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MULTISECTORAL MODELS AND JOINT PRODUCTION

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§.1. Introduction*

In this paper, within the theme of linear economic models, we pay particular attention to joint production and a "modern" treatment of Sraffa's (1960) classic work: Production of Commodities by means of Commodities. We start with a few notes on some relevant mathematics. Consider the mapping $f(x,\lambda)$ where $f:\Delta\times Z \to Y$ and Δ is an open set in a Banach space. Z,Y are Banach spaces and f and its first Frechet derivatives are continuous. In the study of:

$$f(x,\lambda) = 0 \ , \quad \lambda \in \Delta \ , \quad x \in Z \ ; \tag{1.1}$$

a solution of (1.1) is given by $(x,\lambda) \in \Delta\times Z$ such that (1.1) is satisfied. Let $M \subset \Delta\times Z$ denote the set of solutions of (1.1) and $\forall\lambda \in \Delta$, put:

$$M_\lambda = \{x \in X;\ x,\lambda \in M\} \tag{1.2}$$

It has not been sufficiently well recognized that a study of the dependence of the set M_λ on λ can be a unifying analytical framework for important problems in economics, whether static or dynamic, whether within an optimizing system or not. The study of the structure of zeroes of (1.1) via (1.2) by interpreting Δ as a parameter set together with appropriate equivalence relations in open subsets of Z leads,

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*This paper is dedicated to Richard M. Goodwin in honour of his 70th birthday. Many of the tools and concepts we have employed were first introduced to economists by Richard Goodwin. In particular, the Perron-Frobenius theorem, now so familiar, was first brought to the attention of the economics community by him in his celebrated discussion with John Chipman in the pages of the Economic Journal.

as special cases, to classic results in comparative statics in economic analysis. On the other hand, when f is restricted to be a vector field with suitable equivalence relations, the analysis of the above equation(s) includes, as special cases, problems in economic dynamics. An example of the former is the nonlinear eigenvalue problem arising, naturally, in Joint Production Systems. Almost all nonlinear macrodynamic formulations in economics can be subsumed as examples of the latter.

Taking a particular realization of (1.1) and (1.2) as $Ax \varepsilon \lambda Bx$, we can approach the study of joint production and multisectoral models in economics--and their associated balanced growth paths, implied labour values and the resulting nonsubstitution results--as generalized eigenvalue problems and problems in the applications of so-called generalized theorems of the alternative. It will be clear then that the above membership relation when replaced by (in-)equality relations results in applications of Perron-Frobenius type theorems and theorems of the alternative, and hence can easily be interpreted in terms of well-known formulations of linear multisectoral models of single-product economic systems. In this paper we attempt to present a particular unifying analysis of classic economic problems encountered in the economic theories of growth, value and distribution in terms of mathematical theorems of the above type.

The mathematical formalizations of problems of joint production and multisectoral models in the economics of growth, value and distribution have been determined largely by two sets of issues. On the one hand the extent to which so-called intrinsic joint production, for example of the wool-mutton or iron-coke variety, has been emphasized, has determined, as in mainstream neoclassical theory (cf. Marshall, 1920; Edgeworth, 1881; Jevons, 1871; Samuelson, 1966; Kuga, 1973), the production function approach. The emphasis on the problems of intrinsic joint production, and hence its formalization in terms of production functions, <u>implied</u> accounting concepts sharply contrasting stock and flow dimensions (particularly in the case of durable capital goods). On the other hand, where the causality ran the other way round, that is from the needs of the <u>accounting and political arithmeticians</u>, it was natural that only flow concepts were emphasized--again, in the particular case of durable goods. The accountant and the political arithmetician had to solve a valuation problem determined by a (periodic) time interval that did not necessarily coincide with the physically determined length of, in particular, durable capital goods. It was therefore not possible to ignore the fact that there were classes of goods that appeared and re-appeared for several of the actuarial and fiscal time periods. The analytical device of considering durable (or fixed) capital goods lasting more than one time period as many different goods appeared as an almost natural solution to this valuation problem. The result was that flow concepts dominated the scheme. Sraffa (1960) and von Neumann (1937) (and their pre-neoclassical predecessors) developed this line of analysis to an outstanding degree of perfection in relation to theories of growth, value and distribution. A third approach, to some extent a hybrid of the above two, resulted from the so-called Austrian or Neo-Austrian models where accounting relations and intrinsic joint production jointly determined the formalization (cf. in particular Hicks, 1956, 1973) and, thus, stock and flow concepts were equally emphasized. The necessity of an analytical scheme, capable of handling the intricacies of durable capital goods and the complexities of joint production, in one unified format, led, almost naturally, to generalizations of Böhm-Bawerk's ideas of input and output flows characterized as distributions. The distributions, as such, characterized the flows and the moments defined measures of the stock concepts of the production process.

In addition to the economic rationale that led to the three different approaches we have outlined above, there are at least three other inter-related issues that have dominated the mathematical and economic formalization of problems in multisectoral and joint production models. Briefly, they relate to, in mathematical terms, problems of:

(a) homogeneous vs. inhomogeneous systems;

(b) mathematical relations in terms of inequalities as against in terms of equalities;

(c) the representations of linear operators as matrices with square vs. rectangular dimensions.

Economically, the above problems translate respectively to (a) so-called closed vs. open economic systems, (b) whether choice of technique is explicitly considered or not and, finally, (c) whether there is equality between the number of processes and of commodities. Though we treat both homogeneous and inhomogeneous systems, that is closed and open multisectoral models, we have not been as catholic with regard to cases (b) and (c). We have, in the case of (b) and (c), chosen to concentrate on representing economic relations in production in terms of equalities with the associated linear operator given by a square matrix. One major, mathematical, reason is the following: we have relied on a few simple mathematical tools, well known and extensively utilized in economics, to demonstrate most of the important economic propositions. More specifically, we have relied on the topological method of theorems of the alternative and the algebraic methods of Perron-Frobenius theorems. Thus, a unified methodological and conceptual presentation becomes feasible if the formalization of the economic models is in terms of equalities and square matrices. It seems to us that the study of the structure of zeroes of mappings have a generic feature which is increasingly evident in several branches of modelling.

There are other, strictly economic and accounting, reasons for choosing to represent relations in terms of inequalities and rectangular matrices. Indeed, a controversy over this issue has, not very long ago, even enlivened the pages of leading economic journals. It is not clear, however, whether the controversy resulted in any light being shed on the advantages of any one choice. At any rate, these issues are brought to the foreground immediately in any problem involving joint production. We ourselves are, therefore, compelled to discuss, however briefly, these problems as soon as we consider joint production.

In our analysis of production in multisectoral models of the economy we take as the prototype the second of the above three approaches for illustrating the applicability of mathematical methods in elucidating problems in the theory of economic growth, value and distribution. Thus, in section 2 after formalizing the concept of production in multisectoral models we go on to consider the special case of single product industries from the value and quantity aspects. In section 3 the general case of multiple product industries or joint production in multisectoral models is analysed. In section 4, first the concept of joint production is utilized to analyse the problem of the choice of the economic lifetime of durable capital goods; and, secondly, some discussion of the nonsubstitution theorem in a generalized form is presented. Finally, in a concluding section some directions and hints for explorations along the lines developed at the beginning of this introduction (and other directions) are discussed.

§.2. Linear Multisectoral Models Without Joint Production

2.1: Simple No Joint Production System (NJPS)

From a mathematical point of view, the NJPS is a special 'limiting' case in a class of models: a proper interpretation of its mathematical properties and their consequences for the analysis of growth and distribution, is given in section 3 below. Economically, it is interpreted as a model of industries. Each industry is represented by a triple (a^j, b^j, l_j), where a^j is a column vector of inputs of material goods, b^j is a column of outputs where only one entry is assumed to be non-zero and finally l_j is the labour input coefficient (the amount of labour required to produce one unit of the j-th good). There is therefore a strict symmetry between 'industries', identified with the production processes, and goods produced.

It is such a symmetry that is lost in the general joint production model (JPS), where a process (a^j, b^j, l_j) may produce a positive amount of more than one good. From a purely theoretical point of view, this latter framework therefore must be considered as more general.

In capital theory, the NJP model is interpreted as a model of circulating capital, in the sense that only capital goods lasting one period of production are allowed. More precisely, it is assumed that the economic and technological lifetimes of all capital goods are equal to one another and to a common length of time, say the normalized 'year' (see Sraffa (1960)). On the contrary the JP model is interpreted as the model where the presence of fixed capital goods is allowed for. The j-th capital good, installed in the k-th process of production, is characterized by a technological parameter T_k^j representing the maximal length of time, as a multiple of some basic period, for which the capital good may be used, and by a new economic variable t^j, the economic lifetime for which, taking into account the other economic variables, it is profitable to use it. The problem of choosing the optimal economic lifetime is one of choice of techniques, a technique now being defined as a mixture of feasible lifetimes for the set of capital goods. From this point of view, the NJP model is again to be seen as a simplified framework where no such choice is considered.

Let us start with Sraffa's (1960) example of 'an extremely simple society' producing just enough to maintain itself and where the necessary commodities are produced by separate industries:

$$240 \text{ gr of wheat} \oplus 12 \text{ t iron} \oplus 18 \text{ pigs} \rightarrow 450 \text{ gr wheat}$$
$$90 \text{ gr of wheat} \oplus 6 \text{ t iron} \oplus 12 \text{ pigs} \rightarrow 21 \text{ t iron}$$
$$120 \text{ gr of wheat} \oplus 3 \text{ t iron} \oplus 30 \text{ pigs} \rightarrow 60 \text{ pigs}$$

Let Q be the diagonal matrix of gross output, and let M be the matrix of flows on the left of the above relations. Then, the system state can be written as

$$M'i = Q\, i \tag{2.1}$$

where i is the unit sum vector, $i = (1,1,\ldots,1)'$, and M' denotes the transpose of M.

It is then supposed that these produced commodities are exchanged at the annual market which leads Sraffa to make the assertion that:

> There is a unique set of exchange values which, if adopted by the market, restores the original distribution of the products and makes it possible for the process to be repeated; such values spring directly from the methods of production. (Sraffa, 1960, p. 3.)

We have here two formal propositions: (a) there is a unique set of exchange values, and (b) these values spring directly from the methods of production. Introducing the prices of the commodities as unknowns, the above example may be rewritten in matrix form as

$$pT = p \tag{2.2}$$

where T is the matrix given by $T = Q^{-1}M'$ and p is the row-vector of unknown exchange values. The coefficient t_{ij} represents the <u>share</u> of the output of the i-th good that is being employed in the production of the j-th commodity.

Propositions (a) and (b) above can now be stated as:
(a)' there is a unique price such that (2.2) is satisfied and
(b)' this unique vector p depends on the matrix T.

