Stochastic travelling waves driven by one-dimensional Wiener processes

by

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Contents

Acknowledgments

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I also wish to express my gratitude to Dr. Roger Tribe, certainly without whom this thesis wouldn't have been possible, for his continued support, encouragement, enthusiasm and friendship, as well as teaching me the beauty of being a probabilist.

If I have seen further than others, it is by standing upon the shoulders of giants.
Sir Isaac Newton

Imagination is more important than knowledge. For knowledge is limited to all we now know and understand, while imagination embraces the entire world, and all there ever will be to know and understand.
Albert Einstein

Declaration

This thesis, and the material in it, is my own work. It has not been submitted for a degree at any other university.

Abstract

In this thesis we aim to show the existence of a stationary travelling wave of a generalised stochastic KPP equation driven by a one dimensional Wiener process. Chapter 1 discusses the background of the deterministic KPP equation and some interesting properties when consider Stratonovich noise and convert to the Itô noise. Chapter 2 covers preliminaries and background information that will be required throughout the entire thesis. Chapter 3 defines stretching, an important concept throughout this thesis. We show that for any two initial conditions, one more stretched than the other, stretching is preserved with time. We also show that stretching defines a pre-order on our solution space and that the solution started from the Heaviside initial condition converges. In Chapter 4 we show that the limiting law lives on a suitable measurable subset of our solution space. We conclude Chapter 4 by proving that the limiting law is invariant for the process viewed from the wavefront and hence a stationary travelling wave. Chapter 5 discusses domains of attraction and an implicit wave speed formula is shown. Using this framework we extend our previous results, which have concentrated upon Heaviside initial conditions, to that of initial conditions that can be trapped between two Heaviside functions. We show that for these "trapped" initial conditions, the laws converge (in a suitable sense) to the same law as that for the Heaviside initial condition (up to translation). Chapter 6 discusses phase-plane analysis for our equation and we restate the stretching concept within this framework.

Chapter 1

Background

1.1 Introduction

Wave front solutions to scalar reaction diffusion equations have been studied in the context of mathematical biology for many years (for further details the interested reader is referred to [19] and references therein for an excellent introduction into this topic). Work in this area dates back to the papers of Kolmogorov et al [14] and Fisher [10] in 1937 on what is now known as the Kolmogorov-Petrovskii-Piscuinov (KPP) equation

$$\frac{\partial u}{\partial t} = u_{xx} + f(u) \tag{1.1}$$

with $f(u) = ku(1-u)$. A wave front solution to this equation has the form $u(t,x) = U(x-ct)$ where c denotes the wave speed and the function $U : \mathbb{R} \to [0,1]$ is smooth and satisfies,

$$\begin{aligned}
&U_{zz} - cU_z + f(U) = 0;\\
&U(-\infty) = 1,\ U(\infty) = 0;\\
&U_z(z) < 0.
\end{aligned}$$

Examples of scalar reaction diffusion models of wave phenomena include:

- Insect and animal dispersal (See [12], [22]).

  In these applications, u describes the density of a species in space and time and the waves correspond to the spread of the species. These types of equation have been primarily used for modelling and managing ecological invasions.

- Combustion equation (See [21], [34]).

  In these applications u describes the temperature in space and time and the wave corresponds to the spread of the heat through the domain. In this case f is of a special form where there exists a constant $\sigma \in (0, 1)$ such that $f(u) = 0$ for $u \in [0, \sigma]$ and $f(u) > 0$ for $u \in (\sigma, 1)$. The constant σ is known as the ignition temperature.

- Wound healing (See [26]).

  Here u describes the cell density and the wave corresponds to the pattern of wound healing.

- Propagation of Calcium ions (See [28]).

  Here u describes the concentration of Ca^{2+} ions which are important intracellular messengers. The waves correspond to intra- and intercellular movement of these ions. It is thought that these waves serve to synchronise the actions of a cell or group of cells.

Many coupled systems of reaction diffusion equations have been used to study wave front phenomena in the dynamics of "multiple species". In these systems, one or more of the equations will include a diffusion term. Examples include:

- Waves of pursuit and evasion in predator-prey systems (See [5]).

  In this instance two coupled equations are generally used, one describing the predators and one the prey, both of which will include a diffusion term, although the rates of diffusion will generally be different. The waves correspond to pursuit of prey by predators and evasion of predators by prey.

- The spatial spread of epidemics. Here compartmental models are used which have at least two compartments: susceptible and infected. Some examples of epidemics that can be modelled by reaction diffusion systems are:

- Swine flu now taking the world by storm;
- Rabies epizootic spreading across Europe in the late 1990's;
- The black death.

For more applications the interested reader is referred to [13].

Stochastic partial differential equations (SPDEs) are a key ingredient of mathematical modelling in fields like physics, chemistry, biology and engineering. Many problems, like wave propagation in random media, turbulence, dispersion of flows in porous media, evolution of biological populations, can be now understood using SPDEs. More recently the range of applications has been extended to oceanography, image analysis and mathematical finance among others. The introduction of the noise term might arise from allowing for the uncertainty in the coefficients in the equation or any other fluctuation in a coupled quantity.

In this thesis we treat only the case of non-spatial noise where the random function affects all values of $u(t,x)$ equally, for example temperature fluctuations. We do not cover noises which are inhomogeneous in space but feel that some of our results, for example Corollary 31, would pass over unchanged to a white in time, spatially translation invariant noise; Other results, for example Corollary 33, we feel would not.

1.2 Deterministic Review

Here we consider a simple model equation, that is a scalar reaction-diffusion equation of one-space dimension $x \in \mathbb{R}$ for $t \geq 0$

$$u_t = u_{xx} + f(u) \tag{1.2}$$

with the condition

$$f(0) = f(1) = 0 \tag{1.3}$$

so that solutions $u(t,x)$ are sought with values in $[0,1]$.

Definition 1. ***KPP type***

We say f is of the KPP type *if:*

$$f'(0) > 0,\ f'(1) < 0,\ f(x) > 0 \ \textit{for}\ x \in (0,1).$$

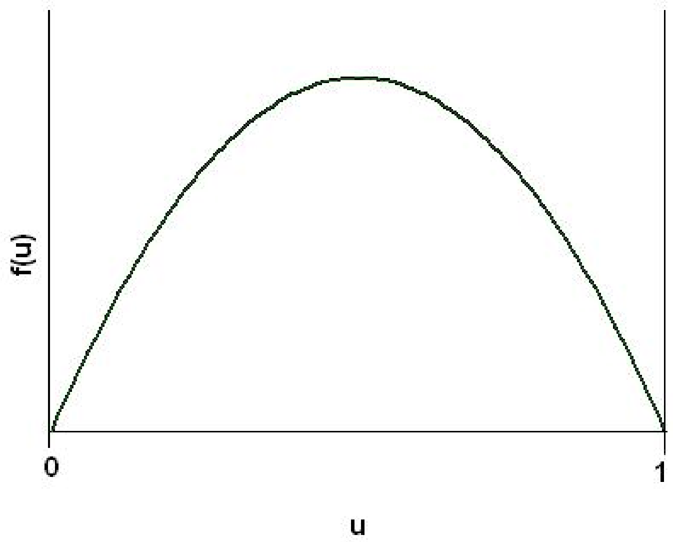

Figure 1.1: A typical example where f is of KPP type.

Remark. In the case that f is of KPP type, there exists a family of travelling wave speeds $c \geq c_{critical} = 2\sqrt{f'(0)}$ and in [14], it was shown that starting from the Heaviside initial condition ($\mathbb{I}_{[x\leq 0]}$), the limiting solution satisfies $c = c_{critical}$. Bramson [3] subsequently extended this analysis to give necessary and sufficient criteria on the initial condition to approach one of the travelling waves.

Definition 2. ***Nagumo type***

We say f is of the Nagumo type *if:*

$f'(0) < 0,\ f'(1) < 0,$ *there exists* $a \in (0,1)$ *so that* $f(a) = 0,\ f'(a) > 0$ *and*

$f(x) \neq 0$ *for all* $x \neq 0, a, 1.$

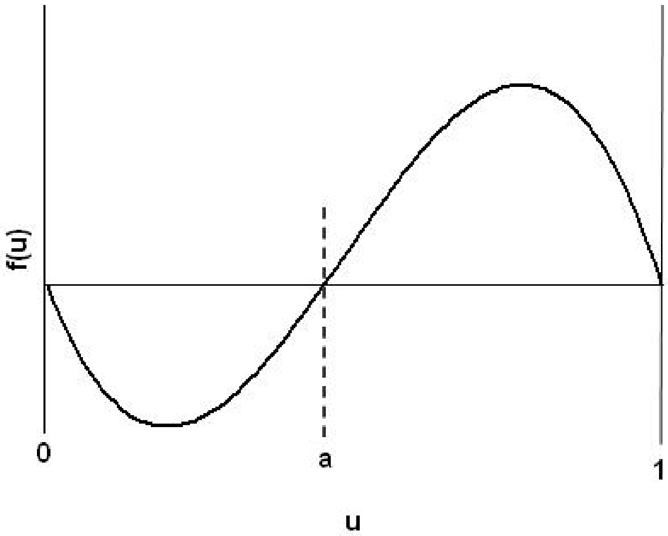

Figure 1.2: A typical example where f is of Nagumo type.

Remark. When f is of Nagumo type, there exists a unique travelling wave.

Definition 3. ***Unstable type***

We say f is of Unstable type *if $-f$ is of Nagumo type, that is:*

$f'(0) > 0,\ f'(1) > 0,$ *there exists* $a \in (0,1)$ *so that* $f(a) = 0,\ f'(a) < 0$ *and*

$f(x) \neq 0$ *for all* $x \neq 0, a, 1.$

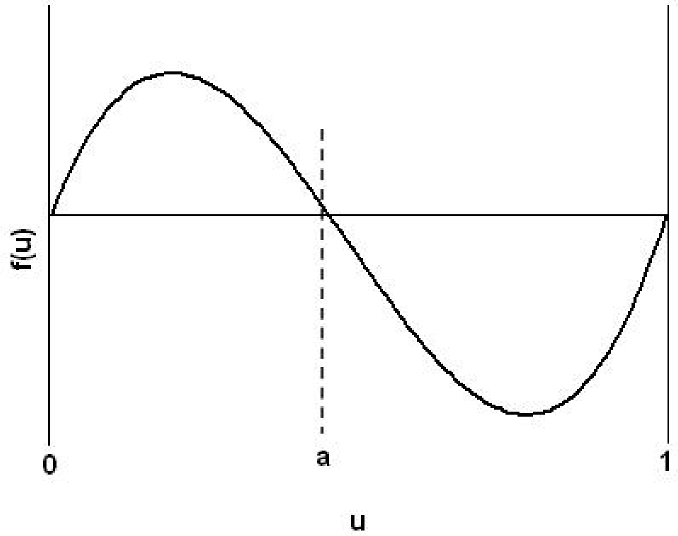

Figure 1.3: A typical example where f is of Unstable type.

Remark. When f is of Unstable type, solutions to (1.2) may evolve as two separate waves, the first travelling to the left (that is the speed, c, is negative) and the second travelling to the right (that is the speed, c, is positive) and a near flat-ish patch at level a between them (See figure 1.4).

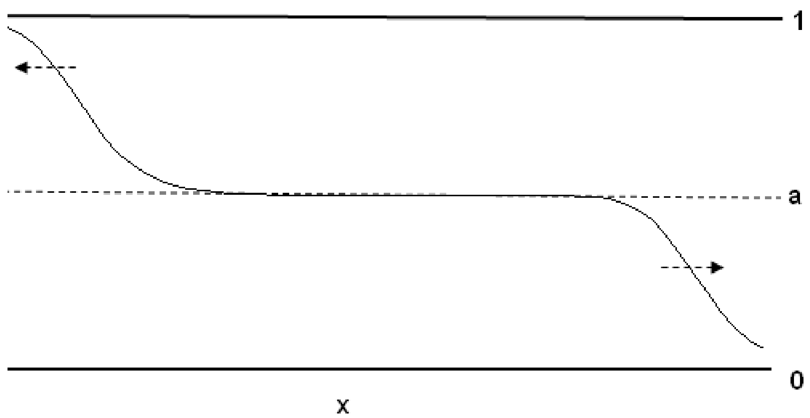

Figure 1.4: For f of Unstable type, solutions may evolve as two separate waves.

1.3 Itô versus Stratonovich noise

A large amount of work to date has concentrated upon how the addition of multiplicative or additive space-time white noise to the KPP equation (equation 1.2) affects the wave dynamics. In this thesis we concentrate upon how a one-dimensional time dependent Brownian motion affects the wave dynamics. This has the advantage over the space-time white noise case given that our equation is not only differentiable in space but a large number of the space-time white noise results hold true in our more simple setting. We concentrate upon the equation

$$du = u_{xx}dt + f(u)dt + g(u) \circ dW \tag{1.4}$$

in the Stratonovich form.

In the majority of modelling cases it is the Stratonovich noise rather than the Ito noise that is used. The use of Stratonovich noise is more intuitive in these cases and this has been largely supported by the work by Wong and Zakai ([33]) where the limit to the noise approximation is the Stratonovich rather than the Itô noise. However, the approximation we describe requires the modeller to already have a Brownian path and then take discrete points to form part of the approximation. We will discuss this further in Chapter 3.

Although our initial set up is via the Stratonvich noise due to our intuition from biology and physics models, the primary step in the majority of our arguments will be to convert from the Stratonovich noise to the Itô noise with the addition of the quadratic variation term, that is for $J(s, \cdot) \in \mathbb{L}^2$,

$$\int_0^t J(s,u) \circ dW_s = \int_0^t J(s,u)dW_s + \frac{1}{2}\left[J(\cdot,u), W_{\cdot}\right]_t$$

where $[\cdot,\cdot]$ is the quadratic covariation or brackets process (See [24] for further details). This change allows us to benefit from some of the key properties of the Itô integral (isometry principle, zero in expectation) but means we have to allow for the additional term when we move from Stratonovich to Itô noise in our analysis. This means that although we can rewrite the drift term to incorporate the correction term ($\bar{f}(u) = f(u) +$

$\frac{1}{2}g(u)g'(u)$) when moving from Stratonovich noise, this thesis will give conditions on the combined drift term ($\bar{f}(u)$) rather than each individual component making up the revised drift term ($f(u)$ and $g(u)g'(u)$). As such, this thesis can be considered to be concerned with the Itô form of the equation and although our hypotheses remain unchanged by doing this, the additional $\frac{1}{2}g(u)g'(u)$ term may have a significant impact on whether $\bar{f}(u)$ is of KPP, Nagumo or Unstable type. Indeed, if $f(u)$ is of KPP type then $\bar{f}(u)$ may be of Unstable type and if $f(u)$ is of Nagumo type, then $\bar{f}(u)$ may be of KPP type. We explore some of these properties in the two following examples.

Example 1. **f is of KPP type but $\bar{f}$ is of Unstable type.** Take $f = \epsilon u(1-u)$ and $g = u(1-u)$. Then whilst f is of KPP type, $\bar{f} = u(1-u)\left(\epsilon + \frac{1}{2} - u\right)$ is of Unstable type when $\epsilon \in (0, \frac{1}{2})$ but still of KPP type if $\epsilon > \frac{1}{2}$.

Remark. Letting $a(t) = \inf\{x : u(t,x) < 1\}$ and $b(t) = \sup\{x : u(t,x) > 0\}$, we define the width of the solution to be $b(t) - a(t)$. Given that solutions when $\bar{f}$ is of KPP type evolve as one wave but solutions when $\bar{f}$ is of Unstable type may evolve as two separate waves (possibly travelling in different directions), we may question whether the width of the solution when $\bar{f}$ is of KPP type is less volatile compared with when $\bar{f}$ is of Unstable type. This can be investigated in the above example as the behaviour of $\bar{f}$ is determined by the size of ϵ.

Example 2. **f is of Nagumo type and $\bar{f}$ is of KPP type.** For fixed $a \in (0,1)$ take $f = \epsilon u(1-u)(u-a)$ and $g = u(1-u)$. Then whilst f is of Nagumo type, if $\epsilon \in \left(\frac{1}{2(1-a)}, 1\right)$ then $\bar{f} = u(1-u)\left(\frac{\frac{1}{2}-\epsilon a}{1-\epsilon} - u\right)$ is of KPP type.

Remark. The above example makes us question, given the uniqueness of a travelling wave in the Nagumo type but a whole family of travelling waves in the KPP type, whether the uniqueness of the stationary travelling wave changes as ϵ passes above $\frac{1}{2(1-a)}$.

We suspect, however, that it is the Stratonovich drift that determines the existence and uniqueness of the stationary travelling wave rather than the Itô drift. We do not offer a proof of this within the thesis but this is something we hope to cover in a later paper.

1.4 Thesis overview

In this thesis we show that when $\bar{f}$ is of KPP, Nagumo or Unstable type, and solutions to (1.4) start from the Heaviside initial condition, there exists a unique (up to translation) stationary travelling wave. This is a new result and confirms our intuition that the addition of noise in the KPP and Nagumuo type equations does not destroy the existence of a stationary travelling wave. It also proves that the addition of noise in the Unstable type stabilises the solution. This is intuitive given that the condition $g(a) \neq 0$ means if any large flat patch at the wave marker level a were to form, then this would be destroyed by the noise. We do not conduct any computer simulations of these results but explore equation (1.4) in a purely theoretical way and whilst a large number of our arguments are soft, the results are extremely informative.

Chapters 3 and 4 concentrate on proving the existence of a stationary travelling wave. To do this we extend a Feynman-Kac result shown by McKean ([18]) and Bramson ([3]) when concentrating upon the KPP equation, equation (1.2) to our setting where the forcing term is allowed to also depend on t and x. This result states that if two wavefronts start from positions where one crosses over the other, then this property is preserved with time. To allow for the noise we prove an enhanced version of the Wong-Zakai Theorem (a new result, see Chapter 3) to extend the deterministic result of McKean/Bramson to our setting. Combining these two key components will yield the stretching corollary, Corollary 33. The intuition behind this result comes from considering solutions as a piece of elastic and the stretching result shows that for two solutions, one more stretched (or flatter) than the other, then the stretching property is preserved over time. If we start from the Heaviside initial condition we can think of this as solutions become flatter over time.

We will spend some time developing the concept of stretching and prove equivalent definitions which will prove useful in later chapters. Chapter 3 continues exploring stretching as a pre-order on our solution space. This enables us to use a result from Strassen ([29]) and Lindvall ([17]) to show that for a countable chain of ordered laws there exist random elements, all on the same probability space and each element corresponding to one of the laws, such that the ordering is preserved.

Given that the law of our solutions starting from the Heaviside initial condition can be ordered through the concept of stretching, we use this result to construct random elements (termed realisations of our solution) such that this ordering is preserved, that is the realisations become more stretched with time. We then centre the wavefront and show that the centred realisations are increasing for $x \geq 0$ and decreasing for $x < 0$. This is then used to prove the key result of Chapter 3, the Stretching Theorem, Theorem 51, showing that the law of our centred solutions starting from the Heaviside initial condition converges.

This result does not preclude the possibility that the stationary travelling wave is a large flat front at the height of the centring point. Much of Chapter 4 is spent proving that this is, almost surely, not the case.

Chapter 5 extends the results of Chapters 3 and 4 for any initial condition which can be trapped between two Heaviside functions but only when $\bar{f}$ is of KPP type (given that $\bar{f}(z) \geq 0$ for all $z \in [0, 1]$) but we will present an extension to this argument in a later paper. The main results of this chapter rely upon an implicit wave speed formula and the property, from Chapter 3, that solutions which start more stretched remain more stretched with time. Intuitively, the flatter or more stretched the wavefront the faster it becomes and we were able to show that for any initial condition which starts off trapped (see Chapter 5), the limiting wave speed is the same as that for the wavefront as started from the Heaviside initial condition. This is an immediate corollary of the Stretching result. Having one wavefront more stretched than the other but both having the same limiting wave speed allows us to conclude that the wavefront started from the trapped initial condition must have the same limiting law as that of the wavefront started from the Heaviside initial condition.

Much of the early analysis on reaction-diffusion equations was conducted by considering the dynamics in the phase-plane. There were obvious benefits of this but the benefit of doing this here, in this thesis, is that stretching becomes a more intuitive, easy to verify concept where standard comparison results can be used. Unfortunately, we were unable to progress this intuition to the depth we would have liked without making further assumptions on the regularity at end points. We will discuss this further in Chapter 6 and develop the intuition behind it throughout the thesis.

