Hamiltonicity, Pancyclicity and Full Cycle Extendability in Multipartite Tournaments *

Zan-Bo Zhang[1,3†], Xiaoyan Zhang[2,3‡], Gregory Gutin[4§], Dingjun Lou[5¶]

[1]School of Information Technology, Guangdong Industry Polytechnic, Guangzhou 510300, China

[2]Institute of Mathematics & School of Mathematical Science, Nanjing Normal University, Nanjing, 210023, China

[3] Faculty of EEMCS, University of Twente, P.O. Box 217, 7500 AE Enschede, The Netherlands

[4] Department of Computer Science, Royal Holloway, University of London, Egham, TW20 0EX, UK

[5] Department of Computer Science, Sun Yat-sen University, Guangzhou 510275, China

Abstract

A digraph D with n vertices is Hamiltonian (pancyclic, vertex-pancyclic, respectively) if D contains a Hamilton cycle (a cycle of every length $3, 4, \ldots n$, for every vertex $v \in V(D)$, a cycle of every length $3, 4, \ldots n$ through v, respectively.) It is well-known that a strongly connected tournament is Hamiltonian (Camion 1959), pancyclic (Harary and Moser 1966), and vertex pancyclic (Moon 1968). A digraph D is cycle extendable if for every non-Hamiltonian cycle C of D, there is a cycle C' such that C' contains all vertices of C plus another vertex of D. A cycle extendable digraph is fully cycle extendable if for every vertex $v \in V(D)$, there exists a cycle of length 3 through v. Note that full cycle extendability is a stronger property than vertex pancyclicity. Hendry (1989) showed that not every strongly connected tournament is fully cycle extendable and characterized an infinite wide class of strongly connected tournaments, which are not fully cycle extendable.

A k-partite tournament is an orientation of a k-partite complete graph (for $k = 2$, it is called a bipartite tournament). Gutin (1984) and Häggkvist and Manoussakis (1989) characterized Hamiltonian bipartite tournaments. A bipartite digraph D with n vertices is even pancyclic (even vertex pancyclic, respectively) if D contains a cycle of every even length $4, 6, \ldots, n$ (a cycle of every even length $4, 6, \ldots, n$ through v for every $v \in V(D)$, respectively). Beineke and Little (1982) and Zhang (1984) proved that every bipartite tournament is even pancyclic and even vertex pancyclic, respectively, if and only if it is Hamiltonian and does not belong to a well-defined infinite class of regular bipartite tournaments. We prove that unlike the case of tournaments, every even pancyclic bipartite tournament is fully cycle extendable. We show that this result cannot be extended to k-partite tournaments for any fixed $k \geq 3$ (where we naturally replace even vertex pancyclicity by vertex pancyclicity).

Key words: hamiltonicity, full cycle extendability, pancyclicity, vertex pancyclicity, bipartite tournaments, multipartite tournaments.

1 Introduction

In this paper, we consider directed and undirected graphs. When it is clear from the context which kind of graphs is considered or the definition/result is applicable to both kinds of graphs, we will omit the adjectives directed and undirected. In digraphs, all considered cycles and paths will be directed and thus we will simply call them cycles and paths.

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†Email address: eltonzhang2001@gmail.com

‡Corresponding Author. Email address: royxyzhang@gmail.com, zhangxiaoyan@njnu.edu.cn

§Email address: g.gutin@rhul.ac.uk

¶Email address: issldj@mail.sysu.edu.cn

A *Hamiltonian* cycle is a cycle in a graph which passes through all vertices. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. A graph G of order n is *pancyclic* if it contains cycles of lengths $3, 4, \ldots n$, and G is *vertex-pancyclic* if every vertex of G is contained in cycles of lengths $3, 4, \ldots n$. A graph G is *cycle extendable* if G is not acyclic and for every non-Hamiltonian cycle C in G there is a cycle C' whose vertices are all vertices of C plus another vertex. We will call G *fully cycle extendable* if it is cycle extendable and for every $v \in V(D)$, there is a cycle of length 3 through v. Full cycle extendability is a significantly stronger property than pancyclicity and vertex pancyclicity since full cycle extendability requires several vertices rather than none or just one to be on the cycle.

Tournaments and Multipartite Tournaments It is well-known that a tournament is Hamiltonian (pancyclic, vertex pancyclic, respectively) if it is strongly connected as proved by Camion [7] (Harary and Moser [24], Moon [25], respectively). In what follows, we will often denote the fact that xy is an arc by $x \to y$. Moreover, for vertex subsets or subgraphs X and Y of a digraph, $X \to Y$ will mean that $x \to y$ for every $x \in X$ and $y \in Y$. Hendry [18] showed that a strongly connected (and thus Hamiltonian) tournament is fully cycle extendable unless it belongs to the following family $\mathcal{T}$ of tournaments: a tournament $T = (V, A)$ is in $\mathcal{T}$ if V can be partitioned into three non-empty sets W, X, and Y such that $|W| \geq 2$, $T[W]$ is strong, $W \to X$ and $Y \to W$. Note that $\mathcal{T}$ is quite a wide class of tournaments. For more information on tournaments, see, e.g., Chapter 2 of a recent edited volume [4].

In this paper, we focus on multipartite tournaments, which are generalizations of tournaments. A *multipartite tournament* or *k-partite tournament* is an orientation of a k-partite complete graph, and thus k-partite tournaments are an extension of tournaments to k-partite graphs. When $k = 2$, k-partite tournaments are called *bipartite tournaments*. Many results on paths and cycles in multipartite tournaments (including bipartite tournaments) are collected in Chapter 7 of a recent edited volume [28].

Hendry's characterization of fully cycle extendable tournaments in [18] implies that for every $k \geq 5$ there exists a finite positive number of k-partite tournaments which are pancyclic but not cycle extendable. The following result extends this fact to $k \geq 3$ and infinite number of such k-partite tournaments. We will see shortly that the case of $k = 2$ is quite different.

Theorem 1.1. *For every $k \geq 3$, there is an infinite number of k-partite tournaments that are pancyclic but not cycle extendable.*

In bipartite graphs G, the analog of pancyclicity is *even pacyclicity*, where only cycles of length $4, 6, \ldots, n$ are required (n is the order of G). Similarly, one defines *even vertex pacyclicity* and *full even cycle extendability*. A *cycle factor* is a disjoint collection of cycles covering all vertices of the graph. Clearly, a Hamiltonian cycle is a cycle factor with just one cycle. Jackson [19] proved a sufficient condition for a bipartite tournament to be Hamiltonian. Hamiltonian bipartite tournaments were characterized by Gutin [13] and, independently, by Häggkvist and Manoussakis [16] as follows.

Theorem 1.2. *Any bipartite tournament T is Hamiltonian if and only if it is strong and has a cycle factor.*

One of the most important implications of Theorem 1.2 is that Hamiltonicity in bipartite tournaments can be decided in polynomial time [13, 14, 16, 22].

Let $T(r, r, r, r)$ be a bipartite tournament whose vertex set can be partitioned into four sets V_0, V_1, V_2 and V_3, each of size r, such that $V_0 \cup V_2$ and $V_1 \cup V_3$ are the parts of the bipartition, and $V_i \to V_{i+1}$ for $i \in \{0, 1, 2, 3\}$ where $V_4 = V_0$. The following characterization of even pancyclic and even vertex-pancyclic bipartite tournaments were obtained by Beineke and Little [5] and K. M. Zhang (see, e.g., [2, 28]), respectively. (The even vertex-pancyclicity result was proved independently by Häggkvist and Manoussakis [16].)

Theorem 1.3. *A bipartite tournament is even pancyclic as well as even vertex-pancyclic if and only if it is Hamiltonian and is not isomorphic to $T(r, r, r, r)$ for any $r \geq 2$.*

The following theorem is the main result of this paper. It shows that, in the sharp contrast to tournaments and k-partite tournaments with $k \geq 3$, every even pancyclic bipartite tournament is fully cycle extendable. Theorem 1.4 is a significant strengthening of Theorem 1.3.

Theorem 1.4. *A bipartite tournament is fully cycle extendable if and only if it is Hamiltonian and is not isomorphic to $T(r, r, r, r)$ for any $r \geq 2$.*

Results on Other Classes of Graphs Apart from those on k-partite tournaments, there have been many results in the literature linking Hamiltonicity, pancyclicity, vertex pancyclicity, and/or fully cycle extendability. There are many results on Hamiltonian and pancyclic undirected graphs, see e.g. survey papers [10, 11, 12] of Gould. There are less such results on digraphs (see e.g. [2, 20, 29]). One of the first sufficient conditions for Hamiltonicity was Dirac's theorem [9], which asserts that every graph of order $n \geq 3$ and minimum degree at least $n/2$ is Hamiltonian. This theorem was generalized by Bondy [6], who showed that the same assumptions imply that either G is pancyclic or n is even and G is isomorphic to $K_{n/2,n/2}$. Hendry [17] showed that Dirac's condition also implies cycle extendability (in undirected graphs), with some exceptional classes that can be characterized. Hendry [17] stated, as an open problem the following question: is a Hamiltonian chordal graph fully cycle extendable? Positive results were obtained for special classes of chordal graphs in [1] and [8], but unfortunately, in general, the answer to the question is proved to be negative [21].

Apart from the papers mentioned above, cycle extendability in digraphs was studied in [23, 26, 30].

Discussion of Theorem 1.4 From Theorem 1.3 and Theorem 1.4, we see that the same condition imply even pancyclicity, even vertex-pancyclicity and full even cycle extendibility for bipartite tournaments. This is somewhat unexpected since similar results does not hold for tournaments. As we have mentioned above, while strong connectivity in tournaments implies pancyclicity, it can only imply cycle extendability if we exclude the wide class of tournaments $\mathcal{T}$.

Since full cycle extendability is a much stronger property than pancyclicity, it is unsurprising that proving Theorem 1.4 brings new challenges. In particular, the techniques used in the proof of Theorem 1.3 in [5] and [16] are not sufficient for our proof. We introduce a new concept of the in-out graph of a digraph and use its properties to prove Theorem 1.4.

