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Efficient Risk Sharing with Limited Commitment and Storage*

Árpád Ábrahám[†] and Sarolta Laczó[‡]

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Abstract

We extend the model of risk sharing with limited commitment (Kocherlakota, 1996) by introducing both a public and a private (non-contractible and/or non-observable) storage technology. Positive public storage relaxes future participation constraints and may hence improve risk sharing, contrary to the case where hidden income or effort is the deep friction. The characteristics of constrained-efficient allocations crucially depend on the storage technology's return. In the long run, if the return on storage is (i) moderately high, both assets and the consumption distribution may remain time-varying; (ii) sufficiently high, assets converge almost surely to a constant and the consumption distribution is time-invariant; (iii) equal to agents' discount rate, perfect risk sharing is self-enforcing. Agents never have an incentive to use their private storage technology, i.e., Euler inequalities are always satisfied, at the constrained-efficient allocation of our model, while this is not the case without optimal public asset accumulation.

Keywords: risk sharing, limited commitment, hidden storage, dynamic contracts

JEL codes: E20

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1 Introduction

The literature on incomplete markets either restricts asset trade exogenously, most prominently by allowing only a risk-free bond to be traded (Huggett, 1993; Aiyagari, 1994), or considers a deep friction which limits risk sharing endogenously. With private information as the friction, a few papers (Allen, 1985; Cole and Kocherlakota, 2001; Ábrahám, Koehne, and Pavoni, 2011) have integrated these two strands of literature by introducing a storage technology. This paper considers limited commitment (Thomas and Worrall, 1988; Kocherlakota, 1996), and makes a similar contribution by introducing both a public and a private storage technology.

The model of risk sharing with limited commitment has been applied in several economic contexts. In these applications, agents are likely to have a way to transfer resources intertemporally, both jointly and privately. However, typically neither public nor hidden storage is considered. For example, in the context of village economies (Ligon, Thomas, and Worrall, 2002), households may keep grain or cash around the house for self-insure purposes, and there also exist community grain storage facilities. Households in the United States (Krueger and Perri, 2006) may keep savings in cash or ‘hide’ their assets abroad. Spouses within a household (Mazzocco, 2007) accumulate both joint assets and savings for personal use. Countries (Kehoe and Perri, 2002) may also have joint savings (in a stability fund, such as the European Stability Mechanism, for example) in addition to their individual asset balances. Consequently, the theoretical insights we derive in this paper can be useful to the further development of these applications.

Our starting point is the two-sided lack of commitment framework of Kocherlakota (1996). Risk-sharing transfers have to be such that each agent is at least as well off as in autarky at each time and state of the world. The storage technology we introduce allows the planner and the agents to transfer resources from one period to the next and earn a net return r , $-1 \leq r \leq 1/\beta - 1$, where β is agents’ subjective discount factor.¹

We first add only public storage. We assume that agents are excluded from the returns of the publicly accumulated assets, an endogenous Lucas tree, when they default, as in Krueger and Perri (2006).² This implies that the higher the level of public assets is, the lower the incentives for default are in this economy.

¹Note that with $r = -1$ we are back to the Kocherlakota (1996) setting.

²Publicly stored assets may be protected by community enforcement (guards for public grain storage facilities in villages), by contracts (divorce law for couples), or by international organizations (for countries). Alternatively, one may think of an outside financial intermediary implementing public storage, as in our decentralization, see Section 2.5. Note that Karaivanov and Townsend (2014) also assume the presence of a financial intermediary for Thai villages.

The characteristics of constrained-efficient allocations, such as long-run asset and consumption dynamics, crucially depend on the return on storage. First, we show that public storage is used in equilibrium as long as its return is sufficiently high and risk sharing is partial in the basic model. Further, if the return on storage is moderately high, assets remain stochastic and the consumption distribution varies over time in the long run. If the return on storage is sufficiently high, assets converge almost surely to a constant and the consumption distribution is time-invariant. Risk sharing remains partial as long as the storage technology is inefficient, i.e., $r < 1/\beta - 1$, and perfect risk sharing is self-enforcing in the long run if the return on storage is equal to agents' discount rate.

To understand how public storage matters, note that limited commitment makes markets endogenously incomplete, i.e., individual consumptions are volatile over time. This market incompleteness triggers precautionary saving/storage motives for the agents and the planner. At the same time, higher public assets reduce default incentives, thereby reducing consumption dispersion and in turn the precautionary motive for saving. Further, agents would like to front-load consumption as long as $\beta(1+r) < 1$. Optimal public asset accumulation is determined by these conflicting forces. If $\beta(1+r) = 1$, it is optimal for the planner to fully complete the market by storage in the long run. This is because the trade-off between imperfect insurance and an inefficient intertemporal technology is no longer present. Note that there is no aggregate income risk in our environment, hence storage would never be used in the first-best allocation. In other words, the aggregate asset dynamics in our model are due to incomplete markets generated by the limited-commitment friction.

The introduction of public storage has new qualitative implications for the dynamics of consumption predicted by the model when assets are stochastic in the long run. First, the amnesia property, i.e., that whenever an agent's participation constraint binds the consumption allocation depends only on his current income (Kocherlakota, 1996), does not hold. Second, the persistence property of the basic model, i.e., that for 'small' income changes consumption is constant, does not hold either. There is a common intuition behind these results: the past history of shocks affects current consumptions through aggregate assets. Data on household income and consumption support neither the amnesia, nor the strong persistence property of the basic model (see Broer, 2013, for an extensive analysis). Hence, these differences are steps in the right direction for the limited-commitment framework to explain consumption dynamics.

We also show that constrained-efficient allocations can be decentralized as competitive equilibria with endogenous borrowing constraints (Alvarez and Jermann, 2000) and a competitive financial intermediation sector which runs the storage technology (Ábrahám and

Cárceles-Poveda, 2006). In this environment, equilibrium asset prices take into account the externality of aggregate storage on default incentives. In this sense, our paper provides a joint theory of endogenous borrowing constraints and an endogenously growing (and shrinking) asset/Lucas tree in equilibrium.

We then consider hidden (non-contractible and/or non-observable) storage as well. In contrast to the basic model, agents no longer have an incentive to store at the constrained-efficient allocation in our model with public storage.³ In other words, with optimal public asset accumulation the social planner preempts the agents' storage incentives. This is true because the planner has more incentive to store than the agents. First, the planner stores for the agents, because she inherits their consumption smoothing preferences. Second, storage by the planner makes it easier to satisfy agents' participation constraints in the future. In other words, the planner internalizes the positive externality generated by accumulated assets on future risk sharing.

This result means that the characteristics of constrained-efficient allocations in a model with both public and private storage and a model with only public storage are the same. This result also implies that agents' Euler inequalities are always satisfied in our model with limited commitment and public storage. The Euler inequality cannot be rejected in micro data from developed economies, once labor supply decisions and demographics are appropriately accounted for (Attanasio, 1999). Therefore, we bring limited commitment models in line with this third observation about consumption dynamics as well.

In a private-information environment with full commitment, Cole and Kocherlakota (2001) show that public storage is never used and agents' private saving incentives are binding in equilibrium, eliminating any risk sharing opportunity beyond self-insurance.⁴ When the deep friction is limited commitment as opposed to private information, the results are very different: first, public storage is used in equilibrium, and second, private storage incentives do not bind. The main difference between the two environments is that in our environment more public storage helps to reduce the underlying limited commitment friction, while with private information public asset accumulation would make incentive provision for truthful revelation more costly.

Finally, we study what are the overall effects of access to storage on welfare. These crucially depend on the return on storage. The availability of storage increases the value of autarky, which reduces welfare, while accumulated public assets improve long-run welfare,

³Note that this result does not hinge on how agents' outside option is specified precisely: they may or may not be allowed to store in autarky, and they may or may not face additional punishments for defaulting.

⁴See also Allen (1985) and Ábrahám, Koehne, and Pavoni (2011).

both by decreasing consumption dispersion and increasing available resources. When the return on storage is sufficiently high, there are welfare gains in the long run, because the economy gets close to perfect risk sharing and aggregate consumption is higher than in the basic model. When the return on storage is lower, the negative effect of a better outside option dominates the positive effect of public assets on welfare. In the short run, public asset accumulation also has costs in terms of foregone consumption. To see whether access to storage improves welfare taking into account the transition from the moment storage becomes available, we propose an algorithm to solve the model numerically. For the parameterizations we have considered, the short-term welfare losses dominate the long-run gains for all returns on storage. However, given private storage, public asset accumulation always improves welfare.

A few papers have considered limited-commitment economies with an intertemporal technology, either pure storage or capital accumulation. In [Marcet and Marimon \(1992\)](#) and [Kehoe and Perri \(2002\)](#), the social planner allocates capital to the agents, which in turn increases their outside option. In contrast, in our model agents cannot expropriate the public assets upon default. In a risk sharing framework with limited commitment, [Ligon, Thomas, and Worrall \(2000\)](#) consider observable and contractible individual storage and no public storage. In their environment individual storage is used in equilibrium, in contrast to our framework. [Krueger and Perri \(2006\)](#) introduce public assets of a constant size and show that its presence increases the amount of risk sharing, as in our model. As opposed to our paper, they do not endogenize the size of the asset/Lucas tree. Finally, [Ábrahám and Cárceles-Poveda \(2006\)](#) introduce public capital and derive the recursive form of the problem, similar to ours, but then they focus on the decentralization of the constrained-optimal allocation as a competitive equilibrium. None of these papers, or any other in the limited-commitment literature, to our knowledge, considers hidden storage.

The rest of the paper is structured as follows. Section 2 introduces and characterizes our model with public storage. Section 3 shows that agents' private storage incentives are eliminated under optimal public asset accumulation. Section 4 presents some computed examples. Section 5 concludes with a summary and provides a broader perspective on our results.

2 The model with public storage

We consider an endowment economy with two types of agents, $i = \{1, 2\}$, each of unit measure, who are infinitely lived and risk averse. All agents are ex-ante identical in the sense

that they have the same preferences and are endowed with the same exogenous random endowment process. Agents in the same group are ex-post identical as well, meaning that their endowment realizations are the same at each time t .⁵ Let $u()$ denote the utility function. We assume that it is characterized by harmonic absolute risk aversion (HARA), i.e., $u'(c) = (a + c)^{-\sigma}$, where $a \geq 0$ and $\sigma > 0$.⁶ Note that HARA utility functions satisfy prudence, i.e., $u'''() > 0$. We further assume that inverse marginal, $1/u'()$, is convex, that is, $\sigma \geq 1$. Some of our results, in particular, those relating to the long run, hold under weaker assumptions. The common discount factor is denoted by β .

Let s_t denote the state of the world realized at time t and s^t the history of endowment realizations, that is, $s^t = (s_1, s_2, \dots, s_t)$. Given s_t , agent 1 has income $y(s_t)$, while agent 2 has income equal to $Y - y(s_t)$, where Y is the aggregate endowment. Note that there is no aggregate risk in the sense that the aggregate endowment is constant.⁷ However, the distribution of income varies over time. We further assume that income has a discrete support with J elements, that is, $y(s_t) \in \{y^1, \dots, y^j, \dots, y^J\}$ with $y^j < y^{j+1}$, and is independently and identically distributed (i.i.d.) over time, that is, $\Pr(y(s_t) = y^j) = \Pr(y^j) = \pi^j, \forall t$. The assumptions that there are two types of agents and no aggregate uncertainty impose some symmetry on both the income realizations and the probabilities. In particular, $y^j = Y - y^{J-j+1}$ and $\pi^j = \pi^{J-j+1}$. The i.i.d. assumption can be relaxed, we only need weak positive dependence, i.e., that expected future income is weakly increasing in current income.

Suppose that risk sharing is limited by two-sided lack of commitment to risk sharing contracts, i.e., insurance transfers have to be voluntary, or, self-enforcing, as in [Thomas and Worrall \(1988\)](#), [Kocherlakota \(1996\)](#), and others. Each agent may decide at any time and state to default and revert to autarky. This means that only those risk-sharing contracts are sustainable which provide a lifetime utility at least as great as autarky after any history of endowment realizations for each agent. We assume that the punishment for deviation is exclusion from risk-sharing arrangements in the future. This is the most severe subgame-perfect punishment in this context. In other words, it is an optimal penal code in the sense of [Abreu \(1988\)](#) ([Kocherlakota, 1996](#)). Note that so far our setting is identical to that of [Kocherlakota \(1996\)](#).

We introduce a storage technology, which makes it possible to transfer resources from today to tomorrow. Assets stored earn a net return r , with $-1 \leq r \leq 1/\beta - 1$. Note that if

⁵We will refer to agent 1 and agent 2 below. Equivalently, we could say type-1 and type-2 agents, or agents belonging to group 1 and group 2.

⁶Note that relative risk aversion is constant for $a = 0$, and we get exponential utility with $a = 1$ and σ approaching infinity.

⁷In [Section 5](#) we discuss the case with aggregate risk.

$r = -1$ we are back to the basic limited commitment model of [Kocherlakota \(1996\)](#). In this section we only allow for public storage, to which defaulting agents do not have access (as in [Krueger and Perri, 2006](#)). In [Section 3](#) we also allow agents to store both in autarky and along the equilibrium path in a hidden way.

The constrained-efficient risk-sharing contract is the solution to the following optimization problem:

$$\max_{c_i(s^t)} \sum_{i=1}^2 \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u(c_i(s^t)), \quad (1)$$

where λ_i is the (initial) Pareto-weight of agent i , $\Pr(s^t)$ is the probability of history s^t occurring, and $c_i(s^t)$ is the consumption of agent i at time t when history s^t has occurred; subject to the resource constraints,

$$\sum_{i=1}^2 c_i(s^t) \leq \sum_{i=1}^2 y_i(s_t) + (1+r)B(s^{t-1}) - B(s^t), \quad B(s^t) \geq 0, \quad \forall s^t, \quad (2)$$

where $B(s^t)$ denotes public storage when history s^t has occurred, with $B(s^0)$ given; and the participation constraints,

$$\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) \geq U_i^{au}(s_t), \quad \forall s^t, \forall i, \quad (3)$$

where $\Pr(s^r | s^t)$ is the conditional probability of history s^r occurring given that history s^t occurred up to time t , and $U_i^{au}(s_t)$ is the expected lifetime utility of agent i when in autarky if state s_t has occurred today. In mathematical terms,

$$U_1^{au}(s_t) = u(y(s_t)) + \frac{\beta}{1-\beta} \sum_{j=1}^J \pi^j u(y^j) \quad (4)$$

$$\text{and } U_2^{au}(s_t) = u(Y - y(s_t)) + \frac{\beta}{1-\beta} \sum_{j=1}^J \pi^j u(y^j).$$

The above definition of autarky assumes that agents cannot use the storage technology in autarky. Note, however, that the qualitative results remain the same under different outside options as long as the strict monotonicity of the autarky value in current income is maintained. For example, agents could save in autarky (as in [Krueger and Perri, 2006](#), and in [Section 3](#)), or they might endure additional punishments from the community for defaulting (as in [Ligon, Thomas, and Worrall, 2002](#)).

2.1 Characterization preliminaries

We focus on the characteristics of constrained-efficient allocations. Our characterization is based on the recursive-Lagrangian approach of [Marcet and Marimon \(2011\)](#). However, the same results can be obtained using the promised-utility approach ([Abreu, Pearce, and Stacchetti, 1990](#)).