1.5 Notation

Throughout the thesis we will use the following notation and write:
$\mathbb{R}$ to the set of real numbers, $\mathbb{Q}$ the set of rational numbers and $\mathbb{N}$ the set of natural numbers (including 0). We will define $C^{\alpha,\beta}(A \times \mathbb{R})$ where $\alpha, \beta \in \mathbb{N}$ and A is an interval in $\mathbb{R}$ to be the space of functions $\mathcal{J}(s,x)$ which are continuous and differentiable in both variables s and x for all orders up to and including α and β respectively, that is $\mathcal{J}(s,x) \in C^{\alpha,\beta}(A \times \mathbb{R})$ if $\mathcal{J}(s,\cdot)$ is $C^{\alpha}(A)$ for $A \subset \mathbb{R}$ and $\mathcal{J}(\cdot,x)$ is $C^{\beta}(\mathbb{R})$ for $x \in \mathbb{R}$. We will also define the $\mathbb{L}^p_{loc}(\mathbb{R})$ metric by $d(f,g) = \sum_{n\geq 1} 2^{-n} \left(\int_{-n}^{n} |f(x) - g(x)|^p \, dx \right)^{\frac{1}{p}}$ for any two functions f and g in our state space. We will write a sequence $\{s_n\}$ converges in $\mathbb{L}^2$, $s_n \xrightarrow{\mathbb{L}^2} s$ if $\mathbb{E}\left[|s_n - s|^2 \right] \to 0$ as $n \to \infty$.

Chapter 2

Preliminaries

Introduction. *In this chapter we define what is meant by a solution to equation (1.4) as well as the state space on which our solutions exist. We also discuss existence, uniqueness and regularity of solutions and other properties that will be used throughout the thesis. We will also cover the running assumptions on f and g although primarily, in later chapters, conditions will be given on the adjusted drift $\bar{f}$. We will also introduce the concept of the wave marker, intuitively the point at which the wave front first reaches a specified height, and develop semi-martingale decompositions of this process.*

2.1 Mild and classical solutions

In this thesis we will primarily concern ourselves with the following SPDE

$$du(t,x) = [u_{xx}(t,x) + f(u(t,x))]\,dt + g(u(t,x)) \circ dW_t \tag{2.1}$$

where (W_t) is a one dimensional Brownian motion adapted to a filtered probability space $(\Omega, (\mathcal{F}_t), P)$. Let $f, g : [0,1] \to \mathbb{R}$ be measurable functions and we consider solutions to the formal equation (2.1) in the mild form in the sense of the following definition.

Definition 4. ***Mild solution*** *Given a filtered space $(\Omega, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ and a $(\mathcal{F}_t)$ Brownian motion W, a random field $u : [0, \infty) \times \mathbb{R}$ which is $\mathbb{P} \times \mathcal{B}$ measurable, where $\mathbb{P}$ are the progressively measurable sets in (t, ω) and $\mathcal{B}$ are the Borel measurable sets in $\mathbb{R}$, is called a mild solution to the stochastic PDE*

$$du(t,x) = [u_{xx}(t,x) + f(u(t,x))]\, dt + g(u(t,x)) \circ dW_t \tag{2.2}$$

if, for each t and x $\mathbb{P}$-almost surely,

$$\begin{aligned} u(t,x) \;=\; & \int_{\mathbb{R}} \Gamma_t(x-y)u(0,y)dy + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) f(u(s,y))\, dyds \\ & + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g(u(s,y)) dy \circ dW_s \end{aligned} \tag{2.3}$$

where $\Gamma_t(x)$ is the Green's kernel for $\frac{\partial^2}{\partial x^2}$ on $\mathbb{R}$.

Remark. Throughout the thesis we will suppress the dependence of u on $\omega \in \Omega$.

Remark. We could also define the Itô form of equation (2.3) by absorbing the correction term $\frac{1}{2} g(u) g'(u)$ into f and defining a new drift term $\bar{f}(u) = f(u) + \frac{1}{2} g(u) g'(u)$:

$$\begin{aligned} u(t,x) \;=\; & \int_{\mathbb{R}} \Gamma_t(x-y)u(0,y)dy + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \bar{f}(u(s,y))\, dyds \\ & + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g(u(s,y)) dy dW_s. \end{aligned}$$

Definition 5. ***Classical Solution*** *We will also consider solutions that are classical (in the PDE sense), that is those solutions that have two continuous spatial derivatives. Thus we define a classical solution on $[t_0, \infty)$, for some $t_0 \geq 0$, with respect to an adapted Brownian motion W on a filtered space $(\Omega, (\mathcal{F}_t), P)$, to be an adapted random* *field* $(u(t,x) : t \geq t_0, x \in \mathbb{R})$, *with values in* $[0,1]$ *satisfying* $u(t,x) \in \mathcal{C}^{0,2}([t_0,\infty) \times \mathbb{R})$ *and, for all* $x \in \mathbb{R}$ *and* $t \geq t_0$,

$$u(t,x) = u(t_0,x) + \int_{t_0}^{t} u_{xx}(s,x)ds + \int_{t_0}^{t} f(u(s,x))ds + \int_{t_0}^{t} g(u(s,x)) \circ dW_s. \qquad (2.4)$$

2.2 Running assumptions on the functions f and g

We will assume that the functions $f, g : [0,1] \to \mathbb{R}$ satisfy the following conditions throughout the thesis:

(**H1**) $f(0) = f(1) = g(0) = g(1) = 0$ and f has at most one zero in $(0,1)$;

(**H2**) $f, g \in \mathcal{C}^3([0,1])$.

Remark. These are certainly not the most general conditions possible but are chosen to cover the cases we meet.

Remark. Although not discussed in this thesis, f having more than three zeros on $[0,1]$ may give rise to a stack of travelling waves.

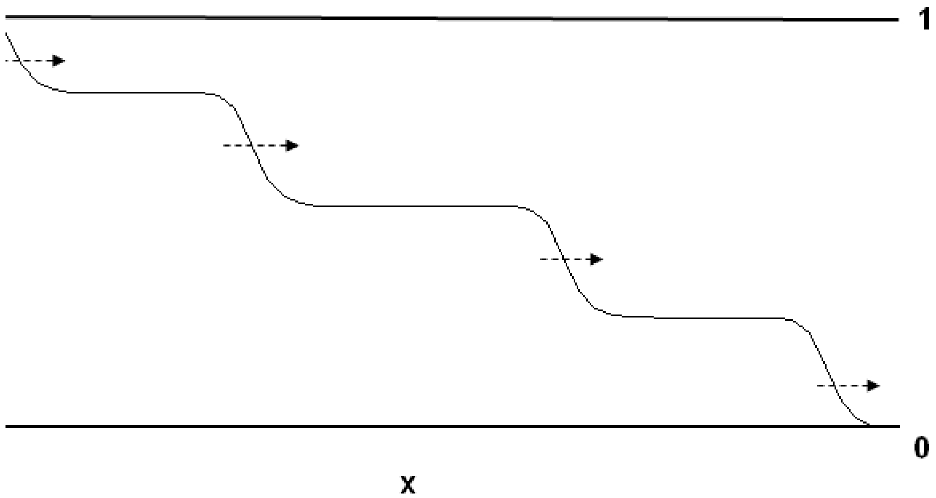

Figure 2.1: A stack of travelling waves.

2.3 $\mathbb{B}_{dec}$ and $\mathbb{B}_{dec}^{0,1}$

Introduction. *In this section we will define the spaces on which solutions to our equation exist as well as some key properties of these spaces. The main space we will concentrate upon is $\mathbb{B}_{dec}^{1,0}$ although the bigger space $\mathbb{B}_{dec}$ will be used in showing convergence in Chapter 3.*

$$\begin{aligned} \mathbb{B}_{dec} &= \{h : \mathbb{R} \to [0,1] : h \text{ is right continuous and decreasing}\}; && (2.5) \\ \mathbb{B}_{dec}^{1,0} &= \{h \in \mathbb{B}_{dec} : h(-\infty) = 1, h(\infty) = 0\}. && (2.6) \end{aligned}$$

Remark. Throughout this thesis we will use the term decreasing to mean, for $x_0 \geq 0$ $f(x) \geq f(x + x_0)$ for all $x \in \mathbb{R}$.

Remark. We give $\mathbb{B}_{dec}$ the topology that arises from the $\mathbb{L}^1_{loc}(\mathbb{R})$ metric and note that $\mathbb{B}_{dec}^{1,0}$ is a measurable subset of $\mathbb{B}_{dec}$.

Definition 6. *A space $\mathcal{E}$ is called Polish if it metrizable to a separable, complete metric space.*

Definition 7. *Let E be a measurable space. Define $\mathcal{M}(E)$ as the set of all finite measures on E and the subset $\mathcal{M}_1(E)$ as the set of all probability measures on E.*

Lemma 8. *$\mathbb{B}_{dec}$ and $\mathbb{B}_{dec}^{1,0}$ are Polish spaces.*

Proof. The proof of this is straightforward and we only present a sketch of the proof here and leave the details to the reader. For $h \in \mathbb{B}_{dec}$ define $\mu^h \in \mathcal{M}(\mathbb{R})$ by $\mu^h((a,b]) = h(a) - h(b)$. This defines a map from $\mathbb{B}_{dec}$ to $\mathcal{M}(\mathbb{R})$ and we note that $\mu^h(\mathbb{R}) \leq 1$ but if $h \in \mathbb{B}_{dec}^{1,0}$ then $\mu^h \in \mathcal{M}_1(\mathbb{R})$. We note the similarity with this definition and the definition of a distribution function, see [2]. However, although the map $h \mapsto \mu^h$ is a bijection on $\mathbb{B}_{dec}^{1,0}$ it is not a bijection on $\mathbb{B}_{dec}$ (consider two functions $h, g \in \mathbb{B}_{dec}$ such that $h(x) = g(x) + constant$). Defining a relation $\sim$ on $\mathbb{B}_{dec}$ by $h \sim g$ if there exists a constant C such that $h(x) = g(x) + C$ it is easy to show $\sim$ defines an equivalence relation on $\mathbb{B}_{dec}$. Now the map from the quotient space $\frac{\mathbb{B}_{dec}}{\sim}$ to $\mathcal{M}(\mathbb{R})$ is a bijection. Furthermore, both maps can be shown to be homeomorphisms. We can now use the Prohorov metric (See [4]) on $\mathbb{B}_{dec}^{1,0}$ or the $\mathbb{L}_1^{loc}(\mathbb{R})$ on $\mathbb{B}_{dec}$ to conclude that the these metrics, respectively, induce the weak topology on $\mathcal{M}_1(\mathbb{R})$ and the vague topology on $\mathcal{M}(\mathbb{R})$. It is easy to check that both spaces are separable and, relative to the above metrics, complete. $\blacksquare$

Lemma 9. *For $\varphi_n, \varphi \in \mathbb{B}_{dec}$, the following types of convergence are equivalent:*

(1) $\varphi_n \xrightarrow{a.e.} \varphi$

(2) $\varphi_n \xrightarrow{\mathbb{L}^1_{loc}(\mathbb{R})} \varphi$

Proof. (1) $\Rightarrow$ (2):

This is easy by the Dominated Convergence Theorem.

(2) $\Rightarrow$ (1):

Claim $\varphi_n(x_0) \to \varphi(x_0)$ for any continuity point x_0 of φ (such a continuity point exists from standard analysis given that that φ is bounded and decreasing and, hence, there are only a finite number of discontinuities). If $\varphi_n(x_0) \nrightarrow \varphi(x_0)$ then there exists a $\delta > 0$ and a subsequence n' such that $|\varphi_{n'}(x_0) - \varphi(x_0)| \geq \delta$ for all n'. Now choose η so that $|\varphi(y) - \varphi(x_0)| \leq \frac{\delta}{10}$ for $|y - x_0| \leq \eta$. Then $\int_{x_0-\eta}^{x_0+\eta} |\varphi_{n'}(x) - \varphi(x)|\, dx \nrightarrow 0$ (using the decreasing nature of paths) contradicting $\mathbb{L}^1_{loc}(\mathbb{R})$ convergence. $\blacksquare$

Lemma 10. $\mathbb{B}_{dec}$ ***is compact***

Proof. Take $\varphi_n \in \mathbb{B}_{dec}$. Since $\varphi_n(x) \in [0,1]$ the sequence $(\varphi_n(x))$ is compact in $\mathbb{R}$ for any $x \in \mathbb{R}$. This shows there exists a subsequence n' such that $\psi(x) = \lim_{n' \to \infty} \varphi_{n'}(x)$ exists for all $x \in \mathbb{Q}$. Note $x \mapsto \psi(x)$ is decreasing. Let $\varphi(x) = \lim_{\substack{y \downarrow x \\ y \in \mathbb{Q}}} \psi(y)$. We claim that $\varphi_{n'} \to \varphi$ in the $\mathbb{L}^1_{loc}(\mathbb{R})$ metric. It is enough to check $\varphi_{n'}(x_0) \to \varphi(x_0)$ at x_0, a continuity point of φ by Lemma 9. Fix $\epsilon > 0$. Choose δ so that $|\varphi(x) - \varphi(x_0)| \leq \epsilon$ where $|x - x_0| \leq \delta$. Now choose δ' such that $|\psi(x) - \varphi(x_0)| \leq 2\epsilon$ when $|x - x_0| \leq \delta'$, $x \in \mathbb{Q}$. Choose $x_1 \in (x_0 - \delta', x_0) \cap \mathbb{Q}$, $x_2 \in (x_0, x_0 + \delta') \cap \mathbb{Q}$ and choose N so that $|\varphi_{n'}(x_1) - \varphi(x_0)| \leq 3\epsilon$ and $|\varphi_{n'}(x_2) - \varphi(x_0)| \leq 3\epsilon$ for $n' > N$. Then by decreasing paths $\varphi_{n'}(x_0) \in [\varphi_{n'}(x_2), \varphi_{n'}(x_1)]$ so that $|\varphi_{n'}(x_0) - \varphi(x_0)| \leq 3\epsilon$ if $n' \geq N$. ■

Proposition 11. *If the metric space* (X, d) *is compact then* $(\mathcal{M}(X), \rho)$ *where* ρ *is the Prohorov metric is also compact.*

Proof. See [6], page 101. ■

Definition 12. *For a metric space* (X, d)*, a probability measure* $\mu \in \mathcal{M}_1(X)$ *is said to be tight if for each* $\epsilon > 0$ *there exists a compact set* $K \subset X$ *such that* $\mu(K) \geq 1 - \epsilon$*. A family of probability measures* $M \subset \mathcal{M}_1(X)$ *is tight if for each* $\epsilon > 0$ *there exists a compact set* $K \subset X$ *such that* $\inf_{\mu \in \mathcal{M}} \mu(K) \geq 1 - \epsilon$*.*

Proposition 13. *Tightness is a necessary and sufficient condition that for every sequence of probability measures* $\{\mu_{n_k}\}$ *there exists a further subsequence* $\{\mu_{n_{k(j)}}\}$ *and a probability measure* μ *such that* $\mu_{n_k} \xrightarrow{D} \mu$ *as* $j \to \infty$*.*

Proof. See [2], page 336. ■

Definition 14. *For two real random variables A, B we write $A \overset{D}{=} B$ if A and B are equal in distribution (law).*

2.4 Existence, uniqueness and moment bounds of solutions

The following theorem summarises the known existence, uniqueness and regularity results for equation (2.1).

Theorem 15. ***Existence, uniqueness and properties of solution***

*Fix a probability space $(\Omega_t, (\mathcal{F}_t), \mathcal{F}, \mathbb{P})$ and an $(\mathcal{F}_t)$ Brownian motion W. Fix an $\mathcal{B} \times \mathcal{F}_0$ measurable initial condition $u_0 : \mathbb{R} \times \Omega \to [0,1]$. Suppose that f and g satisfy hypotheses (**H1**) and (**H2**). Then there exists $\mathcal{C}^{0,3}\left((0,\infty) \times \mathbb{R}\right)$ mild solutions to equation (2.1) with values in $[0,1]$ satisfying:*

(i) Solutions are pathwise unique and they are classical solutions on $[t_0, \infty)$ for any $t_0 > 0$;

(ii) Two solutions as in part (i) started such that $u_0 \leq v_0$ satisfy $u(t,x) \leq v(t,x)$ for all $t \geq 0$ and $x \in \mathbb{R}$, almost surely;

(iii) If $u_0 \in \mathbb{B}_{dec}$ (respectively $\mathbb{B}^{1,0}_{dec}$) then $u(t,\cdot) \in \mathbb{B}_{dec}$ ($\mathbb{B}^{1,0}_{dec}$) all $t \geq 0$ almost surely;

(iv) If u_0 is not identically zero, then $u(t,x) > 0$ for all $t > 0$ and $x \in \mathbb{R}$, almost surely. If $x \mapsto u_0(x)$ is decreasing and not identically constant then $u_x(t,x) < 0$ for all $t > 0$ and $x \in \mathbb{R}$, almost surely;

(v) The laws of solutions form a Markov family;

(vi) There exists a constant $C(T)$ such that

$$\mathbb{E}\left[|u(t,x)-u(s,x)|^2\right] \leq C(T)\left(|t-s| + \left|\frac{t-s}{s}\right|^2\right)$$

for all $x \in \mathbb{R}$ and $0 < s \leq t \leq T$ satisfying $|t-s| \leq 1$;

(vii) Suppose the initial condition $u_0 \equiv 0$ for all large x and $u_0(x) \equiv 1$ for all large $-x$ then for $0 < t_0 < T$ and $p, \eta > 0$

$$E\left[|u_x(t,x)|^p + |u_{xx}(t,x)|^p\right] \leq C(u_0,\eta,p,t_0,T)e^{-\eta|x|} \quad \text{for all } x \in \mathbb{R} \text{ and } t \in [t_0,T]$$

and hence

$$\sup_{t\in[t_0,T]} E\left[\sup_x |u_x(t,x)|^p\right] < \infty.$$

Moreover, for all $t \geq 0$, $|u_x(t,x)| \to 0$ as $|x| \to \infty$ almost surely.

Proof. The proof of the above properties are not contained in any one source but we will point to the main sources for each item and the details will be left to check by the reader. For existence of solutions we use Picard iteration (See [32]). Uniqueness follows from standard ordinary differential equations (ODEs) or stochastic differential equations (SDEs) arguments through the use of a Gronwall argument to the difference of two solutions (See [24]). Part ii) follows using standard coupling arguments, see [23] which proves a comparison result through smoothing $max(0, u-v)$ and applying Itô's lemma where u and v are solutions one lying above the other. Part (iii) follows from part (ii) where we define, for some $\delta > 0$, $u(t,x) = v(t, x+\delta)$. For part (iv) we use a large deviation estimate and Donsker's result (See [24]) to write the Itô integral as time changed Brownian motion and then bound this using the reflection principle. We then use the argument presented in [27] but note it is easier in our setting as we concentrate upon a one-dimensional Brownian motion and not space-time white noise. For the second part apply the same argument as presented in the first part of iv) to the equation $v_t = v_{xx} + \bar{f}'(u)v + g'(u)vdW_t$ where $v = u_x$ noting the same required assumptions for f and g also apply to $\bar{f}'(u)v$ and $g'(u)v$ although these are random coefficients. Part v) follows from standard stochastic differential equation arguments writing, for $s > 0$, $W_{t+s} - W_t \overset{D}{=} \hat{W}_s$ where $\hat{W}$ represents another Brownian motion and using the uniqueness of solution (see [1]). Part (vi) follows from using the Green's function representation of the solution and the various integral bounds (see Appendix). To show parts (i) and (vii) we note that for fixed (t, x) we can use standard Green's function estimates, although a little care is needed since we allow for arbitrary initial conditions; indeed the pth moment of the kth derivative blow up like $t_0^{-\frac{pk}{2}}$ where $t \geq t_0 > 0$. We now develop equations for u_x and u_{xx} through the mild form of the solution and make use of the Mean Value Theorem: $u_x(t, x) = u_x(t, 0) + \int_0^x u_{xx}(t, y)\, dy$, and noting we can bound this by $\sup |u_x(t, x)|^p \leq C(p, \eta) \left(|u_x(t, 0)|^p + \int_{\mathbb{R}} \exp\left(+\eta|x|\right) |u_{xx}(t, y)|^p \, dy \right)$, the first part of (vii) is complete. The last part of (vii) follows from the use of the differentiated form of the mild solution and taking limits. ■

2.5 Wave markers γ_t^a

Introduction. *When looking at travelling waves we are often concerned with how the shape of the wave front changes over time. One unfortunate problem with this is that* *the wave position moves. To overcome this problem the idea of a wave marker, $\gamma_t^a(u)$, is introduced. This ensures that the wave always remains centred such that the height of the wave when $x = 0$ is always defined to be a. To use this definition we formulate the idea of a centred SPDE which we will denote by the use of a tilde on top of the u (see definitions below).*

Definition 16. ***Wave Markers***

For a function $\varphi : \mathbb{R} \to (0,1)$ and $a \in [0,1]$, define $\Gamma^a(\varphi) = \inf\{x : \varphi(x) \leq a\}$ and set $\inf(\emptyset) = -\infty$.

Definition 17. *For a solution u of equation (2.1) define γ_t^a as $\gamma_t^a = \Gamma^a(u(t))$ (See figure 2.2).*

Example 3. For any $a \in (0,1)$ we have for the Heaviside initial condition $u(0,x) = \mathbb{I}_{[x \leq 0]}$ $\gamma_0^a(u) = 0$.