The *in-out graph* of a digraph D is a graph that takes the arc set of D as its vertex set, and in which two vertices are joined by a red (green) edge if they share a common head (tail) in D. The concept looks similar to that of the line graph and actually, in-out graphs can be viewed as line graphs of a certain class of bipartite graphs (see Section 4). Let C_0 and C_1 be two arc-disjoint Hamiltonian cycles on the same vertex set, and denote the digraph formed by these two cycles by $C_0 \cup C_1$. Let L be the in-out graph of $C_0 \cup C_1$. It is not hard to see that the independent sets of L correspond to the path-cycle subgraphs of $C_0 \cup C_1$. Here path-cycle subgraphs refer to subgraphs consisting of disjoint paths and cycles. Therefore, we can use L to construct and analyze the path-cycle spanning subgraphs of $C_0 \cup C_1$, for instance, as in the basic result of Theorem 4.1. The condition of our theorem guarantees the existence of a Hamiltonian cycle H. In our proof, we need to consider even extendability of another cycle C. We perform some contraction operations on $C \cup H$ to map C and H into arc-disjoint Hamiltonian cycles C_0 and C_1 on the same vertex set. Next, we construct L and by analyzing L, we derive many structural properties of $C \cup H$, which imply even extendability of C. These ideas and techniques play crucial role in our proof of the main result.

Paper Organization Section 2 contains additional terminology and notation. Theorem 1.1 is proved in Section 3. In Section 4, as an application of in-out graphs, we prove Theorem 4.1. In Section 5, we give a proof of Theorem 1.4, where Theorem 4.1 is used as a tool. We conclude the paper in Section 6 with some open problems.

2 Terminology and Notation

In this section, we provide most of the terms and notations used in this paper, while a few others will be introduced when used in the sequel, for convenience. The concepts that are not explicitly defined follow those of [2].

We often use D to denote a digraph, and T to denote a bipartite tournament. The vertex set and arc set of a digraph D are denoted by $V(D)$ and $A(D)$, respectively. Let $X \subseteq V(D)$, the subgraph of D induced by X is denoted by $D[X]$. We use $d^-(x)$ and $d^+(x)$ to denote the in-degree and out-degree of a vertex $x \in V(T)$, respectively.

A *k-path-cycle subgraph* F of a digraph D is a collection of k paths and m cycles such that all paths and cycles are pairwise disjoint. If F spans D, we say that it is a *k-path-cycle factor* of D. When $k = 0$ in the above definitions, we call F a *cycle subgraph* and a *cycle factor*, respectively.

Let D be a digraph with n vertices. If D contains cycle of length k for every $3 \leq k \leq n$, we say that D is *pancyclic*. D is *vertex-pancyclic* (*arc-pancyclic*) if it has a cycle of length k containing v (a), for every $3 \leq k \leq n$ and every vertex $v \in V(D)$ (every arc $a \in A(D)$). The concept of *even pancyclicity*, *even vertex-pancyclicity* and *even arc-pancyclicity* are defined analogously, but in this cases only cycles of (all) even length(s) are required.

3 Proof of Theorem 1.1

Firstly we describe the construction. By T_k, we denote a tournament with vertex set $\{v_i : 0 \leq i \leq k-1\}$ and arc set

$$\{v_{k-1}v_0\} \cup \{v_iv_j : \ 0 \leq i < j \leq k-1 \text{ and } (i,j) \neq (0, k-1)\}.$$

Let T_k^r be a k-partite tournament obtained from T_k by replacing every vertex v_i with a set of $r \geq 3$ vertices, $V_i = \{v_{i,0}, v_{i,1}, v_{i,2}, \ldots, v_{i,r-1}\}$, and replacing every arc v_iv_j with all the arcs $v_{i,s}v_{j,t}$, where $0 \leq i,j \leq k-1$, $0 \leq s,t \leq r-1$, and all V_i's are mutually disjoint. Furthermore, let $T_3^{r\prime}$ be obtained from T_3^r by reversing the arcs $v_{0,r-2}v_{1,r-2}$ and $v_{0,r-1}v_{1,r-1}$, and let $T_4^{r\prime}$ be obtained from T_4^r by reversing the arc $v_{0,r-1}v_{1,r-1}$. Finally let $\mathcal{T}^r = \{T_k^r : k \geq 5, r \geq 3\} \cup \{T_3^{r\prime}, T_4^{r\prime} : r \geq 3\}$. We prove that all digraphs in $\mathcal{T}^r$ are pancyclic (and thus Hamiltonian) but not cycle extendable.

In T_k^r, we denote the path $v_{0,i}v_{1,i}\ldots v_{k-1,i}$ by P_i, and let $P_i[v_{s,i}, v_{t,i}]$ denote the subpath of P_i from $v_{s,i}$ to $v_{t,i}$, for $0 \leq i \leq r-1$ and $0 \leq s < t \leq k-1$.

We firstly show that T_k^r is pancyclic for any integer $k \geq 5$, by listing a cycle of length l for every $3 \leq l \leq rk = n$ in Table 1.

Length	Cycle
$3 \leq l \leq k$	$v_{0,0}P_0[v_{k-l+1,0}, v_{k-1,0}]v_{0,0}$
$l = jk+i$, for $3 \leq i \leq k$ and $1 \leq j \leq r-1$	$v_{0,0}P_0[v_{k-i+1,0}, v_{k-1,0}]P_1 \ldots P_jv_{0,0}$
$l = k+1$	$v_{0,0}P_0[v_{3,0}, v_{k-1,0}]v_{0,1}v_{k-2,1}v_{k-1,1}v_{0,0}$
$l = jk+1$, for $2 \leq j \leq r-1$	$v_{0,0}P_0[v_{3,0}, v_{k-1,0}]v_{0,1}v_{k-2,1}v_{k-1,1}P_2 \ldots P_jv_{0,0}$
$l = k+2$	$v_{0,0}P_0[v_{2,0}, v_{k-1,0}]v_{0,1}v_{k-2,1}v_{k-1,1}v_{0,0}$
$l = jk+2$, for $2 \leq j \leq r-1$	$v_{0,0}P_0[v_{2,0}, v_{k-1,0}]v_{0,1}v_{k-2,1}v_{k-1,1}P_2 \ldots P_jv_{0,0}$

Table 1: Cycles of every length from 3 to $n = rk$ in T_k^r for $k \geq 5$

To prove that T_k^r is not cycle extendable, it suffices to find a non-Hamiltonian cycle of T_k^r which is not extendable. We firstly show that every cycle Z of length at least $2k+1$ in T_k^r must contain at least three vertices from V_0 and three vertices from V_{k-1}. Since $T_k^r - V_0$ is acyclic, Z must contain at least a vertex from V_0. Note that any segment $v_{0,i}\ldots v_{0,j}$ ($0 \leq i,j \leq r-1$ and possibly $i = j$) of Z between two vertices from V_0 but without an internal vertex from V_0 is of length at most k. Hence if Z contains at most two vertices from V_0, then its length must be at most $2k$, contradicting the assumption on its length. Thus Z contains at least three vertices from V_0. Then, Z must contain three arcs from V_{k-1} to V_0, so it must contain three vertices from V_{k-1} as well. Now consider the cycle $Z_1 = P_0P_1v_{0,0}$ which is of length $2k$. If Z_1 is extendable to a cycle Z_2 of length $2k+1$, then Z_2 must contain three vertices from V_0 and three vertices from V_{k-1} by above discussion. However, Z_1 contains two vertices from V_1 and two vertices from V_{k-1}, thus $|V(Z_2)\backslash V(Z_1)| \geq 2$, a contradiction. Thus, Z_1 is not extendable. So, T_k^r with $k \geq 5$ is pancyclic, but not cycle extendable.

Now we consider $T_4^{r\prime}$. Again, we list a cycle of length l in $T_4^{r\prime}$, for every $3 \leq l \leq 4r$, in Table 2, to prove that $T_4^{r\prime}$ is pancyclic.

Length	Cycle
$l = 3$	$v_{0,0}v_{2,0}v_{3,0}v_{0,0}$
$l = 4$	$P_0v_{0,0}$
$l = 5$	$v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{3,r-2}v_{0,r-2}$
$l = 6$	$v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{2,r-2}v_{3,r-2}v_{0,r-2}$
$l = 4j - 1$, for $2 \leq j \leq r$	$v_{0,0}v_{2,0}v_{3,0}P_1 \ldots P_{j-1}v_{0,0}$
$l = 4j$, for $2 \leq j \leq r$	$P_0 \ldots P_{j-1}v_{0,0}$
$l = 4j + 1$, for $2 \leq j \leq r - 1$	$P_0 \ldots P_{j-2}v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{3,r-2}v_{0,0}$
$l = 4j + 2$, for $2 \leq j \leq r - 1$	$P_0 \ldots P_{j-2}v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{2,r-2}v_{3,r-2}v_{0,0}$

Table 2: Cycles of every length from 3 to $4r$ in $T_4^{r\prime}$

Next we prove that any cycle Z_3 of length 5 in $T_4^{r\prime}$ must contain the arc $v_{1,r-1}v_{0,r-1}$. Suppose that Z_3 does not contain $v_{1,r-1}v_{0,r-1}$; then it is also a cycle in T_4^r. Similar to the discussion on T_k^r with $k \geq 5$, we can conclude that Z_3 contains at least two vertices from V_0 and two vertices from V_3. So traversing Z_3 we at least go from V_0 to V_3 twice. However there is no arc from V_0 to V_3. Thus, to go from V_0 to V_3 we must pass at least one vertex in $V_1 \cup V_2$. Therefore, we need at least two more vertices from $V_1 \cup V_2$ on Z_3. But Z_3 is of length 5 and we can add only one more vertex from $V_1 \cup V_2$, a contradiction. Therefore Z_3 must contain $v_{1,r-1}v_{0,r-1}$. But then no cycle extends any cycle on four vertices which avoids both $v_{0,r-1}$ and $v_{1,r-1}$, say, $P_0v_{0,0}$. Therefore, $T_4^{r\prime}$ is not cycle extendable.

Finally we prove that $T_3^{r\prime}$ is pancyclic but not cycle extendable. Pancyclicity of $T_3^{r\prime}$ is proved by a list of cycles of length l for $3 \leq l \leq 3r$ in Table 3.