Let $\beta^t \Pr(s^t) \mu_i(s^t)$ denote the Lagrange multiplier on the participation constraint, (3), and let $\beta^t \Pr(s^t) \gamma(s^t)$ be the Lagrange multiplier on the resource constraint, (2), when history s^t has occurred. The Lagrangian is

$$\begin{aligned} \mathcal{L} = & \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \left\{ \sum_{i=1}^2 \left[\lambda_i u(c_i(s^t)) \right. \right. \\ & \left. \left. + \mu_i(s^t) \left(\sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) - U_i^{au}(s_t) \right) \right] \right. \\ & \left. + \gamma(s^t) \left(\sum_{i=1}^2 (y_i(s_t) - c_i(s^t)) + (1+r)B(s^{t-1}) - B(s^t) \right) \right\}, \end{aligned}$$

with $B(s^t) \geq 0$. Note that our problem is convex, because the objective function is clearly concave, the resource constraint is linear, and it has been shown that the participation constraints define a convex set in models with limited commitment and capital accumulation as long as autarky utility does not depend on the capital stock, B here, see [Sigouin \(2003\)](#).⁸ Therefore, the first-order conditions we derive below are both necessary and sufficient, and the solution is unique. Note also that existence is guaranteed as well, because the constraint set is compact. This requires B to be bounded, which we establish below.

Using the ideas of [Marcet and Marimon \(2011\)](#), we can write the Lagrangian in the form

$$\begin{aligned} \mathcal{L} = & \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) \left\{ \sum_{i=1}^2 \left[M_i(s^t) u(c_i(s^t)) - \mu_i(s^t) U_i^{au}(s_t) \right] \right. \\ & \left. + \gamma(s^t) \left(\sum_{i=1}^2 (y_i(s_t) - c_i(s^t)) + (1+r)B(s^{t-1}) - B(s^t) \right) \right\}, \end{aligned}$$

where $M_i(s^t) = M_i(s^{t-1}) + \mu_i(s^t)$ and $M_i(s^0) = \lambda_i$. The first-order condition with respect to agent i 's consumption when history s^t has occurred is

$$\frac{\partial \mathcal{L}}{\partial c_i(s^t)} = M_i(s^t) u'(c_i(s^t)) - \gamma(s^t) = 0. \quad (5)$$

Combining such first-order conditions for agent 1 and agent 2, we have

$$x(s^t) \equiv \frac{M_1(s^t)}{M_2(s^t)} = \frac{u'(c_2(s^t))}{u'(c_1(s^t))}. \quad (6)$$

⁸See also [Thomas and Worrall \(1994\)](#) and [Ábrahám and Cárceles-Poveda \(2006\)](#).

Here $x(s^t)$ is the temporary Pareto weight of agent 1 relative to agent 2.⁹ Defining

$$v_i(s^t) = \frac{\mu_i(s^t)}{M_i(s^t)}$$

and using the definitions of $x(s^t)$ and $M_i(s^t)$, we can obtain the law of motion of x as

$$x(s^t) = x(s^{t-1}) \frac{1 - v_2(s^t)}{1 - v_1(s^t)}. \quad (7)$$

The planner's Euler inequality, i.e., the optimality condition for $B(s^t)$, is

$$\gamma(s^t) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) \gamma(s^{t+1}), \quad (8)$$

which, using (5), can also be written as

$$M_i(s^t) u'(c_i(s^t)) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) M_i(s^{t+1}) u'(c_i(s^{t+1})).$$

Then, using (6) and (7), the planner's Euler becomes

$$u'(c_i(s^t)) \geq \beta(1+r) \sum_{s^{t+1}} \Pr(s^{t+1}|s^t) \frac{u'(c_i(s^{t+1}))}{1 - v_i(s^{t+1})}, \quad (9)$$

where $0 \leq v_i(s^{t+1}) \leq 1, \forall s^{t+1}, \forall i$. Given the definition of $v_i(s^{t+1})$ and equation (7), it is easy to see that (8) represents exactly the same mathematical relationship for both agents.

Equation (9) determines the choice of public storage, $B(s^t)$. It is clear that, first, the higher the return on storage is, the more incentive the planner has to store. Second, whenever risk sharing is not perfect, that is, $c_i(s^{t+1})$ varies over s^{t+1} for a given s^t , the planner has a precautionary motive for storage, a typical motive for saving in models with (endogenously or exogenously) incomplete markets. Third, the new term compared to standard models is $1/(1 - v_i(s^{t+1})) \geq 1$. This term is strictly bigger than 1 for states when agent i 's participation constraint is binding. Hence, binding future participation constraints amplify the return on storage. This is the case because higher storage makes the participation constraints looser in the future by reducing the relative attractiveness of default. The planner internalizes this effect when choosing the level of public storage.

Next, we introduce some useful notation and show more precisely the recursive formulation of our problem. This recursive formulation is going to be the basis for both the analytical characterization and the numerical solution procedure. Let $\mathcal{C}()$ and y denote, respectively,

⁹To reinforce this interpretation, notice that if no participation constraint binds in history s^t for either agent, i.e., $\mu_1(s^t) = \mu_2(s^t) = 0$ for all subhistories $s^t \subseteq s^t$, then $x(s^t) = \lambda_1/\lambda_2$, the initial relative Pareto weight of agent 1.

the consumption function and current income of agent 1, $\mathcal{V}()$ his value function, and \mathcal{B}' the function determining public assets. The following system is recursive with $X = (y, B, x)$ as state variables:

$$x'(X) = \frac{u'(Y + (1+r)B - \mathcal{B}'(X) - \mathcal{C}(X))}{u'(\mathcal{C}(X))} \quad (10)$$

$$x'(X) = x \frac{1 - v_2(X)}{1 - v_1(X)} \quad (11)$$

$$u'(\mathcal{C}(X)) \geq \beta(1+r) \sum_{y'} \Pr(y') \frac{u'(\mathcal{C}(X'))}{1 - v_1(X')} \quad (12)$$

$$u(\mathcal{C}(X)) + \beta \sum_{y'} \Pr(y') \mathcal{V}(X') \geq U^{au}(y) \quad (13)$$

$$u(Y + (1+r)B - \mathcal{B}'(X) - \mathcal{C}(X)) + \beta \sum_{y'} \Pr(y') \mathcal{V}(Y - y', \mathcal{B}'(X), 1/x'(X)) \geq U^{au}(Y - y) \quad (14)$$

$$\mathcal{B}'(X) \geq 0. \quad (15)$$

The first equation, (10), where we have used the resource constraint to substitute for the consumption of agent 2, says that the ratio of marginal utilities between the two agents has to be equal to the current relative Pareto weight. Equation (11) is the law of motion of the co-state variable, x . Equation (12) is the social planner's Euler inequality, which we have derived above. Equations (13) and (14) are the participation constraints of agent 1 and agent 2, respectively. Finally, equation (15) makes sure that public storage is never negative.

Given the recursive formulation above, and noting that the outside option $U^{au}()$ is monotone in current income and takes a finite set of values, the solution can be characterized by a set of state-dependent intervals on the temporary Pareto weight. This is analogous to the basic model, where public storage is not considered (see [Ljungqvist and Sargent, 2012](#), for a textbook treatment). The key difference is that these optimal intervals on the relative Pareto weight depend not only on current endowment realizations but also on B . The following lemma will be useful for specifying the optimal state-dependent intervals, and hence for characterizing the dynamics of our model.

Lemma 1. $\mathcal{C}(\tilde{y}, B, \tilde{x}) = \mathcal{C}(\hat{y}, B, \hat{x})$, $\mathcal{B}'(\tilde{y}, B, \tilde{x}) = \mathcal{B}'(\hat{y}, B, \hat{x})$, and $\mathcal{V}(\tilde{y}, B, \tilde{x}) = \mathcal{V}(\hat{y}, B, \hat{x})$ for all $(\tilde{y}, \tilde{x}), (\hat{y}, \hat{x})$ such that $x'(\tilde{y}, B, \tilde{x}) = x'(\hat{y}, B, \hat{x})$. That is, for determining consumptions, public storage, and agents' expected lifetime utilities, the current relative Pareto weight, x' , is a sufficient statistic for the current income state, y^j , and last period's relative Pareto weight, x .

Proof. Once we know x' , equations (10) and (12), which do not depend on x , give consumption and public assets. Then, the left hand side of (13) gives lifetime utility. \square

Lemma 1 implies that we can express consumptions, public storage, and agents' lifetime utility in terms of accumulated assets and the current Pareto weight. That is, we can write consumption by agent 1, public assets, and the value function as $c(B, x')$, $B'(B, x')$, and $V(B, x')$, respectively.

The following conditions define the lower and upper bounds of the optimal intervals in state y^j as a function of B :

$$V(B, \underline{x}^j(B)) = U^{au}(y^j) \quad \text{and} \quad V\left(B, \frac{1}{\bar{x}^j(B)}\right) = U^{au}(Y - y^j). \quad (16)$$

Hence, given the inherited Pareto weight, x_{t-1} , and accumulated assets, B , the updating rule is

$$x_t = \begin{cases} \bar{x}^j(B) & \text{if } x_{t-1} > \bar{x}^j(B) \\ x_{t-1} & \text{if } x_{t-1} \in [\underline{x}^j(B), \bar{x}^j(B)] \\ \underline{x}^j(B) & \text{if } x_{t-1} < \underline{x}^j(B) \end{cases} . \quad (17)$$

The ratio of marginal utilities is kept constant whenever this does not violate the participation constraint of either agent. When the participation constraint binds for agent 1, the relative Pareto weight moves to the lower limit of the optimal interval, just making sure that this agent is indifferent between staying and defaulting. Similarly, when agent 2's participation constraint binds, the relative Pareto weight moves to the upper limit of the optimal interval. Thereby, it is guaranteed that, ex ante, as much risk sharing as possible is achieved while satisfying the participation constraints.

Note that, given that the value of autarky is strictly increasing in current income and the value function is strictly increasing in the current Pareto weight, $\bar{x}^j(B) > \bar{x}^{j-1}(B)$ and $\underline{x}^j(B) > \underline{x}^{j-1}(B)$ for all $J \geq j > 1$ and B . It is easy to see that, unless autarky is the only implementable allocation, we have that $\bar{x}^j(B) > \underline{x}^j(B)$ for some j .

Given the utility function, the income process, r , and B , the intervals for different states may or may not overlap depending on the discount factor, β . The higher β is, the wider these intervals are. By a standard folk theorem (Kimball, 1988), for β sufficiently high all intervals overlap, that is, $\bar{x}^1(B) \geq \underline{x}^J(B)$, hence perfect risk sharing is implementable at the given asset level. At the other extreme, when β is sufficiently low, agents stay in autarky.

As public assets are accumulated (or decumulated) these optimal intervals change. The intervals are wider when B is higher. This is easy to see from (16). Take the first equality. The right hand side is independent of B , and the value function on the left hand side is increasing in both its arguments (available resources and own relative Pareto weight), hence

as B increases $\underline{x}^j(B)$, the lower limit of the optimal interval in state y^j , must decrease. Similarly, from the second inequality in (16), $1/\bar{x}^j(B)$ must decrease, hence $\bar{x}^j(B)$, the upper limit, must increase. Moreover, $\bar{x}^j(B)$ is strictly increasing and $\underline{x}^j(B)$ is strictly decreasing in B for all j , as long as the length of the j -interval is not zero.

In order to better understand some key characteristics of the dynamics of this model, we now focus on the case where public storage is constant over time. Then, from the next section, we study in detail the joint dynamics of consumption dispersion and assets. However, as we show later, under some conditions the economy will converge (almost surely) to a constant level of public assets. Below we refer to some key properties introduced here. Note also that the basic model is a special case of this economy with $B' = B = 0$.

It is easy to see that consumption is monotone in the current relative Pareto weight in the constant assets case as in the basic model, because aggregate resources are constant at $Y + rB^*$, where B^* denotes the constant level of assets. Then, we can implicitly define the limits of the optimal consumption intervals as

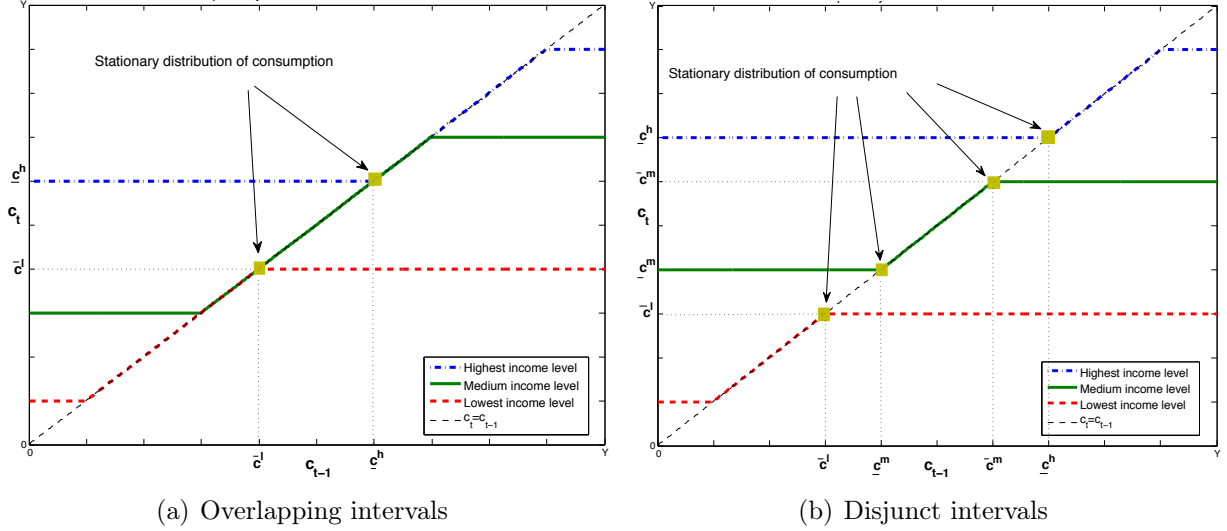
$$\bar{c}^j(B^*) : \bar{x}^j(B^*) = \frac{u'(Y + rB^* - \bar{c}^j(B^*))}{u'(\bar{c}^j(B^*))} \quad \text{and} \quad \underline{c}^j(B^*) : \underline{x}^j(B^*) = \frac{u'(Y + rB^* - \underline{c}^j(B^*))}{u'(\underline{c}^j(B^*))}.$$

We consider scenarios where the long-run equilibrium is characterized by imperfect risk sharing. That is, we assume from now on that $\bar{x}^1(B^*) < \underline{x}^J(B^*)$, or, equivalently, that $\bar{c}^1(B^*) < \underline{c}^J(B^*)$. We do this both because there is overwhelming evidence from several applications (households in a village or in the United States, spouses in a household, countries) about less-than-perfect risk sharing, and because the case of the (unconstrained-)efficient allocation of constant individual consumptions over time is theoretically not interesting. It is not difficult to see that for a constant B the law of motion described by (17) implies that, in the long run, risk sharing arrangements subject to limited commitment are characterized by a finite set of consumption values determined by the limits of the optimal consumption intervals. It turns out that considering two scenarios is enough to describe the general picture: (i) each agent's participation constraint is binding only when his income is highest, and (ii) each agent's participation constraint is binding in more than one state.¹⁰ Given this, to describe the constrained-efficient allocations in these two scenarios, it is sufficient to consider three income states. Hence, for all our graphical and numerical examples, we set $J = 3$.

Consider an endowment process where each agent gets y^h , y^m , or y^l units of the consumption good, with $y^h > y^m > y^l$, with probabilities π^h , π^m , and π^l , respectively. Symmetry implies that $y^m = (y^h + y^l)/2$ and $\pi^h = \pi^l = (1 - \pi^m)/2$.