At this point some assumptions are introduced in order to solve (2.2):

A.1: T is a nonnegative square matrix

A.2: T is characterized by the fact that for each row, the sum of the coefficients along the row, is equal to unity.

A.3: T is indecomposable

Note that A.1 derives from the very nature of the input-output coefficients, t_{ij}. A.2 instead comes from the assumption that the system is a closed system, with no inflows or outflows of goods (so that all means of production are produced within the system itself), and from the balance relation defining the 'self-reproducing' state of the system. A.3 comes from the idea of focusing on the set of 'basic' goods only, that is the set of goods that are used and produced by themselves. Formally, then, Sraffa's proposition can be summarized in the statement:

THEOREM 2.1. Given T and A.1, A.2 and A.3, there exists a price vector which is positive and unique up to a scale factor.

We note that this theorem has been proved also by Gale (1960) for Remak's model of equilibrium in an exchange economy and for a formally analogous model of international trade. Sraffa's matrix shares with Remak's (and the matrix of incomes in international trade) the fact that, not only is it nonnegative, but it can be so transformed that row sums are all equal to unity. Matrices of this sort are usually called stochastic, transition or, in Gale's terminology, exchange matrices.

Proof. We first prove the existence of a semi-positive price vector without assuming indecomposability of T. Then we assume an indecomposable T and prove that the semi-positive price vector is in fact a unique set of positive relative prices. From (2.2) we have

$$p(I - T) = 0 \tag{2.3}$$

By a theorem of the alternative (cf. Gale, 1960, pp. 42-49), a variant of a separating hyperplane theorem or Farkas Lemma, we know that (2.3) has no solution $p \geq 0$ only if there exists a vector x such that

$$(I - T)x > 0 \ , \qquad (2.4)$$

or

$$x_i > \sum_{j=1}^{n} t_{ij} x_j \ , \quad \forall i = 1,2,\dots,n \ . \qquad (2.5)$$

Let $v = \text{Min } x_i$, say $v = x_1$ without loss of generality. Sum the vectors $t^j = (t_{1j},\dots,t_{nj})'$ to give

$$\sum_{j=1}^{n} t_{ij} = 1 \ , \quad \forall i = 1,2,\dots,n \ . \qquad (2.6)$$

Multiply (2.6) by x_1 and subtract from (2.5)

$$x_i - x_1 > \sum_{j=1}^{n} t_{ij}(x_j - x_1) = \sum_{j=2}^{n} t_{ij}(x_j - x_1) \ , \quad \forall i$$

and, in particular, $\sum_{j=2}^{n} t_{1j}(x_j - x_1) < 0$. But this is impossible because both $(x_j - x_1)$ and t_{1j} are nonnegative, $\forall j$. Thus no such vector x exists, that is

$$\exists \text{ a semi-positive vector } p \text{ s.t } pT = p.$$

Now let T be indecomposable. Assume that the price vector p is not positive. This means that there exists $p_j = 0$ for some submatrix of T and $p_i > 0$, $\forall i$ in the complement. Call the submatrix for which the prices are positive M and its complement M'. Then, for all indices $i \in M'$, we have

$$pt^i = p_i = 0 = \sum_{j \in M} t_{ji} p_j$$

and, therefore, $t_{ji} = 0, j\varepsilon M, i\varepsilon M'$. This means that M is independent of the rest of T. But, by assumption, T is indecomposable. Hence $M = T$ or $p_j > 0$, $\forall j = 1,2,\ldots,n$, that is $p > 0$.

Finally, to prove uniqueness, let p,p' be linearly independent solution vectors. Let $\lambda = \text{Min}(p_i/p_i')$ and assume $\lambda = p_1/p_1'$. Then, as both $p,p' > 0$, $p^* = p - \lambda p'$ is a nonnegative vector. As any linear combination of independent solutions to homogeneous equations is a solution itself, p^* has to be a solution too, $p^*T = p^*$, but $p^* \not> 0$ by hypothesis. (In fact $p_1^* = 0$.) Therefore, in $p^*[I - T] = 0$, $p^* = 0$ or $p = \lambda p'$, that is they differ only by a scalar factor, and the theorem is proved.

As a realization of a multisectoral model, the above system is very simple in a number of ways:

(i) from the point of view of its structure, it is not 'complete' at least in the sense that there is no production to be decided upon, for production has already taken place 'yesterday';

(ii) there is no surplus and therefore no positive rate of profit, nor

(iii) a positive rate of growth;

(iv) finally, each sector is assumed to produce only one commodity (no joint production).

To go further into multisectoral modelling in terms of relaxing the above limitations, we have: (i) to construct a quantity system; (ii) to allow for the possibility of a positive rate of profit and/or of growth; finally, (iii) to allow for joint production. We will proceed to this by steps.

The quantity system corresponding to system (2.2) takes the form:

$$Ax = x \; , \qquad (2.7)$$

where, by a similarity transformation, A is related to T but with a crucial difference. As long as we are interested in the properties of (2.2) from the point of view of the existence, positiveness and uniqueness of the set of 'exchange values', we need no assumptions on the technology, for example on the returns to scale, involved. The only technological assumption is that the system is closed. The problem involved in matrix equation (2.7) is different. We now want to determine a set of activity levels x such that for each commodity the balance relation (2.7) is satisfied. The quantity system (2.7) involves a set of true unknowns, so that it is logically different from (2.2) which instead states an assumption.

Duality of the quantity and the price systems is obtained only if we show that matrix A and matrix T collect the same coefficients. This means that $t_{ij} = \Delta^i a_{ij}$, where $\Delta^i > 0$, is a scalar. Coefficients a_{ij} are defined in the usual way (each a_{ij} represents the amount of the i-th good that is technologically required to produce one unit of commodity j). This implies the assumption that the technique represented by (A,I) is linear and with constant production coefficients. In other words, coefficients a_{ij} <u>do not</u> change with the level of the output. Note that (2.6) implies that outputs, instead of being measured by their natural units (tons, pairs, etc.), <u>were normalized to one</u>. This represents a mere change of units, while (2.7) implies a <u>law of returns</u>. By introducing this law, the structure of our system is enriched. We can now not only find out the equilibrium 'exchange values' but also determine <u>equilibrium levels of production</u>. Note also that, if we 'complete' the price system with quantity equations of the type (2.7) and in the latter accept the assumption on the returns to scale, matrices T and A are one and the same thing, so that (2.2) <u>and</u> (2.7) <u>are dual systems</u>. The data of a closed system of production characterized by constant returns to scale (in the form of constant production coefficients) will be summarized by the couple (A,I), that is by a set of 2n vectors with n entries (in $\mathbb{R}^n$).

The existence proof we need is the following:

THEOREM 2.2. <u>Dual systems (2.2) and (2.7) have nonnegative solutions x and p such as to satisfy the condition</u>

$$pIx > 0$$

<u>under identical assumptions</u>.

The last qualification allows us to interpret the theorem in the following way. If one of the above systems has a nonnegative solution, then the other has it also. Moreover, any economically meaningful solution must imply that a positive value of the output is produced. We frame it in this way for we want to take advantage of the theorem of the alternative we have used before.

Let us introduce the following assumptions on matrix A as we did on T:

A.1': A is nonnegative square

A.2': each row in A sums up to unity

A.3': A is indecomposable

A.1' needs no explanation; A.3' states that there are only basic goods (cf. Sraffa (1960)), and is made for convenience (we will come back briefly to the decomposable case in a different context). Once A.2' is justified, we can rely on Theorem 2.1 to assert at least the existence of p' and draw the implications. (A dual proof along the same line would complete the reasoning in the opposite direction: conditions ensuring the existence of x will ensure the existence of p.) This is easily done by noting that, if condition A.2' is satisfied, the system is capable of a self-reproducing state (the one corresponding to the vector $i = (1,1,...,1)'$ or any scalar multiple of it).

Under the above assumptions, it has already been proved that there is a positive vector p, which moreover is unique up to scalar multiplication. We have to prove that this implies that (i) there exists a nonnegative x solving (2.7) and (ii) that for any such x, pIx is positive.

Proof

(i) That (2.7) has a semi-positive solution, implies that the set $X = \{x | (I - A)x > 0\}$ is empty. Geometrically this means that the cone intersections $\Delta(I - A) \cap \mathbb{R}^n_+$ and $\Delta(A - I) \cap \mathbb{R}^n_+$ are both empty. Therefore, the cone spanned by column vectors in (I - A) is really a k-dimensional linear subspace in $\mathbb{R}^n$. In other words, the equation

$$(I - A)z = 0$$

has at least one non-zero, nonnegative solution z.

(ii) As the solution to (2.7) is strictly positive, $pIz > 0$ for all $z \geq 0$ solving equation (2.7).

A completely parallel reasoning will show that under the above assumptions, the vector x solving (2.7) is itself positive and unique up to a scalar multiple (these latter two properties are a result of A.3').

Assumption A.2' is in a sense too particular, since we can make a more general statement with a minimum of assumptions.

THEOREM 2.3

(i) Assume that matrix A satisfies A.1' and the following condition replacing A.2':

A.4. There exists a positive vector x such that $Ax = x$.

Then there is a semi-positive price vector p satisfying the dual equation $pA = p$.

(ii) Vice versa, assume that together with A.1', the following condition is satisfied:

A.5. There exists a positive price vector such that $pA = p$.

Then there is a semi-positive solution to the quantity equation $Ax = x$.

Note that we know that, under indecomposability, both the price and the quantity vectors are strictly positive and unique, in the sense defined. The weaker formulation we are using would serve as a bridge to the so-called complementary slackness conditions of a formulation in terms of linear programming.

In the formulation above, the properties of the dual systems are linked together very closely, and have an economic interpretation: in the formulation (i), the statement says that, if the system is capable of reproducing itself running all activities (A.4), then there is a set of semi-positive prices that satisfy the zero profit condition of competitive markets; in the formulation (ii) vice versa, it says that, if there is a set of competitive, zero-profit prices, then the system admits of a self-reproducing state.

We see that in both cases, given the semi-positivity of one of the vectors and the positivity of the other, we always obtain solutions (x,p) that also satisfy the condition $pIx > 0$.

Theorem 2.3 will be used in the following section to study the existence of nonnegative solutions for more general closed models, before introducing an alternative approach based on the well-known Perron-Frobenius theorem.

2.2: <u>More general closed models of production</u>

(i) Consider the following input matrix

$$\hat{A} = (A + hcl) \ , \qquad (2.8)$$

where h is a scalar indicating a (given) level of real wages in terms of standard wage basket c, A is the technological matrix, and for any given h, cl is a dyadic matrix of consumption goods for labour. Matrix $\hat{A}$ is called _Morishima's augmented matrix_ of inputs (cf. Morishima, 1971; Brody, 1970). The dual systems of equations are now

$$p\hat{A} = p \qquad p \geq 0 \tag{2.9}$$

and

$$\hat{A}x = x \qquad x \geq 0 \ , \tag{2.10}$$

where the data is the matrix $\hat{A}$ and the unknowns are the relative prices and activity levels, p and x. The matrix $\hat{A}$ is really a _matrix of functions of the scalar h_, $\hat{A}(h)$, but for any given value of h, it is a matrix of constant coefficients, and we will consider it as such now.