Length	Cycle
$l = 3$	$P_0v_{0,0}$
$l = 4$	$v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{0,r-2}$
$l = 5$	$v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{2,r-2}v_{0,r-2}$
$l = 3j$, for $2 \leq j \leq r$	$P_0 \ldots P_{j-1}v_{0,0}$
$l = 3j + 1$, for $2 \leq j \leq r - 1$	$P_0 \ldots P_{j-2}v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{0,0}$
$l = 3j + 2$, for $2 \leq j \leq r - 1$	$P_0 \ldots P_{j-2}v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{2,r-2}v_{0,0}$

Table 3: Cycles of every length from 3 to $3r$ in $T_3^{r\prime}$

Now we prove that the only cycle of length 4 in $T_3^{r\prime}$ is $v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{0,r-2}$. Let Z_4 be a cycle of length 4 in $T_3^{r\prime}$. If $V(Z_4) \cap V_2 \neq \emptyset$, then Z_4 contains a path $Q = uvw$, where $u \in V_1$, $v \in V_2$ and $w \in V_0$. However, to form Z_4 we need a path of length 2 from w to u, which does not exist. Thus $Z_4 \cap V_2 = \emptyset$, and $V(Z_4) \subseteq V_0 \cup V_1$. Then Z_4 must contain two arcs from V_1 to V_0, therefore the arcs $v_{1,r-2}v_{0,r-2}$ and $v_{1,r-1}v_{0,r-1}$ must be on Z_4. So $Z_4 = v_{0,r-2}v_{1,r-1}v_{0,r-1}v_{1,r-2}v_{0,r-2}$. Observe that any cycle of length 3 in Z_4 must have exactly one vertex from V_i for $i \in \{0, 1, 2\}$. Hence none of them is extendable. So $T_3^{r\prime}$ is pancyclic but not cycle extendable.

Thus, we have proved that all digraphs in $\mathcal{T}^3$ are pancyclic, but not cycle extendable. □

4 In-out graph, and path-cycle factors of two arc-disjoint Hamiltonian cycles

Let D be a digraph. The *in-out graph* of D is defined as a 2-edge-colored graph, which takes the arc set of D as its vertex set, and two vertices are adjacent by a red edge, if they have a common head in D or by a green edge, if they have a common tail in D. We denote the in-out graph of D as $L_{io}(D)$.

In-out graphs are closely related to line graphs. The *line graph* $L(G)$ of an undirected graph G takes the edge set of G as its vertex set, and two vertices are connected in $L(G)$ if and only if they have a common end-vertex in G. A generalization of line graphs to digraphs is the concept of a line digraph. The *line digraph* $L_d(D)$ of D takes the arc set of a digraph D as its vertex set, and there is an arc directed from u to v in $L_d(D)$, if and only if the head of u coincides with the tail of v in D. Let D be a digraph and G its *underlying graph*, obtained from D by omitting all orientations and removing multiple edges, if any. Normally, there are more edges in $L(G)$ than arcs in $L_d(D)$. If we omit all orientations in $L_d(D)$ and view it as an undirected graph, we have an interesting observation that compared with $L(G)$, the edges that are missing in $L_d(D)$ is exactly the edges of $L_{io}(D)$, that is, $E(L(G)) = E(L_d(D)) \cup E(L_{io}(D))$, where $E(H)$ stands for the edge set of an undirected graph H. Thus, the concept of in-out graph can be viewed as another way of generalizing line graphs.

Furthermore, $L_{io}(D)$ is isomorphic to the line graph $L(G)$ of the *associated bipartite graph* G of D, where G is defined as a bipartite graph with vertex set $V' \cup V''$, where $V' = \{v' : v \in V(D)\}$ and $V'' = \{v'' : v \in V(D)\}$, and edge set $\{u'v'' : uv \in A(D)\}$. If we further color every edge ef of $L(G)$ with the color green or red, according to the common endvertex of e and f in G being in V' or V'', we have the same coloring as in $L_{io}(D)$.

Next, we consider two arc-disjoint Hamiltonian cycles C_0 and C_1 on the vertex set $\{0, 1, \ldots, k-1\}$, where $k \geq 3$. We denote an arc from vertex i to vertex j by (i, j). A cycle C or path P on p vertices is denoted by $(i_0, i_1 \ldots, i_{p-1}, i_0)$ and $(i_0, i_1 \ldots, i_{p-1})$ respectively, where $i_t \in \{0, 1, \ldots, k-1\}$, $0 \leq t \leq p-1 \leq k-1$, and i_{t+1} is the successor of i_t on C for $0 \leq t \leq p-1$ and $i_p = i_0$ (on P for $0 \leq t \leq p-2$, respectively). Without loss of generality, we assume that $C_0 = (0, 1, \ldots, k-1, 0)$.

Consider the in-out graph of $C_0 \cup C_1$, and let $L = L_{io}(C_0 \cup C_1)$. Since every arc in $C_0 \cup C_1$ has a common head with exactly one arc, and a common tail with exactly one arc, the corresponding vertex has degree two in L and is incident to one red edge and one green edge. Therefore, L consists of some mutually disjoint even cycles, the edges of which are red and green, alternately. A vertex of L corresponds to an arc in $C_0 \cup C_1$, and is denoted by (i, j) if it is from vertex i to j in $C_0 \cup C_1$. An edge of L connecting two vertices (i_0, j_0) and (i_1, j_1) is denoted by $(i_0, j_0) - (i_1, j_1)$.

The following theorem establishes connections between L and some spanning subgraphs of $C_0 \cup C_1$.

Theorem 4.1. *Let $k \geq 3$ be an integer and let C_0 and C_1 be two arc-disjoint Hamiltonian cycles on the vertex set $\{0, 1, \ldots, k-1\}$. Let $C_0 = (0, 1, \ldots, k-1, 0)$ and $C_1 = (i_0, i_1, \ldots, i_{k-1}, i_0)$, where $i_{t+1} \neq i_t + 1$, $0 \leq t \leq k-1$ ($i_k = i_0$). Let $L = L_{io}(C_0 \cup C_1)$. For any $0 \leq i, j \leq k-1$, let (i, j') and (i', j) be distinct arcs of C_1.*

(1) If (i, j') and (i', j) are on different cycles of L, then there are two arc-disjoint cycle factors F_0 and F_1 of $C_0 \cup C_1$, each of which contains exactly one of (i, j') and (i', j).

(2) If (i, j') and (i', j) are on the same cycle of L, then there is a 1-path-cycle factor F of $C_0 \cup C_1$, in which the path is a (j, i)-path.

Proof. (1) Since L is the disjoint union of even cycles, we can properly color the vertices of L with two colors; fix such a coloring. Observe that the subgraph of $C_0 \cup C_1$ consisting of the arcs corresponding to all the vertices of the same color in L is a cycle factor of $C_0 \cup C_1$. Indeed, for every arc (i, p) of $C_0 \cup C_1$, there is exactly one arc (p, q) of $C_0 \cup C_1$ whose color in L is the same as that of (i, p). If the corresponding vertices of two arcs of C_1 are in different cycles of L, we can always color the vertices differently, so that we obtain two arc-disjoint cycle factors of $C_0 \cup C_1$, each of which contains exactly one of these two arcs.

(2) Now suppose that (i, j') and (i', j) are on the same cycle Q of L. Delete edges $(i, j') - (i, i+1)$ and $(i', j) - (j-1, j)$ from L and denote the resulting graph by L'. In L', instead of Q, we have two paths, each of which contains an odd number of vertices. Now properly color all vertices of L' with colors l_0 and l_1, such that the vertices[1] (i, j'), $(i, i+1)$, (i', j) and $(j-1, j)$ are colored l_0. Now take the subgraph F of $C_0 \cup C_1$ consisting of all arcs that are colored l_1 in L'. In F, $d^-(i) = 1$, $d^+(i) = 0$, $d^-(j) = 0$, $d^+(j) = 1$, and the indegree and outdegree of all the other vertices are 1. Therefore, F is a 1-path-cycle factor of $C_0 \cup C_1$ in which the path is a (j, i)-path. □

5 Proof of Theorem 1.4

If a bipartite tournament T is fully cycle extendable, then we can start from a cycle of length 4 in T, repeat the operation of cycle extension until we get a Hamiltonian cycle. Thus T is Hamiltonian. Also note that every $T(r, r, r, r)$ is Hamiltonian.

[1] The vertices $(i, i+1)$ and $(j-1, j)$ may coincide if $j = i+1$.

Next we prove that if T is Hamiltonian, then either T belong to $T(r,r,r,r)$ or T is fully cycle extendable.

By Theorem 1.3, every vertex of a Hamiltonian bipartite tournament is on a cycle of length 4. Thus, we only need to prove that if T is Hamiltonian then it is even cycle extendable, unless it is isomorphic to $T(r,r,r,r)$ for some $r \geq 2$. Let T be a Hamiltonian bipartite tournament with bipartition (W, B). Clearly, T is *balanced*, i.e. $|W| = |B|$.

Firstly, we prove the theorem for T of order $|T| \leq 8$. If $|T| = 4$, then there is no cycle to be extended. If $|T| = 6$, then the only possible non-Hamiltonian cycles are of length 4, which can be extended to the Hamiltonian cycle of length 6. Thus, let $|T| = 8$. Let C be a non-Hamiltonian cycle of T. If $|C| = 6$, then C can be extended to a Hamiltonian cycle. The only case left is that of $|C| = 4$.

Suppose that C is not extendable. $T - V(C)$ is a balanced bipartite tournament with four vertices. Up to isomorphism, there can be four such bipartite tournaments, as shown in Figure 1. Note that in Figure 1, (1) is a 4-cycle, while (2), (3) and (4) are acyclic with the length of the longest path being three, two and one, respectively.

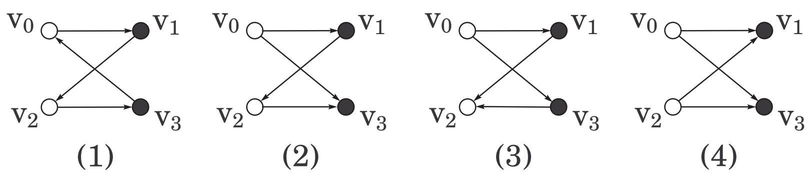

Figure 1: The four balanced bipartite tournaments on four vertices

We let $C = u_0u_1u_2u_3$ and label the vertices of $T - V(C)$ as in Figure 1. Furthermore, without loss of generality, we may assume that the vertices with even (odd) subscripts are in W (B). We will often need the following facts (1) and (2).

