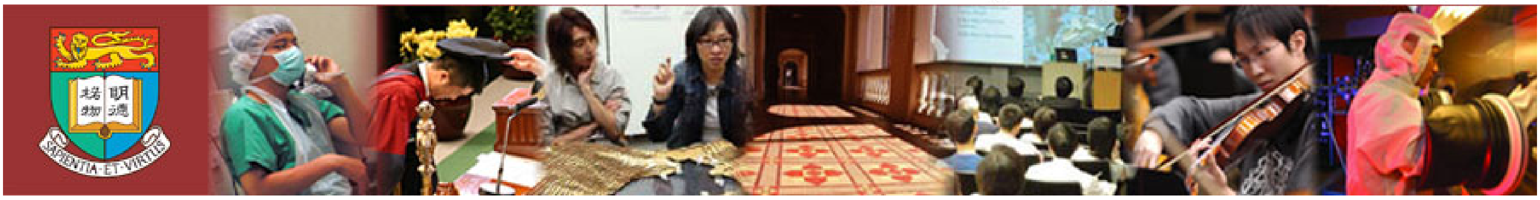

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Breathers and rogue waves for a third order nonlocal partial differential equation by a bilinear transformation

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ABSTRACT

Breathers and rogue waves as exact solutions of a nonlocal partial differential equation of the third order are derived by a bilinear transformation. Breathers denote families of pulsating modes and can occur for both continuous and discrete systems. Rogue waves are localized in both space and time, and are obtained theoretically as a limiting case of breathers with indefinitely large periods. Both entities are demonstrated analytically to exist for special classes of nonlocal equations relevant to optical waveguides.

Keywords: Nonlocal evolution equations; Rogue waves.

1. Introduction

Nonlocal equations refer here broadly to classes of partial differential equations where the spatial rates of change of a function at any point, as well as its time evolution, are related to values at a finite distance from the point under investigation. Such equations have been studied frequently in applied disciplines, e.g. spatiotemporal solitary waves [1] and Hermite-Gaussian beams [2] are considered in nonlocal optical media.

Theoretically, many equations in the theory of nonlinear waves and solitons can be studied through the perspective of nonlocal equations. As illustrative examples, a coupled system of Burgers equations can be rewritten as a single component differential-integral Burgers equation with a translational kernel [3]. Similarly, Boussinesq equations with rational nonlinearity can also be formulated as a single component nonlocal form and soliton expressions are deduced [4]. Indeed many cases studied consist of a diffusion or Schrödinger type differential equation combined with an integral operator with a translational kernel, e.g. (A = complex valued envelope),

$$iA_t + A_{xx} + N(I)A = 0, \quad I = \text{intensity} = |A|^2, \; N(I)A = A \int R(x' - x)I(x')dx' , \qquad (1)$$

where $R(x)$ is a response function. If $R(x) = \delta(x)$ (a delta function), this nonlinear term will reduce to a local evolution model, the conventional nonlinear Schrödinger equation,

$$iA_t + A_{xx} + |A|^2 A = 0. \qquad (2)$$

Eq. (1) and its variants occur in many applications, e.g. in life science phenomena where the emergence and evolution of biological species are critical issues [5].

The focus here is on a class of nonlocal nonlinear Schrödinger equations. The goal in earlier works is to examine collapse prevention and soliton stabilization [6]. Recently the attention tends to be placed on equations with 'parity-time symmetry' [7]. These '*PT*-symmetric' systems are important as theoretically the self-induced potential is then invariant. In terms of applications, wave propagation in symmetric waveguides and photonic lattices has been demonstrated experimentally [7]. An external potential can also be incorporated [8]. A general framework for coupled nonlinear Schrödinger equations can be formulated [9]. Finally, this whole idea can be extended to equations with two or more spatial variables [10].