THEOREM 2.4. _For matrix $\hat{A}$ evaluated at a given h, and corresponding dual systems, existence of nonnegative economic solutions is established via assumptions A.1', A.3' and the dual characterization established in Theorem 2.3._

(ii) We re-define the coefficients of our matrix: the best known way is due to Leontief (1941). Matrix A is made up of n columns and n rows, so that neither labour nor consumption has been explicitly incorporated. The simple model describes flows of commodities in a sort of automatized economy. We now 'augment' A by one column and one row, so as to obtain a matrix A^* of order n+1, where the last column is made up of n coefficients of consumption by each worker plus a zero at the n-1 entry (i.e. c = $(c_1,\ldots,c_n,0)$). The (n + 1)-th column describes in this way the 'wage rate' in real terms as a consumption basket. The (n + 1)-th row, instead, is made up of n labour input coefficients (i.e. $a_{n+1} = (l_1,\ldots,l_n,0)$). The price vector is now given by the row-vector $\hat{p} = (p_1,p_2,\ldots,p_n,w)$, where w stands for the nominal wage rate. The activity level vector is now given by $\hat{x} = (x_1,x_2,\ldots,x_n,L)'$, where L stands for the level of employment. With this interpretation the price system $\hat{p}A^* = \hat{p}$ may be split into

$$pA + wa_{n-1} = p \quad \text{and} \quad pc = w \ . \tag{2.11}$$

The dual system is now $A^*\hat{x} = \hat{x}$ or

$$Ax + Lc = x \quad \text{and} \quad g_{n+1} = L \ . \tag{2.12}$$

We can now use the fact that, if equations (2.11) and (2.12) have non-zero solutions, they determine only relative prices and relative activity levels, $\hat{p}$ and $\hat{x}$. We can therefore fix a normalization rule for both of them by setting w = 1 and L = 1, which is the Leontief closed model. It is obvious that A can be obtained as a particular case of either $\hat{A}$ or A^*. Theorem 2.4 then applies here as well.

2.3: Perron-Frobenius theorems and applications

We have, thus far, considered the solution to homogeneous equations associated with matrices A, A^* and $\hat{A}$ from a purely topological point of view. In the language of topology a solution vector like p or x is a fixed point of a linear map, but there is also an algebraic approach to the above set of problems characterized by a set of theorems associated with the names of Perron and Frobenius. Consider (2.2) and (2.7). From an economic point of view, we are interested primarily in non-zero, and, if possible, nonnegative solutions. If there is one such solution x (or p), it is called an eigenvector (or characteristic vector) of the matrix A. If there is more than one vector x (respectively p) solving equation (2.2) (respectively 2.7), the set of such solutions span a subset of a space in $\mathbb{R}^n$ called the eigenspace.

Let us now consider a more general system

$$p(I - (1 + r)A) = 0$$

and

$$(I - (1 + g)A)x = 0 \ ,$$

where g is the rate of balanced growth and r is the uniform rate of profit. The above form a dual system only if we consider the golden rule rate of growth, $g = r$. Set $\beta = 1/(1 + g)$ and $\rho = 1/(1 + r)$ and we obtain

$$p(\rho I - A) = 0 \tag{2.13}$$

$$(I\beta - A)x = 0 \ . \tag{2.14}$$

The value system (2.13) and the quantity system (2.14) unless connected via some choice-theoretic paradigm, could be solved independently. Parametrizing equations (2.13) and (2.14) in terms of ρ and β, it is clear that (2.2) and (2.7) are special cases of the above for $\rho = \beta = 1$. However, we wish to consider the problem when ρ and β instead of being parameters are unknowns. There are non-trivial solutions if and only if β and ρ are eigenvalues of A. But the solutions, to make economic sense, must not only be non-zero but also nonnegative. Considering for simplicity, only the quantity side, we can apply the Perron-Frobenius theorem. Note that the eigenvalues of A and A' are the same, so that (2.13) also has solutions that are positive and unique in the sense defined by the theorem.

THEOREM 2.5. Equations (2.13), (2.14) for an indecomposable matrix have positive solutions $\hat{p}$, $\hat{x}$ if and only if we set $\beta = \rho = \hat{\beta}$, where $\hat{\beta}$ is the maximal eigenvalue of A.

COROLLARY. The solution vectors $(\hat{p},\hat{x})$ satisfy the requirement $\hat{p}I\hat{x} > 0$.

By using the Perron-Frobenius theorem, we may establish an existence result for equations (2.2)~(2.7) and (2.9)~(2.10).

THEOREM 2.6. Let A (or $\hat{A}$, A^*) be indecomposable. Then the equations have positive solutions p and x, each unique up to a scale factor, if and only if the maximal eigenvalue of A (or $\hat{A}$, A^*) is equal to one.

There is a version of the Perron-Frobenius theorem for general decomposable matrices, but the results are for obvious reasons weaker. Among other things, we obtain that the maximal eigenvalue is just nonnegative, the associated eigenvector is simply nonnegative and the maximal root can appear more than once so that there is a whole space of eigenvectors associated with it. The condition that the eigenvalue be equal to one is only a part of a more complex sufficient condition to ensure the nonnegativeness of the solution (e.g.

Berman and Plemmons, 1979).

Associated with the above homogeneous equations, are the resolvent equations for the 'open systems'

$$(1 + g)Ax + c = x \tag{2.15}$$

$$(1 + r)pA + wl = p \tag{2.16}$$

whose solutions are

$$x = [I - (1 + g)A]^{-1}c$$

$$p = wl[I - (1 + r)A]^{-1}$$

where the inverse matrices $[I - (1 + r)A]^{-1}$ and $[I - (1 + g)A]^{-1}$ are the resolvents. If the resolvents exist, the solutions are uniquely determined. If the resolvents are nonnegative, the solutions, being obtained by multiplying the inverses by nonnegative vectors c, wl, will also be nonnegative: precisely results analogous to those we have previously obtained via theorems of the alternative. We now state two results, the first one rather trivial.

(i) The resolvent exists if and only if the reciprocals of the scalars $(1 + g)$ and $(1 + r)$ are not eigenvalues of A.

(ii) Let $\hat{\beta}$ be the maximal eigenvalue of matrix A. By Perron-Frobenius it is nonnegative; then, for any scalar μ larger than β, the matrix $(\mu I - A)$ has a nonnegative inverse (Debreu and Herstein, 1953).

(iii) If A is indecomposable, then $(\mu I - A)^{-1}$ is positive.

Now, interpreting $\mu = 1/(1+g)$ and $\beta = 1/(1 + g_{max})$ (and likewise for the rates of profit), the two results establish that there are unique nonnegative solutions to equations (2.15), (2.16) only for all those values of the rate of growth and/or of the rate of profit that are less than the maximal rates obtained by solving the homogeneous systems (2.13), (2.14). Finally, are these values of the rates of profit and growth, for which nonnegative price and quantity solutions exist, nonnegative themselves? This is so only if the maximal rates r_{max} and g_{max} are positive themselves, that is if the Perron-Frobenius root, $\hat{\beta}$, is strictly less than one. This means that the matrix A is <u>productive</u> (Nikaido, 1970), that is there is a nonnegative vector of activity levels x such that (I - A)x is semi-positive. This implies that the system (A,I) can produce a surplus of at least one good. There are various equivalent formulations of the conditions ensuring that a matrix like A is productive, and the best known goes under the name of Hawkins-Simon.

We may summarize the above discussion in the following statement: solutions of the inhomogeneous systems (2.15), (2.16) are uniquely determined and nonnegative corresponding to nonnegative rates of profit and of growth whenever the system represented by input matrix A and output matrix I is productive.

To enrich the economic content of the immediately preceding discussion and as an unusual exercise in the usefulness of the Perron-Frobenius theorem we now use these results and concepts in deriving the linear wage-profit curve (cf. Sraffa, 1960). For the above open (inhomogeneous) systems (2.15), (2.16), in addition to the earlier assumptions, assume also that A has n distinct eigenvalues. Then, there exists a similarity transformation $H^{-1}AH = \Lambda$, where Λ is a diagonal matrix made up of the n distinct eigenvalues (λ_i) and the columns of H are made up of the eigenvectors corresponding to each of the distinct eigenvalues. Now, define $p^d H = p$, $l^d H = l$, $H^{-1}x^d = x$ and $H^{-1}c^d = c$. Then (2.15) and (2.16) can be rewritten as

$$p^d[\bar{I} - (1 + r)\bar{\Lambda}] = wl^d \tag{2.17}$$

and

$$[\bar{I} - (1 + g)\bar{\Lambda}]x^d = c^d \tag{2.18}$$

Note that out of the original n commodities, n new composite commodities have been formed. Corresponding to this set of composite commodities, we have a totally decoupled system. However, to the n eigenvalues λ_i, there corresponds only one positive eigenvector (by Perron-Frobenius) and this vector defines Sraffa's multipliers that will reduce the 'actual' system to a Standard System (Sraffa, 1960, pp. 24-28). In the decoupled system each of the composite commodities is a 'standard commodity'--but only the one corresponding to the maximum eigenvalue can refer to Sraffa's 'standard system'. Now, define unit labour cost, $\gamma_i = wl_i^d p_i^d$, and unit consumption, $\delta_i = c_i^d x_i^d$. Then, (2.18) and (2.17) can be written as

$$1 = \lambda_i + \delta_i + g\lambda_i \ , \tag{2.19}$$

i.e. materials + consumption + investment, and

$$1 = \lambda_i + \gamma_i + r\lambda_i \ , \tag{2.20}$$

i.e. materials + wages + profits. Resubstituting γ_i and redefining the composite good which corresponds to the dominant eigenvalue λ^* to be the numeraire, one obtains

$$1 = \lambda_i^* + wl_i^d + r\lambda_i^* . \qquad (2.21)$$

When the units are chosen so that: when $r = 0$, $w^* = 1$ and when $w = 0$, $r^* = 1$, that is $w^* = wl_i^d/(1 - \lambda_i^*)$ and $r^* = r\,{}_i^*/(1 - \lambda_i^*)$, we get $r^* = 1 - w^*$. This is similar to Sraffa's famous linear relation between the distributive variables $r = (1 - w)r_{max}$ except that we have redefined units so that r_{max} is always unity. The dual consumption-growth curve can be derived in a similar way.

