The resulting formula is a bom-SAT instance as described above.*

In the reduction, given a bom-SAT formula ϕ in n variables $X_1, \ldots X_n$ with t 3-clauses and m 2-clauses, the construction of a 3-uniform 3-partite hypergraph G^ϕ with 3 vertex partitions U, V and W proceeds as follows. First of all we sort all the clauses such that all the 3-clauses precede all the 2-clauses. Next we rename all the variables such that the variables in the r-th 3-clause ($r \in t$) are $Y_{3(r-1)+1}, Y_{3(r-1)+2}$ and $Y_{3(r-1)+3}$ corresponding to the first, second and the third position of the clause respectively. We also say that $Y_{3(r-1)+1}, Y_{3(r-1)+2}$ and $Y_{3(r-1)+3}$ belong to the same triple of variables.
Now, we have a gadget G_k^ϕ corresponding to each variable Y_k, $k \in [n]$. G_k^ϕ consists of nodes $(i,j)^k$ and $\overline{(i,j)}^k$, $i,j \in \{1,2,3\}$. Here $(i,j)^k$ refers to the node corresponding to the i-th occurrence of the variable Y_k, and it occurs at the j-th position in the clause in which it appears. $\overline{(i,j)}^k$ refers to the negation of Y_k in its i-th occurrence at the j-th position in the clause. We will drop the superscript k, when it is clear from the context. Clearly, there are 18 such *literal nodes* in a gadget G_k^ϕ, which are ordered along a circle (see the outer circle in Figure 1). Since Y_k appears exactly thrice in ϕ, exactly 3 out of these 18 nodes will correspond to some occurrence of Y_k in ϕ. G_k^ϕ also consists of 18 other nodes, which we call *free nodes* (as they do not correspond to any literal), that are useful in the construction (see the inner circle in Figure 1). We have hyperedges connecting two literal nodes and a free node. There are total 18 hyperedges in G_k^ϕ each consisting of three vertices that form a triangle in Figure 1. Note that every literal node appears in exactly 2 hyperedges, while a free node appears in exactly one of them. We partition the set of nodes in 3 parts, as illustrated in the figure. Among the literal nodes, the nodes corresponding to the first-occurrences ($j = 1$) go to the set U, the ones corresponding to the second-occurrences ($j = 2$) go to the set V, while the

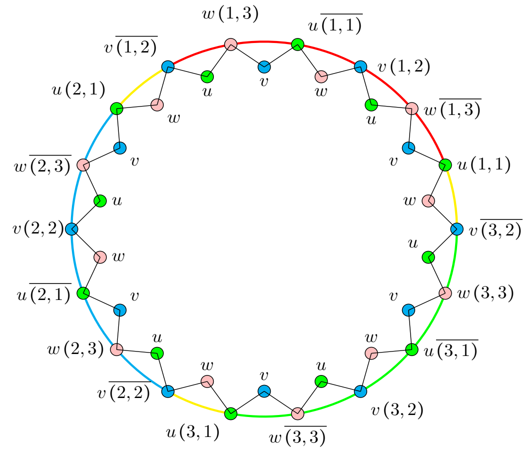

Figure 1: A variable gadget G_k^ϕ corresponding to the variable Y_k in ϕ. Nodes sharing the red, cyan and green arcs correspond to the first, second, and third occurrence of Y_k in a clause respectively. Exactly 3 out of 18 literal nodes are used in clause hyperedges. Nodes with an overline indicate that the negation of Y_k appeared in the corresponding clause. Nodes in the inner circle correspond to the free nodes.

ones corresponding to third occurrences ($j = 3$) go to the set W. We distribute the free nodes equally among the three sets, while maintaining the property of being 3-partite (see Figure 1).

Additionally, we have clause hyperedges, which for a 3-clause, connect the nodes corresponding to the three literals present in it. For every 2-clause, we first introduce another free node to the graph, added to set W (as there are no literals at the third position in a 2-clause). Now, there is an hyperedge for every 2-clause as well, connecting the two nodes corresponding to its literals and a free node. We refer to the hyperedges in a variable gadget either as *variable hyperedges* or *local hyperedges*. We refer to the hyperedges corresponding to the clauses as *clause hyperedges* or *global hyperedges*. We illustrate the set up with an example. See Figure 3.

The following two lemmas together finish the reduction and hence, prove Theorem 5.1.

Lemma 5.2. *The size of the minimum vertex cover of the hypergraph* G^ϕ *is at most* $9n$ *if and only the* bom-SAT *instance* ϕ *is satisfiable.*

The proof of this lemma follows very closely the proof of hardness of hypergraph minimum vertex cover problem (see [27, Lemma 5.3]), which was itself inspired by the proof of NP-hardness of 3-dimensional matching given in Garey and Johnson [22]. We give a sketch here.

Proof. Let ϕ be satisfiable with ν being a satisfying assignment on the variables $Y_1, \dots, Y_n$. Now, we construct the vertex cover set S for G^ϕ of size $9n$ as follows. If $\nu(Y_k) = 0$, we add all the 9 overlined nodes from G_k^ϕ to S, otherwise we add the other 9 nodes to S. Note that S covers all the local hyperedges. Since ν is a satisfying assignment, all the clause hyperedges are also covered by S as well.

