From Large to Infinite Random Simplicial Complexes

by

Lewis Mead

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School of Mathematical Sciences
Queen Mary, University of London
United Kingdom

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Declaration

I, Lewis Mead, confirm that the research included within this thesis is my own work or that where it has been carried out in collaboration with, or supported by others, that this is duly acknowledged below and my contribution indicated. Previously published material is also acknowledged below.

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Details of collaboration and publications:

Chapter 2 is a minor tweak of a publication written in close collaboration with Michael Farber and Tahl Nowik, "Random simplicial complexes, duality and the critical dimension", published in 2019 in the Journal of Topology and Analysis.

For Chapter 3 author would like to thank Andrew Newman for the ideas inspiring the proofs of Proposition 3.2.4 and Corollary 3.2.9, to Justin Ward for revealing the existence of k-trees, and to William Raynaud and Lewin Strauss for insightful discussions. The work of this chapter has been recently submitted for publication and differs slightly from the current arXiv:1912.02078 preprint "Enumerating the class of minimally path connected simplicial complexes".

Chapter 4 is based on joint work with Michael Farber and resulted in the 2020 publication "Random simplicial complexes in the medial regime" in Topology and its Applications.

Chapter 5 was a close collaboration with Michael Farber and fellow PhD Lewin Strauss. The work of this chapter has been recently submitted for publication and differs slightly from the current arXiv:1912.02515 preprint "The Rado simplicial complex".

Chapter 6 was a close collaboration with Chaim Even-Zohar and Michael Farber. An earlier version of this work has been submitted for publication, this chapter differs slightly from that and the current arXiv:2012.01483 preprint "Ample simplicial complexes".

List of publications

Many of the ideas, chapters and sections of this thesis are based on manuscripts which are either published or in preparation, these are listed below in order of completion.

1. Random simplicial complexes, duality and the critical dimension, with M. Farber and T. Nowik, Journal of Topology and Analysis, 2019.
2. Random simplicial complexes in the medial regime, with M. Farber, Topology and its Applications, 2020.
3. Enumerating the class of minimally path connected simplicial complexes, preprint. arXiv:1912.02078, 2019.
4. The Rado simplicial complex, with M. Farber, L. Strauss, arXiv:1912.02515, 2019.
5. Ample simplicial complexes, with C. Even-Zohar, M. Farber, arXiv:2012.01483, 2020.

Abstract

Random simplicial complexes are a natural higher dimensional generalisation to the models of random graphs from Erdős and Rényi of the early 60s. Now any topological question one may like to ask raises a question in probability - i.e. what is the chance this topological property occurs? Several models of random simplicial complexes have been intensely studied since the early 00s. This thesis introduces and studies two general models of random simplicial complexes that includes many well-studied models as a special case. We study their connectivity and Betti numbers, prove a satisfying duality relation between the two models, and use this to get a range of results for free in the case where all probability parameters involved are uniformly bounded. We also investigate what happens when we move to infinite dimensional random complexes and obtain a simplicial generalisation of the Rado graph, that is we show the surprising result that (under a large range of parameters) every infinite random simplicial complexes is isomorphic to a given countable complex X with probability one. We show that this X is in fact homeomorphic to the countably infinite ball. Finally, we look at and construct finite approximations to this complex X, and study their topological properties.

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Everything for Selin. The meaning to my abstraction.

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Chapter 1

Background

In this thesis we investigate and study two general models of large random simplicial complexes, describe what happens when one moves to infinite random simplicial complexes, and introduce a family of simplicial complexes that can meaningfully be described as quasirandom.

In more detail, in Chapter 2 we introduce the *lower* and *upper* models of random simplicial complexes, study some of their typical topology and explore the relationship between them. Chapter 3 further explores the topology of the upper model, in particular when it is path connected and does so by studying a class of simplicial complexes called *minimal connected covers*. Chapter 4 studies both models when all parameters involved are uniformly bounded. Chapter 5 asks what happens with lower model infinite random complexes, and proves that "almost all" infinite random simplicial complexes are actually isomorphic to a given complex X. Chapter 6 looks at finite approximations to this X, shows that "almost all" random complexes are in fact such approximations, as well as constructing an explicitly deterministic family of these approximations.

The purpose of this chapter is to introduce all of the basic terminology, notation, and simple results that will be used for the remainder of this text.

1.1 Combinatorial topology

1.1.1 Simplicial complexes

A *hypergraph* on a (possible) vertex set V is any subset of the power set 2^V. A *simplicial complex* Y on a vertex set V is a hypergraph on V that is downward closed, i.e. if $S \in \Delta$ and $T \subseteq S$ then $T \in \Delta$.

We will let $[n] = \{1, \ldots, n\}$ and $\Delta_n = 2^{[n]}$ throughout, and call Δ_n the *complete simplex* on $[n]$. An element of a simplicial complex $\sigma \in Y$ will be called *simplex* of *dimension* $\dim \sigma = |\sigma| - 1$, we will sometimes let $F(Y)$ denote the set of simplices of Y and $F_k(Y)$ those of dimension k. The k*-dimensional skeleton* of a simplicial complex Y is the k-dimensional subcomplex $Y^{(k)} = \{\sigma \in Y : \dim \sigma \leq k\}$.

Given two disjoint simplices σ, τ we define their *join* $\sigma * \tau$ as the complete simplex on vertex set $V(\sigma) \cup V(\tau)$ – we may sometimes write just $\sigma\tau$ to denote the join. We call the join $v\sigma$, with v a vertex not in σ, the *cone* over σ. Given a simplicial complex Y and a simplex $\sigma \in Y$ we define the *link* of σ in Y by

$$\mathrm{Lk}_Y(\sigma) = \{\tau \in Y : \tau \cap \sigma = \varnothing, \sigma\tau \in Y\}.$$

We observe that the vertex set of the link $V\left(\mathrm{Lk}_Y(\sigma)\right)$ is a subset of $V(Y) - V(\sigma)$.

For a simplicial subcomplex $Y \subseteq \Delta_n$ we denote by $E(Y)$ the set of *external simplices*, i.e. simplices $\sigma \in \Delta_n$ such that $\sigma \notin Y$ but the boundary $\partial\sigma$ is contained in Y. $M(Y)$ denotes the set of *maximal simplices* of Y, i.e. $\sigma \in Y$ such that for every $\tau \supset \sigma$ one has $\tau \notin Y$.

1.1.2 Homology

One of the primary tools of interest throughout this text will be that of *homology.* Loosely speaking homology counts the number and type of "holes" in a simplicial complex. We refer to Section 2 of Hatcher [39] for a proper introduction.

Given a finite simplicial complex Y and an abelian group G we define the *chain groups* of Y

$$C_k(Y;G) = \left\{ \sum_{\sigma \in F_k(Y)} g_\sigma \sigma : g_\sigma \in G \right\}.$$

Informally, $C_k(Y;G)$ consists of formal sums of k-dimensional simplices of Y and we call it's elements *k-chains*. It is clear that $C_k(Y;G) \cong G^{f_k}$ where $f_k = |F_k(y)|$ denotes the number of k-dimensional simplices in Y. To every k-dimensional simplex $\sigma = [v_0, \ldots, v_k] \in Y$ we define its *boundary* as

$$\partial_k \sigma = \sum_{i=0}^{k} (-1)^i [v_0, \ldots, \hat{v_i}, \ldots, v_k]$$

where $[v_0, \ldots, \hat{v_i}, \ldots, v_k]$ denotes the $(k-1)$-dimensional simplex obtained by removing vertex v_i from σ. This gives a map on all chains

$$\partial_k : C_k(Y;G) \to C_{k-1}(Y;G)$$

by extending linearly.

The kernel of the *boundary map* is denoted $Z_k(Y;G) = \ker \partial_k$ and called the *group of k-cycles of Y*. The image of the *boundary map* is denoted $B_k(Y;G) = \operatorname{im} \partial_{k+1}$ and called the *group of k-boundaries of Y*. It is a simple exercise to show that $\partial_k \circ \partial_{k+1} = 0$, so one has $B_k(Y) \subseteq Z_k(Y)$ and we can therefore define the *k-th homology group of Y* as cycles modulo boundaries, that is

$$H_k(Y;G) = \frac{Z_k(Y;G)}{B_k(Y;G)}.$$

When $G = \mathbb{F}$ is a field it can be shown that each homology group $H_k(Y;\mathbb{F})$ is in fact a vector space over the field $\mathbb{F}$ – it therefore makes sense to talk of the dimension of $H_k(Y;\mathbb{F})$ as an $\mathbb{F}$-vector space. Let $b_k(Y)$ denote the dimension of $H_k(Y;\mathbb{Q})$ over the rationals $\mathbb{Q}$, the $b_k(Y)$ are called the *Betti numbers* of Y.

Computing the typical Betti numbers of various models of random simplicial complexes will be a primary interest explored in both Chapter 2 and 4. These two chapters will make use of the following inequality.

Lemma 1.1.1 (Morse inequality.). *Let Y be a simplicial complex, the following inequality holds:*

$$f_k(Y) - f_{k+1}(Y) - f_{k-1}(Y) \leq b_k(Y) \leq f_k(Y),$$

where $f_k(Y)$ denotes the number of k-dimensional simplices in Y.

Proof. We will let $H_k(Y)$ denote $H_k(Y;\mathbb{Q})$, similarly for $Z_k(Y)$ and $B_k(Y)$. Then by definition

$$b_k(Y) \leq \dim Z_k = \dim\ker\partial_k \leq f_k.$$

As $\partial_k \circ \partial_{k+1} = 0$ we have, by the rank-nullity theorem, that $f_k(Y) = \dim\ker\partial_k + \dim \mathrm{im}\ \partial_{k+1}$, and therefore we have

$$\begin{aligned} b_k(Y) &= \dim Z_k - \dim B_k \\ &= (\dim\ker\partial_k - \dim \mathrm{im}\ \partial_{k+1}) - \dim \mathrm{im}\ \partial_k \\ &\geq f_k(Y) - f_{k+1}(Y) - f_{k-1}(Y). \end{aligned}$$

□

1.2 Basic probability

1.2.1 Notation

Given a sequence of measures $\mathbb{P}_n$ and an event E we say that E happens *asymptotically almost surely* (a.a.s) if

$$\lim_{n\to\infty} \mathbb{P}_n(E) = 1.$$

Let f, g be two functions taking values in $\mathbb{R}$. We will make use of the following standard notations throughout this text.

- $f = o(g)$ if $\lim_{x\to\infty} \frac{|f(x)|}{g(x)} = 0$.

- $f = O(g)$ if $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$.
- $f = \Omega(g)$ if $\liminf_{x\to\infty} \frac{f(x)}{g(x)} > 0$.

Let $\Omega_1 \subset \Omega_2 \subset \dots$ be an increasing collection of related objects (e.g. graphs or simplicial complexes). Given a model of randomness returning elements of Ω_n with a probability parameter $p(n)$ and a property $\mathcal{P}$ that makes sense for elements of Ω_n (e.g. for graphs, is it connected?) then a function $f = f(n)$ is called a *coarse threshold* for property $\mathcal{P}$ if

$$p = \begin{cases} o(f) \text{ then } \neg\mathcal{P} \text{ happens a.a.s.} \\ \Omega(f) \text{ then } \mathcal{P} \text{ happens a.a.s.} \end{cases}$$

Let $p(n) = \alpha(n)f(n)$, for some $\alpha(n) \geq 0$, we will call f is a *threshold* for then property $\mathcal{P}$ if

$$\begin{cases} \lim_{n\to\infty} \alpha(n) < 1 \text{ then } \neg\mathcal{P} \text{ happens a.a.s.} \\ \lim_{n\to\infty} \alpha(n) > 1 \text{ then } \mathcal{P} \text{ happens a.a.s.} \end{cases}$$

In Chapter 3 we will investigate some connectivity thresholds in the *upper* model.

1.2.2 Basic inequalities

There are a few elementary general results from probability theory we will make use of time and again. For the following results we will always let X be a random variable taking values in $\{0, 1, 2, \dots\}$.

Theorem 1.2.1 (Markov's inequality)**.** *For any* $a \in \{1, 2, \dots\}$

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}.$$

Proof.

$$\begin{aligned}\frac{\mathbb{E}(X)}{a} &\geq \sum_{n=a}^{\infty} \frac{n}{a} \cdot \mathbb{P}(X = n) \\ &\geq \sum_{n=a}^{\infty} \mathbb{P}(X = n) \\ &= \mathbb{P}(X \geq a)\end{aligned}$$

□

Corollary 1.2.2 (First moment method)**.** $\mathbb{P}(X = 0) \geq 1 - \mathbb{E}(X)$.

Theorem 1.2.3 (Chebychev's inequality)**.** *Let* $a > 0$*. If both* $\mathbb{E}(X)$ *and* $\mathrm{Var}(X)$ *are finite then*

$$\mathbb{P}\left(|X - \mathbb{E}(X)|\right) \geq a \cdot \mathrm{Var}(X)) \leq \frac{1}{a^2}.$$

Proof. Let $Y = (X - \mathbb{E}(X))^2$ and apply Markov's inequality to the random variable Y. □

Corollary 1.2.4 (Second moment method)**.**

$$\mathbb{P}(X > 0) \geq \frac{(\mathbf{E}X)^2}{\mathbf{E}(X^2)}.$$

Proof.

$$\begin{aligned}\mathbb{P}(X > 0) &= 1 - \mathbb{P}(X = 0) \\ &\geq 1 - \mathbb{P}\left(|X - \mathbb{E}(X)| \geq \mathbb{E}(X)\right) \\ &\geq 1 - \frac{\mathrm{Var}(X)}{(\mathbb{E}(X))^2} \\ &= \frac{\mathbb{E}\left(X^2\right)}{(\mathbb{E}(X))^2}.\end{aligned}$$

□

Heuristically, Chebychev gives a bound on how far a random variable can deviate from its expectation – lower variance means lower deviation from the mean as one expects.¬¬

1.3 Random simplicial complexes

1.3.1 Random graphs and first generalisations

The predecessor to all models of random simplicial complexes are Erdős-Rényi random graphs [31], where one includes independently at random every possible edge from vertex set $[n]$ with probability p, the resulting random graph is denoted $G(n, p)$. One tends to care about what happens to such graphs as n grows large and $p \to 0$, for many interesting properties there exists a *threshold function* $f(n)$ such that if $p \ll f(n)$ some property doesn't happen in $G(n, p)$ with probability tending to one and if $p \gg f(n)$ then that property does happen with probability tending to one.

Two natural generalisations to more general simplicial complexes emerged in the 2000s. The first by Linial, Meshulam [58] and Meshulam, Wallach [62] where one begins with the $(k-1)$-dimensional skeleton of the full simplex on n vertices, $\Delta_n^{(k-1)}$, and includes k-dimensional simplexes independently at random with probability p to obtain a random complex $Y \in Y_k(n, p)$. The second is random clique complexes introduced by Kahle [46], where one constructs the Erdős-Rényi random graph $G(n, p)$ and obtains a simplicial complex $X(n, p)$ by filling in all cliques with simplexes, i.e. every induced subgraph isomorphic to the complete graph K_r forms an $(r-1)$-dimensional simplex.

1.3.2 Multiparameter random simplicial complexes

The models mentioned in the above both have just a single parameter of randomness, p. It's a natural idea that a simplicial complex may have independent randomness in every dimension given it's inherent downward closed structure. The multiparameter model of random simplicial complexes of Costa and Farber [27] was the first to study this in detail. It should also be mentioned that concurrent to the work of Costa and Farber, Fowler [35] also began the study of multiparameter random simplicial complexes looking at higher dimensional threhsolds for the vanishing of cohomology. Their model builds random simplicial complexes from the bottom up, beginning with a set $\{1, \dots, n\}$ one includes every possible vertex with probability p_0, from this selection of vertices one adds every edge independently at random with probability p_1, now from this random graph there will be some triangles which we can now fill in with a 2-dimensional simplex with probability p_2, etc. up to the top dimensional simplexes. This gives a random simplicial complex $Y \in Y(n, (p_0, p_1, p_2, \dots))$, in the notation of [27]. It is this idea of multiparameter random simplicial complexes that will be studied and generalised in this thesis.

This multiparameter model has been well studied [26–29] with similar types of threshold results known as for $Y_k(n,p)$ and $X(n,p)$. Of course now any threshold type result will be rather more complex as the probability parameter is higher dimensional.

In [29] the authors introduced the *critical dimension*, k_*, of $Y \in Y(n, (p_0, p_1, p_2, \dots))$ to study their typical Betti numbers and give conditions under which they vanish. A similar independent study was carried out in [35] by Fowler.

More precisely, the critical dimension k_* satisfies the following properties asymptotically almost surely:

1. The Betti number $b_{k_*}(Y)$ in the critical dimension is large,

$$b_{k_*}(Y) \sim C \cdot n^{a_{k_*}},$$

   where $a_{k_*} > 0$, $C > 0$ are constants.

2. The reduced Betti numbers $\tilde{b}_j(Y)$ in all dimensions below the critical dimension $j < k_*$ vanish.
3. The Betti numbers $b_j(Y)$ in dimensions above the critical dimension $j > k_*$ are "significantly smaller" than $b_{k_*}(Y)$. That is, $\frac{b_j(Y)}{b_{k_*}(Y)} \to 0$ a.a.s.
4. If the critical dimension is positive then the random complex Y is connected.
5. If the critical dimension is greater than 2 then Y is simply connected.
6. The critical dimension k_* and the exponents a_{k_*} can be explicitly calculated through the probability parameters p_i.

By the final point above we note that the p_i determine the value of the critical dimension k_*. The transition from $k_* = k$ to $k_* = k + 1$ corresponds to the probability parameter

$(p_0, p_1, \ldots)$ crossing a higher dimensional threshold.

Chapter 2

Lower and upper models of random simplicial complexes

2.1 Introduction

In this section we introduce and study two very general probabilistic models of random simplicial complexes which we call the *lower* and *upper* models.

Lower model random complexes are constructed in fundamentally the same manner as in the case of *multiparameter random simplicial complexes* (see Section 1.3.2) with the added complexity of every simplex having its own probability parameter (not necessarily dependent on dimension). In more detail, one builds the random simplicial complex inductively, step by step, selecting each vertex v independently at random with probability p_v, then adding each edge e between the selected vertices at random each with probability p_e, and on the following step adding randomly 2-simplices σ with probability p_σ to the random graph obtained on the previous stage, and so on. We will see that upon restricting each p_σ to depend only on its dimension (i.e. $p_\sigma = p_i$ where $i = \dim \sigma$) we will recover the *multiparameter random simplicial complexes* studied in [26–29].

In the upper model one selects every simplex σ at random and includes it and every face $\tau \subset \sigma$ with probability p_σ to obtain a simplicial complex. Simultaneous to the results presented here, the upper model has been studied in the papers of Cooley, Kang et al. [22–25].

In this thesis we show that the lower and upper models are *Alexander dual* to each other, see Theorem 2.9.9. More precisely, the upper random simplicial complex is homotopy equivalent to the complement of the lower random simplicial complex in the $(n-1)$-dimensional simplicial sphere $\partial\Delta_n$. Under the duality correspondence the probability parameters p_σ should be replaced by $q_{\hat{\sigma}} = 1 - p_{\hat{\sigma}}$ where $\hat{\sigma}$ is the simplex spanned by the complement of the set of vertices of σ. We see that the duality matches *a sparse* lower model (when $p_\sigma \to 0$) with *a dense* upper model (when $p_\sigma \to 1$) and vice versa.

In the recent paper [29] (see also Section 1.3.2) Costa and Farber established an interesting pattern of behaviour of the Betti numbers of random simplicial complexes in the lower model. As discussed in Section 1.3.2, there exists a specific dimension k_* called *critical dimension* such that the Betti number $b_{k_*}(Y)$ dominates and the Betti numbers $b_j(Y)$ vanish for $0 < j < k$.

One of our goals was to investigate the Betti numbers of random simplicial complexes in the upper model. In Section 2.10 we define the notion of the *critical dimension* k^* and *the spread* s and show that the exponential growth rate of the face numbers $f_\ell(Y)$ is maximal and constant in dimensions ℓ satisfying $k^* \leq \ell \leq k^* + s$. We investigate the Betti numbers of upper model random complexes in Section 2.11. We show that in the case when the spread is zero $s = 0$ we show that the critical dimension k^* behaves similarly to the lower model: the Betti number $b_{k^*}(Y)$ is large and maximal, the Betti numbers $b_j(Y)$ vanish for $0 < j < k^*$ and $b_j(Y)$ is significantly smaller than $b_{k^*}(Y)$ for $j > k^*$. We remark that the following recent papers [22–25] explore the Betti number behaviour of the upper model.

2.2 Random hypergraphs

We shall consider *hypergraphs* X with vertex sets contained in $[n] = \{1, \ldots, n\}$; each such hypergraph X is a collection of non-empty subsets $\sigma \subseteq [n]$. We let Ω_n denote the set of all such hypergraphs.

We will define a probability measure on Ω_n. Let

$$p_\sigma \in [0,1]$$

be a probability parameter associated with each non-empty subset $\sigma \subseteq [n]$. Using these parameters p_σ we may define a probability function $\mathbb{P}_n$ on Ω_n via the formula

$$\mathbb{P}_n(X) = \prod_{\sigma \in X} p_\sigma \cdot \prod_{\sigma \notin X} q_\sigma. \tag{2.1}$$

Here $q_\sigma = 1 - p_\sigma$. Formula (2.1) can be described by saying that each simplex $\sigma \subseteq [n]$ is included into a random hypergraph X with probability p_σ independently of all other simplices. $\mathbb{P}_n$ is essentially a Bernouilli measure on the set of all non-empty subsets of $[n]$.

2.3 Lower and upper random simplicial complexes

Let $\Omega_n^* \subseteq \Omega_n$ denote the set of all simplicial complexes on the vertex set $[n]$. Recall that a hypergraph X is a simplicial complex if it is closed with respect to taking faces, i.e. if $\sigma \in X$ and $\tau \subseteq \sigma$ imply that $\tau \in X$. The set Ω_n^* is the set of all subcomplexes of Δ_n.

There are two natural surjective maps which are the identity on Ω_n^*

$$\overline{\rho}, \underline{\rho} : \Omega_n \to \Omega_n^* \tag{2.2}$$

which are constructed as follows. Given a hypergraph $X \in \Omega_n$ we define

$$\underline{\rho}(X) = \underline{X} = \max_{Y \subseteq \Delta_n} \{Y \subseteq X : Y \in \Omega_n^*\}$$

to be the largest (with respect to subset inclusion) simplicial complex in Ω_n^* contained in X. A simplex $\tau \subseteq [n]$ belongs to $\underline{X}$ if and only if every simplex $\sigma \subseteq \tau$ belongs to X.

We say that $\underline{X}$ is the *downward closure* of X. Similarly, we define

$$\overline{\rho}(X) = \overline{X} = \min_{Y \subseteq \Delta_n} \{Y \supseteq X : Y \in \Omega_n^*\}$$

to be the smallest (with respect to subset inclusion) simplicial complex in Ω_n^* containing X. A simplex $\tau \in \Delta_n$ belongs to $\overline{X}$ if and only if for some $\sigma \in X$ one has $\sigma \supseteq \tau$. We say that $\overline{X}$ is the *upward closure* of X

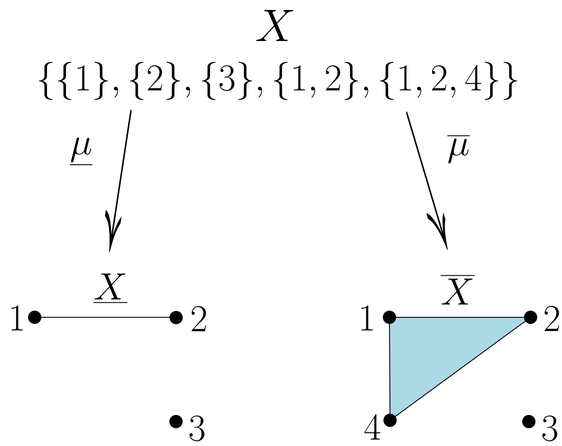

Figure 2.1: A hypergraph supported on $\{1, 2, 3, 4\}$ mapped to different simplicial complexes under $\underline{\rho}$ and $\overline{\rho}$.

It's clear from their definition that one has the following inclusion of sets

$$\underline{X} \subseteq X \subseteq \overline{X}. \tag{2.3}$$

We shall denote by

$$\underline{\mathbb{P}}_n = \underline{\rho}_*(\mathbb{P}_n) \quad \text{and} \quad \overline{\mathbb{P}}_n = \overline{\rho}_*(\mathbb{P}_n) \tag{2.4}$$

the two probability measures on the space of simplicial complexes Ω_n^* obtained as the push-forwards of the measure (2.1) with respect to the maps (2.2). We call $\underline{\mathbb{P}}_n$ the *lower* *measure* and $\overline{\mathbb{P}}_n$ the *upper measure*. Explicitly, for a simplicial complex $Y \subseteq \Delta_n$ one has

$$\underline{\mathbb{P}}_n(Y) = \sum_{X \in \Omega_n,\, \underline{X} = Y} \mathbb{P}_n(X) \quad \text{and} \quad \overline{\mathbb{P}}_n(Y) = \sum_{X \in \Omega_n,\, \overline{X} = Y} \mathbb{P}_n(X).$$

Remark 2.3.1. If $p_\sigma = 0$ for some σ then a random hypergraph X contains σ with probability 0, hence we see that $\sigma \notin \underline{X}$ with probability 1. Thus, if $p_\sigma = 0$, the lower measure $\underline{\mathbb{P}}_n$ is supported on the set of simplicial subcomplexes $Y \subseteq \Delta_n - \mathrm{St}(\sigma)$. Moreover, if $p_\tau = 0$ for every simplex $\tau \supseteq \sigma$ then $\sigma \notin \overline{X}$ with probability one and the measure $\overline{\mathbb{P}}_n$ is supported on the set of simplicial subcomplexes $Y \subseteq \Delta_n - \mathrm{St}(\sigma)$. The symbol $\mathrm{St}(\sigma)$ denotes the star of the simplex σ, i.e. the set of all simplices containing σ.

Remark 2.3.2. Consider now the opposite extreme, $p_\sigma = 1$. Then a random hypergraph X contains σ with probability 1. This implies that $\sigma \in \overline{X}$ with probability 1. Moreover, if $p_\tau = 1$ for every $\tau \subseteq \sigma$ then $\sigma \in \underline{X}$ with probability 1 and the measure $\underline{\mathbb{P}}_n$ is supported on the set of simplicial complexes $Y \subseteq \Delta_n$ containing σ.

Later in Corollary 2.5.7 we shall establish the following explicit formulae. For a simplicial subcomplex $Y \subseteq \Delta_n$ one has

$$\underline{\mathbb{P}}_n(Y) = \prod_{\sigma \in Y} p_\sigma \cdot \prod_{\sigma \in E(Y)} q_\sigma, \quad \text{and} \quad \overline{\mathbb{P}}_n(Y) = \prod_{\sigma \in M(Y)} p_\sigma \cdot \prod_{\sigma \notin Y} q_\sigma, \tag{2.5}$$

where the symbols $E(Y)$ and $M(Y)$ denote the set of external and maximal simplices respectively (see Section 1.1.1).

2.4 Duality between the lower and upper models

In this section we present a duality relation between the lower and upper models; this theme will continue in Section 2.9 where we shall show that the simplicial complexes produced by the lower and upper models are *Alexander dual* to each other.

Recall that $\partial\Delta_n$ is the simplicial complex with vertex set $[n] = \{1, \ldots, n\}$ in which simplices are all nonempty subsets $V \subset [n]$, except $V = [n]$. For a set $\sigma \in \partial\Delta_n$ we define $\hat{\sigma} = [n] - \sigma$. For a hypergraph $X \subseteq \partial\Delta_n$ we denote by $i(X)$ the image of X under the map $\sigma \mapsto \hat{\sigma}$, i.e. $i(X) = \{\hat{\sigma} : \sigma \in X\}$. Since $\sigma \mapsto \hat{\sigma}$ is an involution, $i : \Omega_n \to \Omega_n$ is also an involution. We have

$$i(X \cap Y) = i(X) \cap i(Y), \quad i(X \cup Y) = i(X) \cup i(Y),$$

and $X \subseteq Y$ if and only if $i(X) \subseteq i(Y)$.

Since $\sigma \subseteq \tau$ if and only if $\hat{\sigma} \supseteq \hat{\tau}$, we have that a hypergraph X is a simplicial complex if and only if $i(X)$ is an "anti-complex", by which we mean that if $\sigma \in i(X)$, and $\tau \supseteq \sigma$ then $\tau \in i(X)$.

A second involution on the set of hypergraphs is the map $j : \Omega_n \to \Omega_n$ defined by

$$j(X) = X^c = \{\sigma \in \partial\Delta_n : \sigma \notin X\}.$$

We have $X \subseteq Y$ if and only if $j(X) \supseteq j(Y)$, and by De Morgan's rules we have

$$j(X \cap Y) = j(X) \cup j(Y), \quad j(X \cup Y) = j(X) \cap j(Y).$$

Again, we have that X is a simplicial complex if and only if $j(X)$ is an anti-complex. Since $\sigma \mapsto \hat{\sigma}$ is a bijection we have $i(X^c) = (i(X))^c$ which means $i \circ j = j \circ i$, and so $i \circ j$ is again an involution. Finally, for a hypergraph $X \subset \partial\Delta_n$ we define the *dual* hypergraph

$$c(X) = i \circ j(X).$$

Combining the properties of i and j mentioned above we get the following properties of $c : \Omega_n \to \Omega_n$.

Lemma 2.4.1. *For hypergraphs $X, Y \subseteq \partial\Delta_n$ we have:*

1. *$\sigma \in X$ if and only if $\hat{\sigma} \notin c(X)$.*
2. *$c(c(X)) = X$.*
3. *$X \subseteq Y$ if and only if $c(X) \supseteq c(Y)$.*
4. *$c(X \cap Y) = c(X) \cup c(Y)$ and $c(X \cup Y) = c(X) \cap c(Y)$.*

5. *X is a simplicial complex if and only if $c(X)$ is a simplicial complex.*

When X is a simplicial complex $c(X)$ is sometimes known as its Björner–Tanner dual, see [7] and Section 2.9.

Lemma 2.4.2. *If $Y \subseteq \partial\Delta_n$ is a simplicial complex then a simplex σ is an external simplex of Y if and only if $\hat{\sigma}$ is a maximal simplex of $c(Y)$, and vice versa.*

Proof. An external simplex of Y is by definition a *minimal* simplex *not* in Y. Thus the statement follows from Lemma 2.4.1(1) and the fact that $\sigma \subseteq \tau$ if and only if $\hat{\sigma} \supseteq \hat{\tau}$. $\square$

The following results describe how the dual map $c : \Omega_n \to \Omega_n$ interacts with our other maps of interest. In particular, Lemma 2.4.3 describes the interaction with the downward and upward closures to a simplicial complex, and Proposition 2.4.4 details how c beahves with our probability measures $\mathbb{P}_n$, $\underline{\mathbb{P}}_n$, $\overline{\mathbb{P}}_n$ described in (2.1) and (2.4).

Lemma 2.4.3. *For every hypergraph $X \subseteq \partial\Delta_n$ we have $c(\overline{X}) = \underline{c(X)}$ and similarly $c(\underline{X}) = \overline{c(X)}$.*

Proof. Since $\underline{X} \subseteq X \subseteq \overline{X}$ we have $c(\overline{X}) \subseteq c(X) \subseteq c(\underline{X})$ and hence

$$c(\overline{X}) \subseteq \underline{c(X)} \subseteq c(X) \subseteq \overline{c(X)} \subseteq c(\underline{X}), \tag{2.6}$$

using properties (3) and (5). Applying the operator c to the inclusion $c(\overline{X}) \subseteq \underline{c(X)}$ and replacing X by $c(X)$ we get $\overline{c(X)} \supseteq c(\underline{X})$ which is the inverse to the right inclusion in (2.6). Thus, $\overline{c(X)} = c(\underline{X})$. Replacing here X by $c(X)$ and applying the operator c to both sides we obtain $c(\overline{X}) = \underline{c(X)}$. $\square$

Proposition 2.4.4. *Given $\mathbb{P}_n$ defined on Ω_n by probabilities $\{p_\sigma\}_{\sigma\in\partial\Delta_n}$, define a new probability measure $\mathbb{P}'_n$ on Ω_n by probabilities $\{p'_\sigma\}_{\sigma\in\partial\Delta_n}$ where*

$$p'_\sigma = q_{\hat{\sigma}} = 1 - p_{\hat{\sigma}}.$$

Then

1. *For every hypergraph* $X \subseteq \partial\Delta_n$,

$$\mathbb{P}_n(c(X)) = \mathbb{P}'_n(X).$$

2. *For every simplicial complex* $Y \subseteq \partial\Delta_n$,

$$\overline{\mathbb{P}}_n(c(Y)) = \underline{\mathbb{P}}'_n(Y) \quad \textit{and} \quad \underline{\mathbb{P}}_n(c(Y)) = \overline{\mathbb{P}}'_n(Y).$$

Proof. (1) By definition of $\mathbb{P}_n$ and by Lemma 2.4.1(1) we have

$$\begin{aligned}\mathbb{P}_n(c(X)) = \prod_{\sigma \in c(X)} p_\sigma \cdot \prod_{\sigma \notin c(X)} q_\sigma = \prod_{\hat{\sigma} \notin X} p_\sigma \cdot \prod_{\hat{\sigma} \in X} q_\sigma = \prod_{\sigma \notin X} p_{\hat{\sigma}} \cdot \prod_{\sigma \in X} q_{\hat{\sigma}} \\ = \prod_{\sigma \notin X} q'_\sigma \cdot \prod_{\sigma \in X} p'_\sigma = \mathbb{P}'_n(X).\end{aligned}$$

(2) By (1) and by Lemma 2.4.3, for every simplicial complex Y,

$$\overline{\mathbb{P}}_n(c(Y)) = \sum_{\overline{X} = c(Y)} \mathbb{P}_n(X) = \sum_{c(\overline{X}) = Y} \mathbb{P}'_n(c(X)) = \sum_{\underline{c(X)} = Y} \mathbb{P}'_n(c(X)) = \underline{\mathbb{P}}'_n(Y).$$

□

2.5 The sandwich formulae

Let $A \subseteq B \subseteq \partial\Delta_n$ be two simplicial complexes. In both the lower and upper probability measures $\underline{\mathbb{P}}_n$ and $\overline{\mathbb{P}}_n$, we ask what is the probability that a random simplicial complex Y satisfies $A \subseteq Y \subseteq B$. That is, we are interested in finding the probability

$$\underline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \sum_{A \subseteq Y \subseteq B} \underline{\mathbb{P}}_n(Y) = \sum_{A \subseteq \underline{X} \subseteq B} \mathbb{P}_n(X). \tag{2.7}$$

Here Y denotes a simplicial subcomplex $Y \in \Omega_n^*$ and X denotes a hypergraph $X \in \Omega_n$. Similarly, we want to calculate explicitly the quantities

$$\overline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \sum_{A \subseteq Y \subseteq B} \overline{\mathbb{P}}_n(Y) = \sum_{A \subseteq \overline{X} \subseteq B} \mathbb{P}_n(X). \tag{2.8}$$

Note that for hypergraphs the answer to the analogous question is simple:

$$\mathbb{P}_n(A \subseteq X \subseteq B) = \prod_{\sigma \in A} p_\sigma \cdot \prod_{\sigma \notin B} q_\sigma,$$

where A, B are fixed hypergraphs and X is a random hypergraph.

Recall that for a simplicial complex B, the symbol $E(B)$ denotes the set of all external simplices of B, i.e. simplices $\sigma \in \partial\Delta_n$ such that $\sigma \notin B$ but $\partial\sigma \subseteq B$.

Proposition 2.5.1 (Sandwich formula for the lower model)**.** *Let $A \subseteq B \subseteq \partial\Delta_n$ be two simplicial complexes. For every subset $S \subseteq E(B)$ let A_S be the set of all simplices $\tau \notin A$ such that $\tau \subseteq \sigma$ for some $\sigma \in S$. Let*

$$P_S = \prod_{\tau \in A_S} p_\tau$$

and

$$\tilde{P} = \prod_{\tau \in A} p_\tau.$$

Then

$$\underline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \tilde{P} \cdot \sum_{S \subseteq E(B)} (-1)^{|S|} P_S, \tag{2.9}$$

where by definition $P_\varnothing = 1$.

Proof. Since A and B are simplicial complexes, a hypergraph X satisfies $A \subseteq \underline{X} \subseteq B$ if and only if $X \supseteq A$ and $X \not\supseteq A_{\{\sigma\}}$ for all $\sigma \in E(B)$. So we have

$$\begin{aligned}\{X : A \subseteq \underline{X} \subseteq B\} &= \{X : X \supseteq A\} \cap \bigcap_{\sigma \in E(B)} \{X : X \not\supseteq A_{\{\sigma\}}\} \\ &= \bigcap_{\sigma \in E(B)} \{X : X \supseteq A, X \not\supseteq A_{\{\sigma\}}\}.\end{aligned}$$

To evaluate the probability of this event we use the inclusion-exclusion formula with ambient set $\{X : X \supseteq A\}$, so the event $\{X : X \supseteq A, X \not\supseteq A_{\{\sigma\}}\}$ is the complement of the event $\{X : X \supseteq A, X \supseteq A_{\{\sigma\}}\} = \{X : X \supseteq A \cup A_{\{\sigma\}}\}$. We thus get

$$\begin{aligned}\mathbb{P}_n(A \subseteq \underline{X} \subseteq B) &= \sum_{S \subseteq E(B)} (-1)^{|S|} \mathbb{P}_n\Big(\bigcap_{\sigma \in S} \{X : X \supseteq A \cup A_{\{\sigma\}}\}\Big) \\ &= \sum_{S \subseteq E(B)} (-1)^{|S|} \mathbb{P}_n\big(X \supseteq A \cup A_S\big) \\ &= \sum_{S \subseteq E(B)} (-1)^{|S|} \prod_{\tau \in A \cup A_S} p_\tau = \sum_{S \subseteq E(B)} (-1)^{|S|} \tilde{P} P_S.\end{aligned}$$

The second equality holds since $A_S = \bigcup_{\sigma \in S} A_{\{\sigma\}}$. □

Using the duality results of Section 2.4, we obtain the following dual result for $\overline{\mathbb{P}}_n$.

Proposition 2.5.2 (Sandwich formula for the upper model)**.** *Let $A \subseteq B \subseteq \partial\Delta_n$ be two simplicial complexes. For every $S \subseteq M(A)$ let B_S be the set of all simplices $\tau \in B$ such that $\tau \supseteq \sigma$ for some $\sigma \in S$. Let*

$$Q_S = \prod_{\tau \in B_S} q_\tau$$

and

$$\tilde{Q} = \prod_{\tau \notin B} q_\tau.$$

Then

$$\overline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \tilde{Q} \cdot \sum_{S \subseteq M(A)} (-1)^{|S|} Q_S, \tag{2.10}$$

where by definition $Q_\varnothing = 1$.

Proof. This follows from Proposition 2.5.1 via the dual measure $\mathbb{P}'_n$ using Proposition 2.4.4 and Lemma 2.4.2. $\square$

As a corollary to the proof of Proposition 2.5.1, we get the following characterization of the probability measures $\underline{\mathbb{P}}_n$ and $\overline{\mathbb{P}}_n$.

Corollary 2.5.3 (Intrinsic characterisation of the upper and lower measures)**.** *Let λ be a probability measure on the set of simplicial complexes $Y \subseteq \partial\Delta_n$. Let $\{p_\sigma\}_{\sigma\in\partial\Delta_n}$ be a fixed assignment of numbers $0 \le p_\sigma \le 1$, and denote $q_\sigma = 1 - p_\sigma$.*

1. *We have $\lambda = \underline{\mathbb{P}}_n$ if and only if for every simplicial complex K, $\lambda(Y \supseteq K) = \prod_{\sigma\in K} p_\sigma$.*
2. *We have $\lambda = \overline{\mathbb{P}}_n$ if and only if for every simplicial complex K, $\lambda(Y \subseteq K) = \prod_{\sigma\notin K} q_\sigma$,*

Proof. The "only if" direction follows immediately from the definition of the lower and upper models, and is also a special case of Propositions 2.5.1, 2.5.2. We show the "if" direction of (1) and (2) as follows.

(1) The only place in the proof of Proposition 2.5.1 where the probability measure was used was in the equality $\mathbb{P}_n\big(X \supseteq A \cup A_S\big) = \prod_{\sigma\in A\cup A_S} p_\sigma$. We note however that $K = A \cup A_S$ is a *simplicial complex*, and so $X \supseteq K$ if and only if $\underline{X} \supseteq K$, so $\mathbb{P}_n(X \supseteq K) = \mathbb{P}_n(\underline{X} \supseteq K) = \underline{\mathbb{P}}_n(Y \supseteq K)$. So in fact we only needed to know that $\underline{\mathbb{P}}_n(Y \supseteq K) = \prod_{\sigma\in K} p_\sigma$.

(2) Define another probability measure λ' on simplicial complexes by $\lambda'(Y) = \lambda(c(Y))$. Then for every simplicial complex K, $\lambda'(Y \supseteq K) = \lambda(Y \subseteq c(K)) = \prod_{\sigma\notin c(K)} q_\sigma = \prod_{\hat\sigma\in K} q_\sigma = \prod_{\sigma\in K} q_{\hat\sigma} = \prod_{\sigma\in K} p'_\sigma$, where as in Proposition 2.4.4 we define $p'_\sigma = q_{\hat\sigma}$ and the corresponding $\underline{\mathbb{P}}'_n$. By (1) applied to $\{p'_\sigma\}_{\sigma\in\partial\Delta_n}$ we get $\lambda' = \underline{\mathbb{P}}'_n$ and so by Proposition 2.4.4, $\lambda = \overline{\mathbb{P}}_n$. $\square$

Next we consider a few special cases where simplified versions of the sandwich formulae hold.