Definition 18. *For $\varphi \in \mathbb{B}_{dec}^{1,0}$, define $\tilde{\varphi}^a(\cdot) = \varphi(\cdot + \Gamma^a(\varphi))$. When the value of a is unimportant we often omit it and write $\tilde{\varphi}$.*

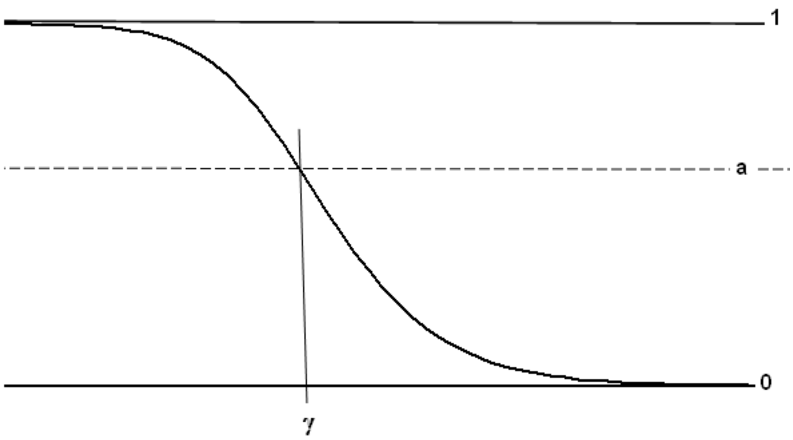

Figure 2.2: Wave marker at height a for a general wavefront.

Definition 19. *For a function $T : \mathbb{R} \to \mathbb{R}$ and for any fixed $a \in \mathbb{R}$ define the transformation $\tau^a T$ to mean $T(\cdot - a)$.*

Remark. For any $\varphi \in \mathbb{B}^{1,0}_{dec}$ and any $a \in \mathbb{R}$, $\tilde{\varphi} = \widetilde{\tau^a \varphi}$.

Remark. For all $x \in \mathbb{R}$ and $t \in \mathbb{R}^+$ note that $\tilde{u}(t,x) = u(t, x + \gamma_t^a)$, where γ_t^a is defined in Definition 17, describes the wave as viewed from the wave front marker.

Remark. In the transformation of the u equation to the centred u equation we note, given the stochastic u equation and the fact that γ_t^a is a semi-martingale (see Theorem 21), the following identities hold:

(**Id1**) $d\tilde{u}(t,x) = du(t, x + \gamma_t^a) + u_x(t, x + \gamma_t^a) \circ d\gamma_t^a$;

(**Id2**) $\tilde{u}_x(t,x) = u_x(t, x + \gamma_t^a)$ and similarly for $\tilde{u}_{xx}(t,x)$.

Also note that in the deterministic equivalent of our stochastic equation, identity one (**Id1**) becomes

$$\tilde{u}_t(t,x) = u_t(t, x + \gamma_t^a) + u_x(t, x + \gamma_t^a)\dot{\gamma}_t^a.$$

The above transformations will be used in the sequel without further note.

Lemma 20. *The map $\varphi \mapsto \Gamma^a(\varphi)$ is measurable on $\mathbb{B}^{1,0}_{dec}$.*

Proof. Consider

$$\{\varphi : \Gamma^a(\varphi) < x\} = \cup_{\substack{y \in \mathbb{Q} \\ y < x}} \{\varphi : \varphi(y) < a\}$$

which indicates $\varphi \mapsto \Gamma^a(\varphi)$ will be measurable providing $\varphi \mapsto \varphi(y)$ is measurable for all y. To show this we consider the map $\varphi \mapsto \Gamma_\epsilon * \varphi$, the convolution of φ with the Gaussian function $\Gamma_\epsilon = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{x^2}{2\epsilon}\right)$. This map is clearly $\mathbb{L}^1_{loc}(\mathbb{R})$ continuous from

$\mathbb{B}^{1,0}_{dec} \to \mathbb{B}^{1,0}_{dec}$. Also $\Gamma_\epsilon * \varphi(y) \to \varphi(y)$ as $\epsilon \to 0$ for almost all y. Hence, writing $\varphi(y) = \lim_{\substack{x\in\mathbb{Q}\\ x\downarrow y}} \overline{\lim}_{n\to\infty} \Gamma_{1/n} * \varphi(x)$ shows that $\varphi \mapsto \varphi(y)$ is measurable as a map $\mathbb{B}^{1,0}_{dec} \to \mathbb{R}$ as required. ■

2.6 γ_t^a dynamics

Theorem 21. *Let u be a solution to (2.1) with $u_0 \in \mathbb{B}^{1,0}_{dec}$. Then for each fixed $t > 0$ the inverse of the map $x \mapsto u(t,x)$ exists. Denoting this inverse function by m, m solves strongly the SPDE:*

$$dm = (m_{xx}/m_x^2)dt - fm_x dt - gm_x \circ dW_t \tag{2.7}$$

for $(t,u) \in (0,\infty) \times (0,1)$ and hence, m is a semi-martingale.

Proof. We define $x \mapsto m(t,x)$ for $x \in (0,1)$ as the inverse, at each fixed $t > 0$, of the map $x \mapsto u(t,x)$, that is via the equation $u(t, m(t,x)) = x$. The fact that $x \mapsto u(t,x)$ is strictly decreasing (Theorem 15, Part (iv))and takes all values in $(0,1)$ ensures that m is well defined for all $t > 0$ and $x \in (0,1)$. By the inverse function theorem m has as many continuous spatial derivatives as u and given that f and g are $\mathcal{C}^3$, so is u and hence so is m. Note, by the inverse function theorem, that $m_x(t,x) = 1/u_x(t,m(t,x)) < 0$. Equation (2.7) would follow by the chain rule if $\dot{W}$ were a smooth function of t. Indeed we believe a proof using the Wong-Zakai framework should give the same result. We present, however, another derivation following a slightly circuitous route to establish it for these Itô processes. Choose $\varphi : (0,1) \to \mathbb{R}$ smooth and compactly supported. Then from the substitution $x \mapsto u(t,x)$ we have, for $t>0$,

$$(m,\varphi) = \int_{\mathbb{R}} m(t,x)\varphi(x)dx = \int_{\mathbb{R}} x\varphi(u(t,x))u_x(t,x)dx = (\varphi(u)u_x, x).$$

(Here we are writing x for the function $x \mapsto x$). Expanding $\varphi(u(t,x))u_x(t,x)$ via Itô's formula, in its Stratonovich form, we obtain

$$\begin{aligned} d(m,\varphi) &= d(\varphi(u)u_x, x) \\ &= (\varphi(u)(u_{xxx} + f'(u)u_x), x)\, dt + (\varphi_x(u)(u_{xx} + f(u))u_x, x)\, dt \\ &\quad + (\varphi(u)g'(u)u_x + \varphi_x(u)g(u)u_x, x) \circ dW. \end{aligned}$$

To assist in our notation we let $\hat{u}$, $\hat{u}_x, \hat{u}_{xx}, \ldots$ denote the composition of the maps $x \mapsto u, u_x, u_{xx}$ with the map $x \to m(t,x)$ (e.g. $\hat{u}_x(t,x) = u_x(t, m(t,x))$). Using this notation we have, for $x \in (0,1), t > 0$,

$$\hat{u}(t,x) = x, \quad \hat{u}_x m_x = 1, \quad \hat{u}_{xx} m_x^2 + \hat{u}_x m_{xx} = 0. \tag{2.8}$$

We continue by using the reverse substitution $x \mapsto m(t,x)$ to reach

$$\begin{aligned} d(m,\varphi) &= (\hat{u}_{xxx} m m_x + f'm, \varphi)\, dt + (\hat{u}_{xx} m + fm, \varphi_x)\, dt \\ &\quad + (g'm, \varphi) \circ dW + (gm, \varphi_x) \circ dW \\ &= -(\hat{u}_{xx} m_x + f m_x, \varphi)\, dt - (g m_x, \varphi) \circ dW \\ &= \big((m_{xx}/m_x^2) - f m_x, \varphi\big)\, dt - (g m_x, \varphi) \circ dW. \end{aligned}$$

In the second equality we have integrated by parts. In the final equality we have used the identities in (2.8). Now letting φ approach a delta function at x reveals the desired equation for $m(t,x)$. Given that $m(t,x)$ has a martingale part and a part of bounded variation, it is clear that it is a semi-martingale. ■

Remark. Note that $\gamma_t^a = m(t,a)$.

Lemma 22. *For the variable $p(t,x) = -u_x(t,m(t,x))$ we have*

$$dp = \left(p^2 p_{xx} + f'p - f p_x\right) dt + (g'p - g p_x) \circ dW$$

Proof. From the derivation of the m SPDE above we can now develop the equation for $p(t,x) = -u_x(t,m(t,x)) = -\hat{u}_x(t,x)$ by applying the Itô-Ventzel formula (see [15] Theorem 3.3.2) and the decompositions for $du_x(t,x)$ and $dm(t,x)$. We find, fixing $x \in (0,1)$, $t > 0$,

$$\begin{aligned} dp(t,x) &= -du_x(t,z)|_{z=m(t,x)} - u_{xx}(t,z)|_{z=m(t,x)} \circ dm(t,x) \\ &= -\left(\hat{u}_{xxx} + f'\hat{u}_x\right) dt - \left(\hat{u}_{xx}\left(\frac{m_{xx}}{m_x^2} - f m_x\right)\right) dt - \left(g'\hat{u}_x - g\hat{u}_{xx}m_x\right) \circ dW. \end{aligned}$$

As u has three continuous spatial derivatives we may differentiate the definition $p(t,x) = -u_x(t,m(t,x))$ to obtain

$$p = -\hat{u}_x, \quad p_x = -\hat{u}_{xx}m_x, \quad p_{xx} = -\hat{u}_{xxx}m_x^2 - \hat{u}_{xx}m_{xx}. \tag{2.9}$$

Combining these with (2.8) we obtain the desired equation. ■

Remark. This is the starting point for the approach in Fife and McLeod for the deterministic KPP equation [9], see Chapter 6, which we started to discuss in Chapter 2. We will revisit these ideas throughout the thesis.

Corollary 23. *For the variable $\tilde{u}(t,x) = u(t, x + \gamma_t^a) = u(t, x + m(t,a))$ we have*

$$d\tilde{u} = \tilde{u}_{xx}dt + f(\tilde{u})dt + g(\tilde{u}) \circ dW + \tilde{u}_x \circ d\gamma^a$$

Proof. By applying the Itô-Ventzel formula and using the decompositions for $du(t,x)$ and $dm(t,x)$ we find, fixing $x \in (0,1)$, $t > 0$,

$$\begin{aligned} d\tilde{u}(t,x) &= du(t,z)|_{z=x+m(t,a)} + u_x(t,z)|_{z=x+m(t,a)} \circ dm(t,a) \\ &= \tilde{u}_{xx}(t,x)dt + f(\tilde{u}(t,x))dt + g(\tilde{u}(t,x)) \circ dW_t + \tilde{u}_x(t,x) \circ d\gamma_t^a \end{aligned}$$

which gives us our desired result. ∎

Chapter 3

Existence of a stretched limit

Introduction. *In this chapter we show that for two initial conditions, one more stretched (in a sense that we will define) than the other, the stretching property is preserved for all time. As a corollary to this we will show that a solution started from the Heaviside initial condition becomes more stretched as time tends to infinity. The central Theorem of this chapter, Theorem 51 and what we will refer to as the Stretching Theorem, proves that the laws of the solution started from the Heaviside initial condition converge on the bigger space* $\mathbb{B}_{dec}$.

3.1 Deterministic stretching lemma

Introduction. *Informally, the key lemma in the Kolmogorov [14] paper shows when we consider the difference of two solutions to* $u_t = u_{xx} + J(u)$ *started from the Heaviside initial condition, then provided the initial conditions has at most one sign change, the difference has at most one sign change for all time. This is the fundamental concept of* *stretching or being more stretched which we will define in Section 3.4. In this section however, we extend the Kolmogorov idea when J is also allowed to depend on t and x and use the probabilistic arguments of McKean [18]. Although, as we will show in Chapter 6, stretching may be more intuitively considered from the phase-plane where to show one solution is more stretched than another standard comparison techniques can be used, these techniques cannot be used here in the $t - x$ space given the additional requirement that the result hold true for any translate. As such, we proceed using a Feynman-Kac argument rather than the standard Gronwall argument.*

Definition 24. *For functions $\varphi : \mathbb{R} \to \mathbb{R}$ define $\theta(\varphi) = \inf\{x : \varphi(x) > 0\}$ and we set* $\inf(\emptyset) = +\infty$.

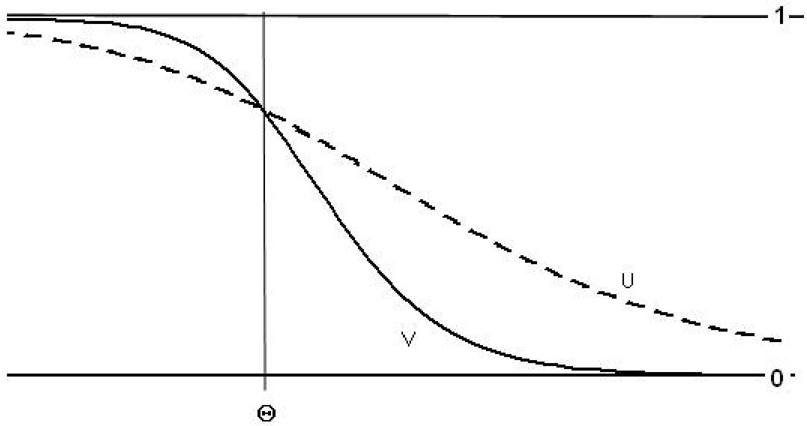

Figure 3.1: $\theta(U - V)$, first positivity point.

Proposition 25. *Consider the heat equation*

$$u_t(t, x) = u_{xx}(t, x) + J(u(t, x), t, x) \tag{3.1}$$

for $t \geq 0$, $x \in \mathbb{R}$ where $J : [0, 1] \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ is a measurable function satisfying:

(P1) *J is Lipschitz in the first variable, that is:*

$$|J(r,s,x) - J(q,s,x)| \leq K|r-q| \text{ for all } r,q \in [0,1],\ x \in \mathbb{R} \text{ and } s \in [0,T];$$

(P2) *$J(0,t,x) = J(1,t,x) = 0$ for all $t \in [0,T]$, $x \in \mathbb{R}$.*

Suppose r and q are mild solutions of (3.1), taking values in $[0,1]$, and $r,q \in \mathcal{C}^{1,2}((0,T] \times \mathbb{R})$. Suppose the initial conditions satisfy $\theta(r(0)-q(0)) \in (-\infty,\infty)$ and $r(0,x) \geq q(0,x)$ for all $x \geq \theta(r(0)-q(0))$. Then, for all $t \in [0,T]$, $r(t,y) > q(t,y)$ for all $y > \theta(r(t)-q(t))$.

Proof. Consider $z : [0,T] \times \mathbb{R} \to [-1,1]$ defined as $z(t,x) = r(t,x) - q(t,x)$. Defining

$$R(t,x) = \left(\frac{J(r(t,x),t,x) - J(q(t,x),t,x)}{r(t,x) - q(t,x)}\right) \mathbb{I}_{(r \neq q)}$$

we can write a differential equation for the differences $z_t = z_{xx} + zR$. It is easy to verify that, by the Lipschitz property of H, R is bounded and given that it is the quotient of measurable functions, it is itself measurable. We will now use these properties in the Feynman-Kac formulation for z. Let us define $B = (B(t) : t \geq 0)$ as an adapted Brownian motion defined on the probability space $(\Omega, (\mathcal{F}_t : t \geq 0), (P_x : x \in \mathbb{R}))$, where under P_x, B starts at x. Fix $t > 0$. Due to R being bounded and measurable, noting that $s \to \int_0^s R(t-r,B(r))\,dr$ is of bounded variation, it is straightforward to demonstrate using Itô calculus that, for $s \in [0,t)$,

$$M(s) = z(t-s,B(s)) \exp\left\{\int_0^s R(t-r,B(r))\,dr\right\}$$

is a bounded $(\mathcal{F}_s)$ martingale (to see this we use Itô calculus on $M(s)$ and show that all dt terms vanish as z solves $z_t = z_{xx} + zR$ and the only remaining terms are the dB

terms), bounded by $\exp\{\|R\|_\infty s\}$ ensuring it is almost surely convergent as $s \uparrow t$ when we take limits below. Using the fact that $z(r,x) \to z(0,x)$ for almost all x one can deduce that $z(t-s, B(s)) \to z(0, B(t))$ $\mathbb{P}_x$ almost surely. Let $s \uparrow t$ to get

$$z(t,x) = \mathbb{E}_x \left[z(0, B(t)) \exp\left\{ \int_0^t R(t-r, B(r))\, dr \right\} \right].$$

Given that M is an $(\mathcal{F}_s)$-martingale, the above argument is also valid if we take expectations at an $(\mathcal{F}_s)$ stopping time τ satisfying $\tau \le t$. Considering the expression at 0 with stopping time $\tau \wedge s$ and letting $s \uparrow t$ gives

$$M(0) = z(t,x) = \mathbb{E}_x\left[M(\tau)\right] = \mathbb{E}_x \left[z(t-\tau, B(\tau)) \exp\left\{ \int_0^\tau R(t-r, B(r))\, dr \right\} \right]. \quad (3.2)$$

Now, consider the stopping time

$$\tau = \inf_{0\le z\le t} \{ z \,|\, M(z) = 0 \} \wedge t$$

and pick x_1 such that $z(t, x_1) > 0$, in particular $x_1 \ge \theta(z(t))$. Then

$$\begin{aligned} \mathbb{E}_{x_1}[M(\tau)] &= \mathbb{E}_{x_1}\left[M(\tau)\mathbb{I}_{(\tau<t)}\right] + \mathbb{E}_{x_1}\left[M(\tau)\mathbb{I}_{(\tau\ge t)}\right] \\ &= 0 + \mathbb{E}_{x_1}\left[M(\tau)\mathbb{I}_{(\tau\ge t)}\right] \\ &= z(t, x_1) > 0 \end{aligned}$$

by construction. Hence, $\mathbb{P}[\tau = t] \ge \mathbb{P}[M(\tau) > 0] > 0$. From this we can construct a deterministic continuous path $(\xi(s) : s \in [0,t])$ such that $\xi(0) = x_1$ and

$$z(t-s, \xi(s)) > 0 \text{ for } 0 \le s \le t.$$

Now let us consider another stopping time defined by

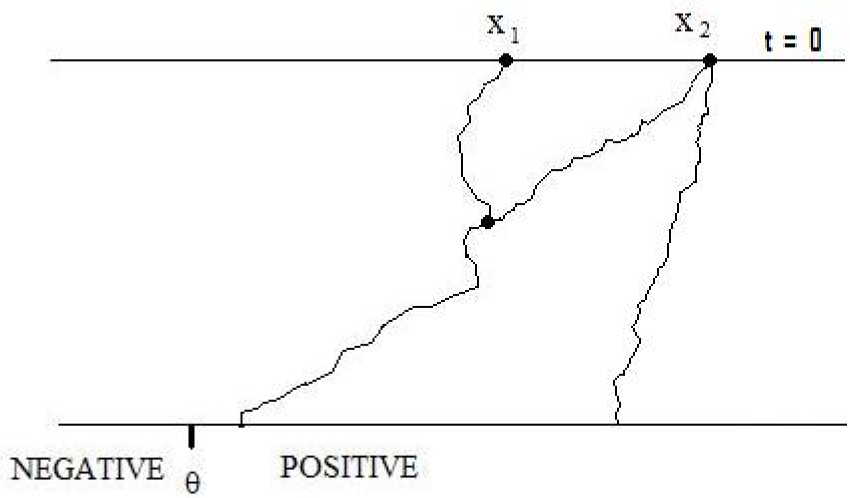

Figure 3.2: Construction of a deterministic curve.

$$\tau^* = \inf_{0\leq z\leq t}\{z\,|B(z) = \xi(z)\} \wedge t$$

and apply (3.2) when $x = x_2 > x_1$ with τ replaced by τ^*. We claim $M(\tau^*) \geq 0$.

To see this, note that given $\{\omega : M(\tau^*) = 0\} \subseteq \{\omega : z(t-\tau^*, B(\tau^*)) = 0\} \subseteq \{\tau^* = t\}$, equality to 0 occurs only when $\tau^* = t$. If $\tau^* = t$ then $B(\tau^*) = B(t) \geq \xi(t)$ but $z(0, \xi(t)) > 0$ and given the assumption on the initial condition, $z(0, B(t)) \geq 0$ and hence $M(\tau^*) \geq 0$. The strict inequality $M(\tau^*) > 0$ occurs whenever we meet the constructed continuous curve ξ, this is whenever $\tau^* < t$.