From the gross output equation (2.19) we can see that one unit of gross product needs λ_i units of itself and, therefore, $l_i^d\lambda_i$ units of labour--and so on. Thus the total direct and indirect labour content in one unit of output will be given by

$$l_i^d(1 + \lambda_i + \lambda_i^2 + \cdots) = \frac{l_i^d}{1 - \lambda_i} \qquad (2.22)$$

Now, the importance of the 'productivity' assumption (in another form also known as the Hawkins-Simon conditions) becomes evident. The maximal eigenvalue of a productive matrix A is less than one, so that for all eigenvalues the inequality $0 < |\lambda_i| < 1$ holds, which makes (2.22) meaningful. From (2.20) we have

$$p_i^d = \frac{wl_i^d}{1 - \lambda_i} + \frac{r}{r_{max}} \qquad (2.23)$$

Clearly prices are not simple multiples of labour values, although prices are <u>proportional</u> to labour values when $r = 0$.

As a final <u>corollary</u> to the Perron-Frobenius theorem, we have the following very useful result: for rates of growth and of profit less than the maximal ones, the resolvents are not only nonnegative but such that each coefficient is a continuous non-decreasing function of the rates g, r. This result is strengthened if the matrix A is indecomposable for then the coefficients are strictly increasing functions (of the exogenously given g, r). This result is very useful in comparative statics, that is, when we compare price and/or quantity solutions corresponding to different levels of the rate of profit and/or the rate of growth. As the price vector is obtained by multiplying the resolvent to the left by the labour input vector times the wage rate, the price vector is a vector of increasing functions of the rate of profit, once the nominal wage rate is given: in a different, more 'classical' jargon, if we take the wage rate as numeraire, i.e. if we set $w = 1$, then 'labour-commanded prices' $\hat{p} = p/w$ all increase, though at different relative speeds, if the rate of profit increases. Likewise, if the vector of final demand is positive and/or the matrix A is indecomposable, any increase of the exogenously given rate of growth g will require an increase in at least one entry of the solution vector x of equation

We may briefly compare the results obtained so far. By appealing to one of the theorems of the alternative, we have given a simple prototype of an existence proof for nonnegative solutions to our dual systems of linear equations. This proof establishes certain local properties as the rate of profit and the rate of growth are treated as given parameters. The Perron-Frobenius theorems and their corollaries, instead, allow us not only to establish analogous existence results, under conditions that lend themselves to an easy economic interpretation, but also to treat such solutions as vectors of functions of r and/or g. We can make use of this important information to derive two central tools for the modern theory of growth and distribution.

Let us specify the wage rate in terms of a bundle of goods in fixed proportions, that is the nonnegative vector c. For any given set of prices, the scalar $(pc)^{-1}$ is an index of the real wage rate. As a consequence of the preceding corollary, this index is a function of the rate of profit treated as exogenous, and the first derivative of the price vector is positive, if we introduce the natural assumption of a positive labour input vector. Thus the real wage rate $(p(r)c)^{-1}$ in terms of any predetermined c, is a decreasing function of the rate of profit, via p(r). This allows us to draw the real wage rate/rate of profit curve associated with the given technique, as a strictly decreasing curve. The envelope of the set of curves associated with all available techniques is the so-called Wage-Profit Frontier of the given technology.

On the other hand, let c be a representative consumption basket. Given the rate of growth, how many of these baskets are left available to each member of society? This will depend on the amount of labour that has to be spent to produce one such basket as net consumption: that is, on 'labour productivity' in terms of net output above the preassigned accumulation rate. Given this, to obtain the vector c, the amount lx^c of labour has to be expended. Therefore, the number of baskets per capita will be equal to $(lx^c)^{-1}$. Remembering that, for given c, $(lx^c)^{-1}$ is a function of g via $x^c(g)$, we obtain that the level of consumption per capita is a decreasing function of the rate of growth. By repeated calculation of $(lx^c(g))^{-1}$ for different values of g, with x constrained to range over a given technique, we derive the Consumption-Investment curve of a given technique. The envelope of the curves for all available techniques is the C-I Frontier of the technology. An analogous result holds if we assume, following Marx and von Neumann, that the wage rate is advanced by capitalists before the production process takes place (see Morishima, 1973).

§.3. General Joint Production Systems

We have so far generalized our basic production model in various directions, but the structure of a non-joint production model was retained. We now remove this last restriction introducing the joint production model in its full generality. The output matrix is, therefore, a matrix B, which if diagonal gives the NJPS as a special case. Dimensions are kept the same, so that both output and input matrices A, B are square of order n. We will start with the open system while the closed one is discussed in the last section from the point of view of a generalized Perron-Frobenius problem.

The quantity- and price-systems are represented by

$$Bx = (1 + g)Ax + c \tag{3.1}$$

and

$$pB = (1 + r)pA + wl \ , \tag{3.2}$$

where the following

A.1* A and B are nonnegative square matrices

A.2* there exists a positive vector $x:(B - A)x \geq 0$

A.3* the labour-input vector is positive

are assumed. To simplify notation let us introduce the symbols $M(r) = (B - (1 + r)A)$ and $M(g) = (B - (1 + g)A)$, where $M(r)$ and $M(g)$ are matrices parameterized by r and g. Likewise, $M^{-1}(\cdot)$ stands for the inverse (or resolvent) matrix, when it is defined (and $M^{-1,j}(\cdot)$ is its j-th column). On the basis of the above assumptions one has been able to establish a number of remarkable results for the NJPS. Firstly, there A.2* implies that 1 is not an eigenvalue of matrix A, and therefore, rank $(I - A) = n$ and the net-output matrix is invertible. Secondly, as A.2* implies that the Perron-Frobenius root of A is less than one, $(I - (1 + r)A)$ and $(I - (1 + g)A)$ are nonnegatively invertible for all $r,g:0 \leq r,g < r_{max}, g_{max}$. So, on the basis of A.2* only, we could get solutions in x and p which are unique and moreover nonnegative for a closed interval of values for the rate of profit and the rate of growth including z.

Here also, A.2* implies that 1 is not a root of the equation $\det(\beta B - A) = 0$ (it is not a generalized eigenvalue of A versus B) so that rank $(B - A)$ is equal to n and the inverse of the net-output matrix exists. However, we do not have the same closed interval as before, which can be seen by transforming $\det(\beta B - A) = \det(\beta I - B^{-1}A)$, if $\det B \neq 0$, or $\det(\beta B - A) = \det(\beta A^{-1}B - I)$ if $\det A \neq 0$ and noticing that the matrix $B^{-1}A$ (matrix $A^{-1}B$, respectively) is not in general nonnegative and therefore has complex eigenvalues. $M(r)$ and $M(g)$ are invertible for all scalars r and g such that $(1 - r)^{-1}$ and $(1 + g)^{-1}$ are not roots of the characteristic equation: for this set of scalars, formal solutions to (3.1) and (3.2) can be found via the resolvent equations

$$\hat{p} = lM^{-1}(r) \quad , \quad \hat{p} = p/w \tag{3.3}$$

$$x = M^{-1}(g)c \tag{3.4}$$

for any given l,c. Solutions x and $\hat{p}$ are, just like in the NJPS, uniquely determined via the resolvents, but the above assumptions are not sufficient to establish that they are also nonnegative.

However, both equations (3.3) and (3.4) have nonnegative solutions if and only if the vector of final demand and the vector of labour-input coefficients satisfy, at the given values of the parameters, the following independent conditions (Manara, 1968):

(i) $l \in \Delta M(r)$, where $\Delta[M(r)] = \{y \in R^n | y = \hat{p}M(r) \ , \ \hat{p} \geqq 0\}$

(ii) $c \in \Delta M(g)$, where $\Delta[M(g)] = \{z \in R^n | z = M(g)x \ , \ x \geqq 0\}$

$\Delta[M(r)]$ and $\Delta[M(g)]$ are convex polyhedral cones spanned by matrices $M'(r)$ and $M(g)$; if $r = g = \lambda$, λ a common scalar, the matrices are the transpose of each other so that $\Delta[M(r)]$ is spanned by the rows and $\Delta[M(g)]$ by the columns of the same matrix $\Delta M(\lambda)$.

In the NJPS, the assumption that the technique is productive implies that the whole nonnegative orthant is contained in the net output cone spanned by M(g) for all $g:0 \leq g < g_{max}$ where $g_{max} = 1 - \hat{\beta}/\hat{\beta}, \hat{\beta}$ the Perron-Frobenius root of **A** (while for $g = g_{max}$, M(g) spans a k-1 linear subspace such that the nonnegative orthant lies completely on one side of it). Therefore, condition (ii) is naturally satisfied by all nonnegative vectors c. On the other hand, productiveness implies also that there is a nonnegative vector $p:pM(r) > 0$ for all $r:0 \leq r < r_{max} = g_{max}$. In other words, the nonnegative orthant is included in the cone spanned by M(r), $0 \leq r < r_{max}$ (and, again, for $r = r_{max}$, $pM(r_{max})$ spans a hyperplane through the origin supporting the nonnegative orthant). Therefore, condition (i) is naturally satisfied by all nonnegative vectors l. This relation of cone inclusion is the topological meaning of the nonnegative invertibility of both M(r) and M(g).

In the joint production model, the above conditions place additional requirements upon sets of data (vector c and net output matrix M(g), vector l and matrix M(r)). To illustrate why negative solutions may arise, let us assume for the moment, that the rate of growth and the rate of profit are equal to each other (say λ). We can thus use only one cone, $\Delta[M(\lambda)]$, and the solutions to the quantity system will lie on an affine subspace on the cone (in general, a hyperplane) while the corresponding price solutions will be a (cone of) vector(s) normal to that affine subspace. Joint Production assumption A.2* (the technique is productive) merely implies that the intersection between $\Delta[M(\lambda)]$ and the interior of the nonnegative orthant is non-empty, but does <u>not</u> necessarily imply that the whole nonnegative orthant is included in $\Delta[M(\lambda)]$.

To emphasize the contrast with NJPS, Figure 1 represents a situation that can typically arise only in the Joint Production model, with a cone of net-output spanned by $M(\lambda)$ lying in the interior of the nonnegative orthant. Now, as long as the given vector of final demand c belongs to the interior of $\Delta[M(\lambda)]$, it can be produced by using both processes with positive activity levels, while if it is collinear either with $M^1(\lambda)$ or with $M^2(\lambda)$, only one process is run and the other stays idle (i.e. it fetches a zero activity level). However, goods cannot be produced exactly in proportions like c' or c" that lie <u>outside</u> $\Delta[M(\lambda)]$ as one of them would be overproduced. Proportions like c', c" could be obtained only 'notionally' by giving one of the two processes a negative activity level (see Sraffa, 1960, p. 43).

