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DEPARTMENT OF ECONOMICS

Essays in Networks and Applied Microeconomic Theory

Kenan Huremović

Thesis submitted for assessment with a view to obtaining the degree of Doctor of Economics of the European University Institute

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Abstract

This thesis contains three papers which examine the role of networks and social structure in different modes of socio-economic interactions.

The first chapter focuses on purely competitive strategic bilateral interactions - contests. I analyse situations in which agents, embedded in a network, simultaneously play interrelated bilateral contest games with their neighbours. The network structure uniquely determines the behaviour of agents in the equilibrium. I also study the formation of such networks, finding that the complete k-partite network is the unique stable network topology. This implies that agents will endogenously sort themselves in partitions of friends, competing with members of other partitions. The model provides a micro-foundation for the structural balance concept in social psychology, and the main results go in line with theoretical and empirical findings from other disciplines, including international relations, sociology and biology.

The second chapter is joint work with my supervisor Fernando Vega-Redondo. We study a competitive equilibrium model on a production network of firms, identifying the measure of centrality in the network that determines the profit of a firm, and network structures that maximize social welfare. The significant part of this chapter focuses on how the network mediates the effects of revenue distortions on profits of firms and social welfare. The results are that the effects of distortions propagate both upstream and downstream through the network. The centrality of the affected firm determines the magnitude of the downstream effect, and the upstream effect is determined by the intercentralities of suppliers of the affected firm. Increasing the density of the network by adding links has a non-monotonic effect on welfare. Adopting a more complex production technology can increase but also decrease the profit of a firm, depending on the network structure; while finding a new buyer will always increase the profit of a firm.

In the third paper I analyse the interaction between formal legal enforcement of cooperation and the role of reputation in a heterogeneous population. By choosing to cooperate, even when the quality of the formal institution is not high, an agent signals that he has high work ethics, thereby earning reputation as a better match for future interactions. When there is reputation benefit, the welfare-maximizing quality of the enforcement institution is generally not the one that maximizes cooperation. Depending on the distribution of types in society, the effect of the increase in quality of enforcement on cooperation can be crowded in or crowded out by reputation concerns. When the institutional quality is determined endogenously, the equilibrium quality of the institution will generically be higher than the optimal quality.

Table of contents

List of figures

Chapter 1

Rent Seeking and Power Hierarchies: A Noncooperative Model of Network Formation with Antagonistic Links

There are a number of situations in which agents can increase the probability of favourable outcome of an competitive interaction by means of certain costly actions. For instance, in the case of armed conflict, this can be investment in weapons; or in the case of litigation, the costly action can be interpreted as hiring lawyers or bribing judges. We refer to this type of interaction as a contest. To be more precise, a contest is an interaction in which players can exert costly effort in order to extract resources from other players (transferable contest); or to receive a larger share of a pie to be divided. An agent does not always compete with just one opponent, but often with several different opponents simultaneously. Contests that an agent is involved in at the same time are related (i.e. an agent spends same costly resources for each contest) which creates (local) spillovers. Another type of spillovers (global) comes from the fact that an agent's opponent also can be involved in more than one contest, as well as his opponents and so on. Since how much will the agent spend in a particular contest depends on how much his opponents spend, this creates global spillovers that propagate throughout the network.

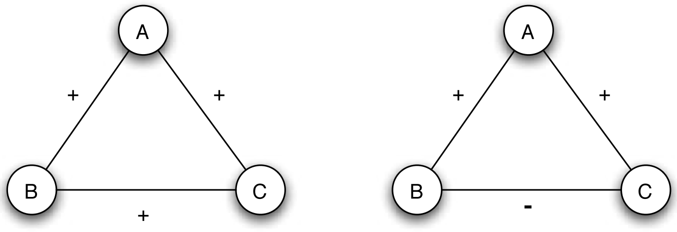
This environment can be represented and explained more effectively in the language of networks, letting $G = G(N, L)$ be a network, in which link $g_{ij} \in L$ between two agents $i, j \in N$ indicates the presence contest between i and j. In the paper we shall focus on the case where contests are transferable, and discuss the implications of different choices of modelling contest game in Section 1.5 and Appendix B.

There are a number of situations that can be described in this way. An example is the network of patent litigations and antitrust disputes discussed in (Sytch and Tatarynowicz, 2013). These types of lawsuits are often very intense and have significant consequences on the future of the company. They arise from the plaintiff's claims that an infringement has been made (patent litigation), or that a firm has adopted unfair competitive practices, including attempts to monopolize the market. The U.S. Federal District Courts registered about 10000 antitrust and 29000 patent infringement cases from 2000 to 2010 (Sytch and Tatarynowicz, 2013). These types of litigation have consequences for both conflicting parties. A plaintiff demands that a defendant refrains from injurious acts and to be be compensated for the losses due to patent infringement. On the other hand, the plaintiff risks being counter-sued (which often happens), even losing property rights. The costs of litigation are very high, reaching more that 5 millions USD per lawsuit, excluding damages and royalties. The transfers to be paid reach sums which are considerably higher than this. The firms can be, and usually are, involved in more than one litigation process at the same time. For example, in 2003 Lucent Technologies (acquired by Alcatel in 2006) filed suit against Gateway and Dell in U.S. District Court, San Diego, concerning violation of patent rights. Microsoft joined the lawsuit later that year. After this lawsuit was filed, Microsoft and Lucent have filed additional patent lawsuits against each other. Finally, the court ruled that the amount of damages to be awarded is in total 1.53 billion[1]. Apart from this, Microsoft has fought numerous legal battles against other firms. These include litigation processes with Apple, Netscape, Intel, Sun Microsystems, Stac Electronics and many others.

Mapping firms as nodes, and litigation processes as a links (indicating contests) we can construct a contest network. Other examples of contest networks include networks of international conflict, patent races, lobbying, Massive Multi-player Role Playing Games (MMORPG), school violence etc.

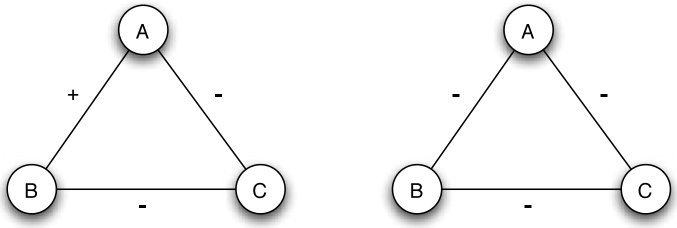
In this paper we first study a model on a fixed network, providing results for existence and uniqueness of the equilibrium. This is very important in order to study a network formation model, which is the main focus of this paper. In a formation model agents can form both positive links (friendship) and negative links

[1] Details available in (McDougall, 2007)



(a) *A, B, and C are mutual friends: balanced.*

(b) *A is friends with B and C, but they don't get along with each other: not balanced.*



(c) *A and B are friends with C as a mutual enemy: balanced.*

(d) *A, B, and C are mutual enemies: not balanced.*

Fig. 1.1 Strong structural Balance, (Easley and Kleinberg, 2010)

(antagonism, contest, conflict). We focus on the negative links, and interpret positive links as a self enforcing commitment of agents not to engage in a contest (the absence of a negative link). A negative link indicates that agents play a bilateral contest game. As the network changes, the effort that players exert in each particular contest will in general change. Thus, in a dynamic model of network formation, we study coupled evolution of network topology and play on the network.

The results of the formation model have important implications for the structural balance theory from social psychology. The theory of (strong) structural balance, originated in (Heider, 1946), applies to situations in which relations between agents can be either negative (antagonistic) or positive (friendship). It states that in groups of three agents, the only socially and psychologically stable structures are those in which all three agents are friends (all links are positive) or two of them are friends with the third being a common enemy (one positive and two negative links). In other words, a friendship relation is transitive. Figure 1.1 graphically illustrates Heider's theory.

As defined in (Heider, 1946), the structural balance can be seen as a local property of a network. A natural question then is: 'What are the global properties of networks that satisfy structural balance?'. That is, given a complete network, how can we sign links (indicating positive and negative) such that all triads of nodes in the network are structurally balanced. The Cartwright-Harary Theorem (Cartwright and Harary, 1956) provides the answer to this question. It states that there are two network structures that satisfy structural balance property: (i) all agents are friends (all links are positive) or (ii) agents are divided into two groups, and links within groups are positive and links across groups are negative. With respect to positive links, a network that satisfies structural balance will be the complete network or a network with two components that are cliques. With respect to negative links, it will be either the empty network or a complete bipartite network.

Extending on Heider's work, it has been argued in (Davis, 1967) that in many contexts we may witness a situation in which all links in a triad are negative. To encompass this type of configuration, he proposed the concept of weak structural balance. The implication for the global structure when allowing for this type of triads is an emergence of the additional balanced network structure. With respect to positive links this is a network with more than 2 components, and each component is a clique. With respect to negative links it is a complete k-partite network.

There are number of empirical papers that support (weak) structural balance in the real world networks. For example (Antal et al., 2006; Sytch and Tatarynowicz, 2013; Szell et al., 2010). On the theoretical side, there is no micro-founded model that explains the emergence of balanced networks. The exception is (Hiller, 2011), who provides a network formation model which results in balanced networks. However, the interaction between agents in his paper is modelled differently than in our paper. In (Hiller, 2011) agents do not make a decision how much to invest in negative relations and thus the paper does not tell anything about intensities of contests. Furthermore, the equilibrium concept used in (Hiller, 2011) is different than in this paper. Given the differences it is very interesting to note that qualitative results from (Hiller, 2011) are in line with qualitative results from this paper, which goes in favour of the robustness of the results from both papers.

This paper provides a micro-founded model of network formation that results with stable networks that are always weakly balanced(satisfy weak structural balance). The strong structural balance is satisfied in particular cases. It is important to note that the structural balance is a concept concerned only with the sign of links, but does not say anything about the intensities/weights assigned to links. Our model results with signed and weighted networks, and thus provides implications that go beyond the structural balance theory.

1.1 Literature Review

The paper is related to the several different streams of literature which we review in separate subsections.

1.1.1 Games on a Fixed Network

The common issues that arise when studying games on networks are the multiplicity of equilibria (even in very simple games), and intractability of analysis due to the complexity of the interaction structure.

One way to deal with these problems is to try to characterize the equilibria for specific classes of games. This is the approach used for example in (Ballester et al., 2006), which considers a class of games with quadratic payoff function. Another approach is to assume that players have incomplete information about network structure, which can sometimes simplify the analysis. This approach is, for example, pursued in (Galeotti et al., 2010). In this paper we consider a game with complete information, in which payoff function and best response functions of an agents are non-linear. The closest paper to ours is (Franke and Öztürk, 2009). Section 1.3 of this paper is can be seen as a generalization of the analysis in (Franke and Öztürk, 2009). However, they do not say anything about the network formation, which is the central issue stressed in this paper.

1.1.2 Network Formation

The main interest of our paper is the model where agents not only decide how much to invest in the bilateral contests but also with whom to play contest game. Thus, the paper is related to network formation literature, of which prominent examples are (Jackson and Wolinsky, 1996), hereafter JW and (Bala and Goyal, 2000), hereafter BG.

How to model a process of the network formation depends strongly on the link formation protocol which is adopted. In 'JW type models', links are formed bilaterally and destroyed unilaterally. The 'BG type models' assume unilateral formation and destruction. The link formation protocol, of course, depends on the interpretation of links. We propose a model in which the link formation protocol does not coincide with any of the two mentioned above, since the nature of links is fundamentally different. In our model links are formed unilaterally and destroyed bilaterally (only if both agents agree to do so). This is a natural link formation protocol given that the links represent transferable contest game. For example, for starting a war it is enough that one party declares it (or just attacks). To make a peace, both parties must commit not to fight.

There is a strong connection between the model considered in this paper and other network formation models, as we are interested in the same general questions - stability and efficiency of network structures. However, this paper considers more complex model since agents also make strategic decision on the investment in each link that is created. This makes the paper close to the literature on the formation of the weighted networks, but also to the literature that jointly considers the network formation and playing game on a network. For example, (Bloch and Dutta, 2009) consider a model of formation of communication network where agents derive positive benefits from the players they are connected to (both directly and indirectly). In their model, homogeneous agents have some fixed endowment and they need to decide how to allocate this endowment creating undirected links (with potentially different capacities) with others. Links can be created and destroyed unilaterally. This model is extended in (Deroïan, 2009) to the case of directed networks.

1.1.3 Contest Games

Informally, a contest is a game in which players decide (simultaneously or sequentially) on the level of effort in order to increase the probability of winning the (endogenous or exogenous) prize. The Contest Success Function (CSF)is a function that describes how the efforts determine the probability of winning the contest. An example is lobbying, where the prize can represent the value of a certain public policy that need to be adopted.

There are two prominent ways to model CSF. The first is to assume that the probability of winning is a function of ratios of efforts, which is introduced in (Buchanan et al., 1980). This is the approach we use in this paper. The second way to model CSF assumes that the probability of winning is a function of difference between effort levels. This approach is introduced in (Hirshleifer, 1989).

A nice, albeit dated, overview of literature on contests can be found in (Corchón, 2007). In this paper we consider transferable contests as introduced in (Hillman and Riley, 1989) using the variant of Tullock's specification introduced in (Nti, 1997). An alternative model, which is offered in Appendix B, gives a model formulation as a colonel Blotto game with Tullock CSF. There is a vast literature on Blotto games and we shall not review it here.

1.2 Bilateral Contest Game

In this section we introduce the bilateral contest game which will serve as a building block of the model. There are two players, i and j competing over a prize with exogenous size R. In order to increase the probability of wining, players choose a non-negative action (effort, investment). The strategy space is thus given with the set of non-negative real numbers $\mathbb{R}_0^+ := [0, +\infty)$. The effort is transformed into the contest specific resource by means of technology function ϕ. One can think of this function as an analogue to production function in a classic market setting. Here we assume that the technology function $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is the function that satisfies the following properties:

Assumption 1.1. *Technology function* $\phi : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ *is:*

(i) Continuous and twice differentiable

(ii) increasing and (weakly) concave ($\phi' > 0,\ \phi'' \leq 0$)

(iii) $\phi(0) = 0$

The first two assumptions are standard, while the third one states that zero effort implies zero contest input. The actions determine the probabilities of wining the prize trough the contest success function. We choose the Tullock ratio form specification of CSF suggested in (Nti, 1997), assuming that the probability that player i, when taking action s_{ij}, will win the contest against player j is:

$$p_{ij} = \frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} \tag{1.1}$$

In (1.1) $r \in \mathbb{R}_0^+$ determines the probability of a draw (no player wins the prize). In the paper we shall maintain the assumption that r is small.

Following (Hillman and Riley, 1989) we consider transferable contest game, that is a game in which the prize is transfer from loser to winner. Assuming a fixed prize, the payoff function of player i is given with

$$\pi_{ij} = p_{ij}R - p_{ji}R - c(s_{ij})$$

where R is a transfer from loser to winner. We assume that the transfer from i to j is the same as the transfer from j to i, although of course in general this does not have to be the case. The cost function $c : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is assumed to be continuously differentiable, increasing and convex.

The bilateral contest game has the unique and symmetric NE in pure strategies, which is interior for r low enough. In this case, the equilibrium strategy of player i is defined with the following implicit function:

$$\phi'(s_{ij}^*)R = (r + 2\phi(s_{ij}^*))c'(s_{ij}^*)$$

1.3 Game on a Fixed Network

Let $G = (N, L)$ be an undirected and unweighed network with set of nodes N and set of links L. The nodes represent players, and link $g_{ij} \in L$ indicates contest relation between players (when $g_{ij} = 1$). Let us also denote the set of agent i as N_i, and let $d_i = |N_i|$ denote the degree of node i. Strategy space of player i is the set $S_i = {\mathbb{R}_0^+}^{d_i}$. A (pure) strategy of player i is d_i-tuple of levels investments $\mathbf{s}_i = (s_{ij_1}, ..., s_{ij_{d_i}}) \in S_i$. We assume that the size of the transfer R is independent of the network structure and the same for every contest g_{ij}. [2], We normalize $R = 1$.

This paper focuses on the negative links. The absence of negative links can be interpreted as a commitment not to initiate contest and thus a positive (friendly) link.

The payoff of player i is given with:

$$\pi_i(\mathbf{s_i}, \mathbf{s_{-i}}\ G) = \sum_{j \in N_i} \left(\frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - \frac{\phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r} \right) - c(A_i) \qquad (1.2)$$

In 1.2, $A_i = \sum_j s_{ij}$ is the total effort of player i, and $\mathbf{s}_{-i}$ denotes strategies of players other than i. Such specification of cost function generates externalities between the contest that agent i is involved in, making it more interesting to study this model on a network.

It is clear that the payoff function π_i is twice differentiable on its domain. Furthermore, the payoff function of player i is concave in $\mathbf{s}_i$. To see this, note that

[2] We use g_{ij} when we talk about link $g_{ij} \in L$ but also when referring to contest between players i and j

$$\frac{\partial^2 \pi_i}{\partial {s_{ij}}^2} = \frac{(r + 2\phi(s_{ji}))\left(\phi''(s_{ij})(r + \phi(s_{ij}) + \phi(s_{ji})) - 2\phi'(s_{ij})^2\right)}{(r + \phi(s_{ij}) + \phi(s_{ji}))^3} - c''(A_i) < 0 \quad (1.3)$$

$$\frac{\partial^2 \pi_i}{\partial s_{ij} \partial s_{ik}} = c''(A_i) < 0 \; \forall j, k \in N_i$$

The inequality in 1.3 holds due to THE properties of function ϕ stated in Assumption 1.1, and the strict convexity of function c. Thus, the Hessian H_i of function π_i with respect to $\mathbf{s}_i$ is the sum of diagonal matrix H_{i1} with diagonal elements equal to:

$$\frac{(r + 2\phi(s_{ji}))\left(\phi''(s_{ij})(r + \phi(s_{ij}) + \phi(s_{ji})) - 2\phi'(s_{ij})^2\right)}{(r + \phi(s_{ij}) + \phi(s_{ji}))^3} < 0$$

and matrix H_{i2} which has all the elements equal to $-c''(A_i) < 0$. Matrix H_{i1} is negative definite and matrix H_{i2} is negative semidefinite, thus Hessian $H_i = H_{i1} + H_{i2}$ is negative definite. To be able to study network formation, we need to know if the equilibrium strategies on a fixed network are uniquely determined. In this section we prove the uniqueness of the equilibrium on a fixed network.

We shall prove two propositions. The first states that the equilibrium of the considered game is unique. The second will give conditions for the interior equilibrium. The first proposition relies on the results from (Rosen, 1965). For the sake of the presentation let us first introduce the following definition:

Definition 1.1. *A game is n persons concave game if (i) Strategy space of game S is the product of closed, convex and bounded subsets of m dimensional Euclidian space, $S = \{S_1 \times S_2 \times ... \times S_n | S_i \subset E^{m_i}\}$ [3] and (ii) payoff function of every player* $_2, ..., \mathbf{s}_n)$*, and concave in $\mathbf{s}_i \in S_i$, for each fixed value $\mathbf{s}_{-i} \in S_{-i}$*

Let us also introduce the function $\sigma : S \times \mathbb{R}_0^{+n} \to \mathbb{R}$ assigned to n persons concave game given with $\sigma(\mathbf{s}, \mathbf{z}) = \sum_{i=1}^{n} z_i \pi_i(\mathbf{s})$. Then, as proved in (Rosen, 1965):

1. There exists a pure strategy equilibrium of n persons concave game
2. If function σ is diagonally strictly concave for some $\mathbf{z} \geq 0$ then the equilibrium is unique

[3] Rosen actually proved more general result when strategy space is 'coupled', that is when $S \subset E^m = E^{m_1} \times E^{m_2} \times ... \times E^{m_n}$ is closed, convex and bounded set. Here we consider special case when strategy space is 'uncoupled'

Proposition 1.1. *There exists unique pure strategy Nash equilibrium of contestgame on a network.*

Proof. See Appendix A □

When the probability of draw is very small, players will always exert a positive level of effort in the equilibrium. The following proposition states exactly that

Proposition 1.2. *The equilibrium is interior when $r > 0$ is small enough*

Proof. See Appendix A □

In what follows, we will assume that r is chosen such that the interiority of the equilibrium is guaranteed. Note also that the above results imply that the equilibrium of the game on a fixed network is defined with FOC system of equations.

Consider now any two connected players i and j. The first order conditions that characterize their behaviour in a contest g_{ij} in the equilibrium are given with:

$$\left(\frac{(r+2\phi(s_{ji}))\phi'(s_{ij})}{(r+\phi(s_{ij})+\phi(s_{ji}))^2} - c'(A_i) = 0\right) \wedge \left(\frac{(r+2\phi(s_{ij}))\phi'(s_{ji})}{(r+\phi(s_{ij})+\phi(s_{ji}))^2} - c'(A_j) = 0\right) \tag{1.4}$$

From (1.4) we get:

$$\frac{(r+2\phi(s_{ji}))\phi'(s_{ij})}{(r+2\phi(s_{ij}))\phi'(s_{ji})} = \frac{c'(A_i)}{c'(A_j)} \tag{1.5}$$

As $\phi' > 0$ and $\phi'' \leq 0$ and $c'' > 0$ we have: $A_i > A_j \Leftrightarrow \frac{c'(A_i)}{c'(A_j)} > 1 \Leftrightarrow \frac{(r+2\phi(s_{ji}))\phi'(s_{ij})}{(r+2\phi(s_{ij}))\phi'(s_{ji})} \Leftrightarrow s_{ji} > s_{ij}$ where the last equivalence is due to the fact that ϕ is increasing and ϕ' is a decreasing function.

This means that in the equilibrium a player with lower total spending will win a contest with the higher probability. This observation reflects the fact that the more 'exhausted' player (one who spends more resources in the equilibrium) has the worst performance in a particular contest. It is because the additional unit of resources is more costly for him (his marginal costs are higher). Note that this does not necessarily mean that a player involved in more contests will have a higher total spending in the equilibrium, although the total spending is increasing with the number of opponents (keeping everything else fixed). Rather, the player with the worst performance will be a player who has higher number of contests and the contests that he is involved in are more intensive. Which contest will be more intensive, depends on the global position of the players in the network. The identification of the characteristic of a node in the network that would determine the total spending of that node in the equilibrium proved to be a very challenging task. A property of a node that would determine his total spending in the equilibrium is a nonlinear measure of (global)centrality of node in a network. Finding such a measure, although interesting, is a very complex task.

The equation (1.5) gives us an another interesting insight. Each link g_{ij} has two actions assigned to it: s_{ij} and s_{ji}. We can interpret these actions as a weights assigned to the directed links $i \to j$ and $j \to i$, respectively.

1.4 Network Formation

The fact that a player with higher total spending in the equilibrium, loses in expectation from a player with lower equilibrium spending (given that they play the game), gives some hints on how agents behave when contests are determined endogenously. But one must note that the results from the previous section are ex-post, and cannot be directly used in a network formation model. This is because the fact that $A_i^* < A_j^*$ in the equilibrium on network G does not imply that we will still have $A_i^* < A_j^*$ in the equilibrium on network $G + g_{ij}$ (where + denotes addition of the link g_{ij} to the network). When link g_{ij} is created, players i and j will, in general, change their efforts in all other contests that they are involved in. This will, furthermore, result in changes in the equilibrium actions of all opponents of i and j in all of their contests; all according to the system of nonlinear equations defined with (1.4). Given the general structure of the network, one can see why the effects of a link creation, which is a 'fundamental' action of network formation game, are in general case very hard to completely characterize. An example of the global effects that adding a link causes in a very simple network is given in Appendix C.

Because of the complex spillover effects, we shall assume that agents are not able to fully take into account these effects when making the decision to create or sever a link. Informally, we assume that when deciding on creating or destroying a link, agents do not take into the account the complex adjustment in actions that will occur in all other contests, due to the change of the network topology. Instead, they assume that all other actions in the network will remain constant when making this decision. If the action is to create a link, the assumption is that equilibrium efforts of that particular contest game will be according to the NE of the bilateral contest game discussed in section 1.2, keeping all other actions in the network fixed. We believe that the bounded rationality assumption here is justified due to the very complex nature of the spillovers.

In what follows we assume that r is sufficiently small, so that the equilibrium of the game on a fixed network is always interior. We shall also for simplicity assume that ϕ is identity mapping. However, all results hold when ϕ has a general specification from the previous section.

We consider two coupled dynamics processes. The first, which happens on the 'slow' scale, governs the evolution of network topology. The second, on a much faster scale, is what we call the action adjustment process. It is the process that describes how actions of players adjust to the new NE when network changes. The reason for the second process is to be consistent with the assumption of bounded rationality that we made in the network formation process.

Let us now be more precise. Time is indexed with $t \in \mathbb{N} \cup \{0\}$. In period $t = 0$ an arbitrary contest network $G(N, L)$ is given[4].

We say that network G is in the *actions equilibrium* when all players play the equilibrium strategy of a contest game on a fixed network described in Section 1.3.

Definition 1.2 (Actions equilibrium)**.** *A network $G(N, L)$ is in actions equilibrium if all actions s_{ij} and s_{ji} assigned to every link $g_{ij} \in L$ are part of equilibrium of a game on a fixed network.*

Given the definition we can describe the dynamics process that we consider:

For every period t:

(i) At the beginning of period t the network from $t-1$ is in the actions equilibrium

(ii) Random player i is chosen and updates her links according to the link formation protocol, resulting with network G_{t+1}

(iii) The second dynamic process (on the fast scale) starts, in which all agents update their strategies according to the process formally described in Subsection 1.4.1 (better reply dynamics), until the actions equilibrium is reached

[4]Due to the 'zero sum like' nature of the game, the empty network will always be stable in our model. In order to describe the dynamic process that leads to the non-empty stable networks we assume that, because of some non modelled mutation or a tremble, the initial conditions are given with the non-empty arbitrary network

Steps (ii) and (iii) deserve some further explanation. First let us define the link formation protocol.

Definition 1.3 (Link formation protocol)**.** *A link g_{ij} will be formed if player i or j decide to form it. A link g_{ij} will be destroyed if both i and j agree to destroy it.*

This means that the link formation is unilateral and the link destruction is a bilateral action. It is natural to define a link formation protocol for the antagonistic (purely competitive) relations in this fashion. A decision to start a contest (i.e. war, litigation) is unilateral by nature, and the 'attacked' player, whether she decides to fight back or not, cannot change that. To end a contest, it is necessary that both parties agree to do it. To our knowledge, this is the first paper that considers such a link formation protocol.

We assume that in each period t a random player can update his linking strategy according to the link formation protocol defined above. Given this, we define the stability concept as follows:

Definition 1.4 (myopically stable network)**.** *A network $G = G(N, L)$ is a myopically stable network if for any player i and any two (possibly empty) sets of nodes $A \subset N$ and $B \subset N$.*

$$\pi_i(G + \{g_{ij}\}_{j \in A} - \{g_{ij}\}_{j \in B}) > \pi_i(G) \Rightarrow (\exists j \in B) : \pi_j(G - g_{ij}) < \pi_j(G)$$

$$\pi_i(G + \{g_{ij}\}_{j \in A}) < \pi_i(G)$$

This definition assumes that no player wishes to change linking strategy - to destroy or create links. The possibility of replacing link is essential for the results. However it does not matter if a player can only replace one or more of his links or destroy/create one or more links at the same time. The results will (qualitatively) hold if we would consider a process in which an agent in a single period can only create a link, destroy a link or replace a link. That is if we would consider the following definition of stability:

Definition 1.5 (myopically stable network - alternative)**.** *A network $G = G(N, L)$ is Myopically stable network if the following conditions hold:*

$$\pi_i(G - g_{ij}) > \pi_i(G) \Rightarrow \pi_j(G - g_{ij}) < \pi_j(G) \ (\forall i, j \in N)$$

$$\pi_i(G + g_{ik} - g_{ij}) > \pi_i(G) \Rightarrow \pi_j(G - g_{ij}) < \pi_j(G) \ (\forall i, j, k \in N)$$

$$\pi_i(G + g_{ij}) < \pi_i(G) \ (\forall i \in N)$$

Let us now clarify what we mean exactly when we say that agents update the connections myopically. When deciding on his connections agent i knows the total spending of all players in the existing network. The effort levels (s_{ij}, s_{ji})[5] assigned to newly formed link are determined as the solution of a bilateral contest game, keeping all other actions in the network fixed. For example, in case of quadratic cost function $c(x) = \frac{1}{2}x^2$. When link g_{ij} is created the corresponding actions $s_{ij}$and s_{ji} are determined as the equilibrium actions of a bilateral contest game between players i and j keeping the spending of these two players fixed in all other contest.

$$\frac{2s_{ji} - r}{(s_{ij} + s_{ji} + r)^2} = (A_i + s_{ij}) \quad \wedge \quad \frac{2s_{ij} - r}{(s_{ij} + s_{ji} + r)^2} = (A_j + s_{ji})$$

The solution of this system is given with:

$$s_{ij} \;=\; \frac{2 + A'_i\left(A'_i + A'_j - \sqrt{4 + (A'_i + A'_j)^2}\right)}{2\sqrt{4 + (A'_i + A'_j)^2}} > 0 \tag{1.6}$$

and symmetric for s_{ji}. Here $A'_i = A_i - r/2$. Player i will wish to form g_{ij} link when:

$$\frac{s_{ij} - s_{ji}}{(s_{ij} + s_{ji} + r)} + A_i^2 - (A_i + s_{ij})^2 > 0 \tag{1.7}$$

and (s_{ij}, s_{ji}) are determined with (1.6), and analogously for player j.