It is easy to see there is a non-zero probability that a Brownian path started at x_2 will intersect with the deterministic path ξ before time t. This shows $\mathbb{P}[M(\tau^*) > 0] > 0$

and hence

$$z(t, x_2) = \mathbb{E}_{x_2}[M(\tau^*)] > 0.$$

This demonstrates that if $z(t, x_1) = r(t, x_1) - q(t, x_1) > 0$ and $x_1 < x_2$ then $z(t, x_2) = r(t, x_2) - q(t, x_2) > 0$. If $\theta(r(t) - q(t)) \in (-\infty, \infty)$ then we can choose x_1 arbitrarily close to $\theta(r(t) - q(t))$ and the proof is finished. In the case where $\theta(r(t) - q(t)) = \pm\infty$ the proof is easier. ■

3.2 Polygonal Approximation to the Noise W_t

Introduction. *From Chapter 2, W_t defines a one-dimensional Brownian motion. In this section we define a piecewise approximation to the noise W_t by W_t^ϵ, see figure 3.3 below. The idea mimics that of the original Wong-Zakai Paper ([33]) and is to take a polygonal approximation of the noise and show the corresponding solution of the approximated equation converges, in mean-square, to the solution of the true equation. This is somewhat counter intuitive given that to construct the approximation, full information of the original noise is required. Despite this, such approximations are sufficient for classical solutions as we have defined.*

Remark. To our surprise we could not find this result in the literature. The closest results we could find were the following:

1. In [30] convergence in distribution is established in a setting that would cover our equation on a finite interval of $\mathbb{R}$, with suitable boundary conditions (one of the authors of this paper agreed that some work was needed to extend the method to

the case of the whole real line);

2. In [20] almost sure results are established but for equations on Hilbert spaces. This, however, requires differentiability properties of the drift and diffusion coefficients that do not hold for our equation.

Our point, however, is that a proof of the result follows along the same lines as the original Wong-Zakai ([33]) result for one-dimensional SDEs. We give the details at the end of this chapter.

Definition 26. *W_t^ϵ is defined such that, for $\epsilon > 0$ and $t \in [k\epsilon, (k+1)\epsilon]$ for some $k \in \mathbb{N} \cup \{0\}$, we have $W_t^\epsilon = W_{k\epsilon} + (t - k\epsilon)\left(W_{(k+1)\epsilon} - W_{k\epsilon}\right)/\epsilon$.*

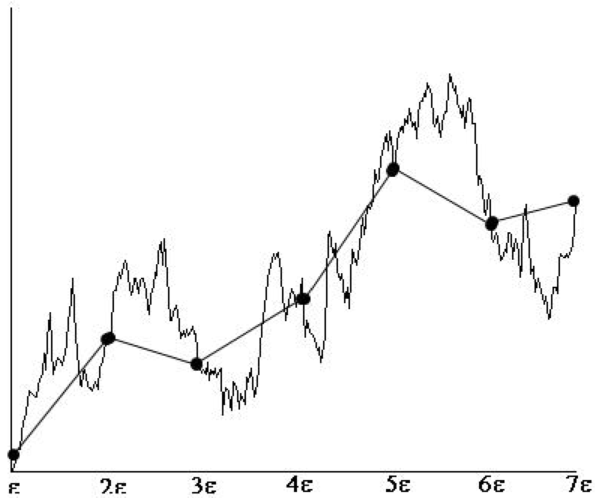

Figure 3.3: Approximating a Brownian motion using a polygonal approximation.

Remark. From the above definition of W_t^ϵ it is clear that for $\epsilon > 0$ and $t \in [k\epsilon, (k+1)\epsilon]$ for some $k \in \mathbb{N} \cup \{0\}$, we have $\dot{W}_t^\epsilon = \left(W_{(k+1)\epsilon} - W_{k\epsilon}\right)/\epsilon$.

Theorem 27. *Consider the two equations*

$$du(t,x) = u_{xx}(t,x)dt + f(u(t,x))dt + g(u(t,x)) \circ dW_t \tag{3.3}$$

$$u^\epsilon_t(t,x) = u^\epsilon_{xx}(t,x) + f(u^\epsilon(t,x)) + g(u^\epsilon(t,x))\,\dot{W}^\epsilon_t \tag{3.4}$$

*for $t \geq 0$ and $x \in \mathbb{R}$ where f and g satisfy hypotheses (**H1**) and (**H2**) from Chapter 2 and W_t, $\dot{W}^\epsilon_t$ are as defined above. Suppose also that the initial conditions are equal $u_0 = u^\epsilon_0 \in [0,1]$, possibly random. Then for fixed $x \in \mathbb{R}$, $t \geq 0$ we have*

$$u^\epsilon(t,x) \xrightarrow[\epsilon \to 0]{\mathbb{L}^2} u(t,x). \tag{3.5}$$

3.3 Solutions cross at most once

Introduction. *We have shown that in the deterministic case, equation (3.1), if two solutions u and v start with u more stretched than v, then u remains more stretched than v for all time. In this section we define stretching in a more rigorous way and expand last sections stretching result to the stochastic setting by applying a polygonal approximation to the noise term.*

Lemma 28. *Let r and q be solutions to equation (3.3) and r^ϵ and q^ϵ be solutions to equation (3.4). Suppose also that r and r^ϵ have the same initial condition as do q and q^ϵ all living in $\mathbb{B}_{dec}$. Then, for fixed $t > 0$,*

$$\limsup_{\epsilon \to 0} \theta(r^\epsilon(t) - q^\epsilon(t)) \leq \theta(r(t) - q(t)) \ \mathbb{P}\text{-almost surely.} \tag{3.6}$$

Proof. By Theorem 27, for fixed $t > 0$, $z \in \mathbb{R}$,

$$r^\epsilon(t,z) \xrightarrow{\mathbb{L}^2} r(t,z),\ q^\epsilon(t,z) \xrightarrow{\mathbb{L}^2} q(t,z) \text{ as } \epsilon \to 0. \tag{3.7}$$

We now fix $z \in \mathbb{Q}$ and apply a diagonalisation argument to prove, along a suitable subsequence $\{\epsilon_n\}$, the above result is true for almost all $\omega \in \Omega$ and then we take a further sub-subsequences $\{\epsilon_n(j)\}$ to prove this result is in fact true for almost all $\omega \in \Omega$ and all $z \in \mathbb{Q}$, that is

$$r^{\epsilon_{n(j)}}(t, z, \omega) \to r(t, z, \omega),\ q^{\epsilon_{n(j)}}(t, z, \omega) \to q(t, z, \omega) \tag{3.8}$$

as $j \to \infty$ tends to zero. Fix such an ω and suppose that $x > \theta(r(t) - q(t))$. Then there must exist a $y \in \mathbb{Q}$ with $y < x$ such that $r(t, y, \omega) > q(t, y, \omega)$. Given (3.8), we can choose N so that, for all $n \geq N$, $r^{\epsilon_n}(t, y, \omega)$ is sufficiently close to $r(t, y, \omega)$ and $q^{\epsilon_n}(t, y, \omega)$ is sufficiently close to $q(t, y, \omega)$ resulting in $r^{\epsilon_n}(t, y, \omega) > q^{\epsilon_n}(t, y, \omega)$, that is $\theta(r^{\epsilon_n} - q^{\epsilon_n}) \leq y$. ■

Theorem 29. *Let r and q be solutions to equation (3.3) with (possibly random) initial conditions $r(0), q(0) \in \mathbb{B}_{dec}$ satisfying $r(0, x) \geq q(0, x)$ for all $x > \theta(r(0) - q(0))$ almost surely. Then $r(t, x) \geq q(t, x)$ for all $x \geq \theta(r(t) - q(t))$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. Fix $t > 0$ and fix $\omega \in \Omega$ as in (3.8). Suppose now $x \in \mathbb{Q}$ and $x > \theta(r(t)-q(t))$ then, by Lemma 28, $x > \theta(r^{\epsilon_n}(t)-q^{\epsilon_n}(t))$ for sufficiently small ϵ_n where r^{ϵ_n} and q^{ϵ_n} are solutions to equation (3.4). Set $J(r^{\epsilon_n}(t, x), t, x) = f(r^{\epsilon_n}(t, x)) + g(r^{\epsilon_n}(t, x))\dot{W}_t^{\epsilon_n}$ in Proposition 25. Then, as f and g satisfy hypothesis (**H1**) and (**H2**), it is clear that conditions (**P1**) and (**P2**) of Proposition 25 are satisfied as well as $r^{\epsilon_n} \in \mathcal{C}^{1,2}((0, \epsilon_n] \times \mathbb{R})$ for some $\epsilon_n > 0$. Similarly for q^{ϵ_n}. Applying Proposition 25 to r^{ϵ_n} and q^{ϵ_n} we deduce, $r^{\epsilon_n}(t, x, \omega) > q^{\epsilon_n}(t, x, \omega)$ for $x > \theta(r^{\epsilon_n}(t) - q^{\epsilon_n}(t))$. Hence, taking limits, we have $r(t, x, \omega) \geq q(t, x, \omega)$. Given that solutions r and q are continuous in x, the above argument can be extended from the set of rational numbers $\mathbb{Q}$ to the whole of the real line $\mathbb{R}$. We then repeat the argument over the interval $[k\epsilon_n, (k+1)\epsilon_n]$ for $k = 1, 2, 3, ...$ and continue in this way to cover the whole region $(0, T]$. ■

3.4 The concept of stretching

Introduction. *In this section we introduce the notion of stretching on our solution space $\mathbb{B}_{dec}$. First, however, we introduce the concept of crossing which will be fundamental to the stretching definition.*

Definition 30. *Crosses*

Consider two functions U and $V : \mathbb{R} \to [0,1]$. Define U **crosses** *V by $U(x) \geq V(x)$ for all $x > \theta(U - V)$.*

To fully appreciate this definition and develop our intuition, let's consider some examples. In the first example (figure 3.4) it is clear that U crosses V as for $x > \theta$, that is for any x beyond the crossing point, U lies above V. That is there is one unique point at which U cuts V.

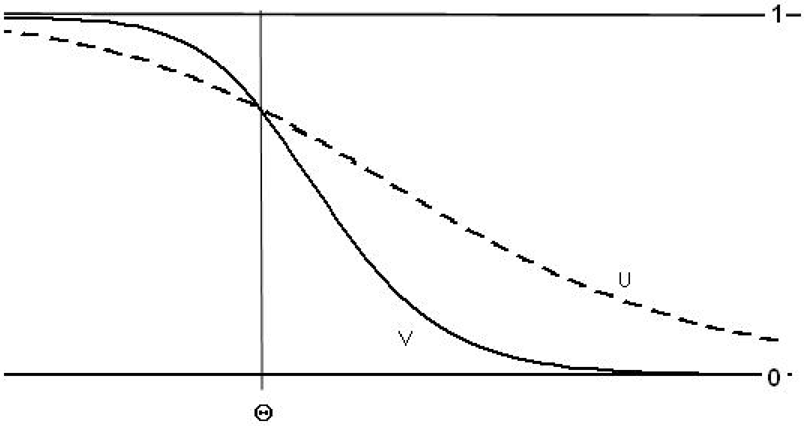

Figure 3.4: First example of the definition U crosses V.

In our second example we cover a subtlety in our crossing definition. Figure 3.5 shows that although U crosses V, this does not mean that U has to lie strictly above V at all later positions, in fact U may merge with V. Our third example, figure 3.6, is that of

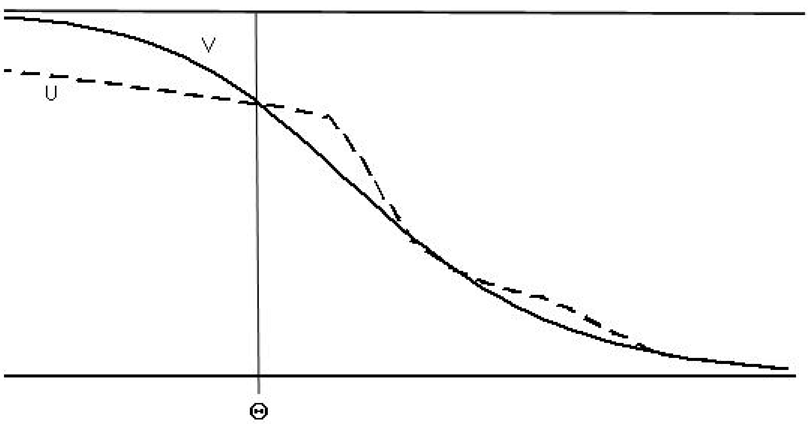

Figure 3.5: Second example of the definition U crosses V.

functions which do not cross. As indicated in our second example above, although U may cross V this does not mean that U has to lie above V for all x beyond the crossing point θ. However, if U does cross V, then it cannot cut V again at any point $x > \theta$.

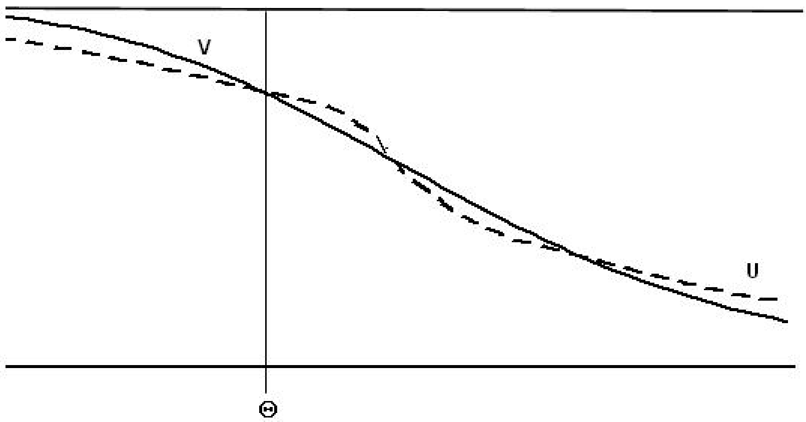

Figure 3.6: An example where U does not cross V.

Remark. If $\theta(r-q) = +\infty$ then $r \leq q$ on $\mathbb{R}$. If $\theta(r-q) = -\infty$ and r **crosses** q then $r \geq q$ on $\mathbb{R}$.

We can now restate the results from section 3.3 as follows:

Corollary 31. *Let r and q be solutions to equation (3.3) with, possibly random, initial conditions $r(0), q(0) \in \mathbb{B}_{dec}$. If $r(0)$ **crosses** $q(0)$ $\mathbb{P}$-almost surely then $r(t)$ **crosses** $q(t)$ $\mathbb{P}$-a.s. for all $t \geq 0$.*

Proof. The proof of this is a direct consequence of Theorem 29 and the definition of **crosses** (Definition 30). ■

Remark. Although we only concentrate on the one-dimensional non-spatial noise in this thesis, we believe the above result would pass over unchanged in the case of a spatially homogenous, white in time noise.

Definition 32. ***Stretched***

Consider two functions U and V in $\mathbb{B}_{dec}$. Define $\mathcal{R}(U)$ to be the open interval $(U(-\infty), U(\infty))$

and similarly for $\mathcal{R}(V)$. Then U is said to be more **stretched** *than V, written $U \overset{s}{\succ} V$, if $\mathcal{R}(U) \cap \mathcal{R}(V) \neq \emptyset$ and $U(\cdot - a)$* **crosses** *$V(\cdot)$ for all $a \in \mathbb{R}$.*

Remark. In the above definition, the requirement $U(\cdot - a)$ ***crosses*** $V(\cdot)$ for all $a \in \mathbb{R}$ can be expressed equivalently as $U(\cdot)$ ***crosses*** $V(\cdot - b)$ for all $b \in \mathbb{R}$ as shown in figure 3.7 below.

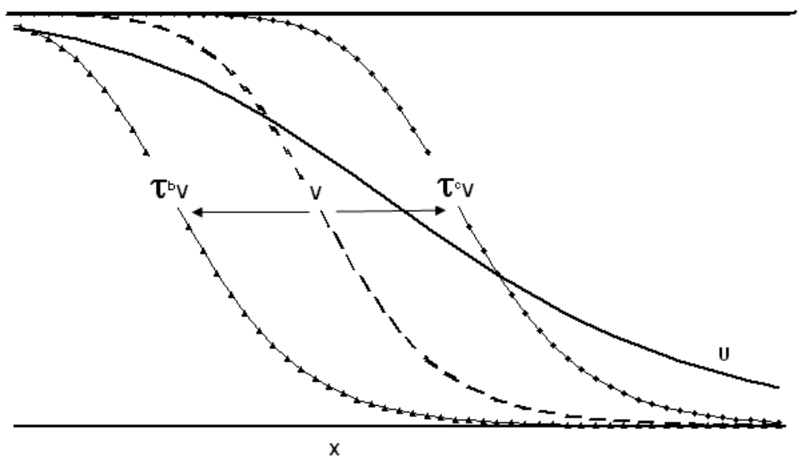

Figure 3.7: U is more stretched than V.

Example 4. Consider two functions $U(x) = \mathbb{I}_{[x \leq 0]}$ and V with V defined as **any** map in $\mathbb{B}^{1,0}_{dec}$. Then V is more stretched than U.

Example 5. Consider two differentiable functions U and V on $\mathbb{B}_{dec}$ satisfying, for some $K > 0$ and $\delta \in (0, \frac{1}{2})$, $U'(x) \leq -K$ for all x such that $U(x) \in [\delta, 1-\delta]$, and $V(x) \in [\delta, 1-\delta]$ and $V'(x) > -K$ for all x. Then V is more stretched than U.

Remark. The concept of stretching is less restrictive than the reader may initially believe as the following lemmas will show.

We will now prove that for two solutions r and q, if q is less **stretched** than r at time 0, q will be less stretched than r at all further times $t > 0$.

Corollary 33. *Let r and q be solutions to (3.3) such that*

$$q(0) \stackrel{s}{\prec} r(0) \ \mathbb{P}\text{-almost surely.} \tag{3.9}$$

Then

$$q(t) \stackrel{s}{\prec} r(t) \ \mathbb{P}\text{-almost surely for all } t > 0. \tag{3.10}$$

Recall the notation: $\tau^a T$ to mean $T(\cdot - a)$.

Proof. The key to this proof is the fact that, for any $a \in \mathbb{R}$, any translate $(\tau^a q(t) : t \geq 0)$ still represents a solution. Given that $q(0) \stackrel{s}{\prec} r(0)$ we have, for each $a \in \mathbb{R}$, $q(0, x - a) \leq r(0, x)$ for all $x \geq \theta(r(0) - \tau^a q(0))$. Now we can rewrite this as, for each $a \in \mathbb{R}$, $\tau^a q(0, x) \leq r(0, x)$ for all $x \geq \theta(r(0) - \tau^a q(0))$. As $\tau^a q$ is also a solution to (3.3), by Theorem 29 we have that $\tau^a q(t, x) \leq r(t, x)$ for all $x \geq \theta(r(t) - \tau^a q(t))$. This conclusion holds for each $a \in \mathbb{R}$ and consequently, for each $a \in \mathbb{R}$, $q(t, x - a) \leq r(t, x)$ for all $x \geq \theta(r(t) - \tau^a q(t))$ and hence $q(t) \stackrel{s}{\prec} r(t)$ as required. ■

Remark. We do not believe this result would pass over in the case of spatially homogenous noises given that the translation would have an impact upon the noise. However, one idea in the extension to a wider variety of noises would be to restate the above result in terms of the laws of the solution. We hope to explore this further in a later paper.