¹⁰It will become clear below that assets can only be optimally constant in this case if they are zero.

Figure 1: Consumption dynamics in the long run with constant assets



Notes: In panel (a) the interval for state y^m overlaps with the intervals for state y^h and state y^l . In panel (b) all three state-dependent intervals are disjunct.

Given constant assets, the consumption intervals become wider if either β increases for a given B^* , as in the basic model, or B^* increases for a given β . Both changes make autarky less attractive. This is true in the former case because agents put higher weight on insurance in the future, and in the latter because agents are excluded from the benefits of more public assets upon default. If partial insurance occurs, there are two possible scenarios depending on the level of the discount factor and public assets. For higher levels of β and/or B^* , $\bar{c}^m(B^*) \geq \underline{c}^h(B^*) > \bar{c}^l(B^*) \geq \underline{c}^m(B^*)$. This means that the consumption interval for state y^m overlaps with both the interval associated with state y^h and the one associated with state y^l . This is the case where each agent's participation constraint binds for the highest income level only. Panel (a) in Figure 1 presents an example satisfying these conditions.

Suppose that the initial consumption level of agent 1 is below $\underline{c}^h(B^*)$. When agent 1 draws a high income realization (which occurs with probability 1 in the long run), his consumption jumps to $\underline{c}^h(B^*)$. Then it stays at that level until his income jumps to the lowest level. At that moment, agent 2's participation constraint binds, because he has high income, and consumption of agent 1 drops to $\bar{c}^l(B^*)$. Then we are back to where we started from. A very similar argument can be used whenever agent 1's initial consumption is above $\underline{c}^h(B^*)$. This implies that consumption takes only two values, $\underline{c}^h(B^*)$ and $\bar{c}^l(B^*)$, in the long run. When consumption changes, it always moves between these two levels, and the past history of income realizations does not matter. This is the *amnesia property* of the basic model

(Kocherlakota, 1996). When state y^m occurs after state y^h or state y^l , the consumption allocation remains unchanged. That is, consumption does not react at all to this ‘small’ change in income. This is the *persistence property* of the basic model. Note that consumption also remains unchanged over time if the sequence (h, m, h) or the sequence (l, m, l) takes place.

The key observation here is that, although individuals face consumption changes over time, the consumption distribution is time-invariant. In every period, half of the agents consume $\underline{c}^h(B^*)$ and the other half consume $\bar{c}^l(B^*)$. Finally, note that this happens for any J as long as $\bar{c}^2(B^*) \geq \underline{c}^J(B^*) > \bar{c}^1(B^*) \geq \underline{c}^{J-1}(B^*)$.

For lower levels of β and/or B^* , none of the three intervals overlap, i.e., $\underline{c}^h(B^*) > \bar{c}^m(B^*) > \underline{c}^m(B^*) > \bar{c}^l(B^*)$. Panel (b) in Figure 1 shows an example of this second case. When all three intervals are disjunct, consumption takes four values in the long run. Notice that the participation constraint of agent 1 may bind for both the medium and the high level of income. That is, whenever his income changes his consumption changes as well, and similarly for agent 2.

In this second case, in state y^m the past history determines which agent’s participation constraint binds, therefore consumption is Markovian. However, current incomes and the identity of the agent with a binding participation constraint fully determine the consumption allocation. The dynamics of consumption exhibit amnesia in this sense here. Further, consumption responds to every income change, hence the persistence property does not manifest itself.

The key observation for later reference is that the consumption distribution changes between $\{\underline{c}^m(B^*), \bar{c}^m(B^*)\}$ and $\{\bar{c}^l(B^*), \underline{c}^h(B^*)\}$. That is, the cross-sectional distribution of consumption is different when state y^m occurs from when an unequal income state, y^h or y^l , occurs. If there are $J > 3$ income states, the cross-sectional consumption distribution changes over time whenever $\bar{c}^2(B^*) < \underline{c}^J(B^*)$ and $\bar{c}^1(B^*) < \underline{c}^{J-1}(B^*)$.¹¹

2.2 The dynamics of public assets and the consumption distribution

In general, aggregate consumption varies as $(1+r)B - B'(x', B)$ varies over time, which depends on x' . Hence, an increase in the current relative Pareto weight, in principle, may imply a sufficiently large decrease in aggregate consumption so that agent 1’s consumption decreases, unlike in the case where assets and hence aggregate consumption are constant.

¹¹The number of income states and the number of states where a participation constraint binds determine the possible number of long-run consumption levels, and consequently the persistence property may appear. Note also that, as we show below, if public assets are allowed to vary, they will not stay constant when the consumption distribution is changing over time.

For now we state the intuitive property that c increases in x' as an assumption.

Assumption 1. *If $\tilde{x}' > \hat{x}'$ then $c(B, \tilde{x}') > c(B, \hat{x}')$, $\forall B$. That is, consumption by agent 1 is strictly increasing in his current relative Pareto weight.*

Next we continue our characterization under Assumption 1. In particular, we prove Claim 1 and Proposition 1 using this assumption. However, to prove Proposition 1, we only need Assumption 1 to hold in the case where r is such that assets are constant in the long run. Afterwards, using the results in Claim 1 and Proposition 1 and the uniqueness of the solution established above, we prove that the property stated in Assumption 1 holds when assets converge to a constant level in the long run, see Lemma 2. Further, we verify numerically that this assumption holds also in the case when public assets are stochastic in the long run.

We can describe the dynamics of the model with similar optimal intervals and updating rule on consumption as on the relative Pareto weight, as for the constant-asset case, under Assumption 1. Using (10) we can implicitly define the limits of the optimal intervals on consumption as

$$\begin{aligned} \bar{c}^j(B) : \bar{x}^j(B) &= \frac{u'(Y + (1+r)B - B'(\bar{x}^j(B), B) - \bar{c}^j(B))}{u'(\bar{c}^j(B))} \\ \text{and } \underline{c}^j(B) : \underline{x}^j(B) &= \frac{u'(Y + (1+r)B - B'(\underline{x}^j(B), B) - \underline{c}^j(B))}{u'(\underline{c}^j(B))}. \end{aligned} \quad (18)$$

Symmetry implies that $\bar{c}^j(B) = Y + (1+r)B - B'(\bar{x}^j(B), B) - \underline{c}^{J-j+1}(B)$.

The next proposition provides a key property of the aggregate storage decision rule and characterizes the short-run dynamics of assets. It shows how public storage varies with the consumption and income distribution.

Claim 1. *Under Assumption 1, $B'(B, x')$ is increasing in x' for $x' \geq 1$ and $B'(B, x') > 0$. That is, the higher cross-sectional consumption inequality is, the higher public asset accumulation is. Further, $\mathcal{B}'(y^j, B, x) \geq \mathcal{B}'(y^k, B, x)$, $\forall (B, x)$, where $j \geq J/2 + 1$, $k \geq J/2$, and $j > k$. That is, aggregate asset accumulation is increasing with cross-sectional income inequality.*

Proof. In Appendix A.

The intuition for Claim 1 is coming from two related observations. Higher inequality in the current period implies higher expected consumption inequality/risk next period. Under convex inverse marginal utility, the planner has a higher precautionary motive for saving whenever she faces more risk tomorrow.¹²

¹²Note that with log utility B' is weakly increasing in $x' \geq 1$, i.e., in cross-sectional consumption inequality, since $1/u'$ is linear in this case, while for CRRA utility functions with a coefficient of relative risk aversion strictly greater than 1, the empirically more plausible range, $1/u'$ is strictly convex, hence B' is strictly increasing in $x' \geq 1$.

We are now ready to characterize the long-run behavior of public assets and the consumption distribution.

Proposition 1. *Assume that β is such that agents obtain low risk sharing in the sense that the consumption distribution is time-varying without public storage.*

- (i) *There exists r_1 such that for all $r \in [-1, r_1]$, public storage is never used in the long run.*
- (ii) *There exists a strictly positive $r_2 > r_1$ such that for all $r \in (r_1, r_2)$, B remains stochastic but bounded, and the consumption distribution is time-varying in the long run.*
- (iii) *For all $r \in [r_2, 1/\beta - 1)$, B converges almost surely to a strictly positive constant, B^* , which is independent of the initial level of assets, and where the consumption distribution is time-invariant, but perfect risk sharing is not achieved.*
- (iv) *Whenever $r = 1/\beta - 1$, B converges almost surely to a strictly positive constant and perfect risk sharing is self-enforcing.*

If β is such that the consumption distribution is time-invariant without public storage, then $r_1 = r_2$, hence only (i), (iii), and (iv) occur.

Proof. In Appendix A.

The intuition behind Proposition 1 is that the social planner trades off two effects of increasing aggregate storage: it is costly as long as $\beta(1+r) < 1$, but less so the higher r is, and it is beneficial because it reduces consumption dispersion in the future. The level of public assets chosen just balances these two opposing forces. The relative strength of these two forces naturally depends on the return on storage, r . When the cross-sectional consumption distribution is time-varying (case (ii)), the relative strength of the two forces determining asset accumulation changes over time. This implies that assets cannot settle at a constant level in this case.

When the return on storage is sufficiently high (case (iii)), assets are accumulated so that participation constraints are only binding for agents with the highest income in the long run, and the consumption distribution becomes time-invariant. In this case, there is a unique constant level of assets, B^* , which exactly balances the trade-off between impatience and the risk-sharing gains of public storage. Decreasing public assets by a small amount would decrease future risk sharing more than the gain coming from the decrease in the inefficient transfer of resources to the future. Finally, in the limiting case of $\beta(1+r) = 1$ (case (iv)), the planner does not face a trade-off between improving risk sharing and using an inefficient

intertemporal technology, hence assets are accumulated until the level where full insurance is enforceable.

We now show that Assumption 1 holds, i.e., that consumption is monotone in the current relative Pareto weight, in the case where assets converge to a constant level in the long run. We first show that Claim 1 and Proposition 1 imply that the property stated in Assumption 1 holds. Then, given the uniqueness of the solution, this implies that the solution is characterized by the property stated in Assumption 1 along the transition as well when r is such that assets are constant in the long run.

Lemma 2. *If $\tilde{x}' > \hat{x}'$ then $c(B, \tilde{x}') > c(B, \hat{x}')$, $\forall B$, as long as assets converge to a constant level in the long run. That is, consumption by agent 1 is strictly increasing in his current relative Pareto weight.*

Proof. In Appendix A.

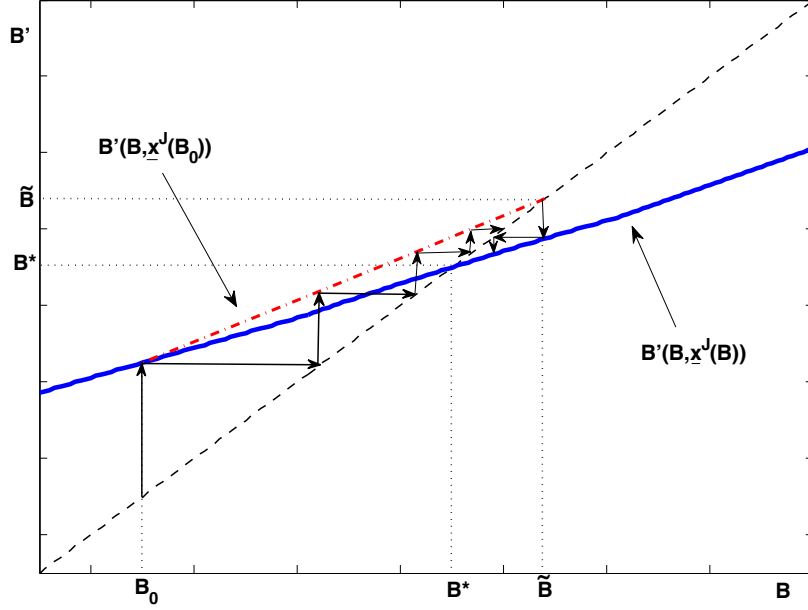
The intuition for Lemma 2 is the following. As a response to increasing inequality, it cannot be optimal to increase public storage so much that both agents have lower consumption. That would contradict the optimal intertemporal smoothing behavior of the planner.¹³

We illustrate the dynamics of assets in our model on two figures. First, Figure 2 shows the short-run dynamics of assets in the case where they converge to a constant level in the long run (case (iii) of Proposition 1). We assume that at B_0 the participation constraint binds only when an agent has the highest possible income. The solid line represents $B'(B, \underline{x}^J(B))$, i.e., we compute B' assuming that the relevant participation constraint is binding. It is easy to see from the figure that at $B = B^*$ assets remain constant in the long run, since $B' = B = B^*$.

Now, we explain how assets converge to B^* . Suppose that state y^J occurs when inherited assets are at the initial level $B_0 < B^*$. Then public storage is $B'(B_0, \underline{x}^J(B_0))$. Next period, if any state y^j with $j \geq 2$ occurs, no participation constraint is binding, hence, according to Claim 1, assets are $B'(B, \underline{x}^J(B_0)) > B'(B, \underline{x}^J(B))$, because given $B > B_0$ we have $\underline{x}^J(B) < \underline{x}^J(B_0)$. The dynamics of assets in states y^j with $j \geq 2$, i.e., when no participation

¹³Note that the analytical proof can be extended to some values of r which imply that assets are stochastic in the long run, in particular, values which are close to the threshold above which assets are constant in the long run or to the threshold below which zero public storage is optimal. Consider an $r < r_2$ in a small neighborhood of r_2 from Proposition 1, the threshold above which assets are constant in the long run. Since c is strictly increasing in x' for r_2 , c must be at least weakly increasing in x' for r sufficiently close to r_2 by continuity. Then we know that in the previous period c is strictly increasing in x' . Now if the original B is part of the stationary distribution, then it will occur other times as well, so c must be strictly increasing in x' there too. Similarly, we can consider $r > r_1$ in a small neighborhood of r_1 from Proposition 1, the threshold below which zero public storage is optimal.

Figure 2: Short-run asset dynamics when assets are constant in the long run



constraint binds, is represented by the dot-dashed line. As long as state y^1 does not occur, assets are determined by this line and would eventually converge to the level $\tilde{B} > B^*$. However, state y^1 occurs almost surely before \tilde{B} is reached. If the level of assets when y^1 occurs is above B^* , then assets are determined by the solid line, and they have to decrease. If a participation constraint continues to bind, which happens in both state y^1 and state y^J , assets converge to B^* along the solid line. If no participation constraint binds, then assets decrease even more, according to Claim 1. This may result in the asset level dropping below B^* . Then the same dynamics start again but in a tighter neighborhood around B^* . This argument implies that, although almost-sure convergence is guaranteed, it does not happen in a monotone way generically.

Figure 3 illustrates both the short- and long-run dynamics of public assets in the case where they are stochastic in the long run. For simplicity, we consider three income states. We have generated the figure numerically and verified that Assumption 1 holds. This means that there are two types of states: two with high income and consumption inequality (states y^h and y^l) and one with low income and consumption inequality (state y^m). The solid line represents $B'(B, \underline{x}^h(B))$, i.e., storage in state y^h (or y^l) when the relevant participation constraint is binding. Similarly, the dot-dashed line represents $B'(B, \bar{x}^m(B))$, i.e., storage in state y^m when the relevant participation constraint is binding. Starting from B_0 , if state y^m occurs repeatedly, assets converge to the lower limit of their stationary distribution, denoted \underline{B} . The relevant participation constraint is always binding along this path, because inherited

assets keep decreasing.