On the other hand, existing link g_{ij} will be destroyed if both players agree to destroy it, that is when $\pi_i(\mathbf{s}_i, \mathbf{s}_{-i}, G - g_{ij}) > \pi_i(\mathbf{s}_i, \mathbf{s}_{-i}, G)$ and $\pi_j(\mathbf{s}_j, \mathbf{s}_{-j}, G - g_{ij})) > \pi_i(\mathbf{s}_j, \mathbf{s}_{-j}, G))$. This will be the case when:

$$A_i^2 - (A_i - s_{ij})^2 - \frac{s_{ij} - s_{ji}}{(s_{ij} + s_{ji} + r)} \geq 0 \wedge A_j^2 - (A_j - s_{ji})^2 - \frac{s_{ji} - s_{ij}}{(s_{ij} + s_{ji} + r)} \geq 0$$

A decision to destroy a link is, again, made assuming that all other actions in the network will remain fixed. The creation and the destruction of more links simultaneously is defined analogously. We also assume that a player will create a link only if it is strictly beneficial to do so. If a player is indifferent to keeping or destroying link, the link will be destroyed. So, a player prefers to have a smaller number of links incident to him. This could be justified by saying that there is some

[5] We omit time index t

infinitesimal fixed cost associated to maintaining a ink, which can be easily included in the model. The tie-breaking rule does not affect the results.

If after some period t^* no player wishes to destroy or create link we say that the process has reached the steady state. Thus, a network is stable when no player can myopically improve his payoff by changing his linking strategy.

Let us consider a network G which is in the actions equilibrium. We can sort the nodes in increasing order with respect to their total spending ($A_1 < A_2 < ... < A_K$), $K \leq n$ where K is the number of different total spending levels in a network. Note that we use A_i to denote both the total spending of player i and the $i-th$ smallest level of total spending in the network. From context it will be always clear what A_i stands for. Recall also that the equation (1.5) implies that in any bilateral contest, a node that has a larger overall spending loses in expectation.

Denote with $\mathcal{A}_i$ the class of nodes that have total spending A_i. Let $K \leq n$ denote the number of classes in network G. When a player $i \in N$ has the total spending A_i we denote that as $i \in \mathcal{A}_i$. We say that player i has control over link g_{ij} if it is beneficial for player j to destroy link g_{ij}. Thus, when player i is in control over a link it is completely up to him if the link will be destroyed.

If $A_i > A_j$ in the actions equilibrium we will say that player j is stronger than player i or that player i is weaker than player j. We shall refer to A_i as a strength of player i (higher A_i implies weaker player i). It is clear that when i is stronger than j then i controls link g_{ij}. Furthermore, both players i and j shall have control over link g_{ij} if this link is not beneficial for both of them. A link g_{ij} is said to be beneficial for player i if the creation of this link (if it does not exist) makes player i better off and if a destruction of this link (if it does exist) makes player i worse off.

In what follows we provide a characterization of stable networks. We proceed by stating and proving a series of propositions and lemmas. Abusing the notation, let $\pi_i(s^*_{ij}, g_{ij}) = \frac{s^*_{ij} - s^*_{ji}}{(s^*_{ij} + s^*_{ji} + r)} - c(A^*_i)$ denote the equilibrium payoff of player a from link g_{ab} in actions equilibrium. Then the following holds:

Proposition 1.3. *Let $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $c \in \mathcal{A}_k$ and $i < j < k$. Then $s^*_{ab} > s^*_{ac}$, $s^*_{ba} > s^*_{ca}$ and furthermore $\pi_a(s^*_{ab}, g_{ab}) < \pi_a(s^*_{ac}, g_{ac})$*

Proof. See Appendix A □

The previous proposition implies that the contest between two players who are more equal in strength is more costly to win. A strong player spends less when competing with weaker player and has higher payoff from that contest. The results of this proposition illustrates the incentive that a strong player has to compete with weak players, given that the transfer for every contest is the same. This effect is self-reinforcing in the sense that it further weakens the weak player making him a more likely target for other strong players.

For the sake of the exposition, let us state the following definition.

Definition 1.6. *Player $a \in \mathcal{A}_i$ is an attacker (winner) if he has all of his links with players from family of classes $\overline{\mathcal{A}}_i = \{\mathcal{A}_j | j > i\}$. Player $a \in \mathcal{A}_i$ is mixed type if there exist players b and c such that $g_{ab}, g_{ac} \in G$ and $A_b > A_a > A_{c.}$ Player $a \in \mathcal{A}_i$ is victim (loser) if he has all of his links with players from classes $\underline{\mathcal{A}}_i = \{\mathcal{A}_j | j < i\}$*

It is clear that every player i must be one of these types. Note also that in a stable network all attackers must have a positive payoff. If this is not true for some attacker i then, since he controls all of his links, he could deviate destroying his links and this deviation would be profitable. Furthermore, there cannot be a node in a stable network that isn't stronger than all opponents of some attacker, as an attacker always has an incentive to attack the weakest players. Lemma 1.1 and Corollary 1.1 explain why.

Lemma 1.1. *Let $a \in \mathcal{A}$ and $\mathcal{A}$ is the class of attackers. Let b and c be two nodes in the network such that $A^*_b \leq A^*_c$, $g_{ab} \in G$ and $g_{ac} \notin G$. Then the deviation of player a in which he replaces contest g_{ab} with g_{ac} is payoff improving.*

Proof. See Appendix A □

From Lemma 1.1 we have:

Corollary 1.1. *If in a stable network player $a \in \mathcal{A}_i$ has a link with player $b \in \mathcal{A}_j$ then she has a link with every player $c \in \mathcal{A}_{j+k}$ $k = 1, 2, ..K - j$*

Proof. See Appendix A □

Lemma 1.1 implies that if there is a player k in the network which is not stronger than at least one opponent j of player i, and if player i has control over g_{ij} link he will have an incentive to replace link g_{ik} with g_{ij}. Thus, there is always an incentive for a player to create a link (attack) a weaker player than one of his current opponents, if such player exists in the network. This furthermore implies that there cannot be more than one component in a stable non-empty network. Indeed, by Corollary 1.1 all nodes that have at least one link in the stable network must have a link towards the weakest player (except that player, of course) - as he is the most attractive player to extract resources from. This will result with connected network - in the simplest case with the star network. The following lemma formalizes this intuition.

Lemma 1.2. *A stable network must be connected if not empty*

Proof. See Appendix A □

From now on, we always talk about connected network. The previous lemmas stressed implications of the attractiveness of the weakest player as a victim. Let us focus now on the attackers. By definition, an attacker has a control over all of his links, that is up to him to destroy any of his links. Thus it is possible for an attacker to mimic a strategy of another attacker. Building on this observation we can state and prove the following:

Lemma 1.3. *If in a stable network two players belong to the same class of attackers $\mathcal{A}$ than they have the same neighbourhood*

Proof. See Appendix A □

Since all attackers in the same class have the same neighbourhood it must be that they have the same payoff in a stable network. Suppose that there is more than one class of attackers and that members of different classes have different payoffs. Since attackers have a control over their links, then members of the class with lower payoff have an incentive to update their linking strategy to match the strategy of members of the class with higher payoff. This process will eventually lead to the situation when all attackers in the network have the same payoff. Furthermore, the incentive to attack weaker players will push all attackers to have the same neighbourhood, and thus to become members of the unique class of attackers. The next Lemma states exactly that.

Lemma 1.4. *There is only one class of attackers in a stable network*

Proof. See Appendix A □

Note that Lemma 1.4 and Corollary 1.1 imply that the members of the unique class of attackers are connected to all other nodes in the network. This is due to the fact that class $\mathcal{A}_2$ must be a class of mixed types or losers. In either case, Lemma 1.4, together with the fact that two players from the same class cannot be connected in a stable network, implies that all members of $\mathcal{A}_1$ and $\mathcal{A}_2$ are connected. Then

Corollary 1.1 implies that attackers are connected to all other nodes in the network which don't belong to $\mathcal{A}_1$.

Let us now say something about mixed types in a stable network. Following the same reasoning as in the case of the attackers, we can conclude that all members of a mixed class must have the same neighbours with respect to players that are weaker than they are. The same thing will hold with respect to the neighbours that are stronger than they are. We prove this and formulate the result in Lemma 1.5.

Lemma 1.5. *In a stable network all members of all existing mixed type classes $\mathcal{A}$ are connected to all other nodes in the network except nodes belonging to their class.*

Proof. See Appendix A □

It is now immediate:

Corollary 1.2. *There is only one class of victims and all victims have same neighborhood*

Let us say something about the size of partitions in a stable network. Results from Lemma 1.1, Lemma 1.5 and the previous corollary imply that players will be connected to all players in the network except members of own class. In this case, one could expect that a player who belongs to larger class (has more friend) has a higher payoff in the equilibrium. This intuition is correct, and the following Lemma holds.

Lemma 1.6. *Let $|\mathcal{A}_k|$ denote the number of nodes that belong to class $\mathcal{A}_k$. Then $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ $\forall k \in \{1, ..., K\}$*

Proof. See Appendix A □

It is clear that $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ is not a sufficient condition for the stability of the network. The difference $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ must be large enough so that members of the stronger class do not find it payoff improving to delete links with members of the weaker class. Previous Lemmas imply the following proposition, which is the main result of this section.

Proposition 1.4. *A stable network is either an empty network or a complete k-partite network with partitions of different sizes. The payoff of members of a partition is increasing with the size of the partition and a total spending per node is decreasing with the size of a partition.*

Recall that the complete k-partite network is the only network topology that satisfies weak structural balance property, as discussed before. When the cost function is too steep, or the transfer size is too small, the only stable network would be a complete bipartite network. The complete bipartite network (with respect to negative links) is the only network topology that satisfies the strong structural balance property.

Not all complete k-partite networks will be stable. In order for them to be stable, no player must have an incentive to create or destroy a link. As only links that can be created are between players from the same partition, no player will wish to create a link. This is because link g_{ij} between players i and j such that $A_i = A_j = A$ cannot be profitable (they will exert the same effort in the equilibrium and thus win and lose contest with the same probability, and since the effort is costly, have a negative net payoff from contest g_{ij}). No player will wish to destroy a link if all links bring a positive payoff to the winner. Combining equilibrium conditions for players i and j we get that in equilibrium [6]

$$\left(s_{ij} = \frac{c'(A_j)}{c'(A_i)} s_{ji} \wedge \frac{2s_{ji}}{(s_{ij}+s_{ji})^2} = c'(A_i) \right) \Rightarrow s_{ij} = \frac{2c'(A_j)}{(c'(A_i)+c'(A_j))^2} \tag{1.8}$$

Using (1.8) we can express the sufficient conditions for stability of a network in terms of the total spending in the eqilibrium, that is we have that a complete k-partite network will be stable when for any contest g_{ij} we have:

$$2\frac{c'(A_j)-c'(A_i)}{c'(A_j)+c'(A_i)} > c(A_i) - c\left(\frac{2c'(A_j)}{(c'(A_i)+c'(A_j))^2}\right)$$

Let us consider a particular example of the complete bipartite network. Note that in this case (due to symmetry) agents that belong to the same partition will play the same strategy in every contest g_{ij} they are involved in. Then all contests g_{ij} will result with a positive payoff for winners iff members of the larger partition have a (total) positive payoff in equilibrium. Denote the two partitions with X and Y, and sizes of partitions with x and y respectively, and let $x > y$. Then total efforts of members of two partitions can be written as $A_X = ys_X$ and $A_Y = xs_Y$, where $s_i,\ i \in \{X, Y\}$ is the equilibrium effort level in each particular contest of members

[6] To simplify calculations we consider the case $r \to 0$

of partition i. Using (1.8) we get that:

$$\pi_X(s_X, s_Y) > 0 \Leftrightarrow y\frac{c'(A_Y) - c'(A_X)}{c'(A_Y) + c'(A_X)} - c(A_X) > 0 \Leftrightarrow \frac{c'(A_Y)}{c'(A_X)} > \frac{y + c(A_X)}{y - c(A_Y)} \tag{1.9}$$

With cost function $c(x) = \frac{1}{2}x^2$, $s_X = \sqrt{\frac{\sqrt{x}}{\sqrt{y}\left(\sqrt{x}+\sqrt{y}\right)^2}}$

The payoff of an agent from partition X is then:

$$\pi_X(s_X, s_Y) = b\frac{s_X - s_Y}{s_X + s_Y} - (bs_X)^2 = \frac{x\left(x - y - \sqrt{xy}\right)}{\sqrt{x} + \sqrt{y}} \tag{1.10}$$

and from here

$$\pi_X(s_X, s_Y) > 0 \Leftrightarrow x > y\left(\frac{3 + \sqrt{5}}{2}\right) \tag{1.11}$$

Thus, we have proved the following proposition:

Proposition 1.5. *For $c(x) = \frac{1}{2}x^2$ and $\phi(x) = x$ a complete bipartite network will be stable when $x > y\left(\frac{3+\sqrt{5}}{2}\right)$ where x and y are sizes of partitions.*

The payoff of players in the larger partition will be increasing with the size of the larger partition, and increasing in the number of players in the smaller partition for $b \leq \frac{b}{6}\left(14 + \sqrt[3]{1475 + 8\sqrt{41}} + \sqrt[3]{1475 - 8\sqrt{41}}\right) \approx 6.07b$, and decreasing otherwise. There are two effects on payoff of members of larger partition when increasing the number of players in the smaller partition. The first one is that the contests become more costly, as the members of smaller partition become 'stronger'. The second effect is that there are more opportunities to extract rents. Depending on which effect dominates, payoff of an agent from a larger partition will increase or decease with the size of partition.

1.4.1 Actions Adjustment Process

As we have discussed in the beginning of Section 1.4, after network structure is changed, players update their strategies in a myopic way until the actions equilibrium on the new networks is reached. In this subsection we describe this process and prove that it is globally asymptotically stable. The actions adjustment process is defined as follows:

$$\frac{d\mathbf{s}_i}{dt} = \alpha\nabla_i\pi_i(\mathbf{s}),\ \alpha > 0,\ i = 1, .., n \tag{1.12}$$

where $\pi_i(\mathbf{s}) = \pi_i(\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_i, ...\mathbf{s}_n)$ and $\nabla_i \pi_i(\mathbf{s}) = \begin{pmatrix} \frac{\partial \pi_i}{\partial s_{i1}} & \frac{\partial \pi_i}{\partial s_{i1}} & ... & \frac{\partial \pi_i}{\partial s_{id_i}} \end{pmatrix}$ is gradient of the payoff function with respect to $\mathbf{s_i}$. It is clear that Nash equilibrium is a steady state of this dynamics. We prove in what follows that NE is the globally asymptotically stable state of this dynamic system. Let us define a function $J : \prod_i [0, M]^{d_i} \rightarrow \prod_i [0, M]^{d_i}$ with:

$$J(\mathbf{s}) = \begin{pmatrix} \nabla_1 \pi_1(\mathbf{s}) \\ \nabla_2 \pi_2(\mathbf{s}) \\ ... \\ \nabla_n \pi_n(\mathbf{s}) \end{pmatrix} \tag{1.13}$$

Denoting with G the Jacobian of J with respect to $\mathbf{s}$, we can write system (1.13) in a more compact form

$$\dot{\mathbf{s}} = \alpha J(\mathbf{s}) \tag{1.14}$$

To prove global stability we need to show that rate of change of $||J|| = JJ'$ is always negative (and equal to 0 in equilibrium). So let us check $\frac{d}{dt}||J||$. We get:

$$\frac{d}{dt}JJ' = (G\dot{\mathbf{s}})'J + J'G\dot{\mathbf{s}} = (J'G'J + J'GJ) = J'(G' + G)J$$

As proved in (Goodman, 1980) the conditions (i)-(iii) discussed in the proof of Proposition 1 imply that $(G' + G)$ is a negative definite matrix. This implies that $\frac{d}{dt}JJ' < 0$ which is what we need to prove.

Thus, if every player adjusts his action according adjustment process in (1.14), the action adjustment process converges, irrespectively of the initial conditions. The process (1.13) can be made discrete without losing the convergence properties. The discussion from above proves the following proposition:

Proposition 1.6. *The action adjustment process is globally asymptotically stable*

1.4.2 Efficiency

It is easy to show that the unique network that maximize the total utility of society is the empty network. This is a direct consequence of the transferable nature of contest game as the rent seeking effort is wasteful. Indeed, the total payoff that society obtains from network G can be expressed as:

$$\begin{aligned} U(G) &= \sum_i \pi_i(\mathbf{s}_i, \mathbf{s}_{-i}; G) \\ &= \sum_i \sum_{j\in N_i} \left(\frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - \frac{\phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - c(A_i) \right) \\ &= -\sum_i c(A_i) \end{aligned} \tag{1.15}$$

From (1.15) we have:

Proposition 1.7. *The efficient network is an empty network*

1.5 Final Remarks

The model in some sense leads to a self-referential characterization of power in antagonistic relationships i.e. a player will be stronger in the equilibrium if his enemies are weaker (recall that we refer to the total spending of a node in the equilibrium as the strength of a player). The previous sentence illustrates the recursive nature of a node's strength in the network emphasised in this model. This is a common feature of global centrality/prestige measures in networks (i.e. Katz centrality, Bonacich centrality, PageRank).

There are two distinctive features of node strength in this model. The first is the negative relation between the strength of a node and its neighbours. That is, strong odd order neighbours (can be reached in path of odd length) will make an agent weaker, while the opposite is true for even order neighbours. The second feature is that strength is nonlinear measure of a centrality - the strength of a node is a nonlinear function of strength of its neighbours. If it were linear, the strength of a node would be a global linear centrality measure (like Katz-Bonacich centrality) but with negative decay factor β. In this model, although the logic is similar, the strength of a node is a non-linear centrality of a node defined with the FOC system of equations.

Replacing transferable contest with the 'classic' contest (i.e. one in which players compete to get a larger share of a pie) would not change the existence and uniqueness results from Section 1.3. However, as two types of contest have different interpretations, the link formation protocol in the formation model needs to be adjusted. In this case it is not clear why link destruction should be a bilateral decision, if the pie exists independently of the contest. Innovation contest/patent race is a situation which more natural to model in this way (Baye and Hoppe, 2003).This approach could be naturally extended to hypergraphs. For example, consider a situation in which there are n firms and m markets (possible contests) in which firms can innovate. Then a linking strategy of a firm would be to decide in which of these m contests to participate, creating a hyperlink to other participants in these contests. The results for existence and uniqueness for a fixed hypernetwork will hold if we specify a contest success function for market k as:

$$p_{ik} = \frac{\phi(s_{ik})}{\sum_{j \in (N^k)} \phi(s_{jk})}$$

where N^k is set of players competing in contest k, p_{ik} is the probability with which i wins the contest k and the rest of the notation is analogue to what we have from before. Ans finally, preliminary results indicate that when modelling positive links explicitly so that positive link g_{ij}^{+} has an effect of reducing the marginal costs of effort in the contest, the qualitative results of the paper will not change.

1.6 Appendix A: Proofs

Proof of Proposition 1.1: As discussed Section 1.3, the payoff function of every player i is continuous and concave in $\mathbf{s}_i$. The strategy space is in general unbounded, but since the transfer R is finite, and the cost function c is strictly increasing, there will exits a point $M \in \mathbb{R}$ such that $c(M) > R$. No player will ever wish to exert an effort larger than M. Therefore we can bound the strategy space from above. Thus there exist the pure strategy equilibrium of the game on a network as defined in the beginning of Section 1.3, as the game is n players concave game. To prove the uniqueness we will use the following specification of diagonally strictly concave function proposed in (Goodman, 1980). The function $\sigma(\mathbf{s}, \mathbf{z})$ will be diagonally strictly concave if the payoff functions are such that for every player i: (i) $\pi_i(\mathbf{s})$ is strictly concave in $\mathbf{s}_i$, (ii) $\pi_i(\mathbf{s})$ is convex in $\mathbf{s}_{-i}$ and (iii) $\sigma(\mathbf{s}, \mathbf{z})$ is concave in $\mathbf{s}$ for some $\mathbf{z} \geq 0$.

For the game that we are considering we have already shown above that π_i has a negative definite Hessian with respect to $\mathbf{s}_i$. We also have that:

$$\frac{\partial^2 \pi_i}{\partial s_{ji}^2} = \frac{(r + 2\phi(s_{ij}))\left(2\phi'(s_{ji})^2 - \phi''(s_{ji})(r + \phi(s_{ij}) + \phi(s_{ji}))\right)}{(r + \phi(s_{ij}) + \phi(s_{ji}))^3} > 0$$

when there is link g_{ij}. Furthermore, $(\forall g_{jk} \in L : k \neq i)$, $\frac{\partial^2 \pi_i}{\partial s_{jk}^2} = 0$ and $\frac{\partial^2 \pi_i}{\partial s_{jk} \partial s_{lt}} = 0$ for any other combination of players, j, k, l and t. Thus, the Hessian of π_i with respect to $\mathbf{s}_{-i}$ is a diagonal matrix with all entries positive or zero and therefore positive semi-definite.

To prove the concavity of $\sigma(\mathbf{s}, \mathbf{z})$ in $\mathbf{s}$ we choose $\mathbf{z} = \mathbf{1}$. Then:

$$\sigma(\mathbf{s}, \mathbf{1}) = \sum_i \sum_{j \in N_i} \left(\frac{\phi(s_{ij})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - \frac{\phi(s_{ji})}{\phi(s_{ij}) + \phi(s_{ji}) + r} - c(A_i) \right) = -\sum_i c(A_i)$$

This equality holds since the every summand $\frac{\phi(s_{ij})}{\phi(s_{ij})+\phi(s_{ji})+r}$ appears exactly once with a positive sign (as a part of payoff function π_i) and exactly once with a negative sign (as a part of function π_j). Function $-\sum\limits_i c(A_i)$ is strictly concave due to the strict convexity of function c. □

Proof of Proposition 1.2: Consider two arbitrary connected players i and j. Let us first prove that in the equilibrium it cannot be $s_{ij} = s_{ji} = 0 \ \forall r > 0$[7]. Assume

[7] We omit * with equilibrium actions in the rest of the proof, but it is clear when s_{ij} denotes

otherwise. Then the payoff for both players in contest g_{ij} will be 0. Consider the deviation of player i from $s_{ij} = 0$ to $s_{ij} = r$. Now the probability of wining for player i becomes $p_{ij} = \frac{\phi(r)}{\phi(r)+r} = \alpha > 0$ and the probability of losing is still 0. This deviation will be profitable as long as $c(\tilde{A}_i) - c(A_i) < \alpha$, where $\tilde{A}_i = A_i + r$. As c is continuous, we can always find such r so that $|c(\tilde{A}_i) - c(A_i)| < \alpha$ when $|\tilde{A}_i - A_i| \leq r$ Therefore, in this case we can always find such r so that the deviation from $s_{ij} = 0$ to $s_{ij} = \alpha$ is profitable. Thus, it cannot be that $s_{ij} = s_{ji} = 0$ in the equilibrium.

Let us now prove that for two arbitrary connected players i and j it cannot be that $s_{ij} \neq 0 \wedge s_{ji} = 0 \; \forall r > 0$. Again, suppose this is the case. Then the necessary conditions imply that in the equilibrium we have

$$\frac{\partial \pi_i}{\partial s_{ij}}|_{(s_{ij},0)} = \frac{(r + 2\phi(0))\phi'(s_{ij})}{(r + \phi(s_{ij}) + \phi(0))^2} - c'(A_i) = \frac{r\phi'(s_{ij})}{(r + \phi(s_{ij}))^2} - c'(A_i) = 0 \qquad (1.16)$$

We can always find r small enough such that (1.16) cannot hold for any value $s_{ij} > 0$ and $A_i > 0$. Indeed, since the reward is finite and the number of nodes in the network is finite, then A_i must be finite for any node i in the network. For any cost function $c \in C^2$ satisfying assumptions for cost function stated in Section 1.2, we can find $U > 0$ such that $c'(A_i) < U$ for every A_i. We can always choose $r > 0$ small enough such that $\frac{r\phi'(s_{ij})}{(r+\phi(s_{ij}))^2} > U \; \forall s_{ij} \in [0, M]$, since $\frac{r\phi'(s_{ij})}{(r+\phi(s_{ij}))^2} \to \infty$ when $r \to 0$ for any fixed s_{ij}. □

Proof of Proposition 1.3: Recall that FOC for any a, b contest are given as:

$$\frac{2s^*_{ab} + r}{(s^*_{ab} + s^*_{ba} + r)^2} = c'(A^*_b) \text{ and } \frac{2s^*_{ba} + r}{(s^*_{ab} + s^*_{ba} + r)^2} = c'(A^*_a) \qquad (1.17)$$

Expressing s^*_{ab} and s^*_{ba} from (1.17) we get that, in equilibrium:

$$s^*_{ab} = \frac{2c'(A^*_b)}{(c'(A^*_a) + c'(A^*_b))^2} - \frac{r}{2}, \; s^*_{ba} = \frac{2c'(A^*_a)}{(c'(A^*_a) + c'(A^*_b))^2} - \frac{r}{2} \qquad (1.18)$$

The function $f(x, y) = \frac{2c'(x)}{(c'(y)+c'(x)^2} - \frac{r}{2}$ is strictly decreasing in x as long as $x > y$ and strictly increasing when $x < y$. f is always strictly decreasing in y

$$\frac{\partial f}{\partial x} = \frac{2\left(-c'(x) + c'(y)\right)c''(x)}{\left(c'(x) + c'(y)\right)^3} \lesseqgtr 0 \text{ when } x \gtreqless y \text{ and } \frac{\partial f}{\partial y} = -\frac{4c'(x)c''(y)}{\left(c'(x) + c'(y)\right)^3} < 0$$

action in equilibrium

This, together with (1.18) and $A_a^* < A_b^* < A_c^*$ implies that $s_{ab}^* > s_{ac}^*$ and $s_{ba}^* > s_{ca}^*$.

To prove that $\pi_a(s_{ab}^*, g_{ab}) < \pi_a(s_{ac}^*, g_{ac})$ we use (1.18) and (after some algebra)get:

$$\pi_a(s_{ab}^*, g_{ab}) = \frac{s_{ab}^* - s_{ba}^*}{(s_{ab}^* + s_{ba}^* + r)} = 1 - \frac{2c'(A_a)}{c'(A_a) + c'(A_b)}$$

It is clear that π_a is strictly increasing in A_b due to strict convexity of function c. Thus, $A_c^* > A_b^* > A_a^* \implies \pi_a(s_{ac}^*, g_{ac}) > \pi_a(s_{ab}^*, g_{ab})$ □

Proof of Lemmma 1.1: From (1.18) we have that s_{ac} in case of the deviation is given with:

$$s_{ac}^* = \frac{2c'(A_c^* + s_{ca}^*)}{(c'(A_a^* - s_{ab}^* + s_{ac}^*) + c'(A_c^* + s_{ca}^*))^2} - \frac{r}{2}$$

and we can write:

$$s_{ab}^* = \frac{2c'(A_b^* - s_{ba}^* + s_{ba}^*)}{(c'(A_a^* - s_{ab}^* + s_{ab}^*) + c'(A_b^* - s_{ba}^* + s_{ba}^*))^2} - \frac{r}{2}$$

Because of the interiority of the equilibrium, $A_c^* + s_{ca}^* > A_c^* \geq A_b^* > A_b^* - s_{ba}^*$. Since $A_c^* + s_{ca} > A_b^*$ the Proposition 1.3 implies that this deviation is profitable. □

Proof of Corollary 1.1: Let us assume otherwise. If link g_{ab} is not profitable for player a then, as noted before, it is not profitable for player b. Then link g_{ab} cannot be part of a stable network. So it must be that link g_{ab} is profitable for player a. Let $c \in \mathcal{A}_{j+k}$ be a node such that link g_{ac} does not exist. Then, from the Lemma1.1 the deviation of player a in which she destroys link g_{ab} and creates link g_{ac} will be profitable. □

Proof of Lemma 1.2: Again, we use a proof by contradiction. So assume not [8]. Then there are at least two components. Choose two arbitrary components from the network and denote them with C_1 and C_2. Let us denote two players with the highest total spending in these components with $h_1 \in \mathcal{A}_{c_1}$ and $h_2 \in \mathcal{A}_{c_2}$. Assume, without the loss of generality, that $A_{c_1} \geq A_{c_2}$. Then, for any player in that attacks player h_2 (and there must be at least one) it is profitable to attack player h_1 instead. □

Proof of Lemma 1.3: Consider two nodes $a, b \in \mathcal{A}$. Let us first prove that they must have an equal degree. Suppose that this is not true. So suppose that the network is stable and W.L.O.G., that $d_b > d_a$ where d_i denotes degree of a node i. Let N_i

[8] We omit *, but it is clear from context that we are considering the equilibrium strategies

denote the neighbourhood of player i. It cannot be that $N_a \subset N_b$ because then the total spending of a and b could not be equal (they would not belong to the same class). If $N_a = N_b$, the proof is completed. If no,t there must be some node $h \in N_a \backslash N_b$ and some node $k \in N_b \backslash N_a$. Suppose, W.L.O.G., that $A_k \geq A_h$. Than it would be better for player a to replace link g_{ah} with link g_{ak} according to the Lemma 1.1. This is a profitable deviation which is a contradiction to the assumption that the network is stable. So it must be $d_a = d_b$.