Since T is Hamiltonian, we have

$$d^-(x),\ d^+(x) \geq 1 \text{ for all } x \in W \cup B. \tag{1}$$

Let xy be an arc in $T - V(C)$. Note that in the formulas below, and also in the sequel, the subscripts are taken modulo 4.

$$\text{If } u_i \to x, \text{ then } u_{i+1} \to y; \text{ and if } y \to u_i, \text{ then } x \to u_{i+3}. \tag{2}$$

For, if any statement of (2) does not hold, then either the cycle $u_ixyu_{i+1}u_{i+2}u_{i+3}u_i$ extends C, or the cycle $yu_iu_{i+1}u_{i+2}u_{i+3}xy$ extends C, contradicting that C is not extendable.

Suppose $T - V(C)$ is isomorphic to Figure 1(1). Since $T - V(C)$ is a cycle, by (2), if $u_i \to v_j$ for any $0 \leq i, j \leq 3$ where $i + 1 \equiv j \pmod 2$, then we must have $u_{i+1} \to v_{j+1}$. Repeatedly applying this argument, we obtain

$$u_i \to v_j \Rightarrow u_{i+r} \to v_{j+r}, \text{ for } r \in \{0,1,2,3\}. \tag{3}$$

And similarly,

$$v_j \to u_i \Rightarrow v_{j+r} \to u_{i+r}, \text{ for } r \in \{0,1,2,3\}. \tag{4}$$

If $u_0 \to \{v_1, v_3\}$ ($\{v_1, v_3\} \to u_0$), then by (3) and (4), all arcs between C and $T - V(C)$ are from C to $T - V(C)$ (from $T - V(C)$ to C), contradicting that T is Hamiltonian. Hence, without loss of generality, we may assume that $v_3 \to u_0 \to v_1$. Then, again by (3) and (4), we have $v_0 \to u_1 \to v_2$, $v_1 \to u_2 \to v_3$, and $v_2 \to u_3 \to v_0$. But then T is isomorphic to $T(2,2,2,2)$.

Suppose $T - V(C)$ is isomorphic to Figure 1(2). By (1), there must be an arc from v_3 to C, say $v_3 \to u_0$. By (2), $\{v_0, v_2\} \to u_3$. Since $v_2 \to u_3$, by (2), $v_1 \to u_2$. Applying (1) for v_0, we have $u_1 \to v_0$. Then, C can be extended to the cycle $v_1u_2u_3u_0u_1v_0v_1$, a contradiction.

Suppose $T - V(C)$ is isomorphic to Figure 1(3). By (1), there must be an arc from C to v_0, say $u_1 \to v_0$. By $u_1 \to v_0$ and (2), $u_2 \to \{v_1, v_3\}$. By $u_2 \to v_1$ and (2), we have $u_3 \to v_2$. Applying (1) for v_2, we have $v_2 \to u_1$. By $v_2 \to u_1$ and (2), we have $\{v_1, v_3\} \to u_0$. However, both v_2 and u_0 have only one out-neighbor u_1, contradicting the Hamiltonicity of T.

Suppose $T-V(C)$ is isomorphic to Figure 1(4). By (1), there must be an arc from v_3 to C, say $v_3 \to u_0$. By $v_3 \to u_0$ and (2), $\{v_0, v_2\} \to u_3$. Applying (1) for v_0 and v_2, we have $u_1 \to \{v_0, v_2\}$. However, both v_0 and v_2 have only one in-neighbor u_1, contradicting the Hamiltonicity of T.

Thus, we either proved that T is in the exceptional class or get a contradiction. This finishes the proof of the theorem for $|T| \leq 8$.

Now we assume that $|T| \geq 10$. Suppose that T is not even cycle extendable, but all Hamiltonian bipartite tournaments of order less than $|T|$ are even cycle extendable, or belong to the exceptional class of bipartite tournaments. Let C be a longest non-even extendable cycle in T. Since T is Hamiltonian, $|C| \leq |T| - 4$.

Claim 1. *$V(C)$ is not contained in any non-Hamiltonian cycle C', such that $|C'| \geq |C| + 2$.*

Proof. Assume that such a cycle C' exists. Since C is not even extendable, $|C'| \geq |C| + 4$. Let $T' = T[V(C')]$; T' is a bipartite tournament with a Hamiltonian cycle C'. By our induction hypothesis, T' is even cycle extendable, or isomorphic to $T(r', r', r', r')$, for some integer $r' \geq 2$. However, C is not even extendable in T, and hence not even extendable in T'. Therefore, T' is not even cycle extendable, and so $T' = T(r', r', r', r')$ for some integer $r' \geq 2$. By definition, $V(T')$ can be partitioned into 4 parts $(V_0,\ V_1,\ V_2,\ V_3)$ where $|V_i| = r'$ and $V_i \to V_{i+1}$ for $i \in \{0, 1, 2, 3\}$ and $V_4 = V_0$. Without loss of generality, we may assume that $V_0,\ V_2 \subseteq W$ and $V_1,\ V_3 \subseteq B$.

By our selection of C, cycle C' is even extendable in T. Suppose that C' can be extended to a cycle C'' where $V(C'') = V(C') \cup \{w, b\}$, $w \in W$ and $b \in B$.

Suppose that w and b are adjacent in C'', say $wb \in A(C'')$. Denote the predecessor of w and the successor of b on C'' by b_0 and w_0, respectively. We must have $b_0 \in V_{2i-1}$ and $w_0 \in V_{2i}$ for $i = 1$ or 2 for if we traverse C'' from w_0 to b_0, we go through $4r'$ vertices, and the vertices must be in V_{2i}, V_{2i+1}, V_{2i+2} and $V_{2i+3} = V_{2i-1}$ for $i = 1$ or 2, successively and recursively. Without loss of generality, we may assume that $b_0 \in V_3$ and $w_0 \in V_0$ as in (A) of Figure 2.

Now suppose that w and b are not adjacent in C'' and denote the predecessor and the successor of w (b) by b_0 and b_1 (w_0 and w_1). First assume that the predecessor and the successor of w or b are in the same V_i for some $0 \leq i \leq 3$, say $b_0, b_1 \in V_1$. We traverse C'' from b_1 to w_0 to obtain a path P_0, and from w_1 to b_0 to obtain a path P_1. Then, $V(P_0) \cup V(P_1) = V(C')$. Since $|V_1| = |V_3|$, and their vertices appear on P_i $(i = 0, 1)$ alternatively, by $b_0, b_1 \in V_1$ we have that the predecessor of w_0 on P_0 and the successor of w_1 on P_1 must be in V_3. Therefore $w_0 \in V_0$ and $w_1 \in V_2$, as in (B) of Figure 2.

Suppose that the predecessor and the successor of w or b are in different V_i for some $0 \leq i \leq 3$, say $w_0 \in V_0$ and $w_1 \in V_2$. Let P_0 and P_1 be defined as above. Since $|V_1| = |V_3|$, and their vertices appear on P_i $(i = 0, 1)$ alternatively, and since the predecessor of w_0 on P_0 and the successor of w_1 on P_1 are both in V_3, we can conclude that $b_0, b_1 \in V_1$, as in (B) of Figure 2. Therefore, it suffices to consider the two cases in Figure 2.

Note that as C is a cycle in T', $|C|$ must be divisible by 4, and the vertices of C must be in V_0, V_1, V_2 and V_3 successively. We discuss the possible direction of the arcs between $\{w,\ b\}$ and $V(C)$ below, and extend C in all cases, thus contradicting that C is not even extendable. In Figure 3 and Figure 4, we use the shadowed region to denote the vertices of C' that are also in C.

Consider (A) of Figure 2. We first consider the case that there exists $b'_0 \in V(C) \cap V_3$ such that $b'_0 \to w$. Suppose that there exists $w'_0 \in V(C) \cap V_0$ such that $b \to w'_0$, as in (A.1) of Figure 3. We may assume that w'_0 is the successor of b'_0 on C and thus we can construct a cycle with vertex set $V(C) \cup \{w, b\}$ by inserting the arc wb between b'_0 and w'_0 on C. Suppose that we cannot find any vertex $w'_0 \in V(C) \cap V_0$ such that $b \to w'_0$. Then $V_0 \cap V(C) \to b$, and the successor w_0 of b on C' is in $V_0 \backslash V(C)$, as in (A.2) of Figure 3. Since $w_0 \to V_1 \cap V(C)$, we can extend C by inserting the arc bw_0 between two consecutive vertices in V_0 and V_1 on C. The case that we cannot find any vertex $b'_0 \in V(C) \cap V_3$ such that $b'_0 \to w$ can be handled similarly.

Now suppose that C' is extended as in (B) of Figure 2. Assume that we can choose w_0, w_1, b_0 and b_1 in such a way that they all be on C, as in (B.1) of Figure 4. Then $|V_i| \geq 2$, and we can further find $w'_0 \in V(C) \cup V_0 \setminus \{w_0\}$, $w'_1 \in V(C) \cap V_2 \setminus \{w_1\}$ and $b'_0, b'_1 \in V(C) \cap V_3$. Let P be a Hamiltonian path starting from a vertex in V_0 and ending at a vertex in V_3 in the subgraph $T[V(C) \setminus U]$, where $U = \{w_i, b_i, w'_i, b'_i :\ i = 0, 1\}$. Then $w_0 b w_1 b'_0 w'_0 b_0 w b_1 w'_1 b'_1 P w'_0$ is a cycle with set $V(C) \cup \{w, b\}$.

Now suppose that we cannot choose at least one of w_0, w_1, b_0 or b_1 so that it is on C.

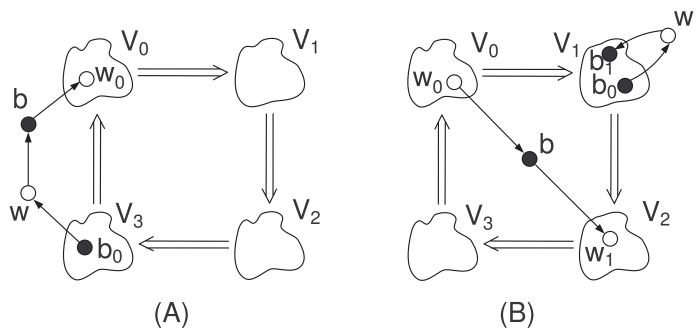

Figure 2: The two possible ways to extend C'

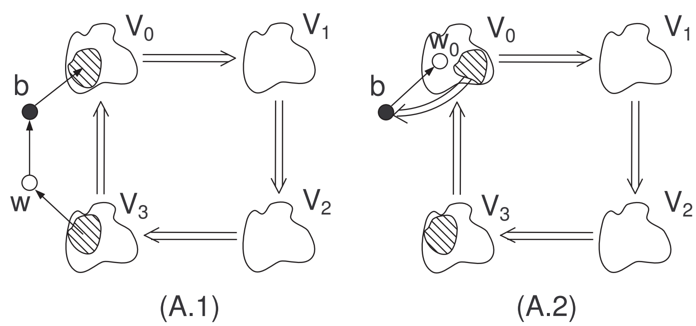

Figure 3: Based on (A) of Figure 2 to extend C', we extend C.