Corollary 2.5.4. *Let $A \subseteq B \subseteq \partial\Delta_n$ be two simplicial complexes.*

1. *In the notation of Proposition 2.5.1, if the sets $A_{\{\sigma\}}$ for $\sigma \in E(B)$ are disjoint,*

then

$$\underline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \tilde{P} \cdot \prod_{\sigma \in E(B)} (1 - P_{\{\sigma\}}).$$

2. *In the notation of Proposition 2.5.2, if the sets $B_{\{\sigma\}}$ for $\sigma \in M(A)$ are disjoint, then*

$$\overline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \tilde{Q} \cdot \prod_{\sigma \in M(A)} (1 - Q_{\{\sigma\}}).$$

Proof. (1) Since the sets $A_{\{\sigma\}}$ are disjoint, for every $S \subseteq E(B)$ we have $P_S = \prod_{\sigma \in S} P_{\{\sigma\}}$. Thus

$$\prod_{\sigma \in E(B)} (1 - P_{\{\sigma\}}) = \sum_{S \subseteq E(B)} (-1)^{|S|} \prod_{\sigma \in S} P_{\{\sigma\}} = \sum_{S \subseteq E(B)} (-1)^{|S|} P_S.$$

(2) is shown almost identically, we include it here for completeness. As the sets $B_{\{\sigma\}}$ are disjoint, for each $S \subseteq M(A)$ we have $Q_S = \prod_{\sigma \in S} Q_{\{\sigma\}}$. Thus

$$\prod_{\sigma \in M(A)} (1 - Q_{\{\sigma\}}) = \sum_{S \subseteq M(A)} (-1)^{|S|} \prod_{\sigma \in S} Q_{\{\sigma\}} = \sum_{S \subseteq M(A)} (-1)^{|S|} Q_S.$$

□

Example 2.5.5. For a simplex σ we have

$$\overline{\mathbb{P}}_n(\sigma \in Y) = 1 - Q_{\{\sigma\}} = 1 - \prod_{\tau \supseteq \sigma} q_\tau.$$

This may be seen from Corollary 2.5.4(2) taking $A = \{\tau : \tau \subseteq \sigma\}$, and noting that $M(A) = \{\sigma\}$.

It also follows immediately from the definition of the upper model that

$$\overline{\mathbb{P}}_n(\sigma \notin Y) = \prod_{\tau \supseteq \sigma} q_\tau.$$

Corollary 2.5.6. *Let $A \subseteq B \subseteq \partial\Delta_n$ be two simplicial complexes.*

1. *If $E(B) \subseteq E(A)$ then $\underline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \prod_{\sigma \in A} p_\sigma \cdot \prod_{\sigma \in E(B)} q_\sigma$.*
2. *If $M(A) \subseteq M(B)$ then $\overline{\mathbb{P}}_n(A \subseteq Y \subseteq B) = \prod_{\sigma \notin B} q_\sigma \cdot \prod_{\sigma \in M(A)} p_\sigma$.*

Proof. (1) Since $E(B) \subseteq E(A)$ we have for every $\sigma \in E(B)$, $A_{\{\sigma\}} = \{\sigma\}$. Thus $P_{\{\sigma\}} = p_\sigma$, and so the factors $1 - P_{\{\sigma\}}$ of Corollary 2.5.4 reduce to $1 - p_\sigma = q_\sigma$. (2) is similar. $\square$

Finally we also obtain an explicit formula for $\underline{\mathbb{P}}_n$ and $\overline{\mathbb{P}}_n$ themselves:

Corollary 2.5.7. *Let $Y \subseteq \partial\Delta_n$ be a simplicial complex. Then*

1. $\underline{\mathbb{P}}_n(Y) = \prod_{\sigma \in Y} p_\sigma \cdot \prod_{\sigma \in E(Y)} q_\sigma$.
2. $\overline{\mathbb{P}}_n(Y) = \prod_{\sigma \notin Y} q_\sigma \cdot \prod_{\tau \in M(Y)} p_\sigma$.

Proof. Apply Corollary 2.5.6 with $A = B = Y$. $\square$

2.6 Links as random complexes

Consider random simplicial complexes Y containing a fixed simplex $\sigma \subset [n]$. The link of σ in Y,

$$\mathrm{Lk}_Y(\sigma) = L \subseteq \Delta_{n-\sigma},$$

is a random simplicial subcomplex of the simplex $\Delta_{n-\sigma}$, where $\Delta_{n-\sigma}$ denotes the simplex spanned by the vertices $[n] - \sigma$. Recall that by the definition a simplex $\tau \in \Delta_{n-\sigma}$ lies in the link $\mathrm{Lk}_Y(\sigma)$ if and only if the simplex $\sigma\tau$ belongs to Y. Here $\sigma\tau$ denotes the simplex $\sigma \cup \tau$ which geometrically is represented by the join $\sigma\tau = \sigma * \tau$.

In this section we shall consider the probability measures on the set of simplicial subcomplexes of $\Delta_{n-\sigma}$ which arise as the push-forwards of the conditional probability measures

$$\underline{\lambda} := \frac{\underline{\mathbb{P}}_n(Y)}{\underline{\mathbb{P}}_n(\sigma \in Y)} \quad \text{and} \quad \overline{\lambda} := \frac{\overline{\mathbb{P}}_n(Y)}{\overline{\mathbb{P}}_n(\sigma \in Y)} \tag{2.11}$$

under the map $Y \mapsto \mathrm{Lk}_Y(\sigma)$.

One is motivated to study links in random simplicial complexes primarily with the view of being able to apply Garland's method, see [4] for details, which uses the structure of the links to find when homology vanishes.

2.6.1 Links in the lower model

Theorem 2.6.1. *The measure $\underline{\lambda}$ is the lower probability measure on the subcomplexes of $\Delta_{n-\sigma}$ with parameters*

$$p'_\tau = p_\tau \cdot \prod_{\nu \subseteq \sigma} p_{\nu\tau}. \tag{2.12}$$

where $\tau \in \Delta_{n-\sigma}$.

Proof. We wish to compute probability that the link L contains a given subcomplex $A \subseteq \Delta_{n-\sigma}$, i.e.

$$\underline{\lambda}(A \subseteq L) = \sum_{A \subseteq L} \underline{\lambda}(A).$$

Using Corollary 2.5.3(1), we find

$$\begin{aligned}
\underline{\lambda}(A \subseteq L) &= \underline{\mathbb{P}}_n(\sigma \in Y)^{-1} \cdot \underline{\mathbb{P}}_n(\sigma * A \subseteq Y) \\
&= \left(\prod_{\nu \subseteq \sigma} p_\nu\right)^{-1} \cdot \left(\prod_{\nu \subseteq \sigma} p_\nu \cdot \prod_{\tau \subseteq A} p_\tau \cdot \prod_{\nu \subseteq \sigma, \tau \subseteq A} p_{\nu\tau}\right) \\
&= \prod_{\tau \subseteq A} \left[p_\tau \cdot \prod_{\nu \subseteq \sigma} p_{\nu\tau}\right] = \prod_{\tau \subseteq A} p'_\tau.
\end{aligned}$$

Our statement now follows from the intrinsic characterisation of the lower measure, see Corollary 2.5.3(1). □

We may also extend this slightly to the case where we consider the link of a vertex set. That is, suppose $V(Y)$ is a fixed vertex set and $\mathrm{Lk}_Y(V)$ is the simplicial complex defined to be the union of all σ disjoint from V such that the join $v\sigma$ is contained in Y for every $v \in V$. Clearly $\mathrm{Lk}_Y(V) \subseteq \Delta_{n-V}$ where Δ_{n-V} denotes the complete simplex on $[n]-V$. The following Lemma 2.6.2 will be used in Section 4.5.2 to study the intersection of links of lower model random complexes in an application of the Nerve Lemma.

Lemma 2.6.2. *Let Y be a random simplicial complex with respect to the lower measure with probability parameters $\{p_\sigma\}$ containing the set of vertices V and consider the link*

$\mathrm{Lk}_Y(V)$ *as a random simplicial subcomplex of* Δ_{n-V}*. Then* $\mathrm{Lk}_Y(V)$ *is a random simplicial subcomplex with respect to the lower probability measure with the set of probability parameters*

$$p'_\tau = p_\tau \cdot \prod_{v \in V} p_{v\tau},$$

where $\tau \in \Delta_{n-V}$*.*

Proof. Our proof follows almost identically to the above. Define the following probability function on the set of all subcomplexes $L \subset \Delta'$

$$\underline{\lambda}(L) = \underline{\mathbb{P}}_n(V \subset Y)^{-1} \cdot \sum_{V \subset Y \& \mathrm{Lk}_Y(V)=L} \underline{\mathbb{P}}_n(Y).$$

Here $\underline{\mathbb{P}}_n(V \subset Y)^{-1} = \left(\prod_{v \in V} p_v\right)^{-1}$ is a normalising factor. We want to compute probability that $\mathrm{Lk}_Y(V)$ contains a given subcomplex $L \subset \Delta'$, i.e.

$$\begin{aligned}
\underline{\lambda}(\mathrm{Lk}_Y(V) \supset L) &= \underline{\mathbb{P}}_n(V \subset Y)^{-1} \cdot \sum_{V \subset Y \& \mathrm{Lk}_Y(V) \supset L} \underline{\mathbb{P}}_n(Y) \\
&= \underline{\mathbb{P}}_n\left(V \subseteq Y\right)^{-1} \cdot \underline{\mathbb{P}}_n\left(V * L \subset Y\right) \\
&= \left(\prod_{v \in V} p_v\right)^{-1} \cdot \prod_{\sigma \in V * L} p_\sigma \\
&= \left(\prod_{v \in V} p_v\right)^{-1} \cdot \left(\prod_{v \in V} p_v \cdot \prod_{\tau \in L} p_\tau \cdot \prod_{v \in V, \tau \in L} p_{v\tau}\right) \\
&= \prod_{\tau \in L} \left[p_\tau \cdot \prod_{v \in V} p_{v\tau}\right] = \prod_{\tau \in L} p'_\tau.
\end{aligned}$$

The statement of Lemma 2.6.2 now follows again from the intrinsic characterisation of the lower probability measure of Corollary 2.5.3(1). □

Example 2.6.3. *Consider the special case when the probability parameters* $p_\tau = p_i$ *depends only in the dimension* $i = \dim \tau$*. Since*

$$\dim(\nu\tau) = \dim \nu + \dim \tau + 1,$$

and there are $\binom{k+1}{j+1}$ *simplices* $\nu \subseteq \sigma$ *of dimension* $j = \dim \nu$*, where* $k = \dim \sigma$*, we see that formula (2.12) can be rewritten as follows*

$$p'_i = p_i \cdot \prod_{j=0}^{k} p_{i+j+1}^{\binom{k+1}{j+1}}, \tag{2.13}$$

This is consistent with Lemma 3.2 from [26] that details the model of links in the multi-parameter random complex.

2.6.2 Links in the upper model

Next we describe the measure $\overline{\lambda}$ as defined in Equation (2.11). It is the push-forward of the conditional probability measure on the set of simplicial complexes $Y \subset \Delta_n$ containing a given simplex σ with respect to the map $Y \mapsto \mathrm{Lk}_Y(\sigma)$.

Theorem 2.6.4. *Let* $Y \subseteq \Delta_n$ *be a random simplicial complex distributed with respect to the upper measure* $\overline{\mathbb{P}}_n$ *with the set of probability parameters* p_σ*. Assume that* Y *contains a fixed simplex* $\sigma \in \Delta_n$*. Then* $\overline{\lambda}$ *equals*

$$c_\sigma \cdot \overline{\mathbb{P}}' + (1 - c_\sigma) \cdot \lambda_\varnothing, \tag{2.14}$$

where $\overline{\mathbb{P}}'$ *denotes the upper probability measure on subcomplexes of* $\Delta_{n-\sigma}$ *with the set of probability parameters* $p'_\tau = p_{\sigma\tau}$*,* $\lambda_\varnothing$ *is the measure supported on the empty complex, and* $c_\sigma = \left(1 - \prod_{\tau \supseteq \sigma} q_\tau\right)^{-1}$ *is a constant.*

Proof. We have

$$\overline{\lambda}(L) = \frac{\overline{\mathbb{P}}_n(\sigma \in Y \ \&\ \mathrm{Lk}_Y(\sigma) = L)}{\overline{\mathbb{P}}_n(\sigma \in Y)} = \frac{\overline{\mathbb{P}}_n(\sigma * L \subseteq Y \subseteq \sigma * L \cup (\partial\sigma * \Delta_{n-\sigma}))}{\overline{\mathbb{P}}_n(\sigma \in Y)}$$

Assuming that $L \neq \varnothing$ we see that the maximal simplices of $\sigma * L$ are of the form $\sigma * \tau = \sigma\tau$ where τ is a maximal simplex of L. These are also maximal simplices of $\sigma * L \cup \partial\sigma * \Delta_{n-\sigma}$.

Hence applying Corollary 2.5.6(2) we find (assuming that $L \neq \varnothing$)

$$\overline{\lambda}(L) = c_\sigma \cdot \prod_{\tau \in M(L)} p_{\sigma\tau} \cdot \prod_{\tau \in \Delta_{n-\sigma} - L} q_{\sigma\tau} = c_\sigma \cdot \prod_{\tau \in M(L)} p'_\tau \cdot \prod_{\tau \in \Delta_{n-\sigma} - L} q'_\tau = c_\sigma \cdot \overline{\mathbb{P}}'(L),$$

where $c_\sigma = \overline{\mathbb{P}}_n(\sigma \in Y)^{-1}$. Besides, for $L = \varnothing$ we have

$$\overline{\lambda}(\varnothing) = \frac{\overline{\mathbb{P}}_n(\sigma \subseteq Y \subseteq \sigma \cup (\partial\sigma * \Delta_{n-\sigma}))}{\overline{\mathbb{P}}_n(\sigma \in Y)} = c_\sigma p_\sigma \prod_{\tau \in \Delta_{n-\sigma}} q_{\sigma\tau} = c_\sigma p_\sigma \overline{\mathbb{P}}'_n(\varnothing).$$

Thus, noting $c_\sigma = (1 - q_\sigma \overline{\mathbb{P}}'_n(\varnothing))^{-1}$, we obtain (2.14). □

Note that $\overline{\lambda}$ is an upper type probability measure with anomaly at $\varnothing$.

2.7 Combining random simplicial complexes

2.7.1 Intersections in the lower model

The following Lemma generalises Lemma 4.1 from [26].

Lemma 2.7.1. *Consider two sets of probability parameters $p_\sigma, p'_\sigma \in [0,1]$ associated to each simplex $\sigma \subseteq \Delta_n$. Let $\underline{\mathbb{P}}$ and $\underline{\mathbb{P}}'$ denote the lower probability measures determined by the probability parameters p_σ and p'_σ. Suppose that $Y, Y' \subseteq \Delta_n$ are two independent random simplicial complexes where Y is described according to the probability $\underline{\mathbb{P}}$ and Y' is sampled according to $\underline{\mathbb{P}}'$. Then the intersection $Y \cap Y' \subseteq \Delta_n$ is a random simplical complex which is described by the lower probability measure with respect to the set of probability parameters $p_\sigma \cdot p'_\sigma$.*

Proof. Let $A \subseteq \Delta_n$ be a simplicial complex. Clearly $A \subseteq Y \cap Y'$ is equivalent to $A \subseteq Y$ and $A \subseteq Y'$. Since Y and Y' are independent we see that the probability that the intersection $Y \cap Y'$ contains A equals the product

$$\underline{\mathbb{P}}(A \subseteq Y) \cdot \underline{\mathbb{P}}'(A \subseteq Y') = \prod_{\sigma \in A} p_\sigma \cdot \prod_{\sigma \in A} p'_\sigma = \prod_{\sigma \in A} \left(p_\sigma \cdot p'_\sigma\right). \tag{2.15}$$

Our statement now follows from Corollary 2.5.3(1). □

2.7.2 Unions in the upper model

Lemma 2.7.2. *Consider the union $Y \cup Y'$ of two independent random simplicial complexes $Y, Y' \subseteq \Delta_n$ where Y is sampled according to the upper probability measure $\overline{\mathbb{P}}$ with respect to a set of probability parameters q_σ and Y' is sampled according to the upper probability measure $\overline{\mathbb{P}}$ with respect to a set of probability parameters q'_σ. Then the union $Y \cup Y' \subseteq \Delta_n$ is a random simplical complex which is described by the upper probability measure with respect to the set of probability parameters $q_\sigma \cdot q'_\sigma$. In other words, the union $Y \cup Y'$ is an upper random simplicial complex with the set of probability parameters*

$$\sigma \mapsto p_\sigma + p'_\sigma - p_\sigma \cdot p'_\sigma.$$

where $p_\sigma = 1 - q_\sigma$ and $p'_\sigma = 1 - q'_\sigma$.

Proof. Let $B \subseteq \Delta_n$ be a simplicial complex. Clearly $Y \cup Y' \subseteq B$ is equivalent to $Y \subseteq B$ and $Y' \subseteq B$. Since Y and Y' are independent, the probability that the union $Y \cup Y'$ is contained in B equals the product

$$\overline{\mathbb{P}}(Y \subseteq B) \cdot \overline{\mathbb{P}'}(Y' \subseteq B) \;=\; \prod_{\sigma \notin B} q_\sigma \cdot \prod_{\sigma \notin B} q'_i \;=\; \prod_{\sigma \notin B} \left(q_\sigma \cdot q'_\sigma\right). \tag{2.16}$$

Our statement now follows from Corollary 2.5.3(2). □

2.8 Pure random complexes

In this section we consider an interesting example of a random simplicial complex; the result of this section will be used in the proof of Theorem 2.11.3.

We fix a positive integer $k > 0$ and consider an upper random simplicial complex with probability parameters

$$p_\sigma = \begin{cases} p & \text{if } \dim \sigma = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2.17}$$

Here $p \in (0, 1)$ is a positive parameter, which typically depends on n. A random complex in this model is built by randomly selecting k-dimensional simplices $\sigma \in \Delta_n$ independently at random with probability p, and adding all faces of the selected simplices.

Lemma 2.8.1 gives the conditions on the probability parameter p that the random pure k-dimensional complex contains the full l-dimensional skeleton $\Delta_n^{(\ell)}$, where $0 \leq \ell < k$. The proof follows via a standard application of the first and second moments.

Lemma 2.8.1. *The threshold probability for a k-dimensional pure random simplicial complex to contain the complete ℓ-skeleton is*

$$p = \frac{(\ell+1)\log n}{\binom{n-\ell}{k-\ell}}. \tag{2.18}$$

Proof. For $\sigma \in \Delta_n$, $\dim \sigma = \ell$, let X_σ be a random variable which equals 1 if $\sigma \notin Y$ and 0 if $\sigma \in Y$. Then $X = \sum X_\sigma$ is the random variable counting the number of ℓ-simplices not in Y. We have $\mathbb{E}(X_\sigma) = q^{\binom{n-\ell-1}{k-\ell}}$ (where, as usual, $q = 1 - p$) and

$$\mathbb{E}(X) = \binom{n}{\ell+1} \cdot q^{\binom{n-\ell-1}{k-\ell}}.$$

We will show that under the assumption

$$p = \frac{(\ell+1)\log n + \omega}{\binom{n-\ell-1}{k-\ell}} \tag{2.19}$$

we have $\mathbb{E}(X) \to 0$. Indeed,

$$\mathbb{E}(X) \leq \binom{n}{\ell+1} e^{-p\binom{n-\ell-1}{k-\ell}} \leq \exp\left((\ell+1)\log n - p\binom{n-\ell-1}{k-\ell}\right) \sim e^{-\omega} \to 0.$$

It now follows via the first moment method (see Corollary 1.2.2) that under (2.19) one has $\Delta_n^\ell \subset Y$ a.a.s.

We now wish to show conversely that under the assumption

$$p = \frac{(\ell+1)\log n - \omega}{\binom{n-\ell-1}{k-\ell}}, \tag{2.20}$$

then Y does not contain the ℓ-skeleton $\Delta_n^{(\ell)}$, a.a.s.

To do this we will apply the inequality of the second moment method, see Corollary 1.2.4. We will show that under (2.20) $\frac{\mathbb{E}(X)^2}{\mathbb{E}(X^2)} \to 1$. We have

$$X^2 = \sum_{(\sigma,\tau)} X_\sigma X_\tau,$$

where (σ, τ) runs over all pairs of ℓ-dimensional simplices of Δ_n, and

$$\mathbb{E}(X_\sigma X_\tau) = \begin{cases} q^{2\binom{n-\ell-1}{k-\ell} - \binom{n-x-1}{k-x}} & \text{if} \quad x \leq k, \\ q^{2\binom{n-\ell-1}{k-\ell}} & \text{if} \quad x > k. \end{cases} \tag{2.21}$$

where $x = \dim(\sigma \cup \tau)$. Both cases in this formula can be written as in the upper row since $\binom{r}{s} = 0$ for $s < 0$. To explain formula (2.21) we note that $\mathbb{E}(X_\sigma X_\tau)$ equals probability that neither of the simplices σ, τ are included in Y. There are $\binom{n-\ell-1}{k-\ell}$ simplices of dimension k containing σ and the same number of k-simplices contain τ. However in this count we include the k-simplices containing both σ and τ twice, and this fact is reflected in the term $\binom{n-x-1}{k-x}$.

Denoting

$$d = \dim(\sigma \cap \tau) = 2\ell - x$$

we obtain

$$\mathbb{E}(X^2) \quad = \quad \sum_{d=-1}^{\ell} \binom{n}{\ell+1} \cdot \binom{\ell+1}{d+1} \cdot \binom{n-\ell-1}{\ell-d} \cdot q^{2\binom{n-\ell-1}{k-\ell} - \binom{n-x-1}{k-x}}$$

and hence

$$\frac{\mathbb{E}(X^2)}{\mathbb{E}(X)^2} = \sum_{d=-1}^{\ell} \frac{\binom{\ell+1}{d+1}\binom{n-\ell-1}{\ell-d}}{\binom{n}{\ell+1}} \cdot q^{-\binom{n-x-1}{k-x}}. \tag{2.22}$$

Here $x = 2\ell - d$.

The term of the sum (2.22) with $d = \ell$ and $x = \ell$ is

$$\binom{n}{\ell+1}^{-1} q^{-\binom{n-\ell-1}{k-\ell}} = \mathbb{E}(X)^{-1}.$$

We show below that assumption (2.20) implies that $\mathbb{E}(X) \to \infty$ and hence this term tends to 0. There exists $C > 0$ and $N > 0$ such that for any $n > N$ one has $\binom{n}{\ell+1} > Cn^{\ell+1}$. Hence

$$\begin{aligned}
\log \mathbb{E}(X) &> (\ell+1)\log n + \binom{n-\ell-1}{k-\ell}\log(1-p) + C' \\
&> (\ell+1)\log n - \binom{n-\ell-1}{k-\ell} p(1+p) + C' \\
&> \omega(1+p) - (\ell+1)\cdot p \cdot \log n + C'.
\end{aligned}$$

It is easy to see that our assumption (2.20) and also $\ell < k$ imply that $p \log n \to 0$. Hence, we see that the summand of (2.22) with $d = \ell$ tends to zero.

Consider now the term of (2.22) with $d = -1$ and $x = 2\ell + 1$; it equals

$$\frac{\binom{n-\ell-1}{\ell+1}}{\binom{n}{\ell+1}} q^{-\binom{n-2\ell-2}{k-2\ell-1}}.$$

We show below that this term tends to 1 as $n \to \infty$. For $k \le 2\ell+1$ our claim is obvious since the coefficient $\frac{\binom{n-\ell-1}{\ell+1}}{\binom{n}{\ell+1}}$ tends to 1. We shall assume that $k > 2\ell + 1$, and observe that (2.20) implies that

$$p\binom{n-2\ell-1}{k-2\ell-1} \sim pn^{k-2\ell-1} \to 0$$

and therefore we obtain

$$q^{\binom{n-2\ell-2}{k-2\ell-1}} = 1 - p\binom{n-2\ell-2}{k-2\ell-1} + O\left(p^2\binom{n-2\ell-2}{k-2\ell-1}^2\right)$$

which converges to 1.

It remains to show that any summand of (2.22) with $-1 < d < \ell$ tends to zero. If the symbol S_d represents this summand, then

$$S_d^{-1} = \frac{\binom{n}{\ell+1}}{\binom{\ell+1}{d+1}\binom{n-\ell-1}{\ell-d}} \cdot q^{\binom{n-x-1}{k-x}}$$

and we show that $S_d^{-1} \to \infty$. Using the inequalities $\binom{n}{\ell+1} > \frac{n^{\ell+1}}{(\ell+1)^{\ell+1}}$ and $\binom{n-\ell-1}{\ell-d} < n^{\ell-d}$ we obtain

$$\begin{aligned} \log(S_d^{-1}) &> (\ell+1)\log n - (\ell-d)\log n + \binom{n-x-1}{k-x}\log(1-p) + C \\ &> (d+1)\log n - 2\binom{n-x-1}{k-x}p + C, \end{aligned}$$

where $C = -(\ell+1)\cdot\log(\ell+1) - \log\binom{\ell+1}{d+1}$ is a constant. As $d \geq 0$ we have $(d+1)\log n \to \infty$. On the other hand, since $x > \ell$ we have $\binom{n-x-1}{k-x}p \sim pn^{k-x} \to 0$. This completes the proof. □

We get the following immediate special cases as corollaries.

Corollary 2.8.2. $p = \dfrac{\log n}{\binom{n}{k}}$ *is the threshold probability for a k-dimensional pure random simplicial complex to contain the complete vertex set $[n]$.*

Corollary 2.8.3. *For any $\omega = \omega(n) \to \infty$ if $p = \dfrac{2\cdot\log n + \omega}{\binom{n}{k-1}}$ then the k-dimensional pure random simplicial complex is connected a.a.s.*

We will see in Theorem 3.5.2 that the probability parameter in Corollary 2.8.3 is far from being the threshold probability for connectivity in pure random simplicial complexes.

Remark 2.8.4. Equation (2.18) can equivalently be written as

$$p = \frac{(\ell+1)\cdot(k-\ell)!\cdot\log n}{n^{k-\ell}}. \tag{2.23}$$

2.9 Alexander duality

In this section we continue the study of Section 2.4 by showing that random simplicial complexes in lower and upper model are dual to each other in the sense of Spanier and

Whitehead [70]. This will imply that homology and cohomology of the lower and upper complexes satisfy an Alexander duality relation,

Theorem 2.9.1. *[39, Corollary 3.45] Let X be a compact, locally contractible, non-empty, proper subspace of the sphere S^n, then*

$$\widetilde{H}_i(S^n - X) \cong \widetilde{H}^{n-i-1}(X).$$

2.9.1 The dual simplicial complex

In this section we describe a combinatorial duality construction for simplicial complexes. More precisely, for a simplicial subcomplex $X \subseteq \partial\Delta_n$ we construct a simplicial complex $X' \subseteq \Delta_n$ which is homotopy equivalent to the complement of the geometric realisation of X in the geometric realisation of the combinatorial sphere $\partial\Delta_n$.

Let $X \subseteq \partial\Delta_n$ be a simplicial subcomplex. Define *the dual complex* X' as an abstract simplicial complex with the vertex set $E(X)$ (the set of all external faces of X) and a set of external faces $\sigma_1, \ldots, \sigma_k \in E(X)$ of X forms a $(k-1)$-simplex of X' if the union of their vertex sets is a proper subset of $[n]$, i.e.

$$\cup_{i=1}^{k} V(\sigma_i) \neq [n].$$

Proposition 2.9.2. *The geometric realisation of the simplicial complex X' is homotopy equivalent to the complement $\partial\Delta_n - X$.*

Proof. For any $\sigma \in E(X)$ let $\mathrm{St}(\sigma)$ denote $\mathrm{St}(\sigma) = \mathrm{St}_{\partial\Delta_n}(\sigma)$ – the star of σ viewed as a subcomplex of $\partial\Delta_n$. Recall that $\mathrm{St}(\sigma) = \mathrm{St}_{\partial\Delta_n}(\sigma)$ is defined as the union of all open simplices $\tau \subseteq \partial\Delta_n$ whose closure contains σ.

The family of stars $\mathcal{U} = \{\mathrm{St}(\sigma)\}_{\sigma \in E(X)}$ forms a contractible open cover of the complement $\partial\Delta_n - X$. Indeed, for $\sigma \in E(X)$ we obviously have $\mathrm{St}(\sigma) \cap X = \varnothing$ which gives the inclusion

$$\bigcup_{\sigma \in E(X)} \mathrm{St}(\sigma) \subseteq \partial\Delta_n - X.$$

This is in fact an equality, i.e. for any open simplex $\tau \subseteq \partial\Delta_n$ with $\tau \nsubseteq X$ there is a face $\sigma \in E(X)$ such that $\tau \subseteq \mathrm{St}(\sigma)$. Indeed, given $\tau \notin X$ let $\sigma \subseteq \tau$ be a minimal face of τ not in X. Then $\sigma \in E(X)$ and $\tau \subseteq \mathrm{St}(\sigma)$.

Note that the cover $\mathcal{U}$ has the property that each intersection

$$\mathrm{St}(\sigma_1) \cap \cdots \cap \mathrm{St}(\sigma_k) = \mathrm{St}(\sigma)$$

is a star of a simplex σ, where σ has the vertex set $V(\sigma) = \cup_{i=1}^{k} V(\sigma_k)$. Thus, every such intersection is either contractible or empty, and it is empty precisely when $\cup_{i=1}^{k} V(\sigma_k) = [n]$. The result now follows by noting that the nerve of $\mathcal{U}$ is exactly the simplcial complex X' and then applying the Nerve Theorem, see [39], Corollary 4G.3. $\square$

Example 2.9.3. *For $n = 3$ let $X \subseteq \partial\Delta_3$ be its vertex set, i.e. $X = \{1, 2, 3\}$. It is a 0-dimensional subcomplex whose complement $\partial\Delta_3 - X$ is a circle with 3 punctures; it has 3 connected components, each of which is contaractible. The set of external simplices $E(X)$ consists of all edges,*

$$E(X) = \{(ij); i < j, \quad i, j \in [3]\}, \quad |E(X)| = 3.$$

The dual complex X' has no edges, i.e. X' is a 3 point set.

Application of Theorem 2.9.1 combined with Proposition 2.9.2 and the fact that $\partial\Delta_n$ is a simplicial $(n-2)$-dimensional sphere we obtain:

Proposition 2.9.4. *For any proper simplicial subcomplex $X \subseteq \partial\Delta_n$ and for any abelian group G one has*

$$H^j(X'; G) \cong H_{n-3-j}(X; G), \quad \text{where} \quad j = 0, 1, \ldots, n-3.$$

2.9.2 The dual complex $c(X)$ of Björner and Tancer

Recall that in Section 2.4 we defined its *combinatorial dual* $c(X)$ for any simplicial subcomplex $X \subseteq \partial\Delta_n$. The maximal simplices of $c(X)$ are in bijective correspondence with the external faces of X, $E(X)$. More precisely, we have the following:

Lemma 2.9.5. *Let $Y \subseteq \partial\Delta_n$ be a simplicial subcomplex. Then*

$$f_i(Y) + f_{n-1-i}(c(Y)) = \binom{n}{i+1}, \quad i = 0, 1, \ldots, n-1. \tag{2.24}$$

Here $f_i(Y)$ denotes the number of i-dimensional simplices in Y. A simplex $\sigma \subseteq \Delta_n$ is an external simplex for Y if and only if the dual simplex $\widehat{\sigma}$ is a maximal simplex of the complex $c(Y)$. In particular we have

$$e_i(Y) = m_{n-i-1}(c(Y)), \quad i = 0, \ldots, n-1, \tag{2.25}$$

where $e_i(Y)$ denotes the number of external i-dimensional faces of Y and $m_j(Y)$ denotes the number of j-dimensional maximal simplices of Y.

Proof. The map $\sigma \mapsto \hat{\sigma}$ is a bijection between the set of i-dimensional non-simplices of Y and the set of $(n-i-1)$-dimensional simplices of the dual $c(Y)$; this proves (2.24). By Lemma 2.4.2 this map is a bijection between the set $E_i(Y)$ the set of i-dimensional external simplices of Y and the set of maximal simplices of $c(Y)$ of dimension $n-i-1$; this proves (2.25). □

Lemma 2.9.6. *The nerve of the cover of $c(X)$ by its maximal simplices is isomorphic to the simplicial complex X' (as defined in Section 2.9.1).*

Proof. Let $\mathcal{M}$ denote the cover of $c(X)$ by maximal simplices. Consider a set of maximal simplices $\{\sigma_1, \ldots, \sigma_k\}$, where $\sigma_i \in \mathcal{M}$. Each dual simplex $\hat{\sigma}_i$ is external for X. The intersection $\cap_{i=1}^k \sigma_i$ is a simplex with the vertex set $\cap_{i=1}^k V(\sigma_i)$ and the intersection $\cap_{i=1}^k \sigma_i$ is non-empty if and only if $\cap_{i=1}^k V(\sigma_i) \neq \varnothing$. We see that any nonempty intersection is contractible. Since $V(\hat{\sigma}_i)$ is the complement of $V(\sigma_i)$, we obtain that $\cap_{i=1}^k V(\sigma_i) = \varnothing$ if and only if $\cup_{i=1}^k V(\hat{\sigma}_i) = [n]$. Therefore, we see that the nerve of $\mathcal{M}$ can be described as the simplicial complex with the vertex set $E(X)$ where a set of external simplices forms a simplex if and only if the union of their vertex sets is not equal $[n]$. This complex coincides with X' as defined in Section 2.9.1. □

Corollary 2.9.7. *For a simplicial subcomplex $X \subseteq \partial\Delta_n$, the geometric realisation of the simplicial complex $c(X)$ is homotopy equivalent to X' and to the complement $\partial\Delta_n - X$.*

Proof. The cover $\mathcal{M}$ by maximal simplices satisfies the conditions of the Nerve Lemma, see [39], Corollary 4G.3. The first claim follows from the previous Lemma. The second claim follows from Proposition 2.9.2. □

Corollary 2.9.8. *For any simplicial subcomplex $X \subseteq \partial\Delta_n$ and for any abelian group G one has*

$$H^j(c(X); G) \simeq H_{n-3-j}(X; G), \quad \textit{where} \quad j = 0, 1, \ldots, n-3.$$

Taking here $G = \mathbb{Q}$ we obtain equality for the Betti numbers:

$$b_j(c(X)) = b_{n-3-j}(X), \quad j = 0, 1, \ldots, n-3.$$

Next we restate Proposition 2.4.4 as follows:

Theorem 2.9.9. *For a fixed n consider two probability spaces $(\Omega_n^*, \overline{\mathbb{P}}_n)$ and $(\Omega_n^*, \underline{\mathbb{P}}'_n)$ where the probability measure $\overline{\mathbb{P}}_n$ is defined with respect to a set of probability parameters p_σ and the probability measure $\underline{\mathbb{P}}'_n$ is defined with respect to a set of probability parameters p'_σ satisfying*

$$p'_\sigma = q_{\hat{\sigma}} = 1 - p_{\hat{\sigma}}.$$

The map $c : (\Omega_n^*, \overline{\mathbb{P}}_n) \to (\Omega_n^*, \underline{\mathbb{P}}'_n)$, where $X \mapsto c(X)$, is an isomorphism of probability spaces. For an integer $j \in [n]$, consider the j-dimensional Betti number

$$b_j : \Omega^* \to \mathbb{Z}$$

and its distribution functions $F_j^{\overline{\mathbb{P}}_n}(x)$ and $F_j^{\underline{\mathbb{P}}'_n}(x)$ with respect to the measures $\overline{\mathbb{P}}_n$ and $\underline{\mathbb{P}}'_n$ correspondingly. Then

$$F_j^{\overline{\mathbb{P}}_n}(x) \equiv F_{n-3-j}^{\underline{\mathbb{P}}'_n}(x). \tag{2.26}$$

Proof. This follows by combining Corollary 2.9.8 and Propositon 2.4.4. ☐

Note that the distribution function $F_j^{\overline{\mathbb{P}}_n}$ is defined by the equality

$$F_j^{\overline{\mathbb{P}}_n}(x) = \overline{\mathbb{P}}_n(b_j(Y) \leq x)$$

and similarly,

$$F_j^{\underline{\mathbb{P}}'_n}(x) = \underline{\mathbb{P}}'_n(b_j(Y) \leq x).$$

Summarising, we see that for a fixed n, studying Betti numbers in the upper model reduces to studying Betti numbers in the lower model and vice versa. However, in the limit when $n \to \infty$ one needs to deal with the dimension shift $i \to n-2-i$ which creates an additional technical difficulty.

2.10 Critical dimension and spread in the upper model

In this section we introduce the notion of a *critical dimension* and of *a spread* for the upper model and explore its relevance to the face numbers of such a random simplicial complex. This generalises the notion of the critical dimension for multiparameter random complexes discussed in Section 1.3.2. We shall consider the upper probability measure $\overline{\mathbb{P}}_n$ on Ω_n^* under the following assumptions on the probability parameters p_σ:

(a) all probability parameters $p_\sigma = 0$ vanish for $\dim \sigma > r$ where $r \geq 0$ is a fixed integer.

(b) For $i \leq r$ one has $p_\sigma = n^{-\alpha_i}$ where $i = \dim \sigma$ and $\alpha_i > 0$ is a fixed positive real number.

(c) The exponents α_i are not integers, $\alpha_i \notin \mathbb{Z}$, where $i = 0, 1, \dots, r$.

(d) All the differences $\alpha_i - \alpha_j \notin \mathbb{Z}$ are not integers, where $i \neq j$, $i, j = 0, 1, \dots, r$.

We note that (b) in particular requires that p_σ depends only on the dimension of simplex σ. The assumptions (c) and (d) are satisfied for a "generic" set of exponents $\alpha_0, \dots, \alpha_r$.

By Remark 2.3.2 we know that the measure $\overline{\mathbb{P}}_n$ is supported on the set of all r-

dimensional simplicial complexes $Y \subseteq \Delta_n$.

Next we introduce some notation. Let

$$\beta_i = i + 1 - \alpha_i$$

and

$$\beta^* = \max\{\beta_0, \beta_1, \ldots, \beta_r\}, \quad i = 0, 1, \ldots, r. \tag{2.27}$$

We set

$$k^* = \lfloor \beta^* \rfloor. \tag{2.28}$$

Note that $k^* = k^*(\alpha)$ is an integer depending on the initial vector of exponents $\alpha = (\alpha_0, \ldots, \alpha_r)$. We remark that,

$$k^* < \beta^* < k^* + 1,$$

with the strict inequalities holding due to our assumption (c).

Definition 2.10.1. The integer $k^* = k^*(\alpha)$ will be called the *critical dimension* of the random simplicial complex Y in the upper model.

Due to our assumption (d) there exists a single index $i^* \in \{0, \ldots, r\}$ such that $\beta_{i^*} = \beta^*$.

The following observation will be useful:

Lemma 2.10.2. *One has $k^* \leq i^*$.*

Proof. This follows from the inequalities

$$k^* = \lfloor \beta^* \rfloor < \beta^* = \beta_{i^*} < i^* + 1.$$

□

Example 2.10.3. *Let us show that the condition $k^*(\alpha) < 0$ is equivalent to the property* *that* $\overline{\mathbb{P}}_n(\varnothing) = 1$, *a.a.s. Indeed,*

$$\overline{\mathbb{P}}_n(\varnothing) = \prod_{\sigma \in \Delta_n} q_\sigma = \prod_{i=0}^{r} q_i^{\binom{n}{i+1}} = \prod_{i=0}^{r} (1 - n^{-\alpha_i})^{\binom{n}{i+1}}.$$

One has $k^* < 0$ *if and only if* $\beta_i < 0$ *for any* $i = 0, 1, \ldots, r$. *Since* $\binom{n}{i+1} n^{-\alpha_i} = n^{\beta_i} \cdot \left(\frac{1}{(i+1)!} + o(1) \right)$ *we may apply Remark 2.10.4 to obtain*

$$\overline{\mathbb{P}}_n(\varnothing) = 1 - \sum_{i=0}^{r} \frac{n^{\beta_i}}{(i+1)!} + o(1) = 1 + o(1).$$

On the other hand, suppose that $\overline{\mathbb{P}}_n(\varnothing) \to 1$. *Since*

$$\overline{\mathbb{P}}_n(\varnothing) = \prod_{i=0}^{r} (1 - n^{-\alpha_i})^{\binom{n}{i+1}}$$

we see (since each term in this product is smaller than 1) that for each $i = 0, \ldots, r$ *one must have* $(1 - n^{-\alpha_i})^{\binom{n}{i+1}} \to 1$ *which implies* $\binom{n}{i+1} n^{-\alpha_i} \to 0$, *i.e.* $\beta_i < 0$. ☐

Remark 2.10.4. We have used the following fact: If $N \to \infty$ and $Nx \to 0$, $x > 0$ then

$$(1 - x)^N = 1 - Nx + (xN)^2 (1/2 + o(1)). \tag{2.29}$$

For any $\ell = 0, 1, \ldots, r$ consider the function

$$f_\ell : \Omega_n^* \to \mathbb{Z}$$

which assigns the number of ℓ-dimensional faces $f_\ell(Y)$ to a random subcomplex $Y \subseteq \Delta_n$. Using Example 2.5.5 we see that

$$\mathbb{E}(f_\ell) = \sum_{\dim \sigma = \ell} \left(1 - \prod_{\tau \supseteq \sigma} q_\tau \right) = \binom{n}{\ell+1} \cdot \left(1 - \prod_{i=\ell}^{r} q_i^{\binom{n-\ell}{i-\ell}} \right), \tag{2.30}$$

where

$$q_i = 1 - n^{-\alpha_i}, \quad i = 0, 1, \ldots, r.$$

Lemma 2.10.5. *Let k^* denote the critical dimension as defined in Definition 2.10.1. Then for any $\ell < k^*$ a random complex Y contains the full ℓ-dimensional skeleton of Δ_n, a.a.s. More precisely, one has*

$$f_\ell(Y) = \binom{n}{\ell+1}$$

with probability at least

$$1 - n^c \exp\left(-n^{\{\beta^*\}}\right), \tag{2.31}$$

where $\{\beta^*\} > 0$ denotes the fractional part of β^*, i.e. $\{\beta^*\} = \beta^* - k^*$ and $c > 0$ is a positive constant.