Let us now prove that there must be $N_a = N_b$. Again, assume this is not the case. Than we can find two nodes $h \in N_a \backslash N_b$ $k \in N_b \backslash N_a$ such that, W.L.O.G., $A_k \geq A_h$. But then it would be better for player a to replace link g_{ah} with link g_{ak} according to the Lemma 1.1. Thus, network G cannot be stable. The assumption that $N_a \neq N_b$ led us to a contradiction and thus must be rejected. □

Proof of Lemma 1.4: We again use the proof by contradiction. Suppose there are two different classes of attackers and denote them with $\mathcal{A}_1$ and $\mathcal{A}_2$ and let $A_2 > A_1$. Since players in $\mathcal{A}_1$ and $\mathcal{A}_2$ are attackers they have control over all of their links. Since Lemma 1.3 implies that all members of the same class of attackers have the same neighbourhood, we restrict our attention to the representative nodes $a \in \mathcal{A}_1$ and $b \in \mathcal{A}_2$. Let us first prove that it must be $\pi_a = \pi_b$. Assume this is not the case. Then it must be that $N_a \neq N_b$. Since $A_2 > A_1$ there are two possible situations that we need to consider.

(i) $N_a \subset N_b$ then then if $\pi_a > \pi_b$ player b could mimic player a (as he is the attacker), and if $\pi_b > \pi_a$ the opposite will hold.[9]

(ii) $N_a \not\subset N_b \implies (\exists k \in N_a \backslash N_b \wedge \exists h \in N_b \backslash N_a)$. But then, if $A_k \geq A_h$ Lemma 1.1 implies that b has a profitable deviation, and if not, same Lemma implies that a has a profitable deviation.

We have proved that in a stable network it must be $\pi_a = \pi_b$. Since $A_2 > A_1$ then it must be that $d_b > d_a$ or the distribution of total spending $a's$ and $b's$ opponents is different. We show that in both cases there exist a possible deviation which makes one of the players better off.

Let us first consider the case when $d_b > d_a$. If $N_a \subset N_b$ we have (i) from above. So there must exist nodes $k \in N_a \backslash N_b$ and $h \in N_b \backslash N_a$. If $A_k \geq A_h$ then player b would be better off by replacing contest g_{bd} with g_{bc}. If not, player a can make an analogue profitable deviation.

[9] Recall that we assume that when a player is indifferent between two actions he prefers to have less links.

If $d_a = d_b$ then, since $A_2 > A_1$, the strengths (total equilibrium spending) of $a's$ opponents are different than strengths of $b's$ opponents. Let q be the strongest node from $(N_a \cup N_b) \backslash (N_a \cap N_b) \neq \emptyset$. If link g_{aq} exists, then it is profitable for a to switch from q to any node in the set $N_b \backslash N_a$. If g_{bq} exists, then the profitable deviation is switching from q to some node in $N_a \backslash N_b$, and the proof is completed. □

Proof of Lemma 1.5. If there are only two classes of nodes in network $\mathcal{A}_1$ and $\mathcal{A}_2$ then there are no mixed types. Suppose there are more than two classes in the network. Consider first the strongest mixed type class ($\mathcal{A}_2$). A node $m \in \mathcal{A}_2$ must be connected to all of the nodes in the class of winners $\mathcal{A}_1$ This is because as a mixed type m must be connected with at least one stronger player, which must be a winner because of the choice of m. Lemma **??** implies then that m must be connected to all players from the class $\mathcal{A}_1$. Let us now prove that all members of the class $\mathcal{A}_2$ have the same neighborhood. Suppose not. Let $\{m_1, m_2\} \subset \mathcal{A}_2 \wedge N_{m_1} \neq N_{m_2}$. We have $(\mathcal{A}_1 \subset N_{m_1} \wedge \mathcal{A}_1 \subset N_{m_2}) \implies ((N_{m_1}/N_{m_2}) \cup (N_{m_2}/N_{m_1})) \cap \mathcal{A}_1 = \emptyset$. Thus, if they differ, neighborhoods of m_1 and m_2 must differ only in the part where m_1 and m_2 have control over their links. It cannot be $N_{m_1} \subset N_{m_2} \vee N_{m_2} \subset N_{m_1}$ because then it cannot be $A_{m_1} = A_{m_2}$. Consider two nodes, $k \in N_{m_1} \backslash N_{m_2}$ and $l \in N_{m_2} \backslash N_{m_1}$. Note that sets $N_{m_1} \backslash N_{m_2}$ and $N_{m_2} \backslash N_{m_1}$ cannot be empty. If $A_k \geq A_l$ then m_2 has a profitable deviation switching from $g_{m_2 l}$ to $g_{m_2 k}$. If not, then m_1 has an analogue profitable deviation.

Let $\mathcal{A}_3$ be the third strongest class in the network. If this is the weakest class (if $K = 3$) then, by definition, all players from $m \in \mathcal{A}_2$ must be connected to some of the players of $\mathcal{A}_3$, because otherwise they would not be mixed types. Note that if player $i \in \mathcal{A}_3$ is connected to some player from class $\mathcal{A}_2$ that he is connected to all players from class $\mathcal{A}_2$ since we have shown that all members of class $\mathcal{A}_2$ have the same neighborhood. If there exists some player $j \in \mathcal{A}_3$ who is not connected to a player from $\mathcal{A}_2$ then he is connected only to players from $\mathcal{A}_1$ but then it cannot be $A_i = A_j$, that is i and j cannot belong to the same class. Thus, if $K = 3$ the claim holds.

If not, then $\mathcal{A}_3$ is a mixed type class. Corollary 1.1 implies that all members of $\mathcal{A}_1$ must be connected to all members of $\mathcal{A}_3$ since they are connected to all the members of $\mathcal{A}_2$ and $A_2 < A_3$. Suppose that there does not exist link g_{ij} such that $i \in \mathcal{A}_2$ and $j \in \mathcal{A}_3$. Since all players from $\mathcal{A}_2$ have the same neighborhood there aren't any links between members of class $\mathcal{A}_2$ and $\mathcal{A}_3$. This means that players from $\mathcal{A}_3$ lose only in contest with players from $\mathcal{A}_1$, so they have control over all of their links except those that connect them to players $\mathcal{A}_1$. Furthermore, $A_2 < A_3 \implies N_i \neq N_j$. As before, we first consider the case when $\pi_i \neq \pi_j$.

(i) $N_i \subset N_j$ then j can destroy links towards all players N_j/N_i and have same the payoff as i (if $\pi_i \geq \pi_j$), or player i can create links to all players in N_j/N_i (if $\pi_i < \pi_j$)

(ii) $N_i \not\subset N_j \implies (\exists k \in N_i \backslash N_j \wedge \exists h \in N_j \backslash N_h)$. But then, if $A_k \geq A_h$ Lemma 1.1 implies that j has a profitable deviation, and if not, same Lemma implies that i has a profitable deviation.

If $\pi_i = \pi_j$ since $A_2 > A_1$ then it must be that $d_j > d_i$ or that the distribution of total spending of $i's$ and $j's$ opponents is different. We show that in the both cases there is possible deviation which makes one of the players better off.

Let us first consider the case when $d_i > d_j$. If $N_i \subset N_j$ we have (i) from above. If not we have analogue of (ii).

If $d_i = d_j$ then, since $A_2 > A_1$, the strengths (total equilibrium spending) of $i's$ opponents are different than the strength of $j's$ opponents. Let q be the strongest node from $(N_a \cup N_b) \backslash (N_a \cap N_b) \neq \emptyset$. If link g_{iq} exists, then it is profitable for i to switch from q to any node in the set $N_j \backslash N_i$. If g_{jq} exists, then the profitable deviation is switching from q to some node in $N_i \backslash N_j$. We have shown that it cannot be that there are no links between $\mathcal{A}_2$ and $\mathcal{A}_3$, thus every player from $\mathcal{A}_2$ is connected to every player from $\mathcal{A}_3$.

Proceeding in the same way, we can show that all players from $\mathcal{A}_k$ must be connected to all players from $\mathcal{A}_{k+1}$. Since the number of nodes is finite, the number of classes is finite and this procedure reaches $\mathcal{A}_K$ in a finite number of steps[10]. □

Proof of Lemma 1.6: Suppose not. Note that FOC imply that $s^*_{ij} = s^*_{ih}$ $\forall \{i,j,h\} \in N \wedge \{j,h\} \in \mathcal{A}_l$. If $|\mathcal{A}_k| < |\mathcal{A}_{k+1}|$ Lemma1.5 implies that $A_k = \sum\limits_{i \neq k} |\mathcal{A}_i| s_{ki}$ and $A_{k+1} = \sum\limits_{i \neq k+1} |\mathcal{A}_i| s_{k'i}$ for any two nodes $k \in \mathcal{A}_k$ and $k' \in \mathcal{A}_{k+1}$. Recall that s^*_{ij} is strictly decreasing in A^*_i which implies that $s^*_{kj} > s^*_{k'j}$ $\forall j \in \{1,..,K\} \backslash \{k,k'\}$. Also, $A_k < A_{k+1} \implies s_{kk'} > s_{k'k}$. But then $|\mathcal{A}_k| < |\mathcal{A}_{k+1}| \implies (A_k = \sum\limits_{i \neq k} |\mathcal{A}_i| s_{ki} > A_{k+1} = \sum\limits_{i \neq k+1} |\mathcal{A}_i| s_{k'i})$, contradiction! It must be $|\mathcal{A}_k| > |\mathcal{A}_{k+1}|$ □

[10] If n is not finite the claim is easily proved using mathematical induction

1.7 Appendix B: An Alternative Formulation

Suppose that, instead of a general convex cost function, we have a Blotto type game. That is, each player is endowed with the equal amount of resources (time) and the strategy is how to distribute the resources across different contests. Note that this also defines the 'cost' function to be convex, as resources are free up to some point and then prohibitively costly.

So, suppose for simplicity $\sum_{j \in N_i} s_{ij} = 1 \ \forall i, j$. Keeping the same CSF the existence, uniqueness and interiority guaranteed by the results from (Rosen, 1965). Let λ_i denote the Lagrange multiplier associated to the budget constraint for agent i. The first order conditions that characterize behaviour in a contest g_{ij} are given with:

$$\begin{aligned}
\frac{(r + 2\phi(s^*_{ji}))\phi'(s^*_{ij})}{(r + \phi(s^*_{ij}) + \phi(s^*_{ji}))^2} - \lambda_i &= 0 \\
\frac{(r + 2\phi(s^*_{ij}))\phi'(s_{ji})^*}{(r + \phi(s^*_{ij}) + \phi(s^*_{ji}))^2} - \lambda_j &= 0 \\
\sum_{k \in N_i} s^*_{ik} &= 1 \\
\sum_{k \in N_j} s^*_{jk} &= 1
\end{aligned}$$

and from here, we get:

$$\frac{(r + 2\phi(s^*_{ji}))\phi'(s^*_{ij})}{(r + 2\phi(s^*_{ij}))\phi'(s^*_{ji})} = \frac{\lambda_i}{\lambda_j} \tag{1.19}$$

Thus, the role of λ_i is analogous to the role of A^*_i. Higher A^*_i implies a higher marginal cost of additional unit of effort, and λ_i is the shadow price of the resource for player i in this formulation.

1.8 Appendix C: Numerical Example

Let us consider the following example to illustrate the complexity of global effects in the network.

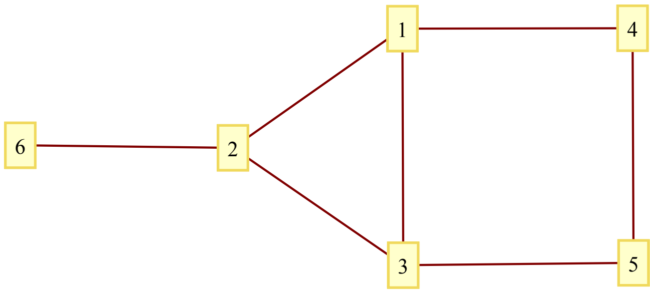


Fig. 1.2 Actions equilibrium - initial network

Calculating the equilibrium actions, we get that the matrix of the equilibrium efforts S is given with:

$$S = \begin{pmatrix} 0 & 0.289 & 0.289 & 0.286 & 0 & 0 \\ 0.292 & 0 & 0.292 & 0 & 0 & 0.269 \\ 0.289 & 0.289 & 0 & 0 & 0.286 & 0 \\ 0.350 & 0 & 0 & 0 & 0.354 & 0 \\ 0 & 0 & 0.350 & 0.354 & 0 & 0 \\ 0 & 0.479 & 0 & 0 & 0 & 0 \end{pmatrix}$$

And the assigned payoffs are:

$$\pi = (-0.854, -0.999, -0.854, -0.395, -0.395, 0.050)$$

Deleting link g_{13} we get a network with

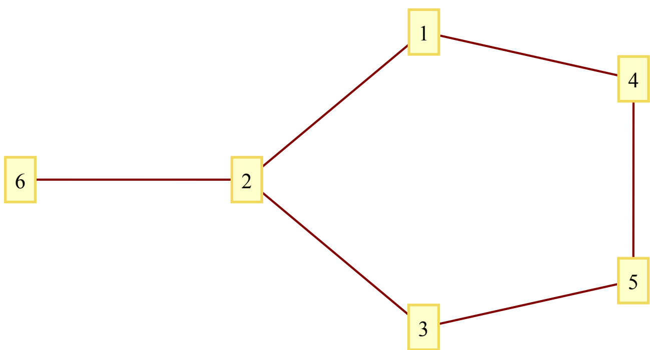


Fig. 1.3 Actions equilibrium - resulting network

and

$$\bar{S} = \begin{pmatrix} 0 & 0.351 & 0 & 0.354 & 0 & 0 \\ 0.290 & 0 & 0.290 & 0 & 0 & 0.270 \\ 0 & 0.351 & 0 & 0 & 0.354 & 0 \\ 0.353 & 0 & 0 & 0 & 0.353 & 0 \\ 0 & 0 & 0.353 & 0.353 & 0 & 0 \\ 0 & 0.480 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and payoffs

$$\pi = (-0.402, -1.193, -0.402, -0.501, -0.501, 0.048)$$

Chapter 2

Production Networks

with Fernando Vega Redondo

2.1 Introduction

The network nature of interconnections between firms by means of buyer−supplier relationships is receiving more attention in recent years in studying various kinds of economic phenomena; including transmission of economic shocks, trade, spread of innovation etc. Admitting the importance of the network structure, however, is not a new idea in economics. Studying input-output linkages historically has a long tradition, and it is usually associated with works of Wassily Leontief and A .O. Hirschmann (Hirschman, 1958).

Recently, the importance of intermediate goods has been emphasised in literature on economic fluctuations (Gabaix, 2011; Horvath, 1998). The main question in this literature is if and how microeconomic shocks to firms or sectors can cause significant macroeconomic fluctuations. The importance of the production network topology in explaining when a sizeable aggregate volatility can result from sectoral idiosyncratic technology shocks, in an environment similar to the baseline model in this paper, has been studied in (Acemoglu et al., 2012).

Our work is also related to (Jones, 2011) who, building on (Long and Plosser, 1983), studies how misallocation on the sectoral level affects GDP.

This paper studies a general equilibrium model on the production network, building on (Long and Plosser, 1983). We are primarily interested in answering the following questions: (*i*) How does the position of a firm in the network affect the profit of a firm? (*ii*) What kind of networks maximize social welfare? (*iii*) Which firms are the most important for the profits of other firms? (*iv*) How distortions propagate through the network and affect the profit of firms and social welfare? (*v*) How do changes in the network topology affect the distribution of profits across firms and social welfare?

The role of the network in mediating the effects of distortions is very important in the context of economic development. Promoting a firm or a sector has an indirect effect on other firms in the economy through backward and forward linkages in the production network. The backward linkages are demand-driven stimuli by which a sector generates demand (pull effect) that helps developing other sectors. In contrast, the forward linkages are associated with the supply effect that a sector may have in lowering the production costs of other sectors which rely on the former for intermediate inputs (push effect). The backward and forward linkages have an important role in determining the effects of a firm or a sector level shocks/policies on the performance of the economy. A key insight of the unbalanced growth theory as a strategy for economic development (propounded by A.O. Hirschmann in (Hirschman, 1958) among others) is the need for policies that promote 'strategic sectors' of the economy instead of all the sectors simultaneously. The other sectors would automatically develop themselves through the linkage effects. Different kinds of policies favouring some sectors over others, will in general generate different pull and push effects. Therefore, understanding the role of forward and backward linkages is crucial when designing a policy to promote economic development. The model in this paper provides a natural framework for the identification of 'strategic sectors' and for understanding the role of the production network in mediating effects of different kinds of policies and shocks.

The interest in the distribution of profits comes from the fact that profit is one of the most significant factors that influence a firm's survival in the market (Hopenhayn, 1992). Thus profits determine which firms will leave the market, and therefore the network topology that will materialize in the long run.

This paper is organized as follows. In Section 2.2 we present the benchmark model and basic results, including the relation between Bonacich centrality and the profit of a firm in the equilibrium. We also identify the welfare maximizing network topologies. In Section 2.3 we study how different kinds of distortions affect profit distribution and welfare. Subsections 2.3.1 and 2.3.2 provide results regarding the effects of revenue distortions on the distribution of profits across firms and the welfare. In Subsection 2.3.3 we study effects of changing the network topology.

Section 2.4 discusses the Hirschmann's pull vs. push dichotomy, explaining the channels through which different distortions propagate through the network. Section 2.5 contains the general formulation of the model, accommodating heterogeneity across multiple dimensions.

2.2 The Model and Basic Results

In this paper we study economy as a network of production relationships among competitive firms which in the end delivers some net flow of final goods to consumers. The model builds on the model proposed in (Long and Plosser, 1983).

There are infinitely many consumers that can be represented with a representative consumer. Consumers are owners of firms and of the only primary resource, which we simply identify with labour. Each firm produces single commodity that can be consumed or used as an input (intermediate good) for the production of some other good. Denote the set of commodities by $N = \{1, 2, ..., n\}$ and let $M \subseteq N$ be the set of consumption goods. A good is the consumption good if it generates the positive utility to the consumer. In general, we don't require that all produced goods are necessarily consumption goods.

The consumption side of economy is modelled through a representative consumer who has a Cobb-Douglas utility function $U : \mathbb{R}^s :\to \mathbb{R}$ given by:

$$U(\mathbf{c}) = A_c(m) \prod_{i=1}^{m} c_i^{\gamma_i} \tag{2.1}$$

In (2.1) $m = |M|$ is the number of the consumption goods, and γ_i is the share of good i in the consumption basket. We shall normalize $\sum_{i=1}^{m} \gamma_i = 1$. For the simplicity we shall often focus on the case $\gamma_i = \frac{1}{m}$ $\forall i \in M$, that is the case when all consumption goods are symmetric. The analysis goes through without the symmetry assumption, and qualitatively the results remain the same. Note that the leisure is not an argument in the utility function defined above, so the total amount of labour will be supplied inelastically to the market. It might also be interesting to include leisure into the arguments of function (2.1), since this would endogenize the amount of the primary resource devoted to the production, as a function of the effectiveness of the production system in addressing consumption preferences. In the benchmark model however, we adopt the simplifying assumption of inelastic supply of labour, and normalize the total amount of labour to be 1. $A_c = A_c(m)$ captures how the scope of the market impinges on consumer preferences reflecting the 'taste for variety' of the representative consumer in a similar fashion as in (Benassy, 1996). As discussed in (Benassy, 1996) the 'taste for variety' parameter is the most often in almost arbitrary manner identified with the degree of substitutability between goods in CES utility function. In order to disentangle between the 'taste for variety' as a preference for having more consumption goods, and the degree of substitutability between goods, (Benassy, 1996) introduces an analogue term to $A_c(m)$ in the utility function. This way, the taste for variety can be analysed independently of the degree of substitution between goods. The similar approach has been adopted in subsequent works, for example in (Benassy, 1996) and (Acemoglu et al., 2007). Following these papers we shall assume specific form $A_c(m) = m^{\xi}$, where parameter $\xi > 0$ captures the extent of the taste for variety. Values $\xi > 1$ imply that the consumer prefers to have more consumption goods in the basket.

As noted above, we assume that there is a one-to-one correspondence between firms and goods (thus, in particular, we rule out joint production). The production of each good k takes place under decreasing returns to scale, and requires both labour and intermediate produced goods as inputs. The set of intermediate inputs firm k uses in the production is denoted with N_k^+ and $n_k = |N_k^+|$. Let l_k be the amount of labour and $(z_{jk})_{j\in N_k^+}$ the amount of intermediate goods used for production of good k. Firm k produces amount y_k of good k using the production function $f_k : \mathbb{R}^{n_k} \times \mathbb{R} \to \mathbb{R}$ which has the following Cobb-Douglas formulation:

$$y_k = f_k\left((z_{jk})_{j\in N_k^+}; l_k\right) = A_k l_k^{\beta} \left(\prod_{j\in N_k^+} z_{jk}^{g_{jk}}\right)^{\alpha} \tag{2.2}$$

where we assume $\sum_{j\in N_k^+} g_{jk} = 1$ and $\alpha + \beta < 1$ [1]. The first assumption is just a normalization, while the second implies decreasing returns to scale.

To capture the idea that more advanced (productive) technology involves a greater range of intermediate inputs, and thus a higher degree of specialization, we follow (Benassy, 1998) and set $A_k = n_k^{\alpha+\kappa}$. With this formulation, parameter κ captures the extent to which input diversity (or production complexity/sophistication) enhances productivity. To see this heuristically, suppose that firm k has a total amount of U euros to spend among n_k intermediate inputs used in the production

[1] In general we shall allow α and β to differ across firms, i.e. we shall allow for firm specific α_k and β_k and also for any other pattern of asymmetry across goods in Section 2.5

of good k. Then, if the price of each input is equal, firm k splits U equally among the n_k inputs. Fixing the amount of labour used, this implies that production is proportional to $n_k^{\alpha+\kappa}U^\alpha\left(\frac{1}{n_k}\right)^\alpha = n_k^\kappa U^\alpha$. Thus, $\kappa > 0$ indicates that there are benefits from sophistication, and parameter κ captures this effect. A way capture the idea that production complexity enhances productivity is to use the CES production function $f(\mathbf{x}) = (\sum_i^n x^\rho)^{\frac{1}{\rho}}$. However, in this representation ρ determines both elasticity of substitution between inputs and the elasticity of output with respect to the complexity of the production technology (number of inputs). To avoid this issue authors often include a term analogous to A_k defined here, in order discriminate between these two (see for example Benassy (1998)).

The set N_k^+ of inputs used in the production of good k is represented as a set of in-neighbours in a weighted adjacency matrix of the production network $G(N, L) = (g_{ij})_{i,j\in N}$. Each vertex in G corresponds to a firm (good) in the economy. A directed edge $(i, j) \in L$ means that firm j uses a good i as an input. The weight of a link, g_{ij}, represents the share of good i among intermediate goods that firm j uses for the production[2].

Given the matrix G prevailing at some point in time Walrasian equilibrium $[\mathbf{p}, \mathbf{c}, \mathbf{y}, (z_{ij}), \mathbf{l}, w]$ is obtained. Here, $\mathbf{p}$,$\mathbf{c}$,$\mathbf{z}$ and $\mathbf{l}$ are vectors of prices, consumptions, outputs and labour demands respectively. Matrix $(z_{ij})_{i,j\in N}$ is a demand matrix, where z_{ij} represents a demand for input i by firm j. The wage is denoted with w. Given this description of economy, there exists the unique equilibrium. The uniqueness result is easily established as the vector of excess demand functions satisfies the gross substitutability condition.

The formulation of the production function (2.2) has the implication that labour

[2]Conceptually, one can think of a good (a node in the network) as defining a sector having several (possibly many) firms producing perfect substitutes, identical with respect to their production technology. Then a firm's production function can be written in form of:

$$y_k = f_k\left((\mathbf{z_{jk}})_{j\in N_k^+}; l_k\right) = A_k l_k^\beta \left(\prod_{j\in N_k^+}\left(\sum_{i\in S_j} z_{jk}^i\right)^{g_{jk}}\right)^\alpha$$

where S_j is the set of firms (sector) producing the same good as firm j; and i is a superscript indicating firms in that sector. Then, given that every good is produced by many firms, the competitive equilibrium is a natural equilibrium concept to be used. In the competitive equilibrium all goods from sector S_j will have the same price p_j. The exact demands for goods produced by firms in sector S_j by firm k denoted with $\mathbf{z_{jk}}^*$ are then undetermined in the equilibrium, but the total demand for goods provided by that sector is uniquely determined. In this paper, for the simplicity, we treat all the firms that produce the same good as a single firm and use the terms good and firm interchangeably, keeping this note in mind.

and at least one intermediate good are essential inputs for production. This implies that no firm can be active (i.e. display a positive production in the equilibrium) unless it relies on some other active firm. This imposes a natural 'systemic balance' condition for an economic system to be viable at all. It also requires that, from an originally empty technological structure, a viable 'innovation' can materialize only if there arise complementary (balancing) innovations in the production of some of its inputs. Similar requirements can be found in the definition of auto-catalytic sets used in evolutionary biology and chemistry, see for example (Jain and Krishna, 1998).

For the sake of the formal convenience, let us denote with $\hat{G}$ the adjacency matrix of a graph created from the production network by adding a node c representing the consumer. The consumer node has an outlink to every node in graph, except to itself, since the consumer supplies labour to every firm in the economy. On the other hand, node c has an inlink from every node that produces a consumption good.

Let $dp(i,j)$ denote a directed path from i to j in the directed graph $\hat{G}$. We define the notion of a strongly connected component of graph $\hat{G}$ as follows:

Definition 2.1 (SCC). *A set of nodes $K \subset N$ is a strongly connected component (SCC) in a directed graph $\Gamma(N, L)$ if $\forall (i \in K \wedge j \in K) \, \exists \, dp(i,j)$*

We shall say that a firm is active if it produces a nonnegative output in the equilibrium. Then from the definition of the production function the following result is straightforward:

Lemma 2.1. *A firm i is active the corresponding node in network $\hat{G}$ satisfies $\exists (j \in SCC) \, : \, \exists (dp(j,i) \wedge dp(i,c))$*

Proof. Follows directly from the fact that both labor and at least one intermediate good from an active firm is essential for the production □

Figure 2.1 illustrates active firms (red) non-active firms (yellow) and the consumer node (grey node 11) in a simple network:

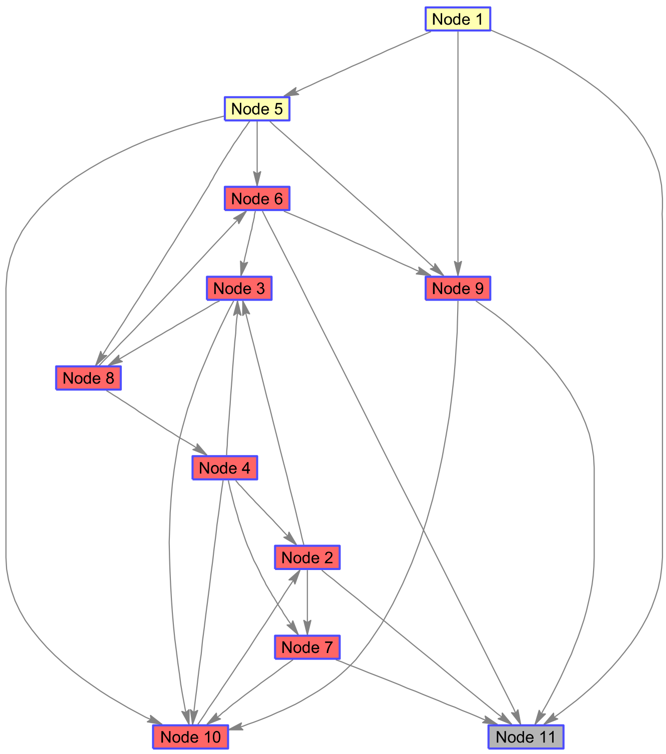


Fig. 2.1 An illustration of Lemma 2.1

In what follows we will consider networks of active firms. That is, the condition from Lemmma 2.1 is satisfied for every node in the network. If there are non-active firms in equilibrium, they can be simply ignored together with incident links without any consequence.