Lemma 34. *Suppose $U, V : \mathbb{R} \to [0, 1]$ are continuous and strictly decreasing functions such that $U \stackrel{s}{\succ} V$. Then $\mathcal{R}(U) \subseteq \mathcal{R}(V)$.*

Proof. Before starting we note that given that U and V are continuous and strictly decreasing, $\mathcal{R}(U)$ and $\mathcal{R}(V)$ are the open intervals $(U(-\infty), U(\infty))$ and $(V(-\infty), V(\infty))$

respectively. If $\sup \mathcal{R}(U) > \sup \mathcal{R}(V)$ then $\theta(U - V) = -\infty$ which also holds true for any translation of U, $\theta(\tau^a U - V) = -\infty$. By definition of stretching, this means $U(x - a) = \tau^a U(x) \geq V(x)$ for all $x \in \mathbb{R}$, for all $a \in \mathbb{R}$. We can rewrite this as $U(y) \geq V(x)$ for all $y, x \in \mathbb{R}$ and it is clear $\mathcal{R}(U) \cap \mathcal{R}(V) = \emptyset$ given that $\mathcal{R}(U)$ and $\mathcal{R}(V)$ are open intervals. This contradicts the definition of stretching. Similarly, if $\inf \mathcal{R}(U) < \inf \mathcal{R}(V)$ then given $U \overset{s}{\succ} V$, U cannot have crossed V at an $x \in \mathbb{R}$ and hence $\theta(U - V) = \infty$. Again, this holds for all translates of U, $\theta(\tau^a U - V) = \infty$. By definition of stretching this means $U(x - a) = \tau^a U(x) \leq V(x)$ for all $x \in \mathbb{R}$, $a \in \mathbb{R}$. We can rewrite this as $U(y) \leq V(x)$ for all $y, x \in \mathbb{R}$ and it is clear $\mathcal{R}(U) \cap \mathcal{R}(V) = \emptyset$, again a contradiction. Combining the above we have that $\sup \mathcal{R}(U) \leq \sup \mathcal{R}(V)$ and $\inf \mathcal{R}(U) \geq \inf \mathcal{R}(V)$. From this it is clear $\mathcal{R}(U) \subseteq \mathcal{R}(V)$ as required. ■

Lemma 35. *Suppose $U, V : \mathbb{R} \to [0, 1]$ are two continuously differentiable functions with $U_x, V_x < 0$ on $\mathbb{R}$. Then U is more stretched than V if-and-only-if $\mathcal{R}(U) \cap \mathcal{R}(V) \neq \emptyset$ and whenever $U(x_1) = V(x_2)$, then $U_x(x_1) \geq V_x(x_2)$.*

Proof. We will first prove **Necessity:**

Suppose $U(x_1) = V(x_2)$. Choose $a \in \mathbb{R}$ such that $a = x_2 - x_1$ then, by construction,

$$\tau^a U(x_2) = U(x_1) = V(x_2) \tag{3.11}$$

and if $x_2 \geq \theta(\tau^a U - V)$

$$\begin{aligned}
U_x(x_1) &= \lim_{h \downarrow 0} \left[\frac{U(x_1 + h) - U(x_1)}{h} \right] \\
&= \lim_{h \downarrow 0} \left[\frac{\tau^a U(x_2 + h) - U(x_1)}{h} \right] \text{ by definition of } a \\
&= \lim_{h \downarrow 0} \left[\frac{\tau^a U(x_2 + h) - V(x_2)}{h} \right] \text{ by assumption (3.11)} \\
&\geq \lim_{h \downarrow 0} \left[\frac{V(x_2 + h) - V(x_2)}{h} \right] \\
&= V_x(x_2)
\end{aligned}$$

and similarly, for $x_2 < \theta(\tau^a U - V)$:

$$\begin{aligned}
U_x(x_1) &= \lim_{h \downarrow 0} \left[\frac{U(x_1) - U(x_1 - h)}{h} \right] \\
&= \lim_{h \downarrow 0} \left[\frac{V(x_2) - \tau^a U(x_2 - h)}{h} \right] \\
&\geq \lim_{h \downarrow 0} \left[\frac{V(x_2) - V(x_2 - h)}{h} \right] \\
&= V_x(x_2).
\end{aligned}$$

Sufficient To prove this direction we will rewrite our assumption in terms of inverse functions. Given that U and V are continuously differentiable such that U_x and V_x are less than zero, we can write, for $x \in \mathcal{R}(U)$,

$$U_x(U^{-1}(x)) \left(U^{-1}\right)_x (x) = 1 \tag{3.12}$$

and similarly for V. Suppose $U(a) = V(a) = \theta_0 \in (0, 1)$ for some $a \in \mathbb{R}$, that is $U^{-1}(\theta_0) = V^{-1}(\theta_0) = a$ (See figure 3.8). Note that this does not mean U and V cross at a, only touch. Given that $\mathcal{R}(U)$, $\mathcal{R}(V)$ are open intervals we can pick $\theta_1 < \theta_0 < \theta_2$ such that $(\theta_1, \theta_2) \subseteq \mathcal{R}(U) \cap \mathcal{R}(V)$. Suppose $\theta_2 > \theta > \theta_0$. Let us now integrate over the

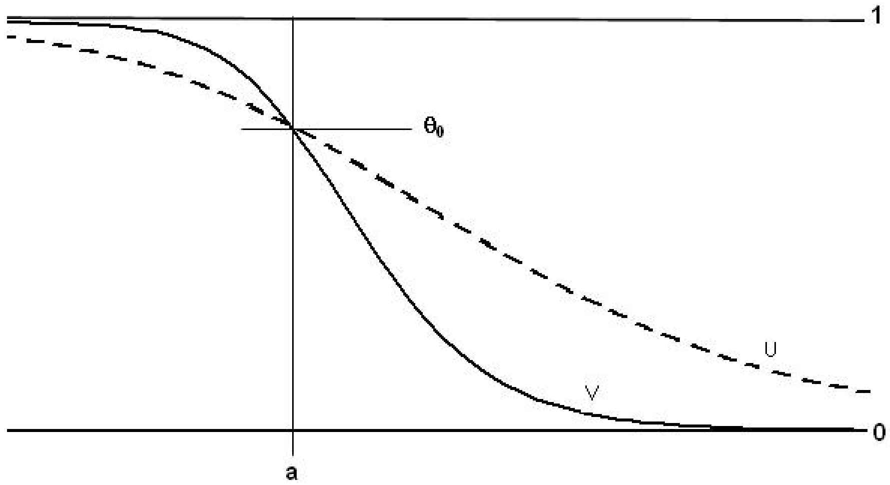

Figure 3.8: θ_0 in $(0,1)$.

range $\theta_0 \leq x \leq \theta$ to get

$$\begin{aligned} U^{-1}(\theta) - U^{-1}(\theta_0) &= \int_{\theta_0}^{\theta} (U^{-1}(x))_x \, dx \\ &\leq \int_{\theta_0}^{\theta} (V^{-1}(x))_x \, dx \\ &= V^{-1}(\theta) - V^{-1}(\theta_0) \end{aligned}$$

where we have used equation (3.12) for both U and V. By definition of θ_0 we can rewrite this as

$$U^{-1}(\theta) \leq V^{-1}(\theta) \text{ for } \theta_2 > \theta > \theta_0. \tag{3.13}$$

Similarly we have $U^{-1}(\theta) \geq V^{-1}(\theta)$ for $\theta_0 > \theta > \theta_1$ (See figure 3.9).

Note that $U^{-1}(\theta) \to -\infty$ as $\theta \uparrow U(-\infty)$ and $U^{-1}(\theta) \to \infty$ as $\theta \downarrow U(\infty)$. We can choose $\theta_1 = U(\infty) \vee V(\infty)$ and $\theta_2 = U(-\infty) \wedge V(-\infty)$. By (3.13) it is clear that $U(\infty) \geq V(\infty)$ and similarly $U(-\infty) \leq V(-\infty)$ and hence, $\theta_1 = U(\infty)$ and

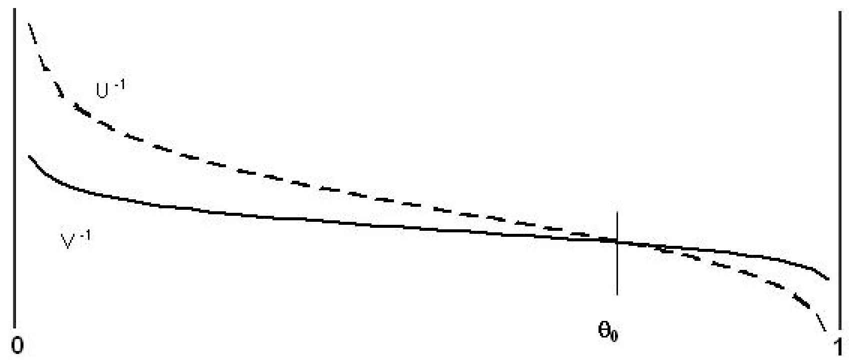

Figure 3.9: Stretching when we consider inverse functions.

$\theta_2 = U(-\infty)$. This shows that $\mathcal{R}(U) \subseteq \mathcal{R}(V)$. We can then rewrite (3.13) as

$$U^{-1}(\theta) \leq V^{-1}(\theta) \text{ for all } \theta \geq \theta_0,\ \theta \in \mathcal{R}(U)$$

$$U^{-1}(\theta) \geq V^{-1}(\theta) \text{ for all } \theta \leq \theta_0,\ \theta \in \mathcal{R}(U).$$

Inverting these maps gives

$$U(x) \geq V(x) \text{ for all } x \geq a \tag{3.14}$$

$$U(x) \leq V(x) \text{ for all } x \leq a. \tag{3.15}$$

We will now consider the three possibilities for $\theta(U - V)$ to show U crosses V. If $\theta(U - V) = -\infty$ then, by a contradiction argument, if U and V cross at $x^* > -\infty$ then $\theta(U - V) > -\infty$. Hence, $U(x) \geq V(x)$ for all $x > -\infty$. If $\theta(U - V) \in \mathbb{R}$ then U and V must cross and by taking a in the above proof as this crossing point, U crosses V as required. If $\theta(U - V) = \infty$ then there is nothing to prove. The above argument holds for all translations of U and we can write, if $\tau^b U(a) = V(a)$,

$$\tau^b U(x) \geq V(x) \text{ for all } x \geq a$$

$$\tau^b U(x) \leq V(x) \text{ for all } x \leq a.$$

This holds for all $b \in \mathbb{R}$ and hence, $U \overset{s}{\succ} V$. ■

Corollary 36. *Suppose $U, V : \mathbb{R} \to [0,1]$ are two continuously differentiable functions with U_x, $V_x < 0$, satisfying U is more stretched than V. Then if $U(x) = V(x)$ then $U(y) \geq V(y)$ for all $y \geq x$.*

Proof. This corollary is a direct consequence of the sufficiency proof in Lemma 35 and shows equations (3.14) and (3.15) where a is defined as any point such that $U(a) = V(a)$. This completes the proof. ■

Remark. As suggested in the introduction, stretching may be reformulated by thinking of U and V in the phase-plane. If $U \overset{s}{\succ} V$ and U, V are sufficiently smooth then in the phase-plane U lies below V, an easier property to test and a more intuitive representation of stretching where more standard comparison arguments can be applied. This intuition was developed by the papers of Fife and McLeod (See [9]) and will be explored in Chapter 6.

3.5 Stochastic Ordering

Introduction. *In this section we show that stretching defines a closed pre-order on $\mathbb{B}_{dec}$. This will be important when we consider the limiting law of solutions and the proof* *that the law of the solution to equation (2.1), started from the Heaviside initial condition, converges. We will also show that for stretching, associativity fails as given two functions p and q satisfying $p \overset{s}{\succ} q$ and $q \overset{s}{\succ} p$ may be translates of one another rather than be equal.*

Definition 37. ***Partial-order*** *Consider some set $\mathcal{A}$ and a binary relation $\sim$ on $\mathcal{A}$. Then $\sim$ is a partial-order if it is reflexive, associative and transitive, that is*

(1) $p \sim p$ for all $p \in \mathcal{A}$ (Reflexive)

(2) If $p \sim q$ and $q \sim p$ for some $p, q \in \mathcal{A}$ then $q = p$ (Associative)

(3) If $p \sim q$, $q \sim r$ for some $p, q, r \in \mathcal{A}$ then $p \sim r$ (Transitive).

Definition 38. ***Pre-order*** *Consider some set $\mathcal{A}$ and a binary relation $\sim$ on $\mathcal{A}$. Then $\sim$ is a pre-order if it is reflexive and transitive, as defined above.*

Lemma 39. *The set $M = \{(f,g) \in \mathbb{B}^{1,0}_{dec} \times \mathbb{B}^{1,0}_{dec} : f \overset{s}{\succ} g\}$ is closed in the product topology on $\mathbb{B}^{1,0}_{dec} \times \mathbb{B}^{1,0}_{dec}$.*

Proof. For $n \in \mathbb{N}$ suppose we have functions f_n and g_n in $\mathbb{B}^{1,0}_{dec}$ such that, $f_n \xrightarrow{\mathbb{L}^1_{loc}} f$ and $g_n \xrightarrow{\mathbb{L}^1_{loc}} g$ and, for each n, $f_n \overset{s}{\succ} g_n$. To demonstrate that M is closed we have to show that, in the limit, $f \overset{s}{\succ} g$. Given $f_n \to f$ and $g_n \to g$, both in $\mathbb{B}^{1,0}_{dec}$, then it is clear by Lemma 9 that $f_n \to f$ a.e. and $g_n \to g$ a.e.. Define $\mathbb{D}^c$, the complement of a set $\mathbb{D}$, as the set of all points for which convergence fails in the above. Given $\delta > 0$ there exists $x \in (\theta(f-g), \theta(f-g)+\delta)$ such that $f(x) > g(x)$. As $\mathbb{D}$ is dense in $\mathbb{R}$, since $\mathbb{D}^c$ has measure zero, we can choose $x \in \mathbb{D}$. Pick n_0 such that $f_n(x) > g_n(x)$ for all $n \geq n_0$. Hence, $x \geq \theta(f_n - g_n)$ and $f_n(x) \geq g_n(x)$ for all $n \geq n_0$ gives us $f(x) \geq g(x)$. Given f

and g are both right continuous we have $f(x) \geq g(x)$ for all $x > \theta(f-g)$ and the proof is complete. ■

Remark. We do not believe the space $\mathbb{B}_{dec} \times \mathbb{B}_{dec}$ is closed.

Lemma 40. *Stretching defines a pre-order on both* $\mathbb{B}_{dec}$ *and* $\mathbb{B}_{dec}^{1,0}$.

Proof. **Reflexive:** Given that either $\theta(p-p) = \pm\infty$ it is easy to check $p \overset{s}{\succ} p$.

Transitivity: We first consider $\mathbb{B}_{dec}^{1,0}$. Define the convolution $p^\epsilon(x) = p(x) * \Gamma_\epsilon(x)$ where Γ_ϵ denotes the Gaussian function $\Gamma_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{x^2}{2\epsilon}\right)$. The resulting p^ϵ is infinitely differentiable and, in the limit as epsilon tends to zero, $p^\epsilon(x) = p(x) * \Gamma_\epsilon(x) \to p(x)$ a.e. and therefore $p^\epsilon \to p$ in $\mathbb{B}_{dec}$. Define q^ϵ, r^ϵ similarly. By Corollary 33 in the special case $f = g = 0$, if $p \overset{s}{\succ} q$ then $p^\epsilon \overset{s}{\succ} q^\epsilon$. Note that p^ϵ, q^ϵ and r^ϵ lie in $\mathbb{B}_{dec}^{1,0}$, are $\mathcal{C}^2$ functions and are strictly decreasing. Using this we will show that if $p^\epsilon \overset{s}{\succ} q^\epsilon$ and $q^\epsilon \overset{s}{\succ} r^\epsilon$ then $p^\epsilon \overset{s}{\succ} r^\epsilon$. To finish we will then take the limit as epsilon tends to zero to reverse the smoothing and show, in the limit, the same conclusion holds. Suppose $p^\epsilon(x_1) = r^\epsilon(x_2)$. Since p^ϵ and r^ϵ are strictly decreasing, this common value lies in (0,1). Then, given that q^ϵ is continuous and onto (0,1) being in $\mathbb{B}_{dec}^{1,0}$, there exists an x^* such that $q^\epsilon(x^*) = p^\epsilon(x_1) = r^\epsilon(x_2)$. By Lemma 35 we conclude, $p_x^\epsilon(x_1) \geq q_x^\epsilon(x^*) \geq r_x^\epsilon(x_2)$. Also by Lemma 35 we know that $p^\epsilon \overset{s}{\succ} r^\epsilon$. We now appeal to Lemma 39 to conclude that, given $p^\epsilon \to p$ and $r^\epsilon \to r$, both on $\mathbb{B}_{dec}^{1,0}$, then $p \overset{s}{\succ} r$. We now consider $\mathbb{B}_{dec}$. Unlike $\mathbb{B}_{dec}^{1,0}$, any two functions (smoothed if necessary as above) in $\mathbb{B}_{dec}$ may not meet at any point given the unspecified behavior at the end points. We mimic the case for functions in $\mathbb{B}_{dec}^{1,0}$ and use the same notation as above to define p^ϵ, q^ϵ and r^ϵ. Given the result of Lemma 34, as $p^\epsilon \overset{s}{\succ} q^\epsilon$ and $q^\epsilon \overset{s}{\succ} r^\epsilon$ we have that $\mathcal{R}(p^\epsilon) \subseteq \mathcal{R}(q^\epsilon)$ and $\mathcal{R}(q^\epsilon) \subseteq \mathcal{R}(r^\epsilon)$. The argument of transitivity now follows directly from the same proof as in the $\mathbb{B}^{1,0}_{dec}$ case. ■

Remark. Associativity fails both in $\mathbb{B}^{1,0}_{dec}$ and $\mathbb{B}_{dec}$ as the following lemma shows.

Lemma 41. *Suppose $p \overset{s}{\succ} q$ and $q \overset{s}{\succ} p$ then, providing $p \neq q$, there exists a translation $b \in \mathbb{R}$ such that $p = \tau^b q$.*

Proof. Consider the convolution of p and q with the Gaussian function, denoted p^ϵ and q^ϵ respectively. This ensures p^ϵ and q^ϵ are infinitely differentiable whilst also retaining the stretching property (use Corollary 33 in the special case $f = g = 0$). Also note, it is automatic that if $p^\epsilon \overset{s}{\succ} q^\epsilon$ then $\widetilde{p^\epsilon} \overset{s}{\succ} \widetilde{q^\epsilon}$ (recall $\tilde{\cdot}$ indicates the centred function, see Definition 18). Now, by centring at 0, the definition of $\widetilde{p^\epsilon} \overset{s}{\succ} \widetilde{q^\epsilon}$ can be expressed as $\widetilde{p^\epsilon}(0) = \widetilde{q^\epsilon}(0)$ and

$$\widetilde{p^\epsilon}(x) \geq \widetilde{q^\epsilon}(x) \text{ for all } x \geq 0$$

$$\widetilde{p^\epsilon}(x) \leq \widetilde{q^\epsilon}(x) \text{ for all } x \leq 0.$$

Similarly, as $q^\epsilon \overset{s}{\succ} p^\epsilon$ we have

$$\widetilde{q^\epsilon}(x) \geq \widetilde{p^\epsilon}(x) \text{ for all } x \geq 0$$

$$\widetilde{q^\epsilon}(x) \leq \widetilde{p^\epsilon}(x) \text{ for all } x \leq 0$$

and hence, $\widetilde{q^\epsilon}(x) = \widetilde{p^\epsilon}(x)$ for all $x \in \mathbb{R}$. As $p \neq q$ this means there exists a translation such that, for some $b \in \mathbb{R}$, $p = \tau^b q$. ■

Remark. Other simple properties of orders do not hold. For example, let $g = \mathbb{I}_{(-\infty,A)}$ and $i = \mathbb{I}_{(-\infty,B)}$ for some $A < B$ and choose $f = h \in \mathbb{B}^{1,0}_{dec}$ with gradient $f'(x) = h'(x) \leq$

$2(A-B)$ for all x such that, for $\delta \in (0, \frac{1}{2})$, $f, h \in [\delta, 1-\delta]$. Then it is clear that $f \overset{s}{\succ} g$ and $h \overset{s}{\succ} i$ but $\frac{f+h}{2} \overset{s}{\not\succ} \frac{g+i}{2}$.

In the next lemma we give an alternative proof of transitivity for functions in $\mathbb{B}_{dec}^{1,0}$ but by using the result from Corollary 36.

Lemma 42. *For three functions p, q and r in $\mathbb{B}_{dec}^{1,0}$ if $p \overset{s}{\succ} q$ and $q \overset{s}{\succ} r$ then $p \overset{s}{\succ} r$.*

Proof. First assume that p, q and r are continuous. Otherwise, mimicking the argument as in Lemma 40, we can define the convolution $p^\epsilon(x) = p(x) * \Gamma_\epsilon(x)$ where Γ_ϵ denotes the Gaussian function $\Gamma_\epsilon(x) = \frac{1}{\sqrt{2\pi\epsilon}} \exp\left(-\frac{x^2}{2\epsilon}\right)$. The resulting p^ϵ is infinitely differentiable and, in the limit as epsilon tends to zero, $p^\epsilon(x) = p(x) * \Gamma_\epsilon(x) \to p(x)$ a.e. and therefore $p^\epsilon \to p$ in $\mathbb{B}_{dec}$. We can define q^ϵ, r^ϵ similarly. Suppose $x \geq \theta(\tau^a p - q)$ for some a and $\theta(\tau^a p - q) < \infty$. Given continuity properties and the fact that p, r and q lie in $\mathbb{B}_{dec}^{1,0}$ we can choose b such that

$$\theta(\tau^a p - q) = \theta(\tau^a p - \tau^b r).$$

It is clear that for all $x \geq \theta(\tau^a p - q)$, $\tau^a p(x) \geq \tau^b r(x)$ and there are two possibilities as to the relationship between $\theta(\tau^b r - q)$ and $\theta^* = \theta(\tau^a p - q) = \theta(\tau^a p - \tau^b r)$:

(1) $\theta(\tau^b r - q) \leq \theta^*$;

(2) $\theta(\tau^b r - q) > \theta^*$.