Proof. Since $\ell < k^* = \lfloor \beta^* \rfloor$ we see that there exists k such that $\beta_k > \ell+1$, i.e. $k - \alpha_k > \ell$. We may assume (without loss of generality) that $\beta_k = \beta^*$. Consider a pure random k-dimensional simplicial complex Z with probability parameter $p = n^{-\alpha_k}$ as defined in Section 2.8. Applying Lemma 2.8.1 we see that Z contains the full ℓ-dimensional skeleton $\Delta_n^{(\ell)}$ with probability at least $1 - e^{-\omega} = 1 - n^c \exp(-n^{k-\ell-\alpha_k})$. Obviously Z is contained in Y, and $k - \ell - \alpha_k \geq \beta^* - k^* = \{\beta^*\}$; thus we see that $f_\ell(Y) = \binom{n}{\ell+1}$ with probability at least (2.31). □

Next we examine the expectation $\mathbb{E}(f_k)$ for $k \geq k^*$.

Lemma 2.10.6. *For any $k \geq k^*$ one has*

$$\mathbb{E}(f_k) = \frac{1}{(k+1)!} \cdot \left(\sum_{i=k}^{r} \frac{n^{\beta_i}}{(i-k)!} \right) \cdot (1 + o(1)). \tag{2.32}$$

Proof. Note that for $k \geq k^*$ one has $k + 1 \geq k^* + 1 > \beta^*$ and hence for any $i = 0, \dots, r$ we have $\beta_i < k+1$ which means that $i + 1 - \alpha_i < k + 1$, i.e. $i - \alpha_i < k$. Next we observe that

$$q_i^{\binom{n-k}{i-k}} = 1 - \frac{n^{i-k-\alpha_i}}{(i-k)!} \cdot (1 + o(1)), \quad i = k, \dots, r,$$

since, as we mentioned above, all the exponents $i - k - \alpha_i$ are negative. Substituting this into (2.30) we obtain (2.32). □

Corollary 2.10.7. *For any $k \geq k^*$ one has*

$$\mathbb{E}(f_k) = \frac{1}{(k+1)!(i_k^* - k)!} \cdot n^{\beta_k^*} \cdot (1 + o(1)), \tag{2.33}$$

where

$$\beta_k^* = \max\{\beta_k, \beta_{k+1}, \dots, \beta_r\}$$

and i_k^* is the unique integer $k \leq i_k^* \leq r$ such that $\beta_{i_k^*} = \beta_k^*$.

Proof. This follows automatically from Lemma 2.10.6. Here we also use our assumption (d) (saying that $\alpha_i - \alpha_j \notin \mathbb{Z}$) which guarantees uniqueness of the maximum. □

Note that $\beta_{k^*}^* = \beta^*$ since $k^* \leq i^*$. Besides, $k^* < \beta^* < k^* + 1$ and for $k > i^*$ one has $\beta_k^* < \beta^*$.

Theorem 2.10.8. *Denoting the rate of exponential growth*

$$\gamma_k := \lim_{n\to\infty} \frac{\log \mathbb{E}(f_k)}{\log n}$$

we have

$$\gamma_k = \gamma_k(\alpha) = \begin{cases} k+1, & \text{for} \quad k < k^*, \\ \beta^*, & \text{for} \quad k^* \leq k \leq i^*, \\ \beta_k^* & \text{for} \quad k > i^*. \end{cases}$$

In particular, the value of γ_k is constant, maximal and is equal to β^* for all k satisfying $k^* \leq k \leq i^*$.

Proof. This follows from Lemma 2.10.6 (for $k \geq k^*$). Besides, for $k < k^*$ we know that $f_k = \binom{n}{k+1}$, a.a.s. □

Definition 2.10.9. We shall call the non-negative integer $i^* - k^* = s = s(\alpha)$ the *spread.*

The spread $s = s(\alpha)$ is the length of the flat maximum of the graph of the function

$k \mapsto \gamma_k$. We have

$$\gamma_0 < \gamma_1 < \cdots < \gamma_{k^*} = \cdots = \gamma_{i^*} > \gamma_{k^*+1} \geq \cdots \geq \gamma_r. \tag{2.34}$$

Note that in the case when the spread is zero, $k^* = i^*$ and the sequence of exponents in (2.34) is unimodal.

Example 2.10.10. *Consider the case when $r = 1$ (random graphs with respect to the upper probability measure). In this case we have two exponents α_0 and α_1. Recall that $\beta_0 = 1 - \alpha_0$, $\beta_1 = 2 - \alpha_1$, $\beta^* = \max\{\beta_0, \beta_1\}$ and $k^* = \lfloor \beta^* \rfloor$. We see that $k^* < 0$ when both $\alpha_0 > 1$ and $\alpha_1 > 2$. We will additionally consider following three cases:*

(a) $k^* = 0$ and $i^* = 0$,

(b) $k^* = 0$ and $i^* = 1$,

(c) $k^* = 1$ and $i^* = 1$.

(a) occurs when $1 - \alpha_0 > 2 - \alpha_1$ and $1 - \alpha_0 > 0$. This can be summarised by $\alpha_0 < 1$ and $\alpha_1 > 1 + \alpha_0$.

(b) can be characterised by the inequalities $0 < \beta_0 < \beta_1 < 1$ which can be rewritten as $\alpha_0 < 1$ and $1 < \alpha_1 < 1 + \alpha_0$.

In the case of (c) we have the inequalities: $1 - \alpha_0 < 2 - \alpha_1$ and $1 < 2 - \alpha_1$. These inequalities reduce to the condition $0 < \alpha_1 < 1$.

Note that in cases (a) and (c) the spread is 0 and in case (b) the spread is 1.

Example 2.10.11. *One can characterise the vectors $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_r)$ with zero spread $s(\alpha) = 0$ as follows. The index $i^* \in \{0, 1, \ldots, r\}$ of the critical dimension satisfies*

$$\beta_{i^*} = \max\{\beta_0, \ldots, \beta_r\}, \quad \lfloor \beta_{i^*} \rfloor = i^*.$$

In view of the definition $\beta_i = i + 1 - \alpha_i$ we see that $s(\alpha) = 0$ is equivalent to

$$\alpha_{i^*} < 1, \quad \text{and} \quad \alpha_{i^*+k} > \alpha_{i^*} + k, \tag{2.35}$$

for all $k = 1, \ldots, r - i^*$.

From (2.35) we see that i^* is the largest index satisfying $\alpha_{i^*} < 1$. However this condition alone is not sufficient.

Example 2.10.12. *Consider the case $r = 2$ (two dimensional random simplicial complexes in the upper model). Using Example 2.10.11 we find that the vectors of exponents $\alpha = (\alpha_0, \alpha_1, \alpha_2)$ with zero spread are as follows:*

1. *If $k^* = 0$ then $s = 0$ is equivalent to $\alpha_0 < 1$ and $\alpha_1 > 1 + \alpha_0$ and $\alpha_2 > 2 + \alpha_0$.*
2. *If $k^* = 1$ then $s = 0$ is equivalent to $\alpha_1 < 1$ and $\alpha_2 > 1 + \alpha_1$.*
3. *If $k^* = 2$ then $s = 0$ is equivalent to $\alpha_2 < 1$.*

2.11 Betti numbers in the upper model

We begin this section by showing that the number of faces in all dimensions concentrates around its expectation for large n.

Theorem 2.11.1. *Consider a random simplicial complex $Y \in \Omega_n^*$ with respect to the upper probability measure $\overline{\mathbb{P}}_n$. We shall assume that the probability parameters p_σ vanish for $\dim \sigma > r$ and for $\dim \sigma \le r$ they have the form $p_\sigma = n^{-\alpha_i}$, where $i = \dim \sigma$, and the exponents $\alpha_i > 0$ satisfy the genericity assumptions (a) - (d), see Section 2.10. We shall also assume that the critical dimension $k^* \ge 0$ is non-negative. Let $f_k : \Omega_n^* \to \mathbb{Z}$ denote the random variable counting the number of k-dimensional faces of a random complex, where $k = 0, 1, \ldots, r$. Then there exists a sequence of real numbers $t_n \to 0$ such that for any $k \ge k^*$ one has*

$$(1 - t_n) \cdot \frac{n^{\gamma_k(\alpha)}}{(k+1)! \cdot (i_k^* - k)!} \le f_k \le (1 + t_n) \cdot \frac{n^{\gamma_k(\alpha)}}{(k+1)! \cdot (i_k^* - k)!}, \qquad (2.36)$$

a.a.s. The exponent $\gamma_k(\alpha)$ is as defined in Theorem 2.10.8 and the integer $i_k^* \in \{k, k+1, \ldots, r\}$ is the index satisfying $\beta_{i_k^*} = \max\{\beta_k, \beta_{k+1}, \ldots, \beta_r\}$.

Proof. Consider a random hypergraph $X \in \Omega_n$ with probability parameters

$$p_\sigma = \begin{cases} n^{-\alpha_i} & \text{if } \dim \sigma = i \leq r, \\ 0 & \text{if } \dim \sigma > r. \end{cases}$$

For $k = 0, 1, \ldots, r$ we will let $g_k : \Omega_n \to \mathbb{Z}$ denote the random variable counting the number of k-dimensional faces. Note that g_k is a binomial random variable $\mathrm{Bi}\left(\binom{n}{k+1}, n^{-\alpha_k}\right)$, hence we obviously have

$$\mathbb{E}(g_k) = \binom{n}{k+1} \cdot n^{-\alpha_k} \sim \frac{n^{\beta_k}}{(k+1)!}, \qquad k \leq r. \tag{2.37}$$

Thus, by the first moment method of Corollary 1.2.2, we see that for $\beta_k < 0$ one has $g_k \equiv 0$, a.a.s.

Note that the assumptions (a)-(d) described in Section 2.10 exclude the possibility $\beta_k = 0$.

Below we shall assume that $\beta_k > 0$. We may use the Chernoff bound, see Theorem 2.1 of [45], which states that for any $\tau \geq 0$,

$$\mathbb{P}(g_k \geq \mathbb{E}(g_k) + \tau) \leq \exp\left(-\frac{\tau^2}{2(\mathbb{E}(g_k) + \tau/3)}\right),$$

and

$$\mathbb{P}(g_k \leq \mathbb{E}(g_k) - \tau) \leq \exp\left(-\frac{\tau^2}{2\mathbb{E}(g_k)}\right),$$

We will apply the Chernoff bound with $\tau = \mathbb{E}(g_k)^{2/3} = t_n \mathbb{E}(g_k)$ where $t_n = \mathbb{E}(g_k)^{-1/3} = o(1)$. This gives us

$$\mathbb{P}(g_k \geq (1 + t_n) \cdot \mathbb{E}(g_k)) \leq \exp\left(-\frac{\tau^2}{2(\mathbb{E}(g_k) + \tau/3)}\right) \leq \exp\left(-\frac{\mathbb{E}(g_k)^{1/3}}{4}\right)$$

and

$$\mathbb{P}(g_k \le (1-t_n)\cdot \mathbb{E}(g_k)) \le \exp\left(-\frac{\mathbb{E}(g_k)^{1/3}}{2}\right).$$

Since $\mathbb{E}(g_k)^{1/3} \to \infty$ we obtain that

$$(1-t_n)\cdot \mathbb{E}(g_k) \;\le\; g_k \;\le\; (1+t_n)\cdot \mathbb{E}(g_k), \tag{2.38}$$

a.a.s. By combining with (2.37) we see that

$$(1-t_n)\cdot \frac{n^{\beta_k}}{(k+1)!} \;\le\; g_k \;\le\; (1+t_n)\cdot \frac{n^{\beta_k}}{(k+1)!}, \tag{2.39}$$

a.a.s., where $t_n \to 0$.

Let $\Omega_{n,r}$ denote the set of hypergraphs $X \subset \Delta_n$ of dimension at most r (i.e. where hyperedges are of cardinality at most $r+1$). Similarly, let $\Omega^*_{n,r}$ denote the set of all simplicial subcomplexes $Y \subset \Delta_n$ of dimension at most r. We have the map

$$\overline{\rho}_r : \Omega_{n,r} \to \Omega^*_{n,r}$$

which is the restriction of the map which appears in (2.2). Recall that for $X \in \Omega_{n,r}$ the simplicial complex $\overline{\rho}(X)$ is the minimal simplicial complex Y containing X. In other words, Y is obtained from X by adding all faces of all simplices of X.

Since we assume that $p_\sigma = 0$ for all simplices σ of dimension $> r$ we obtain that the measure $\mathbb{P}_n$ (given by (2.1)) is supported on $\Omega^*_{n,r} \subset \Omega^*_n$. Hence, we obtain that the upper measure $\overline{\mathbb{P}}_n$ on $\Omega^*_{n,r}$ coincides with the direct image $(\overline{\rho}_r)_*(\mathbb{P}_n)$.

For any $k = 0, 1, \dots, r$ we have two random variables $g_k : \Omega_{n,r} \to \mathbb{Z}$ and $f'_k = f_k \circ \overline{\rho}_r : \Omega^*_{n,r} \to \mathbb{Z}$. As $\overline{\rho}_r(X)$ includes every hyperedge of X (together will its subsets) as a simplex into $\overline{X}$ we obtain

$$\max_{i=k,\dots,r}\left\{\binom{i+1}{k+1} g_i\right\} \;\le\; f'_k \;\le\; \sum_{i=k}^{r} \binom{i+1}{k+1} g_i \tag{2.40}$$

Combining with (2.39) we find

$$(1-o(1))\cdot\frac{n^{\gamma_k}}{(k+1)!\cdot(i_k^*-k)!}\;\le\; f_k'\;\le\;(1+o(1))\cdot\frac{n^{\gamma_k}}{(k+1)!\cdot(i_k^*-k)!},$$

a.a.s. By the definition, $\overline{\mathbb{P}}_n = \overline{\rho_{r*}}(\mathbb{P}_n)$, and hence the above inequality implies (2.36).

□

Using the face estimate of Theorem 2.11.1 together with the Morse inequalities described in Lemma 1.1.1 we obtain the following estimate on Betti numbers.

Theorem 2.11.2. *Consider a random simplicial complex $Y \in \Omega_n^*$ with respect to the upper probability measure $\overline{\mathbb{P}}_n$. Assume that the probability parameters p_σ vanish for $\dim\sigma > r$ and for $\dim\sigma \le r$ they have the form $p_\sigma = n^{-\alpha_i}$, where $i = \dim\sigma$, and the exponents $\alpha_i > 0$ satisfy the genericity assumptions (a) - (d), see Section 2.10. We shall also assume that the critical dimension $k^* \ge 0$ is non-negative and the spread vanishes,*

$$s(\alpha) = 0.$$

Then the Betti number in the critical dimension $b_{k^*}(Y)$ dominates all other Betti numbers, a.a.s. More precisely, for a sequence $t_n \to 0$ one has

$$(1-t_n)\cdot\frac{n^{\gamma_{k^*}(\alpha)}}{(k^*+1)!}\le b_{k^*}(Y)\le(1+t_n)\cdot\frac{n^{\gamma_{k^*}(\alpha)}}{(k^*+1)!}, \tag{2.41}$$

a.a.s. Besides, for any $k > k^*$ there exists $\varepsilon_k > 0$ such that

$$b_k(Y) < n^{-\varepsilon_k}\cdot b_{k^*}(Y), \tag{2.42}$$

a.a.s.

We prove below that under the assumptions of Theorem 2.11.2 the reduced Betti numbers $\tilde{b}_k(Y)$ in dimensions below the critical dimension $k < k^*$ vanish, a.a.s.

Proof. First we apply the Morse inequality $b_{k^*}(Y) \le f_{k^*}(Y)$ and use the right hand side of (2.36); this gives the right inequality (2.41). To prove the left inequality (2.41) we note that

$$b_{k^*}(Y) \geq f_{k^*}(Y) - f_{k^*+1}(Y) - f_{k^*-1}(Y),$$

which combined with (2.36) gives

$$b_{k^*}(Y) \geq (1 - t'_n) \cdot \frac{n^{\gamma_{k^*}(\alpha)}}{(k^*+1)!}, \quad t'_n \to 0.$$

If $k > k^*$ then

$$b_k(Y) \leq f_k(Y) \leq (1 + o(1)) \cdot \frac{n^{\gamma_k}}{(k+1)!}$$

and we see that (2.42) holds with any ε_k satisfying $\gamma_{k^*} - \gamma_k > \varepsilon_k > 0$. □

By Lemma 2.8.1 we are able to relate our upper model random complexes to those studied by Linial, Meshulam, and Wallach [58, 62] and use their classical results to understand when homology vanishes. We remark again that the following recent papers [22–25] explore the Betti number behaviour of the upper model in some detail.

Theorem 2.11.3. *Under the assumptions of Theorem 2.11.2 the reduced Betti numbers of the random complex Y below the critical dimension k^* vanish,*

$$\tilde{b}_j(Y) = 0$$

for all $j < k^*$, a.a.s.

Proof. Consider a random hypergraph X of dimension $\leq r$ with probability parameters $p_i = n^{-\alpha_i}$ where $i = 0, \ldots, r$. We can view X as the disjoint union of pure (uniform) hypergraphs $X = \sqcup_{i=0}^r X_i$ where X_i has dimension i if not empty. Denote by $Y_i = \overline{X}_i$ the smallest simplicial complex containing X_i; it is a pure simplicial complex of dimension i if $Y_i \neq \varnothing$. Clearly, the random complex in the upper model Y can be represented as $Y = Y_0 \cup Y_1 \cup \cdots \cup Y_r$.

Denote $Z_i = Y_0 \cup \cdots \cup Y_i$, obviously, Z_i is a simplicial complex of dimension $\leq i$. Note that the complex Z_{k^*} contains the full (k^*-1)-dimensional skeleton. It follows that the reduced Betti numbers $\tilde{b}_j(Z_{k^*}) = 0$ vanish for $j < k^* - 1$.

Next we show that the Betti number $b_{k^*-1}(Z_{k^*}) = 0$ vanishes a.a.s. Note that Z_{k^*} is a Linial-Meshulam random simplicial complex with probability parameter $p = n^{-\alpha_{k^*}}$ and $\alpha_{k^*} < 1$, see Example 2.10.11, where we use our assumption that the spread is zero. Here we also use Lemma 2.8.1 which implies that the pure upper random complex Z_{k^*} contains the full $(k^* - 1)$-dimensional skeleton. It is well known (see [62]) that in this situation the rational homology in dimension $k^* - 1$ vanishes, i.e. $b_{k^*-1}(Z_{k^*}) = 0$.

The complex Y contains Z_{k^*} as a subcomplex. Since Z_{k^*} contains the full $(k^* - 1)$-skeleton, we see that Y is obtained from Z_{k^*} by adding subsequently simplices of dimension $k^*, k^* + 1, \dots, r$. Hence $\tilde{b}_j(Y) = 0$ for $j < k^* - 1$, a.a.s. In general, adding simplices of dimension k^* may either reduce by 1 the Betti number in dimension $k^* - 1$ or to increase by 1 the Betti number in dimension k^*. However in our case, since $b_{k^*-1}(Z_{k^*}) = 0$, the result may only increase the k^*-dimensional Betti number. Further, adding simplices of dimension $> k^*$ may not affect the $(k^* - 1)$-dimensional homology. Hence we obtain $b_{k^*-1}(Y) = 0$, a.a.s. □

Remark 2.11.4. As one has $\underline{X} \subset \overline{X}$ a possible research direction of interest, suggested by Vidit Nanda, was to study the *relative homology* $H_k(\overline{X}, \underline{X})$ in both the random case and for general hypergraphs X as a measure of the difference between the maps $\underline{\rho}$ and $\overline{\rho}$.

Chapter 3

Minimal connected covers and connectivity of pure random complexes

3.1 Introduction

The motivation behind this chapter was to find the threshold probability for a pure r-dimensional random simplicial complex, described in (2.17), to be path connected. This is a topic that has been studied to great success by Cooley, Kang et al. [22–24] in the last few years. These texts go far beyond anything that we try to achieve here, studying thresholds for the vanishing of cohomology in different dimensions as well as what precisely happens within the phase transition itself. Here we will emulate the classical proof of Erdős and Rényi [33] for computing the threshold for connectivity of a random graph that utilises in a fundamental way Cayley's formula. In order to complete this emulation we introduce a minimally connected higher dimensional analogue to a tree with bounds on its labelled enumeration playing a similarly crucial role to Cayley's formula – see Theorem 3.5.2.

Trees have a rich history of study dating back to the 1860s with their simple enumeration first given by Borchardt [13] that is now commonly referred to as Cayley's formula [18].

Cayley's formula. The number of trees on n labelled vertices is n^{n-2}.

A tree may be uniquely characterised as being a connected and acyclic graph, both of these are topological properties that generalise naturally to higher dimensions and so it made sense to do so using the language of higher dimensional combinatorial structures – i.e. simplicial complexes. This was done so in the groundbreaking paper of Kalai [48] where he introduced $\mathbb{Q}$-*acyclic simplicial complexes.*

Definition. T is an r-dimensional $\mathbb{Q}$-*acyclic simplicial complex* if T is a simplicial complex with full $(r-1)$-dimensional skeleton with both $H_{r-1}(T;\mathbb{Q}) = 0$ and $H_r(T;\mathbb{Q}) = 0$.

We let $\mathcal{T}_r(n)$ denote the class of all such simplicial complexes on a labelled vertex set $[n] = \{1, \ldots, n\}$. By the definition of $\mathbb{Q}$-acyclic complexes it is clear, by an application of the universal coefficient theorem [39], that $H_{r-1}(T;\mathbb{Z})$ is a finite group. Kalai found the following beautiful enumeration for this class of higher dimensional acyclic simplicial complexes that generalises Cayley's formula.

Theorem (Kalai [48]). Let $d < n$ be integers, then

$$\sum_{T \in \mathcal{T}_r(n)} |H_{r-1}(T;\mathbb{Z})|^2 = n^{\binom{n-2}{r}}$$

where $|G|$ denotes the cardnality of the group G.

One need not generalise trees to higher dimensions in such a way however. A tree can also be characterised as a connected graph such that the removal of any edge disconnects it. This is a property that is certainly not generalised via the work the of Kalai. Generalising this notion of being minimally path connected in the sense of removing something and becoming disconnected was introduced by Schmidt-Pruzan and Shamir [44] where they used the language of hypergraphs.

Definition. A *hypertree* (or *h-tree*) is a hypergraph $H = (V, E)$ that is connected and the removal of any hyperedge from E will disconnect V.

It is this flavour of generalisation that we will study in this chapter. We reformulate the definition of h-trees using the language of simplicial complexes – the primary reason for this is that we will also be concerned with both the geometric and homological connectivity of these objects, making simplicial complexes the natural combinatorial framework to work with.

We call a simplicial complex an r-dimensional *minimal connected cover* if it is connected and the removal of any r-dimensional simplex (and all of its dependent subfaces) is either disconnected or covers a smaller vertex set (see Definition 3.2.3).

Let $\mathcal{M}_r(n)$ denote the class of all such simplicial complexes on a labelled vertex set $[n] = \{1, \ldots, n\}$. A primary goal is to estimate the quantity $M_r(n) := |\mathcal{M}_r(n)|$. To this end we show the following (see Proposition 3.3.4 and Corollary 3.4.4):

Main Theorem. Fix $r \geq 1$ any integer. There exists constants $A, B > 0$ such that

$$A^n \cdot n^n \leq M_r(n) \leq B^n \cdot n^n.$$

With the lower bound computed using the original work of [44] and the upper bound computed by relating minimal connected covers to a combinatorial object that have a known enumeration [54]. It is with this estimation that we will compute the threhsold probability for a pure random simplicial complex to be path connected.

It's clear that $\mathbb{Q}$-acyclic simplicial complexes of Kalai are homologically connected up to dimension $r-2$ by the condition upon having full $(r-1)$-skeleton included, i.e. if $i \leq r-2$ then $H_i(T; \mathbb{Z}) = 0$ for all $T \in \mathcal{T}_r(n)$. The same is not necessarily true of minimal connected covers, in fact we show that any finite abelian group can be realised as a homology group of some minimal connected cover – see Corollary 3.2.9.

3.2 Structure of minimal connected covers

3.2.1 Basic definition

Definition 3.2.1. Let X be a simplicial complex. We say that a simplex $\sigma \in X$ is a *maximal simplex* (or a *facet*) if for every $\tau \supseteq \sigma$ then $\tau \in X$ if and only if $\tau = \sigma$. That is, no larger simplices in X contain σ.

For the purposes of this chapter whenever we talk of maximal simplices we will always assume they are of dimension at least 1, i.e. we never consider isolated vertices to be maximal simplices. We say that a simplicial complex X is *pure* if all maximal simplices have the same dimension.

Definition 3.2.2. Let X be a simplicial complex on vertex set $V = V(X)$. Let $M = M(X)$ be the set of maximal simplices. We say that X' is obtained from X by *removing* a maximal simplex $\sigma \in M$ if $V(X') = V$ and $M(X') = M - \{\sigma\}$.

Definition 3.2.3. Let Y be a pure r-dimensional simplicial complex on $[n]$. We call Y a *minimal connected cover* if Y is path connected and the removal of any facet disconnects Y.

Let $\mathcal{M}_r(n)$ denote the set of r-dimensional minimal connected covers on $[n]$ and let $M_r(n) = |\mathcal{M}_r(n)|$.

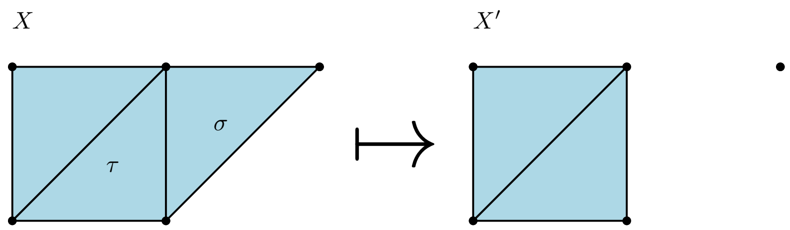

Figure 3.1: A maximal simplex σ is removed from a simplicial complex X. Observe that though removing σ disconnects X it is not true that X a minimal connected cover as there exists a facet, τ, that we could remove without disconnecting X.

3.2.2 Homology of minimal connected covers

By construction we know that every minimal connected cover is path connected, phrased in the language of homology this can expressed as $H_0(Y) = \mathbb{Z}$ for every minimal connected cover $Y \in \mathcal{M}_r(n)$. In this section we show that it is a different story for homology in higher dimensions, with almost any combination of homology possible.

Proposition 3.2.4. *For any $Y \in \mathcal{M}_r(n)$ one has*

$$H_{r-1}(Y;\mathbb{Z})_T = 0 \quad \textit{and} \quad H_r(Y;\mathbb{Z}) = 0.$$

Where given an abelian group G, the notation G_T refers to the torsion subgroup*. That is, G_T is the subgroup of G containing all elements of finite order.*

Proof. Every r-dimensional simplex $\sigma \in Y$ contains at least one free $(r-1)$-dimensional face, if this were not the case when one could remove such a σ from Y without affecting the path connectivity so Y certainly could not have been minimally connected. We may therefore simplicially collapse every maximal r-simplex along this free face to obtain a new complex Y' of dimension $r-1$ that is homotopy equivalent to Y, the statement then follows as Y has homotopy dimension[1] at most $r-1$. $\square$

Notice the difference in homological behaviour of minimal connected covers compared with $\mathbb{Q}$-acyclic simplicial complexes of Kalai. Both require that the top dimensional homology H_r vanishes, but the conditions upon H_{r-1} are quite contrasting coinciding only when it is the trivial group. As we shall see, in lower dimensions their differences increase even further.

Proposition 3.2.5. *For any path connected k-dimensional topological space X with a finite triangulation there exists a minimal connected cover $Y \in \mathcal{M}_r(n)$ for any $r > k$ and some n such that Y is homotopy equivalent to X*

Proof. Let T be a triangulation of X. To every maximal simplex in T of dimension ℓ we may simplicially join it to a new uniquely labelled simplex of dimension $r-\ell-1$, call this new simplicial complex Y. Clearly Y simplicially collapses onto T, in particular Y is homotopy equivalent to T as required. $\square$

[1]The *homotopy dimension* of a topological space X is the minimal dimension realisable up to homotopy. That is, it equals $\min\{\dim Y : Y \simeq X\}$.

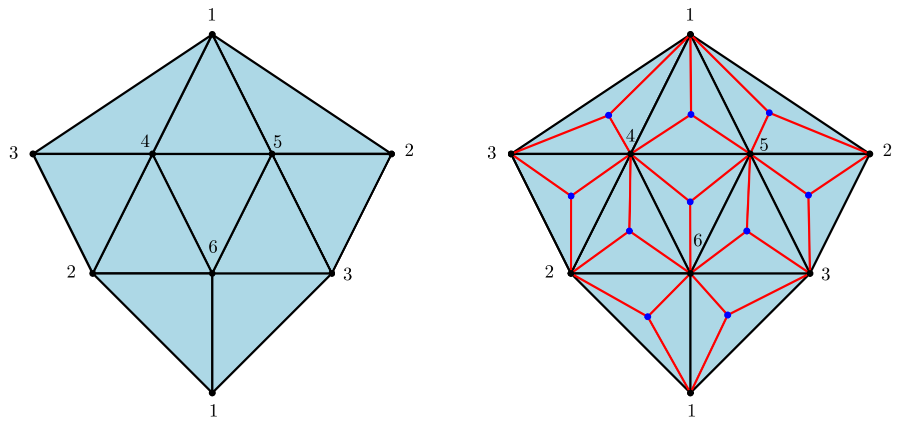

Figure 3.2: Shows the process described in the proof of Proposition 3.2.5. A triangulation of $\mathbb{RP}^2$ is turned into a 3-dimensional minimal connected cover.

Corollary 3.2.6. *For any integer $r \geq 3$, any $m \geq 2$ and all $1 \leq k \leq r-2$ there exists an n such that there is a $Y \in \mathcal{M}_r(n)$ with $H_k(Y) = \mathbb{Z}_m$.*

Proof. Consider the Moore space $M = M(\mathbb{Z}_m, k)$, that is a finite CW complex of dimension $k+1$ such that $H_k(M) = \mathbb{Z}_m$ and $H_i(M) = 0$ for all $i \neq k$.[2] One may then cover M by open sets sufficiently finely so that the conditions of the Nerve Lemma (see Corollary 4G.3 of [39]) are satisfied, the obtained nerve complex of this cover N has the same homotopy type as our Moore space M, in particular $H_k(N) = \mathbb{Z}_m$ and we may conclude by application of Proposition 3.2.5 to this N. □

In the proof of Corollary 3.2.6 one could alternatively consider the vertex minimal construction of a simplicial complex X with prescribed torsion $H_{k-1}(X) = \mathbb{Z}_m$ as constructed in the paper Newman [47]. Perhaps with some work this same construction could be shown to give rise to the vertex minimal minimal connected cover with prescribed torsion.

Corollary 3.2.7. *For all $r \geq 2$, any $1 \leq i \leq r-1$ there exists an n such that there is a $Y \in \mathcal{M}_r(n)$ with $H_i(Y) = \mathbb{Z}$.*

Proof. Apply Proposition 3.2.5 to a triangulation of the i-sphere S^i. □

We make note of the following obvious but useful observation.

[2] M is obtained from the sphere S^k by attaching one $(k+1)$-cell by a map $S^k \to S^k$ of degree m.

Lemma 3.2.8. *Let $Y_1 \in \mathcal{M}_r(n_1)$ and $Y_2 \in \mathcal{M}_r(n_2)$ then their wedge along any two vertices is also a minimal connected cover, in particular $Y_1 \vee Y_2 \in \mathcal{M}_r(n_1 + n_2 - 1)$.*

Corollary 3.2.9. *Fix $r \geq 2$ and let G be any finitely presented abelian group. Then for any $1 \leq k \leq r-2$ there exists an n such that there is a $Y \in \mathcal{M}_r(n)$ with $H_k(Y) = G$.*

Proof. We may write $G = \mathbb{Z}^\zeta \oplus \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_\beta}$ by the Fundamental Theorem of Abelian Groups. The result then follows by applications of Corollary 3.2.6, Corollary 3.2.7, and Lemma 3.2.8. □

3.2.3 Geometry of minimal connected covers

Here we explore the geometry of minimal connected covers; describing their size and internal structure.

Lemma 3.2.10. *If $Y \in \mathcal{M}_r(n)$ then*

$$\left\lceil \frac{n-1}{r} \right\rceil \leq f_r(Y) \leq n - r.$$

Proof. Let $F_r(Y) = \{\sigma_1, \ldots, \sigma_{f_r(Y)}\}$ denote the set of r-dimensional simplices such that $\sigma_i \cap \sigma_{i+1} \neq \varnothing$. Say that σ_i *first covers* a set of vertices V if $V \subset V(\sigma_i)$ and $V(\sigma_j) \cap V = \varnothing$ for all $j < i$. Let V_i be the maximal set of first covered vertices for simplex σ_i.

Clearly,

$$\bigsqcup_{i=1}^{f_r(Y)} V_i = [n],$$

as every vertex must be covered. $|V_1| = r + 1$ as the "first" simplex must cover all of its vertices. Each subsequent simplex σ_i must first cover at least one new vertex otherwise it would not be necessary for connectivity and thus not in the minimal connected cover, similarly each subsequent simplex can first cover at most r vertices as it connects to the one before. That is, we have shown $1 \leq |V_i| \leq r$ for all $i \geq 2$. Combining all of this together we see that

$$
\begin{aligned}
n &= \sum_{i=1}^{f_r(Y)} |V_i| \\
&= r + 1 + \sum_{i=2}^{f_r(Y)} |V_i| \\
&\in [f_r(Y) + r, r f_r(Y) + 1]
\end{aligned}
$$

which upon rearranging gives the inequalities in the statement. □

Definition 3.2.11. Let $Y \in \mathcal{M}_r(n)$ and let $v \in V(Y)$ be a vertex. We say that v is a *leaf* if it is contained in a unique r-simplex that we call the *branch*. We say that v is an *external leaf* if removing its branch from Y leaves a unique connected component and isolated vertices.

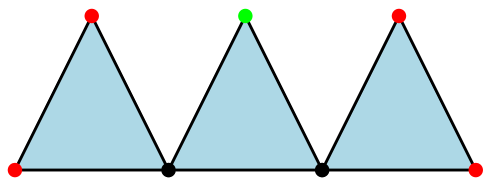

Figure 3.3: An example of $Y \in \mathcal{M}_2(7)$ with external leaves indicated in red and internal leaf in green.

Lemma 3.2.12. *Every $Y \in \mathcal{M}_r(n)$ contains an external leaf.*

Proof. Suppose there are no external leaves. We will show by strong induction that this implies the existence of paths of r-simplices of arbitrary length.

There certainly exists a path of length 1, choose any facet in Y. Suppose there is a path of r-simplices of length k in Y, i.e. there exists $\sigma_1, \sigma_2, \ldots, \sigma_k$ such that $\sigma_i \cap \sigma_{i+1} \neq \emptyset$ and $\sigma_i \cap \sigma_j = \emptyset$ for all $j \neq i, i+1$. If σ_k had no neighbours except for σ_{k-1} then Y certainly contains an external leaf with branch σ_k. If all of the neighbours of σ_k connect to some σ_is then either σ_k is not required for path connectivity or σ_k is again a branch. This is a contradiction since Y is a minimal connected cover, i.e. we are able to extend to a path of facets of length $k+1$. This holds true for $k > n - r$ which is a contradiction by Lemma 3.2.10, so Y must contain an external leaf. □

Lemma 3.2.13. *The* 1*-skeleton of a minimal connected cover determines it uniquely, i.e. if* $Y, Z \in \mathcal{M}_r(n)$ *with* $Y^{(1)} = Z^{(1)}$ *then* $Y = Z$*.*

Proof. Let $Y, Z \in \mathcal{M}_r(n)$ and suppose $Y^{(1)} = Z^{(1)}$ but $Y \neq Z$, then there exists a facet σ which is in Z but is not in Y. Suppose σ is a branch in Z but is not a branch in Y. If this is the case then there exists a vertex v which is a leaf in Z but is not a leaf in Y, in particular v is in one $(r+1)$-clique of Z but at least two in Y. This contradicts $Y^{(1)} = Z^{(1)}$ so the facet σ cannot be a branch of Z.

Therefore σ must be necessary for the connectivity of Z. That is, removing σ from Z must disconnect it and leave no isolated vertices. This can only occur if σ contains at least one edge which is not contained within any other facet. This cannot happen by the assumption that our 1-skeletons are the same and $\sigma \notin Y$. □

3.3 Lower bound on $M_r(n)$

Definition 3.3.1. We call a minimal connected cover $Y \in \mathcal{M}_r(n)$ *treelike* if it is contractible and for every pair of distinct facets σ, τ one has $|\sigma \cap \tau| \leq 1$.

The following is a restatement of Lemma 3.11 from Schmidt-Pruzan and Shamir [44].

Lemma 3.3.2. *Suppose* $n = kr + 1$ *for some integer* k*. Then the number of* $Y \in \mathcal{M}_r(n)$ *such that* Y *is treelike equals*

$$\frac{(n-1)! \cdot n^{k-1}}{k! \cdot r!^k}.$$

This result together with the following Lemma stating that $M_r(n)$ is a non-decreasing function of n will give our lower bound.

Lemma 3.3.3. $M_r(n) \leq M_r(n+1)$.

Proof. Let $Y \in \mathcal{M}_r(n)$, there exists an external leaf in Y by Lemma 3.2.12. Let v be the smallest leaf in Y with branch σ with vertices $\{v, v_1, \ldots, v_r\}$. Let σ' denote a new r-simplex on vertex set $\{v_1, \ldots, v_r, n+1\}$ and define a new simplicial complex $Y' = Y \cup \sigma'$.

It's clear that $Y' \in \mathcal{M}_r(n+1)$. Moreover one sees that $Y' = Z'$ if and only if $Y = Z$. We have therefore constructed an injective map $\mathcal{M}_r(n) \to \mathcal{M}_r(n+1)$ which proves the lemma. □

Proposition 3.3.4. *For any $A < \frac{1}{2er!}$*

$$M_r(n) \geq A^n \cdot n^n.$$

Proof. Let $n = kr + c$ for $c = 0, 1, \ldots, r-1$. Then by Lemma 3.3.3 and Lemma 3.3.2

$$\begin{aligned} M_r(n) \geq \cdots \geq M_r(n-c+1) &\geq \frac{(n-c)! \cdot (n-c+1)^{k-2}}{(k-1)! \cdot r!^{k-1}} \\ &\geq \frac{(n-c)!}{r!^{n-c}} \cdot \frac{(k \cdot r)^{k-2}}{(k-1)!} \\ &\geq \left(\frac{n-c}{e \cdot r!}\right)^{n-c} \cdot \frac{k^{k-2} \cdot r^{k-2}}{(k-1)!} \\ &\geq \left(\frac{n-c}{e \cdot r!}\right)^{n-c} \\ &\geq A^n \cdot n^n, \end{aligned}$$

Where in the final inequality we may take A to be any constant less than $\frac{1}{2er!}$. In the above chain of inequalities we have used the known bound for factorials $\dfrac{n^n}{e^n} \leq n!$. □

3.4 Upper bound on $M_r(n)$

Definition 3.4.1. Given an integer $r \geq 1$ an *r-tree* is a graph which is defined inductively as follows:

- The complete graph on r vertices K_r is an r-tree.
- Let G be an r-tree on n vertices, one may construct a new r-tree G' on $n+1$ vertices by connecting a new vertex to any r vertices that form a clique in G.

Any spanning subgraph of an r-tree is called a *partial r-tree.*

The following is a result of Beineke and Pippert [54] for the enumeration of r-trees.

Theorem 3.4.2. *There are $\binom{n}{r} \cdot [r(n-r)+1]^{n-r-2}$ labelled r-trees on n vertices.*

Proposition 3.4.3. *If $Y \in \mathcal{M}_r(n)$ then $Y^{(1)}$ is a partial r-tree.*

Proof. We want to show that there exists an r-tree G on $[n]$ such that $Y^{(1)} \subset G$.

We will prove this by strong induction on the number of vertices. If $Y \in \mathcal{M}_r(r+1)$ then $Y^{(1)} = K_{r+1}$ the complete graph, which is an r-tree.

Now suppose that the 1-skeleton of every minimal connected cover on less than n vertices is a partial r-tree. Let $Y \in \mathcal{M}_r(n)$, by Lemma 3.2.12 there exists some external leaf with branch σ. When we remove σ from Y we are left with some $Y' \in \mathcal{M}_r(n-k)$ and k isolated vertices for some $k = 1, \ldots, r$ the number of leaves in the branch σ. Let $\tau = Y' \cap \sigma$ be the simplex of dimension $r-k$ and note that there exists an $(r-1)$-dimensional simplex $\tau' \in Y'$ with $\tau \subset \tau'$.