Conversely, assume there is a minimum vertex cover S of G^ϕ of size at most $9n$. Now, since all the free nodes appear in only one hyperedge each, we can assume that S does not contain any free node, since we can always replace them by a literal node of the same hyperedge. Now, for $i \in \{1, \dots, n\}$ if S_i is the subset of S such that S_i only contains the vertices corresponding to the variable gadget G_i^ϕ, it can be easily seen that $|S_i| \geq 9$ for all the variable hyperedges to be covered. This implies that $|S_i| = 9$ since we assumed that $|S| = |\cup_{i=1}^n S_i| \leq 9n$. Thus S_i forms a vertex cover corresponding to the local gadget G_i^ϕ and hence covers the hyperedges in G_i^ϕ. However, there are only two vertex covers of G_i^ϕ of size 9, namely the one set containing all the overlined nodes, i.e., they correspond to $\neg Y_k$, and the other set where none of the nodes are overlined, i.e., they correspond to Y_k. In the first case, we assign the value 0 to Y_k, and we assign 1 in the second case. Thus we construct the assignment ν for $Y_1, \dots, Y_n$. Now, since S is a vertex cover and hence span all the hyperedges including the clause hyperedges, ν satisfies all the clauses of ϕ. ◻

The following lemma ensures that the edge set E of the above constructed graph G^ϕ is indeed an antichain under some labelling.

Lemma 5.3. *For every formula ϕ, there exists a way of labelling of the nodes in hypergraph G^ϕ such that the hyperedge set of G^ϕ is an antichain.*

Proof. We first give the labelling used. We have literal nodes and free nodes. The literal nodes either correspond to the first occurrence, the second occurrence or the third occurrence of a variable. In every gadget, we have 6 nodes corresponding to each occurrence, 2 from each partition U, V and W. The free nodes although do not correspond to any occurrences, we say that they correspond to first occurrence if the two literal nodes that they connect both correspond to the first occurrence. In every gadget, there are 5 such nodes, 2 each belonging to U and V, while one belonging to W. If a free node does not correspond to the first occurrence, we say that it corresponds to the second or third occurrence (we do not make distinction within them as it is not needed).

We first give the labelling corresponding to the nodes corresponding to the second and the third occurrences of variables:

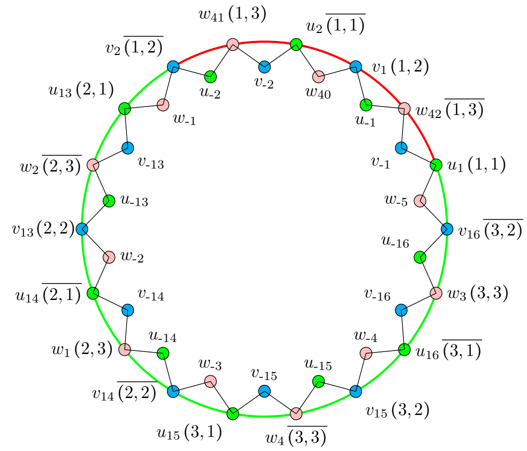

Figure 2: The labelling of variable gadgets G_1^ϕ for $n = 6$. The hyperedges with a red arc correspond to the first occurrence of variables. Notice the difference in labelling of W nodes. Literal nodes are all labelled positive. Free nodes are all labelled negative except the W node connecting the two first occurrence literal nodes.

- The position 1 literal nodes $(i,1)^k$ and $\overline{(i,1)}^k$ in G_k^ϕ are labelled $u_{2n+2(i-2)+4(k-1)+1}$ and $u_{2n+2(i-2)+4(k-1)+2}$, respectively, $\forall k$, for $i = 2, 3$.
- Similarly, the position 2 literal nodes $(i,2)^k$ and $\overline{(i,2)}^k$ are labelled $v_{2n+2(i-2)+4(k-1)+1}$ and $v_{2n+2(i-2)+2(k-1)+2}$, respectively, $\forall k$, for $i = 2, 3$.
- Likewise, the position 3 literal nodes $(i,3)^k$ and $\overline{(i,3)}^k$ are labelled $w_{2n+2(i-2)+4(k-1)+1}$ and $w_{2n+2(i-2)+4(k-1)+2}$ respectively, $\forall k$, for $i = 2, 3$.
- The 4 free U nodes in G_k^ϕ corresponding to the second or third occurrence are labelled $u_{-2n-4(k-1)-\ell}$, $\ell \in [4]$ (see Figure 2 to see which ones exactly).
- Similarly, the 4 such free V nodes in G_k^ϕ are labelled $v_{-2n-4(k-1)-\ell}$, $\ell \in [4]$.
- Finally, the 5 such free W nodes in G_k^ϕ are labelled $w_{-5(k-1)-\ell}$, $\ell \in [5]$.
- All the 2-clauses also correspond to the second and third occurrence of variables. Each such 2-clause will have a corresponding hyperedge. Here we have a freedom to choose the position for the free node. We invariably choose it to be at the third position. Thus the first two nodes of the hyperedges will take

the relevant literals as per the clause, while the W nodes will be free ones. For the s−th 2-clause (under an arbitrary order), $s \in [m]$ label the W nodes as w_{-5n-s}.