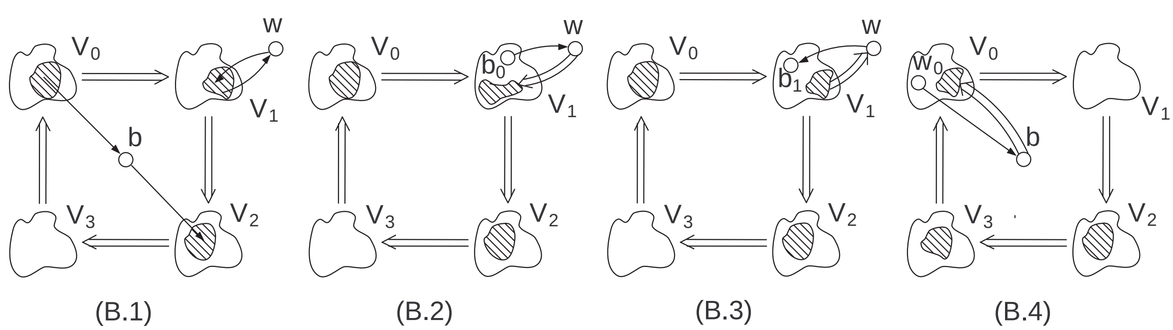

Figure 4: Based on (B) of Figure 2 to extend C', we extend C.

If we cannot choose b_0 so that it is on C, then $w \to V(C) \cap V_1$ and $b_0 \in V_1 \setminus V(C)$, as in (B.2) of Figure 4. However, since $V_0 \cap V(C) \to b_0$, we can extend C by inserting the arc b_0w between two consecutive vertices in V_0 and V_1 on C. If we cannot choose b_1 so that it is on C, then $V(C) \cap V_1 \to w$ and $b_1 \in V_1 \setminus V(C)$, as in (B.3) of Figure 4. Similarly to the above we can extend C by inserting wb_1 between two consecutive vertices in V_0 and V_1 on C. If we cannot choose $w_0 \in V(C) \cap V_0$ such that $w_0 \to b$, then $b \to V(C) \cap V_0$ and $w_0 \in V(C) \setminus V_0$. Similarly, we can extend C by inserting w_0b between two consecutive vertices in V_3 and V_0 on C. The last case that we cannot choose $w_1 \in V(C)$ can be handled similarly. Hence in all cases we can extend C, contradicting that C is not extendable and prove our claim. □

Let $C = u_0u_1 \dots u_{2m-1}u_0$, where $u_{2i} \in W$ and $u_{2i+1} \in B$, $0 \le i \le m-1$.

Claim 2. *If $T - V(C)$ has a spanning cycle Q, then $|Q| = 4$ and $T = T(r, r, r, r)$ for some integer $r \ge 2$.*

Proof. Suppose the condition holds. Let $Q = v_0v_1 \dots v_{2k-1}v_0$, where $v_{2j} \in W$ and $v_{2j+1} \in B$, $0 \le j \le k-1$. Since T is Hamiltonian, there is at least one arc from C to Q. Without loss of generality we may assume that $u_0 \to v_1$.

Firstly we assume that $|Q| \ge 6$, i.e. $k \ge 3$. By Claim 1, there cannot be any non-Hamiltonian cycle longer than C and containing all vertices of C. Therefore,

(a) if we have $u_{2i} \to v_{2j-1}$, we must have (the subscripts of u_i are modulo $2m$ and the subscripts of v_j are modulo $2k$, and the same below)

$$u_{2i+1} \to v_{2j},\ u_{2i+1} \to v_{2j+2},\ \dots,\ u_{2i+1} \to v_{2j-4},\ \text{and}$$

(b) if we have $u_{2i-1} \to v_{2j}$, we must have

$$u_{2i} \to v_{2j+1},\ u_{2i} \to v_{2j+3},\ \dots,\ u_{2i} \to v_{2j-3}.$$

Since $k \ge 3$ and $u_0 \to v_1$, by (a) we have $u_1 \to v_2$ and $u_1 \to v_4$. Then, by (b) we deduce that u_2 sends an arc to every vertex in $V(Q) \cap B$. Again by (a) we have that u_3 send an arc to every vertex in $V(Q) \cap W$. Applying (a) and (b) alternatively, we can finally deduce that every vertex on C sends an arc to every vertex on Q in different color class of it. Then, there is no arc from Q to C, contradicting that T is hamiltonian. Therefore $|Q| = 4$, and so $Q = v_0v_1v_2v_3v_0$. Then (a) and (b) become

(a′) if we have $u_{2i} \to v_{2j-1}$, we must have $u_{2i+1} \to v_{2j}$, and

(b′) if we have $u_{2i-1} \to v_{2j}$, we must have $u_{2i} \to v_{2j+1}$.

Now we prove that m is even, i.e. $|C|$ is divisible by 4.

Suppose that m is odd. Applying (a′) and (b′), by $u_0 \to v_1$ we have $u_1 \to v_2$, and by $u_1 \to v_2$ we have $u_2 \to v_3$. Repeating the process, we then have $u_{2m-1} \to v_2$, since m is odd. And by $u_{2m-1} \to v_2$ we have $u_0 \to v_3$. Applying (a′) and (b′) repeatedly, by $u_0 \to v_1$ and $u_0 \to v_3$ we will finally deduce that every vertex on C sends an arc to every vertex on Q in different color class of it, again contradicting that T is Hamiltonian. Hence m is even.

Let $U_i = \{u_{4t+i},\ 0 \le t \le m/2 - 1\}$, $i \in \{0, 1, 2, 3\}$. By $u_0 \to v_1$, repeatedly applying (a′) and (b′), we have $U_0 \to v_1$, $U_1 \to v_2$, $U_2 \to v_3$ and $U_3 \to v_0$. By above discussion, without loss of generality, we have $v_3 \to u_0$, and by similar arguments we have $v_3 \to U_0$, $v_0 \to U_1$, $v_1 \to U_2$ and $v_2 \to U_3$.

Note that we have $u_0 \to v_1$ and $v_1 \to u_2$. Replacing u_1 with v_1 on C we have a cycle $C_1 = u_0v_1u_2 \dots u_{2m-1}u_0$. Consider the arc u_1v_2. For any $1 \le t \le m/2 - 1$, we have $v_2 \to u_{4t+3}$. If $u_{4t+2} \to u_1$ for some t ($1 \le t \le m/2 - 1$), then we have a cycle

$$u_0v_1u_2 \dots u_{4t+2}u_1v_2u_{4t+3} \dots u_0,$$

which extends C, a contradiction. Therefore, $u_1 \to u_{4t+2}$ for all $1 \le t \le m/2 - 1$. Together with $u_1 \to u_2$, we have $u_1 \to U_2$. By similar arguments we conclude that $U_i \to U_{i+1}$, for all $0 \le i \le 3$ ($U_4 = U_0$). Together with Q and the arcs between Q and C we see that $T = T(r,\ r,\ r,\ r)$, where $r = m/2 + 1$. And by $2m = |C| \ge 4$, we have $r \ge 2$. □

Claim 3. *Let Q be a non-spanning cycle in $T - V(C)$, then either all arcs between C and Q are from C to Q, or all arcs between C and Q are from Q to C.*

Proof. Suppose the conclusion does not hold, then there is at least one arc from C to Q and at least one arc from Q to C. Then $T' = T[V(C) \cup V(Q)]$ is strong. By Theorem 1.2, T' must be Hamiltonian. But a Hamiltonian cycle of T' is a non-Hamiltonian cycle of T, which contains all vertices of C and is longer than C, contradicting Claim 1. □

From now on we assume that $T \neq T(r,r,r,r)$ for any $r \geq 2$.

Let H be a Hamiltonian cycle of T and let $H \cap C$ denote the digraph with vertex set $V(C)$ and arcs belonging to both C and H. Observe that $H \cap C$ consists of disjoint paths, some of which may be just single vertices. We call these paths *common paths* of H and C, or just common paths when no ambiguity is caused. Suppose there are k common paths; we denote them by S_0, S_1, …, S_{k-1}, according to the order in which they appear on C. After removing all arcs and internal vertices of the common paths from C, the remaining arcs are all the arcs from the terminal vertex of S_i to the starting vertex of S_{i+1}, $0 \leq i \leq k-1$ (the subscripts modulo k, and the same below). We call them *C-arcs*, and denote an arc from the terminal vertex of S_i to the initial vertex of S_{i+1} as $a_C(i,i+1)$. Note that two C-arcs may be consecutive arcs on C. Removing all arcs and internal vertices of the common paths from H, we obtain k paths, which are called *H-paths*. An H-path starts with the terminal vertex of S_i and terminates with the initial vertex of S_j, for some $0 \leq i, j \leq k-1$. We denote such an H-path as $S_H(i,j)$. H-paths are internally disjoint, but the initial vertex of one H-path may be the terminal vertex of another H-path. If an H-path contains no internal vertex we say that it is *trivial*, else we say that it is *nontrivial*. Note that the number of C-arcs and the number of H-paths are also k.

Claim 4. *Let $S = u_i v_0 \dots v_{t-1} u_j$, $0 \leq i, j \leq 2m-1$, be a nontrivial H-path, then $t \leq 3$.*

Proof. Suppose to the contrary that $t \geq 4$.