By our inductive hypothesis there exists an r-tree, G, on $n-k$ vertices such that $Y'^{(1)} \subset G$. If $k = 1$ then $G_1 = G \cup \sigma^{(1)}$ is an r-tree such that $Y^{(1)} \subset G_1$. If $k > 1$ let $\{v_1, \ldots, v_k\}$ be the set of leaves and construct a new graph from G as follows:

- Connect v_1 to all of the vertices in τ'.
- Connect v_2 to v_1 and any $r-1$ vertices in τ'.
- ...
- Connect v_j to all vertices $v_1, v_2, \ldots, v_{j-1}$ and any $r-j+1$ vertices in τ'.

Note that at each stage the new edges that are added ensure the graph is an r-tree. Moreover, the graph constructed at the kth step certainly contains $Y'^{(1)} \cup \sigma^{(1)}$ and thus it contains $Y^{(1)}$. ☐

Corollary 3.4.4. *For any $B > 2^{\binom{r+1}{2}} \cdot r$*

$$M_r(n) < B^n \cdot n^n.$$

Proof. Combining Lemma 3.2.13 and Proposition 3.4.3 gives an injective map from $\mathcal{M}_r(n)$ to the set of partial r-trees on n vertices, so we just need a bound on the size of this set and we are done.

It's clear that an r-tree on n vertices has $\binom{r}{2} + r(n-r)$ edges. Therefore by Theo-

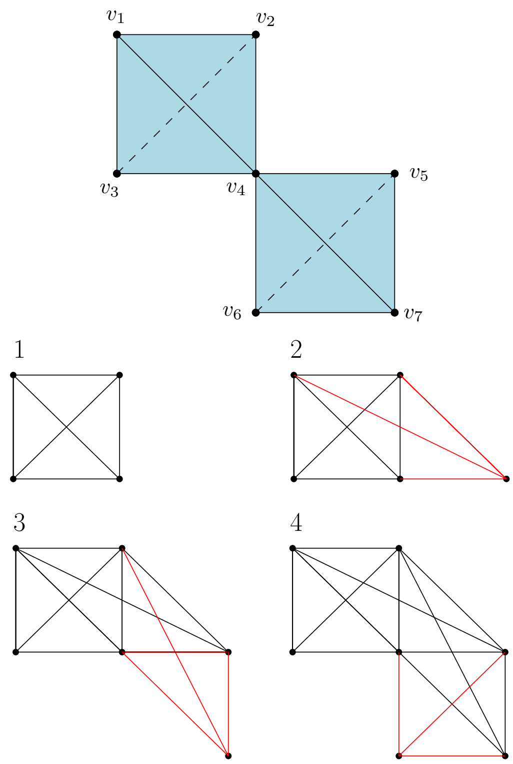

Figure 3.4: Shows the process described in the proof of Proposition 3.4.3. It shows that the 1-skeleton of some $Y \in \mathcal{M}_3(7)$ (shown on the top) is a partial 3-tree by removing the branch $\sigma = [v_4, v_5, v_6, v_7]$ and doing the described process with $\tau = [v_4]$ and $\tau' = [v_1, v_2, v_4]$.

rem 3.4.2 we know that the number of partial r-trees on n vertices equals

$$\begin{aligned} 2^{\binom{r}{2}+r(n-r)} \cdot \binom{n}{r} \cdot [r(n-r)+1]^{n-r-2} &\leq 2^{\binom{r}{2}} \cdot (2^r \cdot r \cdot n)^{n-2} \\ &< B^n \cdot n^n. \end{aligned}$$

Where in the final inequality we may take B to be any constant greater than $2^{\binom{r+1}{2}} \cdot r$. $\square$

3.5 Path connectivity of upper model random simplicial complexes

In this section we consider an extended example that utilises the bound for the number of minimal connected covers found in Corollary 3.4.4. The following is a classical result of Erdős and Rényi [31] about connectivity in random graphs.

Theorem 3.5.1. *If $G = G(n, p)$ is an Erdős-Rényi random graph then*

$$p = \frac{\frac{1}{2}\log n}{n}$$

is the threshold probability for G to contain a connected isolated subgraph on at least two vertices.

Alternatively, one can view this as the threshold for G to have a unique connected component on at least two vertices.

This theorem in particular tells us that as soon as there are no isolated vertices the random graph will almost surely be connected. The goal of this section is of course to generalise this result for random simplicial complexes.

We begin by returning to the study of pure random simplicial complexes initiated in Section 2.8. Throughout Y will be an upper model random complex defined on possible vertex set $[n] = \{1, \dots, n\}$ with each r-dimensional simplex and all of its subsimplices included independently at random with probability p, see defining equation (2.17). One may equivalently view this as a model for random $(r+1)$-uniform hypergraphs.

In contrast with classical random simplicial complexes of Linial, Meshulam, Wallach [58, 62] we observe that such a Y has no condition requiring it to contain the full skeleton of dimension $(r-1)$ so questions about path connectivity cannot be automatically taken as given. The recent paper of Cooley, Del Giudice, Kang, Sprüssel [22] meticulously studies thresholds for homological connectivity of such a random simplicial complex Y and goes far beyond the results presented in this section – this section is not intended to provide any new results but to provide a proof of a result analogous to Theorem 3.5.1 using techniques similar to that of Erdős and Rényi.

Theorem 3.5.2. *Let Y be the pure random simplicial complex on vertex set $[n]$ with each r-dimensional simplex included independently at random with probability p. Then* $p = \dfrac{\frac{r!}{r+1}\log n}{n^r}$ *is the threshold probability for Y to have a unique connected component.*

To show this we will first compute an upper bound for the expected number of connected components on k vertices using our bound on the number of minimal connected covers (Lemma 3.5.3), then we will compute the threshold probability for such a random complex Y to have isolated r-simplices (Lemma 3.5.4) before showing that this is precisely the threshold for the connectivity of Y (Lemma 3.5.5).

Lemma 3.5.3. *The expected number of connected components on k vertices in the random complex Y is bounded above by*

$$C^k \cdot \binom{n}{k} k^k p^{\lceil \frac{k-1}{r} \rceil} (1-p)^{Q(n,k)}$$

where $Q(n,k) = \sum_{i=1}^{r} \binom{k}{i}\binom{n-k}{r-i+1}$ and C is some fixed finite constant.

Proof. The probability that a given k vertices form a connected component is the product of two probabilities: the probability that they are connected and the probability that they do not connect to anything outside. A given set of k vertices V is connected if and only if some minimal covering is present, which occurs with probability bounded above by $C^k k^k p^{\lceil \frac{k-1}{r} \rceil}$ by Corollary 3.4.4 and Lemma 3.2.10.

Such a V is disconnected from the rest of the complex if and only if no simplex with 1, 2, ..., r vertices in V are selected. Therefore there must be

$$Q(n,k) = \sum_{i=1}^{r} \binom{k}{i}\binom{n-k}{r-i+1}$$

r-simplices with are not selected, which occurs with probability $(1-p)^{Q(n,k)}$. Thus the probability of one particular set of k vertices defining a connected component is bounded above by $C^k k^k p^{\lceil \frac{k-1}{r} \rceil}(1-p)^{Q(n,k)}$. Therefore, the expected number of all such connected components on k vertices is at most $C^k \cdot \binom{n}{k} k^k p^{\lceil \frac{k-1}{r} \rceil}(1-p)^{Q(n,k)}$. □

Lemma 3.5.4. $p = \dfrac{\frac{r!}{r+1} \cdot \log n}{n^r}$ *is the threshold probability for the existence of isolated r-dimensional simplices in the random pure complex Y.*

Proof. Let N be the random variable which counts the number of isolated simplices in Y. A simplex is isolated precisely when it is selected and no other simplices with vertices in common are. Thus we must have

$$\begin{aligned} Q(n, r+1) &= \binom{r+1}{1}\binom{n-r-1}{r} + \cdots + \binom{r+1}{r}\binom{n-r-1}{1} \\ &= \frac{r+1}{r!} n^r (1 - O(1/n)) \end{aligned}$$

simplices that are not selected. So the probability of some simplex being isolated is $p(1-p)^{Q(n,r+1)} = p(1-p)^{\frac{r+1}{r!}n^r(1-O(1/n))}$. The expected number of such isolated simplices with $p = \frac{\zeta \log n}{n^r}$ is therefore given by

$$\begin{aligned} \overline{\mathbb{E}}_n(N) &= \binom{n}{r+1} \cdot p(1-p)^{\frac{r+1}{r!}n^r(1-O(1/n))} \\ &\sim \frac{n^{r+1}}{(r+1)!} \cdot \frac{\zeta \log n}{n^r} \cdot \exp\left(-\frac{r+1}{r!} \cdot n^r \cdot \frac{\zeta \log n}{n^r} \cdot (1 - O(1/n))\right) \\ &= \frac{\zeta n \log n}{(r+1)!} \cdot n^{\frac{-\zeta(r+1)}{r!}} \cdot n^{O(1/n)} \\ &\sim \frac{\zeta \log n}{(r+1)!} \cdot n^{1 - \frac{\zeta(r+1)}{r!}}. \end{aligned}$$

If $\zeta > \dfrac{r!}{r+1}$ then this expectation equals $o(1)$, so by Markov's inequality, Theorem 1.2.1, we see that such a random simplicial complex Y has no isolated simplices asymptotically almost surely.

Now suppose that $\zeta < \dfrac{r!}{r+1}$. The probability that two disjoint r-simplices σ and τ are both selected is

$$p^2(1-p)^{2Q(n,r+1)-R(n,r)}$$

where $R(n,r) = \sum_{1 \le i,j \le r} \binom{r+1}{i}\binom{r+1}{j}\binom{n-2r-2}{r+1-i-j} = O(n^{r-1})$. This counts all those simplices which intersect both σ and τ that we do not want to double count. Now

$$\begin{aligned} (1-p)^{R(n,r)} &> 1 - pR(n,r) \\ &= 1 - \frac{\zeta \log n}{n^r} O(n^{r-1}) \\ &= 1 - O(\log n / n) \to 1. \end{aligned}$$

Therefore $(1-p)^{-R(n,r)} = 1 + o(1)$ and so

$$\begin{aligned}
\overline{\mathbb{E}}_n(N^2) &\leq \overline{\mathbb{E}}_n(N) + 2 \sum_{\sigma\cap\tau=\emptyset} p^2(1-p)^{2Q(n,r+1)-R(n,r)} \\
&\leq \overline{\mathbb{E}}_n(N) + 2 \cdot \binom{\binom{n}{r+1}}{2} \cdot p^2(1-p)^{2Q(n,r+1)-R(n,r)} \\
&\leq \overline{\mathbb{E}}_n(N) + \binom{n}{r+1}^2 \cdot p^2(1-p)^{2Q(n,r+1)}(1+o(1)) \\
&= (1+o(1)) \cdot \overline{\mathbb{E}}_n(N)^2
\end{aligned}$$

Where the last equality follows by using the fact that

$$\overline{\mathbb{E}}_n(N) \sim \frac{\zeta \log n}{(r+1)!} \cdot n^{1-\frac{\zeta(r+1)}{r!}} \to \infty$$

for $\zeta < \dfrac{r!}{r+1}$. Therefore by using Chebychev's inequality in the form

$$\overline{\mathbb{P}}_n(N > 0) \geq \frac{\overline{\mathbb{E}}_n(N)^2}{\overline{\mathbb{E}}_n(N^2)}$$

we conclude that Y has isolated simplices with probability converging to one. $\square$

In particular, Lemma 3.5.4 tells us that if $\zeta < \frac{r+1}{r!}$ then the random simplicial complex in the description of Theorem 3.5.2 is disconnected. To complete the proof of Theorem 3.5.2 we will show the following,

Lemma 3.5.5. *If* $p = \dfrac{\zeta \log n}{n^r}$ *with* $\zeta > \frac{r!}{r+1}$ *then* Y *has a unique connected component asymptotically almost surely.*

Proof. We will prove this statement by showing that there are no connected components on k vertices for all $r+1 \leq k \leq n/2$, i.e. if this were true since there are no isolated r-dimensional simplices there is a unique connected component supported by at least $n/2$ vertices and potentially isolated vertices proving the statement.

Let X_k be the random variable which counts the number of connected components on k vertices. We will show that for $p = \dfrac{\zeta \log n}{n^r}$ that the expected number of connected components of size $r+1 \leq k \leq n/2$ is less than $O\left(n^{-\varepsilon} \cdot \log n\right)$ for some positive ε.

We use Lemma 3.5.3 to get the following upper bound

$$\overline{\mathbb{E}}_n(X_k) \leq C^k \binom{n}{k} k^k p^{\frac{k-1}{r}} (1-p)^{Q(n,k)},$$

where we have used the trivial fact that $p^{\frac{k-1}{r}} \geq p^{\lceil \frac{k-1}{r} \rceil}$.

We will further simplify this bound by using the inequalities $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$, and $1 - p \leq e^{-p}$. To complete the argument we need a better understanding of $Q(n,k) = \sum_{i=1}^{r} \binom{k}{i}\binom{n-k}{r-i+1}$. For this we cite the result of Lemma A.1.1 found in Appendix A.1 which tells us that

$$\frac{r}{x} - \frac{\zeta \cdot r \cdot Q(n,x)}{n^r \cdot x}$$

is maximised by $x = r+1$ or $x = n/2$ in the domain $[r+1, n/2]$ for sufficiently large n and that if $\zeta > \dfrac{r!}{r+1}$ then this maximal value is at most $-\varepsilon + O(1/n)$ for some positive constant ε dependent on r and ζ.

Putting this all together we get the following when we substitute $p = \frac{\zeta \log n}{n^r}$,

$$\begin{aligned}
\overline{\mathbb{E}}_n(X_k) &\leq C^k \cdot \left(\frac{en}{k}\right)^k \cdot k^k \cdot \left(\frac{\zeta \log n}{n^r}\right)^{\frac{k-1}{r}} \cdot \exp\left(\frac{-\zeta Q(n,k) \log n}{n^r}\right) \\
&\leq \zeta^{\frac{k}{r}} \cdot C^k \cdot e^k \cdot n^{1-\frac{\zeta Q(n,k)}{n^r}} \cdot \log^{\frac{k-1}{r}} n \\
&\leq \left(\text{const} \cdot n^{r/k - \frac{\zeta r Q(n,k)}{n^r k}} \cdot \log n\right)^{k/r} \\
&= O\left(n^{-\varepsilon + O(1/n)} \cdot \log n\right)^{k/r} \\
&= O\left(n^{-\varepsilon} \cdot \log n\right)^{k/r}.
\end{aligned}$$

Where the final line follows from the fact that $n^{O(1/n)}$ converges to a constant.

Since $\overline{\mathbb{E}}_n(X_k)$ decreases geometrically we see by linearity that $\overline{\mathbb{E}}_n\left(\sum_{k=r+1}^{n/2} X_k\right) = O\left(n^{-\varepsilon} \cdot \log n\right) = o(1)$. The result follows by application of Markov's inequality. $\square$

Theorem 3.5.2 is then proven by application of Lemma 3.5.4 and Lemma 3.5.5.

Consider now the general r-dimensional upper model random simplicial complex Z

now with *complete vertex set* $[n]$. That is, $Z = Z(n, (p_1, \dots, p_r))$ is an upper model random simplicial complex, see Section 2.3, with parameters

$$p_\sigma = \begin{cases} 1 & \text{if } \dim \sigma = 0, \\ p_i & \text{if } \dim \sigma = i, \\ 0 & \text{if } \dim \sigma > r. \end{cases} \tag{3.1}$$

Lemma 3.5.6. *Let the probability parameters satisfy* $p_i = \dfrac{\zeta_i \log n}{n^i}$ *for some* ζ_i. *If* $\sum_{i=1}^r \dfrac{\zeta_i}{i!} > 1$ *then the random simplicial complex* Z *contains an isolated vertex asymptotically almost surely.*

Coversely, if $\sum_{i=1}^r \dfrac{\zeta_i}{i!} < 1$ *then* Z *contains no isolated vertices asymptotically almost surely.*

Proof. A vertex in Z is isolated with probability

$$\begin{aligned} \prod_{i=1}^r q_i^{\binom{n-1}{i}} &\sim \exp\left(-\sum_{i=1}^r p_i \binom{n}{i}\right) \\ &\sim \exp\left(-\sum_{i=1}^r \frac{\zeta_i}{i!} \cdot \log n\right) \\ = n^{-\sum_{i=1}^r \frac{\zeta_i}{i!}}. \end{aligned}$$

Let N be the random variable denoting the number of isolated vertices in Z. By the above we see that

$$\overline{\mathbb{E}}_n(N) \sim n^{1-\sum_{i=1}^r \frac{\zeta_i}{i!}}.$$

We see that if $\sum_{i=1}^r \frac{\zeta_i}{i!} > 1$ then $\overline{\mathbb{E}}_n(N) \to 0$, and so by Markov's inequality we see that under this assumption there are no isolated vertices asymptotically almost surely.

For the converse case, when $\sum_{i=1}^r \frac{\zeta_i}{i!} < 1$ we will use a second moment method argument, see Corllary 1.2.4. We first remark that clearly under this assumption, by the above considerations, that $\overline{\mathbb{E}}_n(N) \to \infty$. We also note that two distinct vertices are *both* isolated with probability $\prod_{i=1}^r q_i^{2\binom{n-1}{i} - \binom{n-2}{i-1}}$. Therefore

$$\begin{aligned}
\overline{\mathbb{E}}_n(N^2) &= \sum_{i,j} \overline{\mathbb{P}}_n(i, j \text{ are isolated}) \\
&= \overline{\mathbb{E}}_n(N) + \sum_{i \neq j} \prod_{i=1}^{r} q_i^{2\binom{n-1}{i} - \binom{n-2}{i-1}} \\
&= n \cdot q_i^{\binom{n-1}{i}} + n(n-1) \cdot \prod_{i=1}^{r} q_i^{2\binom{n-1}{i} - \binom{n-2}{i-1}}.
\end{aligned}$$

It then follows via application of the inequality $\overline{\mathbb{P}}_n(N > 0) \geq \dfrac{\overline{\mathbb{E}}_n(N)^2}{\overline{\mathbb{E}}_n(N^2)}$ that the probability $N > 0$ is bounded below by

$$\frac{n \cdot q_i^{\binom{n-1}{i}} \cdot \prod_{i=1}^{r} q_i^{\binom{n-2}{i-1}}}{\prod_{i=1}^{r} q_i^{\binom{n-2}{i-1}} + (n-1) \cdot q_i^{\binom{n-1}{i}}} \to 1.$$

It follows that the above quantity converges to 1 since $n \cdot q_i^{\binom{n-1}{i}} = \overline{\mathbb{E}}_n(N) \to \infty$ and $\prod_{i=1}^{r} q_i^{\binom{n-2}{i-1}} = \exp\left(-\sum_{i=1}^{r} \frac{\zeta_i}{(i-1)!} \cdot \frac{\log n}{n}\right) \to 1.$ □

This gives the immediate corollary for a general upper model random simplicial complex to be connected with high probability.

Corollary 3.5.7. *Let the probability parameters satisfy* $p_i = \dfrac{\zeta_i \log n}{n^i}$ *for some* ζ_i*. If at least one* $\zeta_i > i!$ *then* Z *is connected asymptotically almost surely.*

Proof. By Lemma 2.7.2 we see that we may decompose $Z = [n] \cup Z_1 \cup \cdots \cup Z_r$ where each Z_i is a pure random simplicial complex with probability parameters satisfying

$$p_\sigma = \begin{cases} p_i & \text{if } \dim \sigma = i, \\ 0 & \text{otherwise.} \end{cases}$$

As some $zeta_i > (i+1)!$ we see that $\sum_{i=1}^{r} \frac{\zeta_i}{i!} > 1$ so by Lemma 3.5.6 Z_i contains the full vertex set $[n]$ a.a.s. Similarly, by Theorem 3.5.2 we see that Z_i is connected a.a.s. It's clear that if some Z_i is connected and contains the full vertex set $[n]$ then Z must be connected. □

In the pure case we get the stronger result with a threshold for connectivity trivially.

Corollary 3.5.8. *Let $Y' = Y \cup [n]$ be the upper model random simplicial complex with probability parameters*

$$p_\sigma = \begin{cases} 1 & \textit{if } \dim \sigma = 0, \\ p & \textit{if } \dim \sigma = r, \\ 0 & \textit{otherwise.} \end{cases}$$

Then $p = \frac{r! \log n}{n^r}$ is the threshold probability for Y' to be connected.

Chapter 4

Random simplicial complexes in the medial regime

4.1 Introduction

In this chapter we continue the study of lower and upper model random simplicial complexes introduced in Chapter 2. There we studied properties of the complexes when the parameters $p_\sigma \to 0$. Here we study the opposite situation: that is we assume that the probability parameters satisfy

$$p \leq p_\sigma \leq P \tag{4.1}$$

for all simplices σ where the numbers $p, P \in (0, 1)$ are independent of n. We call this *the medial regime*. In the medial regime the probability parameters p_σ can approach neither 0 nor 1.

We show that *a lower model* random simplicial complex Y in the medial regime has dimension

$$\dim Y \sim \log_2 \ln n + \log_2 \log_2 \ln n,$$

it is simply connected, and may have nontrivial Betti numbers $b_j(Y)$ only for

$$j \in \left[\log_2 \ln n + c,\ \log_2 \ln n + \log_2 \log_2 \ln n + c'\right], \tag{4.2}$$

where c, c' are constants. A more precise statement is given below as Theorem 4.2.2. The proof uses the Garland method relating the spectral gap of links with the vanishing of the Betti numbers, see [4].

We also describe the topology of a typical random simplicial complex Y with respect to *the upper model* in the medial regime. We show that it has a rather different behaviour: its dimension equals $n-2$, it contains the skeleton $\Delta_n^{(n-d)}$ where

$$d \sim \log_2 \ln n + \log_2 \log_2 \ln n,$$

and the Betti numbers $b_{n-j}(Y)$ vanish except for a range of dimensions of width approximately $\log_2 \log_2 \ln n$. A precise statement is given below as Theorem 4.2.3.

We employ the Alexander duality relation of Theorem 2.9.9, which allows us to deduce the results concerning the upper model from the lower model.

4.2 Definitions and statements of the main results

We shall say that the system of probability parameters $\{p_\sigma\}$ is *in the medial regime* if there exist constants $p, P \in (0,1)$ such that for any simplex $\sigma \in \Delta_n$ one has

$$0 < p \leq p_\sigma \leq P < 1. \tag{4.3}$$

We emphasise that the numbers p, P are independent of n. In other words, in the medial regime the probability parameters p_σ are allowed to approach neither 0 nor 1, as $n \to \infty$. It will be convenient to write

$$p = e^{-a}, \quad P = e^{-A} \tag{4.4}$$

where the $0 < A \leq a$ are constants.

Definition 4.2.1. We call the system of probability parameters $\{p_\sigma\}$ *homogeneous* when they depend only upon their dimension. That is, $p_\sigma = p_\tau$ if $\dim \sigma = \dim \tau$.

Next we state two main results of this Chapter:

Theorem 4.2.2. *Let $Y \in \Omega_n^*$ be a random simplicial complex in the medial regime with respect to the lower measure. Then:*

1. *The dimension of Y satisfies*

$$\lfloor \beta(n,a) \rfloor - 1 \;\leq\; \dim Y \;\leq\; \beta(n,A) - 1 + \varepsilon_0,$$

a.a.s. Here $\varepsilon_0 > 0$ is an arbitrary positive constant and we use the notation

$$\beta(n,y) = \log_2 \ln n + \log_2 \log_2 \ln n - \log_2(y);$$

2. *Y is connected and simply connected, a.a.s;*
3. *If the system of probability parameters p_σ is homogeneous (see Definition 4.2.1) then with probability tending to 1 as $n \to \infty$ the Betti numbers $b_j(Y)$ vanish for all*

$$0 < j \;\leq\; \log_2 \ln n - \log_2 a - 1 - \delta,$$

where $\delta > 0$ is an arbitrary constant.

Thus, under the assumptions of Theorem 4.2.2 a random complex Y may potentially have nontrivial reduced Betti numbers only in dimensions j satisfying

$$\log_2 \ln n - \log_2 a - 1 - \delta < j \leq \log_2 \ln n + \log_2 \log_2 \ln n - \log_2 A - 1 + \varepsilon_0,$$

a.a.s.

To illustrate Theorem 4.2.2, let us assume that the integer n is written in the form $n = e^{2^k}$. Then the dimension of the random complex Y satisfies

$$\dim Y \;\sim\; k + \log_2 k,$$

and the range of potentially nontrivial Betti numbers is roughly

$$k \leq j \leq k + \log_2 k.$$

We see that a lower model random simplicial complex in the medial regime is homologically highly connected with nontrivial Betti numbers concentrated in a thin layer of dimensions near the dimension of the complex.

In the following Theorem we shall describe the properties of the random simplicial complexes in the upper model. If the initial system of probability parameters p_σ is in a medial regime (4.3) then the dual system $p'_\sigma = 1 - p_{\hat{\sigma}}$ (where $\hat{\sigma}$ is as defined in Section 2.4) will also be in the medial regime since

$$0 < 1 - P \leq p'_\sigma \leq 1 - p < 1.$$

We shall need the *dual numbers*

$$0 < a' \leq A'$$

defined by the equations

$$e^{-a} + e^{-a'} = 1 = e^{-A} + e^{-A'}. \tag{4.5}$$

One has $e^{-A'} \leq p'_\sigma \leq e^{-a'}$.

Theorem 4.2.3. *Let $Y \in \Omega_n^*$ be a random simplicial complex with respect to the upper probability measure associated to a system of probability parameters p_σ. Assume that p_σ satisfies*

$$0 < p \leq p_\sigma \leq P < 1,$$

where $p = e^{-a}$ and $P = e^{-A}$ are constant, i.e. the system of probability parameters is in the medial regime. Then, with probability tending to 1, one has:

1. *The dimension* $\dim Y$ *equals* $n - 2$*;*
2. *The maximal dimension d such that Y contains the $(n-d)$-dimensional skeleton*

$\Delta_n^{(n-d)}$ *of the simplex* Δ_n *satisfies*

$$\lfloor \beta(n, A') \rfloor + 1 \leq d \leq \beta(n, a') + 1 + \varepsilon_0.$$

Here $\varepsilon_0 > 0$ *is an arbitratry positive constant.*

3. *If the system of probability parameters* p_σ *is homogeneous then the reduced Betti numbers* $b_j(Y)$ *vanish for all dimensions* j *except possibly*

$$\log_2 \ln n - \log_2 A' + 1 - \delta < n - j \leq \beta(n, a') + 1 + \varepsilon_0.$$

We see that the topology of a typical random simplicial complex Y in the upper model in the medial regime is totally different from one in the lower model. If n is written in the form $n = e^{2^k}$ then Y contains the skeleton $\Delta_n^{(n-d)}$, where

$$d \sim k + \log_2 k$$

and the nontrivial Betti numbers of Y are concentrated in an interval of dimensions of width $\sim \log_2 k$ above the dimension $n - d \sim n - k - \log_2 k$.

Remark 4.2.4. An argument using Morse inequalities suggests that under the assumptions of Theorem 4.2.2 the expected Betti number in one of the dimensions $\sim \log_2 \ln n$ is nonzero and goes to infinity with n.

The proofs of Theorems 4.2.2 and 4.2.3 are given in the following sections. Theorem 4.2.2 is the summary of Proposition 4.4.1, Corollary 4.5.2, Proposition 4.5.4 and Theorem 4.6.1. The proof of Theorem 4.2.3 is given in section §4.7.

4.3 Coupling

In this section we compare the properties of random simplicial complexes Y and Y' in the two models having different probability parameters p_σ and p'_σ. We show that for $p_\sigma \leq p'_\sigma$ one may *"realise"* Y as a subcomplex of Y'. This leads to the conclusion that for any monotone property $\mathcal{P}$ of random simplicial complexes the probability of the event $Y \in \mathcal{P}$ is dominated by the probability of the event $Y' \in \mathcal{P}$.

Next we introduce some notations. We denote by $\underline{\mathbb{P}}_n$ and $\underline{\mathbb{P}}'_n$ the lower probability measures on the set Ω_n of random simplicial complexes $Y \subset \Delta_n$ associated to the systems of probability parameters p_σ and p'_σ correspondingly. We shall denote by $\overline{\mathbb{P}}_n$ and $\overline{\mathbb{P}}'_n$ the corresponding upper measures on Ω_n. Consider also the set $\mathfrak{P}\Omega_n$ of all pairs (X, Y) consisting of a simplicial complex $X \subset \Delta_n$ and one of its subcomplexes $Y \subset X$. There are two projections

$$\pi_1, \pi_2 : \mathfrak{P}\Omega_n \to \Omega_n$$

where $\pi_1(X, Y) = X$ and $\pi_2(X, Y) = Y$.

Theorem 4.3.1. *(A) Suppose that two systems of probability parameters $p_\sigma \le p'_\sigma$ are given. Then there exists a probability measure $\underline{\mu}$ on $\mathfrak{P}\Omega_n$ such that its direct images under the projections π_1, π_2 are*

$$(\pi_1)_*(\underline{\mu}) = \underline{\mathbb{P}}'_n \quad \text{and} \quad (\pi_2)_*(\underline{\mu}) = \underline{\mathbb{P}}_n. \tag{4.6}$$

Similarly, there exists a probability measure $\overline{\mu}$ on $\mathfrak{P}\Omega_n$ such that its direct images under the projections π_1, π_2 are

$$(\pi_1)_*(\overline{\mu}) = \overline{\mathbb{P}}'_n, \quad (\pi_2)_*(\overline{\mu}) = \overline{\mathbb{P}}_n. \tag{4.7}$$

(B) Suppose additionally that $p_\sigma = p'_\sigma$ for any simplex σ of dimension $\le k$, where $k \ge 0$ is an integer. Then the measure $\underline{\mu}$ on $\mathfrak{P}\Omega_n$ is supported on the sets of pairs (X, Y) of simplicial complexes having identical k-dimensional skeleta, i.e. $X^{(k)} = Y^{(k)}$.

(C) If $p_\sigma = p'_\sigma$ for all simplices σ of dimension $> k$ where k is fixed integer then the measure $\overline{\mu}$ is supported on the sets of pairs (X, Y) of simplicial complexes satisfying $X - X^{(k)} = Y - Y^{(k)}$.

Let $\mathcal{P}$ be a property of a simplicial complex which is monotone, i.e. $Y \in \mathcal{P}$ implies $X \in \mathcal{P}$ for a simplicial subcomplex $Y \subset X$.

Corollary 4.3.2. *Under the assumption $p_\sigma \le p'_\sigma$, for any monotone property $\mathcal{P}$ one has*

$$\underline{\mathbb{P}}_n(Y \in \mathcal{P}) \le \underline{\mathbb{P}}'_n(Y \in \mathcal{P}) \quad \text{and} \quad \overline{\mathbb{P}}_n(Y \in \mathcal{P}) \le \overline{\mathbb{P}}'_n(Y \in \mathcal{P}). \tag{4.8}$$

Proof. Applying Theorem 4.3.1 one has

$$\underline{\mathbb{P}}_n(Y \in \mathcal{P}) = \underline{\mu}(\{(X,Y); Y \in \mathcal{P}\}) \le \underline{\mu}(\{(X,Y); X \in \mathcal{P}\}) = \underline{\mathbb{P}}'_n(X \in \mathcal{P}).$$

The case of the upper measure $\overline{\mu}$ is similar. □

As an example we consider the property $\dim Y \ge d$ where d is an integer. Since it is monotone we obtain:

Corollary 4.3.3. *Under the assumption that $p_\sigma \le p'_\sigma$ for every simplex $\sigma \subset [n]$, one has*

$$\underline{\mathbb{P}}_n(\dim Y \ge d) \le \underline{\mathbb{P}}'_n(\dim Y \ge d) \quad \text{and} \quad \overline{\mathbb{P}}_n(\dim Y \ge d) \le \overline{\mathbb{P}}'_n(\dim Y \ge d)$$

for any integer $d \ge 0$. Here $\underline{\mathbb{P}}_n$ and $\underline{\mathbb{P}}'_n$ are lower probability measures on Ω_n associated to the systems of probability parameters p_σ and p'_σ, correspondingly.

The following arguments will be used in the proof of Theorem 4.3.1.

Let S be a finite set and suppose that for each element $s \in S$ we are given a probability parameter $p_s \in [0,1]$. *The Bernoulli measure* ν on the set 2^S of all subsets of S is characterised by the property that for $A \subset S$ one has

$$\nu(A) = \prod_{s \in A} p_s \cdot \prod_{s \notin A} (1 - p_s). \tag{4.9}$$

Consider now another set of probability parameters $p'_s \in [0,1]$ with the property

$$p_s \le p'_s$$

for any $s \in S$; let ν' be the corresponding Bernoulli measure on 2^S, i.e.

$$\nu'(A) = \prod_{s\in A} p'_s \cdot \prod_{s\notin A}(1 - p'_s). \tag{4.10}$$

Lemma 4.3.4. *Let $\mathfrak{P}\Omega_S$ denote the set of all pairs (X, Y) where $Y \subset X \subset S$. Consider the projections $\pi_1, \pi_2 : \mathfrak{P}\Omega_S \to 2^S$ where $\pi_1(X, Y) = X$ and $\pi_2(X, Y) = Y$. There exists a probability measure μ on $\mathfrak{P}\Omega_S$ such that*

$$(\pi_1)_*(\mu) = \nu' \quad \textit{and} \quad (\pi_2)_*(\mu) = \nu. \tag{4.11}$$

If $p_s = p'_s$ for all elements s in a subset $T \subset S$ then the measure μ is supported on the set of pairs (X, Y) of subsets of S satisfying $X \cap T = Y \cap T$.

Proof. We define a probability measure μ on $\mathfrak{P}\Omega_S$ by the formula:

$$\mu(X, Y) = \prod_{s\in X} p'_s \cdot \prod_{s\in S-X}(1 - p'_s) \cdot \prod_{s\in Y}\frac{p_s}{p'_s} \cdot \prod_{s\in X-Y}\left(1 - \frac{p_s}{p'_s}\right). \tag{4.12}$$

The equalities (4.11) are verified directly by computation, we give this in Appendix A.2.

The assumption $p_s \le p'_s$ is used to ensure non-negativity of μ. If there exists an element $s \in X - Y$ which lies in T, then $p_s = p'_s$ and $\mu(X, Y) = 0$ since the last factor in (4.12) vanishes. □

Proof of Theorem 4.3.1. We apply Lemma 4.3.4 with $S = 2^{[n]}$, the set of subsets of the set of vertices $[n]$. The subsets $X \subset S$ can be identified with hypergraphs and we see that the set $\Omega_S = 2^S$ is the set of all hypergraphs with vertices in $[n]$ which in Section 2.3 was denoted Ω_n. The two systems of probability parameters p_σ and p'_σ (where $\sigma \in S$ is a simplex) define two Bernoulli probability measures on $2^S = \Omega_S = \Omega_n$ which we shall denote by ν and ν' correspondingly, see formulae (4.9) and (4.10).

The set of pairs $\mathfrak{P}\Omega_S$ which appears in Lemma 4.3.4 can be viewed as the set of pairs of hypergraphs (X, Y) where Y is a subhypergraph of X. Since for any simplex σ one has $p_\sigma \le p'_\sigma$, we may apply Lemma 4.3.4 to obtain a probability measure μ on $\mathfrak{P}\Omega_S = \mathfrak{P}\Omega_n$

with the property $(\pi_1)_*(\mu) = \nu'$ and $(\pi_2)_*(\mu) = \nu$.

Consider the maps $\underline{\rho}, \overline{\rho} : \Omega_n \to \Omega_n^*$ (see (2.2) in §2.3) where Ω_n^* denotes the set of all simplicial subcomplexes of Δ_n. These maps obviously define maps of pairs $\underline{\rho}, \overline{\rho} : \mathfrak{P}\Omega_n \to \mathfrak{P}\Omega_n^*$ and we define the probability measures $\underline{\mu}, \overline{\mu}$ on $\mathfrak{P}\Omega_n^*$ by the formulae

$$\underline{\mu} = (\underline{\rho})_*(\mu), \quad \overline{\mu} = (\overline{\rho})_*(\mu). \tag{4.13}$$

We have two sets of commutative diagrams

$$\begin{tikzcd} \mathfrak{P}\Omega_n \arrow[r, "\pi_i"] \arrow[d, "\underline{\rho}"] & \Omega_n \arrow[d, "\underline{\rho}"] \\ \mathfrak{P}\Omega_n^* \arrow[r, "\pi_i"] & \Omega_n^*, \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{P}\Omega_n \arrow[r, "\pi_i"] \arrow[d, "\overline{\rho}"] & \Omega_n \arrow[d, "\overline{\rho}"] \\ \mathfrak{P}\Omega_n^* \arrow[r, "\pi_i"] & \Omega_n^*. \end{tikzcd}$$

where $i = 1, 2$. Applying the definitions, we obtain

$$\pi_{1*}(\underline{\mu}) = \pi_{1*}(\underline{\rho}_*(\mu)) = \underline{\rho}_*(\pi_{1*}(\mu) = \underline{\rho}_*(\nu') = \underline{\mathbb{P}}'_n.$$

And similarly

$$\pi_{2*}(\underline{\mu}) = \pi_{2*}(\underline{\rho}_*(\mu)) = \underline{\rho}_*(\pi_{2*}(\mu) = \underline{\rho}_*(\nu) = \underline{\mathbb{P}}_n.$$

This proves formulae (4.6). Formulae (4.7) follow similarly. This proves statement (A).

To prove statement (B) we engage the last statement of Lemma 4.3.4 which claims that the constructed measure μ on $\mathfrak{P}\Omega_n$ is supported on the set of pairs of hypergraphs (X, Y) having identical k-dimensional skeleta. Then obviously the measure $\underline{\mu} = (\underline{\rho})_*(\mu)$ is supported on the set of pairs of simplicial complexes having identical k-skeleta.

The proof of (C) is similar. If $p_\sigma = p'_\sigma$ for any simplex of dimension greater than k then the measure μ is supported on the set of pairs of hypegraphs $(X, Y) \in \mathfrak{P}\Omega_n$ which are identical above dimension k. This implies that the direct image measure $\overline{\mu} = (\overline{\rho})_*(\mu)$ is supported on the set of pairs of simplicial complexes which are identical above dimension k. □

4.4 Dimension of a lower random simplicial complex in the medial regime

In this section we shall consider a random simplicial complex $Y \in \Omega_n^*$ with respect to the lower model and will impose the medial regime assumptions (4.3). We shall write

$$p = e^{-a}, \quad P = e^{-A} \quad \text{where} \quad 0 < A \leq a. \tag{4.14}$$

Let us denote

$$\beta = \beta(n, y) = \log_2 \ln n + \log_2 \log_2 \ln n - \log_2(y). \tag{4.15}$$

Proposition 4.4.1. *Let $\varepsilon_0 > 0$ be a fixed constant. Under the above assumptions the dimension of a random simplicial complex Y satisfies*

$$\lfloor \beta(n, a) \rfloor - 1 \;\leq\; \dim Y \;\leq\; \beta(n, A) - 1 + \varepsilon_0, \tag{4.16}$$

a.a.s.

Remark 4.4.2. Note that the quantity

$$\beta(n, A) - \beta(n, a) = \log_2 \left(\frac{a}{A}\right) = \log_2 \left(\frac{\ln p}{\ln P}\right) \geq 0 \tag{4.17}$$

is constant (independent of n). Hence Proposition 4.4.1 determines the dimension of a random complex Y with finite error (4.17) while the dimension itself $\dim Y$ tends to infinity.

In the special case when $p = P$ and $a = A$ we obtain $\lfloor \beta(n, a) \rfloor - 1 \leq \dim Y \leq \beta(n, a) - 1 + \varepsilon_0$, a.a.s. which nearly uniquely determines the dimension $\dim Y$.

Proof of Proposition 4.4.1. We start by establishing the upper bound in (4.16). Using the monotonicity of dimension we may apply Theorem 4.3.1 and Corollary 4.3.3. Therefore in the proof of the upper bound we may assume without loss of generality that

$$p_\sigma = P = e^{-A}$$

for any simplex σ.

Let $f_\ell : \Omega_n^* \to \mathbb{R}$ denote the number of ℓ-dimensional simplices in Y. Note that as a random variable, $f_\ell = \sum X_\sigma$, where the sum runs over all simplices $\sigma \subset [n]$ of dimension ℓ and X_σ is a random variable which takes values 0 and 1 depending on whether the simplex σ is included into the random complex Y. We have

$$\mathbb{E}(X_\sigma) = \prod_{\nu \subset \sigma} p_\nu = P^{2^{\ell+1}-1}.$$

Then

$$\mathbb{E}(f_\ell) = \binom{n+1}{\ell+1} \cdot P^{2^{\ell+1}-1}.$$

We may estimate the expectation from above as follows

$$\mathbb{E}(f_\ell) \le (n+1)^{\ell+1} \cdot P^{2^{\ell+1}-1} \le \frac{e}{P} \cdot \left(\exp\left[\ln n - A \cdot \frac{2^{\ell+1}}{\ell+1} \right] \right)^{\ell+1}.$$

Since the function $x \mapsto \frac{2^x}{x}$ is monotone increasing for $x \ge 2$ we obtain that for any

$$\ell \ge \beta(n, A) + \varepsilon_0 - 1 = \beta + \varepsilon_0 - 1$$

(where $\varepsilon_0 > 0$ is fixed) one has

$$\ln n - A\frac{2^{\ell+1}}{\ell+1} \le \ln n - A\frac{2^{\beta+\varepsilon_0}}{\beta+\varepsilon_0} = \left[1 - 2^{\varepsilon_0}\frac{\log_2 \ln n}{\beta+\varepsilon_0}\right] \cdot \ln n \le -\frac{1}{2}(2^{\varepsilon_0}-1) \cdot \ln n$$

(for sufficiently large n) implying

$$\mathbb{E}(f_\ell) \le \frac{e}{P} n^{-c(\ell+1)}, \quad \text{where} \quad c = \frac{1}{2} \cdot (2^{\varepsilon_0} - 1) > 0.$$

We obtain

$$\sum_{\ell+1\geq\beta+\varepsilon_0} \mathbb{E}(f_\ell) \leq \frac{e}{P} \sum_{\ell+1\geq\beta+\varepsilon_0} n^{-c(\ell+1)} \leq \frac{e}{P}\cdot\frac{n^{-c(\beta+\varepsilon_0)}}{1-n^{-c}} \to 0.$$

Thus, by the first moment method, Y has no simplices in any dimension $\ell \geq \beta+\varepsilon_0-1$ a.a.s., i.e. we obtain the right inequality of (4.16).