- We take all the hyperedges that include all the above labelled free W nodes. This will include all the 2-clause hyperedges along with 5 hyperedges per variable gadget. Now the tuple of U and V coordinates (u_a, v_b) of these hyperedges will have a partial order among themselves. We shuffle their W coordinates so that the order of the W coordinates becomes the reverse of the order of the tuple (u_a, v_b). We can do this without disturbing other hyperedges because these W nodes are all free and are used in only one hyperedge each.

Now it remains to label the literal nodes corresponding to the first occurrences and the free nodes pertaining to them. They are labelled differently so as to ensure that the antichain property indeed holds when the hyperedges connecting these would be compared with the 3-clause hyperedges. One key difference is that the labels of W nodes for G^{ϕ}_k in this case also depend on whether $k \equiv 1, 2$ or $0 \mod 3$.

- The position 1 literal nodes $(1,1)^k$ and $\overline{(1,1)}^k$ in G^{ϕ}_k are labelled $u_{2(k-1)+1}$ and $u_{2(k-1)+2}$, respectively, $\forall k$.

- The position 2 literal nodes $(1,2)^k$ and $\overline{(1,2)}^k$ are labelled $v_{2(k-1)+1}$ and $v_{2(k-1)+2}$, respectively, $\forall k$.

- The position 3 literal nodes $(1,3)^k$ and $\overline{(1,3)}^k$ get the labels $w_{7n-9(q-1)}$ and $w_{7n-9(q-1)-1}$, respectively, for $k = 3(q-1)+1$, whereas $w_{7n-9(q-1)-3}$ and $w_{7n-9(q-1)-4}$, respectively, for $k = 3(q-1)+2$, and $w_{7n-9(q-1)-5}$ and $w_{7n-9(q-1)-6}$, respectively, for $k = 3(q-1)+3$

- The 2 free U nodes corresponding to the first occurrence of the variable get the labels $u_{-2(k-1)-1}$ and $u_{-2(k-1)-2}$, respectively. Similarly such free V nodes get the labels $v_{-2(k-1)-1}$ and $v_{-2(k-1)-2}$ respectively, whereas the such free W nodes (1 per gadget) get the labels $w_{7n-9(q-1)-2}$ for $k = 3(q-1)+1$ and $w_{7n-9(q-1)-7}$ for $k = 3(q-1)+2$, and $w_{7n-9(q-1)-8}$ for $k = 3(q-1)+3$.

Figure 3 illustrates the labelling for $k = 1, 2, 3$ when $n = 6$.

We now show that with the above ordering, the set of hyperedges E of the hypergraph G^{ϕ} indeed is an antichain.

To simplify the argument, we divide the set of hyperedges in two parts $E = \mathcal{A} \cup \mathcal{B}$:

- Set $\mathcal{A}$: This set consists of local hyperedges in which both the literal nodes correspond to the first occurrence of variables. We also include the 3-clause hyperedges.

- Set $\mathcal{B}$: The set consisting of the remaining hyperedges, i.e., the ones in which at least one of the literal nodes correspond to the second or the third occurrences of variables. We also include the 2-clause hyperedges.

We first argue that the subset $\mathcal{B}$ is an antichain.
We note that in $\mathcal{B}$, the literal nodes are all labelled positive $(2n + 2(i-2) + 4(k-1) + j)$, $i \in \{2,3\}$, $k \in [n]$, $j \in [4]$, while the free nodes are all labelled negative $(-2n - 4(k-1) - \ell)$, $k \in [n]$, $\ell \in [4]$, for U and V nodes, whereas $(-5(k-1) - \ell)$, $k \in [n]$, $\ell \in [5]$ for W nodes, and it is easy to verify that as the labels of the literal node increase, the labels along the free node decrease.
Now we take two arbitrary elements of the set $\mathcal{B}$. Recall that every hyperedge in $\mathcal{B}$ contains exactly one free node. Now the free node will either be in the same partition or in different ones.
If they are in different ones, we are done because we have a pair of coordinates such that, in one of them, one hyperedge is labelled positive while the other is labelled negative, while the opposite happens in the other coordinate. If the free nodes are in the same coordinate, we are done again because as the literal coordinate increases, the free coordinate decreases.
Note that, since we have already shuffled the nodes with free W nodes taking the 2-clause hyperedges into account, the 2-clause hyperedges are also taken care off.

Now, we argue that given an arbitrary hyperedge of the set $\mathcal{A}$, and an arbitrary hyperedge of the set $\mathcal{B}$, they are incomparable too.
For this, we notice that, the labels of the W nodes of all the hyperedges in $\mathcal{A}$ are higher than the labels of all the W nodes of the hyperedges in $\mathcal{B}$. For this, we simply note that range of the W labels of the second and the third occurrence (set $\mathcal{B}$) is $\{-5n, \ldots, 4n\} \setminus \{0\}$, whereas the W labels of the first occurrence ($\mathcal{A}$) has the range from $\{4n+1, \ldots, 7n\}$. Secondly, notice that the labels of the U and V literal nodes at the second and third occurrences, i.e., from the edges of set $\mathcal{B}$ (range $\{2n+1, \ldots, 6n\}$) are all higher than that of the first occurrence, i.e., from the edges of the set $\mathcal{A}$ (range $\{1, \ldots, 2n\}$).