To give a geometrical interpretation to part (i) of the condition, consider the equation on the cone:

$$\hat{p}M(\lambda)x = \bar{K} \quad (3.5)$$

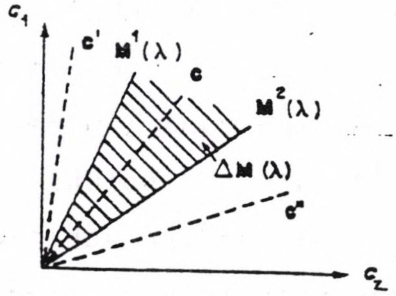

Figure 1: Joint production

Vector $\hat{p} = 1M^{-1}(\lambda)$ is the outward direction coefficient vector, normal to the transformation hyperplane $M(\lambda)x$. In $\mathbb{R}^2$, the equation of the hyperplane is

$$\bar{K} = \hat{p}_1 c_1 + \hat{p}_2 c_2 = 1M^{-1,1}(\lambda) + 1M^{-1,2}(\lambda)c_2 \tag{3.6}$$

whence the slope

$$dc_1/dc_2 = -(\hat{p}_2(\lambda)/\hat{p}_1(\lambda)) = -(1M^{-1,2}(\lambda))/1M^{-1,1}(\lambda) \ . \tag{3.7}$$

In the NJPS, input coefficients a_{ij} are usually standardized with reference to unit gross outputs. However, here this cannot be done (unless the output matrix is diagonal), and we may assume that entries in (A,B) are (input and output) coefficients per unit of labour input (i.e. the labour vector is the unit sum vector i). In this way, each vector $M^j(\lambda)$, $j = 1, 2$, represents the net output, above accumulation at the given rate $\lambda = g$, produced by a unit of labour in the j-th process. Expression (3.7) becomes

$$dc_1/dc_2 = -(iM^{-1,2}(\lambda)/iM^{-1,1}(\lambda)) \ . \tag{3.8}$$

A negative slope of the transformation curve implies that both prices $\hat{p}_1$, $\hat{p}_2$ are positive. However, in the JPS the slope may well be positive and, therefore, the rate of transformation between goods 1,2 may be positive as in the following example. In Figure 2 any vector of final demand such as c can technically be produced with nonnegative (in fact, positive) activity levels and no overproduction would appear. Therefore, corresponding to the set of vectors c belonging to $\Delta[\bar{M}(\lambda)]$ (a cone itself), equation (3.5) would have nonnegative x solutions. However, the net output vectors $M^1(\lambda)$ and $M^2(\lambda)$ stand in the relation $M^1(\lambda) > M^2(\lambda)$, and the hyperplane connecting

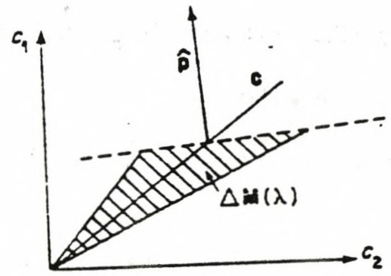

Figure 2: Inefficiency in the allocation of labour

them (the transformation curve over $\Delta[\bar{M}(\lambda)]$) has a positive slope, so that the direction coefficient vector $p(\lambda)$ contains entries with opposite signs. The use of process $M^2(\lambda)$ implies a sort of 'inefficiency' in the allocation of labour among the available processes, and this is indirectly revealed by the appearance of a partly negative price solution in the dual system (3.4).

Within the algebraic approach we have consistently used so far, we may only find certain sufficient conditions to obtain nonnegative solutions and provide for them an economic justification. However, the major justification for this approach comes from the fact that we obtain a full characterization of the NJPS as the special JPS where all such conditions are naturally satisfied.

Taken together, (i) and (ii) ensure that the x solutions corresponding to a given rate of growth and the price solutions corresponding to a rate of profit are nonnegative. They do not impose the so-called golden rule, i.e. they do not require equality between r and g, which we are assuming for simplicity. There are many such conditions, but we briefly discuss only the most important of them. Keeping the assumption of a rate of growth equal to the rate of profit, it can be shown, as a special case, that solutions are simultaneously nonnegative for both (3.4) and (3.5) <u>only if</u> the net output matrix $M(\lambda)$, corresponding to a given rate of growth, satisfies

A.4* There exists a vector $x > 0 : M(\lambda)x \geq 0$ <u>and</u> if $M(\lambda)$ is a <u>Z-matrix</u>, i.e. a matrix with the pattern $M_{ij}(\lambda) \leq 0$, $i \neq j$.

Note that A.4* generalizes the notion of productiveness as stated by A.2*: a system is said to be productive at λ if $M(\lambda)$ satisfies the first half of A.4* (and obviously, it is productive in the usual sense if A.4* is satisfied for $\lambda = 0$). If $M(\lambda)$ is a productive Z-matrix (more technically, a P-matrix, see Berman and Plemmons (1979)), it is nonnegatively invertible and, therefore, both prices and quantities, corresponding to λ are unique and nonnegative for all vectors l and c. The NJPS is the typical, though not the only, representative of joint production systems whose net output matrices are Z-matrices. The important difference, however, is that, for NJPS, $M(\lambda)$ is a Z-matrix for all λs between zero and a maximal rate, $r_{max} = g_{max}$, while this is not in general true for JPS. In this case, $M(\lambda)$ may happen to be a Z-matrix at some given positive $\bar{\lambda}$, and thus it will stay so for $\lambda > \bar{\lambda}$, <u>without</u> this implying that it is a Z-matrix for all $\lambda : 0 \leq \lambda < \bar{\lambda}$.

At this point, it should be emphasized that both conditions (i)-(ii) and the condition that $M(\lambda)$ be a productive Z-matrix share the property that, even if either is satisfied, it still would hold only locally, i.e. for the given rate of profit (equal to the rate of growth, by hypothesis. This is why they are stated with reference to given values of the parameters). Moreover, if the latter condition holds, conditions (i) and (ii) are satisfied automatically, but not vice versa. The only way to avoid this problem would require assuming that the net output matrix is a productive Z-matrix already at $\lambda = 0$. However, nothing new would be gained, since the JPS would behave just like the NJPS, and they could not be distinguished. The price equation could be put into the form:

$$p = rpA(B - A)^{-1} + wl(B - A)^{-1} = rpF + w\hat{l},\ F = A(B - A)^{-1},$$

where F and $\hat{l}$ are nonnegative when $(B - A) = M(0)$ is a productive Z-matrix. Therefore, prices are positive for all $r: 0 \leq r < r_{max}$ with r_{max} equal to the reciprocal of the Perron-Frobenius root of F. Analogous manipulations of the quantity system yield a nonnegative matrix $K = M(0)^{-1}A$, as the vertically integrated capital matrix (see Pasinetti, 1973); for this reason, the x-solutions are positive and unique in a closed interval of g, $[0, g_{max}]$.

We have assumed so far that $r = g$. To complete the list of difficulties arising in an equation approach to general JPS, we now consider briefly the case where the two rates are allowed to differ (and, obviously, $r \geq g$). Then, the net output matrix M(r) may even be a productive Z-matrix at some given $r = \bar{r}$ (so that the corresponding prices are positive), while M(g) is not, for $g \neq \bar{r}$ (hence, the quantity solutions may turn out to be partly negative), unless we are prepared to assume that condition (ii) is also satisfied. This again is not the case with NJPS, where M(r) and M(g) are, naturally, Z-matrices already at r,g = 0. This allowed us to solve, independently, quantity and price systems. In the JPS, the assumption of a golden rule balanced growth is sufficient to ensure that, whenever M(r) is a Z-matrix, so is M(g), and that the x- and $\hat{p}$-solutions to the dual systems of equations are both nonnegative.

One important outcome of the preceding discussion is that it makes clear how in JPS, and in NJPS as a special case, the existence and uniqueness of nonnegative solutions are determined by the properties of the net output matrix, and not by the individual properties of the A,B. The NJPS is 'simpler' just because certain properties of the input matrix are carried over to the net output matrix. The JPS, instead, forces us to discuss the existence and uniqueness of nonnegative solutions by analysing the relations between all our data: namely, the input/output matrices, on one side, and the labour and final demand vectors, on the other.

Our discussion (and examples) of negative solutions can easily be interpreted in terms of the two neoclassical rules of pricing and choice of techniques. The rule of free goods is violated if the quantity system yields some negative activity levels: on the other hand, the rule of efficiency is violated if the price system yields a semi-negative solution. In the neoclassical approach, only techniques whose associated vectors (x,p) satisfy these two rules are entitled to represent a competitive equilibrium. That is a situation where goods in excess supply become free goods and processes yielding less than the uniform rate of profit are discarded.

Again the contrast between JPS and NJPS is sharp. In the NJPS, if a technique is productive, any vector of final demand can be produced exactly in the required proportions. Supply can always be made to match demand and, for equilibrium, no good must fetch a zero price. On the other hand, as sectors are so specialized that they produce a positive net quantity of at most one good, no two vectors can be such that $M^i(\lambda) > M^j(\lambda)$ implying a positive slope of the transformation curve. This means that there are no choices between alternative activities, so that the only choice is between to produce or not (and if to produce, in what amounts) and the question of efficiency does not arise.

The above neoclassical rules can only be introduced as complementary slackness conditions for the two sets of dual inequalities that should replace the above quantity and price equations. In other words, instead of an algebraic approach, as we could use for the NJPS, JPS requires a more clear topological approach and a choice theoretical framework. In fact the interest of JPS lies in the fact that it enriches the simple structure of the linear production model by introducing the salient feature of the general equilibrium model, essentially nonlinear in nature, that is the interdependence between technological choices and patterns of demand. The equation approach, therefore, is not adequate to determine competitive equilibrium solutions for a general joint production model. It fails whenever at least one of the very general conditions discussed in this section is not satisfied. Actually, we have to expect in general either nonnegative activity levels (or prices) coupled with semi-negative prices (or activity levels, respectively), or even two semi-negative solution vectors. They are both nonnegative only by fluke. However, a semi-negative solution in the activity levels has a different economic implication from a (partly) negative solution for prices. In fact, in the former case, the given technique would violate the condition of equilibrium between demand and supply. In the latter, it would violate the postulate of profit maximization on the part of producers. If we do not care about the determination of general equilibrium positions, but are rather interested in whether a technique could or could not be considered as part of their choice set by producers, it is only the condition that prices are nonnegative that is relevant. (No technique showing semi-negative prices at the ruling rate of profit can belong to the producers' choice set). This would only be a sort of partial analysis focusing on 'observable' as opposed to 'equilibrium' techniques. This point of view is properly Sraffa's (the so-called 'production prices approach'), whereas the general equilibrium viewpoint is von Neumann's. The algebraic approach is seen to provide a clear characterization of the set of technological choices open to maximizing producers.