Next we prove the left inequality in (4.16), i.e. the lower bound for the dimension. While doing so we may assume (using Theorem 4.3.1 and the monotonicity of dimension) that $p_\sigma = p = e^{-a}$ for any simplex σ. We assume below that

$$\ell \leq \beta(n,a) - 1 \tag{4.18}$$

and our goal is to show that $f_\ell > 0$ with probability tending to 1 as $n\to\infty$. We shall use the following estimates for the binomial coefficient

$$\frac{1}{3}\left(\frac{ne}{\ell}\right)^\ell \cdot \ell^{-1/2} \leq \binom{n}{\ell} \leq \left(\frac{ne}{\ell}\right)^\ell \cdot \ell^{-1/2} \tag{4.19}$$

which are valid for $1 \leq \ell < n/2$ and n large enough; it follows from Stirling's formula, see page 4 in [10]. Hence we obtain

$$\mathbb{E}(f_\ell) = \binom{n+1}{\ell+1} p^{2^{\ell+1}-1} \geq \left(\frac{n}{\ell}\right)^{\ell+1} p^{2^{\ell+1}-1} = p^{-1}\left[\exp\left(\ln\frac{n}{\ell} - a\frac{2^{\ell+1}}{\ell+1}\right)\right]^{\ell+1}. \tag{4.20}$$

Using (4.18) we find that

$$\frac{a2^{\ell+1}}{\ell+1} \leq \ln n \cdot \frac{\log_2 \ln n}{\log_2 \ln n + \log_2\log_2\ln n - \log_2 a}$$

implying

$$\ln\frac{n}{\ell} - a\frac{2^{\ell+1}}{\ell+1} \geq \frac{\ln n\cdot \log_2\log_2\ln n}{2\cdot\log_2\ln n}.$$

This shows that $\mathbb{E}(f_\ell)\to\infty$.

We shall apply the second moment inequality, Corollary 1.2.4, with $X = f_\ell$ and show that under the assumptions (4.18) the inverse quantity $\frac{\mathbb{E}(f_\ell^2)}{\mathbb{E}(f_\ell)^2}$ tends to 1 as $n \to \infty$. Since we know apriori that $\frac{\mathbb{E}(f_\ell^2)}{\mathbb{E}(f_\ell)^2} \geq 1$, it is enough to show that the ratio $\frac{\mathbb{E}(f_\ell^2)}{\mathbb{E}(f_\ell)^2}$ is bounded above by a sequence tending to 1 as $n \to \infty$.

As above, $f_\ell = \sum X_\sigma$, where the sum runs over all simplices $\sigma \subset [n]$ of dimension ℓ. Hence $f_\ell^2 = \sum_{\sigma,\tau} X_\sigma X_\tau$ and $\mathbb{E}(f_\ell^2) = \sum_{\sigma,\tau} \mathbb{E}(X_\sigma X_\tau)$. We have

$$\mathbb{E}(X_\sigma X_\tau) = \mathbb{P}_n(\sigma \subset Y \,\&\, \tau \subset Y) = p^{2\cdot 2^{\ell+1}-2^i-1}$$

where i denotes the cardinality of intersection $\sigma \cap \tau \subset [n]$. One therefore obtains

$$\mathbb{E}\left(f_\ell^2\right) = \sum_{i=0}^{\ell+1} \binom{n+1}{\ell+1} \cdot \binom{\ell+1}{i} \cdot \binom{n-\ell}{\ell+1-i} \cdot p^{2\cdot 2^{\ell+1}-2^i-1}$$

and since

$$\mathbb{E}(f_\ell) = \binom{n+1}{\ell+1} p^{2^{\ell+1}-1}$$

we obtain

$$\frac{\mathbb{E}(f_\ell^2)}{\mathbb{E}(f_\ell)^2} = \sum_{i=0}^{\ell+1} \frac{\binom{\ell+1}{i} \cdot \binom{n-\ell}{\ell+1-i}}{\binom{n+1}{\ell+1}} \cdot p^{-2^i+1}.$$

We shall denote by r_i the terms in the last sums where $i = 0, 1, \dots, \ell+1$. For the term r_0 we have

$$r_0 = \frac{\binom{n-\ell}{\ell+1}}{\binom{n+1}{\ell+1}} < 1.$$

One goal is to show that the sum of all other terms $r_1 + r_2 + \cdots + r_{\ell+1}$ tends to zero with n. For the term r_1 we have

$$r_1 = \frac{(\ell+1)\binom{n-\ell}{\ell}}{\binom{n+1}{\ell+1}} \cdot p^{-1} \leq \frac{(\ell+1) \cdot n^\ell}{\left(\dfrac{n}{\ell+1}\right)^{\ell+1}} \cdot p^{-1} = \frac{(\ell+1)^{\ell+2}}{n} \cdot p^{-1}.$$

Using our assumption (4.18) and (4.15) we see that $r_1 \to 0$ as $n \to \infty$.

Next we consider the term r_i with $2 \leq i \leq \ell+1$. Since $p^{-1} = e^a$ and taking into account that the function $\dfrac{2^x}{x}$ is increasing for $x \geq 2$ we obtain

$$\begin{aligned} r_i &= \frac{\binom{\ell+1}{i} \cdot \binom{n-\ell}{\ell+1-i}}{\binom{n+1}{\ell+1}} \cdot p^{-2^i+1} \leq \frac{(\ell+1)^{\ell+i+1}}{n^i} \cdot p^{-2^i} \\ &\leq \beta^{2\beta} \cdot \left\{ \exp\left[\frac{a2^i}{i} - \ln n\right]\right\}^i \leq \beta^{2\beta} \cdot \left\{ \exp\left[\frac{a2^\beta}{\beta} - \ln n\right]\right\}^i, \end{aligned}$$

where we have used (4.18) and the following standard inequalities for the binomial coefficients

$$\frac{a^b}{b^b} \leq \binom{a}{b} \leq a^b.$$

One has

$$\frac{a2^\beta}{\beta} - \ln n = -\ln n \cdot \frac{\log_2 \log_2 \ln n - \log_2 a}{\beta} \leq -\ln n \cdot \frac{\log_2 \log_2 \ln n}{2 \cdot \log_2 \ln n}.$$

Denoting

$$\gamma = \gamma(n) = \frac{\log_2 \log_2 \ln n}{2 \cdot \log_2 \ln n}$$

we may write, for $i \geq 2$,

$$r_i \leq \beta^{2\beta} \cdot \{\exp(-\gamma \cdot \ln n)\}^i = \frac{\beta^{2\beta}}{n^{i\gamma}}.$$

Clearly, $\gamma \to 0$. Summing up we obtain

$$\sum_{i=2}^{\ell+1} r_i \leq \beta^{2\beta} \cdot \frac{n^{-2\gamma}}{1 - n^{-\gamma}}.$$

The lower bound estimate in (4.16) would now follow once we know that $n^\gamma \to \infty$ and moreover $\dfrac{\beta^\beta}{n^\gamma} \to 0$. This is equivalent to

$$\beta \cdot \log_2 \beta - \gamma \cdot \ln n \to -\infty.$$

Since $\beta < 2\log_2 \ln n$ it is sufficient to show that

$$2\log_2 \ln n \cdot \log_2(2\log_2 \ln n) - \ln n \cdot \frac{\log_2 \log_2 \ln n}{2\log_2 \ln n} \to -\infty.$$

The above expression can be written in the form

$$2\log_2 \ln n \cdot \left[1 + \log_2 \log_2 \ln n \cdot \left[1 - \frac{\ln n}{4(\log_2 \ln n)^2}\right]\right] \tag{4.21}$$

since $\log_2(2\log_2 \ln n) = 1 + \log_2 \log_2 \ln n$. Obviously $\frac{\ln n}{(\log_2 \ln n)^2} \to \infty$, and therefore we see that (4.21) tends to $-\infty$. This completes the proof of Proposition 4.4.1. □

4.5 Simple connectivity of lower random simplicial complex in the medial regime

In order to establish connectivity and simple connectivity of a lower model random simplicial complex in the medial regime we shall consider the cover by closed stars of vertices and apply the Nerve Lemma.

4.5.1 Common neighbours

Recall that *a common neighbour* of a set $S \subset Y$ of vertices in a simplicial complex Y is a vertex $v \in Y - S$ which is connected by an edge to every vertex of S. The following Lemma estimates the probability for the existence of common neighbours; this information will be used below together with the Nerve Lemma to prove the simple connectivity of a medial regime simplicial complex.

Lemma 4.5.1. *Let $0 < \varepsilon \leq 1$ be fixed. Let $Y \in \Omega_n^*$ be a random simplicial complex with respect to the lower measure in the medial regime. Then any set S of*

$$\left\lfloor \frac{\ln n}{(1+\varepsilon)a} \right\rfloor$$

vertices of Y have a common neighbour with probability at least $1 - C \cdot \exp(-\frac{n^{\varepsilon/2}}{2})$. Here

$C > 0$ is a constant independent of n (which however depends on the value of p).

The number a which appears in the statement is defined in (4.4).

Proof. Let $S \subset Y$ be a set of k vertices. A vertex $v \notin S$ is a common neighbour for S with probability $p_v \cdot \prod_{u \in S} p_{uv}$. Hence, a set $S \subset Y$ has no common neighbours in $Y - S$ with probability

$$\prod_{v \notin S} \left(1 - p_v \cdot \prod_{u \in S} p_{uv}\right) \leq \left(1 - p^{k+1}\right)^{n+1-k}.$$

Let $X_k : \Omega_n^* \to \mathbb{Z}$ be the random variable counting the number of k element subsets $S \subset Y$ having no common neighbours in $Y - S$. Using the above inequality, we see that the expectation $\mathbb{E}(X_k)$ is bounded above by

$$\begin{aligned}
\binom{n+1}{k} \cdot \left(1 - p^{k+1}\right)^{n+1-k} &\leq n^k \cdot \exp\left(-(n+1-k) \cdot p^{k+1}\right) \\
&= \exp\left(k \ln n - (n+1-k) \cdot p^{k+1}\right) \\
&\leq C \cdot \exp\left(k \ln n - n \cdot p^{k+1}\right).
\end{aligned}$$

In the final line we have used the fact that $(k-1)p^k$ is bounded for any $k \geq 2$. For n fixed the function $k \mapsto k \ln n - n \cdot p^{k+1}$ is monotone increasing. Using this we find that for $k \leq \frac{\ln n}{(1+\varepsilon)a}$

$$\mathbb{E}(X_k) \leq C \cdot \exp\left(\frac{(\ln n)^2}{(1+\varepsilon)a} - n \cdot e^{-a\left(\frac{\ln n}{(1+\varepsilon)a}\right)}\right) = C \cdot \exp\left(\frac{(\ln n)^2}{(1+\varepsilon)a} - n^{\frac{\varepsilon}{1+\varepsilon}}\right) \leq C \exp(-\frac{n^{\varepsilon/2}}{2}).$$

Hence we obtain

$$\mathbb{P}_n(X_k > 0) \;\leq\; \mathbb{E}(X_k) \;\leq\; C \exp(-\frac{n^{\varepsilon/2}}{2}).$$

This completes the proof. □

Corollary 4.5.2. *Let $Y \in \Omega_n^*$ be a random simplicial complex with respect to the lower measure in the medial regime. Then the complex Y is connected with probability at least*

$$1 - C \exp\left(-\frac{n^{1/2}}{2}\right),$$

where $C > 0$ is a constant depending on p and independent of n.

Proof. Applying Lemma 4.5.1 with $\varepsilon = 1$ we obtain that any two vertices of Y have a common neighbour in Y with probability at least $1 - C\exp(-\frac{n^{1/2}}{2})$. Then obviously any two vertices can be connected by a path in Y, i.e. Y is path-connected with probability at least $1 - C\exp(-\frac{n^{1/2}}{2})$. □

Remark 4.5.3. Clearly, the connectivity depends only on the 1-skeleton and the 1-skeleton of a medial regime random simplicial complex is a random graph. It is well known (from the classical work of Erdős - Rényi) that such random graphs are connected with probability tending to 1. Corollary 4.5.2 gives a quantitative bound on the probability which will be used later to show simple connectivity.

4.5.2 Simple connectivity

Recall that a simplicial complex X is said to be simply connected if it is connected and its fundamental group $\pi_1(X, x_0)$ is trivial. Our goal is to prove the following statement:

Proposition 4.5.4. *A random simplicial complex $Y \in \Omega_n^*$ with respect to the lower probability measure in the medial regime is simply connected, a.a.s.*

The proof will consist of applying the Nerve Lemma (see [6], Theorem 10.6) to the cover $\mathcal{U}$ of Y formed by the closed stars of vertices. Recall that for a vertex $v \in Y$ the closed star $\mathrm{St}(v) \subset Y$ is the union of all closed simplices $\sigma \in Y$ such that $v \in \sigma$. The nerve $\mathcal{N}(\mathcal{U})$ of this cover is the simplicial complex with vertex set identical to the vertex set of Y and a set S of vertices of Y forms a simplex in $\mathcal{N}(\mathcal{U})$ if and only if the intersection

$$\cap_{v \in S} \mathrm{St}(v) \neq \emptyset \tag{4.22}$$

is not empty. Note that this intersection (4.22) is not empty if the set of vertices S has a common neighbour. Rephrasing Lemma 4.5.1 we obtain:

Corollary 4.5.5. *Let $Y \in \Omega_n^*$ be a random simplicial complex with respect to the lower* *probability measure in the medial regime. Let $\mathcal{U}$ denote the cover of Y formed by the closed stars of vertices of Y. Then for any constant $0 < \alpha < 1$, the nerve complex $\mathcal{N}(\mathcal{U})$ contains the full $\left\lfloor \alpha \cdot \log_{(p^{-1})} n \right\rfloor$-dimensional skeleton of the simplex spanned by the vertex set of Y. In particular, the nerve complex $\mathcal{N}(\mathcal{U})$ is $\left(\left\lfloor \alpha \cdot \log_{(p^{-1})} n \right\rfloor - 1\right)$-connected, a.a.s.*

Recall that the parameter $0 < p < 1$ of Lemma 4.5.5 is the one which appears in the definition of the medial regime, see (4.3).

Proof of Proposition 4.5.4. First we recall the Nerve Lemma, see [6], Theorem 10.6:

Lemma 4.5.6. *If Y is a simplicial complex and $\{S_i\}_{i \in I}$ is a family of subcomplexes covering Y such that for any $t \geq 1$ every non-empty intersection $S_{i_1} \cap \cdots \cap S_{i_t}$ is $(k-t+1)$-connected. Then Y is k-connected if and only if the nerve complex $\mathcal{N}(\{S_i\}_{i \in I})$ is k-connected.*

To prove Proposition 4.5.4 we shall apply Lemma 4.5.6 with $k = 1$ to the cover $\{\mathrm{St}(v)\}$ of Y formed by closed stars of vertices $v \in Y$. Each star $\mathrm{St}(v)$ is contractible and the nerve complex $\mathcal{N}(\{\mathrm{St}(v)\})$ is simply connected (see Corollary 4.5.5), a.a.s. To complete the proof we need to show that any nonempty intersection $\mathrm{St}(v) \cap \mathrm{St}(w)$ is connected, a.a.s.

Note that

$$\mathrm{St}(v) \cap \mathrm{St}(w) = \begin{cases} \mathrm{Lk}_Y(v) \cap \mathrm{Lk}_Y(w), & \text{if} \quad (vw) \notin Y, \\ (\mathrm{Lk}_Y(v) \cap \mathrm{Lk}_Y(w)) \cup \mathrm{St}(vw), & \text{if} \quad (vw) \in Y. \end{cases} \tag{4.23}$$

Here (vw) denotes the edge connecting v and w.

We introduce events $A_n, B_n, C_n \subset \Omega_n^*$ defined as follows. $A_n \subset \Omega_n^*$ denotes the set of all simplicial complexes Y such that for any two vertices $v, w \in Y$ the intersection $\mathrm{Lk}_y(v) \cap \mathrm{Lk}_Y(w)$ is connected. $B_n \subset \Omega_n^*$ is defined to be the set of all simplicial complexes Y which have no edges $e \subset Y$ of degree zero, i.e. every edge $e \subset Y$ is incident to a 2-simplex $\sigma \subset Y$. And finally, $C_n \subset \Omega_n^*$ will denote the set of all simplicial complexes Y such that every triple of its vertices has a common neighbour.

We note that any $Y \in A_n \cap B_n \cap C_n$ is simply connected. Indeed, taking the cover by the closed stars of vertices we see that the intersection $\mathrm{St}(v) \cap \mathrm{St}(w)$ is connected; if $(vw) \not\subset Y$ then it follows from the definition of A_n and if $(vw) \subset Y$ then $\mathrm{St}(vw)$ is contractible (and hence connected) and has nontrivial intersection with $\mathrm{Lk}_Y(v) \cap \mathrm{Lk}_Y(w)$ as follows from our assumption $Y \in B_n$; this shows that $\mathrm{St}(v) \cap \mathrm{St}(w)$ is connected. Finally we apply the Nerve Lemma 4.5.6 using our assumption $Y \in C_n$.

To complete the proof we therefore only need to show that $\underline{\mathbb{P}}_n(A_n) \to 1$ and $\underline{\mathbb{P}}_n(B_n) \to 1$ as Lemma 4.5.1 tells us that $\underline{\mathbb{P}}_n(C_n) \to 1$.

Consider two fixed vertices $v, w \in Y$ and consider the intersection $\mathrm{Lk}_Y(v) \cap \mathrm{Lk}_Y(w)$. By Lemma 2.6.2 this intersection is a random simplicial complex with respect to the lower measure with probability parameters $p'_\tau = p_\tau p_{v\tau} p_{w\tau}$, i.e. it is also a lower model random simplicial in the medial regime. By Corollary 4.5.2 the intersection $\mathrm{Lk}_Y(v) \cap \mathrm{Lk}_Y(w)$ is disconnected with probability at most $C \exp(-\frac{n^{1/2}}{2})$ and hence the expected number of pairs of vertices with disconnected $\mathrm{Lk}_Y(v) \cap \mathrm{Lk}_Y(w)$ is bounded above by

$$Cn^2 \exp(-\frac{n^{1/2}}{2}) \to 0.$$

This proves that $\underline{\mathbb{P}}_n(A_n) \to 1$.

The proof of $\underline{\mathbb{P}}_n(B_n) \to 1$ is similar. By Theorem 2.6.1, the link of an edge $e = (vw) \subset Y$ is a random simplicial complex with respect to the lower model with probability parameters

$$p'_\tau = p_\tau p_{v\tau} p_{w\tau} p_{e\tau} \geq p^4$$

and hence the probability that an edge e has empty link is bounded above by

$$(1 - p^4)^{n-1} \leq \exp(-p^4 n)$$

for n large enough. Thus, the expected number of edges $e \subset Y$ with empty links is at most

$$\binom{n+1}{2} \cdot e^{-p^4 n} \to 0,$$

implying $\mathbb{P}_n(B_n) \to 1$ by the first moment method. This completes the proof of Proposition 4.5.4. $\square$

We will show in Corollary 6.5.6 that lower model medial regime random simplicial complexes are actually 2-connected with probability converging to one. The proof of this fact is rather more involved and uses results we will prove about *ample* complexes.

4.6 Vanishing of the Betti numbers

The main result of this section states that homogeneous (see Definition 4.2.1) lower model random simplicial complexes in the medial regime have trivial rational homology in every dimension not exceeding

$$\log_2 \ln n - \log_2 a - 1 - \delta,$$

where $p = e^{-a}$ as in (4.4) and $\delta > 0$ is any constant.

Theorem 4.6.1. *Let $Y \in \Omega_n^*$ be a homogeneous random simplicial complex with respect to the lower probability measure in the medial regime. Then for any constant $\delta > 0$, the rational homology of Y vanishes,*

$$H_j(Y; \mathbb{Q}) = 0,$$

for all

$$0 < j \leq \log_2 \log_{(p^{-1})} n - 1 - \delta,$$

a.a.s.

The proof of Theorem 4.6.1 given below uses Garland's method as described in [4].

Given a graph G we denote by $\mathcal{L} = \mathcal{L}(G)$ the normalised Laplacian of G. We refer the reader to [4] for the definitions. All eigenvalues of $\mathcal{L}$ lie in $[0, 2]$ and the multiplicity of the eigenvalue 0 equals the number of connected components of G. Let $\kappa(G) > 0$ denote the smallest non-zero eigenvalue of $\mathcal{L}$; the quantity $\kappa(G)$ is known as the *spectral gap* of

G.

Given a simplicial complex X and a simplex $\sigma \in X$, let L_σ denote the 1-skeleton of the link $\mathrm{Lk}_X(\sigma)$ and let $\kappa_\sigma = \kappa(L_\sigma)$ denote the spectral gap of the graph L_σ.

The following result is well-known, see [4]:

Theorem 4.6.2. *Let $\ell \geq 0$ be a non-negative integer. If X is a finite pure $(\ell+2)$-dimensional simplicial complex such that for every ℓ-dimensional simplex $\sigma \in X$ the link L_σ is a non-empty connected graph with spectral gap satisfying*

$$\kappa\left(L_\sigma\right) > 1 - \frac{1}{\ell+2},$$

then

$$H^{\ell+1}(X;\mathbb{Q}) = 0.$$

Recall that by Corollary 4.5.2 the lower random complex Y in the medial regime is connected. Thus, Theorem 4.6.1 follows once we have established:

Lemma 4.6.3. *Let $Y \in \Omega_n^*$ be a homogeneous random simplicial complex with respect to the lower probability measure in the medial regime, see (4.3). Then Y has the following property with probability tending to 1 as $n \to \infty$: for every ℓ-dimensional simplex $\sigma \subset Y$, where*

$$0 \leq \ell \leq \log_2 \log_{(p^{-1})} n - 2 - \delta \tag{4.24}$$

the link L_σ is non-empty, connected and its spectral gap satisfies $\kappa_\sigma > 1 - \frac{1}{\ell+2}$.

Proof. Fix a simplex $\sigma \subset \Delta_n$ of dimension ℓ and let $\Delta_{n-\sigma} \subset \Delta_n$ denote the simplex spanned by those vertices of $[n]$ which are not in σ; clearly $\dim \Delta_{n-\sigma} = n - \ell - 1$. Consider a random simplicial complex $Y \in \Omega_n^*$ containing σ. The 1-skeleton L_σ of the link $\mathrm{Lk}_Y(\sigma)$ is a random subgraph of $\Delta_{n-\sigma}$ and according to Theorem 2.6.1 the graph L_σ is a random graph with respect to the lower probability measure with vertex and edge probability parameters given by the formulae

$$p'_v = p_v \cdot \prod_{\tau \subseteq \sigma} p_{v\tau} \quad \text{and} \quad p'_e = p_e \cdot \prod_{\tau \subseteq \sigma} p_{e\tau}. \tag{4.25}$$

Since $p \le p_\tau \le P$ for every simplex τ we obtain the following bounds on the probability parameters p'_v and p'_e of the graph L_σ

$$p^{2^{\ell+1}} \le p'_v \le P^{2^{\ell+1}} \quad \text{and} \quad p^{2^{\ell+1}} \le p'_e \le P^{2^{\ell+1}}. \tag{4.26}$$

Since by assumption Y is homogeneous it follows that the link L_σ is homogeneous as well, i.e. $p'_v = p'_{v'}$ and $p'_e = p'_{e'}$ for any vertices v, v' and edges e, e' of L_σ.

The function f_0^σ counting the number of vertices of L_σ, $Y \mapsto f_0(L_\sigma)$, is a random variable and its expectation $\mathbb{E}(f_0^\sigma)$ satisfies

$$(n-\ell)p^{2^{\ell+1}} \le \mathbb{E}(f_0^\sigma) \le (n-\ell)P^{2^{\ell+1}}.$$

From now on we shall assume, by (4.24), that

$$\ell + 1 \le \log_2 \log_{(p^{-1})} n - 1 - \delta$$

where $\delta > 0$ is a constant. We can write

$$\ell + 1 = \log_2 \log_{(p^{-1})} n - x$$

where $x = x(\ell) \ge 1 + \delta$. Then

$$p^{2^{\ell+1}} = n^{-2^{-x}} \quad \text{and} \quad P^{2^{\ell+1}} = n^{-\lambda 2^{-x}}$$

where $\lambda = \log_p P \in (0,1)$ is a constant. Thus we see that

$$\mathbb{E}(f_0^\sigma) \ge (n-\ell)n^{-2^{-x}} \ge \frac{1}{2} n^{1-2^{-x}} \tag{4.27}$$

and similarly,

$$\mathbb{E}(f_0^\sigma) \leq n^{1-\lambda 2^{-x}}. \tag{4.28}$$

Since f_0^σ is a binomial random variable we may apply Chernoff's inequality (see Corollary 2.3 in [45]) which states that for any $0 < \varepsilon < 3/2$ the probability that f_0^σ deviates from its expectation $\mathbb{E}(f_0^\sigma)$ by more than $\varepsilon\mathbb{E}(f_0^\sigma)$ is at most $2\exp\left(-\frac{\varepsilon^2}{3}\mathbb{E}(f_0^\sigma)\right)$. Therefore, the probability that f_0^σ is smaller than $\frac{1}{4}n^{1-2^{-x}}$ is bounded above by

$$2 \cdot \exp\left(-\frac{n^{1-2^{-x}}}{24}\right) \leq 2 \cdot \exp\left(-n^{1/2}\right).$$

Similarly, the probability that f_0^σ is larger than $2n^{1-\lambda 2^{-x}}$ is smaller than $2 \cdot \exp(-n^{1/2})$. Hence we see that the probability that for some ℓ satisfying

$$\ell + 1 \leq \log_2 \log_{(p^{-1})} n - 1 - \delta$$

the inequality

$$\frac{1}{4}n^{1-2^{-x}} \ \leq \ f_0^\sigma \ \leq \ 2n^{1-\lambda 2^{-x}} \tag{4.29}$$

is violated is smaller than

$$4e^{-n^{1/2}} \cdot (n+1)^{\log_2 \log_{(p^{-1})} n},$$

it is easy to see that this quantity tends to zero as $n \to \infty$ (its logarithm converges to $-\infty$). Thus, asymptotically almost surely, the graph L_σ is an Erdős-Rényi random graph on a number of vertices $N = f_0^\sigma$ satisfying (4.29). The edge probability ρ of L_σ satisfies the inequalities

$$p^{2^{\ell+1}} \leq \rho \leq P^{2^{\ell+1}}.$$

We shall use the following result about the spectral gap of the Erdős-Rényi random graphs which is a corollary of Theorem 1.1 from [42]. Consider a random Erdős-Rényi graph $G \in G(N, \rho)$ such that

$$\rho \geq \frac{(1+\delta)\log N}{N}, \tag{4.30}$$

for some fixed $\delta > 0$. Then for any $c \geq 1$ there exists an integer $N_{c,\delta}$ such that for any $N > N_{c,\delta}$ the graph G is connected and

$$\kappa(G) > 1 - \frac{1}{c} \tag{4.31}$$

with probability at least $1 - N^{-\delta}$.

We shall apply this statement with $c = \ell + 2$ and $\delta = 3\ell$. Using (4.29) we obtain

$$\frac{\rho N}{\log N} \geq \frac{p^{2^{\ell+1}} f_0^\sigma}{\log f_0^\sigma} \geq \frac{1}{2}\frac{n^{-2^{-x}} n^{1-2^{-x}}}{(1-\lambda 2^{-x})\log n + 1} \geq \frac{1}{4}\frac{n^{1-2^{1-x}}}{\log n} \geq \frac{1}{4}\frac{n^{1-2^{-\delta}}}{\log n} \quad \geq 1 + \delta = 3\ell + 1.$$

Hence we see that for any $n \geq M_0$ (where M_0 is an integer depending only on the value of δ) the inequality (4.30) will be violated for a given simplex σ with probability at most

$$n \cdot N^{-3\ell} \leq n \cdot \left(\frac{1}{4} n^{1-2^{-x}}\right)^{-3\ell} \leq n^{-\frac{3\ell}{2}},$$

provided $N \geq \frac{1}{4} n^{1-2^{-x}}$. Here the factor n takes into account the fact that we are applying inequality (4.30) a number of times, for each possible value of N, and the range of values of N is bounded above by $2n^{1-\lambda 2^{-x}} \leq n$ according to (4.29).

Therefore the expected number of simplices σ with $\dim \sigma \leq \log_2 \log_{(p^{-1})} n - 2 - \delta$, for which (4.30) is violated is bounded above by

$$n^{\ell+1} \cdot \left(4e^{-n^{1/2}} \cdot (n+1)^{\log_2 \log_{(p^{-1})} n} + n^{-\frac{3\ell}{2}}\right),$$

a quantity which obviously tends to zero. Thus, with probability tending to 1, the spectral gap inequality (4.31) will be satisfied for all simplices σ in the indicated range of dimensions. This completes the proof of Lemma 4.6.3. □

4.7 Proof of Theorem 4.2.3

The probability that no $(n-2)$-dimensional simplices is included into Y is

$$\prod_{\dim\sigma=n-2}(1-p_\sigma)\leq(1-p)^n$$

which converges to 0 since $0<p<1$ is a constant. This proves statement (1).

The proofs of statements (2) and (3) are based on Theorem 4.2.2 and the duality relation given by Theorem 2.9.9. Indeed, let Y be a random simplicial complex with respect to the upper model in the medial regime, i.e. we assume that the probability parameters p_σ satisfy

$$0<e^{-a}=p\leq p_\sigma\leq P=e^{-A}<1.$$

Consider the dual system of probability parameters $p'_\sigma=1-p_{\check{\sigma}}$ which satisfies

$$0<e^{-A'}=1-P\leq p'_\sigma\leq 1-p=e^{-a'}<1,$$

where a' and A' are defined in (4.5). Next, we use the isomorphism c of Theorem 2.9.9 and the duality for the Betti numbers Corollary 2.9.8. The complex $c(Y)$ is a random simplicial complex in the lower model with respect to the system of probability parameters p'_σ. Hence by Theorem 4.2.2, the dimension of the complex $c(Y)$ satisfies

$$\lfloor\beta(n,A')\rfloor-1\leq\dim c(Y)\leq\beta(n,a')-1+\varepsilon_0,\tag{4.32}$$

a.a.s. where $\varepsilon_0>0$ is an arbitrary constant. Since the maximal dimension d such that $c(Y)$ contains the skeleton $\Delta_n^{(d)}$ equals $n-2-\dim Y$, the inequality (4.32) implies statement (2) of Theorem 4.2.3.

To prove the third statement we observe that the reduced Betti numbers of $c(Y)$ vanish in all dimensions except possibly

$$\log_2\ln n-\log_2 A'-1-\delta<j\leq\log_2\ln n+\log_2\log_2\ln n-\log_2 a'-1+\varepsilon_0,$$

Since $b_j(Y) = b_{n-2-j}(c(Y))$, Corollary 2.9.8, we obtain that the Betti numbers $b_j(Y)$ vanish except possibly for

$$\log_2 \ln n - \log_2 A' + 1 - \delta \, < \, n - j \, \leq \log_2 \ln n + \log_2 \log_2 \ln n - \log_2 a' + 1 + \varepsilon_0.$$

Which completes the proof.

Chapter 5

The Rado complex and infinite random simplicial complexes

5.1 Introduction

In the 1920's, Urysohn constructed a remarkable complete, separable metric space which is known as the *Urysohn space* $\mathcal{U}$. The space $\mathcal{U}$ is *universal* in the sense that it contains an isometric copy of any complete, separable metric space. Additionally, the Urysohn space $\mathcal{U}$ is *homogeneous* in the sense that any partial isometry between its finite subsets can be extended to a global isometry. The properties of universality and homogeneity determine $\mathcal{U}$ uniquely up to isometry, see [73] for a detailed exposition.

The Rado graph Γ is another notable mathematical object, which can also be characterised by its universality and homogeneity. The graph Γ has countably many vertices, it is universal in the sense that any graph with countably many vertices is isomorphic to an induced subgraph of Γ. Moreover, any isomorphism of between finite induced subgraphs of Γ can be extended to the whole Γ (homogeneity). The properties of universality and homogeneity determine Γ uniquely up to isomorphism. One may mention surprising robustness of Γ: removing any finite set of its vertices and edges produces a graph isomorphic to Γ. We refer to the comprehensive survey of Cameron [17] for detailed exposition.

Erdős and Rényi [32] showed that a random graph on countably many vertices has the following characteristic property with probability one: given finitely many distinct vertices $u_1, ..., u_m, v_1, ..., v_n$ there exists a vertex which is adjacent to $u_1, ..., u_m$ and non-adjacent to $v_1, ..., v_m$. It is not difficult to see that the Rado graph Γ is the unique countable graph possessing the characteristic property and hence a random countable graph is isomorphic to Γ with probability 1; this result explains why Γ is sometimes called "the random graph". In [68] Rado suggested a deterministic construction of Γ in which the vertices are labelled by integers $\mathbb{N}$ and a pair of vertices labelled by $m < n$ are connected by an edge if and only if the m-th digit in the binary expansion of n is 1. This same graph construction previously appeared in a paper of W. Ackermann [1] who studied consistence of the axioms of the set theory.

The Rado graph Γ and the Urysohn space $\mathcal{U}$ are related. Any graph determines a metric on the set of its vertices where the distance between a pair of distinct vertices is either 1 (if they are connected by an edge) or 2 (otherwise). Thus, the Rado graph Γ admits an isometric embedding into $\mathcal{U}$; it can be viewed as a restricted version of the Urysohn space limited on metric spaces with the metric taking values in the set $\{0, 1, 2\}$.

In this chapter we study a high-dimensional generalisation of the Rado graph which we call the *Rado simplicial complex* X. X is *universal* in the sense that any countable simplicial complex is an induced subcomplex of X. Additionally, X is *homogeneous*, i.e. any two isomorphic finite induced subcomplexes are related by an automorphism of X. Moreover, we prove that X is the unique simplicial complex (up to isomorphism) which is both universal and homogeneous. The 1-skeleton of X is the Rado graph. We introduce a characteristic property of the Rado complex which we call *ampleness*, see Definition 5.2.2, which generalises the characteristic property of the Rado graph. In Theorem 5.7.1, we show that a random simplicial complex on countably many vertices is isomorphic to X with probability 1. We also give explicit deterministic constructions of the Rado complex in Section 5.3, and show that the geometric realisation of the Rado complex X is homeomorphic to an infinite dimensional simplex in Theorem 5.5.1.

In Section 5.4 we observe several curious properties of X. Showing that X is "robust" to many changes, for example if the set of vertices of X is partitioned into finitely many parts, the simplicial complex induced on at least one of these parts is isomorphic to X and that the link of any finite simplex of X is isomorphic to X.

The Rado complex X can be viewed as the limit of a finite random simplicial complex in the medial regime, introduced in Chapter 4. Informally, we may view finite random simplicial complexes in the medial regime as subcomplexes of the Rado complex X induced on the first n vertices. We expand on this idea in Chapter 6.

Next we comment on relations with the previously known results. Theorem 3 of Rado [68] suggests a construction of a universal simplicial complex but after close examination one finds that Rado's construction is correct only when it is restricted to the class of simplicial complexes of a fixed dimension ℓ having complete $(\ell - 1)$-dimensional skeleta

The 2013 preprint, [14], applies the methods of mathematical logic and model theory to study the geometry of simplicial complexes; it uses language and methods very different from ours. A well-known general construction of model theory is the Fraïssé limit for a class of relational structures possessing certain amalgamation properties, see [41]. The Fraïssé limit construction, when applied to the class of all finite simplicial complexes, produces a simplicial complex F on countably many vertices which is universal and homogeneous, i.e. it is a Rado complex in the terminology of this chapter. The universality of the Fraïssé limit F is stated with respect to finite simplicial complexes, but this is equivalent to the countable version of universality as appears in Definition 5.2.1, justified in Remark 5.2.10.

In [14], Brooke-Taylor and Testa study the group of automorphisms of F and state that any direct limit of finite groups and any metrisable profinite group embeds into the group of automorphisms of F. [14] also contains a proof that the geometric realisation of F is homeomorphic to an infinite-dimensional simplex, a result which we independently establish below in Section 5.5. The authors of [14] also consider a probabilistic approach and claim a 0-1 law for first order theories. We were unable to fully understand the construction of their measure and the related proofs; we suspect that the measure they consider in their Section 5 is related to a special case of the measure constructed below in Section 5.6.

Although the current text and [14] study the same object, the motivation, language, and methods used here are totally different compared to [14]. For us the Rado complex is a stable and interesting simplicial complex; our notion of ampleness is crucial in illustrating its resilience and for studying its topology.

5.2 The definition of the Rado complex

In this section we introduce the primary definition of interest in this chapter, the *Rado complex*, give characterising properties, and prove its uniqueness up to isomorphism.

5.2.1 Basic terminology

Throughout we will let $\Delta_{\mathbb{N}}$ denote the standard countably infinite simplex, i.e. the simplicial complex with vertex set $\mathbb{N} = \{1, 2, \dots\}$ and all non-empty finite subsets of $\mathbb{N}$ as simplices.

Two simplicial complexes are *isomorphic* if there is a bijection between their vertex sets which induces a bijection between the sets of simplices. That is, simplicial complexes X and Y are isomorphic if there exists a bijection $\varphi : V(X) \to V(Y)$ such that $\{x_0, \dots, x_k\}$ defines a simplex in X if and only if $\{\varphi(x_0), \dots, \varphi(x_k)\}$ defines a simplex in Y.

A simplicial subcomplex $Y \subset X$ is said to be *induced* if any simplex $\sigma \in X$ with all its vertices contained $V(Y)$ is a face of Y. The induced subcomplex $Y \subset X$ is completely determined by the set of its vertices, $V(Y) \subset V(X)$. We shall use the notation $Y = X_U$ where $U = V(Y)$.

5.2.2 Universal, homogeneous, and ample complexes

Definition 5.2.1. (1) A countable simplicial complex X is said to be *universal* if any countable simplicial complex is isomorphic to an induced subcomplex of X. (2) We say that X is *homogeneous* if for any two finite induced subcomplexes $X_U, X_{U'} \subset X$ and for any isomorphism $f : X_U \to X_{U'}$ there exists an isomorphism $F : X \to X$ with $F|X_U = f$. (3) A countable simplicial complex X is a *Rado complex* if it is universal and homogeneous.

It is clear that the 1-skeleton of a Rado complex is a Rado graph; the latter can be defined as a universal and homogeneous graph having countably many vertices, see [17].

The following property is a useful criterion of being a Rado complex:

Definition 5.2.2. We call a countable simplicial complex X *ample* if for any finite subset $U \subset V(X)$ and for any simplicial subcomplex $A \subset X_U$ there exists a vertex $v \in V(X) - U$ such that

$$\mathrm{Lk}_X(v) \cap X_U = A. \tag{5.1}$$

Remark 5.2.3. Condition (5.1) can equivalently be expressed as

$$X_{U'} = X_U \cup (vA), \tag{5.2}$$

where $U' = U \cup \{v\}$ and vA denotes the cone with apex v and base A. In literature the cone vA is also sometimes denoted $v * A$, the simplicial join of a vertex v and complex A.

Remark 5.2.4. Suppose that X is a simplicial complex with a countable set of vertices $V(X)$. One may naturally consider exhaustions $U_0 \subset U_1 \subset U_2 \subset \cdots \subset V(X)$ consisting of finite subsets U_n satisfying $\cup U_n = V(X)$. In order to check that X is ample as defined in Definition 5.2.2 it is sufficient to verify that for every $n \geq 0$ and for any subcomplex $A \subset X_{U_n}$ there exists a vertex $v \in V(X) - U_n$ satisfying $\mathrm{Lk}_X(v) \cap X_{U_n} = A$.

Remark 5.2.5. Suppose that X is an ample simplicial complex. Given finitely many distinct vertices $u_1, \ldots, u_m, v_1, \ldots, v_n \in V(X)$, there exists a vertex $z \in V(X)$ which is adjacent to $u_1, \ldots, u_m$ and nonadjacent to $v_1, \ldots, v_n$. To see this we apply Definition 5.2.2 with $U = \{u_1, \ldots, u_m, v_1, \ldots, v_n\}$ and $A = \{u_1, \ldots, u_m\}$. This shows that the 1-skeleton of a Rado complex satisfies the defining property of the Rado graph [17]. This also shows that ampleness is a high dimensional generalizaton of this graph property.

The following property of ample complexes will be useful in the next section.