The dashed line represents the scenario where state y^h (or state y^l) occurs when inherited assets are at the lower limit of the stationary distribution, \underline{B} , and then the same state occurs repeatedly. This is when assets approach the upper limit of their stationary distribution, denoted \bar{B} . The relevant participation constraint is not binding from the period after the switch to y^h , therefore storage given inherited assets is described by the function $B'(B, \underline{x}^h(\underline{B}))$.

Under Assumption 1 we can also characterize analytically the limits of the long-run stationary distribution of assets as follows.

Claim 2. *Under Assumption 1, the lower limit of the stationary distribution of public assets, \underline{B} , is either strictly positive and is implicitly given by*

$$u'(\bar{c}^m(\underline{B})) = \beta(1+r) \sum_{j=1}^J \pi^j \frac{u'(\mathcal{C}(y^j, \underline{B}, \bar{x}^m(\underline{B})))}{1 - v_1(y^j, \underline{B}, \bar{x}^m(\underline{B}))}, \quad (19)$$

or is zero and (19) holds as strict inequality. The upper limit of the stationary distribution of public assets, \bar{B} , is implicitly given by

$$u'(\mathcal{C}(y^J, \bar{B}, \underline{x}^J(\underline{B}))) = \beta(1+r) \sum_{j=1}^J \pi^j u'(\mathcal{C}(y^j, \bar{B}, \underline{x}^J(\underline{B}))). \quad (20)$$

Proof. In Appendix A.

Finally, assume, without loss of generality, that state y^l occurred many times while approaching \bar{B} , and suppose that state y^h occurs when inherited assets are (close to) \bar{B} . In this case, $x' = \underline{x}^h(\bar{B}) < \underline{x}^h(\underline{B})$, and assets decrease. They then converge to a level \tilde{B} from above with the relevant participation constraint binding along this path. The same happens whenever $B > \tilde{B}$ when we switch to state y^h (or y^l). \tilde{B} is implicitly given by

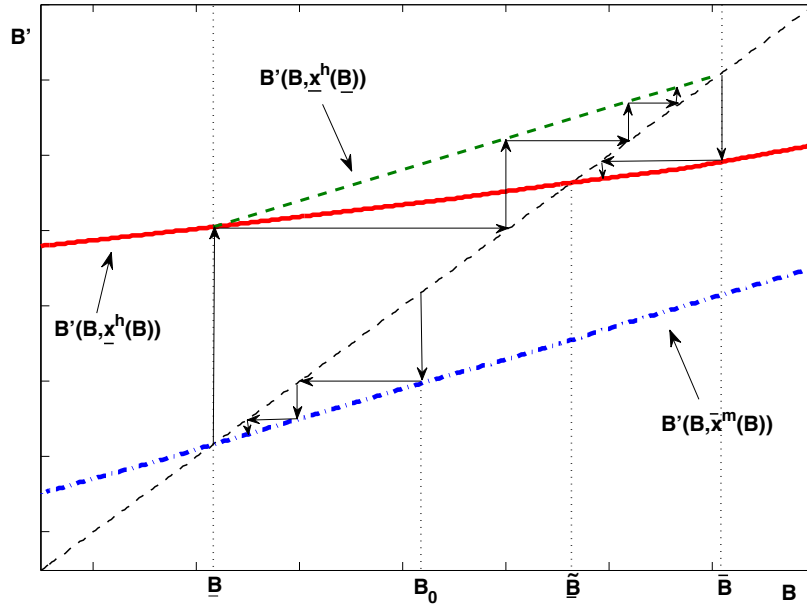
$$u'(\underline{c}^h(\tilde{B})) = \beta(1+r) \sum_{j=\{l,m,h\}} \pi^j u'(\mathcal{C}(y^j, \tilde{B}, \underline{x}^h(\tilde{B}))).$$

Note that as long as only state y^h and y^l occur, assets remain constant at \tilde{B} , similarly as in the previous figure. The key difference is that when the income distribution switches to the most equal one (y^m), a participation constraint binds, triggering a move in x toward 1, hence assets drop according to Claim 1.

2.3 The dynamics of individual consumptions

Having characterized assets, we now turn to the dynamics of consumption. One key property of the basic model is that whenever either agent's participation constraint binds ($v_1(X) > 0$

Figure 3: Asset dynamics when assets are stochastic in the long run



or $v_2(X) > 0$), the resulting allocation is independent of the preceding history. In our formulation, this implies that x' is only a function of y^j and the identity of the agent with a binding participation constraint. This is often called the amnesia property (Kocherlakota, 1996). See further our discussion in Section 2.1. Typically data do not support this pattern, see Broer (2013) for the United States and Kinnan (2012) for Thai villages. Allowing for storage helps to bring the model closer to the data in this respect.

Proposition 2. *The amnesia property does not hold when public assets are stochastic in the long run.*

Proof. x' and hence current consumption depend on both current income and inherited assets, B , when a participation constraint binds. This implies that the past history of income realizations affects current consumptions through B . \square

Another property of the basic model is that whenever neither participation constraint binds ($v_1(X) = v_2(X) = 0$), the consumption allocation is constant and hence exhibits an extreme form of persistence. This can be seen easily: (11) gives $x' = x$, and the consumption allocation is only a function of x' with constant aggregate income. This implies that for ‘small’ income changes which do not trigger a participation constraint to bind, we do not see any change in individual consumptions. It is again not easy to find evidence for this pattern in the data, see Broer (2013). In our model, even if the relative Pareto weight does not change, (10) does not imply that individual consumptions will be the same next

period as in the current period. This is because $(1+r)B - \mathcal{B}'(X)$ is generically not equal to $(1+r)B' - \mathcal{B}''(X')$ when assets are stochastic in the long run.¹⁴

Proposition 3. *The persistence property does not hold generically when public assets are stochastic in the long run.*

Proof. Even though $x' = x$, when neither participation constraint binds, consumption is only constant if net savings are identical in the past and the current period. This is generically not the case when B is stochastic. \square

The last two propositions imply that in our model the dynamics of consumption are richer and closer to the data than in the basic model in a qualitative sense. We leave the study of the quantitative implications of storage on consumption dynamics to future work.

2.4 Welfare

It is clear that access to public storage cannot reduce welfare, because zero assets can always be chosen. Along the same lines, if public storage is strictly positive for at least the most unequal income state, then welfare strictly improves. Proposition 1 implies that this is the case whenever the basic model does not display perfect risk sharing and the return on storage is higher than $r_1 < 1/\beta - 1$.

2.5 Decentralization

Ábrahám and Cárceles-Poveda (2006) show how to decentralize a limited-commitment economy with capital accumulation and production. That economy is similar to the current one in one important aspect: agents are excluded from receiving capital income after default. They introduce competitive intermediaries and show that a decentralization with endogenous debt constraints which are ‘not too tight’ (which make the agents just indifferent between participating and defaulting), as in Alvarez and Jermann (2000), is possible.¹⁵ However, Ábrahám and Cárceles-Poveda (2006) use a neoclassical production function where wages depend on aggregate capital. This implies that there the value of autarky depends on aggregate capital as well.¹⁶ They show that if the intermediaries are subject to endogenously determined

¹⁴The only exceptions are asset levels \underline{B} , \tilde{B} , and \bar{B} in Figure 3 with the appropriate income states occurring. However, the probability that assets settle at these points in the stationary distribution is zero.

¹⁵Note that this holds as long as the implied interest rate is ‘high,’ see Alvarez and Jermann (2000).

¹⁶This is also the case in the two-country production economy of Kehoe and Perri (2004).

capital accumulation constraints, then this externality can be taken into account, and the constrained-efficient allocation can be decentralized as a competitive equilibrium.¹⁷

Public storage can be thought of as a form of capital, B units of which produce $Y + (1 + r)B$ units of output tomorrow and which fully depreciates. Hence, the results above directly imply that a competitive equilibrium corresponding to the constrained-efficient allocation exists. In particular, households trade Arrow securities subject to endogenous borrowing constraints which prevent default, and the intermediaries also sell these Arrow securities to build up public storage. The key intuition is that equilibrium Arrow security prices take into account binding future participation constraints, as these prices are given by the usual pricing kernel. Moreover, agents do not hold any ‘shares’ in public storage, hence their autarky value is not affected. Finally, no arbitrage or perfect competition guarantees that the intermediaries make zero profits in equilibrium. As opposed to [Ábrahám and Cárceles-Poveda \(2006\)](#), capital accumulation constraints are not necessary, because in our model public storage does not affect agents’ outside option.

3 The model with both public and private storage

So far we have assumed that storage is available to the social planner, but agents can use it neither in autarky nor while in the risk sharing arrangement. In this section, we allow agents to use the same storage technology as the social planner. Access to storage both affects agents’ autarky value and enlarges the set of possible actions and deviations. In practice, allowing for private storage requires adding agents’ Euler inequalities as constraints to the problem given by the objective function (1) and the constraints (2) and (3), and modifying the participation constraints, (3).

The social planner’s problem becomes

$$\max_{\{c_i(s^t), B(s^t)\}} \sum_{i=1}^2 \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u(c_i(s^t)) \quad (21)$$

$$\text{s.t.} \quad \sum_{i=1}^2 c_i(s^t) \leq \sum_{i=1}^2 y_i(s_t) + (1 + r)B(s^{t-1}) - B(s^t), \quad B(s^t) \geq 0, \quad \forall s^t, \quad (22)$$

$$(P1) \quad \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) \geq \tilde{U}_i^{au}(s_t), \quad \forall s^t, \forall i, \quad (23)$$

$$u'(c_i(s^t)) \geq \beta(1 + r) \sum_{s^{t+1}} \Pr(s^{t+1} | s^t) u'(c_i(s^{t+1})), \quad \forall s^t, \forall i. \quad (24)$$

¹⁷[Chien and Lee \(2010\)](#) achieve the same objective by taxing capital instead of using a capital accumulation constraint.

The objective function, the resource constraint, and the non-negativity of storage restriction remain the same as before. The participation constraints, (23), change slightly, since $\tilde{U}_i^{au}(s_t)$ (to be defined precisely below) is the value function of autarky when storage is allowed. Agents' Euler inequalities, equation (24), guarantee that agents have no incentive to deviate from the proposed allocation by storing privately. Note that we implicitly assume that private storage is zero at the initial period.

A few remarks are in order about this structure before we turn to the characterization of constrained-efficient allocations. First, agents can store in autarky, but they lose access to the benefits of the public asset.¹⁸ This implies that $\tilde{U}_i^{au}(y^j) = V_i^{au}(y^j, 0)$, where $V_i^{au}(y^j, b)$ is defined as

$$V_i^{au}(y^j, b) = \max_{b'} \left\{ u(y^j + (1+r)b - b') + \beta \sum_{k=1}^J \pi^k V_i^{au}(y^k, b') \right\}, \quad (25)$$

where b denotes private savings. Since $V_i^{au}(y^j, 0)$ is increasing (decreasing) in j for agent 1 (2), it is obvious that if we replace the autarky value in the model of Section 2 (or in the basic model) with the one defined here, the same characterization holds. Note that, unlike in [Bulow and Rogoff \(1989\)](#), state-contingent assets are not available in autarky.¹⁹

Second, we use a version of the first-order-condition approach (FOCA) here. That is, these constraints only cover a subset of possible deviations. In particular, we verify that the agent is better off staying in the risk arrangement rather than defaulting and possibly storing (constraint (23), see also (25)), and that he has no incentive to store given that he does not ever default (constraint (24), agents' consumption-saving optimality condition). It is not obvious whether these constraints are sufficient to guarantee incentive compatibility, because multiple and multi-period deviations are not considered by these constraints.²⁰ In particular, an agent can store in the current period to increase his value of autarky in future periods and default in a later period. For now, we assume that these deviations are not profitable given the contract which solves Problem *P1*. We characterize the solution under this assumption. Then, in [Appendix C](#) we provide a numerical verification algorithm to show that agents indeed have no incentive to use these more complex deviations, unless the return on the storage technology is in a small neighborhood of the efficient level.

¹⁸This is the same assumption as in [Krueger and Perri \(2006\)](#), where agents lose access to the benefits of a tree after defaulting. In our model the tree is endogenous.

¹⁹[Bulow and Rogoff \(1989\)](#) find that access to state-contingent "cash-in-advance contracts" in autarky prevents risk sharing in equilibrium. However, "[t]his conclusion does depend on a sovereign's ability to reproduce any risk-sharing advantages of loan contracts by holding a portfolio of foreign assets" (p. 49).

²⁰In fact, [Kocherlakota \(2004\)](#) shows that in an economy with private information and hidden storage the first-order-condition approach can be invalid.

Third, both the participation constraints, (23), and the Euler constraints, (24), involve future decision variables. Given these two types of forward-looking constraints, a recursive formulation using either the promised-utility approach (Abreu, Pearce, and Stacchetti, 1990) or the recursive-Lagrangian approach (Marcet and Marimon, 2011) is difficult. Euler constraints have been dealt with using the agent's marginal utility as a co-state variable in models with moral hazard and hidden storage, see Werning (2001) and Ábrahám and Pavoni (2008). In our environment, this could raise serious tractability issues, since we would need two more continuous co-state variables, in addition to the co-state variable which makes the participation constraints recursive.

In this paper, we follow a different approach that avoids these complications. In particular, we show that the solution of a simplified problem where agents' Euler inequalities are ignored satisfies those Euler constraints. That is, instead of Problem *P1*, we consider the following simpler problem:

$$\begin{aligned}
 (P2) \quad & \max_{\{c_i(s^t), B(s^t)\}} \sum_{i=1}^2 \lambda_i \sum_{t=1}^{\infty} \sum_{s^t} \beta^t \Pr(s^t) u(c_i(s^t)) \\
 & \text{s.t.} \quad \sum_{i=1}^2 c_i(s^t) \leq \sum_{i=1}^2 y_i(s_t) + (1+r)B(s^{t-1}) - B(s^t), \quad B(s^t) \geq 0, \quad \forall s^t, \\
 & \quad \sum_{r=t}^{\infty} \sum_{s^r} \beta^{r-t} \Pr(s^r | s^t) u(c_i(s^r)) \geq \tilde{U}_i^{au}(s_t), \quad \forall s^t, \forall i.
 \end{aligned}$$

This is the problem we studied in Section 2, the only difference being that the autarky value is different. Now, we are ready to state the main result of this section.

Proposition 4. *The solution of the model with hidden storage, *P1*, corresponds to the solution of the simplified problem, *P2*.*

Proof. We prove this proposition by showing that the allocation which solves *P2* satisfies agents' Euler inequalities (24), the only additional constraints. Note that the planner's Euler, (9), is a necessary condition for optimality for *P2*. It is clear that the right hand side of (9) is bigger than the right hand side of (24), for $i = \{1, 2\}$, since $0 \leq v_i(s^{t+1}) \leq 1, \forall s^{t+1}$. Therefore, (9) implies (24). \square

This result implies that the characteristics of the constrained-efficient allocations of Problem *P1* are the same as those of Problem *P2*, which is the problem we studied in Section 2. Proposition 4 also means that private storage does not matter as long as public asset accumulation is optimal. We have to emphasize, however, that the result that no private storage occurs hinges on the assumption of optimal public asset accumulation with the same return.