One of the first natural questions that arises in the environment described with this model is if and how the position of a firm in a network determines the profit of that firm. To answer this question, let us define the binary vector $\mathbf{h}$ to be an indicator vector such that $h_i = 1$ means that good i is the consumption good, and let m be the number of consumption goods. For simplicity we treat here all consumption goods symmetrically, and the more general formulation is in Section 2.5. The following proposition provides a clear cut characterization of the profits of each firm in the equilibrium:

Proposition 2.1. *The vector of profits of the active firms in the economy is given by:* $\boldsymbol{\pi} = (1-\alpha-\beta)\,\frac{w(1-\alpha)}{m\beta}(I-\alpha G)^{-1}\mathbf{h}$. *If all goods are consumption goods ($m = n$)*

then $\frac{(1-\alpha)}{m}(I-\alpha G)^{-1}\mathbf{h} = \left(\frac{p_i y_i}{\sum_{j=1}^{n} p_j y_j}\right)_{i=1}^{n}$

Proof. Aee Appendix A □

To discuss the intuition behind Proposition 2.1 we first introduce a notion of the Bonacich centrality that we use in this paper. As the Bonacich centrality[3] is defined in the literature in slightly different ways let us state here the definition that we shall employ.

Definition 2.2. *Consider a network with* $n \times n$ *adjacency matrix* G*, scalar* α *and* n*-dimensional vector* $\mathbf{h}$*. The vector of Bonacich centralities of graph* G *with discount factor* α*, weight vector* $\mathbf{h}$ *and scaling parameter* ζ *is given with* $\mathbf{b}(G, \alpha, \zeta, \mathbf{h}) = \zeta(I-\alpha G)^{-1}\mathbf{h}$

Let m_{ij} denote the (i, j) element of matrix $M(\alpha, G) = (I-\alpha G)^{-1} = \sum_{i=0}^{\infty} \alpha^i G^i$ which is well defined for low values of α[4]. Then $m_{ij} = \sum_{k=0}^{\infty} \alpha^k g_{ij}^{[k]}$ which is (i, j) element of matrix $M(\alpha, G)$ counts all paths from i to j in graph G where paths of length k are discounted with α^k.

Then Bonacich centrality of node i, in notation $b_i(\alpha, \zeta, G, \mathbf{h})$ can be written as:

$$b_i(\alpha, \zeta, G, \mathbf{h}) = \zeta \sum_{j=1}^{n} m_{ij} h_j \tag{2.3}$$

The centrality of node i, $b_i(\alpha, \zeta, oG, \mathbf{h})$ is thus the number of paths from i to every node in network G such that paths of length k are discounted with α^k. The total number of paths from i to a node j is weighted by a corresponding coordinate of the weight vector h_j. If G is a weighted network, then the centrality will not count the number of paths, but the weights of paths, where again weight of paths of length k is discounted with α^k. The role of $\mathbf{h}$ remains the same in the case of weighted network. Thus, we can say that m_{ij} measures the contribution of node j to the centrality of node i in network G.

In our model element g_{ij} of matrix G is the share of intermediate good i among the intermediate goods used in the production of j. In the network this means that the direction of the commodity flows is $i \to j$ and the direction of money flows is $j \to i$. The sum of all elements of row i, $(I-\alpha G)^{-1}$ is the total number of paths

[3]We shall use the term centrality throughout the paper when there is no danger of confusion.

[4]Values of α which are smaller than the absolute value of the inverse of the largest eigenvalue of G.

between node i and all other nodes in the direction of flow of goods in the network according to weighted matrix G where paths of length k are discounted by α^k.

Let us now focus on the vector $\mathbf{h}$. We shall refer to it as consumption/taste vector, or weight vector - depending on the context. In context of Proposition 2.1, vector $\mathbf{h}$ indicates which paths will be taken into account when calculating the centrality[5]. If firm j does not produce a consumption good, then paths from i to j will not contribute to the centrality of node i. Only the paths that end with a firm that produces a consumption good will be taken into account. Thus we can say informally:

The profit a firm i is determined by the number of paths that lead from node i to every other node (including i itself) in the network where we count only paths that end with a producer of a final good.

Proposition 2.1 thus describes the profit of a firm as a function of the firm's position in the network. It states that a firm's profit is proportional to the number of paths from that firm to firms that produce consumption good; discounting paths of length k with α^k. Thus, the centrality of firm k in a sense captures the size of the market for a good k. Firms with a larger market make a higher profit in the equilibrium.

It is also important to note that, due to the normalization $\sum_{j=1}^{n} g_{ij} = 1$, matrix G is a column stochastic matrix. This implies that matrix $\frac{1-\alpha}{m}\mathbf{h}\mathbf{1}' + \alpha G$ is also a column stochastic. The vector of centralities $s = \frac{1-\alpha}{n}\frac{w}{\beta}(I - \alpha G)^{-1}\mathbf{h}$ is, when all goods are consumption goods, proportional to the stationary distribution of irreducible Markov chain with transition matrix $\frac{1-\alpha}{m}\mathbf{h}\mathbf{1}' + \alpha G$.

The utility of the consumer depends on the network structure. The following expression gives the log utility of the representative consumer:

$$\begin{aligned}\log U =& (\xi - 1)\log n + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha} \\ & \left((\alpha+\kappa)\sum_i v_i \log r_i - (1-\alpha-\beta)\sum_i v_i \log v_i + \sum_i \sum_j v_i g_{ji} \log g_{ji} \right) \quad (2.4)\end{aligned}$$

where $\mathbf{v} = \frac{\beta}{w}\mathbf{s}$ and r_i the number of intermediate goods firm i uses in the produc-

[5] Vector $\mathbf{h}$ in Proposition 2.1 is created from consumption weights vector ($\boldsymbol{\gamma}$ with i-th element equal to γ_i) associated to the utility function (2.1). Since we assumed symmetry, $\gamma_i = \frac{1}{m}$ if good i is a consumption good, making $\boldsymbol{\gamma} = \frac{1}{m}\mathbf{h}$

tion, and for simplicity we have assumed for that all goods are consumption goods. We can obtain the analogue expression for the case when all goods are not consumption good, but the qualitative nature of the result will not change (see Appendix A for details).

It is interesting to note that the utility increases with the entropy $-\sum_i v_i \log v_i$ of the centrality vector and decreases with 'entropy' of production matrix G weighted by centralities of nodes (-$\sum_i \sum_j v_i g_{ji} \log g_{ji}$). Thus, to maximize the utility, it is better to have the production network in which all firms have the same profit, as this maximizes $-\sum_i v_i \log v_i$. This will, for example, be the case for a subclass of the class of regular networks. On the other hand it is welfare improving when goods are not treated symmetrically in the production function, as this minimizes $\sum_j g_{ji} \log g_{ji}$. The term $(\alpha+\kappa) \sum_i v_i \log r_i$ captures benefits from technology sophistication - stating that it is welfare improving when more central firms have more complex production technology.

It is clear that, keeping the production network fixed, the utility of the consumer is increasing in the number of consumption goods. What is not straightforward is which structure of the production network maximizes the utility of the consumer. Assuming that every firm treats each input symmetrically, i.e. $g_{ji} = g_{ki} \ \forall \Big((j \wedge k) \in N_i^+ \wedge i \in N\Big)$, the following proposition characterizes the optimal network structure:

Proposition 2.2. *For the network $G = G(N, L)$ and $n = |N|$ the utility of the consumer is given with:*

$$\begin{aligned}\log U =&(\xi - 1) \log n + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \\ &+ \frac{1}{1-\alpha}\left((\alpha+\kappa-1)\sum_i v_i \log r_i - (1-\alpha-\beta)\sum_i v_i \log v_i\right)\end{aligned}$$

Furthermore, the complete network maximizes the utility for $\kappa > 1-\alpha$. For $\kappa < 1-\alpha$ the ring network maximizes the utility. For $\kappa = 1-\alpha$ any network with $v_i = v_j \ \forall (i, j \in N)$ maximizes the consumer's utility.

Proof. See Appendix A □

Note that any $\kappa > 0$ implies benefits from production complexity, but only $\kappa > 1-\alpha$ implies that the welfare maximizing network will be the complete network - network in which all firms use the most complex technology possible. This is due to the negative effect of the 'entropy' term $\sum_i \sum_j v_i g_{ji} \log g_{ji}$ which in the symmetric case considered in Proposition 2.2 has the form: $-\sum_j v_i \log r_i$.

Proposition 2.2 states that when the benefits from production sophistication are high enough, the utility maximizing network will be the complete network, in which each firm has the most sophisticated technology possible. If this is not the case, then the utility will be maximized by the minimal network that can support economic activity in which all nodes have the same centrality (earn the same profit) - the ring network. This is because the adverse 'entropy' effect captured by $\sum_i \sum_j v_i g_{ji} \log g_{ji}$ will be minimized in the case of ring (will be equal to 0). At the same time, the ring will maximize the centrality entropy, $-\sum_i v_i \log v_i$ as all nodes in the ring will have the same centrality.

2.3 Distortions and the Role of Network

As we have shown in the previous section, the position of a firm in the production network determines its profit. Furthermore, the network topology has an important effect on social welfare. In this section we focus on how revenue distortions, hitting the whole economy or a specific firm, affect the distribution of profits and social welfare. One can think of these distortions as any kind of a policy or a shock in the economy that might favour some firms over others (taxes, subsidy, regulations, natural catastrophe). We put emphasis in this paper on the role of the network in mediating these effects. A distortion affecting firm i, such as a tax or a subsidy, will not only affect firm i, but will also have an indirect effect on all firms in the system. How large this effect will be depends on the structure of the production network and the share of intermediate goods in the production function, α. The analysis of the effect on the distribution of profits is important because the profit of a firm determines the probability of firm's survival in economy. Changing the distribution of profits will change these probabilities. This will in turn affect the topology of the network in the future. Due to these long run effects, it is important to be able to predict changes in the profit distribution after distortions.

In this section we shall not assume that the revenue decrease resulting from a distortion is transferred to the consumer (as will be the in the case of taxes), or that for financing a subsidy the social planner collects the resources by taxing consumer or other firms. The reason is that we focus on short term effects of general revenue distortions, which for example can be a natural catastrophe, or some other type of shock that affects the revenues of firms. In Appendix C we discuss the case when the overall budget constraint must be satisfied. We argue that imposing the budget balance constraint does not significantly change the analysis.

2.3.1 Common Revenue Distortions

In this subsection we consider a case when the distortion is common to all firms in the economy. The distortion changes the revenue of all firms in the economy in the same proportion τ. One can think of this type of distortion as an aggregate shock which has the same effect on revenue of every firm. After the distortion τ the revenue of a firm i becomes:

$$R_i = (1-\tau)p_i A_i l_i^\beta \left(\prod_{j \in N_i^+} z_{ji}^{g_{ji}} \right)^\alpha \tag{2.5}$$

Parameter τ in (2.5) can take positive and negative values. A possible interpretation in the first case is to think of it as a tax; and in the second, as a subsidy. In what follows, we shall focus on the case $\tau \geq 0$. The case when $\tau < 0$ is completely analogue.

The vector of centralities after common distortion τ is defined (see Appendix A for derivation):

$$\mathbf{s}(\tau) = \frac{1-\alpha}{\beta m} w\mathbf{h} + (1-\tau)\alpha G\mathbf{s}(\tau) \Rightarrow \mathbf{s}(\tau) = \frac{1-\alpha}{\beta m} w(I-(1-\tau)\alpha G)^{-1}\mathbf{h} \tag{2.6}$$

In this subsection we are interested in how the vector $\mathbf{s}(\tau)$ depends on τ.

Since function $\mathbf{s} : [0,1] \to \mathbb{R}^n$ is differentiable, our first step is to examine the properties of $\frac{d\mathbf{s}}{d\tau}$ (the first derivative of vector $\mathbf{s}(\tau)$ with respect to τ). Taking the derivative we get:

$$\frac{d\mathbf{s}}{d\tau} = -\alpha G\mathbf{s}(\tau) + (1-\tau)\alpha G\frac{d\mathbf{s}}{d\tau} \Rightarrow \frac{d\mathbf{s}}{d\tau} = -\alpha(I-(1-\tau)\alpha G)^{-1}G\mathbf{s}(\tau) \tag{2.7}$$

It is clear from (2.7) that the profit of all firms will decrease when $\tau > 0$ increases. The extent of it depends on the structure of the network.

Note that for $\tau \geq 0$ we have, normalizing $w = \beta$, that $\mathbf{1}'\mathbf{s}(\tau) = \frac{1-\alpha}{1-(1-\tau)\alpha}$, so the total decrease of profit will not depend on the structure of the network, but only on share of intermediate inputs in the production function, α and distortion τ. So, as one could expect, the revenue distortion $\tau \geq 0$ will decrease the consumer's income, and the decrease will be equal to $\frac{1-\alpha}{1-(1-\tau)\alpha} \in [0, 1-\alpha]$.

To examine the effect of the common distortion on the distribution of profits (in particular its diverse impact on different firms) it is more useful to do the following exercise. We normalize $w = \frac{1-(1-\tau)\alpha}{1-\alpha}\beta = \beta\left(1 + \frac{\alpha\tau}{1-\alpha}\right)$. Assume also for simplicity that all goods are consumption goods. This is without loss of generality for the analysis below and it only simplifies notation. With this normalization, sum of profits will remain constant (for any level of τ will be equal to 1) and the effect on relative profits will be more clear.

The following proposition holds.

Proposition 2.3. *The marginal effect of a common revenue distortion on the centrality of firm i*

$$\frac{ds_i}{d\tau} = \frac{1}{n(1-\tau)} \sum_{j=1}^{n} m_{ij}(\tau)(1 - ns_j(\tau)) \tag{2.8}$$

is larger for firms that sell higher share of output (directly and indirectly) to firms with high profits. Common distortion will have a non-monotonic effect on relative profits

Proof. See Appendix A □

In Proposition 2.3 $m_{ij}(\tau)$ denotes (i, j)-th element of matrix $(I - (1-\tau)\alpha G)^{-1}$. Recall that m_{ij} measures the contribution of node j to the centrality of node i.

Due to the normalization we use, the sum of the centralities will always be 1. The expression (2.8) in Proposition 2.3 is informative about the relative movements of the centralities caused by distortion τ. First note that the derivative depends on the global properties of the network. The sign of the derivative is ambiguous, and it will be negative if the contribution of nodes with above average centrality[6] to the centrality of node i is larger than the contribution of nodes with below than average centrality. If the profit of firm k depends more on firms that make high profit, then the decrease in demand for good k will be higher due to common revenue distortion, decreasing profit of a firm i more. If, however, the profit of a firm k depends on the firms that earn low profits - the effect of the revenue distortion on firm k's profit

[6]Due to the normalization average centrality is $\frac{1}{n}$

will not be so high. In other words, when the market for a good k is composed of firms which earn high profits, then the effect of the common distortion will be larger compared to the case when the market for a good k is composed of firms with below average profits.

As the centrality of a node depends on the magnitude of distortion τ, the sign of the derivative in (2.42) also depends on τ, which can make $s_i(\tau)$ non-monotonic. This results in an interesting consequence illustrating the complex effect of the network structure - under common distortion, the distribution of profits can significantly change. The following simple example illustrates this effect.

Example 2.1. *Consider the production network in Figure 2.2 (it is assumed that all firms produce a consumption good and the arrows indicate the direction of flow of intermediate goods).*

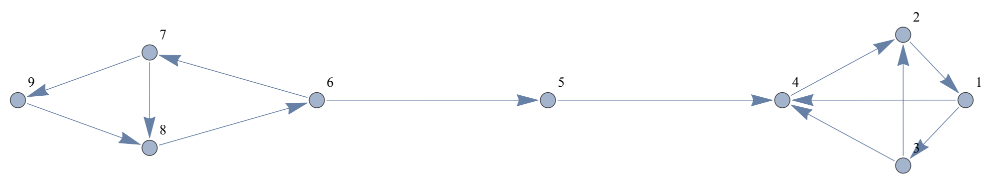


Fig. 2.2 Example common distortion - network

We calculate the centrality for every node for values $\tau \in [0, 1]$ which is graphically represented on Figure 2.3. The graphic illustrates the changes of the relative ranking of centralities when all firms are hit with the common revenue distortion:

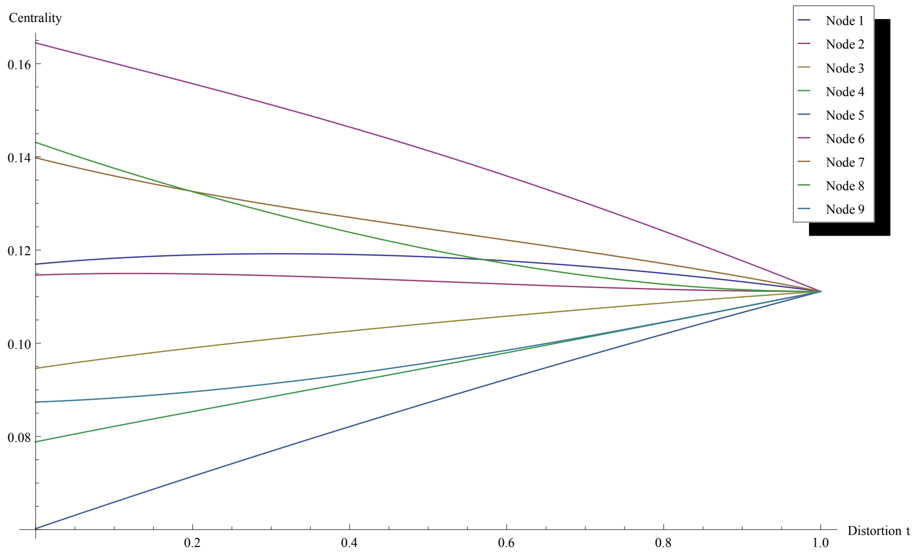


Fig. 2.3 Example common distortion - centralities

It is also interesting to notice that the effect of a common distortion, keeping everything else fixed, can be non-monotonic when focusing on a single firm. That is, the relative profit of a firm can both increase and decrease with the size of common distortion, depending on the size of the distortion. This is illustrated in Figure 2.4, depicting changes of the centrality of node 1 in the network in Figure 2.2.

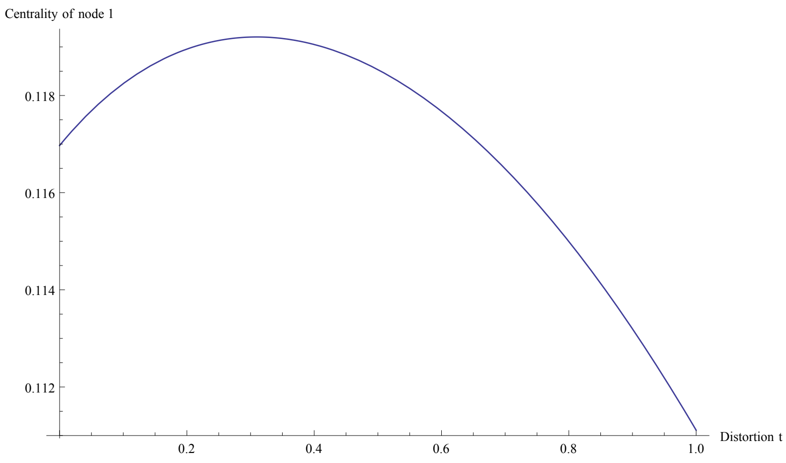


Fig. 2.4 Example common distortion - centrality of node 1

2.3.1.1 Utility

The second question of interest is how welfare behaves with common distortion τ. In Appendix A we derive the expression for the log utility of consumer with common distortion τ, which is given with:

$$\log U(\tau) = (\xi - 1)\log n + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}$$
$$\left((\alpha+\kappa-1)\sum_i v_i(0)\log r_i - (1-\alpha-\beta)\sum_i v_i(0)\log v_i(\tau) + (\alpha+\beta)\log(1-\tau)\right)$$

Where $\mathbf{v} = \frac{\beta}{w}\mathbf{s}$, that is $\mathbf{v}(\tau) = \frac{1-\alpha}{n}\mathbf{h} + (1-\tau)\alpha G\mathbf{v}(\tau)$

To see how the utility changes with distortion τ we take the derivative with respect to τ. Let us first simplify the expression for $\frac{dv(\tau)}{d\tau}$. From (2.7), substituting $\alpha Gv(\tau)$ from the definition of $\mathbf{s}(\tau)$ in we get:

$$\begin{aligned}\frac{dv(\tau)}{d\tau} &= -(I-(1-\tau)\alpha G)^{-1}\left(\frac{1}{1-\tau}\left(\mathbf{v}(\tau) - \frac{1-\alpha}{n}\mathbf{h}\right)\right)\\ &= -\frac{1}{1-\tau}\left((I-(1-\tau)\alpha G)^{-1}\mathbf{v}(\tau) - \mathbf{v}(\tau)\right)\\ &= -\frac{1}{1-\tau}\left((I-(1-\tau)\alpha G)^{-1} - I\right)\mathbf{v}(\tau)\end{aligned}$$

And for particular i:

$$\frac{dv_i(\tau)}{d\tau} = -\frac{1}{1-\tau}\left(\sum_j m_{ij}(\tau)v_j(\tau) - \frac{1-\alpha}{n}\sum_j m_{ij}(\tau)\right) \tag{2.9}$$

Note that $\frac{dv_i(\tau)}{d\tau} < 0$ as $v_i(\tau) > \frac{1-\alpha}{n}$ by the definition of $\mathbf{v}(\tau)$. Finally we can write:

$$\frac{d\log U(\tau)}{d\tau} = \frac{1}{1-\alpha}\left(-(1-\alpha-\beta)\sum_i v_i(0)\frac{v_i'(\tau)}{v_i(\tau)} - \frac{\alpha+\beta}{1-\tau}\right)$$

where $v_i'(\tau)$ given with: (2.9).

It is interesting to note that there will be two opposing effects here. The first one is positive, and comes from the fact that there will be a decrease in the original 'entropy' part of the utility function due to changes in centrality vector, raising utility (see equation (2.4). This effect is captured with expression $-(1-\alpha-\beta)\sum_i v_i(0)\frac{v_i'(\tau)}{v_i(\tau)}$. The second effect is the direct negative effect of the distortion captured with $(\alpha+\beta)\log(1-\tau)$. Numerical calculations indicate that the total effect of distortion will always be negative. However, the magnitude of the effect depends on the network structure. The importance of the network is captured with: $-\sum_i v_i(0)\frac{v_i'(\tau)}{v_i(\tau)}$, which can be interpreted as a measure of resilience of the network to this type of distortions.

2.3.2 Firm Specific Distortions

In this section we consider firm-specific revenue distortions. We discuss the case when the revenue of a particular firm i is distorted, and again denote the distortion with τ. In this case, the centrality equation becomes (see Appendix A)

$$\mathbf{s}(\tau) = \frac{(1-\alpha)w}{\beta n}\mathbf{h} + \alpha\left(G + Q(\tau)\right)\mathbf{s}(\tau) \tag{2.10}$$

Where:

$$Q(\tau) = \begin{pmatrix} 0 & \dots & -g_{1i}\tau & \dots & 0 \\ 0 & \dots & -g_{2i}\tau & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & -g_{ni}\tau & \dots & 0 \end{pmatrix} = \left(\begin{pmatrix} -g_{1i}\tau \\ \vdots \\ -g_{ii}\tau \\ \vdots \\ -g_{ni}\tau \end{pmatrix} \cdot \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \end{pmatrix}\right) = \boldsymbol{\tau}\mathbf{e_i}' \tag{2.11}$$

We use $\boldsymbol{\tau}$ to denote the vector with element j equal to $-g_{ji}\tau$. In the fully symmetric model (all intermediate goods have an equal share in the production function), $g_{ij} = 1/deg^+(j)$ when link (i,j) exists. We shall proceed with general entries g_{ij}, assuming the normalization condition $\sum_i g_{ij} = 1$. The symmetric case will be discussed later. Note that the analysis can be done with the more general formulation considered in section 2.5, as the only crucial part of the analysis is that the distortion hits a single firm, so there is a rank 1 update of the matrix G.

Before we proceed, let us state a known result from Linear Algebra known as the Sherman-Morrison (or sometimes Sherman-Morrison-Woodbury) formula:

Theorem 2.1 (Sherman-Morrison formula). *Let A be a nonsingular n-dimensional* *matrix, and* $\mathbf{c}$, $\mathbf{d}$ *two n-dimensional vectors such that* $1 + \mathbf{d}'A^{-1}\mathbf{c} \neq 0$, *then*

$$(A + \mathbf{c}\mathbf{d}')^{-1} = A^{-1} - \frac{A^{-1}\mathbf{c}\mathbf{d}'A^{-1}}{1 + \mathbf{d}'A^{-1}\mathbf{c}}$$

Proof. See (Sherman and Morrison, 1949) or (Hager, 1989) □

The following result holds:

Proposition 2.4. *Let* τ *be a revenue distortion of firm* i *in network* G*. The change in the vector of centralities due to this distortion is:*

$$\Delta\mathbf{s} = \mathbf{s}(0) - \mathbf{s}(\tau) = \alpha\tau s_i(0)\frac{(I - \alpha G)^{-1}G_{[i]}}{1 + \alpha\tau\sum_{j=1}^{n} g_{ji}m_{ij}} \tag{2.12}$$

and thus determined with intercentralites of the direct suppliers of affected firm.

Proof. See Appendix A □

The expression in the numerator of (2.13) captures the change of centralities of other nodes in the network due to the decrease in the demand of firm i for the production inputs. The elements of the vector $G_{[i]}$ are the shares of intermediate inputs in the production of good i. How large the decrease in demand will be, depends on the direct effect of the distortion τ on the revenue of the firm i ($\tau s_i(0)$), corrected by the term in denominator (which captures the extent in which the suppliers of firm i contribute to the centrality of firm i in the distorted network). If the centrality (revenue) of firm i is more due to its direct suppliers than other nodes in the network, then the effect of the change in the revenue of firm i will have a smaller effect on the centralities of the other nodes in the network.

The effect of the distortion propagates upstream through the network - it is demand driven. This is captured by the expression in the numerator of (2.12) $(I - \alpha G)^{-1}G_{[i]}$.

For a specific firm k, the expression (2.12) has a form:

$$\Delta s_k = s_k(0) - s_k(\tau) = \alpha\tau s_i(0)\frac{\sum_{j=1}^{n} g_{ji}m_{kj}}{1 + \alpha\tau\sum_{j=1}^{n} g_{ji}m_{ij}} \tag{2.13}$$

which in the symmetric case takes a form:

$$\Delta s_k = \alpha\tau s_i(0)\frac{\sum_{j\in N_i^+} g_{ji}m_{kj}}{1+\alpha\tau\sum_{j\in N_i^+} g_{ji}m_{ij}} = \alpha\tau s_i(0)\frac{\frac{1}{deg^+(i)}\sum_{j\in N_i^+} m_{kj}}{1+\alpha\tau\frac{1}{deg^+(i)}\sum_{j\in N_i^+}^{n} m_{ij}}$$
$$= \alpha\tau s_i(0)\frac{\sum_{j\in N_i^+} m_{kj}}{deg^+(i)+\alpha\tau\sum_{j\in N_i^+} m_{ij}}$$

Let us compare the effect of the distortion τ hitting node i on two arbitrary nodes k and l. We get:

$$\Delta s_k - \Delta s_l = \alpha\tau s_i(0)\frac{1}{1+\alpha\tau\sum_{j\in N_i^+} g_{ji}m_{ij}}\left(m_{kj}-m_{lj}\right) \tag{2.14}$$

The equation (2.14) states that the distortion of firm i shall decrease the profit of firm k more than the profit of firm l if in-neighbours of firm i (firms that supply intermediate good to i) have a higher effect on the centrality of firm k than on firm l.

2.3.2.1 Utility

As shown in Appendix A, utility of the consumer after distortion of firm i is given with:

$$\log U = (\xi-1)\log n + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}\left((\alpha+\kappa-1)\sum_i v_i(0)\log r_i\right)$$
$$+\frac{1}{1-\alpha}\left((1-\alpha-\beta)\sum_i v_i(0)\log v_i(\tau) + (\alpha+\beta)v_i(0)\log(1-\tau_i)\right)$$

The negative effect of the distortion will be the highest when firm with the highest centrality is distorted, as captured with $(\alpha+\beta)v_i(0)\log(1-\tau_i) < 0$. The positive effect of the distortion is captured with $-(1-\alpha-\beta)\sum_i v_i(0)\log v_i(\tau) > 0$. Heuristically, this effect will be maximized when the distorted firm has a high effect on the centralities of nodes with high centrality. Thus, the importance of the node, in the sense of its effect on social welfare when distorted, is determined by its centrality and intercentrality (how important this node is to centralities of nodes in the network[7]).

[7] See the definition of intercentrality in Appendix B

2.3.3 Link Updates

Updating weights of the existing links, which happens in the case when the production technology of a firm changes so the shares of inputs in the production function change, can be analysed using the same approach as in the previous subsection. The results will be analogous to the results in Subsection 2.3.3. In this subsection we consider a different type of a link updating - adding/deleting links between firms and adding/deleting links toward the consumer. We characterize the effects of this type of change in the network topology on the profit distribution and welfare.