Lemma 5.2.6. *Let X be an ample complex and let $L' \subset L$ be a pair consisting of a finite simplical complex L and an induced subcomplex L'. Let $f' : L' \to X_{U'}$ be an isomorphism of simplicial complexes, where $U' \subset V(X)$ is a finite subset. Then there exists a finite subset $U \subset V(X)$ containing U' and an isomorphism $f : L \to X_U$ with $f|L' = f'$.*

Proof. It is enough to prove this statement under an additional assumption that L has a single extra vertex, i.e. $V(L) - V(L') = 1$. In this case L is obtained from L' by attaching a cone wA where $w \in V(L) - V(L')$ denotes the new vertex and $A \subset L'$ is a subcomplex (the base of the cone). Applying the defining property of the ample complex to the subset $U' \subset V(X)$ and the subcomplex $f'(A) \subset X_{U'}$ we find a vertex $v \in V(X) - U'$ such that $\mathrm{Lk}_X(v) \cap X_{U'} = f(A)$. We can set $U = U' \cup \{v\}$ and extend f' to the isomorphism $f : L \to X_U$ by setting $f(w) = v$. $\square$

Theorem 5.2.7. *A simplicial complex is Rado if and only if it is ample.*

Proof. Suppose X is a Rado complex, i.e. X is universal and homogeneous. Let $U \subset V(X)$ be a finite subset and let $A \subset X_U$ be a subcomplex of the induced complex. Consider an abstract simplicial complex $L = X_U \cup wA$ which obtained from X_U by adding a cone wA with vertex w and base A where $X_U \cap wA = A$. Clearly, $V(L) = U \cup \{w\}$. By universality, we may find a subset $U' \subset V(X)$ and an isomorphism $g : L \to X_{U'}$. Denoting $w_1 = g(w)$, $A_1 = g(A)$ and $U_1 = g(U)$ we have $X_{U'} = X_{U_1} \cup w_1 A_1$. Obviously, g restricts to an isomorphism $g|X_U : X_U \to X_{U_1}$. By the homogeneity property we can find an isomorphism $F : X \to X$ with $F|X_U = g|X_U$. Denoting $v = F^{-1}(w_1)$ we shall have $X_{U \cup \{v\}} = X_U \cup vA$ as required, see Remark 5.2.3.

Now suppose that X is ample. To show that it is universal consider a simplicial complex L with at most countable set of vertices $V(L)$. We may find a chain of induced subcomplexes $L_1 \subset L_2 \subset \dots$ with $\cup L_n = L$ and each complex L_n has exactly n vertices. Then L_{n+1} obtained from L_n by adding a cone $v_{n+1}A_n$ where v_{n+1} is the new vertex and $A_n \subset L_n$ is a simplicial subcomplex. We argue by induction that we can find a chain of subsets $U_1 \subset U_2 \subset \cdots \subset V(X)$ and isomorphisms $f_n : L_n \to X_{U_n}$ satisfying $f_{n+1}|L_n = f_n$. If U_n and f_n are already found then the next set U_{n+1} and the isomorphism f_{n+1} exist because X is ample: we apply Definition 5.2.2 with $U = U_n$ and $A = f_n(A_n)$ and we set $U_{n+1} = U_n \cup \{v\}$ where v is the vertex given by Definition 5.2.2. The sequence of maps f_n defines an injective map $f : V(L) \to V(X)$ and produces an isomorphism between L and the induced subcomplex $X_{f(V(L))}$.

The fact that any ample complex is homogeneous follows from Lemma 5.2.8 below. We state it in a slightly more general form so that it also implies the uniqueness of Rado complexes. □

Lemma 5.2.8. *Let X and X' be two ample complexes and let $L \subset X$ and $L' \subset X'$ be two induced finite subcomplexes. Then any isomorphism $f : L \to L'$ can be extended to an isomorphism $F : X \to X'$.*

Proof. We shall construct chains of subsets of the sets of vertices $U_0 \subset U_1 \subset \cdots \subset V(X)$ and $U'_0 \subset U'_1 \subset \cdots \subset V(X')$ such that $\cup U_n = V(X)$, $\cup U'_n = V(X')$, $X_{U_0} = L$, $X_{U'_0} = L'$, and $|U_{n+1} - U_n| = 1$, $|U'_{n+1} - U'_n| = 1$. We shall also construct isomorphisms $f_n : X_{U_n} \to X_{U'_n}$ satisfying $f_0 = f$ and $f_{n+1}|X_{U_n} = f_n$. The whole collection $\{f_n\}$ will then define a required isomorphism $F : X \to X'$ with $F|L = f$.

To constructs these objects we shall use the well known back-and-forth procedure. Enumerate vertices $V(X) - V(L) = \{v_1, v_2, \dots\}$ and $V(X') - V(L') = \{v'_1, v'_2, \dots\}$ and start by setting $U_0 = V(L)$, $U'_0 = L'$ and $f_0 = f$. We act by induction and describe U_n, U'_n and f_n assuming that the objects U_i, U'_i and $f_i : U_i \to U'_i$ have been already defined for all $i < n$.

The procedure will depend on the parity of n. For n odd we find the smallest j with $v_j \notin U_{n-1}$ and set $U_n = U_{n-1} \cup \{v_j\}$. Applying Lemma 5.2.6 to the simplicial complexes $L = X_{U_n}$, $L' = X_{U_{n-1}}$ and the isomorphism $f_{n-1} : X_{U_{n-1}} \to X'_{U'_{n-1}}$ we obtain a subset $U'_n \subset V(X')$ containing U'_{n-1} and an isomorphism $f_n : X_{U_n} \to X'_{U'_n}$ extending f_{n-1}.

For n even we proceed in the reverse direction. We find the smallest j with $v'_j \notin U'_{n-1}$ and set $U'_n = U'_{n-1} \cup \{v'_j\}$. Next we applying Lemma 5.2.6 to the simplicial complexes

$L = X'_{U'_n}$, $L' = X'_{U'_{n-1}}$ and the isomorphism $f_{n-1}^{-1} : X'_{U'_{n-1}} \to X_{U_{n-1}}$. We obtain a subset $U_n \subset V(X)$ containing U_{n-1} and an isomorphism $f_n^{-1} : X'_{U'_n} \to X_{U_n}$ extending f_{n-1}^{-1}. □

Corollary 5.2.9. *Any two Rado complexes are isomorphic.*

Proof. This follows from Theorem 5.2.7 with subsequent applying Lemma 5.2.8 with $L = L' = \emptyset$. □

Remark 5.2.10. In Definition 5.2.1 we defined universality with respect to arbitrary countable simplicial subcomplexes. A potentially more restrictive definition dealing only with finite subcomplexes together with homogeneity is in fact equivalent to Definition 5.2.1; this will follow from the arguments used in the proof of Theorem 5.2.7.

5.3 Deterministic constructions of Rado complexes

In Corollary 5.2.9 we prove that if a Rado complex exists then it is unique up to isomorphism. In this section we provide multiple deterministic constructions of the Rado complex.

5.3.1 An inductive construction

One may construct a Rado simplicial complex X inductively as the union of a chain of finite induced simplicial subcomplexes

$$X_0 \subset X_1 \subset X_2 \subset \dots, \quad \cup_{n\geq 0} X_n \;=\; X.$$

Here X_0 is a single point and each complex X_{n+1} is obtained from X_n by first adding a finite set of vertices $v(A)$, labeled by subcomplexes $A \subset X_n$ (including the case when $A = \emptyset$); secondly, we construct the cone $v(A) * A$ with apex $v(A)$ and base A, and thirdly we attach each such cone $v(A) * A$ to X_n along the base $A \subset X_n$. Thus,

$$X_{n+1} = X_n \cup \bigcup_A (v(A) * A). \tag{5.3}$$

To show that the complex $X = \cup_{n\geq 0} X_n$ is ample, i.e. a Rado complex, we refer to Remark 5.2.4 and observe that any subcomplex $A \subset X_n$ the vertex $v = v(A) \in V(X_{n+1})$ satisfies $\mathrm{Lk}_X(v) \cap X_n = A$.

5.3.2 An explicit construction

Here we shall give an explicit construction of a Rado complex X. To describe it we shall use the sequence $\{p_1, p_2, \dots\}$ of all primes in increasing order, where $p_1 = 2$, $p_2 = 3$, etc.

The set of vertices $V(X)$ is the set of all positive integers $\mathbb{N}$. Each simplex of X is uniquely represented by an increasing sequence $a_0 < a_1 < \cdots < a_k$ with certain properties. Subsimplices of $a_0 < a_1 < \cdots < a_k$ are obtained by erasing one or more elements in the sequence.

Definition 5.3.1. (1) A sequence $a_0 < a_1$ is a 1-dimensional simplex of X if and only if p_{a_0}-th binary digit of a_1 is 1. (2) We shall say that an increasing sequence of positive integers $0 < a_0 < a_1 < \cdots < a_k$ represents a simplex of X if all its proper subsequences are in X and additionally the $p_{a_0} p_{a_1} \dots p_{a_{k-1}}$-th binary digit of a_k is 1.

Proposition 5.3.2. *The obtained simplicial complex X is Rado.*

Proof. With any increasing sequence σ of positive integers $0 < a_0 < a_1 < \cdots < a_k$ we associate the product

$$N_\sigma = p_{a_0} p_{a_1} \dots p_{a_k},$$

which is an integer without multiple prime factors. Note that for two such increasing sequences σ and σ' one has $N_\sigma = N_{\sigma'}$ if and only if σ is identical to σ'.

Given a finite subset $U \subset V(X)$ and a simplicial subcomplex $A \subset X_U$, consider the vertex

$$v = \sum_{\sigma \in A} 2^{N_\sigma} + 2^{K_U} \in V(X) \tag{5.4}$$

where

$$K_U = 1 + \prod_{u \in U} p_u.$$

The binary expansion of v has ones exactly on positions N_σ where $\sigma \in F(A)$ and it has zeros on all other positions except an additional 1 at position K_U. Note that $K_U > N_\sigma$ for any simplex $\sigma \subset X_U$. In particular, we see that vertex v defined by (5.4) satisfies $v > w$ for any $w \in U$.

Consider a simplex $\sigma \subset X_U$. By definition, the simplex $v\sigma$ with apex v and base σ lies in X if and only if the N_τ-th binary digit of v is 1 for every $\tau \subseteq \sigma$. We see from (5.4) that this happens if and only if $\sigma \in A$. This means that $\mathrm{Lk}_X(v) \cap X_U = A$ and hence X is a Rado complex. □

5.4 Some simple properties of the Rado complex

Lemma 5.4.1. *Let X be a Rado complex, let $U \subset V(X)$ be a finite set and let $A \subset X_U$ be a subcomplex. Let $Z_{U,A} \subset V(X)$ denote the set of vertices $v \in V(X) - U$ satisfying (5.1). Then $Z_{U,A}$ is infinite and the induced complex on $Z_{U,A}$ is also a Rado complex.*

Proof. Consider a finite set $\{v_1, \dots, v_N\} \subset Z_{U,A}$ of such vertices. One may apply Definition 5.2.2 to the set $U_1 = U \cup \{v_1, \dots, v_N\}$ and to the subcomplex $A \subset X_{U_1}$ to find another vertex v_{N+1} satisfying the condition of Definition 5.2.2. This shows that $Z_{U,A}$ must be infinite.

Let $Y \subset X$ denote the subcomplex induced by $Z_{U,A}$. Consider a finite subset $U' \subset Z_{U,A} = V(Y)$ and a subcomplex $A' \subset X_{U'} = Y_{U'}$. Applying the condition of Definition 5.2.2 to the set $W = U \cup U' \subset V(X)$ and to the subcomplex $A \sqcup A'$ we find a vertex $z \in V(X) - W$ such that

$$\mathrm{Lk}_X(z) \cap X_W = A \cup A'. \tag{5.5}$$

Since $X_W \supset X_U \cup X_{U'}$, the equation (5.5) implies $\mathrm{Lk}_X(z) \cap X_U = A$, i.e. $z \in Z_{U,A}$. Intersection both sides of (5.5) with $X_{U'} = Y_{U'}$ and using $\mathrm{Lk}_Y(z) = \mathrm{Lk}_X(z) \cap Y$ (since Y is an induced subcomplex) we obtain

$$\mathrm{Lk}_Y(z) \cap Y_{U'} = A'$$

implying that Y is Rado. $\square$

Corollary 5.4.2. *Let X be a Rado complex and let Y be obtained from from X by selecting a finite number of simplices F of X and deleting all simplices $\sigma \in X$ which contain simplices from F as their faces. Then Y is also a Rado complex.*

Proof. Let $U \subset V(Y)$ be a finite subset and let $A \subset Y_U$ be a subcomplex. We may also view U as a subset of $V(X)$ and then A becomes a subcomplex of X_U since $Y_U \subset X_U$. The set of vertices $v \in V(X)$ satisfying $\mathrm{Lk}_X(v) \cap X_U = A$ is infinite (by Lemma 5.4.1) and thus we may find a vertex $v \in V(X)$ which is not incident to simplices from the family F. Then $\mathrm{Lk}_Y(v) = \mathrm{Lk}_X(v) \cap Y$ and we obtain $\mathrm{Lk}_Y(v) \cap Y_U = A$. $\square$

Corollary 5.4.3. *Let X be a Rado complex. If the vertex set $V(X)$ is partitioned into a finite number of parts then the induced subcomplex on at least one of these parts is a Rado complex.*

Proof. It is enough to prove the statement for partitions into two parts. Let $V(X) = V_1 \sqcup V_2$ be a partition; denote by X^1 and X^2 the subcomplexes induced by X on V_1 and V_2 correspondingly. Suppose that none of the subcomplexes X^1 and X^2 is Rado. Then for each $i = 1, 2$ there exists a finite subset $U_i \subset V_i$ and a subcomplex $A_i \subset X^i_{U_i}$ such that no vertex $v \in V_i$ satisfies $\mathrm{Lk}_{X^i}(v) \cap X^i_{U_i} = A_i$. Consider the subset $U = U_1 \sqcup U_2 \subset V(X)$ and a subcomplex $A = A_1 \sqcup A_2 \subset X_U$. Since X is Rado we may find a vertex $v \in V(X)$ with $\mathrm{Lk}_X \cap X_U = A$. Then v lies in V_1 or V_2 and we obtain a contradiction, since $\mathrm{Lk}_{X^i}(v) \cap X^i_{U_i} = A_i$. $\square$

Lemma 5.4.4. *In a Rado complex X, the link of every simplex is a Rado complex.*

Proof. Let $Y = \mathrm{Lk}_X(\sigma)$ be the link of a simplex $\sigma \in X$. To show that Y is Rado, let $U \subset V(Y)$ be a subset and let $A \subset Y_U$ be a subcomplex. We may apply the defining property of the Rado complex to the subset $U' = U \cup V(\sigma) \subset V(X)$ and to the subcomplex $A \sqcup \bar{\sigma} \subset X_{U'}$; here $\bar{\sigma}$ denotes the subcomplex containing the simplex σ and all its faces. We obtain a vertex $w \in V(X) - U'$ with $\mathrm{Lk}_X(w) \cap X_{U'} = A \sqcup \bar{\sigma}$ or equivalently, $X_{U' \cup w} = X_{U'} \cup wA$, see Remark 5.2.3. Note that $w \in Y = \mathrm{Lk}_X(\sigma)$ since the simplex $w\sigma$ is in X. Besides, $Y_{U \cup w} = Y_U \cup wA$. Hence we see that the link Y is also a Rado complex. $\square$

5.5 Geometric realisation of the Rado complex

Recall that for a simplicial complex X *the geometric realisation* $|X|$ is the set of all functions $\alpha : V(X) \to [0,1]$ such that the support $\text{supp}(\alpha) = \{v; \alpha(v) \neq 0\}$ is a simplex of X (and hence finite) and $\sum_{v \in X} \alpha(v) = 1$, see [70]. For a simplex $\sigma \in F(X)$ the symbol $|\sigma|$ denotes the set of all $\alpha \in |X|$ with $\text{supp}(\alpha) \subset \sigma$. The set $|\sigma|$ has natural topology and is homeomorphic to the linear simplex lying in an Euclidean space. The *weak topology* on the geometric realisation $|X|$ has as open sets the subsets $U \subset |X|$ such that $U \cap |\sigma|$ is open in $|\sigma|$ for any simplex σ.

Theorem 5.5.1. *The Rado complex is isomorphic to a triangulation of the simplex $\Delta_{\mathbb{N}}$. In particular, the geometric realisation $|X|$ of the Rado complex is homeomorphic to the infinite dimensional simplex $|\Delta_{\mathbb{N}}|$.*

The following general statement about subdivisions of simplicial complexes will be used in the proof of Theorem 5.5.1.

Lemma 5.5.2. *Let (K, L) be a pair consisting of a finite simplicial complex K and its subcomplex L. Then there is a subdivision K_0 of K with the following properties:*

1. *K_0 contains L as a subcomplex, i.e. no simplex of L is subdivided;*
2. *L is an induced subcomplex of K_0;*
3. *The number of new vertices $|V(K_0) - V(K)|$ equals the number of external simplices of L in K of positive dimension.*

Proof. Recall that a simplex $\sigma \subset K$ is said to be an external simplex of L if $\sigma \not\subset L$ but all proper faces of σ lie in L. A characteristic property of an induced subcomplex is that all its external simplices are zero-dimensional. Based on this remark one can prove this result by an inductive argument as follows.

Suppose the number of external simplices of L in K of positive dimension is $N > 0$ and let σ be such a simplex, i.e. $\sigma \subset K$, $\sigma \not\subset L$, $\partial\sigma \subset L$, and $\dim \sigma > 0$. We introduce a new vertex v_σ in the centre of σ and replace the closed star $\mathrm{St}(\sigma)$ by the simplicial cone $v_\sigma * (\mathrm{Lk}(\sigma) * \partial\sigma)$. We obtain a subdivision K_1 of K having one extra vertex (lying outside L) such that the number of external simplices of positive dimension of L in K_1 is $N-1$. Repeating this process N times, we arrive at the desired subdivision. At each step the number of external simplices of positive dimension is reduced by one.

□

Lemma 5.5.3. *Let X be a Rado complex. Then there exists a sequence of finite subsets $U_0 \subset U_1 \subset U_2 \subset \cdots \subset V(X)$ such that $\cup U_n = V(X)$ and for any $n = 0, 1, 2, \ldots$ the induced simplicial complex X_{U_n} is isomorphic to a triangulation L_n of the standard simplex Δ_{n+1} of dimension n. Moreover, for any n the complex L_n is naturally an induced subcomplex of L_{n+1} and the isomorphisms $f_n : X_{U_n} \to L_n$ satisfy $f_{n+1}|X_{U_n} = f_n$.*

Proof. Let $V(X) = \{v_0, v_1, \ldots\}$ be a labelling of the vertices of X. One constructs the subsets U_n and complexes L_n by induction stating from $U_0 = \{v_0\}$ and $L_0 = \{v_0\}$. Suppose that the sets U_i and complexes L_i with $i \leq n$ have been constructed. Consider the subset $U'_{n+1} = U_n \cup \{v_i\} \subset V(X)$ where $i \geq 0$ is the smallest integer satisfying $v_i \notin U_n$. The induced simplicial complex $X_{U'_{n+1}}$ has dimension $\leq n+1$. Clearly, the complex $X_{U'_{n+1}}$ has the form $X_{U_n} \cup (v_i * A)$ for some subcomplex $A \subset X_{U_n}$. By applying Lemma 5.5.2 to the simplicial pair

$$(v_i * X_{U_n}, X_{U_n} \cup (v_i * A)) = \left(v_i * X_{U_n}, X_{U'_n}\right)$$

we obtain a subdivision L_{n+1} of the cone $v_i * X_{U_n}$ which contains $X_{U'_{n+1}}$ as an induced subcomplex. The map

$$\mathrm{id} * f_n : v_i * X_{U_n} i * L_n$$

is a simplicial isomorphism. By induction L_n is a subdivison of a simplex of dimension n and hence the simplicial complex L_{n+1} is a subdivison of a simplex of dimension $n+1$ containing L_n as a subdivision of a face of codimension one.

We shall apply Lemma 5.2.6 to the abstract simplicial complexes $X_{U'_{n+1}}$ and L_{n+1}. It gives a subset $U_{n+1} \subset V(X)$ containing U'_{n+1} and an isomorphism $f_{n+1} : X_{U_{n+1}} \to L_{n+1}$ satisfying $f_{n+1}|X_{U_{n+1}} = f_n$.

Obviously, $\cup U_n = V(X)$. This completes the proof. □

Proof. (Theorem 5.5.1) It follows from the previous Lemma. □

Corollary 5.5.4. *The geometric realisation $|X|$ of the Rado complex is contractible.*

Proof. Corollary follows Theorem 5.5.1 as the infinite simplex is contractible. We also give a short independent proof below.

Let X be a Rado complex. By the Whitehead theorem we need to show that any continuous map $f : S^n \to X$ is homotopic to the constant map. By the Simplicial Approximation theorem f is homotopic to a simplicial map $g : S^n \to X$. The image $g(S^n) \subset X$ is a finite subcomplex. Applying the property of Definition 5.2.2 to the set of vertices U of $g(S^n)$ and to the subcomplex $A = X_U$ we find a vertex $v \in V(X) - U$ such that the cone vA is a subset of X. Since the cone is contractible, we obtain that g, which is equal the composition $S^n \to A \to vA \to X$, is null-homotopic. □

Remark 5.5.5. The geometric realisation of a simplicial complex carries another natural topology, the metric topology, see [70]. The geometric realisation of X with the metric topology is denoted $|X|_d$. While for finite simplicial complexes the spaces $|X|$ and $|X|_d$ are homeomorphic, it is not true for infinite complexes in general. For the Rado complex X the spaces $|X|$ and $|X|_d$ are not homeomorphic. Moreover, in general, the metric topology is not invariant under subdivisions, see [63], where this issue is discussed in detail. We do not know if for the Rado complex X the spaces $|X|_d$ and $|\Delta_{\mathbb{N}}|_d$ are homeomorphic.

5.6 Infinite random simplicial complexes

We show in Section 5.7 that a random infinite simplicial complex is a Rado complex with probability 1, in a certain regime. In this section we prepare the grounds and describe the probability measure on the set of infinite simplicial complexes.

Let L be a finite simplicial complex. Suppose that with each simplex $\sigma \subset L$ one has associated a probability parameter $p_\sigma \in [0, 1]$. We shall use the notation $q_\sigma = 1 - p_\sigma$. Given a subcomplex $A \subset L$ we may consider the set $E(A|L)$ consisting of all simplices of L which are not in A but such that all their proper faces are in A. The simplices of $E(A|L)$ are called *external for A in L*. As an example we mention that any vertex $v \in L - A$ is an external simplex, $v \in E(A|L)$.

With a subcomplex $A \subset L$ one may associate the following real number

$$p(A) = \prod_{\sigma \in F(A)} p_\sigma \cdot \prod_{\sigma \in E(A|L)} q_\sigma \in [0, 1]. \tag{5.6}$$

For example, taking $A = \emptyset$ we obtain $p(\emptyset) = \prod_{v \in V(L)} q_v$, the product is taken with respect to all vertices v of L.

The following result will be used to show that the measure we will construct for infinite random simplicial complexes is *consistent* in the way it projects onto finite simplicial complexes of different dimensions

Lemma 5.6.1. *One has $\sum_{A \subset L} p(A) = 1$, where A runs over all subcomplexes of L, including the empty subcomplex.*

Proof. Given in Appendix A.3. □

Let $\Delta = \Delta_{\mathbb{N}}$ denote the simplex spanned by the set $\mathbb{N} = \{1, 2, \dots\}$ of positive integers. We shall denote by Ω the set of all simplicial subcomplexes $X \subset \Delta$. Each simplicial complex $X \in \Omega$ has a finite or countable set of vertices $V(X) \subset \mathbb{N}$ and any finite or countable simplicial complex is isomorphic to one of the complexes $X \in \Omega$.

Let Δ_n denote the simplex spanned by the vertices $[n] = \{1, 2, \dots, n\} \subset \mathbb{N}$. Let Ω_n denote the set of all subcomplexes $Y \subset \Delta_n$. One has the projection

$$\pi_n : \Omega \to \Omega_n, \quad X \mapsto X \cap \Delta_n.$$

In other words, for $X \in \Omega$ the complex $\pi_n(X) \subset \Delta_n$ is the subcomplex of X induced on the vertex set $[n] \subset \mathbb{N}$.

For a subcomplex $Y \subset \Delta_n$ we shall consider the set

$$Z(Y, n) = \pi_n^{-1}(Y) = \{X \in \Omega; X \cap \Delta_n = Y\} \subset \Omega. \tag{5.7}$$

Note that for $n = n'$ the sets $Z(Y, n)$ and $Z(Y', n')$ are either identical (if and only if $Y = Y'$) of disjoint; for $n > n'$ the intersection $Z(Y, n) \cap Z(Y', n')$ is nonempty if and only if $Y \cap \Delta_{n'} = Y'$ and in this case $Z(Y, n) \subset Z(Y', n')$. Note also that for $n > n'$ and $Y \cap \Delta_{n'} = Y'$ one has

$$Z(Y', n') = \bigsqcup_j Z(Y_j, n) \tag{5.8}$$

where $Y_j \subset \Delta_n$ are all subcomplexes with $Y_j \cap \Delta_{n'} = Y'$; one of these subcomplexes Y_j coincides with Y.

Let $\mathcal{A}$ denote the set of all subsets $Z(Y, n) \subset \Omega$ and $\emptyset$. The set $\mathcal{A}$ is *a semi-ring*, see [51], i.e. $\mathcal{A}$ is $\cap$-closed and for any $A, B \in \mathcal{A}$ the difference $B - A$ is a finite union of mutually disjoint sets from $\mathcal{A}$. We shall denote by $\mathcal{A}'$ the σ-algebra generated by $\mathcal{A}$.

Example 5.6.2. *Let $U \subset \mathbb{N}$ be a finite subset and let L be a simplicial complex with vertex set $V(L) \subset U$. Then the set $\{X \in \Omega; X_U = L\}$ is the union of finitely many elements of the semi-ring $\mathcal{A}$ and in particular, $\{X \in \Omega; X_U = L\} \in \mathcal{A}'$. Indeed, let n be an integer such that $U \subset [n]$ and let $Y_j \subset \Delta_n$, for $j \in I$, be the list of all subcomplexes of Δ_n satisfying $(Y_j)_U = L$; in other words, Y_j induces L on U. Then the set $\{X \in \Omega; X_U = L\}$ is the union $\sqcup_{j \in I} Z(Y_j, n)$.*

Next we define a function $\mu : \mathcal{A} \to \mathbb{R}$ as follows. Fix for every simplex $\sigma \subset \Delta_{\mathbb{N}}$ a probability parameter $p_\sigma \in [0, 1]$. The function

$$F(\Delta_{\mathbb{N}}) \to [0, 1], \quad \sigma \mapsto \{p_\sigma\} \tag{5.9}$$

will be called the *system of probability parameters.* Here σ runs over all simplices $\sigma \in F(\Delta_{\mathbb{N}})$. We shall use the notation $q_\sigma = 1 - p_\sigma$.

For an integer $n \geq 0$ and a subcomplex $Y \subset \Delta_n$ define

$$\mu(Z(Y, n)) = \prod_{\sigma \in F(Y)} p_\sigma \cdot \prod_{\sigma \in E(Y|\Delta_n)} q_\sigma. \tag{5.10}$$

Let us show that μ is additive. We know that the set $Z(Y, n)$ equals the disjoint union

$$Z(Y, n) \;=\; \sqcup_{j \in I} Z(Y_j, n + 1) \tag{5.11}$$

where Y_j are all subcomplexes of Δ_{n+1} satisfying $Y_j \cap \Delta_n = Y$. One of these subcomplexes Y_{j_0} equals Y and the others contain the vertex $(n + 1)$ and have the form

$$Y_j = Y \cup ((n + 1) * A_j)$$

where $A_j \subset Y$ is a subcomplex. In other words, all complexes Y_j with $j \neq j_0$ are obtained from Y by adding a cone with apex $n + 1$ over a subcomplex $A_j \subset Y$. Clearly, any subcomplex $A_j \subset Y$ may occur, including the empty subcomplex $A_j = \emptyset$.

Applying the definition (5.10) we have

$$\mu(Z(Y, n + 1)) = \mu(Z(Y, n)) \cdot q_{n+1},$$

and for $j \neq j_0$,

$$\mu(Z(Y_j, n + 1) = \mu(Z(Y, n)) \cdot p_{n+1} \cdot \prod_{\sigma \in F(A_j)} p'_\sigma \cdot \prod_{\sigma \in E(A_j|Y)} q'_\sigma, \tag{5.12}$$

where $n + 1$ denotes the new added vertex and p'_σ denotes the probability parameter $p_{(n+1)\sigma}$ associated to the simplex $(n + 1) * \sigma$ (the cone over σ with apex $n + 1$); besides,

$q'_\sigma = 1 - p'_\sigma$. Hence we obtain, using Lemma 5.6.1:

$$\begin{aligned}
&\sum_{j \in I} \mu(Z(Y_j, n+1)) \\
&= \mu(Z(Y,n)) \cdot \left\{ q_{n+1} + p_{n+1} \cdot \left[\sum_{A_j \subset Y} \prod_{\sigma \in F(A_j)} p'_\sigma \cdot \prod_{\sigma \in E(A_j|Y} q'_\sigma \right] \right\} \\
&= \mu(Z(Y,n)).
\end{aligned}$$

Thus we see that μ is additive with respect to relations of type (5.11). But obviously, by (5.8), these relations generate all additive relations in $\mathcal{A}$. This implies that μ is additive.

Note that Ω can be naturally viewed as the inverse limit of the finite sets Ω_n, i.e. $\Omega = \varprojlim \Omega_n$. Introducing the discrete topology on each Ω_n we obtain the inverse limit topology on Ω and with this topology Ω is compact and totally disconnected; it is homeomorphic to the Cantor set. The sets $Z(Y,n) \subset \Omega$ are open and closed in this topology, hence they are compact.

Next we apply Theorem 1.53 from [51] to show that μ extends to a probability measure on the σ-algebra $\mathcal{A}'$ generated by $\mathcal{A}$. This theorem requires for μ to be additive, σ-subadditive and σ-finite. By Theorem 1.36 from [51], σ-subadditivity is equivalent to σ-additivity. Recall that σ-additivity means that for $A = \sqcup_i A_i$ (disjoint union of countably many elements of $\mathcal{A}$) one has $\mu(A) = \sum_i \mu(A_i)$. In our case, since the sets $A_i \subset \Omega$ are open and closed and since Ω is compact, any representation $A = \sqcup_i A_i$ must be finite and hence σ-additivity of μ follows from additivity.

For fixed n we have $\Omega = \sqcup Z(Y,n)$ where Y runs over all subcomplexes of Δ_n (including $\emptyset$). Using additivity of μ and applying Lemma 5.6.1, we have $\mu(\Omega) = \sum_{Y \subset \Delta_n} \mu(Z(Y,n)) = 1$. This shows that μ is σ-finite and hence by Theorem 1.53 from [51] μ extends to a probability measure on $\mathcal{A}'$. The extended measure on $\mathcal{A}'$ will be denoted by the same symbol μ.

Example 5.6.3. *As in Example 5.6.2, let $U \subset \mathbb{N}$ be a finite subset and let L be a* *simplicial complex with vertex set $V(L) \subset U$. Then*

$$\mu(\{X \in \Omega; X_U = L\}) = \prod_{\sigma \in F(L)} p_\sigma \cdot \prod_{\sigma \in E(L|\Delta_U)} q_\sigma, \tag{5.13}$$

where Δ_U denotes the simplex spanned by U.

5.7 Every random simplicial complex in the medial regime is Rado

In this section we prove that an infinite random simplicial complex in the medial regime is a Rado complex with probability one.

Recall that a system of probability parameters p_σ, see (5.9), is in the *medial regime* if there exist $0 < p < P < 1$ such that the probability parameter p_σ satisfies $p_\sigma \in [p, P]$ for any simplex $\sigma \in F(\Delta_{\mathbb{N}})$.

Theorem 5.7.1. *A random simplicial complex with countably many vertices in the medial regime is a Rado complex, with probability one.*

Proof. For a finite subset $U \subset \mathbb{N}$ and for a simplicial subcomplex $A \subset \Delta_U$ of the simplex Δ_U consider the set

$$\Omega^{U,L} = \{X \in \Omega; X_U = L\}. \tag{5.14}$$

This set belongs to the σ-algebra $\mathcal{A}'$ and has positive measure, see Example 5.6.3.

Consider also the subset $\Omega^{U,L,A,v} \subset \Omega^{U,L}$ consisting of all subcomplexes $X \in \Omega$ satisfying $X_{U \cup v} = L \cup vA$. Here $A \subset L$ is a subcomplex and $v \in \mathbb{N} - U$.

The conditional probability equals

$$\mu(\Omega^{U,L,A,v}|\Omega^{U,L}) = p_v \cdot \prod_{\sigma \in F(A)} p_{v\sigma} \cdot \prod_{\sigma \in E(A|L)} q_{v\sigma} \geq p^{|F(A)|}(1-P)^{|E(A|L)|} > 0,$$

see (5.13). Note that the events $\Omega^{U,L,A,v}$, conditioned on $\Omega^{U,L}$ for various v, are independent and the sum of their probabilities is ∞. Hence we may apply the Borel-Cantelli Lemma (see [51], page 51) to conclude that the set of complexes $X \in \Omega^{U,L}$ such that $X_{U\cup v} = L \cup vA$ for infinitely many vertices v has full measure in $\Omega^{U,L}$.

By taking a finite intersection with respect to all possible subcomplexes $A \subset L$ this implies that the set $\Omega_*^{U,L} \subset \Omega^{U,L}$ of simplicial complexes $X \in \Omega^{U,L}$ such that for any subcomplex $A \subset L$ there exists infinitely many vertices v with $X_{U\cup v} = L \cup vA$ has full measure in $\Omega^{U,L}$.

Since $\Omega = \cap_U \cup_{L\subset\Delta_U} \Omega^{U,L}$ (where $U \subset \mathbb{N}$ runs over all finites subsets) we obtain that the set $\cap_U \cup_{L\subset\Delta_U} \Omega_*^{U,L}$ has measure 1 in Ω. But the latter set $\cap_U \cup_{L\subset\Delta_U} \Omega_*^{U,L}$ is exactly the set of all Rado simplicial complexes, see Lemma 5.4.1. $\square$

Chapter 6

Ample simplicial complexes

6.1 Introduction

In this chapter we study a special class of (finite) simplicial complexes that are *stable* and *resilient*, in the sense that small alterations have limited impact on its global properties (such as connectivity and higher connectivity). These stable and resilient complexes can be viewed as finite approximations to the Rado complex described in Chapter 5.

We will call such complexes r-ample, where $r \geq 1$ is an integer characterising the level of ampleness. The Rado complex is the only simplicial complex on countably many vertices which is ∞-ample. The finite simplicial complexes which we study here will have a limited amount of ampleness and thus a limited amount of indestructibility. The formal definition of r-ampleness requires the existence of all possible extensions of simplicial subcomplexes of size at most r, details given in Definition 6.2.1.

We will show in Proposition 6.5.1 that the lower model medial regime random simplicial complexes of Chapter 4 are r-ample, with probability tending to one. We compare this to Theorem 5.7.1, which when translated into our new terminology states that every infinite medial regime random simplicial complex is ∞-ample.

It was proven in Chapter 4 that the medial regime (lower model) random simplicial complex is simply connected and has vanishing Betti numbers in dimensions $\leq \ln \ln n$. For these reasons one expects that any r-ample simplicial complexes is highly connected, for large r – this question is discussed in detail in Section 6.4.

Analogues of this ampleness property have been studied for graphs, hypergraphs, tournaments, and other structures, in combinatorics and in mathematical logic. In the literature a variety of terms have been used: *r-existentially completeness*, *r-existentially closedness*, *r-e.c.* for short [12, 19], and also the *Adjacency Axiom r* [8, 9], an *extension property* [34], *property P(r)* [10, 16], as the *Alice's Restaurant Axiom* [71, 75], and sometimes just as *random*. Here we use the term *r-ample*, in keeping with Chapter 5.

The plan is as follows. In Section 6.2 we give the main definition and discuss several examples. In Section 6.3 we discuss the resilience of r-ample complexes; our main result, Theorem 6.3.1, gives a bound on the number of simplices one can remove so that the level of ampleness by at most k. A significant role in this estimate plays the Dedekind number which equals the number of simplicial complexes on k vertices; good asymptotic approximations for the Dedekind number are known, see Section 6.2.

In Section 6.4 we show that r-ample simplicial complexes are simply connected and 2-connected, for suitable values of r. Note that the Rado complex is contractible and hence one expects that any r-ample complex is k-connected for $r > r(k)$, for some $r(k) < \infty$. We do not know if this is true in general, however we are able to analyse the cases $k = 1$ and $k = 2$.

In Section 6.5 we show that for every $r \geq 5$ and for any $n \geq r2^r2^{2^r}$, there exists an r-ample simplicial complex having exactly n vertices via a probabilistic argument, see Proposition 6.5.4. Finally, in Section 6.6 we construct an explicit family, in the spirit of Paley graphs[32], of r-ample simplicial complexes on $\exp(O(r2^r))$ vertices.

6.2 Definitions and basic properties

6.2.1 r-ampleness

We begin by fixing our notation. If $U \subseteq V(X)$ is a subset we denote by X_U the *induced subcomplex* on U, i.e., $V(X_U) = U$ and a subset of vertices of U forms a simplex in X_U if and only if it is a simplex in X.

An *embedding* of a simplicial complex A into X is an isomorphism between A and an induced subcomplex of X.

We define the *join* of two simplicial complexes X and Y, denoted $X * Y$, as the simplicial complex with vertex set $V(X) \sqcup V(Y)$ with the simplices of the join being simplices of the complexes X and Y as well as those of the form $\sigma * \tau$ where $\sigma \in X$ and $\tau \in Y$.

Here is our main definition.

Definition 6.2.1. Let $r \geq 1$ be an integer. A nonempty simplicial complex X is said to be *r-ample* if for each subset $U \subseteq V(X)$ with $|U| \leq r$ and for each subcomplex $A \subseteq X_U$ there exists a vertex $v \in V(X) - U$ such that

$$\mathrm{Lk}_X(v) \cap X_U \;=\; A. \tag{6.1}$$

We say that X is *ample or ∞-ample* if it is r-ample for every $r \geq 1$.

Recall that this is the same as Definition 5.2.2 with the additional condition that the vertex set U be of cardinality at most r. It's clear that r-ampleness depends only on the r-dimensional skeleton.

The condition (6.1) can equivalently be expressed as

$$X_{U \cup \{v\}} \;=\; X_U \cup (v * A). \tag{6.2}$$

Obviously, no finite simplicial complex can be ∞-ample. In Chapter 5 it was shown that there exists a unique, up to isomorphism, ∞-ample simplicial complex X on countably many vertices, see Theorem 5.7.1.

To be 1-ample a simplicial complex must have no isolated vertices and no vertices connected to all other vertices. A 1-ample complex has at least 4 vertices and Figure 6.1 shows two such examples.

A 2-ample complex is connected since for any pair of vertices there must exist a vertex connected to both, i.e. the complex must have diameter ≤ 2. A 2-ample complex is also

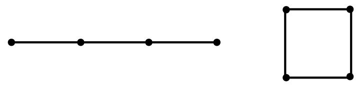

Figure 6.1: 1-ample complexes.

twin-free in the sense that no two vertices have exactly the same link. The following example shows that a 2-ample simplicial complex is not necessarily simply connected.

Example 6.2.2. *Consider a 2-dimensional simplicial complex X having 13 vertices labelled by integers $0, 1, 2, \dots, 12$. A pair of vertices i and j is connected by an edge if and only if the difference $i - j$ is a square modulo 13, i.e. when*

$$i - j \equiv \pm 1, \pm 3, \pm 4 \mod 13.$$

The 1-skeleton of X is a well-known Paley graph of order 13. Next we add 13 triangles

$$i, i+1, i+4, \quad \text{where} \quad i = 0, 1, \dots, 12.$$

We claim that the obtained complex X is 2-ample. The verification amounts to the following: for any two vertices, there exists others adjacent to both, neither, only one, and only the other. Additionally, any edge lies both on a single filled and unfilled triangles. Indeed, an edge $i, i+1$ lies in the triangle $i, i+1, i+4$ (filled) as well as in the triangle $i-3, i, i+1$ (unfilled).

We note that X can be obtained from the triangulated torus with 13 vertices, 39 edges and 26 triangles (see Figure 6.2) by removing 13 white triangles of type $i, i+3, i+4$. From this description it is obvious that X collapses onto a graph and calculating the Euler characteristic we find $b_0(X) = 1$, $b_1(X) = 14$ and $b_2(X) = 0$.

6.2.2 Dimension and size

The following Lemma gives an equivalent criterion for r-ampleness.

Lemma 6.2.3. *A simplicial complex X is r-ample if and only if for every pair (A, B)*

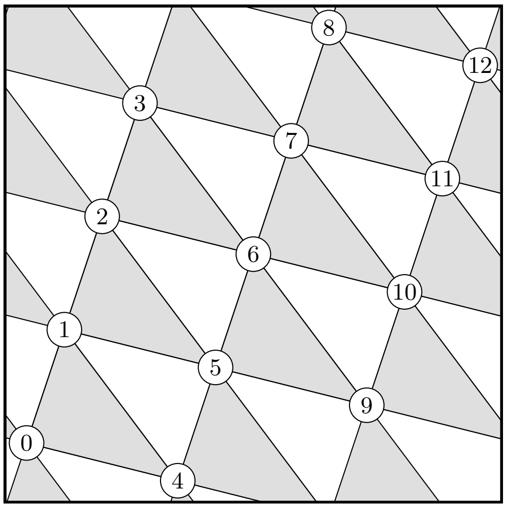

Figure 6.2: The simplicial complex of Example 6.2.2 can be obtained from the triangulated torus with 13 vertices, 39 edges and 26 triangles, by removing 13 triangles of type $\{i, i+3, i+4\}$.

consisting of a simplicial complex A and an induced subcomplex B of A, satisfying $|V(A)| \leq r+1$, and for every embedding f_B of B into X, there exists an embedding f_A of A into X extending f_B.