We are done since for every pair of hyperedges (h_a, h_b), where $h_a \in \mathcal{A}$ and $h_b \in \mathcal{B}$, we have that the W coordinate of h_a will be higher than that of h_b, whereas the among the other two coordinates, whichever is

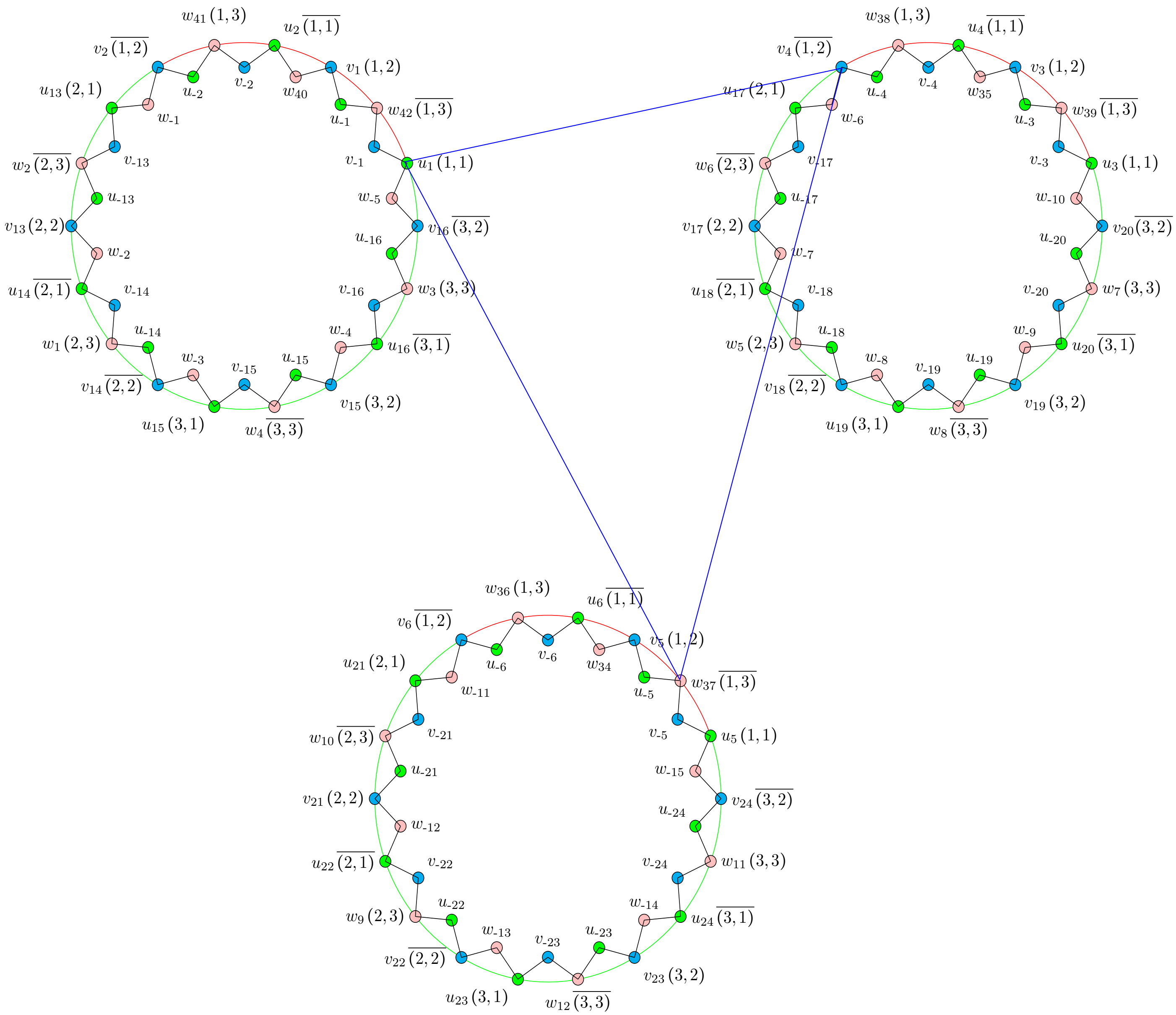

Figure 3: The variable gadgets $G_k^{\phi}, k = 1, 2, 3$ for $n = 6$. The hyperedges with a red arc correspond to the first occurrence of variables. Notice the difference in labelling of W nodes. The clause edge corresponds to the clause $Y_1 \vee \overline{Y_2} \vee \overline{Y_3}$.

positive (i.e., corresponds to a literal node) in h_b will be higher than the correpsonding coordinate in h_a.

Finally we are left to show that $\mathcal{A}$ is also an antichain.

We remind the reader that we have named the variables such that every 3-clause comprises of variables from only one triple of variables, i.e., every 3-clause involves $Y_{3(q-1)+1}, Y_{3(q-1)+2}, Y_{3(q-1)+3}$ at first, second and third position respectively, for some $q > 0$. Now first of all we notice that for a pair of hyperedges which come from a different triple of variables, we are done, because W coordinates of a higher triple are all lower than the W coordinates of a lower triple, since the labels are $(7n - 9(q-1) - \ell), \ell \in \{0, \dots, 8\}$ for q–th triple of variables $Y_{3(q-1)+1}, Y_{3(q-1)+2}, Y_{3(q-1)+3}$, whereas the positive coordinate among U or V will be higher for the higher triple (lables are $4(k-1) + \ell, \ell \in [2]$). When they are in the same triple of variables, it helps to remark that there are three kinds of hyperedges in $\mathcal{A}$, i.e. $\mathcal{A} = \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \mathcal{A}_c$:

- $\mathcal{A}_1$: the ones where the free nodes belong to U or V. These hyperedges have exactly one negative coordinate, which will either be in the U coordinate or the V coordinate.
- $\mathcal{A}_2$: the ones where the free nodes belong to W. All the coordinates are positive.
- $\mathcal{A}_c$: the set of 3-clause hyperedges: All the coordinates are again positive, as all the nodes are literal nodes.