2.3.3.1 Adding and Deleting Links Between Firms

Adding a link in the production network corresponds to a situation in which an innovation causes the need for an additional input, and the new input is acquired. Deleting an in-link corresponds to a situation when the production process changes so the input in question is no more needed for the production. Let us focus on the situation when a link is created. So, consider a situation when link $j \to i$ is created with the weight $\tilde{g}_{ji}$. Then, two effects will take place. First, firm j is now selling its product to firm i directly, and indirectly to all firms downstream from firm i. The second effect is that firm i will change its production technology by adding a new input. This means that, in general, all weights $g_{ki},\ k \in N_i^+$ will change. Therefore, the demand for all other intermediate goods by firm i will change. Let us denote these new weights with $\tilde{g}_{ki},\ k \in N_i^+$.

Due to complex interaction it is not easy to see the effect of the addition of a link to a centrality of a node incident to that link. The following proposition partially answers this question:

Proposition 2.5. *Let G be a production network, and $\tilde{G}$ a matrix created from G by adding link (j, i). Then the profit of firm j will not decrease, while the profit of firm i can either decrease or increase*

Proof. A direct consequence of Theorem 2.4 in Appendix D □

Adding an outlink from firm j basically means increasing the market for good j, and one might expect that in this case the profit of firm j will increase. Adding an inlink to a firm i means adding an additional input for the production of good i, making it more complex. This might, but doesn't necessarily, increase the profit of firm i. The following example illustrates a situation in which adopting a more complex technology decreases the profit of a firm.

Example 2.2. *Consider a change in the network as in Figure 2.5. It is assumed that all nodes produce a consumption good.*

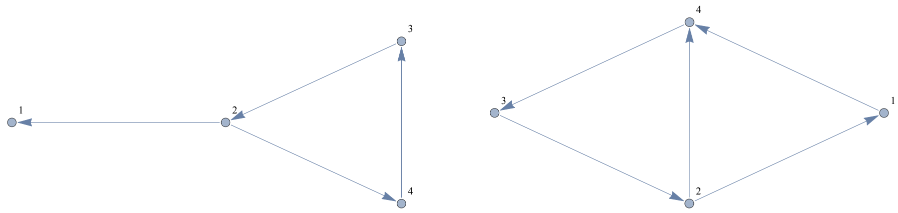


Fig. 2.5 Example adding a link - centrality

The graph on the right hand side is created from the graph on the left hand side by adding link $1 \to 4$. *This change will decrease the centrality of node* 4. *The resulting vectors of centralities are*

$$\mathbf{s_r} = \begin{pmatrix} 0.1000 \\ 0.3265 \\ 0.2960 \\ 0.2775 \end{pmatrix} \quad \mathbf{s_l} = \begin{pmatrix} 0.1790 \\ 0.2863 \\ 0.2718 \\ 0.2630 \end{pmatrix} \tag{2.15}$$

Let us now discuss the exact effect of adding a link on the vector of centralities. When (j, i) link is added, in general all other weights g_{ki} will adjust. Let us denote with $\tilde{G}$ the adjacency matrix after the creation of link (j, i), and write $\tilde{G} = G + Q$ where Q is defined as:

$$Q(\tau) = \begin{pmatrix} 0 & \dots & q_1 & \dots & 0 \\ 0 & \dots & q_2 & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & q_n & \dots & 0 \end{pmatrix} = \left(\begin{pmatrix} q_1 \\ \vdots \\ q_i \\ \vdots \\ q_n \end{pmatrix} \cdot \begin{pmatrix} 0 & \dots & 1 & \dots & 0 \end{pmatrix} \right) = \mathbf{q}\mathbf{e_i}' \tag{2.16}$$

We shall use $\tilde{\mathbf{s}}$ to denote the new vector of centralities after a link (j, i) is added, and use $\mathbf{s}$ to denote the vector of centralities before the addition of (j, i) link. The following result holds:

Proposition 2.6. *Adding entry (i, j) to adjacency matrix G creating matrix $\tilde{G} = G + Q$ where Q is given by equation (2.16) will cause change in the centrality of a node k equal to:*

$$s_k - \tilde{s_k} = -\alpha s_i \frac{\sum_{t \in N_i^+} q_t m_{kt} + q_j m_{kj}}{1 + \alpha \sum_{t \in N_i^+} q_t m_{it} + q_j m_{ij}}$$

Proof. See Appendix A □

In the fully symmetric case (all goods enter the production function with an equal share) $g_{ji} = \frac{1}{deg^+ i}$ and therefore $q_{ki} = \frac{1}{deg^+(i)+1} - \frac{1}{deg^+(i)} = -\frac{1}{deg^+(i)(deg^+(i)+1)}$ for $k \neq i \wedge g_{ki} > 0$, $q_{ji} = \frac{1}{deg^+(i)+1}$ and $q_{ki} = 0$ for $k \neq j \wedge g_{ki} = 0$. Here $deg^+(i)$ is the indegree of node i in network G. In this case (2.52) becomes:

$$\begin{aligned} s_k - \tilde{s_k} &= -\alpha s_i \frac{-\frac{1}{deg^+(i)^2(deg^+(i)+1)} \sum_{t \in N_i^+} m_{kt} + \frac{1}{\mathrm{deg}^+(i)+1} m_{kj}}{1 - \frac{1}{deg^+(i)^2(deg^+(i)+1)} \alpha \sum_{t \in N_i^+} m_{it} + \alpha \frac{1}{\mathrm{deg}^+(i)+1} m_{ij}} \\ &= \alpha s_i \frac{\sum_{t \in N^+(i)} m_{kt} - deg^+(i)^2 m_{kj}}{deg^+(i)^2 (deg(i)+1) - \alpha \sum_{t \in N^+(i)} m_{it} + \alpha deg^+(i)^2 m_{ij}} \end{aligned} \tag{2.17}$$

The expression in the denominator of (2.17) will always be positive, as $\alpha \sum_{t \in N^+(i)} m_{it} < 1$ and $deg^+(i)^2(deg(i)+1) > 1$. The expression in the numerator will be negative if m_{kj} is high enough. This means that if the contribution of node j to the centrality of node k is high in network G, then adding link $j \rightarrow i$ will increase the centrality of node k. As we have seen before, adding link $j \rightarrow i$ will increase the centrality (profit) of firm j. This will, in turn, increase the centralities of all firms which centrality depends on the centrality of firm j. Specifically for firm k, this effect will be proportional to $deg^+(i)^2 m_{kj}$. However, the adjustment of links (t, i) $t \in N_i^+$ (the decrease in weight from g_{ti} to $g_{ti} + q_{ti}$), will have a negative effect on the centrality of firm k through direct suppliers of firm i, as captured by the term $\sum_{t \in N^+(i)} m_{kt}$ in the numerator of (2.17). Depending on which effect dominates, the centrality will increase or decrease.

Utility

The effect adding a link on utility is ambiguous. Although, one would intuitively guess that adding a link increases utility, this is not the case. This will not be the case even when there are benefits from the production complexity ($\kappa > 0$). The reason is that the effect decreasing the entropy $-\sum_{i \in N} v_i \log v_i$ might dominate if the addition of a link makes distribution of centralities 'less uniform'. The following example illustrates this.

Example 2.3. *The network on the right hand side is created from the network on the left hand side by adding* $5 \to 3$

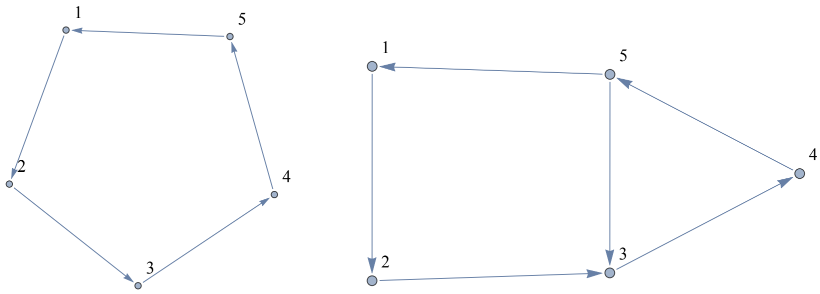


Fig. 2.6 Example adding a link - utility

The consumer's utility will decrease with this change even for $\kappa > (1-\alpha)$ *(recall that when* $\kappa > 1-\alpha$ *the complete network maximizes social welfare, and* $\kappa > 0$ *indicates that there are benefits from the production sophistication).*

2.3.3.2 Adding Links Toward Consumer

Suppose now that firm i starts selling its product to the consumer, so the number of consumption goods increase from m to $m+1$. This will result in a new centrality vector $\tilde{s}$ defined with:

$$\tilde{\mathbf{s}} = \frac{w(1-\alpha)}{\beta(m+1)}(I-\alpha G)^{-1}(\mathbf{h}+\mathbf{e_i}) = \frac{m}{m+1}\mathbf{s} + \frac{1}{m+1}\frac{w(1-\alpha)}{\beta}(I-\alpha G)^{-1}\mathbf{e_i}$$

Where $\mathbf{e_i}$ is vector with 1 on i-th position, and all other elements equal to 0. The new centrality vector $\tilde{\mathbf{s}}$ will be the convex combination of the old vector of centralities, and the vector of the centrality increases due to introduction of the new consumption good i. The second effect is captured with $\frac{1}{m+1}\frac{w(1-\alpha)}{\beta}(I-\alpha G)^{-1}\mathbf{e_i}$ and thus, for some arbitrary firm k, proportional to the contribution of node i to the centrality of node k.

Let $\boldsymbol{\gamma}$ now be a general taste vector with elements γ_i equal to exponents γ_i in utility function (2.1) the following result holds:

Proposition 2.7 (Linearity). *Let* $(\boldsymbol{\gamma}^i)_{i\in F}$ *be a family of taste vectors and* $(\mathbf{s}^i)_{i\in F}$ *be the corresponding family of centralities* $\mathbf{s}^i = \frac{w(1-\alpha)}{\beta}(I-\alpha G)^{-1}\boldsymbol{\gamma}^i$. *Then, the centrality* *induced by convex combination of preference vectors is just a convex combination of corresponding centralities weighted by the same weights*

Proof. Omitted □

The immediate consequence is that if we calculate centrality for vectors $\mathbf{e_i}$ $i = 1 \dots n$ then we have a basis to express centrality for any preference vector from $\mathbb{R}^n$.

In the case when the consumption basket remains the same (no additional consumption good is added), but preferences of consumer change as captured by change of vector $\boldsymbol{\gamma}$ to $\boldsymbol{\gamma} + \epsilon\mathbf{e_i}$, the change in centrality vector is given with:

$$\Delta \mathbf{s} = \frac{w(1-\alpha)}{\beta}(I - \alpha G)^{-1}\epsilon\mathbf{e_i} = \frac{w(1-\alpha)}{\beta}\epsilon\mathbf{m_i} \tag{2.18}$$

where elements of $\mathbf{m_i}$ are m_{ki} $k = 1, ...n$ - contributions of node i to centralities of nodes $k = 1, ...n$.

2.4 Push vs. pull effect

A.O. Hirschman in (Hirschman, 1958) has conceptually stressed the importance of backward and forward linkages in promoting development. Backward linkages are demand-driven stimuli by which a firm (sector) generates demand that helps develop other firms (sectors). In contrast, forward linkages are associated with the supply effect that some firms may have in lowering the costs of production of other firms which rely on the former for inputs. In this section we explore this idea in some detail in the context of our model.

Before we discuss the push and the pull effects of the distortions considered in this paper, let us consider two different types of shocks and the way they propagate trough the network - technology shocks and taste shocks.

2.4.1 Technology Shocks

In this subsection we consider a situation in which firm i is affected by a technology shock which corresponds to a Hicks-neutral technical change. That is, we consider a situation in which A_i changes to $A_i + \epsilon$ for some firm i in (2.2). We first argue that the centrality vector $\mathbf{s}$ will not change as a consequence of this type of shock. To show this, let us define matrix A as:

$$A = \begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_n \end{pmatrix}$$

and rewrite the centrality equation as:

$$A\tilde{\mathbf{s}} = \frac{1-\alpha}{\beta m} w\mathbf{h} + \alpha G A\tilde{\mathbf{s}} \tag{2.19}$$

where $A_i \tilde{s}_i = s_i$. So $\tilde{s}_i$ is a production net to technology-neutral parameter A_i. As we already know, this equation has the unique solution with respect to $\mathbf{s} = A\tilde{\mathbf{s}}$, which is: $\mathbf{s} = \frac{1-\alpha}{\beta m} w(I - \alpha G)^{-1}\mathbf{h}$. For fixed matrix A this uniquely determines the vector $\tilde{\mathbf{s}}$.

Suppose now that a technology shock hits firm i resulting in a change from A_i to $A_i + \epsilon$. Denoting the resulting matrix A with $\bar{A}$, the equation (2.19) becomes:

$$\bar{A}\tilde{\mathbf{s}} = \frac{1-\alpha}{\beta m} w\mathbf{h} + \alpha G \bar{A}\tilde{\mathbf{s}} \tag{2.20}$$

The equation (2.20) also has the unique solution with respect to $\mathbf{s} = \bar{A}\tilde{\mathbf{s}}$. The equations (2.19) and (2.20) are identical with respect to $\mathbf{s}$, implying that change in A_i will only affect $\tilde{\mathbf{s}}$ in the equilibrium, but not the centrality vector $\mathbf{s}$. Furthermore, the demands for inputs $(z_{ji})_{j \in N_i^+}$ and l_i in the equilibrium (as given with equations (2.26) and (2.27)) will not change. This is because $s_i = p_i y_i$ will remain fixed. Since $s_i = p_i y_i$ will not change, y_i is defined with (2.2), and the demands for inputs will remain the same, the direct effect of the technology shock hitting firm i must be reflected only in price p_i. The shock will not propagate upstream through the network, as the demand for inputs will remain fixed. But as p_i changes in the equilibrium, so will the demand for good i by all firms that use i in the production. This effect will propagate further downstream through the network (in the direction of the commodity flows) and will be mediated by the price adjustments.

To show the dynamics of the price adjustment, let $\mathbf{b}$ be the column vector with elements $b_i = -\Big(\log A_i - (1-\alpha-\beta)\log s_i + \alpha \sum_j g_{ij} \log g_{ij}\Big)$. Normalizing $\beta \log w + B = 0$, where $B = \alpha \log \alpha + \beta \log \beta$, from (2.31) we get:

$$\log \mathbf{p} = (I - \alpha G')^{-1}\mathbf{b} \tag{2.21}$$

The equation (2.21) looks like an expression of the vector of Bonacich centralities in the transposed production network (the centralities calculated in the direction of the money flows). But, the vector $\mathbf{b}$ contains vector $\mathbf{s}$ and thus depends on p_i. Therefore, (2.21) does not define the prices as the centrality vector in network G'. However, we are not interested here in an equation that determines the vector of prices as a function of exogenous variables [8], but in the adjustment of prices in the equilibrium as a consequence of the technology shock. As we have shown above, $\mathbf{s}$ will not change in the equilibrium due to the technology shock. Thus, we can use (2.21) to study the effect of the technology shocks on the equilibrium price vector as if $\mathbf{b}$ does not depend on the prices. When A_i changes to $A_i + \epsilon$ the change in the prices of all goods is:

$$\Delta \log \mathbf{p} = (I - \alpha G')^{-1}\boldsymbol{\epsilon} \tag{2.22}$$

where $\boldsymbol{\epsilon}$ is a column vector with ϵ as i-th element, and zeros everywhere else. For a particular good k this change will be equal to $m'_{ki}\epsilon$, where m'_{ki} is the (k,i)-th element of the matrix $(I - \alpha G')^{-1}$. Recall that m'_{ki} is proportional to the effect of node i on the centrality of node k [9] in matrix G'. In other words, the effect of the technology shock of firm i on prices of all other goods in the economy is determined by the upstream intercentrality of node i (the intercentrality of node i in the network G').

Since the prices of all goods in the network will in general change as a response to the technology shock of firm i, and $\mathbf{s}$ will remain fixed in the equilibrium, the outputs of all firms in the economy will in general change. When the price of good k changes from p_k to $p_k + \delta$ in the equilibrium, the equilibrium output will change from y_k to $\frac{p_k}{p_k+\delta} y_k$.

Remark 2.1. *It is useful to note that the Bonacich centrality of a node in network G is equal to its intercentrality in network G'. Indeed, let $\mathbf{s} = \mathbf{h} + \alpha G \mathbf{s}$ define the centrality in network G. Thus we have $\mathbf{s} = (I - \alpha G)^{-1}\mathbf{h}$. Transposing, we get $\mathbf{s}' = \mathbf{h}' + \alpha \mathbf{s}' G'$, and therefore $\mathbf{s}' = \mathbf{h}'(I - \alpha G')^{-1}$, which defines vector of the intercentralities in network G'*

[8] We can obtain such an equation by simply replacing $\mathbf{s}$ with expression $\mathbf{s} = \frac{1-\alpha}{\beta m} w (I - \alpha G)^{-1}\mathbf{h}$

[9] See Section 2.2 for the discussion on the Bonacich centrality, and Appendix B for the definition of the intercentrality adopted in this paper

From Remark 2.1 and the discussion in this subsection it follows that the magnitude of the effect of a technology shock hitting a firm i is determined by its intercentrality in network G' (its centrality in network G). We can thus say that the magnitude of a push effect caused by a technology shock is determined by the centrality of the affected firm.

2.4.2 Taste Shocks

Let us now consider the taste shocks in the consumer's utility function. Suppose the weight of good i in the utility function increases, increasing the demand for good i. This will result in higher revenue of firm i, which will in turn increase the demand of firm i for its inputs. Thus, the taste shock will propagate upstream and will be dominantly mediated through quantity adjustments, increasing the revenues of the firms upstream in the economy. Contrary to technology shocks, taste shocks will have an effect on the distribution of centralities, as captured by equation (2.18). It is also worth recalling that the magnitude of the taste shock will be captured with the intercentrality of the affected good, as discussed in Subsection 2.3.3.2 and visible from (2.18).

2.4.3 Revenue Distrotions

Let us now focus on the revenue distortion of the type we have discussed in Subsection 2.3.2, affecting firm k. One can think of this type of revenue distortion as a combination of a technology and a taste shock. This comes from the fact that the demand for inputs and the production of the firm k will be as if firm k has been hit with a technology shock changing A_k to $A_k(1-\tau)$ (equation 2.43). This will decrease[10] the production of good k triggering the downstream (push) effect mediated by prices as described in Subsection 2.4.1. Contrary to the pure technology shock, there is also a taste shock dimension here, as matrix G will change to $G + Q(\tau)$. The effect of this change is analogous to the effect of a consumer taste shock - as basically the taste of firm i for goods it uses as inputs has changed. This shock will propagate upstream through the network, starting from firm k, creating a pull effect. Thus the revenue shock has both the downstream and the upstream effect, that is, both the pull and the push dimension.

[10] In the case $\tau > 0$, when $\tau < 0$ there will be an increase

The pull effect is visible from equation (2.12) where the part in the numerator $(I-\alpha G)^{-1}G_{[k]}$ captures the upstream propagation of revenue shock that hits firm k. The equation (2.12) describes how this distortion affects centralities of all nodes in the network. Of course, a change in the distribution of centrality will consequently change social welfare, as discussed in Section 2.3.

The downstream effect is not visible from the expression (2.12). This is due to the fact that the technology shock will not change the distribution of centralities. The downstream effect will be mediated through the prices as described in (2.21). When firm k is distorted, entry k of vector $\mathbf{b}$ in (2.21) will change as the term $(\alpha+\beta)\log(1-\tau)$ will be added to $i-th$ coordinate.

In light of Hirshman's discussion of the pull vs. push effect we can say the following: *The revenue distortion of firm i has both pull and push effect. The push (downstream) effect is mediated through prices, and is captured with equation (2.21). The pull (upstream) effect is mainly quantity mediated, and propagates upstream through the network, affecting the centralities (and thus the profits) of firms in the economy in the way described by equation (2.12).*

2.5 General Formulation of the Model

In this section we provide a more general formulation of the model, allowing heterogeneity across nodes in the production function. We only require that the production function is Cobb-Douglas. So we write the production function of firm i as:

$$y_i = A_i l_i^{\beta_i} \left(\prod_{j \in N_i^+} z_{ji}^{g_{ji}} \right)^{\alpha_i} \tag{2.23}$$

and $\alpha_i + \beta_i < 1$.

The utility function has a general form: $U(\mathbf{c}) = A_c \sum_i \gamma_i \log(c_i)$, with $\gamma_i > 0\ \forall i \in N$

Then, the demand of firm i for intermediate good j is:

$$z_{ji} = \frac{p_i \alpha_i g_{ji}}{p_j} y_i$$

and the demand for labor:

$$l_i = \frac{p_i \beta_i}{w} y_i$$

The consumer's demand for consumption good i is:

$$c_i = \gamma_i \frac{\sum_{j=1}^{n} \pi_j + w}{p_i} = \gamma_i \frac{w + \sum_{j=1}^{n}(1-\alpha_j-\beta_j)p_j y_j}{p_i}$$

The profit of a firm i can now be written as:

$$\pi_i = (1-\alpha_i-\beta_i)p_i y_i$$

Market clearing condition for good i states:

$$s_i = \gamma_i w + \gamma_i \sum_{j=1}^{n}(1-\alpha_j-\beta_j)s_j + \sum_{j=1}^{n}\alpha_j g_{ij} s_j \tag{2.24}$$

where $s_i = p_i z_i$.

Before proceeding let us introduce some additional notation:

$$\boldsymbol{\gamma} = \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \vdots \\ \gamma_n \end{pmatrix} \; \boldsymbol{\rho} = \begin{pmatrix} 1-\alpha_1-\beta_1 \\ 1-\alpha_2-\beta_2 \\ \vdots \\ 1-\alpha_n-\beta_n \end{pmatrix} \text{ and } \aleph = \begin{pmatrix} \alpha_1 & 0 & \dots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_n \end{pmatrix}$$

The market clearing condition for every good, in the matrix notation, can be written as:

$$\mathbf{s} = w\boldsymbol{\gamma} + \boldsymbol{\gamma}\boldsymbol{\rho}'\mathbf{s} + G\aleph\mathbf{s} \tag{2.25}$$

and thus:

$$\mathbf{s} = w(I - \boldsymbol{\gamma}\boldsymbol{\rho}' - G\aleph)^{-1}\boldsymbol{\gamma}$$

Let us look at the matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$. Note that the sum of column j of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}'$ is $(1-\alpha_j-\beta_j)\sum_{k=1}^{n}\gamma_i = (1-\alpha_j-\beta_j) < 1$. The sum of column j of matrix $G\aleph$ is $\sum_k = g_{kj}\alpha_j = \alpha_j < 1$. Then the sum of column j of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ is $0 < 1-\beta_j < 1$. Furthermore, it is clear that all elements of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ are positive. Thus, matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ can be seen as a submatrix of a column stochastic matrix. It is implication of Perron Frobenius theorem that the spectral radius of matrix $\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph$ is smaller than 1. This implies that $I - \boldsymbol{\gamma}\boldsymbol{\rho}' - G\aleph$ is invertible (as $(I - \boldsymbol{\gamma}\boldsymbol{\rho}' - G\aleph)^{-1} = \sum_{i=0}^{\infty}(\boldsymbol{\gamma}\boldsymbol{\rho}' + G\aleph)^i$ converges)

Setting $\alpha_i = \alpha \; \forall i \in N$, $\beta_i = \beta \; \forall i \in N$ and $\gamma_i = \begin{cases} \frac{1}{m} & \text{if i is a consumption good} \\ 0 & \text{otherwise} \end{cases}$ equation (2.24) becomes (2.28).

2.6 Conclusion

This paper offers some new insights on the complex effects the network structure has on profits of firms and the welfare in competitive equilibrium.

We show that the position of a firm in the production network determines its revenue and profits in the equilibrium. The profit of a firm is proportional to its Bonacich centrality in the network, where the centrality is calculated in the direction of the commodity flows. The consumer preferences determine the weight vector (taste vector) of the Bonacich centrality. When the benefits from the production sophistication are large enough, the complete network will maximize social welfare.

The network mediates the effects of the distortions on the competitive equilibrium. In the case of aggregate shocks (distortions that affect each firm in the economy in the same way), the welfare and the distribution of profits can significantly change depending on the network structure. Thus, even though a distortion affects each firm in the same way, it will alter the set of market 'winners' and 'losers', and thus the set of firms that will due to the low profit exit the economy in the future. In the long run, the common distortions will therefore significantly alter the production network topology. The same thing happens when a specific firm is distorted. The effect of the distortion of a single firm will propagate through the network both upstream and downstream creating the pull and the push effect. The push effect is mediated by prices and its magnitude is determined by the centrality of a firm in the network. The pull effect is mostly quantity mediated, and its magnitude depends on the intercentralities of the direct suppliers of the affected firm.

An innovation of a firm (a change in the technology sophistication) will not necessarily increase its profit. This will be the case even when the output is increasing with technology sophistication, keeping everything else fixed. Finding a new market (i.e. a firm that will use a good as an input) will always increase the profit of a firm. The effects of the innovation of a firm i will propagate through the network creating the pull and the push effect. These effects will propagate through the network in the same way as the effects of firm specific revenue distortions.

The model in the general case is rich enough to accommodate the heterogeneity of firms in a number of dimensions and can therefore be used in the empirical analysis. The empirical counterpart of this paper is work in progress.

The results of this paper indicate that the distortions, aggregate and idiosyncratic, have an important effect on the distribution of profits in the economy. As a firm's probability of survival in the market is an increasing function of its profit, then the effects of the distortions on the topology of the network and the welfare is large in the long run. It is very important to take this into account when formulating taxes and subsidy policies. The analysis the dynamic aspect seems to be a promising direction for the future.

2.7 Appendix A: Proofs

2.7.1 Benchmark model

Definition 2.3 (Competitive equilibrium:). *A competitive equilibrium of economy* $G = G(N, L)$ *is a set of prices* $(p_1, p_2, ..., p_n, w)$*, consumption bundle* $(c_1, c_2, ..., c_m)$*, and quantities* $(l_i, y_i, (z_{ji})_{j \in N_i^+})$ *such that:*

- *The representative consumer solves*

$$\begin{aligned} &\max_{\mathbf{c}} \quad A_c(m) \prod_{i=1}^{m} c_i^{\frac{1}{m}} \\ &s.t. \quad \sum_{i \in C} p_i c_i = w + \sum_{k=1}^{n} \pi_k \end{aligned}$$

- *Firms maximize profits, that is, solve:*

$$\begin{aligned} &\max_{(z_{ji})_j, l_i} \quad p_i y_i - \sum_{j \in r_i^+} p_j z_{ji} g_{ji} - w l_i \\ &s.t. \quad z_i = A_i l_i^{\beta} \left(\prod_{j \in N_i^+} z_{ji}^{g_{ji}} \right)^{\alpha} \end{aligned}$$

- *Markets for labour and intermediate goods clear:*

$$y_i = \sum_j z_{ij} + c_i \;\; \forall (i \in N)$$

$$\sum_i l_i = 1$$

Proof of Proposition 2.1: The first order conditions for the maximization problem of firm i, with respect to z_{ji} are given with $(\forall j)$

$$A_i p_i \alpha g_{ji} l_i^{\beta} z_{ji}^{\alpha g_{ji}-1} \prod_{k,k\neq j} z_{ki}^{\alpha g_{ki}} = p_j \Rightarrow z_{ji} = \frac{p_i \alpha g_{ji}}{p_j} y_i \tag{2.26}$$

and with respect to labour l_i:

$$A_i p_i \beta l_i^{\beta-1} \prod_{k} z_{ki}^{\alpha g_{ki}} = w \Rightarrow l_i = \frac{p_i \beta}{w} y_i \tag{2.27}$$

and the demand for final goods is $\forall (i \in S)$:

$$c_i = \frac{\sum_{i=1}^{n} \pi_i + w}{m p_i}$$

where m is the number of consumption goods. Profit of firm i is thus

$$\pi_i = p_i y_i - \sum_j \left(p_j \frac{p_i \alpha g_{ji}}{p_j} y_i \right) - w \frac{p_i \beta}{w} y_i = (1-\alpha-\beta) p_i y_i$$

Market clearing for labour gives (using (2.27)):

$$\sum_i l_i = 1 \Leftrightarrow \beta \sum_i p_i y_i = w$$

which gives:

$$c_i = \frac{\sum_{j=1}^{n} (1-\alpha-\beta) p_j y_j + w}{m p_i} = \frac{(1-\alpha) w}{\beta m p_i}$$

and we can write market clearing for good i as:

$$\begin{aligned} z_i &= c_i h_i + \sum_j z_{ij} \Leftrightarrow \\ s_i &= \frac{(1-\alpha) w}{\beta m} h_i + \alpha \sum_j g_{ij} s_j \end{aligned} \tag{2.28}$$

where $s_i = p_i y_i$ is a revenue of a firm i, and $h_i = 1$ if good i is a consumption good (there is a link from firm i toward the consumer); otherwise $h_i = 0$. Note that we assume symmetry in consumption goods here (See Section 2.5 for a more general formulation). Using vector notation, we write the system of market clearing conditions for all intermediate goods as:

$$\mathbf{s} = \frac{(1-\alpha)w}{m\beta}\mathbf{h} + \alpha G\mathbf{s} \Leftrightarrow (I-\alpha G)\,\mathbf{s} = \frac{(1-\alpha)w}{m\beta}\mathbf{h} \Rightarrow$$
$$\mathbf{s} = \frac{(1-\alpha)w}{m\beta}\,(I-\alpha G)^{-1}\,\mathbf{h}$$

where $\mathbf{h}$ is a column vector with elements h_i as defined before. We shall also define $\mathbf{v} = \frac{1-\alpha}{n}\,(I-\alpha G)^{-1}\,\mathbf{h}$, so $\mathbf{s} = \frac{wn}{m\beta}\mathbf{v}$

This gives

$$\boldsymbol{\pi} = (1-\alpha-\beta)\mathbf{s} = (1-\alpha-\beta)\frac{wn}{m\beta}\mathbf{v} \tag{2.29}$$

Suppose now that all goods are consumed (we call this the benchmark case). Then $\mathbf{h} = \mathbf{1}$ Since $\alpha < 1$ and G is stochastic we can write

$$\mathbf{v} = \frac{1-\alpha}{n}\left(\sum_{k=0}^{\infty}\alpha^k G^k\right)\mathbf{1}$$

Note also that $\mathbf{1}'G\mathbf{1} = n$ (the sum of all elements in matrix). Since the product of two stochastic matrices is stochastic, we have that $\mathbf{1}'G^k\mathbf{1} = n\ \forall k \in N$ this gives:

$$\mathbf{1}'\mathbf{v} = (1-\alpha)\sum_{k=0}^{\infty}\alpha^k = \frac{1-\alpha}{1-\alpha} = 1$$

Now:

$$\mathbf{s} = \frac{wn}{nm\beta}\mathbf{v} \Rightarrow \mathbf{1}'\mathbf{s} = \frac{wn}{m\beta} \Leftrightarrow \sum_{i=1}^{n} p_i y_i = \frac{wn}{m\beta}$$

for $j-th$ element we have

$$s_j = \frac{wn}{m\beta}v_j \Rightarrow v_j = \frac{s_j}{\frac{wn}{\beta m}} = \frac{p_j y_j}{\sum_{i=1}^{n} p_i y_i} \tag{2.30}$$

which completes the proof.