We have briefly illustrated where partly negative prices arise, by means of a two good - two processes example, as the case where a process, by employing the same amount of labour, produces a larger net output of all goods than the other process. The 2x2 example is a lucky one for direct vector comparison is possible; but unfortunately, vectors cannot be directly compared in the n x n case and we have to consider linear combinations of them in comparing net outputs. A linear combination, with positive weights, of processes taken from a given (A,B,i) is called a <u>subsystem</u> (obviously, (A,B,i) itself forms a subsystem). The following definition borrows the terminology from game theory:

<u>A technique is dominated at a given rate of profit $\bar{r}$, if it contains a subset of processes, which, taken as a subsystem, could produce a larger net output, above $g = \bar{r}$, of at least one good, using the same amount of labour</u>.

The definition emphasizes that the quantity system corresponding to the golden rule rate of growth, $g = \bar{r}$, is only being used here as an auxiliary construction without implying any proper duality relation with the price system. However, the concept of domination, being related with the (auxiliary) quantity side, looks independent from the valuation side and even prior to it. It can be ascertained before determining prices. Nevertheless, the presence of prices that are not all positive, signals that some processes not only are unnecessary, but should be discarded for a correct choice of the technique. This is formalized in the following theorem:

THEOREM 3.1. <u>Prices corresponding to dominated techniques are not all positive</u>.

<u>Proof</u>. In order to make use of a theorem of the alternative, first, we construct an (augmented) net output matrix, $\bar{M}(\lambda) = M(\lambda) - ci$, where c is a predetermined wage basket (and the

wage rate is $w = pc$). Assume $\lambda = (1 - \beta)/\beta$, where β is a root of $\det(\beta B - A)$ as is obvious. Assume that there is a non-trivial solution $\hat{x}$ for $\hat{M}(\lambda)x = 0$ such that $i\hat{x} = \bar{K}$ (total employment).

Whenever (A,B,i) is dominated, there is a subset of processes $j \in J \subset N = 1,2,...,n$, such that for a vector $x^{(j)}: x_j > 0, j$ in J, and $x_k = 0$ otherwise, the following inequality is satisfied: $\hat{M}(\lambda)x^{(j)} \geq \hat{M}(\lambda)\hat{x}$ with $x^{(j)}: ix^{(j)} = \bar{K}$. Therefore, there is a solution to the inequality: $\hat{M}(\lambda)x \geq 0$, and, by a theorem of the alternative, there is <u>no positive</u> left vector p, such that $p\hat{M}(\lambda) = 0$.

The proof of the preceding theorem looks a bit artificial for it specifies the wage basket. Nevertheless, it can be done for any composition of the wage basket, provided consumption of the workers is not a function of the price vector. Therefore, no loss of generality is involved in assuming, as we did, a particular composition of the wage basket. Unfortunately, we have only established that, whenever a technique is dominated, its prices are <u>not all positive</u>. They can be a zero vector, a semi-positive vector or a vector with <u>some</u> negative entries. (Owing to the assumption that the system is productive, prices cannot all be negative.) In the first case, the solution is trivial, and therefore, economically uninteresting. In the second, some goods are free goods, and the slope of the transformation curve is zero or infinity in some directions; the last case we saw to be associated with the phenomenon of negative prices. The following characterizes dominated techniques (i.e. techniques with a transformation curve of a negative slope in the direction of at least one good):

THEOREM 3.2. <u>Whenever the price vector corresponding to a given value of the rate of profit λ, contains some negative entries the auxiliary quantity system can produce no smaller net output by using less labour</u>.

<u>Proof</u>. We use again a fundamental theorem of the alternative, namely the so-called Farkas lemma. We need not transform the original system into an homogeneous one, hence we use matrix $M(\lambda)$ /and not $\hat{M}(\lambda)$/. Then, the theorem assumes that there is no nonnegative solution vector to the equation $\hat{p}M(\lambda) = i$. Therefore, there is a solution to the inequalities $M(\lambda)z \geq 0$ and $iz < 0$. We interpret z as the vector of subsystem multipliers in the following way. Assume that there is a <u>nonnegative solution</u> x^o to $M(\lambda)x^o = c^o$, $c^o \neq 0$ such that total employment is $ix^o = \bar{K}$. Now increase c^o to $c^o + e_i$, where $e_i' = (0,\ldots,1,\ldots,0)$. Then, as $c^o = 0$, $M(\lambda)$ is invertible, and there is a solution $\hat{x}$ to $M(\lambda)x = c^o + e_i$, whence, by linearity,

$$\hat{x} = M^{-1}(\lambda)c^o + M^{-1}(\lambda)e_i$$

and, letting $z = \hat{x} - x^o$, $M^{-1}(\lambda)z = e_i$. We may repeat the exercise for each good to obtain the set of multiplier vectors

$$S = \{z \in \mathbb{R}^n | M(\lambda)z = e_i \ , \ \ i = 1,2,\ldots,n\} \ .$$

Then, there is at least one $z \in S$, such that $M(\lambda)z \geq 0$ <u>and</u> $iz < 0$, or $i(\hat{x} - x^o) = i\hat{x} < ix^o = \bar{K}$.

Corollary: <u>If, at some $r = \bar{r}$, $\hat{p}_i(\bar{r}) < 0$, then the net out-</u>

<u>put of the i-th commodity, above $g = \bar{r}$, could be increased whilst saving labour.</u>

<u>Proof</u>. Let $\Delta c_i' = (0,\ldots,c_i,0,\ldots,0)$, with $c_i > 0$. Then,

$$M(\lambda)z = \Delta c_i \Leftrightarrow z = M^{-1}(\lambda)\Delta c_i \ .$$

Multiplying both sides by the labour input vector i:

$$iz = iM^{-1}(\lambda)\Delta c_i = \hat{p}_i(\lambda)c_i \ .$$

But $\hat{p}_i(\bar{r}) = iM^{-1}(\lambda)$ is negative, therefore

$$iz = i(\hat{x} - x^o) < 0 \quad \text{or} \quad i\hat{x} < ix^o = \bar{K} \ .$$

It is apparent that the condition for a technique to have all positive prices is that it is not dominated. On the other hand, the choice set that will be considered by maximizing producers is the set of non-dominated techniques. Finally, the definition makes clear that domination is a property depending on the value of the parameter λ. Techniques dominated at a given value of r, need not be so at a different value of the rate of profit (a 'truth' that came out of the debate in capital theory, and re-appears in the debate on negative labour values corresponding to positive production prices (Morishima and Steedman, 1976; Wolfstetter, 1976).

Finally, a generalization of the concept of non-dominated technique to allow for rectangular matrices is at the basis of the generalized Non-Substitution theorem we discuss in section 4.5 below. Only non-dominated techniques can hold the Non-Substitution property. Before doing this, we will show that, notwithstanding all the difficulties it runs into in the general JPS, the equation approach has some justification when applied to fixed capital (as by Sraffa, 1960, and Hicks, 1973).

§.4. Fixed Capital as Joint Product and the Non-Substitution Theorem

4.1. The fixed capital case

Both in Von Neumann and in Sraffa, Joint Production is introduced not only to deal with the 'classical case' of mutton-wool type, but also as a method to deal with problems related with the use of fixed capital goods. For example, assume that there is only one machine or plant and only one output (say, 'steel'). In production, the machine at different stages of wear and tear is combined with, possibly, steel as input, and labour, to produce some output. Let us take the following simple profile of the technological processes:

$$M_o \oplus \text{steel} \oplus \text{labour} \rightarrow \text{steel} \oplus M_1$$
$$M_1 \oplus \text{steel} \oplus \text{labour} \rightarrow \text{steel} \oplus M_2$$
$$M_2 \oplus \text{steel} \oplus \text{labour} \rightarrow \text{steel}$$

(and we note that it is a non-homogeneous system). From the point of view of the 'plant', this is installed new at time zero, M_o, and yields a flow of steel at times 1,2,3. Vintage machines are M_1, M_2 which appear both on the output side, next to steel, as improper products, and on the input side as inputs to produce some more steel, till they die out completely, $M_3 = 0$. We may therefore split the general output submatrix B representing the above profile of the working life of the machine into a matrix of outputs as finished goods (in our case, only steel), indicated by B^*, and a matrix of the machine as output, M_1. Likewise, we split the general input sub-matrix A into the matrix of material inputs currently used as circulating capital A^* and the matrix M_o representing the machine coefficients as inputs. Obviously, $A = A^{*} + M_o$ and $B = B^{*} + M_1$. Likewise, labour coefficients have a time index, that is coefficient l_t refers to the process using the plant at the t-th stage of wear and tear. For our simple example, therefore, the matrices will have the following pattern, if we call b_i and a_j output or input of steel

$$B^{*} = \begin{pmatrix} b_1, & b_2, \dots & b_T \\ 0 & & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \quad \text{and } M_1 = \begin{pmatrix} 0 \cdots 0 & & 0 \\ 0 \cdots 0 & \cdots & 0 \\ M_1 & 0 & \vdots \\ \vdots \ \ddots & \vdots & \vdots \\ 0 \cdots & \ddots & \vdots \\ 0 \cdots \vdots & M_{T-1} \ 0 & 0 \end{pmatrix}$$

under the hypothesis that the machine dies out completely at time T. Similarly,

$$A^{*} = \begin{pmatrix} a_o, & a_1, \dots & a_{T-1} \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \qquad M_o = \begin{pmatrix} 0 & \cdots & 0 \\ M_o & & \vdots \\ 0 & M_1 & \vdots \\ \vdots & \ddots & \vdots \\ 0 & \cdots M_{T-1} & 0 \end{pmatrix}$$

and $l = (l_o, l_1, \dots, l_{T-1})$. We can now produce 'vertically integrated profiles'. The machine has a maximal lifetime T and we have the alternative of running it for any number of periods t not exceeding T. Each length t represents an economic lifetime as conceptually different from the physical/technological lifetime which we take as a datum. The choice as to which of them is the most profitable, depends on the 'ruling rate of profit' we take as given in the open system. Obviously, we have to make flows of inputs and flows of outputs, appearing at different stages, conformable by discounting. This can be done by constructing vertically integrated matrices $\hat{A}(\bar{r})$ and $\hat{B}(\bar{r})$ and vector $\hat{l}(\bar{r})$, where each column of the matrix $\hat{A}(\bar{r})$ is characterized by a given time index, say t, and represents the set of coefficients of inputs over a lifetime equal to t, properly discounted to t = 0 (that is to the stage where the new machine is installed). In other words, each column of $\hat{A}(\bar{r})$ is a column of entries of the type

$$\sum_{\tau=1}^{t} a_t (1 + r)^{-(\tau-1)} \quad , \quad \text{for each good } i = 1,2,\ldots,n$$

for a fixed t. These are vertically integrated coefficients as they represent discounted streams of inputs applied to production. In a similar way, we construct matrix $\hat{B}(\bar{r})$ of properly discounted streams of outputs over alternative feasible lifetimes of the machine or plant, and the corresponding vector of labour coefficients $\hat{l}(\bar{r})$.