The objective here is to show that both breathers and rogue waves, intensively studied topics recently, can be derived analytically for these nonlocal nonlinear Schrödinger equations. Breathers are pulsating modes and rogue waves are unexpectedly large amplitude displacements from a tranquil background [11]. Rogue waves were first noted in the oceans by sailors and researchers in fluid mechanics, but are now being pursued in optics and other fields as well [12]. The Darboux transformations have been frequently used for computing rogue waves for many models [13], e.g. the Hirota equation, a member from the nonlinear Schrödinger family with third order dispersion [14]. Recently, the bilinear method has also been shown to be applicable as well, e.g. for the derivative nonlinear Schrödinger equation [15].

The structure of this paper can now be explained. The new nonlocal, third order partial differential equation is formulated and the background for the bilinear transformation is reviewed (Section 2). The expansion scheme for a breather is given and the rogue wave mode is derived by taking a long wave limit (Section 3). The analogy with other evolution equations exhibiting rogue wave modes is highlighted and conclusions are drawn (Section 4).

2. A nonlocal third order nonlinear Schrödinger equation

Consider the nonlocal equation

$$iA_t + icA_x + A_{xx} + \sigma A\left[A(-x,t)\right]^* A + i\lambda A_{xxx} + i\lambda 3\sigma A\left[A(-x,t)\right]^* A_x = 0, \tag{3}$$

where A is a complex valued wave envelope, the parameters λ, σ, and c are real and '*' denotes the complex conjugate. If $c = \lambda = 0$, Eq. (3) reduces to a nonlocal equation studied earlier [7]:

$$iA_t + A_{xx} + \sigma A[A(-x, t)]^*A = 0, \tag{4}$$

which possesses the usual appealing features of soliton equations, e.g. a Lax pair and an infinite number of conservation laws. Furthermore, the elegant mechanism of inverse scattering is also applicable to Eq. (4). The conventional (local) nonlinear Schrödinger equation is recovered if $-x$ is replaced by x in Eq. (4).

The conventional nonlinear Schrödinger equation has an 'integrable' higher order extension which incorporates a third order derivative, commonly known as the Hirota equation, after the equation was first proposed by R. Hirota in 1973 [16]:

$$iA_t + A_{xx} + \sigma AA^*A + i\lambda(A_{xxx} + 3\sigma AA^*A_x) = 0. \qquad (5)$$

It will be natural to search for the nonlocal extension of Eq. (5) and we propose that Eq. (3) will be the ideal candidate. It will be illuminating to demonstrate that breathers and rogue wave modes also exist for this class of nonlocal evolution equations with peculiar x, t symmetry. Future works will focus on attempts to establish the existence of an infinite number of conservation laws and other benchmarks of 'integrability'. Searching for an inverse scattering transform will also be highly beneficial [7]. A Lax pair, i.e. a compatible pair of linear equations, will identify the eigenfunctions to correlate initial data with scattering data, and will confirm the 'integrability' of the system.

An effective method to find multi-soliton solutions as shown over the years is the bilinear method [17]. We shall prove here that this bilinear method is also applicable to obtain the breathers and rogue waves for nonlocal equations. A dependent variable transformation in rational form is first implemented (ρ = a real amplitude parameter, ω = real angular frequency):

$$A = \rho \exp(-i\omega t) g/f, \quad g \text{ complex and } f \text{ real.} \tag{6}$$

As two independent variables, g and f, are introduced to solve for one unknown (A), one can impose without loss of generality that

$$f(-x,t) = f(x,t). \tag{7}$$

The evolution model Eq. (3) is now rewritten as bilinear equations

$$[iD_t + i(c + 3\lambda\sigma\rho^2)D_x + D_x^2 + i\lambda D_x^3] g \cdot f = 0, \tag{8a}$$

$$D_x^2 f \cdot f = \sigma\rho^2 \left[gg(-x)^* - f^2 \right], \qquad \omega = -\sigma\rho^2, \tag{8b}$$

where D is the bilinear operator:

$$D_x^m D_t^n g.f = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n g(x,t) f(x',t') |_{x'=x, t'=t} . \tag{9}$$

Breathers localized in time (t) but periodic in space (x) will be generated in the next section.