□

Proof of Proposition 2.2: To calculate utility of the consumer we proceed as follows. Plug in the first order conditions for labour and intermediate goods into the production function:

$$y_i = A_i \left(\frac{p_i\beta}{w} y_i\right)^\beta \prod_j \left(\frac{p_i \alpha g_{ji}}{p_j} y_i\right)^{\alpha g_{ji}} \Rightarrow$$
$$\prod_{j\in N_i^+} \left(p_j^{\alpha g_{ji}}\right) y_i = A_i \left(\frac{p_i\beta}{w} y_i\right)^\beta \prod_j \left(p_i \alpha g_{ji} y_i\right)^{\alpha g_{ji}} \Rightarrow$$
$$\frac{\prod_{j\in N_i^+} p_j^{\alpha g_{ji}}}{p_i} s_i = A_i \left(\frac{\beta}{w} s_i\right)^\beta \prod_j \left(\alpha g_{ji} s_i\right)^{\alpha g_{ji}}$$

Taking natural logs we get that for every firm i:

$$\alpha \sum_{j\in r_i} g_{ji} \log p_j + \log s_i - \log p_i = \log A_i +$$
$$\beta \left(\log \beta + \log s_i - \log w\right) + \alpha \sum_j g_{ji} \left(\log \alpha + log g_{ji} \log s_i\right) \Leftrightarrow$$
$$\alpha \sum_{j\in r_i} g_{ji} \log p_j - \log p_i = \log A_i + B - (1-\alpha-\beta) \log s_i - \beta \log w + \alpha \sum_j g_{ji} \log g_{ji} \tag{2.31}$$

Where $B = \alpha \log \alpha + \beta \log \beta$

We consider here the benchmark case ($\mathbf{h} = \mathbf{1}$), and write system of (2.31) in vector notation ($\forall i \in N$). After some rearranging we get:

$$(1-\alpha)\mathbf{1} \log w = (\alpha+\kappa) \log \mathbf{r} - (1-\alpha-\beta) \log \mathbf{v} + \alpha \left(I - \alpha G'\right) \log \mathbf{p} + \mathbf{u} + \alpha H' \mathbf{1} \tag{2.32}$$

where $r_i = \sum_j \mathbb{1}_{g_{ij}}$ is the indegree of node i in the binary adjacency matrix of network G (number of intermediate inputs that firm i uses in the production). Vector $\mathbf{u}$ is a vector with each element equal to: $\alpha \log \alpha + \beta \log \beta + (1-\alpha-\beta) \log \beta$ (and we use that $s_i = \frac{w}{\beta} v_i$, which follows from market clearing conditions). H is the matrix with (i,j) element equal to $g_{ij} \log g_{ij}$. Recall that $A_i = r_i^{\alpha+\kappa}$.
Premultiplying (2.32) with $\mathbf{v}' = \frac{1-\alpha}{n} \mathbf{h}'(I - \alpha G')^{-1}$ we get:

$$(1-\alpha)\mathbf{v}'\mathbf{1}\log w = (\alpha+\kappa)\mathbf{v}'\log\mathbf{r} - (1-\alpha-\beta)\mathbf{v}'\log\mathbf{v} + \alpha\frac{1-\alpha}{n}\mathbf{1}'\log\mathbf{p} + v'\mathbf{1}u + \alpha\mathbf{v}'H'\mathbf{1} \Leftrightarrow$$
$$(1-\alpha)\log w = (\alpha+\kappa)\sum_i v_i\log r_i - (1-\alpha-\beta)\sum_i v_i\log v_i +$$
$$\frac{\alpha-\alpha^2}{n}\sum_i \log p_i + \sum_i\sum_j v_i g_{ji}\log g_{ji} + u$$

We normalize the prices $\sum_i \log p_i = 0$. Furthermore, for the symmetric case $g_{ji} = g_{ki} = 1/r_i \ \forall (j,k \in N_i^+)$, we have:

$$\sum_i\sum_j v_i g_{ji}\log g_{ji} = \sum_i v_i \log\frac{1}{r_i} = -\sum_i v_i\log r_i$$

We can finally write:

$$(1-\alpha)\log w = (\alpha+\kappa-1)\sum_i v_i\log r_i - (1-\alpha-\beta)\sum_i v_i\log v_i + \alpha\log\alpha + (1-\alpha)\log\beta$$

Recall that:

$$c_i = \frac{1-\alpha}{n\beta p_i}w \Rightarrow U(\mathbf{c}) = n^{\xi}\prod_{i=1}^{n}\left(\frac{1-\alpha}{n\beta p_i}w\right)^{\frac{1}{n}} = n^{\xi-1}\frac{1-\alpha}{\beta}w$$

Combining two previous equations, we get:

$$\begin{aligned}\log U &= (\xi-1)\log n + \log(1-\alpha) - \log\beta + \frac{1}{1-\alpha}\left((\alpha+\kappa-1)\sum_i v_i\log r_i\right)\\ &+ \frac{1}{1-\alpha}\left(-(1-\alpha-\beta)\sum_i v_i\log v_i + \alpha\log\alpha + (1-\alpha)\log\beta\right) \Rightarrow \qquad (2.33)\\ \log U &= (\xi-1)\log n + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}\\ &+ \frac{1}{1-\alpha}\left((\alpha+\kappa-1)\sum_i v_i\log r_i - (1-\alpha-\beta)\sum_i v_i\log v_i\right) \qquad (2.34)\end{aligned}$$

For a fixed n, let us find the welfare maximizing network topology. We write

$\Phi(\mathbf{z}, \mathbf{v}) = \sum_{i=1}^{n} v_i \log r_i$, and define function Ψ as:

$$\Psi(\bar{y}) = \max_{(x_i)_{i=1}^n, (y_i)_{i=1}^n} \sum_{i=1}^{n} x_i log y_i$$

s.t.

$$\sum_{i=1}^{n} y_i = \bar{y} \wedge \sum_{i=1}^{n} x_i = 1 \tag{2.35}$$

$$y_i \leq n-1 \wedge y_i \geq 1 \wedge x_i > 0,\ i = 1, ..., n$$

One can easily show that function Ψ is strictly increasing in $\bar{y}$. For any graph with n nodes and $\bar{r}$ links ($\sum_i r_i = \bar{r}$)we have $\Phi(r_i, v_i) \leq \Psi(\bar{r}) < \Psi(n(n-1))$. Note that for complete network $\forall(i, j \in N)(r_i = r_j) \Rightarrow \Phi(v_i, r_i) = \Psi(n(n-1))$. Thus, complete network is the unique maximizer of Φ. The unique minimizer of Φ, in the space of strongly connected networks, is the ring network. The value of expression $-\sum_i^n v_i \log v_i$ which is in fact the entropy measure of simplex vector $\mathbf{v}$ will be maximized when $v_i = v_j \forall(i, j \in N)$, which will incidentally be true in the case of the complete network and the ring network. It fallows now that utility will be maximized at the complete network when $\alpha - \kappa > 1$ and that complete network will be the unique network that maximizes consumers utility. When $\alpha - \kappa < 1$ the ring will maximize social welfare. When $\alpha - \kappa = 1$, then any network such that centrality of each node is equal will maximize social welfare. This will be the case for the subclass of the class of regular strongly connected networks. □

2.7.2 Common Revenue Distortions

In the case of common distortion, demands for labor and intermediate goods (2.26) and (2.27) become:

$$z_{ji} = (1-\tau)\frac{p_i \alpha g_{ji}}{p_j} y_i$$

$$l_i = (1-\tau)\frac{p_i \beta}{w} y_i$$

We can write profit of firm i as:

$$\pi_i(\tau) = (1-\tau)p_i y_i - \sum_{j \in N_i^+} p_j(1-\tau)\frac{p_i \alpha g_{ji}}{p_j} y_i - w(1-\tau)\frac{p_i \beta}{w} y_i = (1-\tau)(1-\alpha-\beta)p_i y_i \tag{2.36}$$

From the labor market clearing condition, we have that $\sum_{i=1}^n p_i y_i = \frac{w}{\beta(1-\tau)}$

Then we can write:

$$c_i = \frac{(1-\tau)\sum_{j=1}^n (1-\alpha-\beta)p_j y_j + w}{m p_i} = \frac{(1-\alpha)w}{\beta m p_i}$$

and we can write market clearing for good i as:

$$\begin{aligned} z_i &= c_i h_i + \sum_j z_{ij} \Leftrightarrow \\ s_i &= \frac{(1-\alpha)w}{\beta m} h_i + \alpha \sum_j (1-\tau) g_{ij} s_j \end{aligned}$$

Using vector notation, we write system of market clearing conditions for all intermediate goods as:

$$\mathbf{s} = \frac{(1-\alpha)w}{m\beta}\mathbf{h} + \alpha(1-\tau)G\mathbf{s} \Rightarrow \mathbf{s} = \frac{(1-\alpha)w}{m\beta}\left(I - \alpha(1-\tau)G\right)^{-1}\mathbf{h}$$

which gives the expression for centrality with common distortion.

As for utility we can write equation (2.31)

$$\begin{aligned} \alpha \sum g_{ji} \log p_j - \log p_i = {} & \log A_i + (\alpha+\beta)\log(1-\tau) + B \\ & - (1-\alpha-\beta)\log s_i(\tau) - \beta \log w + \alpha \sum_j g_{ji} \log g_{ji} \end{aligned} \tag{2.37}$$

and in the vector notation:

$$\begin{aligned} (1-\alpha)\mathbf{1}\log w = {} & (\alpha+\kappa)\log \mathbf{r} + (\alpha+\beta)\log(1-\tau)\mathbf{1} \\ & - (1-\alpha-\beta)\log \mathbf{v}(\tau) + (I-\alpha G)\log \mathbf{p} + \mathbf{u} + \alpha H'\mathbf{1} \end{aligned} \tag{2.38}$$

Premuliplying (2.38) with $v'(0)$ and using the fact that $\mathbf{v}'(0)\mathbf{1} = 1$, we get:

$$
\begin{aligned}
(1-\alpha)\log w = (\alpha+\kappa-1)\sum_i v_i(0)\log r_i - (1-\alpha-\beta)\sum_i v_i(0)\log v_i(\tau) \\
+\alpha\log\alpha + (1-\alpha)\log\beta + (\alpha+\beta)\log(1-\tau)
\end{aligned} \tag{2.39}
$$

where we have used the normalization $\sum_{i=1}^{n}\log p_i = 0$ and assumed that all goods are symmetric in every production function ($g_{ji} = g_{ki}\ i,k \in N_i^+$) as in the no-distortion case. The consumer utility then can be written as:

$$
\log U(\tau) = (\xi-1)\log n + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}
\left((\alpha+\kappa-1)\sum_i v_i(0)\log r_i - (1-\alpha-\beta)\sum_i v_i(0)\log v_i(\tau) + (\alpha+\beta)\log(1-\tau)\right)
$$

Proof of Proposition 2.3: Using the normalization $w = \frac{1-(1-\tau)\alpha}{1-\alpha}\beta = \beta\left(1+\frac{\alpha\tau}{1-\alpha}\right)$ we can write:

$$
\mathbf{s}(\tau) = \frac{1-(1-\tau)\alpha}{n}\mathbf{h} + (1-\tau)\alpha G s(\tau) \Rightarrow \mathbf{s}(\tau) = \frac{1-(1-\tau)\alpha}{n}(I-(1-\tau)\alpha G)^{-1}\mathbf{h} \tag{2.40}
$$

and taking the derivative with respect to τ:

$$
\begin{aligned}
\frac{d\mathbf{s}}{d\tau} &= \frac{\alpha}{n}\mathbf{h} - \alpha G\mathbf{s}(\tau) + (1-\tau)\alpha G\frac{d\mathbf{s}}{d\tau} \Rightarrow \\
\frac{d\mathbf{s}}{d\tau} &= \alpha(I-(1-\tau)\alpha G)^{-1}\left(\frac{1}{n}\mathbf{h} - Gs(\tau)\right) \\
&= \mathbf{s}(\tau)\frac{\alpha}{1-(1-\tau)\alpha} - \alpha(I-(1-\tau)\alpha G)^{-1}G\mathbf{s}(\tau) \\
&= \alpha\left(\frac{1}{1-(1-\tau)\alpha}I - (I-(1-\tau)\alpha G)^{-1}G\right)\mathbf{s}(\tau)
\end{aligned} \tag{2.41}
$$

The expression in the parenthesis in (2.41) determines the sign of the derivative, and the relative movements of the profit due to common distortion τ. We can further expand on (2.41) using (2.40) and write:

$$
\alpha G s(\tau) = \frac{1}{1-\tau}\left(\mathbf{s}(\tau) - \frac{1-(1-\tau)\alpha}{n}\mathbf{h}\right)
$$

and plugging this into (2.41) we get:

$$
\begin{aligned}
\frac{d\mathbf{s}}{d\tau} &= \frac{\alpha}{1-(1-\tau)\alpha}\mathbf{s}(\tau) \\
&\quad - \frac{1}{1-\tau}\left((I-(1-\tau)\alpha G)^{-1}\mathbf{s}(\tau) - \frac{1-(1-\tau)\alpha}{n}(I-(1-\tau)\alpha G)^{-1}\mathbf{h}\right) \\
&= \frac{\alpha}{1-(1-\tau)\alpha}\mathbf{s}(\tau) - \frac{1}{1-\tau}\left((I-(1-\tau)\alpha G)^{-1}\mathbf{s}(\tau) - \mathbf{s}(\tau)\right) \\
&= \frac{1}{1-\tau}\left(\frac{1}{1-(1-\tau)\alpha}I - (I-(1-\tau)\alpha G)^{-1}\right)\mathbf{s}(\tau)
\end{aligned}
$$

and for a particular firm i:

$$
\begin{aligned}
\frac{ds_i}{d\tau} &= \frac{1}{1-\tau}\left(\frac{1}{1-(1-\tau)\alpha}s_i(\tau) - \sum_{j=1}^{n} m_{ij}(\tau)s_j(\tau)\right) \\
&= \frac{1}{1-\tau}\left(\frac{1}{n}\sum_{j=1}^{n} m_{ij}(\tau) - \sum_{j=1}^{n} m_{ij}(\tau)s_j(\tau)\right) \\
&= \frac{1}{n(1-\tau)}\sum_{j=1}^{n} m_{ij}(\tau)(1-ns_j(\tau))
\end{aligned} \tag{2.42}
$$

□

2.7.3 Firm Specific Distortions

Suppose now that distortions differ across firms, and let τ denote the revenue distortion that hits firm i. Then the demand for intermediate goods and labor of firm i can be written as:

$$
\begin{aligned}
z_{ji} &= (1-\tau)\frac{p_i \alpha g_{ji}}{p_j} y_i \\
l_i &= (1-\tau)\frac{p_i \beta}{w} y_i
\end{aligned} \tag{2.43}
$$

and for all other firms as (2.26) and (2.27).

We can write the profit of firm i as: $\pi_i = (1-\tau)(1-\alpha-\beta)p_i y_i$, and for all other firms $j \neq i$ as: $\pi_j = (1-\alpha-\beta)p_j y_j$. From the labor market clearing condition we have that $\sum_{j=1\wedge j\neq i}^{n} p_j y_j + (1-\tau)p_i y_i = \frac{w}{\beta}$. Then we can write:

$$c_k = \frac{(\sum_{j=1\wedge j\neq i}^{n}(1-\alpha-\beta)p_j y_j + (1-\tau)(1-\alpha-\beta)p_i y_i + w)}{mp_i} = \frac{(1-\alpha)w}{\beta m p_k}$$

The market clearing condition for distorted good k is:

$$\begin{aligned} z_k &= c_k h_k + \sum_{j\neq i} z_{kj} + (1-\tau)z_{ki} \Leftrightarrow \\ s_k &= \frac{(1-\alpha)w}{\beta m} h_k + \alpha\left(\sum_j g_{kj}s_j - \tau g_{ki}s_i\right) \end{aligned}$$

Using vector notation, we write the system of market clearing conditions for all intermediate goods as:

$$\mathbf{s} = \frac{(1-\alpha)w}{m\beta}\mathbf{h} + \alpha\left(G + Q(\tau)\right)\mathbf{s} \Rightarrow \mathbf{s} = \frac{(1-\alpha)w}{m\beta}\left(I - \alpha(G+Q(\tau))\right)^{-1}\mathbf{h}$$

and

$$Q(\tau) = \begin{pmatrix} 0 & \dots & -g_{1i}\tau & \dots & 0 \\ 0 & \dots & -g_{2i}\tau & \dots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \dots & -g_{ni}\tau & \dots & 0 \end{pmatrix}$$

As for utility, the equation (2.31) with distorted firm i becomes

$$\begin{aligned} \alpha\sum g_{ji}\log p_j - \log p_k = {} & \log A_i + \delta_{ki}(\alpha+\beta)\log(1-\tau) + B \\ & - (1-\alpha-\beta)\log s_k(\tau) - \beta\log w + \alpha\sum_j g_{jk}\log g_{jk} \end{aligned}$$

where δ_{ik} is the Kronecker delta.

For all firms we get:

$$\begin{aligned} (1-\alpha)\mathbf{1}\log w = {} & (\alpha+\kappa)\log\mathbf{r} + (\alpha+\beta)\log(1-\tau)\mathbf{e_i} - (1-\alpha-\beta)\log\mathbf{v}(\boldsymbol{\tau}) \\ & + \alpha\left(I - \alpha G'\right)\log\mathbf{p} + \mathbf{u} + \alpha H\mathbf{1} \end{aligned}$$

Premultiplying with $\mathbf{v}'(\mathbf{0})$ we get:

$$(1-\alpha)\log w = (\alpha+\kappa-1)\sum_j v_j(0)\log r_j - (1-\alpha-\beta)\sum_j v_j(0)\log v_j(\boldsymbol{\tau}) + v_i(0)\log(1-\tau) + \alpha\log\alpha + (1-\alpha)\log\beta$$

and we can write the consumer utility as:

$$\log U = (\xi-1)\log n + \log(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} + \frac{1}{1-\alpha}\left((\alpha+\kappa-1)\sum_j v_j(0)\log r_j\right) + \frac{1}{1-\alpha}\left(-(1-\alpha-\beta)\sum_j v_j(0)\log v_j(\boldsymbol{\tau}) + (\alpha+\beta)v_i(0)\log(1-\tau)\right)$$

Proof of Proposition 2.4: Using the Sherman-Morrison formula we can write

$$\begin{aligned}\mathbf{s}(\tau) &= \frac{(1-\alpha)w}{\beta m}(I-(\alpha G+\alpha\boldsymbol{\tau}\mathbf{e_i}'))^{-1}\mathbf{h} \\ &= \frac{(1-\alpha)w}{\beta m}\left((I-\alpha G)^{-1}\mathbf{h} + \frac{(I-\alpha G)^{-1}\alpha\boldsymbol{\tau}\mathbf{e_i}'(I-\alpha G)^{-1}}{1-\alpha\mathbf{e_i}'(I-\alpha G)^{-1}\boldsymbol{\tau}}\mathbf{h}\right)\end{aligned} \tag{2.44}$$

We shall use $\mathbf{s}(\tau)$ to denote the vector centralities after distortion τ, as we did before. The first summand of the expression (2.44)

$$\frac{(1-\alpha)w}{\beta m}(I-\alpha G)^{-1}\mathbf{h}$$

is equal to $\mathbf{s}(0)$. Let us focus on the second summand of the expression (2.44):

$$\frac{(1-\alpha)w}{\beta m}\frac{(I-\alpha G)^{-1}\alpha\boldsymbol{\tau}\mathbf{e_i}'(I-\alpha G)^{-1}}{1-\alpha\mathbf{e_i}'(I-\alpha G)^{-1}\boldsymbol{\tau}}\mathbf{h} \tag{2.45}$$

which essentially captures the effect on profits of firms caused by taxation of firm i. First we discuss the expression in the denominator. Note that $\mathbf{e_i}'(I-\alpha G)^{-1}$ is the $i-th$ row in the matrix $(I-\alpha G)^{-1}$. As before, m_{ij} denotes an (i,j) element of the matrix $(I-\alpha G)^{-1}$. We can write:

$$\mathbf{e_i}'(I-\alpha G)^{-1}\boldsymbol{\tau} = -\tau\sum_{j=1}^{n} g_{ji}m_{ij} \tag{2.46}$$

if all goods are symmetric in the production function of good i, that is if $g_{ji} = \frac{1}{deg^+(i)}$ then (2.46) becomes:

$$-\tau \sum_{j=1}^{n} g_{ji} m_{ij} = -\frac{\tau}{deg^+(i)} \sum_{j \in N_i^+} m_{ij} \tag{2.47}$$

Let us now simplify the numerator of (2.45). Since $\frac{(1-\alpha)w}{\beta m}(I-\alpha G)^{-1}\mathbf{h} = \mathbf{s}(0)$ we can write the numerator of (2.45) as: $(I-\alpha G)\alpha\boldsymbol{\tau}\mathbf{e_i}' s(0)$. Furthermore $\mathbf{e_i}'\mathbf{s}(0) = s_i(0)$. Using the definition of $\boldsymbol{\tau}$ we can finally write the numerator of (2.45) as:

$$\frac{(1-\alpha)w}{\beta m}(I-\alpha G)^{-1}\alpha\boldsymbol{\tau}\mathbf{e_i}'(I-\alpha G)^{-1} = -\alpha\tau s_i(0)(I-\alpha G)^{-1} G_{[i]} \tag{2.48}$$

where $G_{[i]}$ is the i-th column of the matrix G, that is:

$$(I-\alpha G)^{-1} G_{[i]} = (I-\alpha G)^{-1} \begin{pmatrix} g_{1i} \\ g_{2i} \\ \vdots \\ g_{n-1i} \\ g_{ni} \end{pmatrix}$$

The $k-th$ element of this vector will be: $\sum_{j=1}^{n} m_{kj} g_{ji}$. This is the sum of contributions of nodes that are suppliers of node i (i.e. $g_{ji} \neq 0$) to the centrality of node k, weighted by g_{ji} - how important j is as a supplier for i. In the symmetric case $\sum_{j=1}^{n} g_{ji} m_{kj} = \frac{1}{deg^+(i)} \sum_{j \in N_i^+} m_{kj}$.

Using equations (2.46) and (2.48) we get:

$$\mathbf{s}(\tau) = \frac{(1-\alpha)w}{\beta m}(I-\alpha G-\alpha\boldsymbol{\tau}\mathbf{e_i}')^{-1}\mathbf{h} = \mathbf{s}(0) - \alpha\tau s_i(0)\frac{(I-\alpha G)^{-1} G_{[i]}}{1+\alpha\tau\sum_{j=1}^{n} g_{ji} m_{ij}} \tag{2.49}$$

Thus the change in the centrality vector after the revenue distortion τ hits only firm i is given with:

$$\Delta\mathbf{s} = \mathbf{s}(0) - \mathbf{s}(\tau) = \alpha\tau s_i(0)\frac{(I-\alpha G)^{-1} G_{[i]}}{1+\alpha\tau\sum_{j=1}^{n} g_{ji} m_{ij}}$$

□

2.7.4 Link Updates

Proof of Proposition 2.6:

$$\begin{aligned}\tilde{\mathbf{s}} &= \frac{(1-\alpha)w}{\beta m}(I-\alpha\tilde{G})^{-1}\mathbf{h} = \frac{(1-\alpha)w}{\beta m}(I-\alpha G-\alpha\mathbf{q}\mathbf{e_i}')^{-1}\mathbf{h} \\ &= \frac{(1-\alpha)w}{\beta m}\left((I-\alpha G)^{-1}\mathbf{h} + \frac{(I-\alpha G)^{-1}\alpha\mathbf{q}\mathbf{e_i}'(I-\alpha G)^{-1}}{1-\alpha\mathbf{e_i}'(I-\alpha G)^{-1}\mathbf{q}}\mathbf{h}\right)\end{aligned} \tag{2.50}$$

Let us focus on the expression:

$$\frac{(1-\alpha)w}{\beta m}\frac{(I-\alpha G)^{-1}\alpha\mathbf{q}\mathbf{e_i}'(I-\alpha G)^{-1}}{1-\alpha\mathbf{e_i}'(I-\alpha G)^{-1}\mathbf{q}}\mathbf{h} \tag{2.51}$$

Proceeding analogue to the analysis in Subsection 2.3.2, we get that the effect of adding a link (j,i) on the centrality of firm k is:

$$s_k - \tilde{s_k} = -\alpha s_i \frac{\sum_{t\in N_i^+} q_t m_{kt} + q_j m_{kj}}{1+\alpha\sum_{t\in N_i^+} q_t m_{it} + q_j m_{ij}} \tag{2.52}$$

where $\tilde{s_k}$ is the centrality of k after adding link (j,i) and N_i^+ is the in-neighbourhood of firm i in network G. □

2.8 Appendix B: Definitions

2.8.1 Profits as a Stationary Distribution of a Markov chain

Matrix G in general has elements such that $\sum_j g_{ji} = 1\ \forall i$ so it is column stochastic matrix. For a column stochastic matrix P we have $P\mathbf{x} = \mathbf{x}$ for stationary distribution $\mathbf{x}$. From (2.30) we have $\sum_i p_i y_i = \mathbf{1}'\mathbf{s} = \frac{w}{\beta}$. Normalizing $w = \beta$ we can write:

$$\mathbf{s} = \frac{1-\alpha}{m}\mathbf{h}\mathbf{1}'\mathbf{s} + \alpha G\mathbf{s} = \left(\frac{1-\alpha}{m}\mathbf{h}\mathbf{1}' + \alpha G\right)\mathbf{s} \tag{2.53}$$

The Matrix $\frac{1-\alpha}{m}\mathbf{h}\mathbf{1}'$ in the benchmark case will be the matrix having sum of columns (and rows) equal to $1-\alpha$. The sum of elements of each column in matrix αG is equal to α. Thus, matrix $\frac{1-\alpha}{m}\mathbf{h}\mathbf{1}' + \alpha G$ is a column stochastic matrix. The vector of profits $\boldsymbol{\pi} = (1-\alpha-\beta)\mathbf{s}$ will be a scaled vector of the stationary distribution of (column) stochastic, irreducible (because we look only at the active firms) Markov chain with the transition matrix $\frac{1-\alpha}{m}\mathbf{h}\mathbf{1}' + \alpha G$. In general case, for an arbitrary $\mathbf{h}$ and weights attached to each consumption good this matrix will still be stochastic.