The intuition behind this result is that the planner has more incentive to store than the agents. She stores for the agents, because she inherits their consumption smoothing preferences. Thereby she can eliminate the agents' incentive to store in a hidden way. Further, comparing (9) and (24) again, it is obvious that the planner has more incentive to store than the agents in all but the most unequal states. In particular, the presence of $1/(1 - v_i(s^{t+1})) > 1$ in the planner's Euler indicates how public asset accumulation helps the planner to relax future participation constraints, and thereby improve risk sharing, or, make markets more complete. In other words, the planner internalizes the positive externality of public asset accumulation on future risk sharing.

Next, we relate the case with both private and public storage to the case with private storage in autarky but without public storage. The following result follows from Proposition 4.

Corollary 1. *The planner stores in equilibrium whenever an agent's Euler inequality is violated at the constrained-efficient allocation of the basic model with no public storage and private storage only in autarky.*

Corollary 1 says that whenever agents have private storage incentives in the basic model, public storage is used in equilibrium. However, this result is only interesting if private storage matters, i.e., agents' Euler inequalities are violated, in the basic model under general conditions. This is what we establish next.

3.1 Does hidden storage matter in the basic model?

In this section, we identify conditions under which agents would store at the constrained-efficient solution of the basic model without public storage. We assume that partial insurance occurs at the solution, because otherwise it is trivial that private storage is never used. If agents' Euler inequalities are violated, the solution is not robust to deviations when private storage is available. Further, Corollary 1 implies that public storage is going to be strictly positive, at least under some histories, whenever this technology is available.

We first consider the benchmark case where agents have access to an efficient intertemporal technology, i.e., storage earns a return r such that $\beta(1 + r) = 1$. Afterwards, we study the general case. We only examine whether agents would use the available hidden intertemporal technology at the constrained-efficient allocation of the basic model. We do not make any assumption about the number of income states, except that income may take a finite number of values and the support of the income distribution is bounded.

Lemma 3. *Suppose that partial insurance occurs and the hidden storage technology yields a return r such that $\beta(1+r) = 1$. Then agents' Euler inequalities are violated at the constrained-efficient allocation of the basic model.*

Proof. We show that the Euler inequality is violated at the constrained-efficient allocation at least when an agent receives the highest possible income, y^j , hence his participation constraint is binding. By the characterization in Section 2.1, it is clear that for all future income levels his consumption will be no greater than his current consumption, i.e., $\mathcal{C}(y^j, 0, \underline{x}^J(0)) \leq \underline{c}^J(0)$. If partial insurance occurs, then it must be that there exists some state y^k where the agent consumes $\mathcal{C}(y^k, 0, \underline{x}^J) < \underline{c}^J(0)$. Then,

$$u'(\underline{c}^J(0)) < \sum_{y^j} \pi^j u'(\mathcal{C}(y^j, 0, \underline{x}^J(0))),$$

that is, the Euler inequality is violated. \square

It is obvious that if the return on storage is low, the constrained-efficient allocation of the basic model satisfies agents' Euler inequalities. The following proposition shows that for all economies with partial insurance one can find a threshold return on storage above which agents' storage incentives bind in the basic model.

Proposition 5. *There exists $\tilde{r} < 1/\beta - 1$ such that for all $r > \tilde{r}$ agents' Euler inequalities are violated at the constrained-efficient allocation of the basic model.*

Proof. \tilde{r} is defined as the solution to

$$u'(\underline{c}^J(0)) = \beta(1 + \tilde{r}) \sum_{y^j} \Pr(y^j) u'(\mathcal{C}(y^j, 0, \underline{x}^J(0))). \quad (26)$$

For \tilde{r} close to -1 , the right hand side is close to zero. For $\tilde{r} = 1/\beta - 1$, the right hand side is greater than the left hand side by Lemma 3. It is obvious that the right hand side is continuous and increasing in \tilde{r} . Therefore, there is a unique \tilde{r} that solves equation (26), and agents' Euler inequalities are violated for higher values of r . \square

The intuition behind this result is that whenever partial insurance occurs, the agent enjoying high consumption in the current period faces a weakly decreasing consumption path. Therefore, if a storage technology with sufficiently high return is available, the agent uses it for self-insurance purposes. We can also show that the threshold \tilde{r} in Proposition 5 can be negative. In particular, we have shown that agents would use a storage technology with $r = 0$ under non-restrictive conditions. A necessary condition is that the consumption distribution is time-varying in the long run. The proofs of these results are available upon request.

3.2 The dynamics of individual consumptions revisited

We have shown in Section 2.3 that, introducing public storage, we overturn two counterfactual properties of consumption dynamics in the basic model, the amnesia and persistence properties. We can improve on the basic model with respect to a third aspect of the dynamics of consumption. In particular, the Euler inequality cannot be rejected in household survey data from developed economies, once household demographics and labor supply are appropriately accounted for (see [Attanasio, 1999](#), for a comprehensive review of the literature). Since in our model with public storage agents' Euler inequalities are satisfied, while they are violated in the basic model, we bring limited commitment models in line with this third observation as well.

Would other extensions of the basic model yield the same three improvements? Two components are necessary: (i) allowing for storage to make sure that agents' Euler inequalities are satisfied, and (ii) an endogenous aggregate variable which makes aggregate consumption vary without aggregate income changing, for which the natural candidate is aggregate saving. Hence, ours is the simplest extension to the basic limited commitment framework which delivers all three properties.

3.3 Welfare revisited

In Section 2.4 we have argued that access to public storage unambiguously reduces consumption dispersion and improves welfare. It is clear that hidden storage counteracts these benefits of storage, because it increases the value of agents' outside option, which in itself increases consumption dispersion and reduces welfare. The overall effects of access to both public and private storage are hence ambiguous in general, and depend on the return to storage, r . We first compare welfare at the long-run stationary distribution of our model with both public and private storage and the basic model without storage. Afterwards, we discuss the effects of the transition from the moment when storage becomes available.

In the following proposition we compare consumption dispersion and (equal-weighted) social welfare in the long-run steady state in two economies. In the first economy neither public nor private storage is available, in the second one both are available. We assume that some risk sharing occurs both with and without access to storage for all r .

Proposition 6.

- (i) *There exists \tilde{r}_1 such that for all $r \in [-1, \tilde{r}_1]$ storage is not used even in autarky, therefore access to storage leaves consumption dispersion unchanged and is welfare neutral.*

- (ii) *There exists $\tilde{r}_2 > \tilde{r}_1$ such that for all $r \in (\tilde{r}_1, \tilde{r}_2]$ storage is used in autarky but not in equilibrium, therefore consumption dispersion increases and welfare deteriorates as a result of access to storage, and strictly so as long as perfect risk sharing is not self-enforcing.*
- (iii) *There exists $\tilde{r}_3 > \tilde{r}_2$ such that for all $r \in (\tilde{r}_2, \tilde{r}_3)$ public storage is (at least sometimes) strictly positive, but access to storage is still welfare reducing and consumption dispersion is higher than in the basic model without storage. Access to storage is welfare neutral in the long run at the threshold $r = \tilde{r}_3$.*
- (iv) *There exists $\tilde{r}_4 > \tilde{r}_3$ such that for all $r \in (\tilde{r}_3, \tilde{r}_4)$ access to storage is welfare improving in the long run, but consumption dispersion is still higher than in the basic model. Consumption dispersion is the same at the threshold $r = \tilde{r}_4$.*
- (v) *For all $r \in (\tilde{r}_4, 1/\beta - 1]$ access to storage is welfare improving in the long run, and consumption dispersion is lower than in the basic model.*

Proof. In Appendix A.

Even when welfare improves in the long run, accumulating public assets has short-run costs in terms of reduced aggregate consumption. This implies that the total gains (losses) from gaining access to storage are lower (higher) than those we have considered in Proposition 6. However, it is not clear whether access to both private and public storage improves welfare. For this reason, we explore this issue using numerical examples in the next section.

4 Computed examples

In this section we solve for the constrained-efficient allocation in economies with limited commitment and access to public and private storage. As in Section 3, agents are allowed to store in autarky. We describe the algorithm we have applied in more detail in Appendix B. We show that aggregate storage can be significant in magnitude. We also illustrate how risk sharing, welfare, and the dynamics of consumption are affected by the availability of storage with different returns $-1 \leq r < 1/\beta - 1$.²¹

We assume that agents' per-period utility function is of the CRRA form with a coefficient of relative risk aversion equal to 1, i.e., $u(\cdot) = \ln(\cdot)$. We assume that the income of both agents is i.i.d. over time and may take three values, with equal probabilities. We normalize aggregate income to 1, hence the middle income realization $y^m = 0.5$. We choose the high

²¹For $r = 1/\beta - 1$, the first-order-condition approach may be invalid, see Appendix C, hence we focus on the case of an inefficient storage technology.

and low income values to match the coefficient of variation of households' income in village economies. In particular, we use data from three Indian villages collected by the International Crop Research Institute for the Semi-Arid Tropics (ICRISAT).²² We find that the median coefficient of variation of households' income is 0.294. Hence, the income values are 0.353, 0.5, and 0.647.

We consider two discount factors, low ($\beta = 0.85$) and high ($\beta = 0.9$). In the former case risk sharing is partial without storage, however, the consumption distribution is time-invariant (i.e., the participation constraint of each agent binds only for the highest income level). In the latter case, perfect risk sharing occurs without access to storage. Note that this does not imply that public and private storage cannot be relevant as access to private storage increases the autarky values and may prevent full insurance with zero public assets. This triggers public asset accumulation if the return on storage is sufficiently high.

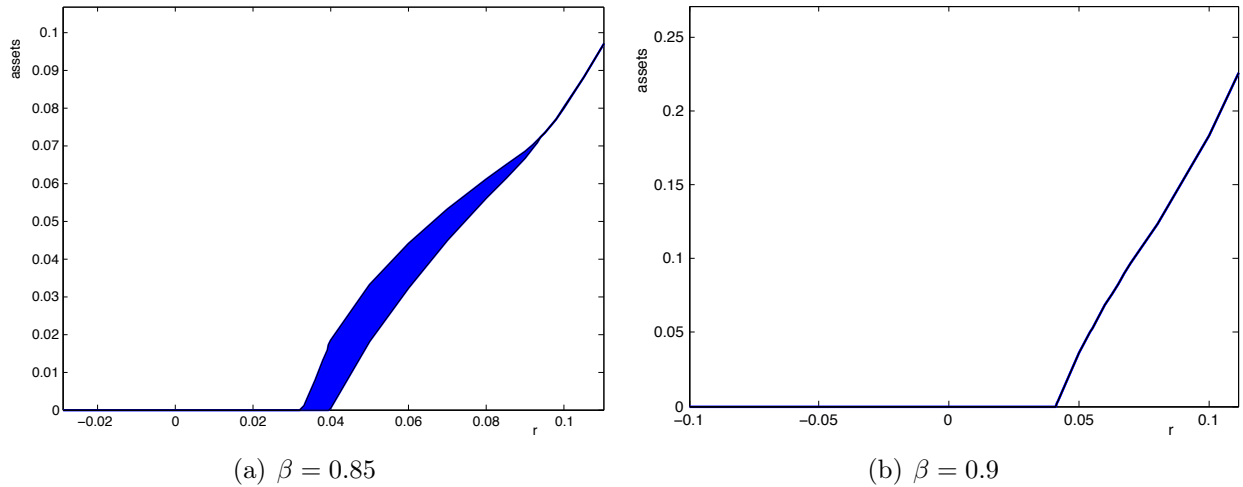
We present the simulation results on a few figures. First, let us look at the behavior of assets in the long run. Figure 4 shows the stationary distribution of assets, the first panel for $\beta = 0.85$ and the second for $\beta = 0.9$. Note the difference in scales in the two panels. Assets in the long run naturally increase with r . When the discount factor is high ($\beta = 0.9$), the participation constraints in state y^m do not bind in the long run, and assets always converge to a constant for any return on storage (case (iii) in Proposition 1). Public storage is strictly positive for $r > 0.041$. For example, with $r = 0.06$ the planner's savings amount to 6.85 percent of aggregate (non-asset) income, with $r = 0.08$ they are 12.37 percent, and with $r = 0.11$ they are at least 22.59 percent.

When $\beta = 0.85$, for intermediate values of r the participation constraints bind in all three states, and assets remain stochastic in the long run (case (ii) in Proposition 1). Public storage is (sometimes) strictly positive for $r > 0.033$. For example, with $r = 0.07$ public assets vary between 4.51 and 5.32 percent of aggregate (non-asset) income. When the interest rate is $r = 0.4$, assets vary between 0 and 1.85 percent. This last example shows that 0 can be part of the stationary distribution of assets when they are stochastic in the long run (see Claim 2). With $r = 0.176$ public storage reaches at least 22.03 percent of income (not shown on the figure).

Figure 5 shows the possible long-run consumption values. Together with Figure 4, this figure reflects the different cases described in Propositions 1 and 6. If $\beta = 0.85$ ($\beta = 0.9$) for returns below $\tilde{r}_1 = -0.143$ ($\tilde{r}_1 = -0.192$) storage does not even affect the value of autarky

²²It is safe to say that it is the most widely-used income-consumption survey from developing countries. In particular, the ICRISAT dataset has been used by many papers studying risk sharing in village economies, including Townsend (1994), Ligon (1998), Ogaki and Zhang (2001), Ligon et al. (2002), Mazzocco and Saini (2012), and Laczó (2014).

Figure 4: Assets in the long run



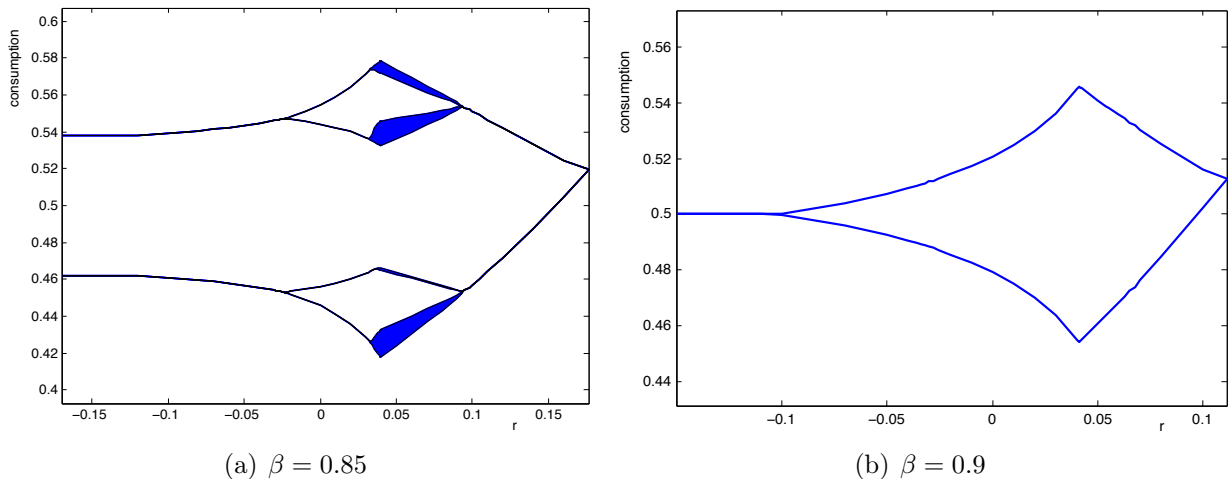
Notes: The stationary distribution of public assets. The aggregate endowment is 1 in each period. Note the difference in scales in the two panels.

and hence it is not used in equilibrium either. In this case, the allocation is not affected by the availability of storage. Given our parameterization, this implies that in the low patience case ($\beta = 0.85$) the consumption distribution has two values, while in the high patience case ($\beta = 0.9$) full risk sharing is enforceable. In fact, for $\beta = 0.9$, perfect risk sharing occurs in the long run for $r \leq -0.104$, since the participations constraints still do not bind in the range $-0.192 < r \leq -0.104$. As long as r is below $\tilde{r}_2 = 0.033$ ($\tilde{r}_2 = 0.041$) for β low (high), public storage is still not used, but storage increases the value of autarky, so consumption dispersion increases with the rate of return on storage.²³ For $r \geq \tilde{r}_2$, as r and aggregate asset accumulation increases, consumption dispersion declines.