Example 4.1. t=2 yields

$$p_1 a_{11}(1 + r) + M_o(1 + r) + l_1 w = p_1 b_1 + p_2 M_1$$

$$p_2 M_1(1 + r) + p_1 a_{12}(1 + r) + l_2 w = p_1 b_2 \ .$$

To discount, multiply the second row by $(1 + r)^{-2}$ and the first by $(1 + r)^{-1}$:

$$p_1 a_{11} + M_o + l_1 w(1+r)^{-1} = p_1 b_1 (1+r)^{-1} + p_2 M_1 (1+r)^{-1}$$

$$p_2 a_{12}(1+r)^{-1} + p_2 M_1 (1+r)^{-1} + l_2 w(1+r)^{-2} = p_1 b_2 (1+r)^{-2}$$

Summing up, we obtain

$$p_1(a_{11} + a_{12}(1 + r)^{-1}) + M_o + w(l_1(1 + r)^{-1} + l_2(1 + r)^{-2}) =$$
$$p_1(b_1(1 + r)^{-1} + b_2(1 + r)^{-2}) \ .$$

Now $(a_{11} + a_{12}(1 + r)^{-1})$ is the vertically integrated input coefficient (of the first good) referred to a lifetime t=2, $(b_1(1 + r)^{-1} + b_2(1 + r)^{-2})$ is the corresponding output coefficient, and, finally, $(l_1(1 + r)^{-1} + l_2(1 + r)^{-2})$ is the labour input coefficient.

In this process of vertically integrating, vintage machines, once appearing as outputs as well as inputs, cancel out, under the assumption that vintage capital goods are fully employed so that the output coefficient at time t is equal to the input coefficient at time t+1. In this way, the net output matrix contains only new goods (machines, circulating capital goods and final goods) and is a Z-matrix, because B(r) is diagonal. If it is also productive in the sense of A.4*, prices of final goods are positive (see Schefold).

In the simple examples above, there are three crucial assumptions. (i) There is no intrinsic joint production, (ii) the fixed capital good is not transferable once it has been installed and (iii) for any feasible economic lifetime $t \leq T$, the old machine appearing as last by-product has a zero coefficient. Under these assumptions, we obtain as many new unknowns, i.e. prices as '<u>accounting values</u>' of the t-1 machines at different stages of wear and tear, as new equations. As long as we accept the assumptions above, for any choice of feasible lifetime of the set of machines, we have square input and output sub-matrices for final plus old capital goods. However, we note that the above description can accommodate any behaviour of the efficiency of the plant or machine (see Böhm-Bawerk, 1889; Hicks, 1973; and Sraffa, 1960).

Rectangular matrices appear as soon as we allow the machine, once it has become old, to move from the production of one good to the production of another. That is, only one good is produced at any one time by the machine as in (i) but the good varies according to the age of the machine. Consider the following example:

$$M_o \oplus a_o \oplus wl_o \ominus\!\!\!\!\rightarrow b \ominus M_1$$
$$M_1 \oplus a_1 \oplus wl_1 \ominus\!\!\!\!\rightarrow b \ominus M_2$$
$$M_2 \oplus a_2 \oplus wl_2 \ominus\!\!\!\!\rightarrow c$$

The machine M_o lasts three periods and may produce in the first two periods good b while at stage 2 it may 'migrate' into the production of c. We have three equations, while the unknowns are p_o, p_1, p_2, p_b, p_c and w, with the rate of profit as given. If we take the wage rate as numeraire and assume that in the rest of the system there is an equation where the price p_o may be determined, we still are left with four unknowns or one too many. In other words, the system becomes rectangular with the number of columns (i.e. price equations) smaller than the number of unknowns. This can also be seen by performing the vertical integration over the lifetime of the machine, T=3. We obtain the equation

$$p_oM_o + p_b(a_o + a_1(1+r)^{-1} + a_2(1+r)^{-2}) + w(l_o + l_1(1+r)^{-1} + l_2(1+r)^{-2}) = p_bb((1+r)^{-1} + (1+r)^{-2}) + p_cc(1+r)^{-3}) .$$

where we have reduced the number of unknowns as ageing capital goods disappear in the process of vertical integration but, after assuming that w = 1 and p_o has an equation of its own, we still have at least two unknowns, p_b and p_c and one single equation to determine them. The system is <u>under-de-</u>

termined on the price side but would be over-determined on the quantity side.

This shows that, if capital goods are transferable over their lifetimes, and we consider their profile over a sufficient length of time as to allow them to produce different goods a kind of intrinsic joint production arises. In this case the 'classical' method of counting the (number of independent) equations fails to attain the objective of proving the existence of a mathematical solution and even if it does succeed, it is in general inadequate to prove that the mathematical solution satisfies also the economic criterion of being nonnegative. This was the method of General Equilibrium analysis till the contributions of Wald and von Neumann in the 1930s. The inadequacy is even more obvious for systems of nonlinear equations. Modern theory uses fixed point theorems as well as other topological and algebraic methods.

4.2. Notes on the Non-Substitution theorem

One approach that we have not been able to develop, but which plays an important role in the treatment of open multisectoral models, is based on the theory of extrema of linear functionals on convex sets. It is not logically independent from our development of the simplest applications of a theorem of the alternative, nor from the Perron-Frobenius treatment, but just a different point of view based on the theory of functions and generalizations of classical results on the extrema of functions in analysis. We could have done without referring to it, but it provides the quickest link with linear programming and its use becomes particularly important as we come to the discussion of the so-called Non-Substitution theorem (on which, however, there will only be an heuristic discussion).

Associated with linear production models is the Non-Substitution theorem, originally established by Arrow, Samuelson, Georgescu-Roegen and others. All these seminal papers appear in a volume edited by T. Koopmans (1951). In the standard version, originally proved for a NJP-system (A,I,l), the theorem states that, under certain assumptions, there is one set of activities or processes, as many as the number of products to be produced for final and intermediate uses, that will be 'efficient' irrespective of the <u>composition</u> of 'demand'. Associated with this technique there is a unique positive price vector, which will clear markets and satisfy the competitive condition of zero profits. The theorem has been extended to allow for a positive rate of profit, equal throughout all industrial sectors, and re-named the <u>dynamic</u> Non-Substitution theorem (Samuelson, 1961). Such a Non-Substitution result is very useful as it simplifies calculations for general equilibrium models where a linear productive structure is incorporated and is important in exercises in comparative statics. The assumptions required to obtain this useful result are:

(i) there is only one 'primary factor of production', usually labour;

(ii) no joint production of the 'intrinsic type' and the obvious assumption we already introduced in § 2 of constant returns to scale; and

(iii) the technology is productive.

Joint production of the intrinsic type has to be excluded, both at a given point of time and at different stages of life of capital goods. The model of fixed capital goods not transferable during their lifetimes satisfies the conditions for the Non-Substitution theorem, a result hinted at by Samuelson (1961) and proved later by Mirrlees (1969), Stiglitz (1970) (under some additional assumptions on the efficiency patterns of the capital goods) and finally, in full generality, by Bliss (1975). However we want to arrive at this simple Non-Substitution result starting from the general Joint Production case.

In § 3, we derived the transformation curve as the curve describing for a given open model (A,B,l) the set of net outputs that can be obtained with the technique subject to the constraint of the availability of labour. No restriction was placed there on the gross output vector arising as a result of the productive activity: we allowed for 'intrinsic' joint production. The matrix $(B - (1 + \bar{g})A)$ spans a cone in R^n, representing the set of net output bundles that can be made available above the material requirements to expand the system at balanced growth rate $\bar{g}$. The constraint set by the total availability of the non-reproducible resource 'labour' sets a bound to the levels of the final uses that can be satisfied. The transformation curve is piecewise linear, with a finite number of kinks, due to the assumption of a linear, constant coefficient structure of the production relations. The rate of transformation between any two goods is therefore constant over whole ranges of net output compositions, as in the example below (see Figure 3) where the transformation curve is a

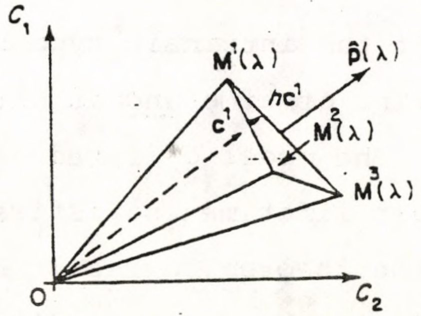

Figure 3: Non-substitution with intrinsic joint production: $\overline{/M^1(\lambda), M^3(\lambda)/}$ is the efficient technique (as $h > 1$).

broken line. Over any given range portion of this broken line, prices are uniquely determined. Therefore, as long as the vector of final uses is restricted to vary only within a given range, one and <u>the same price vector</u> implicit in the technique will be able to clear the markets.

The Non-Substitution theorem deals with 'efficiency'. If the technique is efficient, then there is no need to change it when c changes. In other words, if the technique is efficient in producing a given vector of final demand, it will also be efficient in producing any other vector 'close enough' not to fall out of its own cone of net output. The same price vector (if it was a competitive 'equilibrium' for the original vector c) would still be an equilibrium for a number of alternative configurations. Let (A,B,l) again be a <u>technique</u>, where a competitive set of prices associated with (A,B,l) is a nonnegative vector p such as to ensure a uniform rate of profit, say $\bar{r}$. By definition, the following equation for p is satisfied:

$$p(B - (1 + \bar{r})A) = wl \qquad (4.1)$$

where we may choose w = 1 for the sake of simplicity. Assume now that for (A,B,l) at the rate of growth $\bar{g}$, the given vector of final demand c is being met, i.e.

$$(B - (1 + \bar{g})A)x = c \tag{4.2}$$

with nonnegative vector x. Finally, assume that for price vector p,

$$p(B' - (1 + \bar{r})A') \leqq wl' \tag{4.3}$$

at the same rate of profit and for all other available techniques (A',B',l'). This implies that technique (A,B,l) is a competitive equilibrium technique. If we now assume that the set of net output vectors, above $\bar{g}$, producible by techniques (A',B',l') may also be produced by (A,B,l) then the same (A,B,l) is a competitive equilibrium technique for any vector c in this set. (This is a simplified version of Johansen's (1972) General Non-Substitution theorem). The importance of the theorem lies in that it does not require the Non-Substitution property to hold for all compositions of final demand but only for a subset.

It is clear that the above problem can be interpreted in terms of linear programming. Technique (A,B,l) is formed by the set of K activities that fetch positive activity levels in the optimal vector $\hat{x}$ for the linear program min lx s.t. $(\hat{B} - (1 + g)A)x \geq c$ and $x \geq 0$ while the price vector p is the shadow price vector solving the dual program max pc s.t. $p(\hat{B} - (1 + r)\hat{A}) \geq wl$ for $\bar{r} = g$, and rectangular matrices B and A are formed by augmenting B with B', and A with A' respectively.