Case 1. Suppose $\theta(\tau^b r - q) \leq \theta^*$, see figure 3.10, then for all $x \geq \theta^*$ we have $\tau^b r(x) \geq q(x)$ and $\tau^a p(x) \geq \tau^b r(x)$. Hence, it is clear that for $x \geq \theta^*$, $\tau^a p(x) \geq \tau^b r(x) \geq q(x)$ as required.

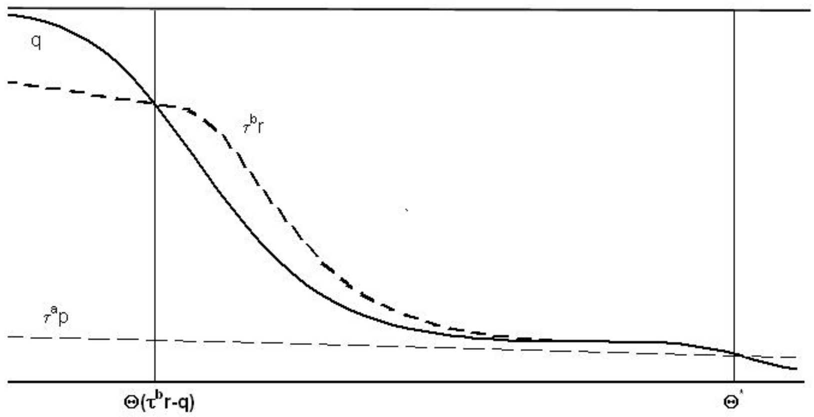

Figure 3.10: Case 1.

Case 2. Suppose $\theta(\tau^b r - q) > \theta^*$, see figure 3.11. By definition we must have that $\tau^b r$

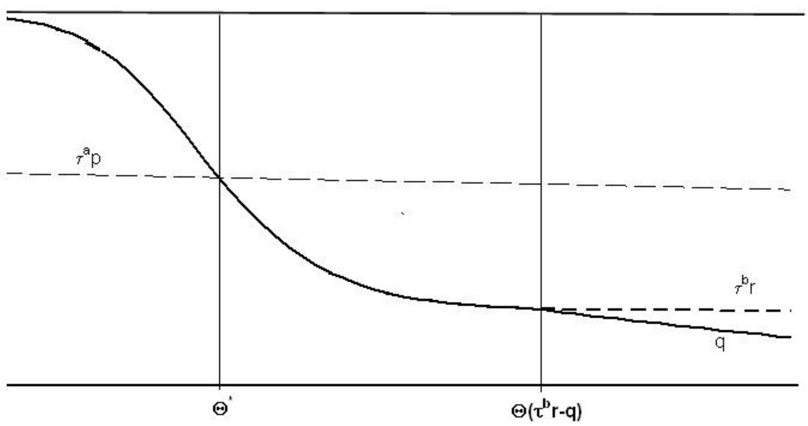

Figure 3.11: Case 2.

and q touch but do not cross at θ^*. Corollary 36 shows us that:

- For $x \in [\theta^*, \theta(\tau^b r - q))$, we have $\tau^a p(x) \geq \tau^b r(x)$ and $\tau^b r(x) = q(x)$. Hence, for all

$x \geq \theta^*$ we have $\tau^a p(x) \geq \tau^b r(x) = q(x)$ as required.

- For $x \geq \theta(\tau^b r - q)$ the argument is clear.

This completes the proof. $\blacksquare$

Remark. **Crossing** does not define a pre-order like **stretching** as in simple cases transitivity fails.

Example 6. Set $\epsilon > 0$. For constants $A, B \in \mathbb{R}$ such that $A < B$ take $p = \mathbb{I}_{(-\infty,A]}$, $q = \mathbb{I}_{(-\infty,B]}$ and $r = \mathbb{I}_{(-\infty,A]} * \Gamma_\epsilon$, that is a smoothed version of p. It is clear that p crosses q, q crosses r but p does not cross r.

Definition 43. *Let $\mathcal{P}$,$\mathcal{Q}$ be probability measures on $\mathcal{B}^{1,0}_{dec}$. Then we define $\mathcal{Q}$ as more stretched than $\mathcal{P}$, written $\mathcal{Q} \overset{s}{\succ} \mathcal{P}$, if there exists random variables u, v on the same probability space satisfying $u \overset{D}{=} \mathcal{Q}$, $v \overset{D}{=} \mathcal{P}$ and $u \overset{s}{\succ} v$ almost surely.*

Definition 44. *We will write $\mathcal{L}(u(t))$ to mean the law of the solution $u(t)$.*

Lemma 45. *If $\mathcal{P}_n, \mathcal{Q}_n \in \mathcal{M}_1(\mathbb{B}^{1,0}_{dec})$ are such that $\mathcal{P}_n \overset{D}{\to} \mathcal{P}$ and $\mathcal{Q}_n \overset{D}{\to} \mathcal{Q}$ and $\mathcal{P}_n \overset{s}{\succ} \mathcal{Q}_n$ for all n then $\mathcal{P} \overset{s}{\succ} \mathcal{Q}$.*

Proof. We can find (u_n, v_n) with $u_n \overset{D}{=} \mathcal{P}_n$ and $v_n \overset{D}{=} \mathcal{Q}_n$ and $u_n \overset{s}{\succ} v_n$ $\mathbb{P}$-almost surely. However, $(\mathcal{L}(u_n))_{n\in\mathbb{N}}$ are tight by Proposition 13 (since the sequence $(\mathcal{L}(u_n))_{n\in\mathbb{N}}$ converges) and similarly $(\mathcal{L}(v_n))_{n\in\mathbb{N}}$ are tight. So $(\mathcal{L}(u_n, v_n))_{n\in\mathbb{N}}$ are tight on $\mathcal{M}_1(\mathbb{B}^{1,0}_{dec} \times \mathbb{B}^{1,0}_{dec})$ and there exists a subsequence n' and copies $(\hat{u}_{n'}, \hat{v}_{n'}) \overset{D}{=} (u_n, v_n)$ so that $(\hat{u}_{n'}, \hat{v}_{n'}) \overset{a.s.}{\to} (u, v)$. Since $\hat{u}_{n'} \overset{s}{\succ} \hat{v}_{n'}$ almost surely we know $u \overset{s}{\succ} v$ by $\overset{s}{\succ}$ being a closed pre-order by Lemmas 40 and 39. Then, as $\mathcal{L}(u) = \lim_{n\to\infty} \mathcal{L}(\hat{u}_n) = \lim_{n\to\infty} \mathcal{L}(u_n) = \lim_{n\to\infty} \mathcal{P}_n \overset{D}{=} \mathcal{P}$ and similarly $\mathcal{L}(v) = \mathcal{Q}$, we are finished. $\blacksquare$

Remark. A similar result holds for three probability measures. That is, suppose $\mathcal{P}_n, \mathcal{Q}_n, \mathcal{R}_n \in \mathcal{M}_1(\mathbb{B}^{1,0}_{dec})$ satisfying $\mathcal{P}_n \overset{s}{\succ} \mathcal{Q}_n \overset{s}{\succ} \mathcal{R}_n$ for all n such that $\mathcal{P}_n \overset{D}{\to} \mathcal{P}$, $\mathcal{Q}_n \overset{D}{\to} \mathcal{Q}$ and $\mathcal{R}_n \overset{D}{\to} \mathcal{R}$. Then $\mathcal{P} \overset{s}{\succ} \mathcal{Q} \overset{s}{\succ} \mathcal{R}$.

Corollary 46. *If $(u_n, v_n) \in \mathbb{B}^{1,0}_{dec} \times \mathbb{B}^{1,0}_{dec}$ satisfies $u_n \overset{s}{\succ} v_n$ $\mathbb{P}$-almost surely for all n and $u_n \overset{D}{\to} u$, $v_n \overset{D}{\to} v$ then $\mathcal{L}(u) \overset{s}{\succ} \mathcal{L}(v)$.*

Proof. Take $\mathcal{P}_n = \mathcal{L}(u_n)$, $\mathcal{Q}_n = \mathcal{L}(v_n)$ in Lemma 45. ■

Theorem 47. *Let E be a Polish space. Let the state space (E, ξ) be equipped with a pre-ordering $\prec$. Let P_1, P_2,... be a sequence of probability measures on (E, ξ). The following are equivalent:*

(i) $P_1 \prec P_2 \prec \ldots$,

(ii) there exist random elements X_1,X_2,... in (E, ξ) such that $X_i \overset{\mathcal{D}}{=} P_i$ and $X_1 \prec X_2 \prec \ldots$ a.s.

Proof. The proof of this when $\prec$ defines a partial order can been found in [17] and relies upon Strassen's Theorem (see [29]). By [16] the conditions for Strassen's Theorem may be relaxed to that of a pre-order rather than a partial order. Using this the remainder of the proof, as found in [17], carries over line by line unchanged. ■

3.6 Convergence in distribution on $\mathbb{B}_{dec}$

Notation. Let us define the notation $\mathcal{Q}^{\mu}_t$ to represent the law at time t of a solution to (3.3) starting from an initial condition whose law on $\mathbb{B}_{dec}$ is μ. We will write $\mathcal{Q}^H_t$ for a solution whose initial condition is the Heaviside function $H(x) = \mathbb{I}_{[x \leq 0]}$.

Lemma 48. *For all $t_0 > 0$, $\mathcal{Q}^H_{t_0} \overset{s}{\succ} \mathcal{Q}^H_0$.*

Proof. The proof of this follows directly from the Heaviside function being less stretched than any other function. ■

Theorem 49. *For all $0 \leq s < t$, $\mathcal{Q}^H_t \overset{s}{\succ} \mathcal{Q}^H_s$.*

Proof. By Lemma 48 we have, for every $t_0 > 0$, $\mathcal{Q}^H_{t_0} \overset{s}{\succ} \mathcal{Q}^H_0$. Now consider two solutions with initial conditions whose distributions are $\mu_1 = \mathcal{Q}^H_{t_0}$ and $\mu_2 = \mathcal{Q}^H_0$. Given $\mu_1 \overset{s}{\succ} \mu_2$ we can apply Corollary 33 to show $\mathcal{Q}^{\mu_1}_t \overset{s}{\succ} \mathcal{Q}^{\mu_2}_t$. By use of the Markov Property (see Theorem 15), $\mathcal{Q}^{\mu_1}_t = \mathcal{Q}^H_{t+t_0}$. It is clear $\mathcal{Q}^{\mu_2}_t = \mathcal{Q}^H_t$. Hence, $\mathcal{Q}^H_{t+t_0} \overset{s}{\succ} \mathcal{Q}^H_t$ for all $t \geq 0$ and the proof is complete. ■

Definition 50. *Let $\tilde{\mathcal{Q}}^{\mu,a}_t$ be the law of $\tilde{u}(t)$ where $\tilde{u}(t)$ is a solution started according to law μ, centred at a such that $\tilde{u}(t,0) = a$. We will write $\tilde{\mathcal{Q}}^{H,a}_t$ as shorthand for $\tilde{\mathcal{Q}}^{\delta_H,a}_t$.*

Remark. A large number of our functions depend critically upon parameter a. However, when the exact level of a is unimportant, we will tend to suppress this for ease and elegance of notation.

Theorem 51 (Stretching Theorem)**.** *$\tilde{\mathcal{Q}}^H_t$ converges in $\mathcal{M}(\mathbb{B}_{dec})$ (given the topology of weak convergence of measures) as $t \to \infty$.*

Proof. Take a sequence $\{t_n\}_{n=1}^{\infty} \uparrow \infty$. By Theorem 49 we know $\mathcal{Q}^H_{t_n} \overset{s}{\succ} \mathcal{Q}^H_{t_m}$ for $n \geq m$. By definition 43 and Theorem 47 we may construct random variables $(\hat{u}_{t_n})$ such that $\hat{u}_{t_n} \overset{D}{=} \mathcal{Q}^H_{t_n}$ and $\hat{u}_{t_n} \overset{s}{\succ} \hat{u}_{t_m}$ for all $n \geq m$, almost surely. Note that, given that the map $\varphi : \mathbb{B}_{dec} \to \mathbb{B}_{dec}$ defined by $\varphi(\cdot) \mapsto \varphi(\cdot + \gamma^a_t(\varphi))$ is a measurable transformation,

$\hat{\hat{u}}_{t_n} \stackrel{D}{=} \tilde{\mathcal{Q}}^H_{t_n}$. Since $\hat{\hat{u}}_{t_n}(0) = \hat{\hat{u}}_{t_m}(0)$ we have $\theta(\hat{\hat{u}}_{t_n}(0) - \hat{\hat{u}}_{t_m}(0)) \geq 0$. Corollary 36 shows that $\hat{\hat{u}}_{t_n}(x) = \hat{\hat{u}}_{t_m}(x)$ for $x \in [0, \theta(\hat{\hat{u}}_{t_n} - \hat{\hat{u}}_{t_m})]$ almost surely. Hence, almost surely, for all $n \geq m$

$$\hat{\hat{u}}_{t_n}(x) \geq \hat{\hat{u}}_{t_m}(x) \text{ for all } x \geq 0$$

$$\hat{\hat{u}}_{t_n}(x) \leq \hat{\hat{u}}_{t_m}(x) \text{ for all } x \leq 0.$$

Let $u_\infty(x) = \overline{\lim}_{n\to\infty} \hat{\hat{u}}_{t_n}(x)$ and define $\bar{u}_\infty(x)$ be the right continuous modification of u_∞ (which changes at most, countably many values). Hence,

$$\hat{\hat{u}}_{t_n}(x) \to \bar{u}_\infty(x) \text{ for almost all } x, \mathbb{P}\text{-almost surely.}$$

Therefore $\hat{\hat{u}}_{t_n} \xrightarrow{\mathbb{L}^1_{loc}} \bar{u}_\infty$, $\mathbb{P}$- almost surely and hence $\tilde{\mathcal{Q}}^H_{t_n} \to \mathcal{L}(\bar{u}_\infty)$ in $\mathcal{M}(\mathbb{B}_{dec})$. We claim that this limit law does not depend upon the choice of sequence $\{t_n\}$. Let $\{s_n\}_{n=1}^\infty \uparrow \infty$ be a second sequence and consider a third sequence, $\{r_n\} \uparrow \infty$ which contains all of the elements of both $\{s_n\}$ and $\{t_n\}$. If $\bar{u}_\infty$ is the almost sure limit constructed from this refinement then we find that

$$\tilde{\mathcal{Q}}^H_{r_n} \to \mathcal{L}(\bar{u}_\infty).$$

However, $\tilde{\mathcal{Q}}^H_{t_n} \to \mathcal{L}(\bar{u}_\infty)$ and $\tilde{\mathcal{Q}}^H_{s_n} \to \mathcal{L}(\bar{u}_\infty)$ and these are subsequences of $\{r_n\}$. Since for any convergent sequence in $\mathcal{M}(\mathbb{B}_{dec})$ we can take versions such that these versions converge almost surely, we have shown that $\tilde{\mathcal{Q}}^H_t$ converges. ■

Remark. We are yet to prove the limit lies within $\mathcal{M}_1\left(\mathbb{B}^{1,0}_{dec}\right)$. This will be covered in the next chapter.

Definition 52. *Let* $\nu = \lim_{t\to\infty} \tilde{\mathcal{Q}}^H_t$. *This will be used throughout the later chapters.*

3.7 Wong-Zakai Extension

Introduction. *In this section we will give the proof of Theorem 27. The argument follows the original method of Wong and Zakai (see [33]) but uses the Green's function representation for the solution to the SPDE.*

To prove this we will proceed in several steps but first we will start with some notation.

Definition 53. *For $\epsilon > 0$ and $s \in I_k(t) = [k\epsilon \wedge t, (k+1)\epsilon \wedge t]$, define $s^-(\epsilon) = k\epsilon,\ s^+(\epsilon) = (k+1)\epsilon$.*

Remark. Given the definition of $\dot{W}^\epsilon_s$ we may write $\dot{W}^\epsilon_s = \frac{W_{s^+} - W_{s^-}}{\epsilon} = \frac{W_{(k+1)\epsilon} - W_{k\epsilon}}{\epsilon} := \frac{\Delta W_k}{\epsilon}$ for $s \in I_k$. Using this formulation it is easy to show the following key property:

$$\mathbb{E}\left[\int_{I_k} \left|\dot{W}^\epsilon_s\right|^2 ds\right] = \int_{I_k} \mathbb{E}\left[\left|\dot{W}^\epsilon_s\right|^2\right] ds = \int_{I_k} \mathbb{E}\left[\left|\frac{\Delta W_k}{\epsilon}\right|^2\right] ds = 1.$$

Consider the equations, all with the same (possibly random) initial condition $u_0 \in [0,1]$:

$$du = u_{xx}\, dt + \bar{f}(u)\, dt + g(u)\, dW \tag{3.16}$$

$$d\bar{u} = \bar{u}_{xx}\, dt + \bar{f}(u^-)\, dt + g(u^-)\, dW \tag{3.17}$$

$$dv = v_{xx}\, dt + f(v)\, dt + g(v)\, \dot{W}^\epsilon dt \tag{3.18}$$

$$d\bar{v} = \bar{v}_{xx}\, dt + \bar{f}(v^-)\, dt + g(v^-)\, \dot{W}^\epsilon dt \tag{3.19}$$

$$d\tilde{v} = \tilde{v}_{xx}\, dt + \bar{f}(v^-)\, dt + g(v^-)\, \dot{W}^\epsilon dt \tag{3.20}$$

where W and W^ϵ are as defined in section 3.2, $\bar{f}(z) = f(z) + \frac{1}{2}gg'(z)$ and, for any $z : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ we define z^- by $z^-(t,x) = z(t^-(\epsilon), x)$. Note that $\bar{u}, v, \bar{v}, \tilde{v}$ all depend upon ϵ.

Remark. Existence and uniqueness hold for all five equations and $u, v \in [0,1]$ for all time.

Remark. $\bar{v}, \tilde{v}, \bar{u}$ need not lie in the interval $[0,1]$ but it is easy to confirm that their second moments are finite.

Remark. Notice the absence of the $\bar{f}$ in equation (3.18). This is due to the fact that in such approximations, equation (3.18) will tend to the same equation with Stratonovich noise rather than Itô noise.

By the triangle inequality and the use of Jensen's inequality we can write

$$\begin{aligned}
|u(t,x)-v(t,x)|^2 &\leq 3\,|u(t,x)-\bar{u}(t,x)|^2 + 3\,|\bar{u}(t,x)-\bar{v}(t,x)|^2 + 3\,|\bar{v}(t,x)-v(t,x)|^2 \\
&\leq 3\,|u(t,x)-\bar{u}(t,x)|^2 + 3\,|\bar{u}(t,x)-\bar{v}(t,x)|^2 + 6\,|\bar{v}(t,x)-\tilde{v}(t,x)|^2 \\
&\qquad +6\,|\tilde{v}(t,x)-v(t,x)|^2 .
\end{aligned}$$

The idea is to show that each term on the right-hand-side are either of order epsilon or an integral copy of the left-hand-side, and to use a Gronwall argument to prove the left hand side tends to zero. We will prove this using a string of lemmas which all refer to equations (3.16)-(3.20). First, however, we will prove a result that will be useful throughout this chapter and is analogous to the Kolmogorov increment estimates for u.

Notation. $C(f,g,T,...)$ will denote a constant whose exact value is unimportant, and may change from line to line, but whose dependence will be indicated.

Lemma 54.

$$\mathbb{E}\left[\left|v(t,x)-v(t^-,x)\right|^2\right] \leq C(f,g,T)\left(\epsilon\left(\ln^2\left(\frac{1}{\epsilon}\right)\vee 1\right)+\left(\frac{\epsilon}{t^-}\right)^2\right)$$

for all $\epsilon \in (0,1)$, $t \in [0,T]$, $x \in \mathbb{R}$.