If t is even, consider the arc between v_0 and v_{t-1}. If $v_0 \to v_{t-1}$, we can replace $v_0 v_1 \dots v_{t-1}$ with the arc $v_0 v_{t-1}$ on H, and obtain a non-Hamiltonian cycle which contains all vertices of C and is longer than C, contradicting Claim 1. If $v_{t-1} \to v_0$, we have a cycle $Q = v_0 v_1 \dots v_{t-1} v_0$ in $T - V(C)$, with one arc from C to Q and one arc from Q to C. By Claim 3, Q must be a spanning cycle of $T - V(C)$. But then by Claim 2, we must have $T = T(r,\ r,\ r,\ r)$ for some integer $r \geq 2$, contradicting our assumption.

If t is odd, consider the arc between v_0 and v_{t-2}, and the arc between v_1 and v_{t-1}. If $v_0 \to v_{t-2}$ or $v_1 \to v_{t-1}$, by arguments similar to the above, we can obtain a non-Hamiltonian cycle which contains all vertices of C and is longer than C, again contradicting Claim 1. Hence we have $v_{t-2} \to v_0$ and $v_{t-1} \to v_1$. Then we have two cycles $Q_0 = v_0 v_1 \dots v_{t-2} v_0$ and $Q_1 = v_1 v_2 \dots v_{t-1} v_1$ in $T - V(C)$, which are not spanning cycles of $T - V(C)$. Since there is one arc from C to Q_0, by Claim 3, all arcs between C and Q_0 are from C to Q_0. Similarly, all arcs between Q_1 and C are from Q_1 to C. However, this is impossible, since v_1 is on both Q_0 and Q_1. □

Therefore, we cannot have an H-path with more than three internal vertices. Since there are at least four vertices in $T - V(C)$, there are at least two nontrivial H-paths.

Claim 5. *Let $S_H(i,j)$, $0 \leq i, j \leq k-1$, be an H-path, then $j \neq i+1$.*

Proof. If $S_H(i,j)$ is trivial and $j = i+1$, then it is also a C-arc $a_C(i,i+1)$, contradicting the definition of an H-path. Suppose that $S_H(i,j)$ is nontrivial, and $j = i+1$. We can replace the C-arc $a_C(i,i+1)$ with $S_H(i,i+1)$ on C, obtaining a cycle C', such that $V(C) \subseteq V(C')$ and $|C'| \geq |C|+2$. Furthermore, by the above discussion, there is at least one more nontrivial H-path, whose internal vertices are not contained in $V(C')$, so C' is non-Hamiltonian. This contradicts Claim 1. So $j \neq i+1$. □

Let $P = u_i v_0 v_1 v_2 u_j$, $0 \leq i, j \leq 2m-1$, be an H-path with three internal vertices. If $v_1 \to u_{i+1}$, then C can be extended to $u_i v_0 v_1 u_{i+1} C u_i$, a contradiction. Therefore $u_{i+1} \to v_1$, and similarly $u_{i+2} \to v_2$, $v_1 \to u_{j-1}$ and $v_0 \to u_{j-2}$. Hence, each of v_0, v_1 and v_2 sends and receives some arcs from C. Similarly, we can prove that every internal vertex of any nontrivial H-paths sends and receives some arcs from C. So, every vertex in $T - V(C)$ sends and receives some arcs from C.

Now we can show the following two claims.

Claim 6. *There is no cycle in $T - V(C)$.*

Proof. Suppose that there is a cycle Q in $T - V(C)$. If Q is spanning, then by Claim 2, $B = T(r, r, r, r)$ for some integer $r \geq 2$, contradicting our assumption. If Q is not spanning, by Claim 3, either all arcs between C and Q are from C to Q, or all arcs between C and Q are from Q to C. However, we have just proved that every vertex in $T - V(C)$ sends and receives some arcs from C. □

Claim 7. *There cannot exist a cycle subgraph F of T, such that $V(C) \subset V(F) \subset V(T)$, where $\subset$ stands for proper inclusion.*

Proof. Suppose such a cycle subgraph F exists. Since every vertex in $T - V(C)$ sends and receives arcs from C, $T[V(F)]$ must be strong. By Theorem 1.2, $T[V(F)]$ is hamiltonian. Then $V(C)$ is covered by the Hamiltonian cycle of $T[V(F)]$, contradicting Claim 1. □

We will use Claim 6 and Claim 7 frequently in our subsequent proof.

We claim that $k \geq 3$. By the above discussion, there are at least two nontrivial H-paths, so $k \geq 2$. If $k = 2$, then there are only two common paths S_0 and S_1. The two C-arcs must be $a_C(0,1)$ and $a_C(1,0)$, and the two nontrivial H-paths must be $S_H(0,1)$ and $S_H(1,0)$, contradicting Claim 5.

We define a contraction operation on $C \cup H$. We contract every common path S_i into a vertex i, $0 \leq i \leq k-1$. Then, we contract every H-path $S_H(i,j)$ into an arc (i,j). The resulting digraph consists of two arc-disjoint cycles on the vertices $\{0, 1, \ldots, k-1\}$. One is $C_0 = (0, 1, \ldots, k-1, 0)$, obtained from C by contracting the common paths. The other one, denoted by C_1, is formed by all arcs obtained by contracting H-paths. Formally, we define a mapping η from the set of common paths, C-arcs and H-paths of $C \cup H$, to the vertex set and arc set of $C_0 \cup C_1$, where

$$\eta(S_i) = i, \ \eta(a_C(i, i+1)) = (i, i+1), \ \text{and} \ \eta(S_H(i,j)) = (i,j).$$

Let F be a subgraph of $C_0 \cup C_1$, we use $\eta^{-1}(F)$ to denote the subdigraph of $C \cup H$, which consists of the preimages of the vertices and arcs of F. We also say that $\eta^{-1}(F)$ is the *preimage* of F.

Let F_0 be a cycle factor of $C_1 \cup C_0$. Then $\eta^{-1}(F_0)$ is a cycle subdigraph of $C \cup H$ which covers $V(C)$. Let F_1 be a 1-path-cycle factor of $C_1 \cup C_0$, in which the path is from vertex i to vertex j. Then $\eta^{-1}(F_1)$ is a 1-path-cycle subgraph of $C \cup H$ covering $V(C)$, in which the path starts with S_i and terminates with S_j.

Let $L = L_{io}(C_0 \cup C_1)$ be the in-out graph of $C_0 \cup C_1$. We will work on L to gain structural properties of $C_0 \cup C_1$ and $C \cup H$ in the rest of our proof. We show in Figure 5 an example of $C \cup H$, $C_0 \cup C_1$ and $L_{io}(C_0 \cup C_1)$.

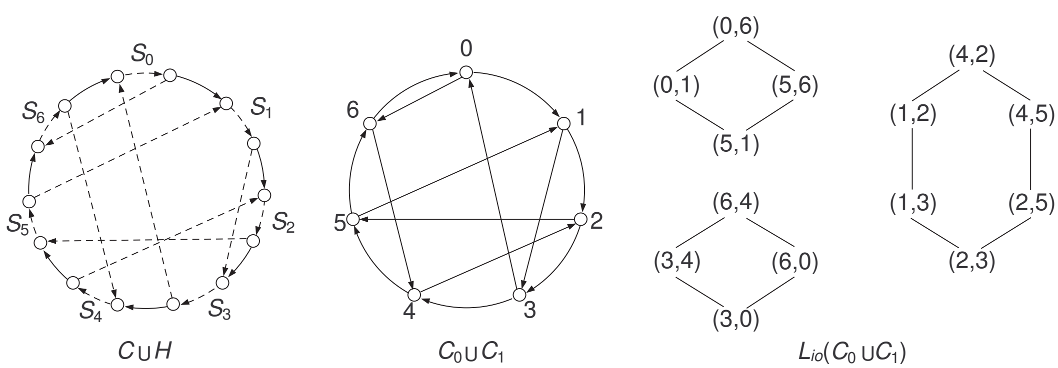

Figure 5: An Example: $C \cup H$, $C_0 \cup C_1$ and $L_{io}(C_0 \cup C_1)$

Claim 8. *All arcs of C_1 whose preimages are nontrivial H-paths must be on the same cycle, denoted by Q, of L.*

Proof. Suppose to the contrary that there exist two arcs a_0 and a_1 of C_1 whose preimages are nontrivial H-paths, where a_0 and a_1 are on different cycles of L. By Theorem 4.1, there exists a cycle factor F of $C_0 \cup C_1$, which contains a_0 but does not contain a_1. However, $\eta^{-1}(F)$ is a cycle subgraph of $C \cup H$, which covers $V(C)$ but does not cover the internal vertices of $\eta^{-1}(a_1)$, contradicting Claim 7. □

Let $S_H(i, j_0)$ and $S_H(i_0, j)$ be two different nontrivial H-paths. Let v be an internal vertex of $S_H(i, j_0)$ and w be an internal vertex of $S_H(i_0, j)$ such that there is an arc from v to w. We traverse $S_H(i, j_0)$ from the initial vertex of it to v, then go through vw, and traverse $S_H(i_0, j)$ from w to the terminating vertex of it to obtain a path, which is uniquely determined by the arc vw and denoted by $P(vw)$ (if the arc is from w to v, the path obtained is denoted by $P(wv)$).

Claim 9. *$P(vw)$ must cover all internal vertices of $S_H(i, j_0)$ and $S_H(i_0, j)$.*

Proof. Let $S_H(i, j_0)$ and $S_H(i_0, j)$ be mapped to the arcs (i, j_0) and (i_0, j) by η, respectively. By Claim 8, both (i, j_0) and (i_0, j) must be on Q. And by Theorem 4.1, we can find a 1-path-cycle factor F of $C_0 \cup C_1$ in which the path is a (j, i)-path. Then, $\eta^{-1}(F)$ is a 1-path-cycle subgraph of $C \cup H$ covering $V(C)$, in which the path P starts with S_j and terminates with S_i. Then $P(vw) \cup P$ is a cycle of $C \cup H$. However, $(F \backslash P) \cup \{P(vw) \cup P\}$ is a cycle subgraph of $C \cup H$ covering $V(C)$, and by Claim 7, it must be a cycle factor of T. Therefore, $P(vw)$ must cover all internal vertices of $S_H(i, j_0)$ and $S_H(i_0, j)$. □

Let $S = u_i v_0 v_1 v_2 u_j$, $0 \le i, j \le 2m-1$, be an H-path with three internal vertices. Without loss of generality, we may assume that $v_1 \in B$. Suppose there exists another nontrivial H-path which contains an internal vertex $w \in W$. Since T is a bipartite tournament, either $v_1 \to w$ or $w \to v_1$. However, the path $P(v_1 w)$ does not cover v_2, and the path $P(w v_1)$ does not cover v_0, both contradicting Claim 9. Hence, all other nontrivial H-paths must contain only one internal vertex which is in B. To keep T balanced, there must be only one such H-path S', the internal vertex of which is denoted by v_3. Applying Claim 9 on S and S', we have $v_3 \to v_0$ and $v_2 \to v_3$. But then $v_0 v_1 v_2 v_3 v_0$ is a cycle in $T - V(C)$, contradicting Claim 6. Therefore, there cannot be a nontrivial H-path with three internal vertices.