Proof. Clearly the property described in Lemma 6.2.3 implies r-ampleness and we only need to show the inverse. Suppose that X is r-ample and let (A, B) be a pair consisting of a simplicial complex A with $|V(A)| \leq r+1$ and its induced subcomplex B. We can find a chain of subcomplexes

$$B = B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_k = A$$

where each subcomplex B_{i+1} is obtained from B_i by adding a vertex v_{i+1} and attaching a cone $v_{i+1} * Y_i$ where $Y_i \subset B_i$ is a subcomplex. Here $V(B_i) \leq r$ for any i. Once $B = B_0$ is identified with an induced subcomplex of X we may apply inductively the definition to extend this embedding to an embedding of A. $\square$

Applying Lemma 6.2.3 in the case when B is a single vertex, we obtain:

Corollary 6.2.4. *If X is r-ample then any simplicial complex on at most $r+1$ vertices can be embedded into X.*

Corollary 6.2.5. *The dimension of an r-ample simplicial complex X is at least r.*

We shall denote by $M'(n)$ the number of simplicial complexes with vertices from the set $\{1, 2, \ldots, n\}$. The number $M'(n) + 1 = M(n)$ is known as *the Dedekind number*, see [50], it equals the number of monotone Boolean functions of n variables and has some other combinatorial interpretations, being also equal to the number of antichains in the set of n elements. A few first values of *"the reduced Dedekind number"* $M'(n)$ are $M'(1) = 2$, $M'(2) = 5$, $M'(3) = 19$. For general n, $M'(n)$ admits estimates

$$\binom{n}{\lfloor n/2 \rfloor} \le \log_2(M'(n)) \le \binom{n}{\lfloor n/2 \rfloor}\left(1 + O\left(\frac{\log n}{n}\right)\right). \tag{6.3}$$

The lower bound in (6.3) is easy: one counts only the simplicial complexes having the full $\lfloor n/2 \rfloor$ skeleton; the upper bound in (6.3) is obtained in [50]. We shall also mention that

$$\binom{n}{\lfloor n/2 \rfloor} \sim \sqrt{\frac{2}{\pi n}} \cdot 2^n, \tag{6.4}$$

as follows from the Stirling formula. Thus,

$$\log_2 \log_2(M'(n)) = n - \frac{1}{2}\log_2 n + O(1). \tag{6.5}$$

Corollary 6.2.6. *An r-ample simplicial complex contains at least*

$$M'(r) + r \ge 2^{\binom{r}{\lfloor r/2 \rfloor}} + r$$

vertices.

Proof. Let X be an r-ample complex. Using Lemma 6.2.4 we can embed into X an $(r-1)$-dimensional simplex Δ having r vertices. Applying Definition 6.2.1, for every subcomplex A of Δ we can find a vertex v_A in the complement of Δ having A as its link intersected with Δ. The number of subcomplexes A is $M'(r)$ and we also have r vertices of Δ which gives the estimate. $\square$

6.3 Resilience of ample complexes

In this section we present a few results characterising *resilience* of r-ample simplicial complexes: small perturbations to the complex reduce its ampleness in a controlled way and hence many geometric properties remain.

The perturbations that we have in mind are as follows. If X is a simplicial complex and $\mathcal{F}$ is a finite set of simplices of X, one may consider the simplicial complex Y obtained from X by removing all simplices of $\mathcal{F}$ as well as all simplices which have faces belonging to $\mathcal{F}$. We shall say that Y is *obtained* from X by *removing* the set of simplices $\mathcal{F}$.

We are interested in situations when Y preserves certain properties of X despite the "damage" caused by removing the family of simplices $\mathcal{F}$. We will characterise the size of $\mathcal{F}$ by two numbers: $|\mathcal{F}|$ (the cardinality of $\mathcal{F}$) and $\dim(\mathcal{F}) = \sum_{\sigma\in\mathcal{F}} \dim\sigma$ (the *total dimension* of $\mathcal{F}$).

Theorem 6.3.1. *Let X be an r-ample simplicial complex and let Y be obtained from X by removing a set $\mathcal{F}$ of simplices. Then Y is $(r-k)$-ample provided that*

$$|\mathcal{F}| + \dim(\mathcal{F}) < M'(k) + k. \tag{6.6}$$

In particular, the complex Y is $(r-k)$-ample if

$$|\mathcal{F}| + \dim(\mathcal{F}) < 2^{\binom{k}{\lfloor k/2\rfloor}} + k. \tag{6.7}$$

Proof. Without loss of generality we may assume that $\mathcal{F}$ forms an anti-chain, i.e. no simplex of $\mathcal{F}$ is a proper face of another simplex of $\mathcal{F}$. Indeed, if $\sigma_1 \subset \sigma_2$, where $\sigma_1, \sigma_2 \in \mathcal{F}$, we can remove σ_2 from $\mathcal{F}$ without affecting the complex Y.

Consider a vertex $v \in V(Y)$ and its links $\mathrm{Lk}_Y(v) \subset \mathrm{Lk}_X(v)$ in Y and in X, correspondingly. Let $\mathcal{F}_v$ denote the set of simplices $\sigma \subset \mathrm{Lk}_X(v)$ such that either $\sigma \in \mathcal{F}$ or $v\sigma \in \mathcal{F}$. It follows directly from the definitions that $\mathrm{Lk}_Y(v)$ is obtained from $\mathrm{Lk}_X(v)$ by removing the set of simplices $\mathcal{F}_v$. Our goal is to be able to pick v such that Definition 6.2.1 is satisfied for any vertex set of cardinality at most $r-k$.

Let W_0 denote the set of 0-dimensional simplices in $\mathcal{F}$, and $W_1 = \{V(\sigma) : \sigma \in \mathcal{F},\ \dim\sigma \geq 1\}$ the vertices of all higher dimensional simplices in $\mathcal{F}$; by our anti-chain assumption we have $W_0 \cap W_1 = \emptyset$. Note that, $V(Y) = V(X) - W_0$ and therefore $W_1 \subset V(Y)$.

Let $U \subset V(Y)$ be a subset, given $v \in V(Y)$ define the following properties of v:

(a.) $v \notin W_1$,

(b.) $\mathrm{Lk}_X(v) \cap X_U$ is a subcomplex of Y_U.

If v satisfies (a.) and (b.) then

$$\mathrm{Lk}_Y(v) \cap Y_U \;=\; \mathrm{Lk}_X(v) \cap X_U. \tag{6.8}$$

Indeed, by (a) we have $\mathrm{Lk}_X(v) \cap Y_U = \mathrm{Lk}_Y(v) \cap Y_U$, and $\mathrm{Lk}_X(v) \cap Y_U = \mathrm{Lk}_X(v) \cap X_U$ by (b). Our goal for the rest of this proof is therefore to find such a vertex satisfying both conditions.

Let k be an integer satisfying (6.6) and $U \subset V(Y)$ a subset with $|U| \leq r-k$. Given a subcomplex $A \subset Y_U$, we want to show the existence of a vertex $v \in V(Y) - U$ such that

$$\mathrm{Lk}_Y(v) \cap Y_U = A. \tag{6.9}$$

This would mean that our complex Y is $(r-k)$-ample.

Note that the induced subcomplex X_U obviously contains A as a subcomplex, and consider the abstract simplicial complex

$$K = X_U \cup (A * \Delta_k),$$

where Δ_k is an abstract full simplex on k vertices. Note that K has at most r vertices, X_U is an induced subcomplex of K and it is naturally embedded into X by r-ampleness. Using the assumption that X is r-ample and by application of Lemma 6.2.3, we can find an embedding of K into X extending the identity map of X_U. In other words, we can find k vertices $U' = \{v_1, \dots, v_k\} \in V(X) - U$ such that for a simplex τ of X_U and for any subset $S \subset \{v_1, \dots, v_k\}$ one has $\cup_{u \in S} u * \tau \in X$ if and only if $\tau \in A$, i.e. every v_i satisfies property (b.). If one of the v_i lies in $V(Y) - W_1$ then, (6.8) holds so we have,

$$\mathrm{Lk}_Y(v_i) \cap Y_U = \mathrm{Lk}_X(v_i) \cap X_U = A$$

and we are done. Thus, we suppose that $U' \subset W_0 \cup W_1$.

Let $Z \subset \Delta_k$ be an arbitrary simplicial subcomplex. We may use the r-ampleness of X and apply Definition 6.2.1 to the subcomplex $A \sqcup Z$ of $X_{U \cup U'}$ to get a vertex $v_Z \in V(X) - (U \cup U')$ satisfying

$$\mathrm{Lk}_X(v_Z) \cap X_{U \cup U'} = A \sqcup Z$$

and in particular,

$$\mathrm{Lk}_X(v_Z) \cap X_U = A. \tag{6.10}$$

For distinct subcomplexes $Z, Z' \subset \Delta$ the points v_Z and $v_{Z'}$ are distinct and the cardinality of the set $\{v_Z; Z \subset \Delta\}$ equals $M'(k)$. Noting that (6.10) is a subcomplex of $Y_U \subset X_U$, so v_z satisfies (b.) we see that our claim will follow by (6.9) and (6.10) if v_Z satisfies (a.) for at least one subcomplex Z, that is if some $v_Z \in V(Y) - W_1$.

Let us assume the contrary, i.e. $v_Z \in (W_0 \cup W_1) - U'$ for every subcomplex $Z \subset \Delta$. The cardinality of the set $\{v_Z\}$ equals $M'(k)$ and the cardinality of the set $(W_0 \cup W_1) - U'$ equals $|\mathcal{F}| + \dim \mathcal{F} - k$ and we get a contradiction with our assumption (6.6). This completes the proof.

(6.7) follows immediately by (6.3). $\square$

We finish this section with the following observation.

Proposition 6.3.2. *The link of a vertex in an r-ample simplicial complex is $(r-1)$-ample. More generally, the link of every k-dimensional simplex in an r-ample complex is $(r-k-1)$-ample.*

Proof. We consider the case $k = 0$ first. Let $v \in V(X)$ be a vertex and let L denote the link of v in X. Let (A, B) be a pair consisting of a simplicial complex A and its induced subcomplex B where $|V(A)| \leq r$. Consider the pair (CA, CB) consisting of cones with apex w. Note that CB is an induced subcomplex of CA and $|V(CA)| \leq r + 1$. Since $v * L \subseteq X$, any embedding $f_B : B \to L$ can be extended to an embedding $f_{CB} : CB \to X$ where w is mapped into v. Since X is r-ample, applying Lemma 6.2.3 we can find an embedding $f_{CA} : CA \to X$ extending f_{CB}. Then the restriction $f_{CA}|A$ is an embedding $A \to L$ extending f_B.

The case when $k > 0$ is similar. Let σ be a k-simplex in X and let L denote its link. Consider a pair (A, B) with $|V(A)| \leq r - k$, an induced subcomplex B of A and an embedding $f_B : B \to L$. Consider the joins $A' = A * \sigma$ and $B' = B * \sigma$ and note that $V(A') \leq r + 1$ and B' is an induced subcomplex of A'. By Lemma 6.2.3 the join embedding $f_{B'} = f_B * 1 : B' = B * \sigma \to L * \sigma$ can be extended to an embedding $f_{A'} : A' \to L * \sigma$ which restricts to an embedding $f_A : A \to L$ extending f_B. $\square$

6.4 Higher connectivity of ample complexes

It is natural to ask whether the geometric realisation of an r-ample simplicial complex is highly connected, i.e. do the homotopy groups below certain dimension all vanish. The motivation for this question comes from the fact that an r-ample finite simplicial complex can be viewed as an approximation to the Rado simplicial complex whose geometric realisation is homeomorphic to an infinite dimensional simplex and is hence contractible, see Theorem 5.5.1.

Recall that a simplicial complex Y is *m-connected* if for every triangulation of the i-dimensional sphere S^i with $i \leq m$ and for every simplicial map $\alpha : S^i \to Y$ there exists a triangulation of the disc D^{i+1} extending the given triangulation of the sphere $S^i = \partial D^{i+1}$ and a simplicial map $\beta : D^{i+1} \to Y$ extending α. A 1-connected complex is also said to be simply connected.

Proposition 6.4.1. *For $r \geq 4$, any r-ample simplicial complex Y is simply connected.*

Moreover, any simplical loop $\alpha : S^1 \to Y$ with n vertices in an r-ample complex Y bounds a simplicial disc $\beta : D^2 \to Y$ where D^2 is a triangulation of the disc having n boundary vertices, at most $\lceil \frac{n-3}{r-3} \rceil$ internal vertices and at most $\lceil \frac{n-3}{r-3} \rceil \cdot (r-1)+1$ triangles.

Proof. If $n \leq r$ we may simply apply the definition of r-ampleness and find an extension $\beta : D^2 \to Y$ with a single internal vertex. If $n > r$ we may apply the definition of r-ampleness to any arc consisting of r vertices, see Figure 6.3. This reduces the length of the loop by $r - 3$ and performing $\lceil \frac{n-r}{r-3} \rceil$ such operations we obtain a loop of length $\leq r$ which can be filled by a single vertex. The number of internal vertices of the bounding disc will be $\lceil \frac{n-r}{r-3} \rceil + 1 = \lceil \frac{n-3}{r-3} \rceil$. To estimate the number of triangles we note that on each intermediate step of the process described above we add $r - 1$ triangles and on the final step we may add at most r triangles. This leads to the upper bound $\lceil \frac{n-r}{r-3} \rceil \cdot (r - 1) + r = \lceil \frac{n-3}{r-3} \rceil \cdot (r - 1) + 1$. $\square$

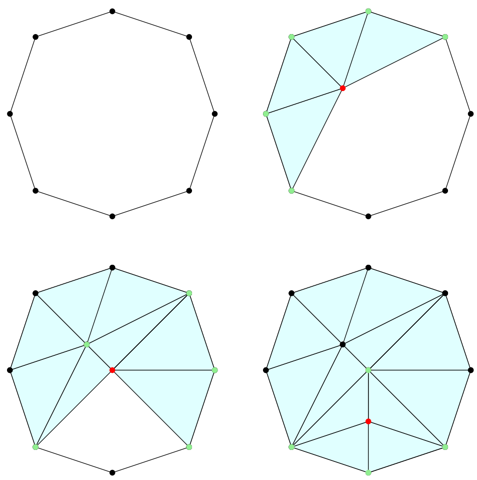

Figure 6.3: The process of constructing the bounding disc in a 5-ample complex as detailed in the proof of Proposition 6.4.1

Currently we're not aware of any examples of a 3-ample complex which is not simply connected. However, the 2-ample complex of Example 6.2.2 has non-trivial first fundamental group and is therefore not simply connected.

Theorem 6.4.2. *For $r \geq 18$, every r-ample simplicial complex is 2-connected.*

In the proof of Theorem 6.4.2 we shall use the following property about triangulations of the 2-sphere.

Lemma 6.4.3. *In any triangulation Σ of the 2-dimensional sphere there exists two adjacent vertices v and w both having degree at most* 11.

Proof. We let d_v denote the (edge) degree of a vertex v of a triangulation Σ of S^2, i.e. it is the number of edges incident to v.

Recall that for any triangulation Σ of the 2-sphere one has the following relation

$$\sum_v \left(1 - \frac{d_v}{6}\right) = 2, \tag{6.11}$$

where v runs over all vertices of Σ and d_v denotes the degree of the vertex v. Formula (6.11) is well-known, it follows from the Euler's formula $V - E + F = 2$ by observing that $E = \frac{1}{2}\sum_v d_v$ and $F = \frac{1}{3}\sum_v d_v$. Formula (6.11) can be viewed as a combinatorial version of the Gauss-Bonnet theorem.

Let A denote the set of vertices $v \in V(\Sigma)$ satisfying $d_v \leq 11$ and let B denote the complementary set consisting of vertices with $d_v \geq 12$. Denote also

$$C_A = \sum_{v \in A} \left(1 - \frac{d_v}{6}\right), \qquad C_B = \sum_{v \in B} \left(1 - \frac{d_v}{6}\right),$$

the contributions of both sets into the sum (6.11). Since $d_v \geq 3$ we have $1 - \frac{d_v}{6} \leq \frac{1}{2}$ and hence

$$C_A \leq \frac{1}{2}|A|.$$

Moreover, $1 - \frac{d_v}{6} \leq -1$ for $v \in B$ and therefore

$$C_B \leq -|B|, \qquad C_A + C_B = 2, \qquad |A| + |B| = V.$$

From these relations one obtains

$$|A| \geq \frac{2}{3}(V + 2). \tag{6.12}$$

Next we claim that there must exist an edge e with both endpoints in A, i.e. having degree ≤ 11. Assuming the contrary, every triangle of the triangulation Σ would have at most one vertex of degree ≤ 11 and since the minimal degree is 3, using (6.12), we obtain that the number of triangles would be at least

$$3 \cdot \frac{2}{3}(V + 2) = 2V + 4.$$

However this contradicts the well-known relation $F = 2V - 4$ for the total number of triangles. $\square$

We shall also need the following simple Lemma:

Lemma 6.4.4. *Let D be a triangulated 2-dimensional disk and let $L = \partial D$ be its boundary circle. Assume that the length (i.e. the number of edges) of L is at least 7 and D has at most one internal vertex. Then there exists a pair of boundary vertices $x, y \in L$ satisfying $d_L(x, y) \geq 3$ such that they can be connected by a simple simplicial arc α in D with $\partial\alpha = \{x, y\} = \alpha \cap \partial D$. Here $d_L(x, y)$ denotes the distance between x and y along the boundary L, i.e. the number of edges in the shortest simplicial path in L connecting x and y.*

Proof. Let us first consider the case when D has no internal vertices. Denoting the length $|L|$ of the boundary by n, we see that there are $n-3$ internal arcs (as follows from the Euler's formula). We want to show that there exists an internal arc such that its end points x, y satisfy $d_L(x, y) \geq 3$. Assuming that $d_L(x, y) = 2$ for any internal arc, we may cut D along an arbitrary internal arc which produces a triangle and a triangulated disk D' with $|L'| = n - 1$ where $L' = \partial D'$. If we knew that our statement was true for D' we could find vertices $x, y \in L'$ satisfying $d_{L'}(x, y) \geq 3$ such that x, y are the endpoints of an internal arc of D'. Then $d_L(x, y) \geq d_{L'}(x, y) \geq 3$. This argument shows that without loss of generality we may assume that the length of L is exactly 7 but in this case one can see that our statement holds by examining a few explicit cases; see left part of Figure 6.4.

Consider now the case when D has a single internal vertex, denoted v. The vertex v is connected to at least 3 other vertices $a, b, c \in L$. Let $d_L(a, c)$ be the maximal among the three numbers $d_L(a,b)$, $d_L(a,c)$, $d_L(b,c)$. Then either $d_L(a,b)+d_L(b,c)+d_L(a,c)=|L|$ or $d_L(a,c)=d_L(a,b)+d_L(b,c)$. In the first case one obtains $d_L(a,c)\geq 4$ (since $|L|\geq 7$) and we are done, as we can take for α the arc $av+vc$. In the second case we may similarly treat the case $d_L(a,b)\geq 3$ and we are left with the possibility $d_L(a,b)=2$ and hence $d_L(a,c)=1$ and $d_L(c,b)=1$. Cutting along the arc $av+vc$ produces two triangulated disks, each with no internal vertices, one having 4 vertices and the other, denoted D', having $|L|$ vertices. We see that $|\partial D'|\geq 7$ and hence we may apply the previous case of the Lemma, i.e. we can find two vertices $x,y\in\partial D'=L'$ connected by an internal arc such that $d_{L'}(x,y)\geq 3$. We are done if none of the points x,y equal v. However if $x=v$ we may consider the pair $y,\ b\in L$ since $d_L(y,b)=d_{L'}(y,v)\geq 3$ and the points y, b are connected by the arc $\alpha=yv+vb$. See Figure 6.4, right. $\square$

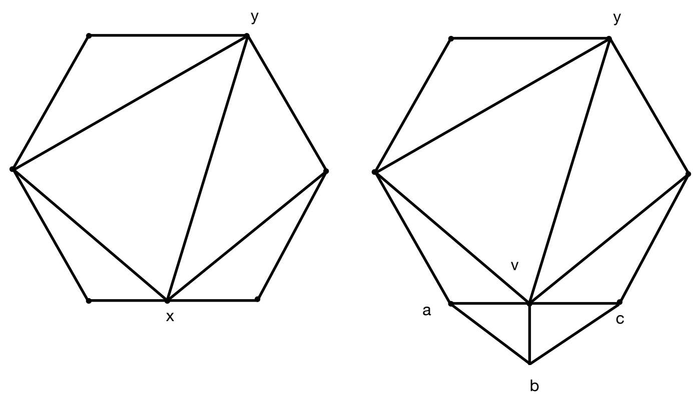

Figure 6.4: Triangulated disks with no internal vertices (left) and one internal vertex (right) illustrating the proof of Lemma 6.4.4.

The following gives us information about local structure of r-ample complexes and will be used below in the proof of Theorem 6.4.2.

Lemma 6.4.5. *Let X be an r-ample simplicial complex. Then any simplicial map $f:K\to X$, with $|V(K)|\leq r$, is null-homotopic.*

Proof. The set $U=f(V(k))\subset V(X)$ has cardinality $\leq r$ and applying Definition 6.2.1 we can find a vertex $v\in V(X)-U$ such that $X_{U\cup\{v\}}=v*X_U$ (cone over X_U). Thus we see that $f:K\to X$ factorises through a map with values in the cone $v*X_U$ which is contractible and hence f is null-homotopic. $\square$

With these results in place we can now prove Theorem 6.4.2.

Proof of Theorem 6.4.2. We shall assume the contrary and arrive to a contradiction. Let Y be an 18-ample simplicial complex which is not 2-connected. From Proposition 6.4.1 we know that Y is simply connected. Let $M(Y)$ denote the smallest number of vertices in a triangulation Σ of the sphere S^2 admitting a simplicial essential (i.e. not null-homotopic) map $f : \Sigma \to Y$. By the well-known Simplicial Approximation Theorem, $M(Y)$ is finite. Lemma 6.4.5 implies $V(\Sigma) \geq 19$ for every simplicial essential map $f : \Sigma \to Y$ and hence $M(Y) \geq 19$.

Let $f : \Sigma \to Y$ be an essential simplicial minimal map, i.e. $|V(\Sigma)| = M(Y)$. We shall use the following geometric property of the triangulation Σ of S^2, its *roundness*, which is described in the next paragraph.

Suppose that $L \subset \Sigma$ is a simple simplicial loop of length $|L| \leq 18$, i.e. L contains at most 18 edges. Clearly, L divides the sphere Σ into two triangulated disks D_1 and D_2 with each of these disks having $|L|$ boundary vertices and possibly a number of internal vertices.

Claim. *At least one of the disks D_1, D_2 has at most one internal vertex, i.e. a "small" loop cannot divide the sphere into two large pieces – we say that Σ is* round.

Proof of claim. Suppose that each of the disks D_1 and D_2 has ≥ 2 internal vertices. Let $D = a * L$ be the cone with apex a and base L. We can form two triangulated spheres $\Sigma_1 = D_1 \cup D$ and $\Sigma_2 = D_2 \cup D$ and each of these spheres has strictly smaller number of vertices than Σ (since D has a single internal vertex and each of the disks D_1, D_2 has at least 2 internal vertices).

Next we observe that each of the spheres Σ_1 and Σ_2 can be mapped simplicially into Y so that at least one of the maps $\Sigma_1 \to Y$ or $\Sigma_2 \to Y$ is essential. Consider the image $f(L) \subset Y$ of the loop L in Y. It is a subcomplex with at most 18 vertices and by the 18-ampleness of Y we can find a vertex $u \in V(Y)$ such that $u * f(L) \subset Y$. Now we may extend the map $f : \Sigma \to Y$ onto the disk $D = a * L$ by mapping a onto u and extending this map onto the cone by linearity. We obtain a simplicial map $g : \Sigma \cup D \to Y$

extending f and the restrictions $g_1 = g|_{\Sigma_1} : \Sigma_1 \to Y$ and $g_2 = g|_{\Sigma_2} : \Sigma_2 \to Y$ are the desired simplicial maps. Since $\Sigma_1 \cup \Sigma_2 = \Sigma \cup D$ and $\Sigma_1 \cap \Sigma_2 = D$ is contractible, we see that g is essential (as $f = g|_\Sigma$ is essential) and hence at least one of the maps g_1, g_2 is essential. Thus, we arrive at a contradiction with the minimality of f. □

Our main idea from here on will be to utilise this claim as follows: we will construct two loops L, L' such that at least one of the pairs of discs defined cannot satisfy the *roundness* property.

Next we invoke Lemma 6.4.3 which gives us two adjacent vertices v and w of Σ, each having degree at most 11. Let e be the edge connecting v and w. Consider the subcomplex U of the surface Σ which is the union of all triangles incident to e. The boundary ∂U is a closed curve (potentially with some identifications, see below) formed by $d_v + d_w - 4 \leq 11 + 11 - 4 = 18$ edges and the interior of U is the union of $d_v + d_w - 2$ triangles. The edge e is incident to two triangles; we shall denote by α and β the vertices

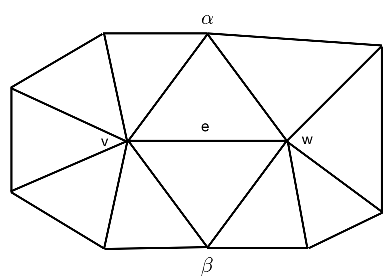

Figure 6.5: Triangles incident to an edge on the surface.

of these two triangles which are not incident to e, see Figure 6.5.

Let us assume first that the links of the vertices v and w satisfy

$$\mathrm{Lk}_\Sigma(v) \cap \mathrm{Lk}_\Sigma(w) = \{\alpha, \beta\}.$$

Then U is a triangulated disk with $\leq 18 + 2 = 20$ vertices, among them 2 are internal, as shown on Figure 6.5.

Suppose now that there exists a vertex $a \in \mathrm{Lk}_\Sigma(v) \cap \mathrm{Lk}_\Sigma(w)$ which is distinct from α and β. Then the path $L = av + vw + wa$ is a simplicial loop on Σ which divides the

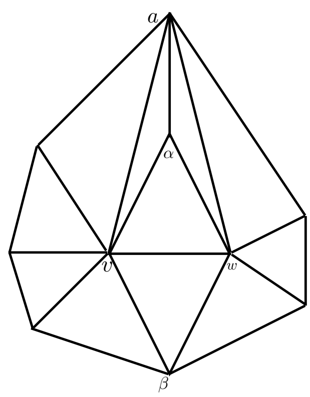

Figure 6.6: Disk U with 3 internal vertices.

surface Σ into two disks. By the roundness property of Σ, one of these two disks must have at most one internal vertex. In fact, the only possibility is that L bounds a disk with one internal vertex and L cannot be the boundary of a triangle: otherwise the edge e would belong to 3 different triangles. It is easy to see that this internal vertex must be either α or β, as there are exactly 2 triangles incident to e, see Figure 6.6. In this case α becomes an internal vertex of U.

For similar reasons it might happen that both vertices α and β are internal vertices of U.

The argument above shows that any vertex lying in $\mathrm{Lk}_\Sigma(v) \cap \mathrm{Lk}_\Sigma(w)$, which is distinct from α and β, belongs to a triangular simplicial loop surrounding either α or β and containing the edge vw (similarly the loop $av + vw + wa$ shown on Figure 6.6). This implies that the intersection $\mathrm{Lk}_\Sigma(v) \cap \mathrm{Lk}_\Sigma(w)$ may contain at most 4 vertices.

Potentially it might happen that $U = \Sigma$, i.e. $\partial U = \emptyset$. Then all vertices of Σ, other than v, w, lie in the intersection $\mathrm{Lk}_\Sigma(v) \cap \mathrm{Lk}_\Sigma(w)$. Using the above arguments, we see that in this case $V(\Sigma) \leq 6$, which contradicts our assumption $V(\Sigma) \geq 19$.

The remaining possibility is that $U \subset \Sigma$ is a subcomplex, it has either 2, 3 or 4 internal vertices and its total number of vertices is at most 20.

The closure of the complement of U in Σ is another disk, U', and applying the roundness property of Σ, we conclude that that U' has at most one internal point.

Thus, we see that the triangulation Σ must have at most 21 vertices in total, and using Lemma 6.4.5 we obtain that $|V(\Sigma)|$ must be equal to one of the three numbers: 19, 20 or 21.

Using this observation we conclude that the length $\ell = |L|$ of the boundary $L = \partial U = \partial U'$ should satisfy $14 \leq \ell \leq 18$.

Finally we show that there must exist a simplicial simple closed curve L' on Σ dividing the sphere Σ into two disks, each having more than one internal points, which will violate the roundness of Σ and gives a contradiction. The curve L' is the union of two arcs $L' = A \cup A'$ where $A \subset U$ and $A' \subset U'$. We first construct the arc $A' \subset U'$; we only must ensure that $(*)$ the endpoints of A' divide the boundary L into two arcs, each of length ≥ 3. The existence of such an arc follows from Lemma 6.4.4 below. Once the arc $A' \subset U'$ satisfying $(*)$ is constructed we connect its endpoints (lying on the boundary $L = \partial U$) by a simple simplicial arc A in U; it is clear from Figures 6.5 and 6.6 that any two points on the boundary can be connected by such an arc in U.

The vertices of L distinct from the two vertices in the boundaries $\partial A = \partial A'$ are internal vertices of the disks on which the sphere Σ is divided by the circle L'; the condition $(*)$ ensures that at least two vertices lie in each connected components of $\Sigma - L'$. This contradicts the roundness property of Σ and completes the proof of Theorem 6.4.2. □

We remark here that there is a simpler proof of a weaker[1] version of Theorem 6.4.2 that uses the *Planar Separator Theorem*, the proof via this method is given in the Appendix A.4.

Question 6.4.6. *For every $k \geq 0$ does there exist $r(k)$ such that every r-ample simplicial complex is k-connected provided that $r \geq r(k)$?*

We know that $r(0) = 1$, $r(1) \leq 4$, and $r(2) \leq 18$ by the results of this section but the further cases remain open. The above proofs do not immediately translate to higher dimensions as there is no analogue to Lemma 6.4.3 that holds for triangulations of the

[1]Weaker in the sense that we require our complex to be at least 79-ample before we can guarantee it is 2-connected

3-sphere and beyond.

We remark that a recent preprint [5] of Barmak (submitted the initial writing of this thesis) seems to answer this question in the positive. In particular, they claim that if a simplicial complex is 6^k-ample then it will be k-connected.

Question 6.4.7. *A further question of interest is to investigate the homology of ample complexes. In particular, as the link of a k-simplex in an r-ample complex stays $(r-k)$-ample one may hope to apply a Garland technique type argument. For this argument to prove fruitful however one would need a good understanding of the spectrum in r-e.c. graphs.*

As a corollary of the Proposition 6.4.1 and Theorem 6.4.2, and Theorem 6.3.1 we are able to state how much destruction one can do to an r-ample complex without breaking higher connectivity.

Corollary 6.4.8. *Let X be an r-ample simplicial complex and let Y be obtained from X by removing a set $\mathcal{F}$ of simplices. Denote by a_i the number of i-dimensional simplices in $\mathcal{F}$ where $i = 0, 1, \dots.$. Then:*

(a) If $r \geq 3$ and $a_0 + 2a_1 < M'(r-2) + r - 2$ then Y is path-connected. In particular, Y is path-connected if

$$a_0 + 2a_1 < 2^{\binom{r-2}{\lfloor r/2 \rfloor - 1}} + r - 2.$$

(b) If $r \geq 5$ and $a_0 + 2a_1 + 3a_2 < M'(r-4) + r - 4$ then Y is simply connected. In particular, Y is simply connected if

$$a_0 + 2a_1 + 3a_2 < 2^{\binom{r-4}{\lfloor r/2 \rfloor - 2}} + r - 4.$$

(c) If $r \geq 19$ and $a_0 + 2a_1 + 3a_2 + 4a_3 < M'(r-18) + r - 18$ then Y is 2-connected. In particular, Y is 2-connected if

$$a_0 + 2a_1 + 3a_2 + 4a_3 < 2^{\binom{r-18}{\lfloor r/2 \rfloor - 9}} + r - 18.$$

Proof. Claim (a) follows from Theorem 6.3.1 and from the observation that a 2-ample complex is connected; claim (b) follows from Theorem 6.3.1 and Proposition 6.4.1; claim (c) follows from Theorems 6.3.1 and 6.4.2. □

6.5 Large random simplicial complexes are ample

In this section we show that a (medial regime) lower model random simplicial complex is r-ample with probability tending to one. This result implies the existence of r-ample finite simplicial complexes, which we use to estimate the minimal number of vertices an r-ample complex must possess.

Recall, from Chapter 2, that the probability of obtaining a simplicial subcomplex $Y \subset \Delta_n$ in the lower model is given by

$$\underline{\mathbb{P}}_n(Y) = \prod_{\sigma \in F(Y)} p_\sigma \cdot \prod_{\sigma \in E(Y)} q_\sigma. \tag{6.13}$$

We shall assume that the parameters p_σ are in the medial regime. For our purposes we will utilise a slightly relaxed condition in place of (4.3) in the following way: if the parameters p_σ are in the medial regime then there exists $p \in (0, 1/2]$ that does not depend on n such that

$$p_\sigma \in [p, 1-p]. \tag{6.14}$$

Note however that in Remark 6.5.2 we will further relax this assumption.

Proposition 6.5.1. *For every integer $r \geq 1$, the probability that a medial regime random simplicial complex is r-ample tends to one, as $n \to \infty$.*

Proof. We estimate probability that a random complex Y is not r-ample. Let us make the following choices: a subset $U \subset [n]$ of cardinality $|U| \leq r$, a subcomplex $Z \subset \Delta_n$ with $V(Z) = U$, a subcomplex $A \subset Z$ and a vertex $v \in [n] - U$. Consider the following events

$$W_U = \{Y \subset \Delta_n \mid U \subset V(Y)\},$$

$$W_{U,Z} = \{Y \subset \Delta_n \mid U \subset V(Y),\, Y_U = Z\},$$

and

$$W_{U,Z,A,v} = \{Y \subset \Delta_n;\, U \cup \{v\} \subset V(Y),\, Y_{U\cup\{v\}} = Z \cup (A * v)\}.$$

Note that

$$W_U = \sqcup_Z W_{U,Z}$$

is the disjoint union where Z runs over all subcomplexes of Δ_n satisfying $V(Z) = U$. Consider also the complement $W^c_{U,Z,A,v}$ of $W_{U,Z,A,v}$ in $W_{U,Z}$, i.e.

$$W^c_{U,Z,A,v} = W_{U,Z} - W_{U,Z,A,v}.$$

A simplicial complex $Y \subset \Delta_n$ belongs to $W^c_{U,Z,A,v}$ if and only if $U \subset V(Y)$ and $Y_U = Z$ and either $v \notin V(Y)$ or $v \in V(Y)$ and $\mathrm{Lk}_Y(v) \cap Y_U \neq A$. We see that for $|U| \leq r$ any complex Y lying in the intersection

$$\bigcap_{v\in[n]-U} W^c_{U,Z,A,v}$$

is not r-ample. Moreover, the set $\mathcal{N}$ of all not r-ample simplicial complexes $Y \subset \Delta_n$, $Y \neq \emptyset$, coincides with

$$\mathcal{N} = \bigcup_{1\leq|U|\leq r} \left(\bigsqcup_{Z} \left(\bigcup_{A\subset Z} \left(\bigcap_{v\in[n]-U} W^c_{U,Z,A,v} \right)\right)\right).$$

We denote

$$\mathcal{N}_{U,Z} = \bigcup_{A\subset Z} \left(\bigcap_{v\in[n]-U} W^c_{U,Z,A,v} \right)$$

and $\mathcal{N}_U = \sqcup_Z \mathcal{N}_{U,Z}$. Then $\mathcal{N} = \cup_U \mathcal{N}_U$, where $|U| \leq r$.

Using definition (6.13) we can compute conditional probability

$$\mathbb{P}_n(W_{U,Z,A,v} \mid W_{U,Z}) = p_v \cdot \prod_{\sigma\in F(A)} p_{v\sigma} \cdot \prod_{\sigma\in E(A|Z)} q_{v\sigma}.$$

Here $v\sigma$ denotes the join $v * \sigma$ and $E(A|Z)$ denotes the set of simplices $\sigma \in F(Z) - F(A)$ such that $\partial\sigma \subset A$. Since $|F(A)| + |E(A|Z)| \leq 2^r - 1$ (note that by definition a simplex is *a nonempty* subset of the vertex set), using the medial regime assumption (6.14), we obtain

$$\underline{\mathbb{P}}_n(W_{U,Z,A,v} \mid W_{U,Z}) \geq p^{2^r}. \tag{6.15}$$

Hence the complement $W^c_{U,Z,A,v}$ of $W_{U,Z,A,v}$ in $W_{U,Z}$ satisfies

$$\underline{\mathbb{P}}_n(W^c_{U,Z,A,v} \mid W_{U,Z}) \leq 1 - p^{2^r}$$

and since for different vertices $v \in [n] - U$ the events $W^c_{U,Z,A,v}$ are conditionally independent over $W_{U,Z}$ we obtain

$$\underline{\mathbb{P}}_n\left(\bigcap_v W^c_{U,Z,A,v} \mid W_{U,Z}\right) \leq (1 - p^{2^r})^{n-|U|} \leq (1 - p^{2^r})^{n-r}$$

and therefore

$$\underline{\mathbb{P}}_n(\mathcal{N}_{U,Z} \mid W_{U,Z}) \ = \ \underline{\mathbb{P}}_n\left(\bigcup_A \bigcap_v W^c_{U,Z,A,v} \mid W_{U,Z}\right) \leq 2^{2^r}(1 - p^{2^r})^{n-r},$$

where A runs over subcomplexes of Z (the number of such subcomplexes is clearly bounded above by 2^{2^r}).

Since $\mathcal{N}_U = \sqcup_Z \mathcal{N}_{U,Z}$ and $\mathcal{W}_U = \sqcup_Z \mathcal{W}_{U,Z}$ we obtain

$$\underline{\mathbb{P}}_n(\mathcal{N}_U) \leq \max_Z \underline{\mathbb{P}}_n(\mathcal{N}_{U,Z} \mid W_{U,Z}) \leq 2^{2^r}(1 - p^{2^r})^{n-r}.$$

And finally, we obtain the following upper bound for the probability of the set $\mathcal{N} = \cup_U \mathcal{N}_U$

of all non-empty simplicial subcomplexes of Δ_n which are not r-ample:

$$\begin{aligned}\mathbb{P}_n(\mathcal{N}) &\leq \sum_{j=1}^{r}\binom{n}{j}\cdot 2^{2^r}\left(1-p^{2^r}\right)^{n-r}\\ &\leq n^r\cdot 2^{2^r}\left(1-p^{2^r}\right)^{n-r}. \end{aligned} \tag{6.16}$$

Clearly, for $n\to\infty$ the expression (6.16) tends to zero. Note also that the probability of the empty simplicial complex is bounded above by $(1-p)^n$ and tends to 0. This completes the proof. ☐

Remark 6.5.2. The above arguments prove that the conclusion of Proposition 6.5.1 holds under a slightly weaker assumption, namely $p_\sigma\in[p,1-p]$ where $p=p(n)>0$ satisfies

$$p^{2^r}=\frac{r\ln n+\nu}{n}$$

with $\nu=\nu(n)$ an arbitrary sequence tending to ∞. Examples satisfying the above condition are $p=1/n^\alpha$ with $\alpha\in(0,2^{-r})$ and $p=1/\ln n$, with the latter choice working for any r.

Remark 6.5.3. The arguments of the proof of Proposition 6.5.1 work without any change if one alters the medial regime assumption by requiring that $p_v=1$ for every vertex $v\in[n]$ while $p_\sigma\in[p,1-p]$ for $\dim\sigma>0$. Formula (6.13) implies that in this case the probability measure is supported on the set of simplicial complexes $Y\subset\Delta_n$ with $V(Y)=[n]$, i.e. having exactly n vertices. This observation will be used below in the Proof of Proposition 6.5.4

Proposition 6.5.4. *For every $r\geq 5$ and for every $n\geq r2^r2^{2^r}$, there exists an r-ample simplicial complex having exactly n vertices.*

Proof. The expression (6.16) is an upper bound of the probability that a medial regime random complex on n vertices is not r-ample. Clearly, if for some n the RHS of (6.16) is smaller than 1 then an r-ample complex exists. The expression (6.16) is bounded above by

$$n^r2^{2^r}e^{-p^{2^r}\cdot(n-r)}=n^re^{-np^{2^r}}\cdot 2^{2^r}e^{rp^{2^r}}$$

and taking the logarithm we obtain the following inequality

$$np^{2^r} - r\ln n > 2^r \ln 2 + rp^{2^r} \tag{6.17}$$

which guarantees the existence of an r-ample complex on n vertices. Below we shall set $p = 1/2$. The function $n \mapsto np^{2^r} - r\ln n$ is monotone increasing for $n > r2^{2^r}$ and therefore we only need to show that (6.17) is satisfied for $n = r2^r2^{2^r}$. The inequality (6.17) turns into

$$r2^r(1-\ln 2) - r^2\ln 2 - r\ln r > 2^r\ln 2 + r2^{-2^r}$$

which is equivalent to

$$r(1-\ln 2) - \ln 2 > \frac{r^2\ln 2 + r\ln r}{2^r} + r2^{-2^r-r}. \tag{6.18}$$

Given that $\ln 2 \simeq 0.6931$ it is easy to see that (6.18) is satisfied for any $r \geq 5$. □

Remark 6.5.5. Even though a random simplicial complex with $2^{\Omega(2^r)}$ vertices is r-ample (as Proposition 6.5.4 claims), $2^{O(2^r/\sqrt{r})}$ vertices do not suffice; this follows from Corollary 6.2.6 and formula (6.5).