3. Breathers and rogue waves

Following our earlier works [15] in the literature, a family of analytical solutions termed breathers (pulsating modes) can typically be obtained through an expansion scheme ($\zeta^{(1)}$, $\zeta^{(2)}$ being arbitrary complex phase factors):

$$g = 1 + a_1 \exp(ipx - \Omega t + \zeta^{(1)}) + a_2 \exp(-ipx - \Omega t + \zeta^{(2)}) + M a_1 a_2 \exp(-2\Omega t + \zeta^{(1)} + \zeta^{(2)}), \quad (10a)$$

$$f = 1 + \exp(ipx - \Omega t + \zeta^{(1)}) + \exp(-ipx - \Omega t + \zeta^{(2)}) + M \exp(-2\Omega t + \zeta^{(1)} + \zeta^{(2)}). \quad (10b)$$

Due to the special nonlocal nature of Eq. (3) one must restrict the parameters p, Ω to be real. The parameters a_1, a_2 (complex) and c, M (real) are given by

$$c = \lambda(p^2 - 3\sigma\rho^2), \quad (11)$$

$$a_1 = a_2 = -\frac{p^2 - i\Omega}{p^2 + i\Omega}, \quad M = \frac{2\sigma\rho^2}{2\sigma\rho^2 - p^2}. \quad (12)$$

Hence one concludes that $|a_1| = |a_2| = 1$, and the connection between Ω and p, i.e. the dispersion relation, is then

$$\Omega^2 = p^2(2\sigma\rho^2 - p^2). \quad (13)$$

The constraint $2\sigma\rho^2 > p^2$ must hold for Ω to be real and $M > 1$. The breather can also be expressed in terms of hyperbolic and trigonometric functions as (g_1 complex, f_1 real and as illustrative example here $\zeta^{(1)} = \zeta^{(2)} = \zeta$ (real))

$A = \rho \exp(-i\omega t) g_1/f_1$, $f_1 = M^{1/2} \cosh\Theta + \cos(px)$,

$g_1 = M^{1/2} [\cos^2\beta \cosh\Theta + \sin^2\beta \sinh\Theta + i \cos\beta \sin\beta (\cosh\Theta - \sinh\Theta)]$

$+ \cos(px) (\cos\beta + i \sin\beta)$, where β, Θ, t_0 are defined by

$a_1 = \exp(i\beta)$, $\Theta = \Omega(t - t_0)$, $\exp(\Omega t_0) = M^{1/2} \exp(\zeta)$.

A popular method to derive the rogue waves theoretically is the Darboux transformation [14], but Eqs. (6–10) have demonstrated that the bilinear method is a feasible scheme in computing breathers (and subsequently rogue waves). This alternative is especially valuable as most soliton systems possess bilinear forms. The breather is given analytically as a combination of Eqs. (6, 10a, 10b, 11, 12, 13), and is illustrated in Fig. 1:

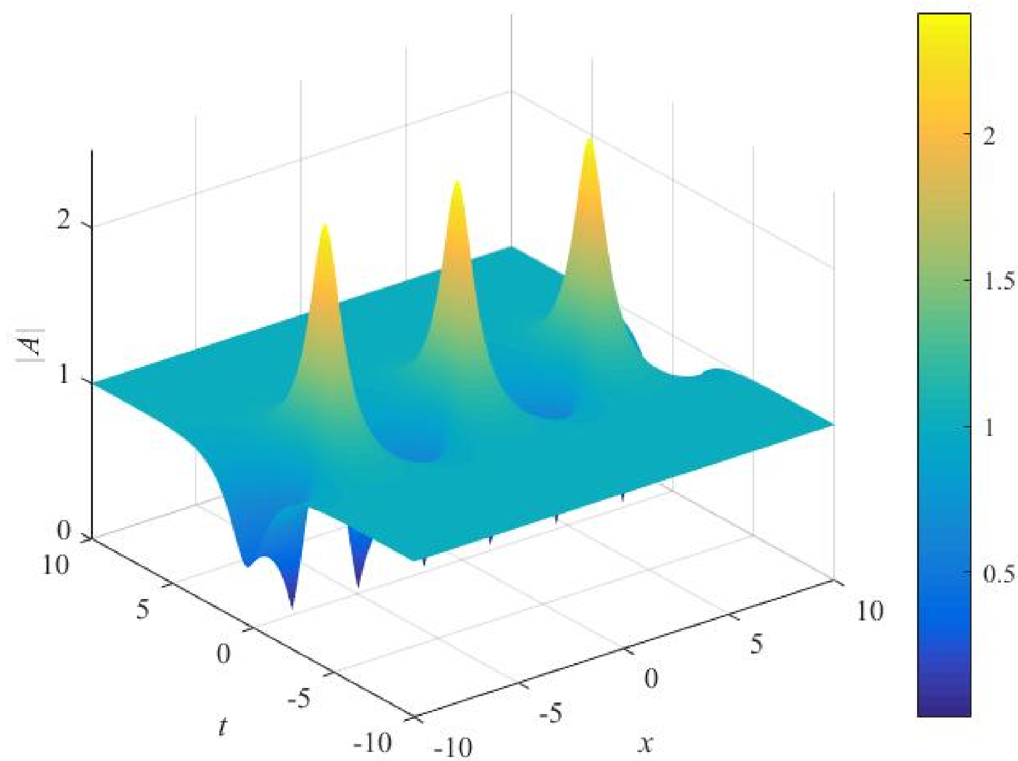

Fig. 1. Breather solution as given in Eqs. (6, 10a, 10b, 11, 12, 13) with σ =1, ρ = 1, λ = 1, p = 1. The amplitude is periodic in x but localized in t.

To generate the rogue waves, a long wave limit (p tending to zero) is now taken with the provision

$\exp(\zeta^{(1)}) = \exp(\zeta^{(2)}) = -1.$

The rogue wave mode as an exact solution of Eq. (3) is given by

$$A = \rho \exp(i\sigma\rho^2 t) \times \left[1 - \frac{2(1 + 2i\rho^2\sigma t)}{\rho^2\sigma\left(x^2 + 2\rho^2\sigma t^2 + \dfrac{1}{2\rho^2\sigma} \right)} \right]. \quad (14)$$

This exact solution is structurally very similar to the Peregrine breather / rogue wave of the intensively studied nonlinear Schrödinger equation, and is localized in both space and time (Fig. 2):