Now, we need to compare the hyperedges of $\mathcal{A}_1, \mathcal{A}_2$ and $\mathcal{A}_c$ with each other and within themselves when they all belong to the same triple of variables, say q–th triple, $Y_{3(q-1)+1}, Y_{3(q-1)+2}, Y_{3(q-1)+3}$ for some $q \in [t]$. We remind the reader that the labelling of the W nodes that appear in $\mathcal{A}$ varies depending on whether the corresponding index $k = 3(q-1)+1$, $3(q-1)+1$, or $3(q-1)+3$.

There are six possible cases:

i. $\mathcal{A}_1$: same proof that was given for the elements of $\mathcal{B}$, where also we had exactly one negative coordinate.

ii. $\mathcal{A}_2$: for the higher variable, the W coordinate is lower (labels are $7n-9(q-1)-2$ for $k = 3(q-1)+1$, $7n-9(q-1)-7$ for $k = 3(q-1)+2$ and $7n-9(q-1)-8$ for $k = 3(q-1)+3$), while the other two coordinates are higher, since both U and V labels are $2(k-1)+1, 2$.

iii. $\mathcal{A}_c$: two different clauses clearly belong to different triple of variables: already taken care of above.

iv. $\mathcal{A}_1 - \mathcal{A}_2$ ($h_{a_1} \in \mathcal{A}_1, h_{a_2} \in \mathcal{A}_2$): Here we have two cases: namely, either h_{a_1} belonging to a higher variable, or h_{a_1} belonging to the same or lower variable as compared to h_{a_2}. In the first case, one of the U or V coordinate of h_{a_1} (whichever is positive) will be higher, while the other coordinate being negative will be lower than that of h_{a_2} (whose all coordinates are positive). In the second case, we note that the W coordinate of h_{a_2} will be lower, since for the same variable, it has the lowest W coordinate (being $7n-9(q-1)-2$ versus $7n-9(q-1)$, $7n-9(q-1)-1$ for $k = 3(q-1)+1$, $7n-9(q-1)-7$ versus $7n-9(q-1)-3$, $7n-9(q-1)-4$ for $k = 3(q-1)+2$ and $7n-3(k-1)-8$ versus $7n-9(q-1)-5$, $7n-9(q-1)-6$ for $k = 3(q-1)+3$), and as we go up the variables, W coordinate decreases, while at least one of the other two coordinate will be higher, i.e., in the coordinate in which h_{a_1} is negative and h_{a_2} is positive.

v. $\mathcal{A}_1 - \mathcal{A}_c$ ($h_{a_1} \in \mathcal{A}_1, h_{a_c} \in \mathcal{A}_c$): When h_{a_1} belongs to $G^{\phi}_{3(q-1)+1}$ or $G^{\phi}_{3(q-1)+2}$, its W coordinate will be higher than that of h_{a_c}, since for the clause hyperedge h_{a_c}, the W node is picked from $G^{\phi}_{3(q-1)+3}$. However, one of the other two coordinates in h_{a_1} is negative. So, it will be lower than that of h_{a_c}. So, we are done. When h_{a_1} belongs to $G^{\phi}_{3(q-1)+3}$, both h_{a_1} and h_{a_c} might share the W coordinate. However, in such h_{a_1}, the positive node among the U and V coordinate will be higher than that of h_{a_c}, since h_{a_1} comes from the highest variable among the triple, and both U and V coordinate increase with higher variables, being labelled $2(k-1)+1, 2$, whereas the negative coordinate will of course be lower than that of h_{a_c} which has no negative coordinate.

vi. $\mathcal{A}_2 - \mathcal{A}_c$ ($h_{a_2} \in \mathcal{A}_2, h_{a_c} \in \mathcal{A}_c$): Here when $h_{a_2} \in G^{\phi}_{3(q-1)+1}$, its V coordinate will be less since $Y_{3(q-1)+1}$ is the lowest variable, whereas the V coordinate of the clause hyperedge h_{a_c} is picked from $G^{\phi}_{3(q-1)+2}$. However, the W coordinate will be higher for h_{a_2} as it is labelled $7n-9(q-1)-2$, whereas the clause gets the W coordinate corresponding to the $G^{\phi}_{3(q-1)+3}$ and hence the label $7n-9(q-1)-5$ or $7n-9(q-1)-6$. Whereas when $h_{a_2} \in G^{\phi}_{3(q-1)+2}$ or $G^{\phi}_{3(q-1)+3}$, the W coordinate will be lower for h_{a_2} (labelled $7n-9(q-1)-7$ or $7n-9(q-1)-8$ respectively) than h_{a_c} (labelled $7n-9(q-1)-5$ or $7n-9(q-1)-6$), whereas the U coordinate of h_{a_2} will be higher, since the clause hyperedge h_{a_c} gets the U coordinate corresponding to variable $Y_{3(q-1)+1}$ which is the lowest variable within the triple and hence has the lowest U coordinate (U labels being $2(k-1)+1, 2$).