Proof. From the Green's formula representation we can write

$$\begin{aligned}
v(t,x) &= \int_{\mathbb{R}} \Gamma_t(y) v_0(y)\,dy + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) f(v(s,y))\,dyds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g(v(s,y))\,dy\dot{W}^\epsilon_s ds
\end{aligned}$$

and

$$\begin{aligned}
v(t,x) - v(t^-,x) &= \int_{\mathbb{R}} v_0(y) \left(\Gamma_t(y) - \Gamma_{t^-}(y)\right) dy \\
&\quad + \int_0^{t^-} \int_{\mathbb{R}} f(v(s,y)) \left(\Gamma_{t-s}(x-y) - \Gamma_{t^- - s}(x-y)\right) dyds \\
&\quad + \int_0^{t^-} \int_{\mathbb{R}} \left(\Gamma_{t-s}(x-y) - \Gamma_{t^- - s}(x-y)\right) g(v(s,y))\, dy\dot{W}_s^\epsilon ds \\
&\quad + \int_{t^-}^{t} \int_{\mathbb{R}} f(v(s,y))\Gamma_{t-s}(x-y)\, dyds \\
&\quad + \int_{t^-}^{t} \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g(v(s,y))\, dy\dot{W}_s^\epsilon ds.
\end{aligned}$$

We will take each of these terms in turn whilst making use of the bound:

$$\int_{\mathbb{R}} \left|\Gamma_{t-s}(y) - \Gamma_{t^- - s}(y)\right| dy \le C\frac{|t-t^-|}{|t^- - s|} \wedge 1 \tag{3.21}$$

for all $0 \le s < t^- \le t < \infty$ (see Appendix). Considering the first term:

$$\begin{aligned}
\left|\int_{\mathbb{R}} v_0(y) \left(\Gamma_t(y) - \Gamma_{t^-}(y)\right) dy\right| &\le \int_{\mathbb{R}} \left|\Gamma_t(y) - \Gamma_{t^-}(y)\right| dy \text{ given that } v \text{ is bounded by } 1 \\
&\le C\frac{\epsilon}{t^-}.
\end{aligned}$$

We will again make use of the fact that $f(v)$ is bounded so that for the second term

$$\begin{aligned}
&\left|\int_0^{t^-} \int_{\mathbb{R}} f(v(s,y)) \left(\Gamma_{t-s}(x-y) - \Gamma_{t^- - s}(x-y)\right) dyds\right| \\
&\le C(f) \int_0^{t^-} \int_{\mathbb{R}} \left|\Gamma_{t-s}(x-y) - \Gamma_{t^- - s}(x-y)\right| dyds \\
&\le C(f) \int_0^{t^- - \epsilon} \frac{\epsilon}{t^- - s} ds + C(f) \int_{t^- - \epsilon}^{t^-} ds \\
&= C(f)\epsilon \left(\ln\left(\frac{1}{\epsilon}\right) + \ln t^- + 1\right) \\
&\le C(f,T)\epsilon \ln\left(\frac{1}{\epsilon}\right)
\end{aligned}$$

for any $t \in [0,T]$. For the third term we have,

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^{t^-}\int_{\mathbb{R}} g(v(s,y))\left(\Gamma_{t-s}(x-y)-\Gamma_{t^--s}(x-y)\right)\,dy\dot{W}_s^\epsilon ds\right|^2\right]\\
&\leq C(\|g\|_\infty)\mathbb{E}\left[\left|\int_0^{t^-}\left|\dot{W}_s^\epsilon\right|\int_{\mathbb{R}}\left|\Gamma_{t-s}(x-y)-\Gamma_{t^--s}(x-y)\right|\,dyds\right|^2\right]\\
&\leq C(\|g\|_\infty)\mathbb{E}\left[\left|\int_0^{t^-}\left|\dot{W}_s^\epsilon\right|\left(\frac{\epsilon}{|t^--s|}\right)\wedge 1\,ds\right|^2\right]\\
&= C(\|g\|_\infty)\mathbb{E}\left[\int_0^{t^-}\int_0^{t^-}\left|\dot{W}_s^\epsilon\right|\left|\dot{W}_r^\epsilon\right|\left(\frac{\epsilon}{|t^--s|}\right)\wedge 1\left(\frac{\epsilon}{|t^--r|}\right)\wedge 1\,dsdr\right]\\
&= C(\|g\|_\infty)\int_0^{t^-}\int_0^{t^-}\mathbb{E}\left[\left|\dot{W}_s^\epsilon\right|\left|\dot{W}_r^\epsilon\right|\right]\left(\frac{\epsilon}{|t^--s|}\right)\wedge 1\left(\frac{\epsilon}{|t^--r|}\right)\wedge 1\,dsdr\\
&\leq C(\|g\|_\infty)\int_0^{t^-}\int_0^{t^-}\frac{1}{\epsilon}\left(\frac{\epsilon}{|t^--s|}\right)\wedge 1\left(\frac{\epsilon}{|t^--r|}\right)\wedge 1\,dsdr\\
&= C(\|g\|_\infty)\frac{1}{\epsilon}\left(\int_0^{t^-}\left(\frac{\epsilon}{|t^--s|}\right)\wedge 1\,ds\right)^2\\
&= C(\|g\|_\infty)\frac{1}{\epsilon}\left(\int_0^{t^--\epsilon}\left(\frac{\epsilon}{|t^--s|}\right)\,ds+\int_{t^--\epsilon}^{t^-}ds\right)^2\\
&= C(\|g\|_\infty)\frac{1}{\epsilon}\left(\left(\epsilon\ln\left(\frac{1}{\epsilon}\right)+\epsilon t^-\right)+\epsilon\right)^2\\
&\leq C(\|g\|_\infty,T)\left(\epsilon\ln^2\left(\frac{1}{\epsilon}\right)\right).
\end{aligned}$$

For the remaining terms:

$$\begin{aligned}
\left|\int_{t^-}^t\int_{\mathbb{R}} f(v(s,y))\Gamma_{t-s}(x-y)\,dyds\right| &\leq C(f)\int_{t^-}^t\left(\int_{\mathbb{R}}\Gamma_{t-s}(x-y)\,dy\right)ds\\
&\leq C(f)\epsilon,\\
\mathbb{E}\left[\left|\int_{t^-}^t\int_{\mathbb{R}}\Gamma_{t-s}(x-y)g(v(s,y))\,dy\dot{W}_s^\epsilon ds\right|^2\right] &\leq C(g)\epsilon\int_{t^-}^t\mathbb{E}\left[\left|\dot{W}_s^\epsilon\right|^2\right]ds\\
&= C(g)\epsilon.
\end{aligned}$$

Combining each of the above terms gives us the required result. ■

Lemma 55. *For $T > 0$*

$$\sup_{x\in\mathbb{R}} \sup_{t\leq T} \mathbb{E}\left[|u(t,x) - \bar{u}(t,x)|^2\right] \to 0 \text{ as } \epsilon \to 0.$$

Proof. Given that u and $\bar{u}$ have the same, possibly random, initial condition in $[0,1]$ we can write

$$\begin{aligned} u(t,x) - \bar{u}(t,x) &= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(u(s^-,y))\right) dyds \\ &\quad + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(g(u(s,y)) - g(u(s^-,y))\right) dydW_s \\ &= I_1 + I_2. \end{aligned}$$

Now, through repeated use of the Cauchy-Schwarz inequality, we have

$$\begin{aligned} \mathbb{E}\left[|I_1|^2\right] &= \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(u(s^-,y))\right) dyds\right|^2\right] \\ &\leq C(T)\mathbb{E}\left[\int_0^t \left(\int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(u(s^-,y))\right) dy\right)^2 ds\right] \\ &\leq C(T)\mathbb{E}\left[\int_0^t \left(\int_{\mathbb{R}} \Gamma_{t-s}(x-y)\, dy\right)\right. \\ &\qquad \left. \times \left(\int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(u(s^-,y))\right) dy\right)^2 ds\right] \\ &\leq C(T)\mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(u(s^-,y))\right)^2 dyds\right]. \end{aligned}$$

However, by the Lipschitz property of $\bar{f}$, given that both f, g and g' are Lipschitz, and the Kolmogorov continuity estimates (see Property (**vi**) Theorem 15) we have

$$\mathbb{E}\left[\left|\bar{f}(u(s,y)) - \bar{f}(u(s^-,y))\right|^2\right] \leq C(f,g,T)\left(\epsilon + \left(\frac{\epsilon}{s^-}\right)^2\right) \tag{3.22}$$

for all $\epsilon \in (0,1)$, $0 \leq s \leq T$. Fix $0 < \epsilon < \frac{1}{10}$. Using equation (3.22) when $s \geq \sqrt{\epsilon}$ and

$$\mathbb{E}\left[\left|\bar{f}(u(s,y)) - \bar{f}(u(s^-,y))\right|^2\right] \leq C(f,g)$$

for $s \in [0, \sqrt{\epsilon}]$ leads to

$$\mathbb{E}\left[|I_1|^2\right] \;\; \leq \;\; C(f, g, T)\sqrt{\epsilon}.$$

We now have to calculate similar estimates for I_2. By use of the Itô Isometry we have:

$$\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(g(u(s,y)) - g(u(s^-,y))\right) dydW_s\right|^2\right] \\
&\quad = \mathbb{E}\left[\int_0^t \left|\int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(g(u(s,y)) - g(u(s^-,y))\right) dy\right|^2 ds\right] \\
&\quad \overset{Cauchy-Schwarz}{\leq} \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left|g(u(s,y)) - g(u(s^-,y))\right|^2 dyds\right].
\end{aligned}$$

which is bounded as for I_1. ■

The next Lemma gives the key approximation.

Lemma 56. *For all $T > 0$,*

$$\sup_{x\in\mathbb{R}} \sup_{t\leq T} \mathbb{E}\left[|v(t,x) - \tilde{v}(t,x)|^2\right] \to 0 \;\; as \; \epsilon \downarrow 0.$$

Proof.

$$\begin{aligned}
v(t,x) - \tilde{v}(t,x) \;\; &= \;\; \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(f(v(s,y)) - \bar{f}(v(s^-,y))\right) dyds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(g(v(s,y)) - g(v(s^-,y))\right) dydW_s^\epsilon \\
&= \;\; \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(f(v(s,y)) - f(v(s^-,y))\right) dyds \\
&\qquad -\frac{1}{2}\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g(v(s^-,y)) g'(v(s^-,y))\, dyds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(g(v(s,y)) - g(v(s^-,y))\right) dydW_s^\epsilon \\
&= \;\; I_1 + I_2 + I_3.
\end{aligned}$$

For the I_1 term it is easy to show, along similar lines to the above arguments, $\mathbb{E}\left[|I_1|^2\right] \le C(T)\sup_{y\in\mathbb{R}} \mathbb{E}\left[\int_0^t |v(s,y) - v(s^-,y)|^2 \, ds\right]$ which converges to 0 as ϵ tends to zero by the use of Lemma 54. Let us next consider the I_3 term. To do this we will make use of Taylor's Theorem with second order remainder. We apply this to the composite function $(g \circ v)(s,y)$ given that $g \in \mathcal{C}^2$ and v is $\mathcal{C}^2$ for s within each $I_k(t)$ to get, for some η lying between s and s^-

$$g(v(s,y)) = g(v(s^-,y)) + \frac{\partial}{\partial s}(g \circ v)(s^-,y)(s-s^-) + \frac{1}{2}\frac{\partial^2}{\partial s^2}(g \circ v)(\eta,y))(s-s^-)^2.$$

Before proceeding with the calculation we will derive explicit forms of these total derivatives. It is clear that, for $t \in [s^-, s)$,

$$\begin{aligned}
\left.\frac{\partial}{\partial t} g(v(t,y))\right|_{t=s^-} &= \left. g'(v(t,y))\,\frac{\partial v}{\partial t}(t,y)\right|_{t=s^-} \\
&= \left. g'(v(t,y))\,\Big(f(v(t,y)) + g(v(t,y))\,\dot{W}^\epsilon_t\Big)\right|_{t=s^-} \\
&= g'(v(s^-,y))\,\Big(f(v(s^-,y)) + g(v(s^-,y))\,\dot{W}^\epsilon_{s^-}\Big).
\end{aligned}$$

Differentiating $\frac{\partial}{\partial t} g(v(t,y))$ again we have

$$\begin{aligned}
\left.\frac{\partial^2}{\partial t^2} g(v(t,y))\right|_{t=\eta} &= \Big(gg'' + g'^2\Big)(v(\eta,y))\,\dot{W}^\epsilon_\eta \Big(f(v(\eta,y))\,dt + g(v(\eta,y))\,\dot{W}^\epsilon_\eta\Big) \\
&\quad + (fg'' + f'g')(v(\eta,y))\Big(f(v(\eta,y)) + g(v(\eta,y))\,\dot{W}^\epsilon_\eta\Big) \\
&= F_1(v(\eta,y)) + F_2(v(\eta,y))\dot{W}^\epsilon_\eta + F_3(v(\eta,y))\left(\dot{W}^\epsilon_\eta\right)^2
\end{aligned}$$

where each of the $F's$ are bounded by hypothesis (**H1**). We will use this formulation later on in our bounds. Define $\eta_s \in [s^-, s]$.

$$
\begin{aligned}
I_3 &= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(g(v(s,y)) - g(v(s^-,y))\right) dy\dot{W}_s^\epsilon ds \\
&= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \\
&\qquad \times \left(\frac{\partial}{\partial t}(g\circ v)(s^-,y)(s-s^-) + \frac{1}{2}(s-s^-)^2\frac{\partial^2}{\partial t^2}(g\circ v)(\eta_s,y)\right) dy\dot{W}_s^\epsilon ds \\
&= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \\
&\qquad \times \left(g'(v(s^-,y))\dot{v}(s^-,y)(s-s^-) + \frac{1}{2}(s-s^-)^2\frac{\partial^2}{\partial t^2}(g\circ v)(\eta_s,y)\right) dy\dot{W}_s^\epsilon ds \\
&= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)g'(v(s^-,y))(s-s^-)f(v(s^-,y))\, dy\dot{W}_s^\epsilon\, ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)g'(v(s^-,y))(s-s^-)g(v(s^-,y))\, dy(\dot{W}_s^\epsilon)^2 ds \\
&\qquad + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\frac{1}{2}\frac{\partial^2}{\partial t^2}(g\circ v)(\eta_s,y)(s-s^-)^2\, dy\dot{W}_s^\epsilon\, ds \\
&=: I_{3,1} + I_{3,2} + I_{3,3}.
\end{aligned}
$$

We will use the properties of the polygonal approximation $\dot{W}_s^\epsilon$, as discussed at the start of this section, in each of the terms above.

$$
\begin{aligned}
I_{3,1} &= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)g'(v(s^-,y))(s-s^-)f(v(s^-,y))\, dy\dot{W}_s^\epsilon\, ds \\
&= \sum_{k=0}^{\infty} \int_{I_k(t)} \int_{\mathbb{R}} \Gamma_{t-s}(x-y)g'(v(s^-,y))(s-s^-)f(v(s^-,y))\, dy\frac{\Delta W_k}{\epsilon}\, ds.
\end{aligned}
$$

Now given that $\int_{I_k(t)}\int_{\mathbb{R}} \Gamma_{t-s}(x-y)g'(v(s^-,y))(s-s^-)f(v(s^-,y))\, dy\frac{\Delta W_k}{\epsilon}\, ds$ are orthogonal variables we can write

$$
\begin{aligned}
&\mathbb{E}\left[|I_{3,1}|^2\right] \\
&\quad = \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g'(v(s^-,y))(s-s^-) f(v(s^-,y))\, dy \frac{\Delta W_k}{\epsilon}\, ds\right|^2\right] \\
&\quad = \mathbb{E}\left[\left|\sum_{k=0}^{\infty} \int_{I_k(t)} \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g'(v(s^-,y))(s-s^-) f(v(s^-,y))\, dy \frac{\Delta W_k}{\epsilon}\, ds\right|^2\right] \\
&\quad = \sum_{k=0}^{\infty} \mathbb{E}\left[\left|\int_{I_k(t)} \int_{\mathbb{R}} \Gamma_{t-s}(x-y) g'(v(s^-,y))(s-s^-) f(v(s^-,y))\, dy \frac{\Delta W_k}{\epsilon}\, ds\right|^2\right] \\
&\quad \leq C(\|f\|_\infty, \|g'\|_\infty) \sum_{k=0}^{\infty} \mathbb{E}\left[\left|\int_{I_k(t)} \left|\frac{\Delta W_k}{\epsilon}\right| (s-s^-)\, ds\right|^2\right] \\
&\quad\ \text{given that } f \text{ and } g' \text{ are bounded} \\
&\quad \leq \epsilon C(\|f\|_\infty, \|g'\|_\infty) \sum_{k=0}^{\infty} \int_{I_k(t)} \mathbb{E}\left[\left(\frac{\Delta W_k}{\epsilon}\right)^2\right] (s-s^-)^2\, ds \\
&\quad\ \text{by the Cauchy-Schwarz inequality} \\
&\quad = C(\|f\|_\infty, \|g'\|_\infty) \sum_{k=0}^{\infty} \int_{I_k(t)} (s-s^-)^2\, ds \\
&\quad \leq C(\|f\|_\infty, \|g'\|_\infty) \epsilon^2 \sum_{k=0}^{\infty} \int_{I_k(t)} ds \\
&\quad \leq C(\|g'\|_\infty, \|f\|_\infty, T) \epsilon^2.
\end{aligned}
$$

We will now prove a result which will aid us in our calculation.

Lemma 57. *For bounded $F : [0,1] \to \mathbb{R}$, $n = 1, 2, 3$,*

$$
\sup_{x \in \mathbb{R}} \sup_{t \leq T} \mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)(s-s^-)^2 F(v(\eta,y))(\dot{W}_s^\epsilon)^n\, dy ds\right|^2\right] \leq C(T)\epsilon^{-n+4}/9
$$

which tends to zero as $\epsilon \to 0$.

Proof.

$$
\begin{aligned}
&\mathbb{E}\left[\left|\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)(s-s^-)^2 F(v(\eta,y))(\dot{W}_s^\epsilon)^n \, dyds\right|^2\right] \\
&\quad \le \mathbb{E}\left[\left|\sum_{k=0}^{\infty}\left|\frac{\Delta W_k}{\epsilon}\right|^n \int_{I_k(t)} (s-s^-)^2 \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\,|F(v(\eta,y))|\, dyds\right|^2\right] \\
&\quad \le C\mathbb{E}\left[\left|\sum_{k=0}^{\infty}\left|\frac{\Delta W_k}{\epsilon}\right|^n \int_{I_k(t)} (s-s^-)^2 \, ds\right|^2\right] \\
&\quad \le C\mathbb{E}\left[\sum_{k=0}^{\infty}\left|\frac{\Delta W_k}{\epsilon}\right|^{2n} \sum_{k=0}^{\infty}\left(\int_{I_k(t)} (s-s^-)^2 \, ds\right)^2\right] \\
&\quad \text{by use of the Cauchy-Schwarz inequality} \\
&\quad = C\mathbb{E}\left[\sum_{k=0}^{\infty}\left|\frac{\Delta W_k}{\epsilon}\right|^{2n} \epsilon^5/9\right] \\
&\quad = C(T)\epsilon^{-n-1}\epsilon^5/9 \\
&\quad = C(T)\epsilon^{-n+4}/9
\end{aligned}
$$

which completes the proof. ■

We will now put this lemma to use in the bound for the $I_{3,3}$ term.

$$
\begin{aligned}
I_{3,3} &= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\frac{1}{2}\frac{\partial^2}{\partial s^2}(g\circ v)(\eta_s,y)(s-s^-)^2 \, dy\dot{W}_s^\epsilon \, ds \\
&= \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)\frac{1}{2}(s-s^-)^2 \\
&\qquad \times \left(F_1(v(\eta_s,y))\dot{W}_s^\epsilon + F_2(v(\eta_s,y))\left(\dot{W}_s^\epsilon\right)^2 + F_3(v(\eta_s,y))\left(\dot{W}_s^\epsilon\right)^3\right) dy\, ds
\end{aligned}
$$

and by Lemma 57, this gives $\mathbb{E}\left[|I_{3,3}|^2\right] = O(\epsilon)$. We will now turn our attention to the terms $I_{3,2} + I_2$ and conclude Lemma 56.

$$
I_{3,2} + I_2 = \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y)g'(v(s_-,y))g(v(s_-,y))\, dy \left((s-s^-)(\Delta W_k^\epsilon)^2 - \epsilon^2/2\right) ds.
$$

Hence, squaring and taking expectations,

$$\begin{aligned}\mathbb{E}\left[|I_{3,2}+I_2|^2\right] &= \mathbb{E}\left[\left|\sum_{k=1}^{\infty}\int_{I_k(t)}\int_{\mathbb{R}}\Gamma_{t-s}(x-y)g'(v(s_-,y))g(v(s_-,y))\,dy\right.\right.\\ &\qquad\left.\left.\times\left((s-s_-)(\Delta W_k)^2-\epsilon^2/2\right)\,ds\right|^2\right].\end{aligned}$$

It is clear that the last term has orthogonal integrals. Defining $\mathcal{F}_{k\epsilon}=\mathcal{B}_{k\epsilon}$ as the Borel sigma algebra at time $k\epsilon$ we can write, for $k\in\mathbb{N}\cup\{0\}$,

$$\mathbb{E}\left[\int_{I_k}(s-s^-)(\Delta W_k)^2-\frac{\epsilon^2}{2}\,ds\,|\mathcal{F}_{k\epsilon}\right]=0.$$

Hence,

$$\begin{aligned}E\left[|I_{3,2}+I_2|^2\right] &= \mathbb{E}\left[\left|\frac{1}{\epsilon^2}\sum_{k=1}^{\infty}\int_{I_k(t)}\int_{\mathbb{R}}\Gamma_{t-s}(x-y)g'(v(s_-,y))g(v(s_-,y))\,dy\right.\right.\\ &\quad\times\qquad\left.\left.\left((s-s_-)(\Delta W_k)^2-\epsilon^2/2\right)\,ds\right|^2\right]\\ &=\frac{1}{\epsilon^4}\sum_{k=1}^{\infty}\mathbb{E}\left[\left|\int_{I_k(t)}\int_{\mathbb{R}}(\Gamma_{t-s}(x-y)g'(v(s_-,y))g(v(s_-,y))\,dy\right.\right.\\ &\qquad\left.\left.\times\left((s-s_-)(\Delta W_k)^2-\epsilon^2/2\right)\right)\,ds\right|^2\right]\\ &\le\frac{C(\|g\|_\infty,\|g'\|_\infty)}{\epsilon^4}\\ &\qquad\times\sum_{k=1}^{\infty}\mathbb{E}\left[\left|\int_{I_k(t)}\left|(s-s_-)(\Delta W_k)^2-\epsilon^2/2\right|\,ds\right|^2\right]\\ &\le\frac{C(\|g\|_\infty,\|g'\|_\infty)}{\epsilon^3}\sum_{k=1}^{\infty}\mathbb{E}\left[\int_{I_k(t)}\left|(s-s_-)(\Delta W_k)^2-\epsilon^2/2\right|^2\,ds\right]\end{aligned}$$

by the Cauchy-Schwarz inequality. Solving for the integral over $I_k(t)$ gives

$$
\begin{aligned}
\epsilon^{-3}\sum_{k=1}^{\infty}\mathbb{E}\left[\int_{I_k(t)}\left|(s-s_-)(\Delta W_k)^2-\epsilon^2/2\right|^2\,ds\right] &= \epsilon^{-3}\sum_{k=1}^{\infty}\mathbb{I}_{\left[k<\frac{t^-(\epsilon)}{\epsilon}\right]}\mathbb{E}\left[\epsilon^3/3(\Delta W_k)^4\right.\\
&\qquad\left.-2\left(\epsilon^2/2\right)(\Delta W_k)^2(\epsilon^2/2)+\epsilon^5/4\right]\\
&= \epsilon^{-4}\left(\epsilon^5-\epsilon^5/2+\epsilon^5/4\right)\\
&\le \epsilon.
\end{aligned}
$$

Putting this all together shows, given that g and g' are bounded, the left-hand-side tends to zero as epsilon tends to zero as required.