Let $S = u_i v_0 v_1 u_j$, $0 \le i, j \le 2m-1$, be an H-path with two internal vertices. Without loss of generality, we may assume that $v_0 \in W$ and $v_1 \in B$. Assume that there is another nontrivial H-path $S' = u_{i'} v_2 v_3 u_{j'}$ $(0 \le i', j' \le 2m-1)$ with two internal vertices. Suppose $v_2 \in B$ and $v_3 \in W$. Since T is a bipartite tournament, either $v_1 \to v_3$ or $v_3 \to v_1$. However, the path $P(v_1 v_3)$ does not cover v_2, and the path $P(v_3 v_1)$ does not cover v_0, both contradicting Claim 9. If $v_2 \in W$ and $v_3 \in B$, applying Claim 9 on S and S', we must have $v_1 \to v_2$ and $v_3 \to v_0$. But then $v_0 v_1 v_2 v_3 v_0$ is a cycle in $T - V(C)$, contradicting Claim 6. Therefore, there is at most one nontrivial H-path with two internal vertices. Furthermore, by $|T| - |C| \ge 4$, there are at least two nontrivial H-paths with one internal vertex.

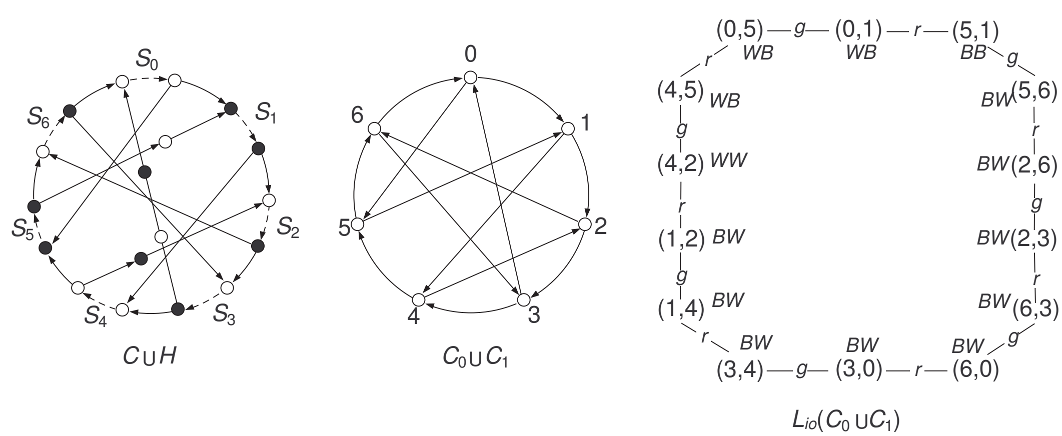

Figure 6: An Example: $C \cup H$ with black and white colors on the vertices, $C_0 \cup C_1$ and $L_{io}(C_0 \cup C_1)$ with labels on the vertices and colors on the edges (denoted by the labels "g" and "r" on the edges)

To further analyze the structure of $C_0 \cup C_1$ and $C \cup H$, we label the vertices of L. Let a be a vertex of L, which is an arc of $C_0 \cup C_1$. The preimage of a, $\eta^{-1}(a)$, is a path (which may degenerate to an arc) in T. We assign the labels l_0 and l_1, denoted as $l_0 l_1$, to a, where l_0, $l_1 \in \{B, W\}$, the initial vertex of $\eta^{-1}(a)$ is in color class l_0 of T and the terminal vertex of $\eta^{-1}(a)$ is in color class l_1 of T. We call l_0 the first label and l_1 the second label of a, and call a vertex with labels $l_0 l_1$ an $l_0 l_1$-vertex. See Figure 6 for an example.

Recall that an edges of L is colored red (green) when the two endvertices of it share a common head (tail) in $C_0 \cup C_1$. Therefore, if two vertices are joined by a red (green) edge, then they have the same second (first) label.

We list some properties of the edge colors and vertex labels in L below. For Property (3), we give a detailed proof.

(1) Since two adjacent vertices in L must have at least one label in common, a BB-vertex can never be adjacent to a WW-vertex, and a BW-vertex can never be adjacent to a WB-vertex.

(2) An arc $a = (i, i+1)$ of C_0 must be labeled BW or WB, $0 \le i \le k-1$ (addition modulo k). A WW-vertex or a BB-vertex of L must be an arc of C_1, whose preimage is an H-path with one internal vertex in $C \cup H$. By Claim 8, all WW- and BB-vertices must be on Q. A WW- or BB-vertex must be adjacent to one WB-vertex and one BW-vertex.

(3) If we traverse Q in one direction, WW-vertices and BB-vertices must appear alternatively. And hence the number of BB-vertices and WW-vertices must be the same on Q.

Proof. If there are no WW- or BB-vertex on Q, then the statement holds. Without loss of generality, we may assume that we have a WW-vertex a_0 on Q, and we traverse Q from a_0 in one direction such that the next vertex on Q is a WB-vertex, which is adjacent to a_0 by a green edge. By (1), a WB-vertex can never be adjacent to a BW-vertex, therefore, we will keep meeting WB-vertex before we meet the next WW- or BB-vertex. By (2), the other neighbor of a_0 is a BW-vertex, so we must have at least one WW- or BB-vertex other than a_0. We denote the first WW- or BB-vertex we meet after a_0 by a_1.

Since the vertices on Q correspond to arcs on C_1 and C_0 alternatively, and all WW- and BB-vertices must correspond to arcs on C_1, a_1 must be at an even distance from a_0 on Q. And since red edge and green edge appear alternatively on Q, a_1 must be adjacent to a WB-vertex by a red edge. But then the second label of a_1 must be B, and therefore it must be a BB-vertex. ☐

(4) If we traverse Q in one direction, by the discussion in the proof of (3), the vertices between a WW-vertex and a BB-vertex that appear consecutively must all be WB- or BW-vertices. We call the segment of Q consisting of all such vertices a WB-path (a BW-path), if all these vertices are WB-vertices (BW-vertices). By (2), a WW- or BB-vertex must be adjacent to one WB-vertex and one BW-vertex. Therefore, WB-paths and BW-paths must appear alternatively on Q.

(5) Let a be a WW-vertex on Q. If we traverse Q from a so that the next vertex is a WB-vertex, then we will meet a WW-vertex, a WB-path, a BB-vertex, and a BW-path successively and recursively, until we return to a. Therefore, take any WW-vertex a_0 and any BB-vertex a_1, if we delete a_0 and a_1 from Q, we have two paths P_0 and P_1, where P_0 starts and terminates with WB-vertices, and P_1 starts and terminates with BW-vertices. We call P_0 the (WB, WB)-path for a_0 and a_1, and P_1 the (BW, BW)-path for a_0 and a_1.

By the above discussion, in $C \cup H$ there are at most one nontrivial H-path with two internal vertices, and at least two nontrivial H-paths with one internal vertex. Therefore, there are at least one WW-vertex and one BB-vertex on Q.

Claim 10. *Let $a_0 = (i_0, j_1)$ be a BB-vertex and $a_1 = (i_1, j_0)$ be a WW-vertex on Q. Denote the internal vertex of $S_H(i_0, j_1)$ ($S_H(i_1, j_0)$) by v_0 (v_1). Then, $v_0 \to v_1$ ($v_1 \to v_0$), if and only if all vertices in $V(Q)\backslash\{a_0, a_1\}$ whose preimages are nontrivial H-paths are on the (BW, BW)-path ((WB, WB)-path) for a_0 and a_1.*

Proof. Firstly, since $|T| - |C| \ge 4$, there are at least one more vertex in $V(Q)\backslash\{a_0, a_1\}$ whose preimage under η is a nontrivial H-path.

Assume that $v_0 \to v_1$. By the proof of Theorem 4.1, we can obtain a 1-path-cycle factor F of $C_0 \cup C_1$ such that the path is from j_0 to i_0. To get F, we delete the edges $(i_0, j_1)-(i_0, i_0+1)$ and $(i_1, j_0)-(j_0-1, j_0)$

from Q. Then, we have two paths P_0, from (i_0, j_1) to (i_1, j_0), and P_1, from (i_0, i_0+1) to (j_0-1, j_0), which is actually the (BW, BW)-path for a_0 and a_1. And we take the arcs in $V(P_0) \cap A(C_0)$ and $V(P_1) \cap A(C_1)$, together with the arcs from other cycles of L to constitute F.

The only path P in $\eta^{-1}(F)$ starts with S_{j_0} and terminats with S_{i_0}. And $P \cup P(v_0v_1)$ is a cycle in $C \cup H$. Then, $F' = (\eta^{-1}(F) \backslash P) \cup \{P \cup P(v_0v_1)\}$ is a cycle subgraph of T, which covers $V(C)$, and contains at least the vertices v_0 and v_1, which are in $V(T) \backslash V(C)$. By Claim 7, F' must be a cycle factor of T. However, F does not contain any arc in $V(P_0) \cap A(C_1)$, therefore F' does not contain the preimage of any arc in $V(P_0) \cap A(C_1)$. So, the preimage of an arc in $V(P_0) \cap A(C_1)$ must not be a nontrivial H-path. In other word, all vertices in $V(Q) \backslash \{a_0, a_1\}$ whose preimages are nontrivial H-paths must be on P_1, which is the (BW, BW)-path for a_0 and a_1.

Similarly, if $v_1 \to v_0$, we can conclude that all vertices in $V(Q) \backslash \{a_0, a_1\}$ whose preimages are nontrivial H-paths must be on the (WB, WB)-path for a_0 and a_1.