∎

6 Slice rank problem as a union of orbit closures

For the necessary mathematical background, the reader is referred [51, 39, 40, 7].

The tensor used to show that the slice rank problem can be phrased as a problem about the union of orbit closures (see Definition 6.2) turns out to be very similar to the tensor $T_{k,n,r}$, which we used to show that the minrank problem is an instance of the orbit closure containment problem in Section 4. Thus, the exposition and the proofs in this section are very similar to that in Section 4.

Let us say we are given a 3-tensor $T \in U \otimes V \otimes W$, and we are interested in finding out if it has slice rank at most r, i.e., if $\text{srk}(T) \leq r$.

In what follows, we phrase this problem geometrically and formulate it as membership testing of T in a union of orbit closures of certain tensors.

Lemma 6.1. *([55, Corollary 2]) Let U, V, W be vector spaces over an algebraically closed field $\mathbb{F}$. The set of all tensors $T \in U \otimes V \otimes W$ with slice rank at most r is a Zariski closed set.*

In fact, they even showed that the set of all tensors $T \in U \otimes V \otimes W$ with slice rank at most r decomposed as (r_1, r_2, r_3) for a fixed tuple (r_1, r_2, r_3) with $r_1+r_2+r_3 = r$ is also Zariski closed.

Definition 6.1. *We call the the affine variety*

$$\mathcal{SV}_{U\otimes V\otimes W,r} = \{T \in U \otimes V \otimes W | \text{srk}(T) \leq r\}$$

the affine slice rank variety *or simply the* slice rank variety*.*

When clear from the context, we drop the index $U \otimes V \otimes W$.

Lemma 6.2. *Let U, V, and W be subspaces of vector spaces U', V', and W', respectively. Then*

$$\mathcal{SV}_{U\otimes V\otimes W,r} = \mathcal{SV}_{U'\otimes V'\otimes W',r} \cap (U \otimes V \otimes W).$$

Proof. A tensor lies in $\mathcal{SV}_{U\otimes V\otimes W,r}$ iff it is an element of the space $U \otimes V \otimes W$ and has slice rank at most r, i.e., lies in $\mathcal{SV}_{U'\otimes V'\otimes W',r}$. ∎

Lemma 6.3. *The slice rank variety $\mathcal{SV}_{U\otimes V\otimes W,r}$ is invariant under the standard action of $GL(U) \times GL(V) \times GL(W)$ on $U \otimes V \otimes W$.*

Proof. If $\text{srk}(T) \leq r$, we have $T = \sum_{i=1}^{r_1} u_{i,1} \otimes_1 T_{i,1} + \sum_{i=1}^{r_2} u_{i,2} \otimes_2 T_{i,2} + \sum_{i=1}^{r_3} u_{i,3} \otimes_3 T_{i,3}$ for some (r_1, r_2, r_3) such that $r_1 + r_2 + r_3 = r$, where $u_{i,1} \in U$, $u_{i,2} \in V$, $w_{i,3} \in W$, and $T_{i,1} \in V \otimes W$, $T_{i,2} \in U \otimes W$, $T_{i,3} \in U \otimes V$. Clearly when $A \otimes B \otimes C \in \mathsf{GL}(U) \times \mathsf{GL}(V) \times \mathsf{GL}(W)$ acts on T, the slice rank remains at most r. ∎

6.1 Slice rank varieties and orbit closures

For every tuple (r_1, r_2, r_3) of non-negative integers such that $r_1+r_2+r_3 = r$, we consider the vector spaces $U'_{(r_1,r_2,r_3)} = \mathbb{F}^{r_1} \oplus (\mathbb{F}^n)^{\oplus(r_2)} \oplus (\mathbb{F}^n)^{\oplus(r_3)}$, $V'_{(r_1,r_2,r_3)} = (\mathbb{F}^n)^{\oplus(r_1)} \oplus \mathbb{F}^{r_2} \oplus (\mathbb{F}^n)^{\oplus(r_3)}$, and $W'_{(r_1,r_2,r_3)} = (\mathbb{F}^n)^{\oplus(r_1)} \oplus (\mathbb{F}^n)^{\oplus(r_2)} \oplus \mathbb{F}^{r_3}$. We will drop the index (r_1, r_2, r_3) in the following.

U' has dimension $s_1(r_1, r_2, r_3) = r_1 + nr_2 + nr_3$, and is decomposed into $1 + r_2 + r_3$ summands, where one summand is of dimension r_1, while the other summands are of dimensions n each. Similarly, V' and W' have dimensions $s_2(r_1, r_2, r_3) = nr_1 + r_2 + nr_3$ and $s_3(r_1, r_2, r_3) = nr_1 + nr_2 + r_3$, respectively, and are decomposed analogously as U', into r_1+1+r_3 summands and $r_1 + r_2 + 1$ summands respectively. We will denote $s_1(r_1, r_2, r_3)$, $s_2(r_1, r_2, r_3)$ and $s_3(r_1, r_2, r_3)$ simply by s_1, s_2, and s_3, respectively. Thus $U' \otimes V' \otimes W' \cong \mathbb{F}^{s_1} \otimes \mathbb{F}^{s_2} \otimes \mathbb{F}^{s_3}$.