2.8.2 Intercentrality

It is important to quantify the contribution of node k to the centrality of other nodes in the network. The vector contributions of node k to centralities of all other nodes in the network is the intercentrality of node k.

Definition 2.4. *Consider a network with $n \times n$ adjacency matrix G, scalars α, ζ and n-dimensional vector $\mathbf{h}$. The vector of intercentralities of graph G with discount factor α and personalization vector $\mathbf{h}$ is given with $\mathbf{m}(G, \alpha, \zeta, \mathbf{h}) = \zeta \mathbf{h}'(I - \alpha G)^{-1}$*

So, if k is not a consumption good, this contribution is 0 for every node in the network. If k is a consumption good, than the intercentrality node k is proportional to the sum of elements of column k of matrix $(I - \alpha G)^{-1}$.

2.9 Appendix C: Budget Constraint

In the paper we have assumed that revenue distortions are exogenous and come from outside the system (i.e. revenue losses from the negative distortion are sunk, and not transferred to the consumer). In other words, we have ignored the budget balance condition when studying the revenue distortion. We have opted for this approach, because our primary focus is on the role of the network when studying revenue distortion of a general kind, and intentionally we were not specific about the interpretation of the distortion. If we talk about specific distortions, such as a tax in a closed economy, the budget balance constraint must be satisfied. In this section we show that imposing this condition will not change the qualitative results from Subsection 2.3.1 and Subsection 2.3.2.

2.9.1 Common Distortions

In the case of common distortion, the income of the consumer when budget constraint is included is:

$$In = (1-\tau)(1-\alpha-\beta)\sum_{i \in N} p_i y_i + (1-\tau)\sum_{i \in N} \beta p_i y_i + \tau \sum_{i \in N} p_i y_i = \frac{w(1-\alpha(1-\tau))}{\beta(1-\tau)}$$

So we can write the centrality equation as:

$$\mathbf{s}(\tau) = \frac{w(1-\alpha(1-\tau))}{\beta(1-\tau)}\mathbf{h} + \alpha(1-\tau)G\mathbf{s}(\tau)$$

The analysis of the derivative (change of the centrality vector) is analogous to the analysis in subsection 2.3.1. For instance, when interested in the relative movement of the centralities (keeping the sum of centralities fixed), the only thing that will change is the normalization of wage w, and the results will be exactly the same. When interested in the absolute movement of the centrality vector (with normalization of wage independent of τ) we get:

$$\begin{aligned}
\frac{d\mathbf{s}(\tau)}{d\tau} &= \frac{w}{\beta(1-\tau)^2}\mathbf{h} - \alpha G\mathbf{s}(\tau) + \alpha(1-\tau)G\frac{d\mathbf{s}(\tau)}{d\tau} \Rightarrow \\
\frac{d\mathbf{s}(\tau)}{d\tau} &= -\alpha(I-\alpha(1-\tau)G)^{-1}G\mathbf{s}(\tau) + \frac{w}{\beta(1-\tau)^2}(I-\alpha(1-\tau)G)^{-1}\mathbf{h} \Rightarrow \\
\frac{d\mathbf{s}(\tau)}{d\tau} &= -\alpha(I-\alpha(1-\tau)G)^{-1}G\mathbf{s}(\tau) + \frac{1}{(1-\tau)(1-\alpha(1-\tau))}\mathbf{s}(\tau) \Rightarrow \\
\frac{d\mathbf{s}(\tau)}{d\tau} &= -\frac{1}{1-\tau}\left(I-\alpha(1-\tau)G\right)^{-1}\mathbf{s}(\tau) + \frac{1}{1-\tau}\mathbf{s}(\tau) + \frac{1}{(1-\tau)(1-\alpha(1-\tau))}\mathbf{s}(\tau) \Rightarrow \\
\frac{d\mathbf{s}(\tau)}{d\tau} &= -\frac{1}{1-\tau}\left(\left(I-\alpha(1-\tau)G\right)^{-1}\mathbf{s}(\tau) - \frac{2-(1-\alpha)\tau}{(1-\tau)(1-\alpha(1-\tau))}\mathbf{s}(\tau)\right) \qquad (2.54)
\end{aligned}$$

The difference from (2.7) is the term $\frac{1}{(1-\tau)(1-\alpha(1-\tau))}\mathbf{s}(\tau)$ as visible in the third line of (2.54)which is the marginal effect of the income change (due to the tax transfer) to centrality vector $\mathbf{s}(\tau)$. This effect will be positive pull effect, increasing the demand for a good and thus its profit (centrality).

As for the analysis of the social welfare with common distortion, we need to take into account now the increase in income due to tax transfer. The utility of the consumer, in this case, can be written as:

$$\log U(\tau) = (\xi-1)\log n + \log(1-\alpha(1-\tau)) - \log(1-\tau) + \frac{\alpha}{1-\alpha}\log\alpha + \frac{1}{1-\alpha}$$
$$\left((\alpha+\kappa-1)\sum_i v_i(0)\log r_i - (1-\alpha-\beta)\sum_i v_i(0)\log v_i(\tau) + (\alpha+\beta)\log(1-\tau)\right)$$

2.9.2 Firm Specific Distortions

Suppose firm k is distorted. Then the income of the consumer becomes:

$$
\begin{aligned}
In &= \sum_{i\in N}(1-\alpha-\beta)\,p_i y_i - \tau\,(1-\alpha-\beta)\,p_k y_k + \beta\sum_{i\in N} p_i y_i - \tau\beta p_k y_k + \tau p_k y_k \\
&= \frac{1-\alpha}{\beta} w + \tau p_k y_k
\end{aligned}
$$

and the centrality equation becomes:

$$
\mathbf{s}(\tau) = \frac{1-\alpha}{\beta m} w\mathbf{h} + \frac{\tau}{m} s_k(\tau) + \alpha\,(G + Q(\tau))\,\mathbf{s}(\tau) \Rightarrow \mathbf{s}(\tau) = \frac{1-\alpha}{\beta m} w\left(I - \alpha\tilde{G}\right)^{-1}\mathbf{h}
$$

The matrix $\tilde{G}$ is created from the original matrix G by adding term $-g_{ik}\tau + \frac{\tau}{m}$ to each element i of column k. This is again a rank one update of the matrix G. Because of this, the analysis of the effect of a single firm revenue distortion can be conducted using the same tools as before.

2.10 Appendix D: Some Useful Results From Markov Chain Theory

Here we state some results from the Markov chain analysis which lead to Proposition 2.5. The discussion of these results can be found in (Chien et al., 2004)

Definition 2.5. *The mean first passage time from i to j denoted with μ_{ij} is the expected number of steps from state i to state j*

The following theorem gives connection between mean first passage time and stationary distribution of a Markov chain.

Theorem 2.2. *Let P be a transition matrix of a regular Markov chain and $\boldsymbol{\pi}$ associated stationary distribution then:*

1. *For any two states i and j we have that $\mu_{ij} = 1 + \sum_{k\neq j} p_{ij}\mu_{kj}$*
2. *For any state i $\pi_i = \frac{1}{\mu_{ii}}$*
3. *For any two states $i \neq j$, a change in the transition probability from j to any other state does not change μ_{ij}*

Proof. See (Chien et al., 2004) □

Definition 2.6. *Fundamental Matrix Z of Markov chain with transition matrix P is defined with $Z = (I - (P - B))^{-1}$ where $B = \lim_{k\to\infty} P^k$*

The following theorem gives a connection between the stationary distribution of Markov chain P and perturbed Markov chain $P + \Delta$

Theorem 2.3. *Let P and $\tilde{P}$ be two transition matrices of a Markov chain such that $\tilde{P} = P + \Delta$ and let $\boldsymbol{\pi}$ and $\tilde{\boldsymbol{\pi}}$ be a corresponding two stationary distributions. Then:*

1. $\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi}\Delta Z + \boldsymbol{\pi}$
2. *Z is diagonally dominant over columns, that is $z_{jj} \geq z_{ij}\ \forall i, j$ and for any two i, j such that $i \neq j$ $z_{jj} - z_{ij} = \mu_{ij}\pi_j$*

The following theorem:

Theorem 2.4. *Let P be a transition matrix of a finite state regular Markov chain and let i and j be two arbitrary states of P. Let Δ be a matrix that is zero everywhere except in row i, the (i, j) element is the only positive element and $\tilde{P} + P + \Delta$ is also a transition matrix of a regular Markov chain. Let $\tilde{\boldsymbol{\pi}}$ denote stationary distribution of $\tilde{P}$, then $\tilde{\pi}_j \geq \pi_j$*

Proof. See (Chien et al., 2004) □

Chapter 3

A Screening Role of Enforcement Institutions

3.1 Introduction

Economic and social activities are governed by a set of formal and informal institutions. These institutions are important because markets and socio-economic activities in general cannot function well without them. As indicated in (Dixit, 2009) there are 3 main goals of governance institutions: (i) securing property rights (ii) enforcement of contracts (iii) resolving collective action problem. Here we focus on (ii), specifically on the problem of contractual opportunism. When the quality of institution that enforces contracts is low (in the sense that there is a low probability of being punished for breaking a contract), contractual opportunism creates a prisoner's dilemma problem and a society may end up in a state with very little or no cooperation at all. Some agents, however, might cooperate and not break a contract even when the quality of the enforcement institution is low due to their innate sense of righteousness or morals. Cooperating in this case signals that an agent has high morals or work ethics (from now on simply a type), which increases agent's reputation and making her a better partner for future ventures. When the enforcement is extremely strong, in a sense that the probability of identifying and punishing the defector makes the defection prohibitively costly, cooperating will not signal anything about the type of an agent. The same will be when the quality of enforcement institution is absent and costs of being defected on are high so that all agents will defect. In this case again no information about types of the players will be revealed. When the quality of legal enforcement is on some intermediate level such that some types choose to cooperate and others to defect, an agent's action will reveal some information about her type.

This mechanism may have a significant effect in different contexts of human interaction. For example, consider a situation in which a population of firms interact in two different ways. The first type of interaction is the the simpler (routine) interaction which can be fully specified and thus monitored and enforced by a formal institution (for example, a delivery of specified products). The second one is a more complex and intangible interaction which cannot be (due to its complexity or simply because it is too costly) monitored or enforced by a formal institution. Examples are R&D projects or joint ventures. In this type of activities an agent prefers having a partner with higher business ethics (better corporate culture, higher morals), as this will imply lower probability of opportunistic behaviour and therefore a higher expected payoff. Cooperating in the first type interaction will signal high business ethics when there is a possibility to defect and get away with it. The screening role of the enforcement institution discussed above can in this case facilitate positive assortative matching between firms with better business ethics in the second type interaction. When the expected pay-off from the second type activity exhibits complementarity, positive assortativeness will increase social welfare. A similar mechanism may be in play within the organization, concerning interaction in teams between employees. Less monitored and rigid work environment will facilitate larger output in more creative activities. Even in everyday interaction between people - not defecting when there is a chance to defect signals high moral values, making an agent a better partner in other socio-economic activities, such as marriage or different kinds of neighbourhood communal activities.

The interaction between formal incentives and signalling concerns has been discussed in the literature in different contexts and can be tracked back to (Titmuss, 1970), who argued that paying blood donors could reduce the supply as it makes donating blood 'less of a good deed' due to monetary incentives. This interaction between intrinsic (being a good, altruistic person) and extrinsic motivation is empirically and experimentally documented in different contexts. For example, (Gneezy and Rustichini, 2000) found that fining parents for picking up their children late from day-care centres resulted in more late arrivals, indicating that extrinsic motivation (fines) has crowded out intrinsic motivation (i.e. signalling being a good parent). In experiments, (Fehr et al., 2001) recorded that subjects provided less effort when the contract specified punishment for bad behaviour, compared to the case when it did not. The effect of extrinsic incentives on prosocial behaviour such as blood donation from theoretical point of view is discussed in (Benabou and Tirole, 2006). In the subsequent paper (Benabou and Tirole, 2011) the same authors in a similar framework analyze how laws and norms interact, specifically in the context of the optimal taxation problem. Even though the questions asked are different, the main idea in these papers is similar to the idea here: "The effect of the extrinsic motivation is substantially determined by the intrinsic motivation of an agent". In this context, we should also mention (Seabright, 2009), who considers a possibility of crowding out of intrinsic motivation in the model as in (Benabou and Tirole, 2006) with explicitly modelled signalling benefits, and (Levy and Razin, 2013) who studies how self and social signalling by being religious can increase the level of cooperation in a society.

In our model, signalling high type does not bring direct benefit, as in (Benabou and Tirole, 2006) for example, but has an instrumental role - it provides better match in the future. In this dimension, this paper has some similarities with the literature on costly signalling and matching (see for example (Hoppe et al., 2009)). However signalling here is not done by paying a cost directly, but by playing a certain strategy in a bilateral game. The game itself, together with the population of players, determines the cost (and the benefits) of signalling. Furthermore, which game will be played is determined by the quality of the enforcement institution. This, together with the distribution of types in the society, determines the cost of signalling.

Our paper is also related to the literature that studies the relationship between formal and informal enforcement institutions. The interaction between formal enforcement and reputation has been discussed in (Dixit, 2003). In his model, the gains from cooperation (honest trade) are greater the larger the distance between the pair of traders. On the the other hand, the frequencies of meetings and spread of information are locally biased. In this setting, reputation can sustain cooperation when the society is not too large. Otherwise, an external enforcement is needed to sustain cooperation. Other related papers that discuss interaction of formal and informal institution mostly focus on repeated interactions. The general conclusion is that better external enforcement crowds out the effect of informal institutions because it weakens reputation incentives (Dhillon and Rigolini, 2011; Kranton, 1996).

The paper is organized as follows. In Section 3.2 we present the basic model.

The analysis of the equilibrium is conducted in Section 3.3, and in Section 3.4 we discuss the effects of the quality of the enforcement institution on the welfare. In Section 3.5 we analyze the quality of enforcement institutions when it is endogenously determined in majority voting elections. Section 3.6 discusses the effect of matching friction, and Section 3.7 concludes.

3.2 Setup

In the paper we are interested in bilateral interactions which, in the absence of enforcement institution, can be represented as a Prisonner's dilemma game (PD). The enforcement institution is modelled as a probability that a defector will be identified and fined. We refer to the magnitude of this probability as to the quality of (enforcement) institution. The quality of institution has a direct effect of increasing cooperation by making defection more costly and therefore less attractive.

We study a game with uncountable set of players. So, there is a population with a continuum of heterogeneous agents which differ in their cost of defecting. The cost of defecting is the type of an agent, and one can think of it as morals of an agent, work ethics, an ability to defect, a psychological cost of cheating or even to a some extent trust and trustworthiness[1].

The interaction happens across three periods. In period 1 players learn their types, and are randomly matched in pairs to play simultaneous move Bayesian game with payoffs as in Table 3.1. When both players cooperate, they receive payoff $\gamma > 0$. When player i defects and player j cooperates, the defector receives payoff $\kappa > \gamma$, but also experiences disutility $-\lambda_i$ (i.e. the psychological cost of immoral action). In addition to this, the defector is identified with probability $\theta \in [0, 1]$ and charged a fine normalized to 1. When this happens, the whole fine is transferred to the cooperator. Thus when player i defects and player j cooperates, the payoff of player i is $\theta(\kappa - 1 - \lambda_i) + (1 - \theta)(\kappa - \lambda_i) = \kappa - \theta - \lambda_i$. The cost of being defected on when cooperating is $c > 0$. In this case player j suffers a loss c and receives transfer from the defector of size 1 with probability θ. We refer to θ as to the quality of the enforcement institution.

Agents don't know the type of the agent they are matched with, but they do know

[1] In this context an agent is said to be more trustworthy if the probability that she will defect conditional on opponent cooperating is low. An agent will be more trusting if it is more likely that she will cooperate

	C	D
C	γ, γ	$\theta - c, \kappa - \theta - \lambda_j$
D	$\kappa - \theta - \lambda_i, \theta - c$	$-\lambda_i, -\lambda_j$

Table 3.1 Two players game

the distribution of types and own type. So type of an agent is private information. In period 2 payoffs from period 1 are realized and actions taken by the every player in the first period are observed by everyone[2]. In period 3, depending on the action in the game from period 1, agents are assortatively matched with other players who acted in the same way (cooperators with cooperators and defectors with defectors[3]). Payoff from the third period is assumed to exhibit complementarity in types in a form standard in the matching literature. When players i and j are matched, the benefit to both of them is $\mu\lambda_i\lambda_j$ where $\mu > 0$ is the parameter measuring the importance of the matching stage. The discount factor is built in μ. There are no strategic decisions in the third period, and one can think of it as a reduced form representation of a potentially long term bilateral project that is intangible or too complex to be enforced by a formal institution (such as an entrepreneurial or R&D project). This formulation captures the idea that expected benefit from the joint project is higher if a partner has higher cost of defection (higher morals, work ethics).

3.2.1 Parameters

Throughout the paper we shall maintain some assumptions on the parameter space which we discuss in this section. First, for simplicity, we assume that the support for type distribution is $\left[\underline{\lambda}, \overline{\lambda}\right] = [0, 1]$. We will also assume that the cooperation is efficient from the single shot game perspective in the first period, even for the worst type($\lambda = \underline{\lambda}$) - that is that $2\gamma > \kappa - c$. We shall also postulate that when $\theta = 0$, meaning that practically there is no legal enforcement, the defection is the optimal strategy for every player. This means that when $\theta = 0$ the game is PD. In terms of parameters this means that: $\kappa - 1 > \gamma \wedge -c < -1 \Rightarrow \kappa > \gamma + 1 \wedge c > 1$

To be more clear let us state the assumptions explicitly:

Assumption 3.1. *We assume the following restrictions on parameters*

[2]Check Section 3.6 for the discussion on the observability assumption
[3]We shall allow for friction with this respect in section 3.6

- $2\gamma > \kappa - c$ - *Cooperation is efficient*
- *The game is PD when* $\theta = 0$

  (i) $\gamma < \kappa - 1$

  (ii) $c > 1$

The assumption 3.1 gives $\kappa - c < 2\gamma < 2\kappa - 2 \Rightarrow \kappa > 2 - c$

For the sake of the discussion in sections below, let us define the concept of strategic substitutes and strategic complements formally, following (Bulow et al., 1985).

Definition 3.1. *A bilateral game is said to be a game of strategic complements (substitutes) if for every two players i and j and their strategies* x_i *and* x_j*:* $\frac{\partial^2 \pi_i}{\partial x_i \partial x_j} > 0$ *(*< 0*)*

From Definition 3.1 it follows that in a game of strategic complements (substitutes) best response functions are upward (downward) sloping. In the model considered in our case, strategic complements (substitutes) will imply that player is more (less) prone to cooperate when the probability of being matched with cooperator is higher.

3.3 Equilibrium

Note that in the third period (matching stage) there are no strategic decisions. Thus, the game here is basically a simultaneous move game with incomplete information, and therefore we shall employ the concept of Bayesian-Nash equilibrium (BNE). Let us consider a match between arbitrary agents i and j, and let p_j denote the probability that player j will cooperate. We can write the expected payoff of cooperation of player i as:

$$\pi(\lambda_i, C) = \gamma p_j + (\theta - c)(1 - p_j) + \mu \lambda_i E(\lambda|C)$$

The expected payoff of defecting is:

$$\pi(\lambda_i, D) = (\kappa - \theta - \lambda_i)p_j + (-\lambda_i)(1 - p_j) + \mu \lambda_i E(\lambda|D)$$

where $E(\lambda|C)$ is the expected type of cooperators, and $E(\lambda|D)$ is the expected type of defectors. Thus the net benefit of cooperating is:

$$\Pi(\lambda_i, \lambda_j) = \pi_i(C) - \pi_i(D) = (\gamma - \kappa + c)p_j + \lambda_i - c + \mu\lambda_i \left(E(\lambda|C) - E(\lambda|D)\right) \quad (3.1)$$

As $\lambda_i \in [0, 1]$, the strategy of player i is a function $\sigma_i : [0, 1] \to \{C, D\}$. We say that player i uses cutoff strategy, if there exists some $x \in [0, 1]$ such that:

$$\sigma_i(\lambda_i) = \begin{cases} \text{C} & \text{if } \lambda_i \geq x \\ \text{D} & \text{otherwise} \end{cases}$$

A cutoff equilibrium is BNE in cutoff strategies.

Suppose that player j uses cut-off strategy with cutoff y ($\sigma_j(\lambda_j) = y$). Than the probability of cooperation of player j is $1 - F(y)$ where F is cdf of the type distribution. We write:

$$\Pi(\lambda_i, y) = (\gamma - \kappa + c)(1 - F(y)) + \lambda_i - c + \mu\lambda_i \left(E(\lambda|C) - E(\lambda|D)\right)$$

For cutoff strategy to be a best response it must be that $\Pi(\lambda_i, y)$ is increasing in the first argument, that is, player i should be more prone to choose C over D the more ethical she is.

We have that $\frac{\partial \Pi(\lambda_i, y)}{\partial \lambda_i} = 1 + \mu \left(E(\lambda|C) - E(\lambda|D)\right)$ which will always be greater than 0 given that all players use the cutoff strategy. Given the positive sign of the partial derivative, we can write the best response of player i to cutoff strategy y of player j as:

$$\sigma_i(y) = \begin{cases} 0 \text{ if} & \Pi(0, y) \geq 0 \\ 1 \text{ if} & \Pi(1, y) \leq 0 \\ x \in (0, 1) & \text{otherwise} \end{cases}$$

where x is the unique solution of equation $\Pi(\lambda, y) = 0$ (so when $\lambda_i > x$ player i plays C and otherwise plays D)

As players are ex ante symmetric, we shall look for the symmetric equilibrium in cutoff strategies (in which all players choose the same cutoff).

The following proposition gives a result regarding existence of the cuotoff equilibria:

Proposition 3.1. *There exist a cutoff equilibrium*

Proof. See Appendix A □

Let us examine the effect of the quality of enforcement institution (θ), and reputation $\mu\Delta(\lambda)$ on the level of cooperation in the equilibrium. Taking the derivative of(3.17) with respect to λ we get:

$$\frac{\partial \lambda^*(\theta)}{\partial \theta} = -\frac{1}{1 + \mu\lambda\Delta'(\lambda) - (\gamma - \kappa + c)F'(\lambda)} \tag{3.2}$$

which for the region of parameters when λ^* is unique (Proposition 3.3) is always positive. Similarly we get that $\frac{\partial \lambda^*}{\partial \mu} > 0$. Thus the increasing quality of the enforcement institution and relative importance of the reputation benefits will increase the cooperation in the first period. However, increasing θ, may actually decrease the reputation benefits for cooperating vs. defecting. Before discussing this, let us first state the result relating density function f with function Δ. Let f be a density function $f : \left[\underline{\lambda}, \overline{\lambda}\right] \to \mathbb{R}$. Let us define function $\Delta : \left[\underline{\lambda}, \overline{\lambda}\right] \to \mathbb{R}$ with

$$\Delta(x) = E(\lambda|\lambda > x) - E(\lambda|\lambda < x)$$

where expectation is according to density function f. Then the following result holds (Jewitt, 2004):

Theorem 3.1 (Jewitt)**.** *If f is everywhere decreasing (increasing) then Δ is everywhere increasing (decreasing). When f is unimodal, Δ is quasiconvex. If f has a unique interior maximum then Δ has an unique interior minimum*

Proof. Omitted □

Following Theorem 3.1 we can state the following result:

Proposition 3.2. *When distribution of types has increasing (decreasing) density function $f : [\underline{\lambda}, \overline{\lambda}] \to \mathbb{R}$, the reputation benefit of cooperating versus defecting in the interior equilibrium will increase (decrease) with the quality of enforcement θ. When f reaches its maximum in the interior of $[\underline{\lambda}, \overline{\lambda}]$, the reputation benefit of cooperating versus defecting will reach its minimum in the interior $[\underline{\lambda}, \overline{\lambda}]$*

Proof. Follows directly from Theorem 3.1. □

For instance, a decreasing density function describes a situation in which there are more 'bad types' (agents with low cost of defecting) relative to the number of 'good types' (i.e. in the case of decreasing density function) in the society. The effect of improving the quality of enforcement institution on the level of cooperation will then be partially crowded out, due to the decrease of the reputation incentive to cooperate vs. defect. Increasing θ will decrease equilibrium threshold $\lambda^*(\theta)$ and increase the mass of agents that cooperate. As f is decreasing, this means that the expected type of cooperator will decrease more than the expected type of defector, causing decrease in $\Delta(\lambda^*)$. This will partially crowd out the effect of increase of θ on the level of cooperation. The strength of the crowding out effect is captured by the term $1+\mu\Delta'(\lambda^*)$ in the equation (3.2). When the density function is increasing, the reputation effect will reinforce the direct effect of θ. This is in essence the same mechanism as in (Benabou and Tirole, 2006) applied to the situation modelled in this paper.

Another, rather intuitive, insight the from equation 3.2 is that the effect of increase θ on the level of cooperation will be smaller in the case of strategic substitutes ($\gamma - \kappa + c < 0$), than in the case of strategic complements ($\gamma - \kappa + c > 0$).

In what follows we shall focus on the parameter space in which the cutoff equilibrium is unique. The following proposition gives a result regarding that:

Proposition 3.3. *The cutoff equilibrium will be unique when:*

$$\mu\Delta(y) + 1 > |\mu\sigma(y)\Delta'(y) - (c + \gamma - \kappa)F'(y)|$$

Proof. See Appendix A □

3.3.1 Analysis of the Equilibrium

In this section we shall assume that the distribution of types is uniform on the segment $[0, 1]$. The indifference condition can be be written as:

$$\Pi(\lambda, \lambda) = (c + \gamma - \kappa)(1 - \lambda) - c + \theta + \frac{\lambda}{2} + \lambda = 0 \tag{3.3}$$

which gives the cutoff point:

$$\lambda^* = \frac{\kappa - \gamma - \theta}{\kappa - \gamma - c + 1 + \frac{\mu}{2}} \tag{3.4}$$

Let us first state the following corollary of Proposition 3.3, defining a region of parameters for which the threshold equilibrium is unique.

Corollary 3.1. *When types are distributed uniformly on the segment* $[0, 1]$ *the equilibrium will be unique when*

$$\frac{1}{2}(\mu + 2) > |\gamma - \kappa + c|$$

Proof. When types are uniformly distributed on $[0, 1]$ we get:

$$\sigma_i'(y) = 2\frac{\gamma - \kappa + c}{\mu + 2} \tag{3.5}$$

and the claim directly follows from Proposition 3.3 □

First note that the threshold defined with equation (3.4) is a linear function of θ. By Proposition 3.3 the denominator of (3.4) will be positive, and therefore the threshold will always decrease with θ (mass of cooperators will increase). The threshold will also decrease with μ as: $\frac{\partial \lambda^*}{\partial \mu} = -\frac{2(\kappa-\gamma-\theta)}{(-2c-2\gamma+2\kappa+\mu+2)^2} > 0$ since by the Assumption 3.1 $\kappa - \gamma - \theta > \kappa - \gamma - 1 > 0$.

The threshold will be interior when $\frac{\mu}{2} + 1 - c > -\theta \Leftrightarrow c < 1 + \theta + \frac{\mu}{2}$, thus when the cost when defected on is not too high, or when the long term benefits, captured with μ are relatively high. Note that due to Assumption 3.1 we have that $\kappa - \gamma > \theta$ as $\theta \in [0, 1]$ so the expression in the numerator of (3.4) is always positive. Thus we have showed that the following lemma holds.

Lemma 3.1. *In the case of uniform distribution of types, the threshold will be interior when*

$$c < 1 + \theta + \frac{\mu}{2} \tag{3.6}$$

In what follows we shall focus on the cases when the threshold is interior.