One important difference between the two cases is that with the lower β at $r = -0.024$ the autarky values become such that a participation constraint binds in state y^m as well. For this reason, in Panel (a) of Figure 5, we see four consumption levels (as in Panel (b) of Figure 1) as long as public storage is not used. As the return reaches $r_1 = \tilde{r}_2 = 0.033$ public storage is used, and assets remain stochastic in the long run until $r_2 = 0.094$ (case (ii) in Proposition 1). This implies that in this case, even in the long run, consumptions not only depend on current income but also on the changing level of assets. Panel (a) of Figure 5 also shows that the stochasticity of assets has small effects on the levels and dispersion of consumption in this example. At $r = 0.094$ the participation constraints stop binding in state y^m , and hence the consumption distribution becomes time-invariant and assets converge to a constant level.

²³For $\beta = 0.9$, the autarky value is affected already for a lower storage return, however at these levels full insurance is still enforceable.

Figure 5: Consumption in the long run

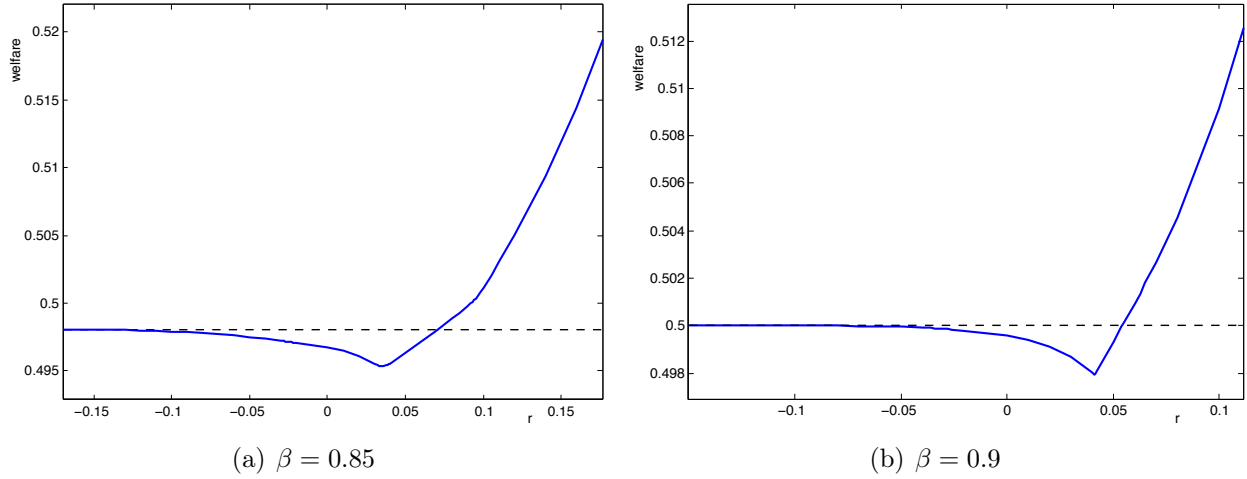


Notes: The stationary distribution of consumption. For $\beta = 0.9$, assets are never stochastic in the long run and consumption may take two values at most for all r . Note the difference in scales in the two panels.

Figure 6 shows long-run welfare expressed in per-period consumption equivalents. We have characterized long-run welfare in Proposition 6. When storage only increases the value of autarky, it decreases welfare. However, when the return is high enough so that it is used by the planner in equilibrium, it may increase welfare in the long run. When $\beta = 0.85$ the threshold return above which long-run welfare improves is $\tilde{r}_3 = 0.070$, when $\beta = 0.9$ it is 0.054. Note that at these thresholds, consumption dispersion is higher than in the case without storage, however aggregate consumption is also higher. As the return on storage approaches the efficient level, consumption dispersion approaches zero, hence welfare is always higher with than without storage in the long run. The welfare gain for r close to the discount rate is approximately equal to a 4.3 percent increase in consumption when $\beta = 0.85$, and to a 2.5 percent increase when $\beta = 0.9$.

Finally, we compute average welfare from the moment the storage technology becomes available. We do this to take into account the costs of asset accumulation. Figure 7 shows the results. In these two examples, access to both public and private storage lowers welfare for all r . The reason is that there are large welfare costs associated with the build-up of aggregate assets, and in our two examples these costs dominate the long-run gains. It is not clear how general this result is, and we leave this investigation to future work due to high computational costs. We know, however, that if perfect risk sharing is self-enforcing without private storage (as with $\beta = 0.9$), public storage is never positive even when it is available. This implies that when we allow for private storage, the feasible set shrinks, and hence welfare deteriorates. Panel (b) of Figure 7 confirms this. With $\beta = 0.85$ risk sharing is

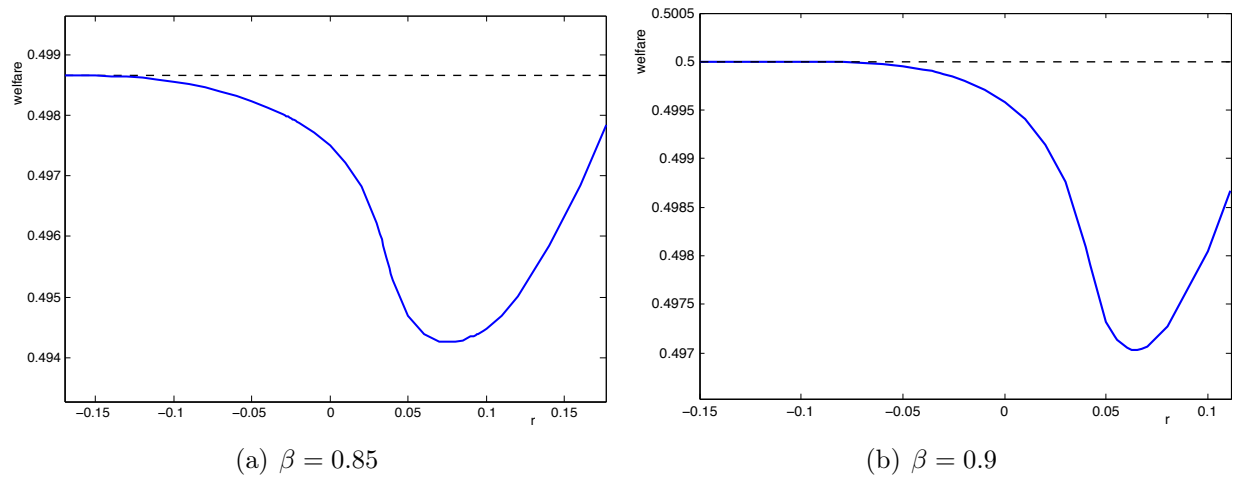
Figure 6: Welfare in the long run



Notes: The solid line shows long-run welfare per period in consumption-equivalent terms with both public and private storage. The dashed line shows long-run welfare per period in consumption-equivalent terms without storage for reference. Note the difference in scales in the two panels.

partial without private storage. Here, public storage would be used and would surely improve welfare if private storage were not allowed. However, private storage reduces risk sharing by improving the outside options of agents. Hence, the overall effect could go either way. We do not see these results as a case against improving storage technologies. If we take hidden private storage unavoidable, then our results indicate that public storage certainly improves welfare.

Figure 7: Welfare including transition



Notes: The solid line shows expected lifetime utility in per-period consumption-equivalent terms from the moment when (both public and private) storage becomes available. The dashed line shows expected lifetime utility in per-period consumption-equivalent terms without storage for reference. Note the difference in scales in the two panels.

5 Summary and discussion

This paper has shown that some implications of the basic risk sharing with limited commitment model with no private or public storage are not robust to hidden storage. When public storage is allowed though, the incentive for private storage is eliminated in the constrained-efficient allocation. The intertemporal technology is used in equilibrium even though the aggregate endowment is constant and the return is lower than the discount rate. Further, when income inequality is not the highest, the planner has more incentive to store than the agents. The reason for additional storage by the planner is that public assets relax future participation constraints and hence improve risk sharing.

The effects of the availability of both public and private storage on asset accumulation, consumption dispersion, and welfare depend on its return. In the long run, (i) for low r , access to storage is welfare neutral, because it is not used, hence we are back to the basic model of [Kocherlakota \(1996\)](#); (ii) for higher r , storage happens only in autarky, therefore, consumption dispersion increases and welfare decreases, but storage does not matter otherwise; (iii) for yet higher r , hidden storage matters in equilibrium in the basic model, public storage is (sometimes) strictly positive, stochastic, and depends positively on consumption inequality (as long as inverse marginal utility is convex), the consumption distribution is time-varying, and many consumption values occur;²⁴ (iv) for yet higher r , public storage becomes positive and constant in the long run, and only two consumption levels occur, i.e., the consumption distribution is time-invariant; (v) for $r = 1/\beta - 1$, public storage is positive and constant, and perfect risk sharing occurs. Long-run welfare improves above some threshold return, which is less than the discount rate. At the same time, there are short-run costs to accumulating assets. However, given access to private storage, public asset accumulation always reduces consumption dispersion and improves welfare.

The dynamics of individual consumptions are richer in our model compared to the basic model when assets are stochastic in the long run. In particular, the amnesia and persistence properties do not hold in general, which brings limited commitment models closer to the data ([Broer, 2013](#)). Further, in our model agents' Euler inequalities hold, which is consistent with empirical evidence from developed countries ([Attanasio, 1999](#)).

Comparing our model with limited commitment and storage to models with hidden income or effort and storage ([Allen, 1985](#); [Cole and Kocherlakota, 2001](#); [Ábrahám, Koehne, and Pavoni, 2011](#)) points to some similarities and remarkable differences. In both models, hidden storage reduces welfare by imposing tighter constraints on risk sharing. In private information

²⁴This third case only occurs for some set of parameter values.

models, public storage cannot mitigate this effect and hence is never used in equilibrium. In contrast, in our model public storage is used in equilibrium and welfare improves if its return is sufficiently high. This is because with limited commitment as the deep friction storage by the planner relaxes the incentive problem, by relaxing future participation constraints, while in the hidden income/effort context aggregate asset accumulation makes incentive provision for truthful revelation more expensive.

Throughout the analysis, we have restricted our attention to a model without aggregate risk. We have done this on purpose to isolate the pure effect of limited commitment on asset accumulation from other motives such as aggregate consumption smoothing. However, we expect our key results to hold with aggregate income uncertainty as well. Clearly, if the return on storage is high enough, public assets would fluctuate even at the first best. It is well known from the literature with exogenous incomplete markets (Huggett, 1993, Aiyagari, 1994), that assets are bounded as long as $\beta(1+r) < 1$.²⁵ This implies that when one combines aggregate income risk and limited commitment, assets will be stochastic and bounded in the long run.

The main mechanism of our paper would operate in a very similar way in the presence of aggregate risk. In fact, the key equation determining public asset accumulation, the planner's Euler equation, (9), would remain the same with the only difference that the histories would include the realizations of the aggregate shock as well. This also means that, as in our model without aggregate risk, the introduction of public storage relaxes the market friction, unlike in incomplete market models with asymmetric information (Cole and Kocherlakota, 2001). Further, if aggregate income is uncorrelated with cross-sectional income inequality, this implies that compared to the first best the constrained-efficient allocation would exhibit more asset accumulation, as it is not only helpful for aggregate consumption smoothing but also for reducing future consumption inequality, as in our model without aggregate risk. If higher aggregate income is correlated with lower cross-sectional income inequality, a potentially empirically relevant case, the two forces determining aggregate asset accumulation go in opposite directions, and we would expect smoother asset behavior than at the first best.

As both the private and public Euler equations remain virtually unchanged, introduction of aggregate risk would not affect another key result either: public storage preempts private storage in equilibrium. This result is particularly useful for solving the model numerically, which would therefore be without difficulty in the presence of aggregate risk in quantitative applications. In terms of welfare, as storage has an intrinsic value even without the limited-

²⁵It would not be the case under $\beta(1+r) = 1$, hence we would expect assets to diverge with limited commitment as well.

commitment friction in this case, we expect that, even though private storage alone reduces welfare, overall the positive effect of public storage could offset these welfare losses. The higher aggregate risk is, the more likely this is to be the case.

Our model could be applied in several economic contexts. The model predicts that risk sharing among households in villages can be improved by a public grain storage facility. Our model also provides a rationale for marriage contracts to specify that some commonly held assets are lost by the spouse who files for divorce. Finally, supranational organizations may help international risk sharing by simply having a jointly held stock of assets. The European Stability Mechanism may serve this purpose. Future work should study the quantitative implications of storage using some of these applications.

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Appendices

A Proofs

Proof of Claim 1. We consider three income states for expositional reasons. Generalizing the proof to more income states is straightforward. Assume indirectly that $B'(B, \tilde{x}') = B'(B, \hat{x}') \equiv B'$.²⁶ This assumption and (10) imply that $u'(c(B, \tilde{x}')) < u'(c(B, \hat{x}'))$.

First, consider \tilde{x}' and \hat{x}' such that $\min\{\underline{x}^h(B'), \bar{x}^m(B')\} \geq \tilde{x}' > \hat{x}' \geq 1$. Let us rewrite (12) as

$$1 \geq \beta(1+r) \sum_{y'} \Pr(y') \frac{u'(C(y', B', x'))}{u'(c(B, x'))(1-v_1(y', B', x'))}. \quad (27)$$

We now detail what happens next period, so that we can compare the right hand side of (27) for \tilde{x}' and \hat{x}' .

- If state y^h occurs, then the participation constraint of agent 1 is binding. Given that B' is the same for \tilde{x}' and \hat{x}' under our indirect assumption, x'' will equal $\underline{x}^h(B')$ and c' will equal $\underline{c}^h(B')$ for both. However, the ratio on the right hand side of (27) differs because $v_1(y', B', \tilde{x}') < v_1(y', B', \hat{x}')$. For $x' = \{\hat{x}', \tilde{x}'\}$ we obtain

$$\frac{u'(\underline{c}^h(B'))}{u'(c(B, x'))(1-v_1(y', B', x'))} = \frac{u'(\bar{c}^l(B'))}{u'(c_2(B, x'))}.$$

where we have combined (10) and (11).

- If state y^m occurs, then no participation constraint is binding, hence the relative Pareto weight does not change. For HARA utility functions, it can be shown using simple algebra that each agent's marginal utility grows at the rate $((2a+c'+c'_2)/(2a+c+c_2))^{-\sigma}$, hence we know that in this case

$$\frac{u'(c(B', \tilde{x}'))}{u'(c(B, \tilde{x}'))} = \frac{u'(c(B', \hat{x}'))}{u'(c(B, \hat{x}'))}.$$

- If state y^l occurs, then the participation constraint of agent 2 is binding. Given that B' is the same for \tilde{x}' and \hat{x}' , x'' will equal $\bar{x}^l(B')$ and c' will equal $\bar{c}^l(B')$ for both. Thus for $x' = \{\hat{x}', \tilde{x}'\}$, we have

$$\frac{u'(\bar{c}^l(B'))}{u'(c(B, x'))}.$$

²⁶If we assume indirectly that $B'(B, \tilde{x}') \leq B'(B, \hat{x}')$ for $\tilde{x}' > \hat{x}' \geq 1$ and $B'(B, \tilde{x}') \geq B'(B, \hat{x}')$ for $1 \geq \tilde{x}' > \hat{x}'$, the steps of the proof are the same, but the algebra is more tedious.