Let us give names to the components: Let L^1 be $(\mathbb{F}^n)^{\oplus(r_1)}$ of dimension nr_1, L^2 be $(\mathbb{F}^n)^{\oplus(r_2)}$, and L^3 be $(\mathbb{F}^n)^{\oplus(r_3)}$, respectively, and we have vector spaces $\tilde{U} = \mathbb{F}^{r_1}$, $\tilde{V} = \mathbb{F}^{r_2}$ and $\tilde{W} = \mathbb{F}^{r_3}$ respectively. Let L_i^k be the i-th summand of $L^k, k \in \{1, 2, 3\}$ with standard basis e_{ij}^k, $j \in [n]$, and let e_i^1, e_i^2 and e_i^3 be the standard basis of $\tilde{U}$, $\tilde{V}$ and $\tilde{W}$. We have $U' = \tilde{U} \oplus L_1^2 \oplus \cdots \oplus L_{r_2}^2 \oplus L_1^3 \oplus \cdots \oplus L_{r_3}^3$ and similar decomposition for V' and W'.

Definition 6.2. *For (r_1, r_2, r_3), we define the* unit slice rank tensor *$S_{n,r_1,r_2,r_3} \in (\tilde{U} \otimes L^1 \otimes L^1) \oplus (L^2 \otimes \tilde{V} \otimes L^2) \oplus (L^3 \otimes L^3 \otimes \tilde{W}) \subseteq U' \otimes V' \otimes W'$ as*

$$S_{n,r_1,r_2,r_3} = \sum_{i=1}^{r_1} \sum_{j=1}^{n} e_i^1 \otimes e_{ij}^1 \otimes e_{ij}^1 + \sum_{i=1}^{r_2} \sum_{j=1}^{n} e_{ij}^2 \otimes e_i^2 \otimes e_{ij}^2 + \sum_{i=1}^{r_3} \sum_{j=1}^{n} e_{ij}^3 \otimes e_{ij}^3 \otimes e_i^3.$$

Along $\tilde{U}$ we have r_1 slices where each slice contains an $n \times n$ identity matrix each in disjoint blocks. Then along $\tilde{V}$, we have r_2 slices with $n \times n$ identity matrices in disjoint blocks. Finally, we have r_3 slices with $n \times n$ identity matrices in disjoint blocks along $\tilde{W}$. Thus S_{n,r_1,r_2,r_3} can be decomposed into three summands $S_{n.r_1} \in \tilde{U} \otimes L^1 \otimes L^1, S_{n,r_2} \in L^2 \otimes \tilde{V} \otimes L^2$ and $S_{n,r_3} \in L^3 \otimes L^3 \otimes \tilde{W}$ such that $S_{n,r_1,r_2,r_3} = S_{n,r_1} \oplus S_{n,r_2} \oplus S_{n,r_3}$. Notice the similarity with the minrank case. In particular, $T_{k,n,r}$ is almost like S_{n,r_1} – the only difference is that the first slice in $T_{k,n,r}$r is of different rank than the rest of its slices. As a consequence, in the spirit, the proof of the following Theorem 6.1 is very similar to the proof of Theorem 4.2.

The group $\mathsf{GL}_{s_1} \times \mathsf{GL}_{s_2} \times \mathsf{GL}_{s_3}$ acts on $U' \otimes V' \otimes W'$ in a natural way. The slice rank variety can be defined as the union of orbit closures of S_{n,r_1,r_2,r_3} under the action of $\mathsf{GL}_{s_1} \times \mathsf{GL}_{s_2} \times \mathsf{GL}_{s_3}$, where the union is taken over (r_1, r_2, r_3) such that $r_1 + r_2 + r_3 = r$.

Theorem 6.1. *Let U, V, and W be n-dim. subspaces of U', V', and W', respectively. Then* $\mathcal{SV}_{U\otimes V\otimes W,r} =$

$$\bigcup_{\substack{r_1,r_2,r_3\\ r_1+r_2+r_3=r}} \overline{(\mathsf{GL}_{s_1} \times \mathsf{GL}_{s_2} \times \mathsf{GL}_{s_3}) S_{n,r_1,r_2,r_3}} \cap (U \otimes V \otimes W).$$

Note that each of the orbit closures is taken in a different ambient space, since each S_{n,r_1,r_2,r_3} lives in a different ambient space. But since we intersect each closure with $U \otimes V \otimes W$, this is fine.

Proof. First of all note that for every such (r_1, r_2, r_3), we have that $S_{n,r_1,r_2,r_3} \in \mathcal{SV}_{U'\otimes V'\otimes W',r}$, simply by the construction of S_{n,r_1,r_2,r_3}, where $U' \cong \mathbb{F}^{s_1}, V \cong \mathbb{F}^{s_2}, W' \cong \mathbb{F}^{s_3}$. Now since by Lemma 6.3, $\mathcal{SV}_{U'\otimes V'\otimes W',r}$ is invariant under the action of $\mathsf{GL}(U') \times \mathsf{GL}(V') \times \mathsf{GL}(W')$, we have that the entire orbit $(\mathsf{GL}_{s_1} \times \mathsf{GL}_{s_2} \times \mathsf{GL}_{s_3}) S_{n,r_1,r_2,r_3}$ lies in it. Also, from Lemma 6.1 (see [55, Corollary 2]), it follows that $(\mathsf{GL}_{s_1} \times \mathsf{GL}_{s_2} \times \mathsf{GL}_{s_3}) S_{n,r_1,r_2,r_3}$ is contained in a Zariski closed subset of $\mathcal{SV}_{U'\otimes V'\otimes W'}$ and hence the orbit closure $\overline{(\mathsf{GL}_{s_1} \times \mathsf{GL}_{s_2} \times \mathsf{GL}_{s_3}) S_{n,r_1,r_2,r_3}}$ also lies in $\mathcal{SV}_{U'\otimes V'\otimes W'}$. Now we apply Lemma 6.2 to get the desired inclusion.