Now assume that all vertices in $V(Q) \backslash \{a_0, a_1\}$ whose preimage are nontrivial H-paths are on the (BW, BW)-path for a_0 and a_1. Since T is a bipartite tournament, exactly one of $v_0 \to v_1$ and $v_1 \to v_0$ holds. If $v_1 \to v_0$ holds, by above discussion, all vertices in $V(Q) \backslash \{a_0, a_1\}$ whose preimages are nontrivial H-paths must be on the (WB, WB)-path for a_0 and a_1, a contradiction. Therefore, $v_0 \to v_1$.

The only case left can be proved similarly. □

Suppose there are at least six WW- or BB-vertices on Q. Let a_0 be a WW-vertex on Q. Traverse Q in one direction from a_0, and denote the first six WW- or BB-vertices we meet by a_0, a_1, a_2, a_3, a_4 and a_5, according to the order they appear. Then, by (3), a_3 must be a BB-vertex. However, there are WW- and BB-vertices on both the (BW, BW)-path and the (WB, WB)-path for a_0 and a_3, contradicting Claim 10. Therefore, there are at most four WW- or BB-vertices on Q. Equivalently, there are at most four nontrivial H-paths with one internal vertex in $C \cup H$.

Suppose there are four WW- or BB-vertices on Q, which are a_0, a_1, a_2 and a_3, according to the order they appear in one direction, say clockwise. Denote the internal vertices of $\eta^{-1}(a_i)$ as v_i, $0 \le i \le 3$. Without loss of generality, we may assume that a_0 and a_2 are WW-vertices, a_1 and a_3 are BB-vertices, and if we traverse Q clockwise, the path between a_0 and a_1 is a (WB, WB)-path for a_0 and a_1. Then, a_2 and a_3 are on the (BW, BW)-path for a_0 and a_1. By Claim 10, we have $v_0 \to v_1$. Similarly, we have $v_1 \to v_2$, $v_2 \to v_3$ and $v_3 \to v_0$. But then we have a cycle $v_0v_1v_2v_3v_0$ in $T - V(C)$, contradicting Claim 6.

Therefore, we can have only one BB-vertex a_0 and one WW-vertex a_1 on Q. By the above discussion, we have one WB- or BW-vertex a_2, whose preimage is a nontrivial H-path with two internal vertices. Denote the internal vertices of $\eta^{-1}(a_0)$ and $\eta^{-1}(a_1)$ by v_0 and v_1, respectively. Then, $v_0 \in W$ and $v_1 \in B$. Without loss of generality, we may assume that a_2 is a BW-vertex, and denote the internal vertices of $\eta^{-1}(a_2)$ by $v_2 \in W$ and $v_3 \in B$, where $v_2 \to v_3$. Applying Claim 9 on a_0 and a_2, we have $v_3 \to v_0$. Applying Claim 9 on a_1 and a_2, we have $v_1 \to v_2$. Further, a_2 is on the (BW, BW)-path for a_0 and a_1, and hence by Claim 10, $v_0 \to v_1$. However, we have a cycle $v_0v_1v_2v_3v_0$ in $T - V(C)$ then, contradicting Claim 6.

Every possible case above has led to contradiction. This completes the proof of Theorem 1.4.

6 Open Problems

As in bipartite tournaments, Hamiltonicity in multipartite tournaments can also be decided in polynomial time [3, 27], but finding a characterization of Hamiltonian multipartite tournaments remains an open problem. It would be interesting to characterize (vertex-)pancyclic multipartite tournaments.

Since multipartite tournaments contain both bipartite and non-bipartite graphs, it does not make sense to study cycle extendability for the whole family of multipartite tournaments. Instead, it would be interesting to characterize Hamiltonian multipartite tournaments T such that for every non-Hamiltonian cycle C in T there is a cycle C' such that $V(C) \subseteq V(C')$, and $|C'| = |C| + 1$ or $|C'| = |C| + 2$.

Note that the (vertex-)pancyclicity problem above was solved for a subclass of multipartite tournaments, that is, extended tournaments. In fact, it was solved for a larger class of extended semicomplete digraphs [15]. A digraph is *semicomplete* if it is obtained from a tournament T by adding to T arcs $\{yx : xy \in A'\}$, where A' is a subset of $A(T)$. An *extended semicomplete digraph* is a digraph obtained from a semicomplete digraph by replacing every vertex x with a set I_x of independent vertices such that the out- and in-neighbors of every vertex in I_x are the same as those of x and $I_x \cap I_y = \emptyset$ as long as $x \neq y$. We say that a digraph D is *triangular* with partition V_0 ,V_1 and V_2, if the vertex set of D can be partitioned into three disjoint sets V_0, V_1 and V_2, with $V_i \rightarrow V_{i+1}$ and there is no arc from V_{i+1} to V_i ($0 \leq i \leq 2$ and the subscripts are taken modulo 3).

Theorem 6.1. *(Gutin [15]) Let D be a Hamiltonian extended semicomplete digraph of order $n \geq 5$ with k partite sets ($k \geq 3$). Then*
(a) D is pancyclic if and only if D is not triangular with a partition V_0 ,V_1 and V_2, two of which induce digraphs with no arcs, such that either $|V_0| = |V_1| = |V_2|$ or no $D[V_i]$ ($i = 0, 1, 2$) contains a path of length 2.
(b) D is vertex-pancyclic if and only if it is pancyclic and either $k > 3$ or $k = 3$ and D contains two cycles Z and Z' of length 2 such that $Z \cup Z'$ has vertices in the three partite sets.

References

[1] Abueida, A., & Sritharan, R. (2006). Cycle extendability and Hamiltonian cycles in chordal graph classes. SIAM J. Discrete Math., 20(3), 669-681.

[2] Bang-Jensen, J., & Gutin, G. (2009). Digraph: theory, algorithms and applications. 2nd Ed., Springer-Verlag, London.

[3] Bang-Jensen, J., Gutin, G., & Yeo, A. (1998). A polynomial algorithm for the Hamiltonian cycle problem in semicomplete multipartite digraphs. J. Graph Theory, 29(2), 111-132.

[4] Bang-Jensen, J. & Havet, F. (2018), Tournaments and Semicomplete Digraphs, Chapter 2 in Classes of Directed Graphs (J. Bang-Jensen and G. Gutin, eds), Springer, 2018.

[5] Beineke, L. W., & Little, C. H. (1982). Cycles in bipartite tournaments. J. Combin. Theory, Ser. B, 32(2), 140-145.

[6] Bondy, J. A. (1971). Pancyclic graphs I, J. Combin. Theory Ser. B 11 (1971) 80-84.

[7] Camion, P. (1959). Chemins et circuits hamiltoniens des graphes complets. Comptes Rendus Hebdomadaires des Séances de l' Académie des Sciences, 249(21), 2151-2152.

[8] Chen, G., Faudree, R. J., Gould, R. J., & Jacobson, M. S. (2006). Cycle extendability of Hamiltonian interval graphs. SIAM J. Discrete Math., 20(3), 682-689.

[9] Dirac, G. A. (1952). Some theorems on abstract graphs. Proc. London Math. Soc., 3(1), 69-81.

[10] Gould, R. J. (1991). Updating the Hamiltonian problem - a survey. J. Graph Theory, 15(2), 121-157.

[11] Gould, R. J. (2003). Advances on the Hamiltonian problem - a survey. Graphs and Combinatorics, 19(1), 7-52.

[12] Gould, R. J. (2014). Recent advances on the Hamiltonian problem: Survey III. Graphs and Combinatorics, 30(1), 1-46.

[13] Gutin, G. (1984). A criterion for complete bipartite digraphs to be Hamiltonian. Vestsī Acad. Navuk BSSR Ser. Fīz.-Mat. Navuk, No. 1, 99-100 (In Russian).

[14] Gutin, G. (1993). Paths and cycles in directed graphs. Ph.D. Thesis, University of Tel-Aviv.

[15] Gutin, G. (1995). Characterizations of vertex pancyclic and pancyclic ordinary complete multipartite digraphs. Discrete Math., 141(1), 153-162.

[16] Häggkvist, R., & Manoussakis, Y. (1989). Cycles and paths in bipartite tournaments with spanning configurations. Combinatorica, 9(1), 33-38.

[17] Hendry, G. R. T. (1985). On paths, factors and cycles in graphs. Doctoral Thesis. Aberdeen University, 1985.

[18] Hendry, G. R. T. (1989). Extending cycles in directed graphs. J. Combin. Theory, Ser. B, 46(2), 162-172.

[19] Jackson, B. (1981). Long paths and cycles in oriented graphs. J. Graph Theory, 5(2), 145-157.

[20] Kühn, D., & Osthus, D. (2012). A survey on Hamilton cycles in directed graphs. European J. Combin., 33(5), 750-766.

[21] Lafond, M., & Seamone, B. (2015). Hamiltonian chordal graphs are not cycle extendable. SIAM J. Discrete Math., 29(2), 877-887.

[22] Manoussakis, Y. & Tuza Z. (1990). Polynomial algorithms for finding cycles and paths in bipartite tournaments. SIAM J. Discrete Math., 3(4), 537-543.

[23] Meierling, D. (2010). Solution of a conjecture of Tewes and Volkmann regarding extendable cycles in in-tournaments. J. Graph Theory, 63(1), 82-92.

[24] Harary, F. and Moser, L. (1966). The theory of round robin tournaments. Amer. Math. Monthly, 73, 231–246.

[25] Moon, J. W. (1968). Topics on Tournaments. Holt, Rinehart and Winston, N.Y..

[26] Tewes, M., & Volkmann, L. (2001). Vertex pancyclic in-tournaments. J. Graph Theory, 36(2), 84–104.

[27] Yeo, A. (1999). A polynomial time algorithm for finding a cycle covering a given set of vertices in a semicomplete multipartite digraph. J. Algorithms, 33(1), 124-139.

[28] Yeo, A. (2018), Semicomplete Multipartite Digraphs, Chapter 7 in Classes of Directed Graphs (J. Bang-Jensen and G. Gutin, eds), Springer, 2018.

[29] Zhang, Z.-B., Zhang, X., & Wen, X. (2013). Directed hamilton cycles in digraphs and matching alternating hamilton cycles in bipartite graphs. SIAM J. Discrete Math., 27(1), 274-289.

[30] Zhang, Z.-B., Zhang, X., Broersma, H. & Lou, D. (2017). Extremal and Degree Conditions for Path Extendability in Digraphs. SIAM J. Discrete Math., 31(3), 1990–2014.