In summary, for $x' = \{\hat{x}', \tilde{x}'\}$ on the right hand side of (27) we have

$$\beta(1+r) \left[\pi^e \frac{u'(\bar{c}^l(B'))}{u'(c_2(B, x'))} + \pi^m \frac{u'(c(B', x'))}{u'(c(B, x'))} + \pi^e \frac{u'(\bar{c}^l(B'))}{u'(c(B, x'))} \right],$$

where $\pi^e = \pi^h = \pi^l$. If this expression is greater for \tilde{x}' than for \hat{x}' , then our indirect assumption is invalidated and B' has to be greater for \tilde{x}' than for \hat{x}' to satisfy (27). The second term is the same in the two expressions. Therefore, the sign of the difference is the sign of

$$\Delta_1(B, \tilde{x}', \hat{x}') \equiv \frac{1}{u'(c_2(B, \tilde{x}'))} + \frac{1}{u'(c(B, \tilde{x}'))} - \left(\frac{1}{u'(c_2(B, \hat{x}'))} + \frac{1}{u'(c(B, \hat{x}'))} \right).$$

Given that $\tilde{x}' > \hat{x}' \geq 1$ implies $c_2(B, \tilde{x}') < c_2(B, \hat{x}') \leq c(B, \hat{x}') < c(B, \tilde{x}')$ by Assumption 1, this difference is (strictly) positive if $1/u'$ is (strictly) convex. So under this condition, B' is (strictly) increasing in x' in the case where $\min\{\underline{x}^h(B'), \bar{x}^m(B')\} \geq x' \geq 1$.

Second, consider \tilde{x}' and \hat{x}' such that $\tilde{x}' > \hat{x}' \geq \bar{x}^m$.

- If state y^l occurs next period, nothing changes compared to the previous case, where $\min\{\underline{x}^h(B'), \bar{x}^m(B')\} \geq \tilde{x}' > \hat{x}' \geq 1$.
- For state y^m the difference between the ratio on the right hand side of (27) for \tilde{x}' and \hat{x}' is

$$\Delta_2(B, B', \tilde{x}', \hat{x}') \equiv \frac{u'(\bar{c}^m(B'))}{u'(c(B, \tilde{x}'))} - \frac{u'(\bar{c}^m(B'))}{u'(c(B, \hat{x}'))} > 0.$$

- In state y^h three cases are possible.
 - The participation constraint of agent 1 is binding for both \tilde{x}' and \hat{x}' . Then we can use the previous case. Note that the difference between the right hand side of (27) for \tilde{x}' and \hat{x}' is given by $\Delta_1(B, \tilde{x}', \hat{x}') + \Delta_2(B, B', \tilde{x}', \hat{x}') > 0$.
 - The participation constraint of agent 1 is not binding for either x' . Then the growth rate of marginal utility is the same for \tilde{x}' and \hat{x}' . In this case, the difference between the right hand side of (27) for \tilde{x}' and \hat{x}' is given by

$$\pi^e u'(\bar{c}^l(B')) \left(\frac{1}{u'(c(B, \tilde{x}'))} - \frac{1}{u'(c(B, \hat{x}'))} \right) + \Delta_2(B, B', \tilde{x}', \hat{x}') > 0.$$

- The participation constraint of agent 1 is binding for \hat{x}' , but not for \tilde{x}' . Then $c_2(B', \tilde{x}') < \bar{c}^l(B')$. Therefore, the difference between the right hand sides of (27) for \tilde{x}' and \hat{x}' is given by

$$\Delta_1(B, \tilde{x}', \hat{x}') + \Delta_2(B, B', \tilde{x}', \hat{x}') + \frac{u'(c_2(B', \tilde{x}')) - u'(\bar{c}^l(B'))}{u'(c_2(B, \tilde{x}'))} > 0.$$

Finally, consider \tilde{x}' and \hat{x}' such that $\tilde{x}' \geq \bar{x}^m(B') > \hat{x}'$. The only difference compared to the previous case is in state y^m . We have $c(B', \hat{x}') < \bar{c}^m(B')$. This implies that

$$\Delta_3(B, B', \tilde{x}', \hat{x}') = \frac{u'(\bar{c}^m(B'))}{u'(c(B, \tilde{x}'))} - \frac{u'(c(B', \hat{x}'))}{u'(c(B, \hat{x}'))} > 0.$$

Hence the same argument as in the previous case follows replacing $\Delta_2(B, B', \tilde{x}', \hat{x}')$ with $\Delta_3(B, B', \tilde{x}', \hat{x}')$.

Since the problem is symmetric, to establish the relationship between B' and $x' \leq 1$, we can consider $1/x' \geq 1$. This means that B' increases as $x' \leq 1$ decreases, i.e., as cross-sectional consumption inequality increases.

If $j > k$, and the optimal intervals for these two states do not overlap given B , then x' must be higher in state y^j than in state y^k , and we have already shown that assets depend positively on cross-sectional consumption inequality. If the optimal intervals overlap given B , then there exists x for which $x' = x$ in both states y^j and y^k . Aggregate savings are identical in the two states in this case. \square

Proof of Proposition 1. Part (i). It is easy to see that r_1 is implicitly defined by the planner's Euler, (12), with equality when agent 1 has the highest possible income. That is, r_1 is implicitly given by

$$u'(C(y^J, 0, \underline{x}^J(0))) = \beta(1 + r_1) \sum_j \pi^j \frac{u'(C(y^j, 0, \underline{x}^J(0)))}{1 - v_1(y^j, 0, \underline{x}^J(0))},$$

If $r > r_1$ public assets will be positive at least when income inequality is highest, while if $r \leq r_1$ public assets will be zero in the long run, and will always be zero if their initial level is zero.

Next, we show that assets are bounded, which we need for parts (ii)-(iv). They are trivially bounded below by 0. It is easy to see that there exists a high level of inherited assets, denoted \widehat{B} , such that perfect risk sharing is at least temporarily enforceable, that is, $\bar{x}^1(\widehat{B}) \geq \underline{x}^J(\widehat{B})$. Therefore, if $r < 1/\beta - 1$, $B'(B, x') < B$ for all $B \geq \widehat{B}$ and $\bar{x}^1(B) \geq x' \geq \underline{x}^J(B)$, i.e., assets optimally decrease; and assets stay constant if $r = 1/\beta - 1$. This implies that assets are bounded above.

We now turn to parts (ii) and (iii). We first show that if the consumption distribution is time-invariant, then there exists a unique constant level of assets, B^* , such that all the conditions of constrained-efficiency are satisfied. Afterwards, we show that assets converge almost surely to B^* starting from any initial level B_0 . Then, we establish that assets remain stochastic when the consumption distribution is time-varying (case (ii)). Finally, we show

that case (iii) occurs when the return on storage is high but less than the discount rate, while assets remain stochastic when the return is below some threshold, denoted r_2 .

Recall that if aggregate assets are constant, the optimal intervals for the relative Pareto weight are time-invariant. Given that each agent's participation constraint binds only for the highest income level in the long run, the optimality condition (10) and $\underline{x}^J(B^*)$ ($\bar{x}^1(B^*)$) uniquely determine $\underline{c}^J(B^*)$ ($\bar{c}^1(B^*)$), the time-invariant high (low) consumption level. Then, using the planner's Euler, we can determine the unique level of B^* such that all optimality conditions are satisfied. The planner's Euler is

$$u'(\underline{c}^J(B^*)) = \beta(1+r) [(1-\pi^e)u'(\underline{c}^J(B^*)) + \pi^e u'(\bar{c}^1(B^*))],$$

where $\pi^e = \pi^J = \pi^1$. Dividing both sides by $u'(\underline{c}^J(B^*))$, we obtain

$$\begin{aligned} 1 &= \beta(1+r) \left[(1-\pi^e) + \pi^e \frac{u'(\bar{c}^1(B^*))}{u'(\underline{c}^J(B^*))} \right] \\ &= \beta(1+r) [(1-\pi^e) + \pi^e \underline{x}^J(B^*)], \end{aligned} \tag{28}$$

where we have used (10). Note that $\underline{x}^J(B^*)$ is monotone and continuous in B^* . Further, at $B^* = 0$ the right hand side of equation (28) is larger than 1 by assumption, and at $B^* = \widehat{B}$ the right hand side of (28) is smaller than 1, because $\underline{x}^J(\widehat{B}) = 1$ and $B^* < \widehat{B}$. Therefore, we know that there exists a unique B^* where the planner's Euler holds with equality by setting $B' = B = B^*$.

Next, we show that assets converge almost surely to B^* starting from any initial level, B_0 . We already know that $B'(B_0, x') < B_0$ for the ergodic range of x' when $B_0 \geq \widehat{B}$, i.e., when perfect risk sharing is (temporarily) self-enforcing, and $B'(0, x') > 0$ for some x' in the ergodic range of x' , since $r > r_1$ by assumption. Consider $B^* < B_0 < \widehat{B}$ first, and assume that state y^J occurs and agent 1's participation constraint is binding. This is without loss of generality, because this happens with probability 1 in the long run, and the problem is symmetric across the two agents. We know that the right hand side of (28) is smaller than 1, because $\underline{x}^J(B_0) < \underline{x}^J(B^*)$. Therefore, marginal utility tomorrow has to increase relative to marginal utility today to satisfy the planner's Euler, therefore $B'(B_0, \underline{x}^J(B_0)) < B_0$. What happens next period? The participation constraint will bind again even if the same state occurs.²⁷ This is because $B'(B_0, \underline{x}^J(B_0)) < B_0$ implies $\underline{x}^J(B'(B_0, \underline{x}^J(B_0))) > \underline{x}^J(B_0)$. Then assets will decrease again. What if some state y^j with $2 \leq j \leq N-1$ occurs? We know that the participation constraints in these states are not binding for any $B \geq B^*$, because they are not binding for B^* . This means that now $x' = x = \underline{x}^J(B_0) < \underline{x}^J(B'(B_0, \underline{x}^J))$. Then, by

²⁷Note that this never happens in the basic model.

Claim 1, storage is lower than when the participation constraint is binding. Note that if states $\{y^2, \dots, y^{J-1}\}$ occur repeatedly, assets converge to a level below B^* . Then we are in the case where $B_0 < B^*$, which we now turn to.

Consider $0 \leq B_0 < B^*$, and suppose again that state y^J occurs and agent 1's participation constraint is binding. We know that $\underline{x}^J(B_0) > \underline{x}^J(B^*)$ in this case. Using (28) again, it follows that $B'(B_0, \underline{x}^J(B_0)) > B_0$. Now, if the same state occurs next period (in fact, any state y^j with $j \geq 2$), then the participation constraint is slack. This means that now $x' = x = \underline{x}^J(B_0) > \underline{x}^J(B'(B_0, \underline{x}^J(B_0)))$. Then, by Claim 1, storage is higher than when the participation constraint is binding. This also implies that if state y^1 does not occur for many periods, assets converge to a level above B^* . Then once y^1 occurs, which happens with probability 1 in the long run, we are back to the case $B_0 > B^*$, and assets start decreasing.²⁸

So far we have shown that when $B_0 < B^*$, assets increase at least in the most unequal states. Unless we are on a path where agents get the highest income shock exactly in turns, assets converge toward a level higher than B^* . We have also shown that whenever $B_0 > B^*$ and an agent's participation constraint binds, assets decrease. Again, unless one of the agents always receives the highest shock, assets converge to a value lower than B^* . This implies that assets oscillate around B^* . Almost-sure convergence is guaranteed because these oscillations shrink whenever a participation constraint binds in the increasing and/or decreasing part, which happens with probability one. To see this, note that from Claim 1 we know that $B'(B, \underline{x}^J(B_1))$ is highest when $\underline{x}^J(B_1)$ is highest. In turn, $\underline{x}^J(B_1)$ is highest when B_1 is lowest. That is, the economy might get close to the highest possible B during the transition if starting with zero public assets state y^1 (or y^J) keeps occurring. Similarly, once we are above B^* , the lowest possible level of B can be reached with a most equal state occurring infinitely many times, if that state starts occurring when assets are at their highest possible level. Note that the upper bound and then lower bound are reached with probability zero. Whenever there is a switch to y^1 or y^J , we get closer to B^* . Then the new possible highest asset level is lower and the lowest asset level is higher (and again the bounds are reached with probability zero). Then, again, the 'circle' shrinks when there is a switch to y^1 or y^J .

Part (ii). Consider the case where in the long run there is a third state in which a participation constraint binds. In this case, each agent's consumption takes at least four different values in the long run. These have to satisfy an additional participation constraint, an additional resource constraint, and an additional Euler, which is generically impossible

²⁸Participation constraints in more states may be binding when B is low, even if they only bind in states y^1 and y^J for B^* . However, with probability 1 assets will reach a level where the participation constraints of the other states are no longer binding.

for constant B .

Finally, we have to show that case (ii) occurs if $r_1 < r \leq r_2$, while case (iii) occurs if $r_2 < r < 1/\beta - 1$. It is easy to see that B^* is lower if r is lower, where B^* can be computed for any r ignoring the participation constraints of states y^j with $2 \leq j \leq N - 1$. However, as assets decrease, the optimal intervals become narrower, and eventually $\bar{c}^2(B) < \underline{c}^J(B)$ and $\bar{c}^1(B) < \underline{c}^{J-1}(B)$. Hence, r_2 is implicitly given by (28) such that B^* is such that $\bar{x}^2(B^*) = \underline{x}^J(B^*)$ (and $\bar{x}^1(B^*) = \underline{x}^{J-1}(B^*)$).

Part (iv). If risk sharing were imperfect in the next period, then it would be that $\underline{c}^J(B') > \bar{c}^1(B')$. Then, from the planner's Euler, (9), with $\beta(1+r) = 1$ we have that $\underline{c}^J(B') > \underline{c}^J(B)$, which implies that $B' > B$. That is, public assets are increasing. This means that as long as a participation constraint binds given B , the planner has an incentive to store more. Hence, in the long run B is constant as in case (iii), and risk sharing is perfect. \square

Proof of Lemma 2. We first show that if c' is weakly increasing in x'' next period, then c is strictly increasing in x' in the current period using Claim 1. Given $\tilde{x}' > \hat{x}'$, six cases are possible in terms of the pattern of binding participation constraints next period in a given income state. Depending on the number of income states, the width of the optimal intervals, and \tilde{x}' and \hat{x}' , not all these types of states necessarily exist.