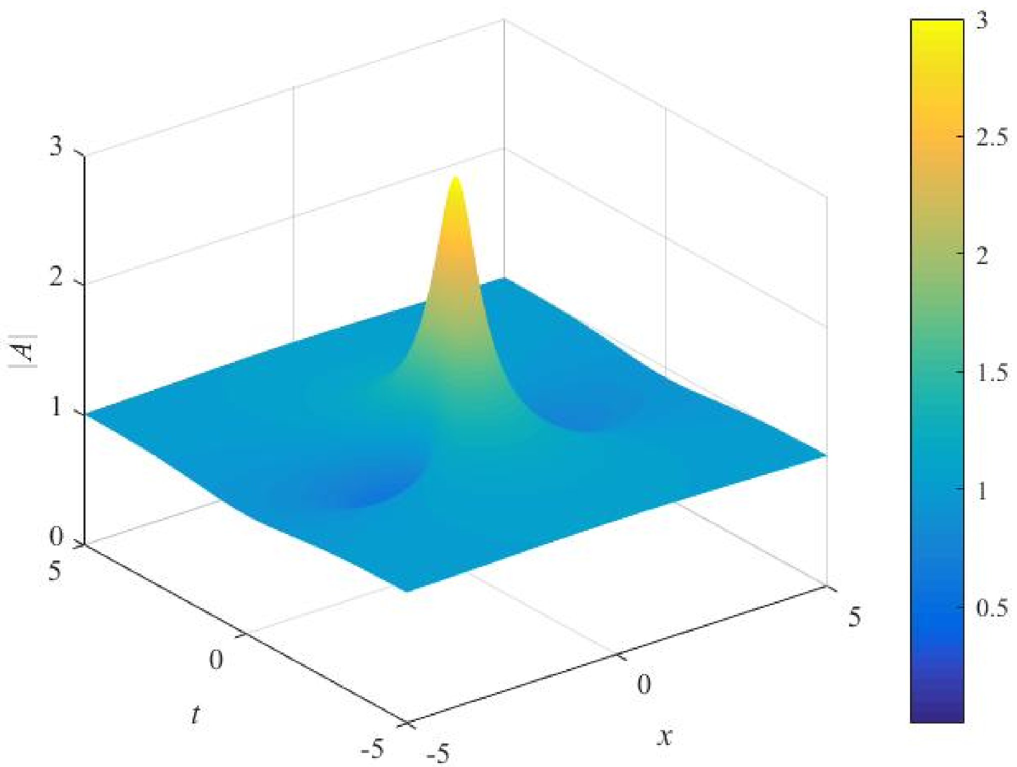

Fig. 2. Rogue waves as given in Eq. (14) with σ = ρ = 1

The main contrast with the nonlinear Schrödinger equation (NLSE), in terms of properties of breathers and rogue waves, is that the expansion scheme of breathers for NLSE permits arbitrary complex wavenumbers and frequencies, whereas p and Ω must be real for the present nonlocal equation (Eqs. (10a, 10b)).

Properties of the rogue wave in terms of the input parameters

Similar to the scenario for the local nonlinear Schrödinger equation (Eq. (2)), Eq. (14) is nonsingular only for $\sigma > 0$. This constraint also guarantees a real Ω in the limit p tending to zero in Eq. (13), consistent with our earlier assumption. The amplification ratio, the ratio of the maximum displacement to the background plane wave, is three again [12]. The parameter c usually denotes physically a measure of the group velocity of the wave packet. Eq. (11) then implies that a constraint must exist between the sign of the cubic nonlinearity (σ), the third order dispersion (λ), the group velocity (c) and the background amplitude (ρ) for the rogue wave to exist. Unlike many similar studies for rogue waves, one cannot draw a direct link between the existence of rogue wave modes and the condition of instability of the plane wave (modulation instability) [15, 18].

4. Discussion and conclusion

Breathers in both continuous and discrete settings are intensively studied in many fields, e.g. optical lattices, Josephson junctions and Bose-Einstein condensates [19]. For a discussion on terminology, *t* in Eqs. (2, 3) will be termed the 'propagation variable' here while *x* will be known as the 'transverse variable'. The associations of *t*, *x* with time and space might be confusing, as they take up different roles in fluid mechanics and optics. Indeed *t* will be slow time / spatial variable, while *x* will denote group velocity coordinate / retarded time in fluid mechanics / optics respectively [20, 21]. In the literature, a breather periodic in the 'transverse variable' is termed an 'Akhmediev' breather [22], and thus the solution in this paper belongs to the category of 'Akhmediev' breather. Because of the nonlocal nature of the governing system as defined by Eq. (3), a Kuznetsov-Ma breather periodic in the 'propagation variable' is not possible [22].

Both breathers and rogue wave modes for a class of nonlocal, third order partial differential equation equations are derived by the bilinear method. These equations arise as counterparts of intensively studied evolution models in soliton theory which satisfy certain symmetry requirements of the independent variables. The motivation of the present work comes from the search for new families of nonlocal equations which permit analytical treatment. Further directions of research will hinge on deriving nonlocal derivative nonlinear Schrödinger and higher dimensional (Davey-Stewartson) equations. Previous studies of nonlocal equations have proven their values for analytical topics in physics like modeling radiating gas and optical waveguides. Hence studying these nonlocal nonlinear Schrödinger models will definitely be fruitful directions in applied mathematics.

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