Let us check now how the payoffs depend on the level of enforcement. For type λ (and interior threshold) the payoff of the cooperator in equilibrium is given with:

$$\pi(\lambda, C) = \frac{2\lambda\mu(c + 2\gamma + \theta - 2\kappa) + 4(c - \theta)(\kappa - \theta) - 2\mu(\gamma + \lambda) - 4\gamma - \lambda\mu^2}{4(c + \gamma - \kappa - 1) - 2\mu} \tag{3.7}$$

and the payoff of defector with type λ at interior threshold is:

$$\pi(\lambda, D) = \frac{-2\lambda(c+\gamma-\kappa-1) + 2(c-\theta-1)(\kappa-\theta) + \mu(\lambda(\gamma+\theta+1) + \theta - \kappa(\lambda+1))}{2(c+\gamma-\kappa-1) - \mu} \tag{3.8}$$

There are a couple of interesting things worth noting here. First let us look at the payoff of cooperators. Taking derivative with respect to θ we get:

$$\frac{\partial \pi(C, \lambda, \theta)}{\partial \theta} = \frac{2(c - 2\theta + \kappa) - \lambda\mu}{-2c - 2\gamma + 2\kappa + \mu + 2} \tag{3.9}$$

We are interested in the behaviour at the unique cutoff equilibrium, thus due to Propostion 3.3, we have that denominator of (3.9) will always be positive. The sign of the numerator $2(c - 2\theta + \kappa) - \lambda\mu$ can be either positive or negative. However, if this expression is negative for some $\lambda = \tilde{\lambda}$ it is negative for all $\lambda > \tilde{\lambda}$. That is, if an increase in the quality of institution decreases payoff of player with type $\tilde{\lambda}$ it will decrease the payoff for every other player that has higher type (higher morals). The expression (3.9) will be negative when $\lambda > \frac{2c-4\theta+2\kappa}{\mu}$. Also, from Assumption 3.1 it follows that $c-2\theta+\kappa > 0$[4]. So for low values of μ this expression will be positive, and it becomes negative when μ is higher. The intuition is that if μ is high, the benefit from the matching stage is high and it becomes more important to be matched with higher type. When θ increases then by (3.4) the threshold will decrease, increasing the number of cooperators, and thus the payoff from the first period. However, this will result in a decrease of the average cooperator type, and in turn decrease the expected payoff of cooperator from the matching stage. Which effect will dominate depends on the size of μ. We can now state the following proposition:

Proposition 3.4. *Increase in the quality of enforcement institution will decrease the benefit of cooperator with type* $\lambda > \frac{2c-4\theta+2\kappa}{\mu}$. *This will be the case when reputation benefits are high enough, so* $\mu \geq \frac{2(c+\gamma-\kappa-1)(c-2\theta+\kappa)}{c+\gamma-\theta}$

Proof. See Appendix A □

Let us look at the payoff of a defector. Partial derivative with respect to θ is

$$\frac{\partial \pi(D, \lambda, \theta)}{\partial \theta} = \frac{2\kappa + 2c - 4\theta - 2 - \mu(1+\lambda)}{-2c - 2\gamma + 2\kappa + \mu + 2} \tag{3.10}$$

[4] As $c > 1 \wedge \kappa > \gamma + 1 > 1 \wedge \theta \in [0, 1]$

As was the case with cooperator, the partial derivative is monotone in λ. When both θ and μ are low, the sign of (3.10) might be positive. This is because an increase in θ will increase the mass of cooperators, by decreasing the threshold. This will increase the expected payoff of defectors from the first period, as it increases the probability that they will be matched with a player who cooperates in the equilibrium. Increase in θ will make the average defector type lower, hence lowering the expected payoff from the second stage for defectors. This effect will not be strong if μ is low enough, so we can state:

Proposition 3.5. *The payoff of defector of type λ can increase with the quality of enforcement institution. This will be the case when type λ is defector and $\lambda < \frac{2\kappa+2c-4\theta-2-\mu}{\mu}$, which is satisfied when reputation benefit is not too high*

Proof. See Appendix A □

What will be the effect of the increase of quality of enforcement institution is determined by the value of the matching in the third stage. If the matching is relatively unimportant, then the main effect of increasing θ for the payoff of cooperators will be through the increase the number of cooperators (thus the probability of meeting a cooperator) in the first stage, which will be positive. As for defectors, increasing the number of cooperators will increase the expected payoff, as it increases the probability of meeting the cooperator. On the other hand, increase in θ will increase the expected fine when defecting. The expected payoff from the third period for both cooperator and defector will be smaller when threshold decreases. This is because increase in the threshold will decrease the expected type of cooperators and the expected type of defectors. If μ is high, this effect will dominate and payoff of every player will decrease with θ (given that threshold is interior). When μ is small, an increase in θ will increase the payoff of cooperators and the effect on the payoff of defector will depend on the behaviour of threshold $\lambda^*(\theta)$. The effect of changing the quality of the enforcement institution thus can have different effects on the payoffs of agents in the society. In the next section we examine what will be the effect on the total welfare.

3.4 Welfare

From the first stage perspective it is optimal to maximize the cooperation, as it is the efficient outcome of the first period game (first stage). However, the matching in the third period is done based on actions taken in the first stage. As the payoff from the matching in period 3 exhibits complementaries, the efficient matching is the perfect positive assortative matching. However, the type of an agent is a private information, and only thing that is observable is the action in the first stage game. Thus the matching will be coarse, with only two classes of agents - cooperators and defectors from the first stage. Two classes seems to be too coarse to get a significant gain in welfare, compared to the purely random matching. However, it has been shown in McAfee (2002) that with two class matching with uniform distribution 75% of the value of using infinitely many classes can be obtained. When both are one-tailed exponential, with two classes 74.62% of possible gains can be obtained. When both are normally distributed, two classes result in about 63% of the possible gains.

In our case, the matching with two classes and a cutoff point of λ^* from the same population means associating values below λ^* with values below, and values above λ^* with values above λ^*. In this case, the social value of the matching is:

$$W_M = F(\lambda^*)\left(\int_{\underline{\lambda}}^{\lambda^*} \lambda \frac{f(\lambda)}{F(\lambda^*)}\,\mathrm{d}\lambda\right)^2 + (1 - F(\lambda^*))\left(\int_{\lambda^*}^{\bar{\lambda}} \lambda \frac{f(\lambda)}{1 - F(\lambda^*)}\,\mathrm{d}\lambda\right)^2 \tag{3.11}$$

Players with type lower than λ^* (defectors) are matched with the players that have type lower than λ^*, and cooperators are matched with cooperators. Thus, the average match value is the probability that a player is defector times the expected value of the defector match, plus the analogue term conditional on both being cooperators (having type higher than λ^*). Here λ^* is defined with (3.4). The matching here is perfect in the sense that cooperators are matched with cooperators and defectors are matched with defectors.

To find the optimal threshold λ^* (optimal θ that induces this λ^*) social planner is facing a trade-off. More cooperation will increase the total payoff from the first stage, but it might not reveal enough information and thus make matching less efficient. Maximizing social gains from matching requires a loss in the first stage, due to the fact that then there must be some defectors in order to better screen out the high types. When μ is small compared to the benefits from cooperation in the first stage, than the optimal λ^* will be the one that maximizes cooperation. When μ is larger, then matching payoff contributes relatively more to the welfare, and thus it will be optimal to have λ^* interior. The following proposition characterizes the optimal value of θ^*

Proposition 3.6. *For* $2\gamma + c - \kappa \geq \frac{\mu}{8}$, $\theta_u^* = 1$ *maximizes welfare. Otherwise the welfare will be maximized with*

$$\theta = \theta_u^* = \frac{3(4\gamma + 1)(c + \gamma - \kappa)}{-8c - 8\gamma + 8\kappa + \mu + 2} + \frac{1}{2}(c + 3\gamma + \kappa) - \frac{\mu}{4}$$

which defines the threshold:

$$\lambda^* = \frac{8c + 16\gamma - 8\kappa - \mu}{2(8c + 8\gamma - 8\kappa - \mu - 2)}$$

Proof. See Appendix A □

When the social benefit from complementarity in the third period is small compared to the benefits from the first stage i.e. the society gain from the cooperation, then it is optimal to minimize mass of defectors in the society, setting $\theta^* = 1$. When μ is large, then it will be optimal to have some 'extra' defectors in the society, in order to more accurately screen out the good types for the matching, and thus $0 < \theta_u^* < 1$

3.5 Endogenous Quality of Enforcement

Having in mind the social game context discussed in the introduction, it is natural to ask what level of θ will emerge endogenously from a process of decentralized social decision making. The question is even more interesting having in mind that the results from Section 3.3.1 state that the payoff of defector and cooperator can be both increasing and decreasing in θ.

When deciding on his preferred level of the enforcement, an agent faces a tradeoff. Conditional on cooperating, a player would like θ to be higher to protect him from being defected on in period 1. On the other hand, he prefers a higher average type of cooperators in the equilibrium as then the reputation benefit will be higher, which happens when θ is lower. Conditional on defecting, an agent prefers lower θ as this implies lower probability of being identified and fined as a defector. On the other hand, he would like θ to be high enough so there are still some cooperators in the society that he can exploit in the first period. Furthermore, as higher θ implies lower average type of defectors, he would prefer θ to be lower, as this will increase his benefits from the third period. Thus, which θ will emerge in a decentralized social decision making process is determined by a complex interaction between reputation concerns and the direct effect of θ in changing rules of interaction in the first period.

To be more concrete, in this paper we shall focus on a simple voting model in which agents vote sincerely on the value of θ, and study how does the equilibrium θ depends on the primitives of the model. Individual preferences over θ are single peaked, and as the voting is done over one-dimensional issue (value of θ) we can apply the median voter theorem. The agents will vote differently conditional on whether they cooperate or defect in the equilibrium.

Type λ_i will prefer different values of θ conditional on cooperating or defecting, and let us denote these two values with θ_i^C and θ_i^D respectively. If payoff of player i is larger when defecting and $\theta = \theta_i^D$ than when cooperating and $\theta = \theta_i^C$, then she will opt for θ_i^D.

The following propostion gives the value of θ in the voting equilibrium:

Proposition 3.7. *The majority rule voting system will chose level of enforcement* $\theta_v^* = \frac{c+\kappa}{2} - \frac{\mu}{8}$

Proof. See Appendix A □

Let us compare θ chosen by the median voter stated in Proposition 3.7 (θ_v^*), with θ that maximizes the social welfare, defined in Proposition 3.6 (θ_u^*). We have that:

$$\begin{aligned}
\theta_v^* - \theta_u^* &= \frac{c+\kappa}{2} - \frac{\mu}{8} - \left(\frac{3(4\gamma+1)(c+\gamma-\kappa)}{-8c-8\gamma+8\kappa+\mu+2} + \frac{1}{2}(c+3\gamma+\kappa) - \frac{\mu}{4} \right) \\
&= -\frac{3(4\gamma+1)(c+\gamma-\kappa)}{-8c-8\gamma+8\kappa+\mu+2} - \frac{3\gamma}{2} + \frac{\mu}{8} \\
&= \frac{-\mu^2 + (8c+20\gamma-8\kappa-2)\mu + 24c + 48\gamma - 24\kappa}{8(8c+8\gamma-8\kappa-\mu-2)} \qquad (3.12)
\end{aligned}$$

The derivative of the expression (3.12) with respect to μ is given with the expression:

$$\frac{3(4\gamma+1)(c+\gamma-\kappa)}{(-8c-8\gamma+8\kappa+\mu+2)^2} + \frac{1}{8} \qquad (3.13)$$

which is always positive in the case of strategic complements $\gamma - \kappa + c > 0$. Furthermore, due to the uniqueness condition in Proposition 3.3 we have that at the left side of the parameter space with respect to μ expression (3.12) will have value:

$\frac{1}{4}(c + 3\gamma - \kappa + 1)$. Thus, (3.12) is increasing in μ and takes positive value at the lowest feasible value of μ. Thus (3.12) will always be positive. Note that in the strategic complements case γ is always relatively low comparing to μ.

In the case of strategic substitutes $\gamma - \kappa + c < 0$ it might as well be that $\theta_v^* < \theta_u^*$, when μ is small relative to γ. When γ is high comparing to μ it becomes more important to have larger mass of cooperators in the society and the screening role of θ becomes less important. This means that θ_u^* will be higher. However, the quality of enforcement institution θ_v^* preferred by the median voter does not depend on γ. When γ is high enough, and μ low, then provided that we are in the case of strategic substitutes, $\theta_u^* \geq \theta_v^*$. As was the case with strategic complements, in the case of strategic substitutes the difference $\theta_v^* - \theta_u^*$ is increasing in μ, as (3.13)will be always larger than zero. Even though the first part of the expression (3.13) is negative, the quadratic equation

$$\frac{1}{8} - \frac{3(4\gamma + 1)(c + \gamma - \kappa)}{(-8c - 8\gamma + 8\kappa + \mu + 2)^2} = 0$$

doesn't have real zeros, and will always have the positive value (in the considered parameter space). As μ increases, the matching stage becomes more important, decreasing the optimal θ_u^* faster than it decreases the optimal θ_v^*, eventually making $\theta_u^* \leq \theta_v^*$

3.6 Matching Frictions

So far we have assumed that there is perfect matching in the third period in a sense that cooperators will always be matched with cooperators and defectors will always be matched with defectors. In this section we discuss the case when this isn't necessarily true. We assume that there is a non-negative probability that a cooperator is matched with defector in the third period. Let $P(C|\tau)$ denote the probability of a player being matched with cooperator in the third period given that his action in the first period is $\tau \in \{C, D\}$. In this case the following balancing condition must hold:

$$(1 - F(\lambda^*))(1 - P(C|C)) = F(\lambda^*)P(C|D) \tag{3.14}$$

The condition (3.14) states just that the mass of defectors that are matched with cooperators is equal to the mass of cooperators matched with defectors. As all players are matched in the third period, it holds that $P(C|C)+P(D|C) = 1$. Let us denote with $\alpha(\lambda^*) = P(C|C) - P(C|D)$. In the matching literature $\alpha(\lambda^*)$ is referred to as 'index of assortativity' (Bergstrom, 2003). Furthermore, let $\zeta = P(C|C)$ and $\xi = P(C|D)$, then $\alpha = \zeta - \xi$. Using this notation we can write(3.14) as:

$$
\begin{aligned}
&(1 - F(\lambda^*))(1 - \zeta)) = F(\lambda^*)\xi \Rightarrow \\
&\xi = \frac{(1 - F(\lambda^*))(1 - \zeta))}{F(\lambda^*)} \Rightarrow \\
&\alpha = \zeta - \frac{(1 - F(\lambda^*))(1 - \zeta))}{F(\lambda^*)} = \frac{\zeta + F(\lambda^*) - 1}{F(\lambda^*)}
\end{aligned}
$$

Then the expected reputation benefit of cooperator and defector of type λ_i denoted with $R(\tau, \lambda)$ $\tau \in \{C, D\}$: can be written as:

$$
\begin{aligned}
R(C, \lambda_i) &= \lambda_i \left(\zeta E(\lambda|C) + (1 - \zeta)E(\lambda|D)\right) \\
R(D, \lambda_i) &= \lambda_i \left(\xi E(\lambda|C) + (1 - \xi)E(\lambda|D)\right) \\
&= \lambda_i \frac{(1 - F(\lambda^*))(1 - \zeta))}{F(\lambda^*)} E(\lambda|C) + \left(1 - \frac{(1 - F(\lambda^*))(1 - \zeta))}{F(\lambda^*)}\right) E(\lambda|D)
\end{aligned}
$$

And reputational benefit from cooperating vs. defecting is then:

$$
\begin{aligned}
R(C, \lambda_i) - R(D, \lambda_i) &= \lambda_i \left((\zeta - \xi)E(\lambda|C) + (1 - \zeta - 1 + \xi)E(\lambda|D)\right) \\
&= \lambda_i(\zeta - \xi) \left(E(\lambda|C) - E(\lambda|D)\right) \\
&= \lambda_i \alpha \left(E(\lambda|C) - E(\lambda|D)\right)
\end{aligned}
$$

where α is the index of assortativity. We shall be interested in the case of positive assortative matching in the third period, thus for the values of $\alpha \in [0, 1]$. Note that then the analysis is parallel with what we had before, with μ being α scaled by some positive factor.

In this context it might be more natural to conduct the analysis with respect to ζ - the probability that an agent will be matched with cooperator given that he cooperates. ζ can be interpreted as a speed of information transmission in the social network, or simply as the quality of informal enforcement institution (reputation). Keeping ζ fixed, increase of θ, will decrease λ^*, $F(\lambda^*)$, and this will make assortativity index higher for a fixed ζ, increasing further on the level of cooperation. This effect captures the complementarity between formal enforcement and reputation throughout a different channel than it has been discussed in context of function Δ.

For example, in the case of uniform distribution (3.3) becomes:

$$\Pi(\lambda, \lambda) = (c + \gamma - \kappa)(1 - \lambda) - c + \theta + \mu \frac{\zeta + \lambda - 1}{\lambda} \frac{\lambda}{2} + \lambda = 0 \Leftrightarrow \tag{3.15}$$

$$\Pi(\lambda, \lambda) = (c + \gamma - \kappa)(1 - \lambda) - c + \theta + \frac{\mu}{2} (\zeta + \lambda - 1) + \lambda = 0 \tag{3.16}$$

and solving we get:

$$\lambda^* = \frac{-2\gamma - \zeta\mu - 2\theta + 2\kappa + \mu}{-2c - 2\gamma + 2\kappa + \mu + 2}$$

from where it is visible that λ^* will decrease with ζ. The further analysis parallel to what has been done in the frictionless matching case in the paper can be conducted in analogous way, and the qualitative nature of the results will not change.

3.7 Conclusion and Extensions

A higher level of enforcement and monitoring will make agents behave according to a prescribed set of rules. However, there are a lot of situations in which the tasks are too complex or intangible to be enforced in such a way. In these situations it is essential to find a partner who will exert high effort, even when not being subject to formal enforcement. When the returns from more complex interactions are higher than from the interactions that can be monitored, it is socially optimal to have lower level of enforcement such that some agents will choose to break the rules, and by doing that signal that they have bad work ethics. This will screen out agents who follow the rules only because of the high enforcement (disutility of being punished) from agents who choose to take a prescribed action as they feel intrinsic disutility from defecting. This will facilitate more efficient matching in tasks that cannot be enforced by a set of formal rules. The model captures this screening role of enforcement institution - emphasising the (nonmonotonic) effect of the strength of the enforcement on welfare and payoffs of agents with high and low work ethics. The set of rules in the society usually arises endogenously through some kind of social decision process. How the equilibrium institutions compare to the optimal depends on the importance of the screening role of the enforcement institution and whether the interaction in the monitored activities has the nature of strategic complements or strategic substitutes.

The formal (θ) and informal (reputation function Δ) enforcement institution interact in determining the level of cooperation and social welfare. By behaving according to the rules an agent earns a reputation to be of high work ethics. The benefit from the good reputation will be higher if a smaller mass of other agents behave the same way - and thus depends and the level of formal enforcement. The matching distortions will decrease the reputation benefit.

The model provides a good framework to study interaction between formal institution (enforcement) and informal institution (reputation) in enhancing cooperation in the society and the effects on the welfare. In relation to this, we find that the effect of the formal enforcement can be both reinforced or diminished by the reputation concerns. When distribution of types is non-decreasing, increase in the quality of institution will increase the reputation benefits, and thus the effect of reputation and institutional quality will go in the same direction, increasing the level of cooperation. When the density function of type distribution is decreasing, then increase of the quality of institution will increase the level of cooperation, but the effect will be partially crowded out by reputation - as net reputation benefit of cooperation will decrease. When the distribution of types is not monotonic, the direction of the reputation effect will depend on the current level of θ. Lower matching frictions, as jet another dimension of reputation as informal institution, imply higher level of cooperation. A higher quality of enforcement institution can increase the quality of matching, increasing the level of cooperation in the society even further.

There are several direction to extend the current model, and some of them are already work in progress. First we have focused on the reputation effect on the intensive margin. The agents do not make decisions to participate in the long term (complex) projects, nor which action to take. The matching stage is just a reduced form representation of this type of interaction, chosen to capture the basic idea that interacting with an agent of higher work ethics will bring higher benefit. Extending the model by allowing agents to choose to engage in non monitored activities that can yield higher returns (i.e. entrepreneurial project), or to interact in enforceable activities, would allow us to examine the information role of the enforcement institution on the level of entrepreneurship in a society on the extensive margin. This can, for example, have an important implications in studying the interaction between quality of enforcement institutions and the level of entrepreneurship in the society. Finally, the level of enforcement affects the utility of cooperators and defectors in a non monotonic way. For some levels of θ material payoff of defectors is higher than payoff of cooperators and vice versa. In a simple indirect evolutionary process, akin to the one in (Ok and Vega-Redondo, 2001), the level quality of enforcement institution will determine the distribution of types in the society. On the other hand, agents choose the level of enforcement in elections as described in the paper, and the equilibrium enforcement level will depend on the distribution of types. Examining the co-evolution of institution and the type distribution in the society seems like interesting avenue for building on this model.

3.8 Appendix A: Proofs

Proof of Propostion 3.1: $\Pi_i(x, y)$ is increasing in x for all $y \in [0, 1]$ and $\Pi_i(\lambda, \sigma(\lambda))$ is continuous in λ. When $\Pi_i(1, 1) \leq 0$ then 1 is the equilibrium cutoff. If $\Pi_i(0, \sigma(0)) \geq 0$, 0 is an equilibrium. Otherwise the equilibrium is defined as a solution of equation $\Pi_i(\lambda, \lambda) = 0$, that is given with:

$$(c + \gamma - \kappa)(1 - F(\lambda)) - c + \theta + \lambda + \lambda\mu\Delta(\lambda) \tag{3.17}$$

□

Proof of Proposition 3.3: To show uniqueness it is sufficient to show that the slope of the best response is in absolute value smaller than 1. It is known that this guarantees that the two best response curves can intersect only once, see (Vives, 2000). The slope of the best response function $\sigma(y)$ we obtain by equating (3.17) to zero and taking implicit derivation. That is we have:

$$\sigma_i'(y) = -\frac{\Pi_2(\sigma_i(y), y)}{\Pi_1(\sigma_i(y), y)} = -\frac{\mu\sigma(y)\Delta'(y) - (c + \gamma - \kappa)F'(y)}{\mu\Delta(y) + 1} \tag{3.18}$$

□

Proof of Proposition 3.4: From (3.9) equilibrium payoff cooperator will decrease with θ when $\lambda \geq \tilde{\lambda} = \frac{2c-4\theta+2\kappa}{\mu}$. Type $\tilde{\lambda}$ will be cooperator when $\tilde{\lambda} \geq \lambda^*$. So the equilibrium payoff of cooperator of type $\tilde{\lambda}$ will decrease with θ when $\frac{2c-4\theta+2\kappa}{\mu} \geq \frac{\kappa-\gamma-\theta}{\kappa-\gamma-c+1+\frac{\mu}{2}}$ which will be the case when $\mu \geq \frac{2(c+\gamma-\kappa-1)(c-2\theta+\kappa)}{c+\gamma-\theta}$. When payoff of $\tilde{\lambda}$ decreases with θ, so does the payoff of type $\lambda > \tilde{\lambda}$. □

Proof of Proposition 3.5: From (3.10) equilibrium payoff of defector of type λ will increase with θ when $\lambda \leq \tilde{\lambda} = \frac{2\kappa+2c-4\theta-2-\mu}{\mu}$. Type $\tilde{\lambda}$ will be defector when $\tilde{\lambda} < \lambda^*$. So the type $\tilde{\lambda}$ will be defector and have payoff which will be increasing function of θ when $\frac{2\kappa+2c-4\theta-2-\mu}{\mu} < \lambda^*$ When payoff of type $\tilde{\lambda}$ is increasing with θ, so is the payoff of every type $\lambda < \tilde{\lambda}$ □

Proof of Proposition 3.6: In our case, the distribution is by assumption uniform on $[0, 1]$ which gives the following expression for social gain from matching in the second stage, where λ^* is the threshold dividing population into two classes.

$$W_M = \mu\left(\lambda^*\left(\int_0^{\lambda^*}\lambda\frac{1}{\lambda^*}\,\mathrm{d}\lambda\right)^2 + (1-\lambda^*)\left(\int_{\lambda^*}^1\lambda\frac{1}{1-\lambda^*}\,\mathrm{d}\lambda\right)^2\right) \tag{3.19}$$

Simplifying (3.19) we get:

$$W_M = \mu\left(\frac{1}{\lambda^*}\int_0^{\lambda^*}\lambda^2\,\mathrm{d}\lambda + \frac{1}{1-\lambda^*}\int_{\lambda^*}^1\lambda^2\,\mathrm{d}\lambda\right) = \frac{\mu}{4}(1+\lambda^*-\lambda^{*2}) \tag{3.20}$$

We can write the total welfare as:

$$\begin{aligned} U &= 2\lambda^*(1-\lambda^*)(-c+\theta-\theta+\kappa) + 2(1-\lambda^*)^2\gamma + \lambda^*\int_0^{\lambda^*} -\frac{\lambda}{\lambda^*}\,\mathrm{d}\lambda + W_M \\ &= \frac{1}{4}\Big((8c+8\gamma-8\kappa-\mu-2)\lambda^{*2} + (-8c-16\gamma+8\kappa+\mu)\lambda^* + (8\gamma+\mu)\Big) \end{aligned}$$

This is a quadratic equation with respect to λ^*, so when the term with λ^{*2} , $8c+8\gamma-8\kappa-\mu-2<0$, the expression will be maximized at the stationary point

$$\lambda_u^* = \frac{8c+16\gamma-8\kappa-\mu}{2(8c+8\gamma-8\kappa-\mu-2)} \tag{3.21}$$

$\lambda_u^* \in [0,1]$ when:

$$\begin{aligned} &8c+16\gamma-8\kappa-\mu<0 \wedge 2(8c+8\gamma-8\kappa-\mu-2) < 8c+16\gamma-8\kappa-\mu \Leftrightarrow \\ &8(\gamma+c-\kappa) < \mu-8\gamma \wedge 8(\gamma+c-\kappa) < \mu+4+8\gamma \Rightarrow \\ &2\gamma+c-\kappa < \frac{\mu}{8} \end{aligned}$$

Thus, the stationary point λ_u^* will be the interior maximizer when: $2\gamma+c-\kappa<\frac{\mu}{8}$. When $2\gamma+c-\kappa>\frac{\mu}{8}\wedge 8c+8\gamma-8\kappa-\mu-2<0$, expression (3.21) will be negative, and the feasible λ^* that maximizes welfare will the left most feasible λ_u^* (the case that maximizes the number of cooperators). This implies $\theta_u^*=1$.

When $8c+8\gamma-8\kappa-\mu-2>0$, welfare will be maximized at the corner; again when λ_u^* takes the lowest possible value - implying $\theta_u^*=1$. So, for $2\gamma+c-\kappa<\frac{\mu}{8}$, $\lambda_u^*\in[0,1]$, and is defined with expression (3.21). The corresponding θ_u^* is defined then with (3.4. Solving for θ_u^* we get: $\theta_u^*=\frac{3(4\gamma+1)(c+\gamma-\kappa)}{-8c-8\gamma+8\kappa+\mu+2}+\frac{1}{2}(c+3\gamma+\kappa)-\frac{\mu}{4}$. For $2\gamma+c-\kappa\geq\frac{\mu}{8}$ the welfare is maximized when $\theta=1$.

□

Proof of Proposition 3.7: To determine preferred θ for every type i, we shall first calculate θ_i^C and θ_i^D. The payoff of cooperator as a function of θ is given equation (3.7). Payoff in (3.7) is strictly concave function of θ. Maximizing with respect to θ we get that when

$$\theta_i^C = \frac{1}{4}(2c - \lambda_i \mu + 2\kappa) \tag{3.22}$$

and payoff of type i when $\theta = \theta_i^C$ is:

$$\pi_i(\theta_i^C) = -\frac{-4\lambda\mu(3c + 4\gamma - 3\kappa) + 4(c-\kappa)^2 + 8\gamma(\mu+2) + \lambda(\lambda+4)\mu^2 + 8\lambda\mu}{16(c+\gamma-\kappa-1) - 8\mu}$$

Which for the median type has the value:

$$\pi_i(\theta_M^C) = -\frac{(-4c + 4\kappa + 3\mu)^2 + 16(4\gamma + \mu)}{64(c+\gamma-\kappa-1) - 32\mu}$$

From (3.8) we have that the preferred level of enforcement institution quality - given that an agent will defect is:

$$\theta_i^D = \frac{1}{2}(c + \kappa - 1) - \frac{1}{4}(\lambda_i + 1)\mu \tag{3.23}$$

and the payoff at this level of enforcement institution for a defector is:

$$\begin{aligned}\pi_i(\theta_i^D) = &\frac{-4\mu(2\gamma\lambda + c(\lambda+1) - \kappa(\lambda+1) + \lambda - 1)}{16(c+\gamma-\kappa-1) - 8\mu} \\ &+ \frac{16\lambda(c+\gamma-\kappa-1) + 4(-c+\kappa+1)^2 + (\lambda+1)^2\mu^2}{16(c+\gamma-\kappa-1) - 8\mu}\end{aligned}$$

Which for the median voter takes a value:

$$\frac{8\mu(3c + 2\gamma - 3\kappa - 1) - 16\left((c-\kappa)^2 + 2\gamma - 1\right) - 9\mu^2}{64(c+\gamma-\kappa-1) - 32\mu}$$

The difference of the median voter's payoffs at the preferred levels of θ conditional on action in the first stage is:

$$\pi_i(\theta_M^C) - \pi(\theta_M^D) = \frac{(2\gamma + 1)(\mu + 2)}{-8c - 8\gamma + 8\kappa + 4\mu + 8} \tag{3.24}$$

The numerator in (3.24) is positive, and as for the denominator, $-8c - 8\gamma + 8\kappa + 4\mu + 8 > 0 \Leftrightarrow \gamma + c - \kappa < 1 + \frac{\mu}{2}$. And this is the condition stated in (3.3). As we are interested only in region of parameters where there is unique equilibrium, $\pi_i(\theta_M^C) - \pi(\theta_M^D)$ will always be positive. The preferred level of enforcement θ for the median voter will thus be: $\frac{c+\kappa}{2} - \frac{\mu}{8}$. □

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