For the other direction, let us assume $T \in \mathcal{SV}_{U\otimes V\otimes W,r}$. Since $\operatorname{srk}(T) \le r$, we have that we have $T = \sum_{i=1}^{r_1} u_{i,1} \otimes_1 T_{i,1} + \sum_{i=1}^{r_2} u_{i,2} \otimes_2 T_{i,2} + \sum_{i=1}^{r_3} u_{i,3} \otimes_3 T_{i,3}$, for some (r_1, r_2, r_3) such that $r_1 + r_2 + r_3 = r$, where $u_{i,1} \in U$, $u_{i,2} \in V$, and $w_{i,3} \in W$ and $T_{i,1} \in V \otimes W$, $T_{i,2} \in U \otimes W$, and $T_{i,3} \in U \otimes V$. Since $\forall i \in [r_1]$, $\operatorname{rk}(T_{i,1}) \le n$, we can write $T_{i,1}$ as $(Q_{i,1} \otimes R_{i,1})(\sum_{j=1}^n e^1_{i,j} \otimes e^1_{i,j})$ for linear maps $Q_{i,1} : L^1_i \to V$ and $R_{i,1} : L^1_i \to W$. Analogously, $T_{i,2} = (P_{i,2} \otimes R_{i,2})(\sum_{j=1}^n e^2_{i,j} \otimes e^2_{i,j})$ for linear maps $P_{i,2} : L^2_i \to U$ and $R_{i,2} : L^2_i \to W$, and $T_{i,3} = (P_{i,3} \otimes Q_{i,3})(\sum_{j=1}^n e^3_{i,j} \otimes e^3_{i,j})$ for linear maps $P_{i,3} : L^3_i \to U$ and $Q_{i,3} : L^3_i \to V$.

Let $Q_1 : L^1 \to V$ and $R_1 : L^1 \to W$ be linear maps which are equal to $Q_{i,1}$ and $R_{i,1}$, respectively, when restricted to the i-th slice L^1_i. Similarly we have maps $P_2 : L^2 \to U$ and $R_2 : L^2 \to W$ whose restrictions to i-th slices are $P_{i,2}$ and $R_{i,2}$, respectively, and $P_3 : L^3 \to U$ and $Q_3 : L^3 \to V$ have their restrictions as $P_{i,3}$ and $Q_{i,3}$.

Finally, we also have linear maps $P_1 : \tilde{U} \to U$ sending e^1_i to $u_{i,1}$, $Q_2 : \tilde{V} \to V$ sending e^2_i to $u_{i,2}$ and $R_3 : \tilde{W} \to W$ sending e^3_i to $u_{i,3}$.

Thus $T = ((P_1 \otimes Q_1 \otimes R_1) \oplus (P_2 \otimes Q_2 \otimes R_2) \oplus (P_3 \otimes Q_3 \otimes R_3)) S_{n,r_1,r_2,r_3}$ for some (r_2, r_2, r_3). The closure of $\mathsf{GL}_{s_1}, \mathsf{GL}_{s_2}$ and GL_{s_3} contains all linear endomorphisms of U', V' and W', respectively, and thus contains $(P_1 \otimes Q_1 \otimes R_1) \oplus (P_2 \otimes Q_2 \otimes R_2) \oplus (P_3 \otimes Q_3 \otimes R_3)$. Therefore, T lies in the closure $\overline{(\mathsf{GL}_{s_1} \times \mathsf{GL}_{s_2} \times \mathsf{GL}_{s_3}) S_{n,r_1,r_2,r_3}}$ for some (r_1, r_2, r_3) with $r_1 + r_2 + r_3 = r$. □

7 Complexity of the orbit containment problem

The orbit containment problem ("$w \in Gv$") can be phrased as a polynomial systems of polynomial size by simply writing out the equations of gv for some generic g, and therefore can reduced to the problem Hilbert's Nullstellensatz HN. To ensure that $\det(g) \neq 0$, we can use a poly-size circuit for det to encode $z\det(g) = 1$ as a poly-size system of equations, where z is a new variable. Thus, we have the following theorem.

Theorem 7.1. *Let $\mathbb{F}$ be a field and K be an effective subfield. Then the orbit containment problem over $\mathbb{F}$ (with coefficients from K) is polynomial-time reducible to Hilbert's Nullstellensatz* HN *over $\mathbb{F}$ (with coefficients from K).*

By the results of Koiran [38], the above theorem implies that the orbit containment problem over the complex numbers is in AM assuming the generalized Riemann hypothesis (GRH), since Koiran's result also assumes GRH.

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