As a byproduct, we also obtain the following result about 2-connectivity of random simplicial complexes in the medial regime.

Corollary 6.5.6. *Every medial regime random simplicial complex is* 2*-connected, asymptotically almost surely.*

Proof. This follows from Proposition 6.5.1 and Theorem 6.4.2. □

Connectivity and simple connectivity of the medial regime random simplicial complexes and vanishing of the Betti numbers was proven in Chapter 4 with arguments involving the Nerve Lemma and Garland's method.

6.6 Explicit constructions of ample complexes

The random construction above shows the existence of r-ample simplicial complexes for every r. However, it does not tell us how to construct an r-ample complex explicitly. In this section we define a deterministic family of complexes that are guaranteed to be r-ample.

6.6.1 Defining iterated Paley complexes

Our construction uses ideas from number theory, and generalises the classical Paley graph, defined below. In Definitions 6.6.3-6.6.6, we introduce an *iterated Paley simplicial complex* $X_{n,p}$ on n vertices, for every odd prime power n and odd prime p dividing $(n-1)$. But first, we state the main theorem of the section.

Theorem 6.6.1. *Let $r \in \mathbb{N}$. Every iterated Paley simplicial complex $X_{n,p}$ with $p > 2^{2^r+2r}$ and $n > r^2 p^{2r}$ is r-ample.*

Theorem 6.6.1 is proven in Section 6.6.2, after the definition of $X_{n,p}$. After the proof, we discuss the selection of the prime parameters n and p, so that r-ample complexes can be constructed for every r. We prove the following corollary:

Corollary 6.6.2. *For every sufficiently large r, there exists an r-ample iterated Paley Complex $X_{n,p}$ on $n = 2^{(2+o(1))r2^r}$ vertices.*

To summarize, $\exp(\Omega(r\,2^r))$ vertices are sufficient for constructing an r-ample complex explicitly, compared to $\exp(\Omega(2^r))$ vertices probabilistically, by Corollary 6.5.4. We do not know how many vertices are really needed for these constructions to be r-ample. However, the lack of r-ample complexes of size $\exp(O(2^r/\sqrt{r}))$, by Corollary 6.2.6, gives a lower bound.

The *Paley graph* and the *Paley tournament* are long known to satisfy the corresponding r-ampleness property for graphs and tournaments respectively [8, 11, 37]. Their vertices are the elements of a finite field $\mathbb{F}_n$ with an edge from x to y if and only if $(y-x)$ is a quadratic residue [65]. More generally, these constructions exhibit numerous important properties typical to their random counterparts, and are accordingly called *pseudorandom* or *quasirandom* [2, 60]. However, these provably r-ample graphs and tournaments are nearly square the size of those probabilistically shown to be r-ample. Understanding such gaps between randomized and explicit solutions is a recurring theme in the study of combinatorial structures and computational complexity.

The most straightforward extension of Paley's graph to higher dimensions is by including a d-dimensional face $[x_0, x_1, \ldots, x_d]$ if $x_0 + x_1 + \cdots + x_d$ is a quadratic residue. Hypergraphs with $(d+1)$-edges constructed by this rule are known to possess *some* quasirandom properties [20, 40, 55]. They also yield large cosystoles in simplicial complexes, pertinent to d-dimensional coboundary expansion over $\mathbb{F}_2$, by Kozlov and Meshulam, see [53]. However, they fail to be ample. Indeed, if four vertices satisfy $a + b = c + d$, then abx is a face if and only if cdx is a face, hence some extensions of such a foursome are not available. All the explicit constructions considered in the study of quasirandom hypergraphs break down when it comes to r-ampleness.

Our new construction combines three generalisations of Paley graphs. First, if $m|(n-1)$ then, rather than quadratic residues in $\mathbb{F}_n$, one may determine adjacency by means of the multiplicative subgroup of mth powers and its cosets. Having similar quasirandom properties [3, 49], such graphs proved useful in Ramsey theory [21, 38, 72]. They appear also in the classification of graphs with strong symmetries [56, 66].

Second, instead of defining hyperedges by summing $x_0 + \cdots + x_d$, one may use the Vandermonde determinant,

$$\Delta(x_0, \ldots, x_d) := \prod_{0 \le i < j \le d} (x_i - x_j)$$

This is an appealing route because such products are compatible with the multiplicative nature of the above subgroups. Hypergraphs produced in this way [36, 52, 67] are known to have several nice properties, but not r-ampleness.

The final and novel ingredient in our construction is the repeated use of Paley-like motifs. Faces are selected according to certain cosets of p-power residues mod n, and those cosets in turn correspond to quadratic residues mod p. For this reason, we name such constructions *iterated Paley.* The need for this double prime construction will be clarified in the ampleness proof.

For the following set of definitions, fix an odd prime power n, an odd prime p that divides $n-1$, and a primitive element g in the finite field $\mathbb{F}_n$.

Definition 6.6.3. For n, p, g as above, let

$$Q_{n,p} \;:=\; \left\{g^{\alpha} \;\mid\; \alpha \equiv \beta^2 \bmod p,\; \text{for } \alpha, \beta \in \mathbb{Z}\right\} \;\subset\; \mathbb{F}_n$$

Remark 6.6.4. As $p|(n-1)$ we have a multiplicative subgroup $H = \langle g^p \rangle$ of index p in $\mathbb{F}_n^{\times} = \langle g \rangle$, and a group isomorphism $\mathbb{F}_n^{\times}/H \to (\mathbb{F}_p, +)$ taking $gH \mapsto 1$. The set $Q_{n,p}$ is the union of H-cosets that correspond to quadratic residues mod p. It therefore contains *about* half the elements of the field,

$$|Q_{n,p}| \;=\; \frac{p+1}{2p}\,(n-1)$$

Definition 6.6.5. The *iterated Paley hypergraph* $H_{n,p}$ has $\mathbb{F}_n$ as its vertex set, and a subset $\{x_1, x_2, \ldots, x_t\}$ forms a hyperedge if

$$\prod_{1 \le i < j \le t} (x_i - x_j) \;\in\; Q_{n,p}$$

Note that $(-1) = g^{(n-1)/2}$, and $(n-1)/2 \equiv 0 \bmod p$ as p is odd, and therefore $(-1) \in H = \langle g^p \rangle$. Therefore, the condition in the definition of $H_{n,p}$ does not depend on the order of the vertices $x_1, x_2, \ldots, x_t$. Note also that all n singletons $\{x\}$ are included as $1 = g^0 \in Q_{n,p}$.

The iterated Paley hypergraph might not be a simplicial complex, as it is not necessarily closed downward. We thus consider the largest simplicial complex contained in it, defined as follows.

Definition 6.6.6. The *iterated Paley simplicial complex* $X_{n,p}$ has $\mathbb{F}_n$ as its vertex set, and a set $\{x_1, x_2, \ldots, x_t\}$ forms a simplex if for every subset $\{x_{s_1}, x_{s_2}, \ldots, x_{s_k}\} \subseteq$

$\{x_1, x_2, \ldots, x_t\}$

$$\prod_{1\le i<j\le k} \left(x_{s_i} - x_{s_j}\right) \in Q_{n,p}.$$

That is, in the notation of Chapter 2 we say $X_{n,p} = \underline{Q_{n,p}}$.

Remark 6.6.7. The definitions of $Q_{n,p}$ and thereby $H_{n,p}$ and $X_{n,p}$ depend on the choice of primitive element $g \in \mathbb{F}_n$. Any other primitive element $h = g^\alpha \in \mathbb{F}_n$ gives the same construction if α is a quadratic residue mod p, and a different one if not. The two constructions are not necessarily isomorphic in general. Our results apply to either choice.

Remark 6.6.8. $H_{n,p}$ and $X_{n,p}$ are invariant under a rather large group of symmetries $\{x \mapsto ax + b \mid a \in H, b \in \mathbb{F}_n\}$.

6.6.2 Iterated Paley complexes are ample

Example 6.6.9. *Before proving Theorem 6.6.1, we sketch the idea of the proof via a simple example: accommodating one 3-ampleness challenge, posed by three vertices $a, b, c \in X = X_{n,p}$. Given a, b, c, suppose that we are looking for another vertex $x \in X$ such that, say, $ax, bx, cx, abx, bcx \in X$ and $acx, abcx \notin X$.*

We find x in two stages. First we decide on three suitable H-cosets $g^\alpha H, g^\beta H, g^\gamma H$, where $H = \langle g^p \rangle$ as before. Then we find $x \in \mathbb{F}_n$ such that $(x - a) \in g^\alpha H$, $(x - b) \in g^\beta H$, and $(x - c) \in g^\gamma H$. Such an x exists by extending the uncorrelation property of squares, from Paley graphs. Specifically, 3 different additive translations of p-power cosets must intersect in $n/p^3 \pm O(\sqrt{n})$ elements.

Without knowing better, we pick $\alpha, \beta, \gamma \in \mathbb{F}_p$ one by one. The requirement $ax \in X$ implies that α must be a square, which gives $\lceil p/2 \rceil$ options. A short calculation shows that $bx, abx \in X$ require both β and $\beta + \delta$ to be squares, where δ is determined by $(a - b)g^\alpha \in g^\delta H$. This has $p/2^2 \pm O(\sqrt{p})$ solutions by the same ampleness property of Paley graphs. The requirements $cx, bcx \in X$ and $acx, abcx \notin X$ give four constraints: γ and $\gamma + \delta_b$ are squares while $\gamma + \delta_a$ and $\gamma + \delta_{ab}$ are nonsquares, where $\delta_a, \delta_b, \delta_{ab}$ are known from a, b, c, α, β. This is satisfied by $p/2^4 \pm O(\sqrt{p})$ elements of $\mathbb{F}_p$, for the same *reason that Paley graphs are 4-ample.*

One has to be a bit careful to avoid contradictions between requirements. For example, $\delta_a = \delta_b$ might mean no solution for γ. The proof will avoid such problematic cases with advance planning. On the other hand, sometimes we can take shortcuts. For example, $acx \notin X$ gives $abcx \notin X$ come for free. We will not rely on such considerations, as they would not simplify the argument in general. This will ensure our proof applies to hypergraphs too.

With the above example in mind, we begin with a formal proof of Theorem 6.6.1. We first formalize the idea that at every step we have an abundance of choices for the witness to ampleness, with differences lying in the necessary cosets.

Lemma 6.6.10. *In a finite field $\mathbb{F}_q$, let $A \subset \mathbb{F}_q^\times$ be a proper multiplicative subgroup of index m. Given d cosets of A,*

$$A_1, A_2, \ldots, A_d \ \in \ \mathbb{F}_q^\times / A$$

and pairwise distinct elements

$$c_1, c_2, \ldots, c_d \ \in \ \mathbb{F}_q$$

the number of elements $x \in \mathbb{F}_q$ satisfying

$$(x - c_1) \in A_1,\ (x - c_2) \in A_2,\ \ldots,\ (x - c_d) \in A_d$$

is at least

$$\frac{q}{m^d} \ - \ (d-1)\sqrt{q} \ - \ \frac{d}{m}$$

This lemma says that different additive translates of cosets of m-power residues are "mutually uncorrelated". Their intersection is of order q/m^d as expected from random subsets, up to an error term of about $d\sqrt{q}$. The proof of Lemma 6.6.10 is given in Appendix A.5.

Proof of Theorem 6.6.1. Let $X = X_{n,p}$ be as in Definition 6.6.6. Consider a set of vertices $\sigma = \{x_1, \ldots, x_d\} \subseteq \mathbb{F}_n = V(X)$. Throughout this proof, σ is assumed to be nonempty. The Vandermonde determinant of σ in $\mathbb{F}_n$ falls into one of the p cosets of $H = \langle g^p \rangle$,

$$\Delta(\sigma) \;=\; \pm \prod_{1 \le i < j \le d} (x_i - x_j) \;\in\; g^{\alpha(\sigma)} H$$

The exponent $\alpha(\sigma) \in \{0, 1, \ldots, p-1\}$ is uniquely determined for each σ, as $\pm 1 \in H$. In view of the Remark 6.6.4, we regard $\alpha(\sigma)$ as an element of $\mathbb{F}_p$.

Recall that a simplex $\sigma \in X$ if and only if for all $\tau \subseteq \sigma$ the Vandermonde determinant $\Delta(\tau) \in Q_{n,p}$. That is equivalent to $\alpha(\tau) \in Q_p \cup \{0\}$, where

$$Q_p \;:=\; \left\{ \beta^2 \;\mid\; \beta \in \mathbb{F}_p^{\times} \right\}$$

Q_p is the multiplicative subgroup of quadratic residues mod p. We let $Q_p^c := \mathbb{F}_p^{\times} \setminus Q_p$ denote the set of quadratic nonresidues.

To verify that X is r-ample, consider a set $U \subset \mathbb{F}_n$ of r vertices, and a subcomplex $Y \subseteq X_U$. We seek a vertex $x \in \mathbb{F}_n \setminus U$ such that for every $\sigma \in X_U$ the join $x\sigma \in X$ if and only if $\sigma \in Y$.

By the above characterisation of the simplices of X, it is sufficient for the desired vertex x to solve the following set of $2^r - 1$ constraints,

$$\forall\, \sigma \subseteq U, \qquad \alpha(x\sigma) \;\in\; \begin{cases} Q_p & \text{if } \sigma \in Y \\ Q_p^c & \text{if } \sigma \notin Y \end{cases} \tag{$\star$}$$

For every possible hypergraph Y on every set of r vertices $U \subset \mathbb{F}_n$, we show that this problem is indeed satisfiable.

We rewrite the above constraints on x. Clearly, the Vandermonde determinant of $x\sigma$ decomposes as follows

$$\Delta(x\sigma) \;=\; \pm \prod_{1 \le i < j \le d} (x_i - x_j) \prod_{v \in \sigma} (x - v) \;=\; \pm \Delta(\sigma) \prod_{v \in \sigma} \Delta(vx).$$

Applying the quotient map $\mathbb{F}_n^\times \to \mathbb{F}_n^\times / H \xrightarrow{\sim} \mathbb{F}_p^+$, where the second map is the isomorphism $gH \mapsto 1$, yielding the following congruence in $\mathbb{F}_p$,

$$\alpha(x\sigma) \;\equiv\; \alpha(\sigma) + \sum_{v\in\sigma} \alpha(vx).$$

We introduce r new variables, $\xi_v \in \mathbb{F}_p$ for each $v \in U$, related to x via $\xi_v = \alpha(vx)$, and obtain an equivalent reformulation of $(\star)$, with the $r+1$ variables $\xi_v \in \mathbb{F}_p$ and $x \in \mathbb{F}_n$.

$$\forall\, \sigma \subseteq U \qquad\qquad \alpha(\sigma) + \sum_{v\in\sigma} \xi_v \;\in\; \begin{cases} Q_p & \text{if } \sigma \in Y \\ Q_p^c & \text{if } \sigma \notin Y \end{cases} \tag{I}$$

$$\forall\, v \in U \qquad\qquad (x - v) \;\in\; g^{\xi_v} H \tag{II}$$

We now show that given any assignment to the r variables ξ_v there exists $x \in \mathbb{F}_n$ that satisfies (II). Indeed, applying Lemma 6.6.10 with $q = n$, $A = H$, $m = p$, and $d = r$, the number of $x \in \mathbb{F}_n$ satisfying $(x - v) \in g^{\xi_v} H$ for every $v \in U$ is at least

$$\frac{n}{p^r} \;-\; (r-1)\sqrt{n} \;-\; \frac{r}{p}.$$

Since $n > r^2 p^{2r}$, this lower bound is positive, and there exists at least one such solution $x \in \mathbb{F}_n \setminus U$. This reduces the problem to finding ξ_v that satisfy (I).

Let $U = \{u_1, \ldots, u_r\}$ in arbitrary order. Given $\sigma \subseteq U$, we call the "last" vertex u_i in this sequence is called the *top vertex* of σ. To be precise, $u_i \in \sigma$ and $u_j \notin \sigma$ for $j > i$. Since the constraints in (I) are labeled by $\sigma \subseteq U$ and include the variables are ξ_v for $v \in \sigma$, we determine $\xi_{u_1}, \ldots, \xi_{u_r}$ inductively, selecting each ξ_{u_i} according to the constraints where u_i is the top vertex. We abbreviate $\xi_i = \xi_{u_i}$ as no confusion can arise.

Supposing $\xi_1, \ldots, \xi_{i-1}$ are determined, the next variable ξ_i has to satisfy the 2^{i-1}

constraints where u_i is the top vertex.

$$\forall\ \{u_i\} \subseteq \sigma \subseteq \{u_1, \ldots, u_i\}, \qquad \xi_i + \delta(\sigma)\ \in\ \begin{cases} Q_p & \text{if } \sigma \in Y \\ Q_p^c & \text{if } \sigma \notin Y \end{cases} \tag{$\star\star$}$$

where

$$\delta(\sigma)\ :=\ \alpha(\sigma) + \sum_{v \in \sigma \setminus u_i} \xi_v\ \in\ \mathbb{F}_p.$$

is already known from the variables determined before ξ_i.

Before invoking Lemma 6.6.10 to show that ξ_i exists, one has to make sure that all $\delta(\sigma)$ in ($\star\star$) are distinct. This requires some care when selecting $\xi_1, \ldots, \xi_{i-1}$. Suppose that $u_i \in \sigma \cap \sigma'$ is the common top vertex of σ and σ', and $u_j \in \sigma \setminus \sigma'$ is the top vertex of their difference $\sigma \triangle \sigma' = (\sigma \setminus \sigma') \cup (\sigma' \setminus \sigma)$. Then the condition $\delta(\sigma) \neq \delta(\sigma')$ takes the form

$$\xi_j\ \neq\ \left(\alpha(\sigma') + \sum_{v \in (\sigma' \setminus \sigma)} \xi_v\right) - \left(\alpha(\sigma) + \sum_{v \in (\sigma \setminus \sigma') \setminus u_j} \xi_v\right)$$

Since the value on the right hand side is known when selecting ξ_j it can be avoided as long as there are enough other options as implied by Lemma 6.6.10. The number of forbidden values for ξ_j is at most the number of such pairs of simplices, $\{\sigma, \sigma'\}$, with common top vertex u_i and top "differentiating" vertex u_j. The number of these pairs is

$$\left(2^{j-1}\right)^2 (2^{r-j} - 1)\ <\ 2^{r+j-2}.$$

To sum up, in order to enable solutions for all ξ_i we will actually find 2^{r+j-2} potential solutions for every variable ξ_j, and proceed with one that evades every issue among $\delta(\sigma)$ as shown above.

We thus assume by induction that $\xi_1, \ldots, \xi_{i-1}$ are given and the 2^{i-1} constraints in ($\star\star$) have distinct translations $\delta(\sigma)$, and apply Lemma 6.6.10 with $q = p$, $A = Q_p$,

$m = 2$, and $d = 2^{i-1}$. This guarantees at least

$$\frac{p}{2^{2^{i-1}}} \;-\; \left(2^{i-1}-1\right)\sqrt{p} \;-\; \frac{2^{i-1}}{2}$$

possible values for the variable ξ_i. Since $p > 2^{2^r+2r}$ and $i \leq r$, this number is greater than

$$2^{2^{r-1}+2r} - (2^{r-1}-1)2^{2^{r-1}+r} - 2^{r-2} \;\geq\; 2^{2^{r-1}+2r-1} \;\geq\; 2^{r+i-2}$$

for all $i \leq r$, as required.

In conclusion, there exists $\xi_1, \ldots, \xi_r \in \mathbb{F}_p$ satisfying $(\star\star)$ for every i, and hence also (I). As shown above, this yields a vertex $x \in \mathbb{F}_n$ that satisfies (II) and hence $\mathrm{Lk}_X(x) \cap X_U = Y$ as required. $\square$

6.6.3 Estimating the smallest r-ample iterated Paley complex

In the rest of this section, we discuss the selection of parameters n and p for r-ample iterated Paley simplicial complexes.

The construction requires two primes satisfying $n \equiv 1 \bmod p$, that are large enough as in Theorem 6.6.1. Given a prime p, the existence of arbitrarily large primes $n \in p\mathbb{N}+1$ is a special case of the classical Dirichlet Theorem [30]. This case actually follows from an elementary argument. For $N > p$, let n be a prime divisor of $M = 1+N!+\cdots+(N!)^{p-1} = ((N!)^p - 1)/(N! - 1)$. As $n|((N!)^p - 1)$ we see that $n > N$. If $N! \equiv_n 1$ then $M \equiv_n p$, which is ruled out by $n|M$. Therefore, $N! \not\equiv_n 1$ while $(N!)^p \equiv_n 1$. By Fermat's little theorem $p|(n-1)$, as desired.

However, in order to establish our quantitative result, Proposition 6.6.2, we need a prime n roughly of order p^{2r}. Dirichlet's theorem asserts that about $1/(p-1)$ of all primes are contained in the arithmetic progression $p\mathbb{N}+1$, in an appropriate sense of asymptotic density [43]. The following lemma uses quantitative estimates of the "error term" to bound the gaps between these primes, which provides such a prime n that is not too large.

Lemma 6.6.11. *There exists a constant P, such that for every prime $p > P$ and every*

$M \geq p^8$ *there exists a prime* $n \equiv 1 \bmod p$ *in the interval*

$$M \ < \ n \ < \ \sqrt{p}\, M$$

Proof. For a and q coprime, the number of primes less than or equal to x that are congruent to $a \bmod q$ is denoted

$$\pi(x; q, a) \ = \ \left|\{n \leq x \ : \ n \text{ is prime}, \, n \equiv a \bmod q\}\right|.$$

Bounds on this number under various assumptions on the relation between q and x are given by the Brun–Titchmarsh theorem [64] and the Siegel–Walfisz theorem [74]. By recent improved bounds due to Maynard [61, Thms 1 & 2], there exists effectively computable positive constants Q and R, such that if $q \geq Q$ and $x \geq q^8$ then

$$\frac{R\, \log q}{\sqrt{q}} \cdot \frac{x}{\phi(q) \log x} \ < \ \pi(x; q, a) \ < \ \frac{2\mathrm{Li}(x)}{\phi(q)}$$

Here $\phi(q) = |\{a < q \ : \ (a, q) = 1\}|$ is Euler's totient function, and the function $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t} \sim \frac{x}{\log x}$ is the Eulerian logarithmic integral. In fact, $\mathrm{Li}(x) < \frac{3x}{2 \log x}$ will suffice for our needs.

Letting $q = p$ and $a = 1$, it follows for any $p > P = \max(Q, \exp(4/R))$ and $M > p^8$ that

$$\pi(M; p, 1) \ < \ \frac{3M}{(p-1)\log M} \ < \ \frac{R\, \log p \sqrt{p}\, M}{\sqrt{p}\,(p-1)\log(\sqrt{p}\, M)} \ < \ \pi(\sqrt{p}M; p, 1)$$

The middle inequality is verified by the observation that

$$\frac{3\log(\sqrt{p}\, M)}{R\, \log p\, \log M} \ < \ \frac{3}{R\, \log p}\left(\frac{1}{8} + 1\right) \ < \ 1$$

Since $\pi(M; p, 1) < \pi(\sqrt{p}M; p, 1)$, there exists at least one prime $n \equiv 1 \bmod p$ between M and $\sqrt{p}M$, as required. $\square$

Proof of Proposition 6.6.2. We now show there exist parameters satisfying the assumptions $p > 2^{2^r+2r}$ and $n > r^2 p^{2r}$ of Theorem 6.6.1, and $n = 2^{(2+o(1))r2^r}$.

In selecting p, we can just rely on *Bertrand's postulate*, i.e., for every $N \in \mathbb{N}$ there exists a prime between N and $2N$. Thus, there exists a prime p in the range

$$2^{2^r+2r} \ < \ p \ < \ 2^{2^r+2r+1}$$

Suppose that r is large enough so that p satisfies Lemma 6.6.11. We pick a prime $n \equiv 1 \bmod p$ in the range

$$r^2 p^{2r} \ < \ n \ < \ r^2 p^{2r+\frac{1}{2}}$$

Therefore, for r sufficiently large, there exists an r-ample iterated Paley simplicial complex $X_{n,p}$, on at most

$$n \ < \ r^2 2^{(2^r+2r+1)(2r+\frac{1}{2})} \ = \ 2^{2r2^r(1+o(1))}$$

vertices. □

Remark 6.6.12. We note that the construction of $X_{n,p}$ is *explicit* at least in the following sense. Given $r \in \mathbb{N}$, one can find suitable primes p and $n = \exp(O(r2^r))$ and a primitive $g \in \mathbb{F}_n$ in poly(n) time. One can also decide whether a given face belongs to the r-dimensional skeleton of $X_{n,p}$ in poly(n) time. These rough estimates leave some room for improvement, as the description of $X_{n,p}$ and such a face are in fact poly-logarithmic in n.

Appendix A

A.1 Bound used in Lemma 3.5.5

Lemma A.1.1. *Let $r \geq 2$ be a fixed integer and define a function*

$$Q_r(n,x) := \sum_{i=1}^{r} \binom{x}{i} \cdot \binom{n-x}{r+1-i},$$

we will think of $Q_r(n,x)$ as a polynomial in (n,x) of bidegree $(r, r+1)$. Define a new function

$$q_r(n,x) := \frac{r}{x} - \frac{\zeta \cdot r \cdot Q_r(n,x)}{n^r \cdot x}$$

for some arbitrary positive constant ζ.

The maxima of $q_r(n,x)$ over $[r+1, n/2]$ is attained at one of the two endpoints for sufficiently large n. Moreover, if $\zeta > \dfrac{r!}{r+1}$ and n is sufficiently large then

$$\max_{x \in [r+1, n/2]} q_r(n,x) \leq -\varepsilon_r + O(1/n) < 0$$

where $\varepsilon_r = \min\left\{ \dfrac{\zeta}{(r-1)!} - \dfrac{r}{r+1}, \sum i = 1^r \frac{2\zeta r}{2^r i!(r+1-i)!} \right\} > 0.$

Proof. The proof will rely on a few simple ideas. We will first show that there is just one $x \geq r+1$ for sufficiently large n such that $q_r'(n,x) = 0$, i.e. a unique positive stationary point. We will then show that $q_r'(n, r+1) < 0$ for large enough n. This then implies that our stationary point is either a minima or point of inflection and moreover that on $[r+1, n/2]$ the maxima is attained at either $x = r+1$ or $x = n/2$, we then just need some good estimates for $q_r(n,x)$ at these points to conclude the proof.

A simple computation shows that

$$q_r'(n,x) = \frac{-r}{x^2} - \frac{\zeta r}{n^r} \cdot \left(\frac{xQ_r' - Q_r}{x^2} \right),$$

which is equal to zero iff

$$\frac{Q_r - xQ_r'}{n^r} = \frac{1}{\zeta}. \tag{A.1}$$

Since $Q_r(n,0) = 0$ we may write $Q_r(n,x) = xP_r(n,x)$ where $P_r(n,x)$ is a polynomial in (n,x) of bidegree (r,r) where the coefficients $a_{i,j}$ of terms $n^i x^j$ vanish if $i + j \geq r + 1$.

Therefore,

$$Q_r - xQ_r' = -x^2 P_r'$$

where P_r' is of bidegree $(r, r-1)$. In fact, since $a_{r,i} = 0$ for all $i \geq 1$ we know that P_r' is of bidegree $(r-1, r-1)$. Therefore, we may rewrite equation (A.1) as

$$-x^2 \cdot \sum_{i+j \leq r-1} b_{i,j} \frac{n^i x^j}{n^r} = \frac{1}{\zeta} \tag{A.2}$$

where $b_{i,j} = (j+1) \cdot a_{i,j+1}$, and $b_{i,j} = 0$ for $i + j \geq r$. Observe that for $i + j \leq r - 1$ and for $x \leq n/2$

$$\frac{x^j}{n^{r-i}} \leq \frac{1}{n}$$

so equation (A.2) is of the form $O\left(\frac{x^2}{n}\right) = \text{constant}$. Therefore, for sufficiently large n there is at most one positive x satisfying (A.1), i.e. there is at most one stationary point of $q_r(n,x)$ in $[r+1, n/2]$.

We now compute

$$\begin{aligned}
q_r'(n, r+1) &= \frac{-r}{(r+1)^2} - \frac{\zeta r}{n^r} \cdot \left(\frac{(r+1)Q_r'(n, r+1) - Q_r(n, r+1)}{(r+1)^2} \right) \\
&\sim \frac{-r}{(r+1)^2} - \zeta r \left(\frac{(r+1)/r! - (r+1)/r!}{(r+1)^2} \right) \\
&= \frac{-r}{(r+1)^2} < 0.
\end{aligned}$$

That is for large enough n we know that $q_r(n,x)$ is initially decreasing, so the critical point found above must be either a minima or a point of inflection. Therefore we know that the largest value of $q_r(n,x)$ on $[r+1, n/2]$ comes at one of the two endpoints.

A simple approximation shows

$$Q_r(n, n/2) = \sum_{i=1}^{r} \frac{n^{r+1}}{2^r i!(r+1-i)!}\left(1 + O(1/n)\right).$$

We let $A_r := \sum_{i=1}^{r} \frac{1}{2^r i!(r+1-i)!}$ and remark that $q_r(n, n/2) = -2\zeta r A_r + O(1/n) \leq -\varepsilon_r + O(1/n)$.

It is easy to see that

$$\begin{aligned} Q_r(n, r+1) &= \sum_{i=1}^{r} \binom{r+1}{i} \cdot \left(\frac{n^{r+1-i}}{(r+1-i)!} + O(n^{r-i})\right) \\ &= \frac{(r+1)\cdot n^r}{r!} + O(n^{r-1}). \end{aligned}$$

Therefore we easily observe that

$$q_r(n, r+1) = \frac{r}{r+1} - \frac{\zeta}{(r-1)!} + O(1/n) \leq -\varepsilon_r + O(1/n),$$

which completes the proof. □

A.2 Computation for Lemma 4.3.4

Recall (4.12)

$$\mu(X,Y) = \prod_{s\in X} p'_s \cdot \prod_{s\in S-X} (1-p'_s) \cdot \prod_{s\in Y} \frac{p_s}{p'_s} \cdot \prod_{s\in X-Y} \left(1 - \frac{p_s}{p'_s}\right). \tag{A.3}$$

where $Y \subset X$ and that we wish to show that

$$(\pi_1)_*(\mu)(A) = \prod_{s\in A} p'_s \cdot \prod_{s\notin A}(1-p'_s) \tag{A.4}$$

$$(\pi_2)_*(\mu)(B) = \prod_{s\in B} p_s \cdot \prod_{s\notin B}(1-p_s) \tag{A.5}$$

where π_1, π_2 are projections onto the first and second complex respectively.

We will show (A.4) and note that (A.5) follows with an identical method,

$$\begin{aligned}
(\pi_1)_*(\mu)(A) &= \sum_{B\subseteq A} \mu(A,B) \\
&= \sum_{B\subseteq A} \prod_{s\in A} p'_s \cdot \prod_{s\notin A} (1-p'_s) \cdot \prod_{s\in B} \frac{p_s}{p'_s} \cdot \prod_{s\in A-B} \left(1-\frac{p_s}{p'_s}\right) \\
&= \prod_{s\in A} p'_s \cdot \prod_{s\notin A} (1-p'_s) \cdot \left[\prod_{s\in A} \left(1-\frac{p_s}{p'_s}\right) \cdot \sum_{B\subseteq A} \prod_{s\in B} \frac{\frac{p_s}{p'_s}}{1-\frac{p_s}{p'_s}}\right] \\
&= \prod_{s\in A} p'_s \cdot \prod_{s\notin A} (1-p'_s) \cdot \left[\prod_{s\in A} \left(1-\frac{p_s}{p'_s}\right) \cdot \sum_{B\subseteq A} (-1)^{|B|} \prod_{s\in B} \left(1-\frac{1}{1-\frac{p_s}{p'_s}}\right)\right] \\
&= \prod_{s\in A} p'_s \cdot \prod_{s\notin A} (1-p'_s) \cdot \left[\prod_{s\in A} \left(1-\frac{p_s}{p'_s}\right) \cdot \prod_{s\in A} \frac{1}{1-\frac{p_s}{p'_s}}\right] \\
&= \prod_{s\in A} p'_s \cdot \prod_{s\notin A} (1-p'_s)
\end{aligned}$$

as required. To go from line 4 to 5 in the above we used the equality

$$\prod_{u\in U} (1-x_u) = \sum_{V\subseteq U} (-1)^{|V|} \cdot \prod_{v\in V} x_v.$$

A.3 Proof of Lemma 5.6.1

Proof. We obviously have

$$1 = \prod_{\sigma\in F(L)} (p_\sigma + q_\sigma) = \sum_{J\subset F(L)} \left(\prod_{\sigma\in J} p_\sigma \cdot \prod_{\sigma\notin J} q_\sigma\right). \tag{A.6}$$

Note that in the above sum, J can be also the empty set. Denote by $A(J) \subset J$ the set of all simplices $\sigma \in J$ such that for any face $\tau \subset \sigma$ one has $\tau \in J$. Note that $A = A(J)$ is a simplicial complex, it is the largest simplicial subcomplex of L with $F(A) \subset J$. We also note that the set of external simplices $E(A|L)$ is disjoint from J.

Fix a subcomplex $A \subset L$ and consider all subsets $J \subset F(L)$ with $A(J) = A$. Any such subset $J \subset F(L)$ contains $F(A)$ and is disjoint from $E(A|L)$. Conversely, any subset

$J \subset F(L)$ containing $F(A)$ and disjoint from $E(A|L)$ satisfies $A(J) = A$.

Denoting $S(A) = F(L) - F(A) - E(A|L)$ and $I = J \cap S(A)$ we see that any term of (A.6) corresponding to a subset J with $A(J) = A$ can be written in the form

$$\left(\prod_{\sigma \in F(A)} p_\sigma \cdot \prod_{\sigma \in E(A|L)} q_\sigma\right) \cdot \left(\prod_{\sigma \in I} p_\sigma \cdot \prod_{\sigma \in S(A)-I} q_\sigma\right) \tag{A.7}$$

and the first factor above is $p(A)$, see (5.6). Hence the sum of all terms in the sum (A.6) corresponding to the subsets J with $A(J) = A$ equals

$$p(A) \cdot \sum_{I \subset S(A)} \left(\prod_{\sigma \in I} p_\sigma \cdot \prod_{\sigma \in S(A)-I} q_\sigma\right) = p(A) \cdot \prod_{\sigma \in S(A)} (p_\sigma + q_\sigma) = p(A). \tag{A.8}$$

We therefore see that the statement follows from (A.6). □

A.4 Sufficiently ample complexes are 2-connected: alternative proof

Theorem A.4.1. *For $r \geq 79$, every r-ample simplicial complex is 2-connected.*

Proof. Let X be a 79-ample simplicial complex, we will show that this X is 2-connected. Let $f : S^2 \to X$ be a continuous map. By the first part of Lemma 6.4.1 X is simply connected, so we are only required to show that the map f is null-homotopic. By the Simplicial Approximation Theorem f is homotopic to a simplicial map $g : \mathcal{T}_n \to X$ where $\mathcal{T}_n$ is some triangulation of the 2-sphere on n vertices. We will prove by induction on n that any such map is null-homotopic. Clearly if $n \leq 79$ we are done, so throughout we suppose $n \geq 80$. Suppose now that we have shown every $T_k \to X$ is null-homotopic for $k < n$.

Remark that the graph $\mathcal{T}_n^{(1)}$ is planar, so we may apply the Planar Separator Theorem [59] which states:

The vertices of an n vertex planar graph can be partitioned into three sets U, V, S such *that no edge joins a vertex in U with a vertex in V, neither U nor V contains more than $2n/3$ vertices, and S contains no more than $2\sqrt{2}\sqrt{n}$ vertices.*

We apply this to find disjoint vertex sets U, V, S that cover $V(\mathcal{T}_n)$ with $|U|, |V| \leq \frac{2n}{3}$ and a separator $|S| \leq 2\sqrt{2}\sqrt{n}$. For S to separate the 2-sphere the induced subcomplex $\mathcal{T}_S = (\mathcal{T}_n)_S$ must be homotopy equivalent to S^1. Let $U', V', S' \subset V(X)$ denote the images under g of U, V, S respectively. Let A, B be the induced complexes $X_{U'}$, $X_{V'}$ with C denoting the 1-cycle formed by the induced complex $X_{S'}$. Clearly A, B are supported by at most $\frac{2n}{3}$ vertices and C by at most $2\sqrt{2}\sqrt{n}$ vertices.

By Lemma 6.4.1, C is the boundary of some 2-disc D on at most $2\sqrt{2}\sqrt{n} + \frac{2\sqrt{2}\sqrt{n}}{79-3} + 1$ vertices. Therefore, the map g, which is equal to the composition

$$\mathcal{T}_n \to A \cup B \cup C \to A \cup B \cup D \to X,$$

is null-homotopic when both inclusions $A \cup D \to X$ and $B \cup D \to X$ are null-homotopic.

Both $A \cup D$ and $B \cup D$ are embedded 2-spheres on at most

$$\frac{2n}{3} + 2\sqrt{2}\sqrt{n} + \frac{2\sqrt{2}\sqrt{n}}{76} + 1$$

vertices. This is strictly smaller than n if

$$72n \cdot \frac{77^2}{76^2} < n^2 - 6n + 9,$$

which holds whenever $n \geq 80$, as we have assumed. By our inductive hypothesis we therefore have that both $A \cup D \to X$ and $B \cup D \to X$ are null-homotopic, which concludes the proof. $\square$

A.5 Proof of Lemma 6.6.10

Following works on Paley graphs and tournaments [8, 11, 37], we use character sums to prove Lemma 6.6.10. A multiplicative *character* of a finite field $\mathbb{F}_q$ is a map $\chi : \mathbb{F}_q \to \mathbb{C}$, such that $\chi(0) = 0$, $\chi(1) = 1$, and $\chi(ab) = \chi(a)\chi(b)$ for every $a, b \in \mathbb{F}_q$. Since χ is a homomrphism between the multiplicative groups, its image is all mth roots of unity, where $m = (q-1)/|\ker \chi|$ is called the *order* of χ.

The following estimate of character sums is based on the work of André Weil [15, 69]. This formulation appears in [57, Thm 5.41] or [43, Thm 11.23].

Theorem A.5.1 (Weil). *Let χ be a character of order $m > 1$ of a finite field $\mathbb{F}_q$, and let $f(x)$ be a polynomial over $\mathbb{F}_q$, that cannot be written as an mth power, $c \cdot g(x)^m$. If $f(x)$ has at most d distinct roots in a splitting field, then*

$$\left| \sum_{x \in \mathbb{F}_q} \chi\left(f(x)\right) \right| \;\leq\; (d-1)\sqrt{q}$$

Proof of Lemma 6.6.10. Let α be a primitive element in $\mathbb{F}_q$, and let $\omega = e^{2\pi i/m}$. In terms of the subgroup A of index m, we define the multiplicative order-m character $\chi(x) = \omega^t$ for every $x \in \alpha^t A$ and $t \in \mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$, and as usual set $\chi(0) = 0$.

Let $A_1, \ldots, A_d$ be A-cosets, and $c_1, \ldots, c_d$ be distinct field elements, as in the lemma. We define $t_1, \ldots, t_d \in \mathbb{Z}_m$ such that $A_i = \alpha^{t_i} A$, and consider the function

$$S(x) \;=\; \prod_{i=1}^{d} \left(\sum_{j=0}^{m-1} \left(\omega^{-t_i} \chi(x - c_i)\right)^j \right)$$

If x satisfies $(x - c_i) \in \alpha^{t_i} A$ then $\chi(x - c_i) = \omega^{t_i}$ and the ith factor equals m. Otherwise it is the sum of all mth roots of unity and therefore vanishes, except in the case $x = c_i$ where it contributes 1. It follows that $S(x) = m^d$ for every x that is counted in the lemma. Any other x attains $S(x) = 0$, apart from $x \in \{c_1, \ldots, c_d\}$ where $|S(x)| \leq m^{d-1}$.

In conclusion, if N is the number of $x \in \mathbb{F}_q$ that satisfy $(x - c_i) \in \alpha^{t_i} A = A_i$ for all $i \in \{1, \ldots, d\}$, then

$$\left| \sum_{x \in \mathbb{F}_q} S(x) \right| \;\leq\; Nm^d + dm^{d-1}$$

On the other hand, we expand the same sum over $S(x)$ into m^d different sums of characters.

$$\begin{aligned}\sum_{x\in\mathbb{F}_q} S(x) &= \sum_{x\in\mathbb{F}_q}\prod_{i=1}^{d}\left(\sum_{j=0}^{m-1}\omega^{-jt_i}\chi(x-c_i)^j\right)\\ &= \sum_{x\in\mathbb{F}_q}\sum_{j_1\cdots j_d}\left(\prod_{i=1}^{d}\omega^{-j_it_i}\right)\prod_{i=1}^{d}\chi(x-c_i)^{j_i}\\ &= \sum_{j_1\cdots j_d}\omega^{-\sum_i j_i\xi_i}\sum_{x\in\mathbb{F}_q}\chi\left(\prod_{i=1}^{d}(x-c_i)^{j_i}\right)\end{aligned}$$

The first term, which corresponds to $(j_1,\ldots,j_d)=(0,\ldots,0)$, is equal to q. Recall that $c_1,\ldots,c_d$ are distinct and $\max_i j_i < m$. By Weil's Theorem, it follows that each one of the other m^d-1 terms is bounded in absolute value by $(d-1)\sqrt{q}$. Therefore, by the triangle inequality,

$$\left|\sum_{x\in\mathbb{F}_q} S(x)\right| \geq q-(m^d-1)(d-1)\sqrt{q}$$

The lemma now follows by combining the two estimates of the sum, and solving for N. □

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