- (i) The participation constraint of agent 1 is binding for both \tilde{x}' and \hat{x}' in state y' next period.²⁹ Let $\tilde{x}''(y') \equiv x''(y', B'(B, \tilde{x}'), \tilde{x}')$, and similarly for $\hat{x}''(y')$, $\tilde{x}''(Y - y')$, and $\hat{x}''(Y - y')$. Given $B'(B, \tilde{x}') > B'(B, \hat{x}')$, we know that $1 < \tilde{x}' < \tilde{x}''(y') = \underline{x}^{y'}(B'(B, \tilde{x}')) < \underline{x}^{y'}(B'(B, \hat{x}')) = \hat{x}''(y')$, which implies $y' > Y/2$. Then, $\tilde{x}' > \hat{x}'$ and (11) imply that

$$\frac{1}{1 - v_1(y', B'(B, \tilde{x}'), \tilde{x}')} < \frac{1}{1 - v_1(y', B'(B, \hat{x}'), \hat{x}')},$$

because x has to increase more from \hat{x}' to \hat{x}'' than from \tilde{x}' to \tilde{x}'' . Now, by symmetry, there is also a state $Y - y' < Y/2$ next period, which occurs with the same probability as state y' . We will show that the consumption allocation next period for this pair of states under current Pareto weight \tilde{x}' has a lower spread and a higher mean than the allocation under current Pareto weight \hat{x}' . For this we have to consider whether PCs bind in state $Y - y'$ next period.

²⁹Clearly, if \tilde{x}' and \hat{x}' are sufficiently high, there will be no such y' .

- First, assume that $\tilde{x}' > \bar{x}^{Y-y'} (B' (B, \tilde{x}'))$ and $\hat{x}' > \bar{x}^{Y-y'} (B' (B, \hat{x}'))$, i.e., the participation constraint of agent 2 is binding in state $Y - y'$ for both \tilde{x}' and \hat{x}' . Then, by symmetry, $\tilde{x}'' (Y - y') = 1/\tilde{x}'' (y') > 1/\hat{x}'' (y') = \hat{x}'' (Y - y')$.
- Second, assume that $\tilde{x}' \leq \bar{x}^{Y-y'} (B' (B, \tilde{x}'))$ and $\hat{x}' \leq \bar{x}^{Y-y'} (B' (B, \hat{x}'))$, i.e., no participation constraint is binding in state $Y - y'$ for either \tilde{x}' or \hat{x}' . Then, $\tilde{x}'' (Y - y') = \tilde{x}' > \hat{x}' = \hat{x}'' (Y - y')$.
- Third, assume that $\tilde{x}' > \bar{x}^{Y-y'} (B' (B, \tilde{x}'))$ and $\hat{x}' \leq \bar{x}^{Y-y'} (B' (B, \hat{x}'))$, i.e, the participation constraint of agent 2 is binding for \tilde{x}' but not for \hat{x}' . It follows that $\tilde{x}'' (Y - y') = \tilde{x}' > \bar{x}^{Y-y'} (B' (B, \tilde{x}')) > \bar{x}^{Y-y'} (B' (B, \hat{x}')) = \hat{x}'' (Y - y')$, where the second inequality holds because $B' (B, \tilde{x}') > B' (B, \hat{x}')$ and the optimal intervals are wider when inherited assets are greater.
- Fourth, assume that $\tilde{x}' \leq \bar{x}^{Y-y'} (B' (B, \tilde{x}'))$ and $\hat{x}' > \bar{x}^{Y-y'} (B' (B, \hat{x}'))$, i.e., the participation constraint of agent 2 is binding for \hat{x}' but not for \tilde{x}' . It follows that $\tilde{x}'' (Y - y') = \bar{x}^{Y-y'} (B' (B, \tilde{x}')) \geq \tilde{x}' > \hat{x}' = \hat{x}'' (Y - y')$.

In all four cases $\hat{x}'' (y') \geq \tilde{x}'' (y') > \tilde{x}'' (Y - y') \geq \hat{x}'' (Y - y')$, hence the consumption allocation given \tilde{x}' has a smaller spread across the states y' and $Y - y'$. It also has a higher mean, because of the higher level of inherited assets and a lower x'' , which implies less storage next period by Claim 1 as long as $\hat{x}'' (y') \geq \tilde{x}'' (y') \geq 1$, which must be the case here. As the mean decreases, expected marginal utility increases. What happens to expected marginal utility as a result of a higher spread? Under prudence, the marginal utility function is decreasing and convex, therefore, expected marginal utility is higher for the more risky process. Finally, the term $1/(1 - v_1())$ further increases the right hand side of (12) given \hat{x}' relative to \tilde{x}' , which implies that c is strictly increasing in x' even if c' is only weakly increasing in x'' .

- (ii) The participation constraint of agent 1 is binding for \hat{x}' but not for \tilde{x}' in state y' next period. In this case, either $\tilde{x}'' (y') \geq \hat{x}'' (y')$ or $\tilde{x}'' (y') < \hat{x}'' (y')$. If $\tilde{x}'' (y') \geq \hat{x}'' (y')$ consumption next period is higher for \tilde{x}' , because of a higher current Pareto weight and more resources than for \hat{x}' . This implies a lower marginal utility tomorrow for \tilde{x}' . In addition, once again the term $1/(1 - v_1())$ further increases the right hand side of (12) given \hat{x}' relative to \tilde{x}' . If $\tilde{x}'' (y') < \hat{x}'' (y')$, then we can use the same argument as in case (i). Since $\tilde{x}'' (y') = \tilde{x}' > \underline{x}^{y'} (B' (B, \tilde{x}'))$, expected marginal utility next period is yet lower given \tilde{x}' for this reason.
- (iii) No participation constraint is binding for \tilde{x}' or \hat{x}' next period. In this case, consumption next period is strictly higher for \tilde{x}' than for \hat{x}' because of a higher B' , so marginal utility

next period is strictly lower for \tilde{x}' than for \hat{x}' , and both $1/(1 - v_1(\cdot))$ s are 1.

- (iv)-(vi) The participation constraint of agent 2 is binding for \tilde{x}' , or for \hat{x}' , or for both next period. In these cases, we can use similar arguments as above to show that $\tilde{x}''(y') > \hat{x}''(y')$, and hence consumption next period is strictly higher for \tilde{x}' than for \hat{x}' .

In all six types of states (or pairs of states), the right hand side of (12) is strictly lower for \tilde{x}' than for \hat{x}' , therefore the left hand side must be strictly lower as well. This means that c must be strictly higher when x' is higher, given that c' depends positively on x'' .

Proposition 1 shows that assets converge to a constant level in the long run almost surely if r is higher than some threshold r_2 . That is, in the long run the characteristics of allocations are the same as in the basic model (while aggregate consumption is $Y + rB^*$ rather than Y), in particular, c strictly increases with x' . Then, moving backwards in time, c must strictly increase with x' in all periods.

Finally, we know that the solution is unique, therefore we can conclude that it is characterized by the consumption of agent 1 increasing in x' for all r such that assets are constant in the long run. \square

Proof of Claim 2. From Claim 1 it is clear that \underline{B} is approached if a least unequal income state, denoted y^m , happens repeatedly, while \bar{B} is approached with state y^J (or y^1) happening many times in a row.

If B is part of the stationary distribution, then it must be that $B \geq \underline{B}$. This means that there are less and less resources available over time while assets approach \underline{B} , hence the relevant participation constraint always binds along this path. The planner's Euler

$$u'(\bar{c}^m(\underline{B})) \geq \beta(1+r) [\pi^e u'(\mathcal{C}(y^l, \underline{B}, \bar{x}^m(\underline{B}))) + (1 - 2\pi^e) u'(\bar{c}^m(\underline{B})) + \pi^e u'(\mathcal{C}(y^h, \underline{B}, \bar{x}^m(\underline{B})))]$$

as equality defines \underline{B} if $\underline{B} > 0$. If at $\underline{B} = 0$ this Euler is satisfied as a strict inequality, then the lower bound is 0.

The upper limit of the stationary distribution, \bar{B} , is approached from below, hence, along that path, the highest shock (state y^J or y^1) happens repeatedly and no participation constraint binds. Let B_1 denote the level of inherited assets when we switch to state y^J (or y^1), and let \tilde{B} denote the level of assets to where B converges. Note that along this path the relative Pareto weight is constant at $\underline{x}^J(B_1)$. Given B_1 , \tilde{B} is the solution to the following system:

$$\frac{u'(\mathcal{C}_2(y^J, \tilde{B}, \underline{x}^J(B_1)))}{u'(\mathcal{C}(y^J, \tilde{B}, \underline{x}^J(B_1)))} = \underline{x}^J(B_1)$$

$$\begin{aligned}
& \mathcal{C}\left(y^J, \tilde{B}, \underline{x}^J(B_1)\right) + \mathcal{C}_2\left(y^J, \tilde{B}, \underline{x}^J(B_1)\right) = Y + r\tilde{B} \\
& u'\left(\mathcal{C}\left(y^J, \tilde{B}, \underline{x}^J(B_1)\right)\right) = \beta(1+r) \sum_{j=1}^J \pi^j u'\left(\mathcal{C}\left(y^j, \tilde{B}, \underline{x}^J(B_1)\right)\right). \tag{29}
\end{aligned}$$

We have to find B_1 such that \tilde{B} is equal to \bar{B} , the upper limit of the stationary distribution. Using Claim 1, we know that $B'(B, \underline{x}^J(B_1))$ is highest when $\underline{x}^J(B_1)$ is highest. In turn, $\underline{x}^J(B_1)$ is highest when B_1 is lowest, i.e., when B_1 is equal to the lower limit of the stationary distribution of assets, \underline{B} . Then, replacing $\underline{x}^J(B_1)$ with $\underline{x}^J(\underline{B})$ and \tilde{B} with \bar{B} in (29) gives (20). \square

Proof of Proposition 6. (i) It is easy to see that storage is never used when its return is close to -1, i.e., as long as it is below some threshold \tilde{r}_1 . (ii) It is similarly easy to see that storage in equilibrium implies storage in autarky. This follows from the fact that the planner's and the agents' saving incentives are the same when income inequality is highest, i.e., when the incentive to store is highest, and agents' Euler inequality is more stringent in autarky than in equilibrium with some risk sharing. Then, if storage only takes place in autarky, the only effect of storage is that the value of agents' outside option increases, which reduces risk sharing and welfare. However, the value of autarky does not matter as long as perfect risk sharing is self-enforcing, hence, as long as that is the case, access to storage is welfare neutral. (iii) As r further increases to above the threshold \tilde{r}_2 , according to Proposition 1 the planner finds public storage optimal. However, by continuity, at this point the negative effect of the increase in the value of autarky dominates the positive effect of the (small) stock of public assets on risk sharing. Therefore, welfare still goes down as a result of access to storage. (iv)-(v) If $r = 1/\beta - 1$, perfect risk sharing occurs and aggregate consumption is $Y + rB^*$ rather than Y , therefore welfare is strictly higher in the long run. Further, consumption dispersion is zero. Then, for any r in a small neighborhood of $1/\beta - 1$, the positive effect of the increase in aggregate consumption dominates the negative effect of the increase in the value of autarky, hence welfare improves. For such r , consumption dispersion is small. By continuity there exists $\tilde{r}_2 < \tilde{r}_3 < 1/\beta - 1$ where the two welfare levels are equalized. At this level of storage return, aggregate consumption has to be higher than in the basic model (at least after some histories). Hence, welfare can be the same only if consumption dispersion is higher than in the basic model. By continuity this should hold above \tilde{r}_3 as well until the threshold $\tilde{r}_4 \leq 1/\beta - 1$. \square

B Computation

We use the recursive system given by equations (10)-(15) to solve the model numerically. We discretize x and B (y is assumed to take a finite number of values). We have to determine x' and B' on a 3-dimensional grid on $X = (y, B, x)$. The initial values for $\mathcal{V}(X')$, $\mathcal{C}(X')$, and $v_1(X')$ are from the solution of a model where the participation constraints are ignored. We iterate until the value and policy functions converge.

As we proceed, we use the characteristics of the solution. In particular, we know that if agent 1's participation constraint binds at \tilde{x} , it also binds at all $x < \tilde{x}$. Similarly, if agent 2's participation constraint binds at \hat{x} , it also binds at all $x > \hat{x}$. At each iteration, at each income state and for each B , we solve directly for the limits \tilde{x} and \hat{x} using (13) and (14) with equality, respectively, first assuming that $B' = 0$. Afterwards, we check whether the planner's Euler is satisfied at the limits. If not, we solve a 2-equation system of (12) and (13) (or (14)), with unknowns B' and x' . Finally, we solve for a new B' at points on the x grid where neither participation constraint binds, i.e., at the interior of the optimal interval for (y, B) of the current iteration.

C Validity of the first-order-condition approach

In Section 3 we assumed that by introducing agents' participation constraints and Euler inequalities (equations (23) and (24), respectively) in Problem *P1* we guarantee incentive compatibility. In other words, we assumed that the constrained-optimal allocation can be obtained by checking that agents have no incentive to default given that they do not have private assets, and that they have no incentive to store given that they never default. In principle, they may still find it optimal to use more complicated 'double' deviations involving both storage and default, potentially in different time periods, given some history of income shocks.

First, note that we have already considered contemporaneous joint deviations, i.e., when the agent defaults and saves at the same time.³⁰ In the participation constraint (23) we use $\tilde{U}_i^{au}(s_t)$, the value of autarky when the agent can store (see equation (25)). Further, note that in autarky the agent is allowed to store whenever this makes him better off. Therefore, the 'default today and store later'-type of double deviations are already taken care of as well.

³⁰In the literature with private information, a similar joint deviation, shirking (or misreporting income) and saving, is the relevant deviation. Detailed discussion of these joint deviations can be found for the hidden income case in Cole and Kocherlakota (2001), and for the hidden action (dynamic moral hazard) case in Kocherlakota (2004) and Ábrahám, Koehne, and Pavoni (2011).

This implies that the only potentially profitable double deviations we still need to consider are those which involve private asset accumulation in the current period and default in a later period.

We can show analytically that as long as the constrained-efficient consumption values do not exceed the autarky consumption value when the agent holds no private assets, ‘default today and store later’-type of double deviations are never profitable for the agents.³¹ However, for high values of r , the highest possible consumption value in equilibrium is higher.³² In fact, [Nozawa \(2013\)](#) shows, in a one-sided limited commitment framework, that there exists a profitable deviation in the case where $\beta(1+r) = 1$, and this deviation involves saving one period and defaulting the next. Note that the deviation happens when the economy has reached its long-run equilibrium and perfect risk sharing occurs, and that the agent’s participation constraint holds with equality when he gets the highest possible income. In our model these conditions are satisfied only when public assets converge to their lowest possible long-run value given an efficient technology, a zero-probability event. Other than in a small neighborhood of this case, we provide a numerical algorithm to verify that ‘store first and default later’-type of double deviations are not welfare-improving.

In order to verify that agents have no incentive to use ‘store first and default later’-type of double deviations, we show numerically that given any level of hidden assets, public assets, incomes, and the inherited relative Pareto weight, agents are better off receiving as endowment the consumptions assigned by the constrained-efficient risk-sharing contract rather than their own incomes today and in the future. In order to see this, along with the autarky consumption-saving problem, we solve the consumption-saving problem of an agent who receives the constrained-efficient consumption process as ‘income.’ Having computed the constrained-efficient policy functions as described in [Appendix B](#), this is without conceptual difficulty, however, the computational cost is rather high, given that there are four state variables, three of which are continuous. We again exploit the characteristics of the solution, namely that the current Pareto weight takes values within an optimal state-dependent interval, in order to shorten computation time.

In examples we have studied, we find that agents are always better off receiving the constrained-efficient consumptions given any level of already accumulated private assets rather than the autarky incomes. Hence, they will never revert to autarky and will never store in a hidden fashion, as long as the first-order conditions are satisfied.

³¹The proof is available upon request.

³²In the long run this only happens for returns ‘close’ to the efficient level, but during the transition this may happen for returns below the threshold r_2 in [Proposition 1](#) as well. This is easy to verify in computed examples.