In other words, matrices (A,B) form a basic solution for the primal programme. This solution may or may not be degenerate according as to whether the number of activities in (A,B), i.e. the number of columns k is less than or equal to the number of goods n. It can be proved that if the solution is a basic non-degenerate one, then except for flukes, the transformation curve has pieces that are flat, that is pieces where the rates of transformation are constant, and therefore prices are determined independently from the composition of the final demand (provided the latter does not vary to the point of 'falling out' of the range on which the technique is optimal). A sufficient condition for this can be stated:

THEOREM 4.1. If technique (A,B) and l has a 'competitive equilibrium' price vector, satisfying (4.1)-(4.3) corresponding to a final demand vector c that belongs to the interior of the cone $\Delta(B - (1 + \bar{g})A)$, then there is a whole set of vectors {c'} for which that same technique represents an optimal solution.

All cases where fixed capital goods are transferable over their lifetimes between the production of various goods fall under this theorem. The vertically integrated input and output matrices referred to profiles where the capital good is transferred, do show more than one positive output entry being positive and therefore are assimilated to the Johansen case.

On the other hand, the theorem (with its local validity) shows that the original, much stronger, version springs from a peculiar property of NJP techniques. If they are productive (i.e., if the feasible region defined by the constraints to the primal programme is non-empty), they are able to produce not just a set of final vectors, c, but final goods in any desired proportions. The whole non-negative orthant (and not just a portion of it) is contained in the net output cone of a NJP technique, so that if it is efficient, it is efficient for the production of any other vector c. In the wording used before, the transformation curve associated with the technique has no kinks and is a hyperplane over the nonnegative orthant with a unique outward direction vector representing competitive prices.

When fixed capital goods are non-transferable, we know that the output matrix $\hat{B}(r)$ is a matrix of coefficients where for each capital good there are streams of a single homogeneous good running. Joint production over lifetime is by assumption banned and the vertically integrated net output matrix $/\bar{\hat{B}}(r) - (1 + r)\hat{A}(r)\}$, if at all productive, contains the nonnegative orthant. Therefore, if a particular choice of lifetimes is efficient for a given bundle of goods c, that same choice will also be efficient for any alternative bundle (Bliss's Dynamic Non-Substitution Theorem).

4.3. Generalized Perron-Frobenius theorems

We have shown by means of very simple examples how theorems of the alternative (and separation theorems) are useful tools to prove the existence of nonnegative solutions. We have dealt briefly with a related result, the Non-Substitution theorem, stating that, under certain conditions, in an open system of production equilibrium prices can be determined independently from final demand. This property is held by a particular choice of processes belonging to the set of techniques with which nonnegative prices are associated. From this point of view, the Non-Substitution theorem is an optimization result linked both with the theory of (mathematical) programming, and with the general body of existence results.

By taking as a prototype a version of Johansen's theorem (1972), our treatment should have made clear that the really crucial assumptions for the Non-Substitution theorem to hold, are a linear structure of production relations and the existence of a unique primary factor. On the contrary, the assumption of no joint production can be disposed of if we are satisfied with local results. Our statement on dominancy provides a necessary condition for a general joint production system to have the Non-Substitution property for only techniques satisfying that condition may represent a competitive equilibrium. This links together two results whose derivation we have sketched.

As an alternative approach, we have been analysing the properties of the characteristic equation associated with the input matrix A. A mathematical result we found very useful in this context, the Perron-Frobenius theorem, states certain properties of the set of eigenvalues of the nonnegative matrix A, and of the corresponding left- and right-eigenvectors, $\hat{p}$ and $\hat{x}$, respectively.

These results allow us to establish readily under which conditions general homogeneous systems like $x = (1 + g_{max})Ax$ and $p = (1 + r_{max})pA$ have a nonnegative solution in r_{max}, g_{max}, x and $\hat{p}$. The triple $(r_{max} = g_{max}, \hat{x}, \hat{p})$ represents a Golden rule balanced growth path. This existence result, originally associated with the name of John von Neumann (1937), is obtained via a mathematical approach of an algebraic, and not of topological, nature. Von Neumann's proof deals with a more general structure, where output matrix I is replaced by B allowing for joint production and matrices are rectangular, and makes use of the mini-max theorem in the theory of games and the earliest version of what came to be known as Kakutani's Fixed Point theorem.

We then have shown the relevance of the above theorem in diagonalizing an open system (A,I,l), a procedure first introduced by R.M. Goodwin (1976). The PF theorem ensures that at least one of the eigensectors (or diagonal sectors) is real. Closing the loop, we have briefly shown the relevance of a corollary of the theorem for establishing the existence of nonnegative solutions still for the open system and in deriving their behaviours as the parameter (r and/or g) changes. The discussion of the roots of the characteristic polynomial seems to provide a more homogeneous and unifying approach to a variety of problems associated with the linear multisectoral models. This provides a good motivation for looking for generalizations of the Perron-Frobenius theorem. This can be done essentially in three directions:

(i) relaxing the assumption that A is square, i.e. have a system whose coefficients are the rectangular matrices

(A,I) by taking B as a matrix of output without joint production;

(ii) we may keep the square dimension and introduce only the general joint production matrix B;

(iii) finally, we may consider (as the most general case) a system with both joint production and rectangular dimensions.

With the approach we have applied consistently throughout this paper, the hypothesis under (iii) would imply treating systems in the von Neumann format with equations rather than inequalities. By introducing appropriate conditions on the matrices, this <u>may be done</u>, and, for purposes of calculating the balanced growth path, <u>it is done</u> (see for instance Thompson and Weil, 1971). The trouble is that, usually, a rectangular dimension renders one of the above dual problems solvable while the other becomes trivial, so that proper duality is lost. Therefore, the equation approach as applied to solving a von Neumann model both in price - rate of profit <u>and</u>, simultaneously, in quantity - rate of growth, seems not very fruitful. (For a full explanation of the analytical reasons, that are rooted in the basic theory of vector spaces, see Mangasarian (1971), and Punzo (1980).)

We have seen that, for the class of models where the only phenomenon allowed of joint production is of non-transferable fixed capital goods, the format of the matrices is naturally square (for any choice of lifetimes of the fixed capital goods, there are as many equations as unknowns). Therefore, the need for assuming rectangular matrices really only arises with transferable fixed capital goods *and* with joint production of the intrinsic type. It is in these two (important) cases that we have no analytical justification, in principle, for taking square matrices, for there is no longer correspondence between processes and products. (Nevertheless, a non-analytical justification can be found in Marshall (1920, Book V, Chapter 6) and Sraffa (1960, p. 43, footnote).) However, we gain something very important from assuming square matrices. In fact, it can be formally proved that, in this case, the systems

$$Ax = \beta Bx$$

and

$$pA = \beta pB$$

may be solved for the same scalar β. Here we can only state one simple result that may be obtained (for a more comprehensive discussion, the reader is invited to consult the references).

THEOREM 4.2 (*Generalized Perron-Frobenius theorem*): *Assume there exists a square matrix X such that either $A = BX$, or $B = AX$. Let in the first case the rank of B be equal to n and in the second case the rank of A be n. Then*

(i) *the set of eigenvalues of X (i.e. the spectrum of X) is also the spectrum of A versus B (i.e. the roots satisfying the generalized determinantal eqn. $|A - \lambda B| = 0$), or of B versus A, respectively.*

(ii) *Let X be a nonnegative matrix. Then, it has a maximal nonnegative eigenvalue β and correspondingly nonnegative left and right eigenvectors. These triples solve the dual homogeneous equations for the system (A,B).*

This statement makes clear that, in order to obtain a generalized result of the Perron-Frobenius type, the two sets of (input and output) proportions must not be independent. In fact, they are required to be related by a third nonnegative matrix X so that either the cone of input proportions is included in the cone of output proportions or vice versa. The NJPS is a special case, when the requirement is naturally satisfied.

In the terminology introduced by Hicks (1965), the two cases considered in the statement above are named 'simple backward' and 'simple forward narrowing' respectively. In either case, the generalized eigensolution$(\hat{\beta}, \hat{x}, \hat{p})$ represents a balanced growth path with full utilization of produced goods. For details and a number of other applications to linear models see Punzo (1978,1980).

§.5. Concluding Remarks

In a paper which aims to survey an exciting but controversial area of economics it is not possible to go into too many details. In particular, important results relating to control, stability and observability of linear economic systems have been completely omitted. We have nowhere introduced the concept of the state space in our discussions (and it is a concept widely used within the framework of the formalization we introduced in the opening paragraphs of this paper; hence not confined to linear systems). It is, nevertheless, implicit in the particular way in which we have, following von Neumann and Sraffa, dealt with the problem of fixed capital in joint production systems. Recent advances in linear (and nonlinear) systems theory enable one to dispense with the somewhat restrictive assumption (that we have employed) of representing linear operators as <u>square</u> matrices. Luenberger's elegant results on the theory of observers (cf. Velupillai, 1979) and Kalman's development of the theory of filtering, controllability and observability are, of course, highly relevant in <u>any</u> theory of <u>linear</u> models.

It is possible that a paper on multisectoral models and joint production could have been better unified in terms of the tools and concepts of systems theory. In economics, in particular, there is a strong case to be made for such an approach, not least because of the obvious possibilities to consider problems in the theory of economic policy (in the form in which Frisch, Bent Hansen and Tinbergen cast it) also as special cases. If such a unifying framework, based on a formal system theoretic paradigm, is used, some of the controversy surrounding problems in the theory of growth, value and distribution could perhaps be defused. This may, however, be an overly mechanistic view of a subject which is essentially a moral science.

In fact Miyao's (1977) reformulation of certain Sraffian themes (linearity of the wage-profit curve, existence of a standard commodity, etc.) is in terms of a generalized controllability matrix (cf. Miyao, op. cit., Lemma 3, definition 27, p. 157). The similarity is, ultimately, due to the essential elements that characterize standard and finite (integer) dimensions.

Finally, important developments in game theory, also of relevance to linear models with joint production, have also been omitted. We have, here, in mind Shapley values and their extensions. Ultimately these are issues that relate to the characteristic Austrian problem of imputation. Mathematically they also can be reformulated as nonlinear eigenvalue problems.

It may well be that the fundamental problem of economics is not the allocation of scarce resources to achieve given ends, but the imputation of joint costs in a system of joint production. The classical economists inherited the latter tradition from the great continental Political Arithmeticians. The Austrians and the Swedes developed it further. We have tried to consider one half of the nexus--the problem of joint production. The whole, however, is composed of more than two halves.

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