Putting all of this together gives the required bound and this completes the proof of Lemma 56. ■

Lemma 58. *For all $T>0$ $\sup_{x\in\mathbb{R}}\sup_{t\le T}\mathbb{E}\left[|\tilde{v}(t,x)-\bar{v}(t,x)|^2\right]\to 0$ as $\epsilon\to 0$.*

Proof. Note that

$$
\tilde{v}(t,x)-\bar{v}(t,x) = \int_0^t\int_{\mathbb{R}}\Gamma_{t-s}(x-y)g(v(s^-,y))\,dy\left(dW_s-\dot{W}_s^\epsilon ds\right).
$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left[|\tilde{v}(t,x)-\bar{v}(t,x)|^2\right]\\
&\quad=\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}}\Gamma_{t-s}(x-y)g(v(s^-,y))\,dy\left(dW_s-\dot{W}_s^\epsilon ds\right)\right|^2\right]\\
&\quad=\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}}\left(\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right)g(v(s^-,y))\,dy\left(dW_s-\dot{W}_s^\epsilon ds\right)\right|^2\right]\\
&\quad\text{since }\textstyle\int_{I_k(t)}dW_s-\dot{W}_s^\epsilon\,ds=0\text{ by definition}\\
&\quad\le C\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}}\left(\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right)g(v(s^-,y))\,dy\,dW_s\right|^2\right]\\
&\qquad+C\mathbb{E}\left[\left|\int_0^t\int_{\mathbb{R}}\left(\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right)g(v(s^-,y))\,dy\,\dot{W}_s^\epsilon ds\right|^2\right].
\end{aligned}
$$

Again taking each of these in turn gives

$$
\begin{aligned}
\mathbb{E}&\left[\left|\sum_{k=0}^{\infty}\int_{I_k(t)}\int_{\mathbb{R}}\left(\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right)g(v(s^-,y))\,dy\,dW_s\right|^2\right] \\
&=\sum_{k=0}^{\infty}\int_{I_k(t)}\mathbb{E}\left|\int_{\mathbb{R}}\left(\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right)g(v(s^-,y))\,dy\right|^2\,ds \\
&\quad\text{by orthogonality} \\
&\leq C(g)\sum_{k=0}^{\infty}\int_{I_k(t)}\left|\int_{\mathbb{R}}\left|\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right|\,dy\right|^2\,ds
\end{aligned}
$$

given that g is bounded. The second term is similar,

$$
\begin{aligned}
\mathbb{E}&\left[\left|\sum_{k=0}^{\infty}\int_{I_k(t)}\int_{\mathbb{R}}\left(\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right)g(v(s^-,y))\,dy\,\dot{W}_s^{\epsilon}ds\right|^2\right] \\
&=\sum_{k=0}^{\infty}\mathbb{E}\left[\left|\int_{I_k(t)}\int_{\mathbb{R}}\left(\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right)g(v(s^-,y))\,dy\,\dot{W}_s^{\epsilon}ds\right|^2\right] \\
&\quad\text{by the orthogonality of } \dot{W}^{\epsilon}(s) \\
&\leq C(\|g\|_\infty)\sum_{k=0}^{\infty}\mathbb{E}\left[\left|\int_{I_k(t)}\int_{\mathbb{R}}\left|\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right|\,dy\,\left|\frac{\Delta W_k}{\epsilon}\right|ds\right|^2\right] \\
&\quad\text{given that } g \text{ is bounded} \\
&\leq C(\|g\|_\infty)\sum_{k=0}^{\infty}\mathbb{E}\left[\int_{I_k(t)}\left(\int_{\mathbb{R}}\left|\Gamma_{t-s}(x-y)-\Gamma_{t-s^-}(x-y)\right|\,dy\right)^2\right. \\
&\qquad\qquad\left.\times\int_{I_k(t)}\left(\frac{\Delta W_k}{\epsilon}\right)^2\,ds\right]
\end{aligned}
$$

by the Cauchy-Schwarz inequality.

For both of these equations we can use equation (3.21) and write, making use of the identity $\int_{I_k(t)}\mathbb{E}\left[\left(\frac{\Delta W_k}{\epsilon}\right)^2\right]=1$ in the second term,

$$
\begin{aligned}
C(g)\sum_{k=0}^{\infty}\int_{I_k(t)} \frac{\epsilon^2}{|t-s|^2}\wedge 1\, ds \\
&= C(g)\int_0^t \frac{\epsilon^2}{|t-s|^2}\wedge 1\, ds \\
&\leq C(g)\epsilon
\end{aligned}
$$

which completes the required bound. ■

Lemma 59. *For $t \in [0,T]$ and for all $x \in \mathbb{R}$*

$$
\mathbb{E}\left[|\bar{u}(t,x)-\bar{v}(t,x)|^2\right] \leq C(f,g,T)\int_0^t \sup_{y\in\mathbb{R}} \mathbb{E}\left[\left|u(s^-,y)-v(s^-,y)\right|^2 ds\right].
$$

Proof. Note

$$
\begin{aligned}
\bar{u}(t,x)-\bar{v}(t,x) &= \int_0^t\int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(\bar{f}(u(s^-,y))-\bar{f}(v(s^-,y))\right) dyds \\
&\quad + \int_0^t\int_{\mathbb{R}} \Gamma_{t-s}(x-y)\left(g(u(s^-,y))-g(v(s^-,y))\right) dydW_s.
\end{aligned}
$$

Applying the Lipschitz property of $\bar{f}$ and g, we deduce the required result. ■

We can now put Lemmas 54 - 59 together to achieve a bound on $|u(s,y)-v(s,y)|^2$ and use Gronwall's inequality to finish the proof. Define $R_t^\epsilon = \sup_{s\leq t}\sup_{y\in\mathbb{R}} \mathbb{E}\left[|u(s,y)-v(s,y)|^2\right]$.

Proof of Theorem 27. By the above lemmas we can write, for all $t \leq T$ and $x \in \mathbb{R}$

$$
\begin{aligned}
\mathbb{E}\left[|u(t,x)-v(t,x)|^2\right] &\leq C(T,\|f\|_\infty,\|g\|_\infty,\|g'\|_\infty)\int_0^t \sup_{y\in\mathbb{R}} \mathbb{E}\left[\left|u(s^-,y)-v(s^-,y)\right|^2\right] ds \\
&\quad + J_\epsilon(T)
\end{aligned}
$$

where $J_\epsilon(T) \to 0$ as $\epsilon \downarrow 0$. Hence, given the definition of R_t^ϵ,

$$
R_t^\epsilon \leq J_\epsilon(T) + C(T,\|f\|_\infty,\|g\|_\infty,\|g'\|_\infty)\int_0^t R_s^\epsilon\, ds.
$$

By the use of a Gronwall argument, this implies $R_t^\epsilon \to 0$ as $\epsilon \downarrow 0$. ■

Chapter 4

Non-triviality of the stationary travelling wave

Introduction. *In this chapter we prove that the stretched limit law ν constructed in Chapter 3 is concentrated on $\mathbb{B}^{1,0}_{dec}$ and is a law of the stationary travelling wave.*

Definition 60. ***Stationary travelling wave*** *A stationary travelling wave is an invariant measure of the centred process. Alternatively, an initial law ν is a stationary travelling wave if $\mathbb{P}\left[\tilde{u}(t,\cdot) \in d\varphi\right] = \nu(d\varphi)$ for all $t \geq 0$.*

4.1 McKean Bound

Introduction. *In this section we prove that although the wavefront becomes more stretched over time, as demonstrated in the previous chapter, it cannot become too flat, see remark below. For the proof we will use the stochastic analogue of a method originally found in the McKean paper ([18]) for the deterministic KPP equation and revisited by Bramson* *(see [3]). We will first review the deterministic KPP case before proving a wave speed bound which will be useful in later chapters and sections.*

4.1.1 Review of the McKean bound for the deterministic KPP equation

The McKean paper [18] considers the KPP equation $u_t = u_{xx} + f(u)$ where $f(u) = u(1-u)$ for $t \geq 0$ and $x \in \mathbb{R}$ with initial condition $u_0(x) = \mathbb{I}_{[x \leq 0]}$. Following the transformation of this equation to the fixed frame of reference using the wave marker γ_t^a, as defined in the previous chapter where we denote the transformed wavefront by $\tilde{u}$, we can write:

$$\partial_t \tilde{u} = \partial_{xx} \tilde{u} + \tilde{u}(1 - \tilde{u}) + \tilde{u}_x(t, x)\dot{\gamma}_t^a,$$

$\dot{\gamma}_t^a$ denoting the true derivative with respect to time $t > 0$.

Remark. Introducing the wave marker into the above equation, although making the analysis from a fixed frame of reference easier, gives rise to an additional term $\tilde{u}_x \dot{\gamma}_t^a$ via the function of a function differentiation rule. In McKean's case γ_t^a is deterministic whereas in the stochastic case, γ_t^a, for $t > 0$, is a semi-martingale (see Chapter 2).

Remark. Recalling the definition of a solutions width (see remark following example 1) and its component drivers, a solution is said to be "too flat" if $b(t) - a(t)$ is sufficiently large such that

$$\lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} \int_{\mathbb{R}} u(s, x)(1 - u(s, x))\, dx ds \nleq \infty.$$

In the deterministic case, McKean [18] integrated both over space (from negative infinity to zero) and then time ($0 < t_0 \leq s \leq t$). Applying Fubini's theorem and appealing to the properties of the initial condition and the value of solutions and their derivatives at the boundary gives rise to the following equation:

$$0 \geq \int_{-\infty}^{0} \left[\tilde{u}(t,x) - \tilde{u}(t_0,x)\right] dx \;=\; \int_{t_0}^{t} \partial_x \tilde{u}(s,0) ds + [a-1]\left(\gamma_t^a - \gamma_{t_0}^a\right) + \int_{t_0}^{t}\int_{-\infty}^{0} \tilde{u}(s,x)\left[1-\tilde{u}(s,x)\right] dxds.$$

It is well known that for the KPP equation, the wave speed is bounded: $\gamma_t^a/t \leq 2\sqrt{f'(0)}$. Using this as well as the properties that the gradient at the centring point is increasing and, for $x \geq 0$, $\tilde{u}(t,x)$ is increasing in t (the proof of both of these things will be extended when we embrace the stochastic case) we can write, for any fixed $t_0 > 0$,

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\int_{t_0}^{t}\int_{-\infty}^{0} \tilde{u}(t,x)(1-\tilde{u}(t,x))dx \leq 2(1-a) - \partial_x \tilde{u}(t_0,0) < \infty.$$

Remark. $\tilde{u}$ depends on the choice of $a \in (0,1)$ but, in the above, we can let $a \to 0$ and use the property that $\partial_x u(t_0,0)$ is bounded below (and hence $-\partial_x u(t_0,0)$ is bounded above) to show that

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\int_{t_0}^{t}\int_{-\infty}^{\infty} u(t,x)(1-u(t,x))dx \leq 2 - \inf_y \partial_x u(t_0,y)$$

where now the dependence on a has disappeared. This bound indicates that in the evolution of $\tilde{u}$ (and given that the introduction of the wave marker is just a transformation of u along the spatial axis, the evolution of u too), the wavefront cannot get too flat for a fixed time $t > 0$ over a large interval $x \in \mathbb{R}$. We aim for a close stochastic analogue of these ideas, although the technical details are considerably more intricate to verify.

4.1.2 McKean bound for the stochastic heat equation

Notation. For $\varphi \in \mathbb{B}_{dec}^{1,0}$ we write $\tilde{\varphi}^a$ for φ centred at $\gamma_t^a(\varphi)$.

Theorem 61. *Consider a solution u to*

$$du = u_{xx}dt + f(u)dt + g(u) \circ dW_t \tag{4.1}$$

*for $t \geq 0$ and $x \in \mathbb{R}$ where f and g satisfy hypotheses (**H1**) and (**H2**). Suppose also that $u(0) = \mathbb{I}_{[x\leq 0]}$ and define $\bar{f}(z) = f(z) + \frac{1}{2}g(z)g'(z)$. If, for some $\epsilon > 0$ and $a \in (0,1)$, $\left|\bar{f}(z)\right| \geq \epsilon z(a - z)$ for $z \in [0, a]$ we have*

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\mathbb{E}\left[\int_0^t \int_{-\infty}^0 \tilde{u}^a(s,x)(a - \tilde{u}^a(s,x))\, dxds\right] < \infty. \tag{4.2}$$

Alternatively, if $\left|\bar{f}(z)\right| \geq \epsilon(1 - z)(z - a)$ for $z \in [a, 1]$ we have

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\mathbb{E}\left[\int_0^t \int_0^\infty (1 - \tilde{u}^a(s,x))(\tilde{u}^a(s,x) - a)\, dxds\right] < \infty. \tag{4.3}$$

We first prove several technical lemmas which will be useful.
In the following Lemmas, 63 - 67, let u define a solution started from the Heaviside initial condition $\mathbb{I}_{[x\leq 0]}$. Fix $a \in (0,1)$ to be the level at which we centre the solution.

Definition 62. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a set of real valued random variables. We will write X_n is stochastically increasing, $X_1 \overset{s}{\prec} X_2 \overset{s}{\prec} \ldots$, if there exist versions $\{\hat{X}_n\}_{n\in\mathbb{N}}$ on the same probability space such that $\hat{X}_n \overset{D}{=} X_n$ for all $n \in \mathbb{N}$ and for all $i, j \in \mathbb{N}$ satsifying $i < j$, $\mathbb{P}\left[\hat{X}_i \leq \hat{X}_j\right] = 1$.*

Lemma 63. ***Gradient around centring point***

$\tilde{u}_x(t, 0)$ is stochastically increasing in t for $t > 0$.

Proof. By Theorem 49 $\mathcal{L}(\tilde{u}(t,\cdot)) \overset{s}{\succ} \mathcal{L}(\tilde{u}(s,\cdot))$ for $0 < s < t$. By definition 43, we may take versions $\hat{\tilde{u}}(t,\cdot) \overset{D}{=} \tilde{u}(t,\cdot)$ and $\hat{\tilde{u}}(s,\cdot) \overset{D}{=} \tilde{u}(s,\cdot)$ so that $\hat{\tilde{u}}(t,\cdot) \overset{s}{\succ} \hat{\tilde{u}}(s,\cdot)$ almost surely. Note $\hat{\tilde{u}}(t,\cdot)$ and $\hat{\tilde{u}}(s,\cdot)$ are $\mathcal{C}^1$ and have strictly negative derivatives almost surely (see

Chapter 2 for regularity result). Applying Lemma 35 with $\hat{\tilde{u}}(t,0) = \hat{\tilde{u}}(s,0) = a$ we find $\hat{\tilde{u}}_x(t,0) \geq \hat{\tilde{u}}_x(s,0)$ almost surely and the lemma follows. ■

Lemma 64. ***Integral bound on*** $\tilde{u}_x(t,x)$

Fix $t_0 > 0$. *Then for* $t_0 \leq t$ *the integral* $\frac{1}{t}\mathbb{E}\left[\int_{t_0}^t \tilde{u}_x(s,0)\,ds\right]$ *is bounded below by* $\mathbb{E}\left[\tilde{u}_x(t_0,0)\right]$ *and bounded above by 0.*

Proof. For $0 < t_0 \leq t \leq T$, given that $\mathbb{E}\left[|\tilde{u}_x(t,0)|\right] < \infty$ by Property (vii) of Theorem 15, we can use Fubini's theorem to exchange the integral and expectation. Noting also that $\mathbb{E}\left[\tilde{u}_x(s,0)\right]$ is increasing by Lemma 63, a simple analysis reveals that

$$\frac{1}{t}\mathbb{E}\left[\int_{t_0}^t \tilde{u}_x(s,0)\,ds\right] = \frac{1}{t}\int_{t_0}^t \mathbb{E}\left[\tilde{u}_x(s,0)\right]\,ds \geq \mathbb{E}\left[\tilde{u}_x(t_0,0)\right](1-\frac{t_0}{t}) \geq \mathbb{E}\left[\tilde{u}_x(t_0,0)\right]$$

and

$$\frac{1}{t}\mathbb{E}\left[\int_{t_0}^t \tilde{u}_x(s,0)\,ds\right] = \frac{1}{t}\int_{t_0}^t \mathbb{E}\left[\tilde{u}_x(s,0)\right]\,ds \leq \mathbb{E}\left[\tilde{u}_x(t,0)\right](1-\frac{t_0}{t}) \leq 0$$

which completes our proof. ■

Lemma 65. ***Bound on Wave-Speed***

Choose $K_1, K_2 \geq 0$ *so that*

$$f(z) + \frac{1}{2}g(z)g'(z) \leq K_1 z$$
$$-f(z) - \frac{1}{2}g(z)g'(z) \leq K_2(1-z)$$

for all $z \in [0,1]$. *Then*

$$\overline{\lim_{t\to\infty}}\,\mathbb{E}\left[\frac{|\gamma_t^a|}{t}\right] \leq 2\sqrt{K_1} + 2\sqrt{K_2}.$$

Remark. Note that the above bounds are always possible given the current hypotheses (**H1**) and (**H2**) on f and g. To show this we make use of Taylor's Theorem:

$$\begin{aligned} f(z)+\frac{1}{2}gg'(z)=r(z) &= r(0)+r'(\eta)z \text{ for } \eta\in(0,z) \\ &\le |r'(\eta)|\, z \text{ given hypothesis } (\mathbf{H2})\ r(0)=0 \\ &\le Cz \text{ as } r' \text{ is bounded.} \end{aligned}$$

Remark. In the case $\bar{f}=f+\frac{1}{2}gg'\geq 0$ and concave, this shows the rate of growth of $\mathbb{E}\left[|\gamma_t^a|\right]$ is at most that of the deterministic wave speed for $u_t=u_{xx}+\bar{f}(u)$, namely $2\sqrt{\bar{f}'(0)}$.

Proof. For intuition we first consider the linearised deterministic equation

$$u_t(t,x)=u_{xx}(t,x)+K_1u(t,x) \tag{4.4}$$

where we have considered the Itô form of the equation, neglected the noise and written $\bar{f}(u)$ as a linear function of u. The Green's function for equation (4.4) can be found easily from standard PDE methods and is $\exp\{K_1t\}\frac{1}{\sqrt{4\pi t}}\exp\{-\frac{(x-y)^2}{4t}\}=\exp\{K_1t\}\Gamma_t(x-y):=\mathcal{H}_t(x-y)$.

Now, fix $t\geq 0$ and multiply both sides of equation (2.2) by a compactly supported, infinitely differentiable (in both variables) function $\varphi(s,x)$, $s\in\mathbb{R}^+, x\in\mathbb{R}$, satisfying $\varphi_x(-\infty)=\varphi_x(\infty)=0$. Integrating over both time and space gives:

$$\begin{aligned} \int_{\mathbb{R}} u(t,x)\varphi(t,x)-u(0,x)\varphi(0,x)\,dx &= \int_0^t u(t,x)\frac{\partial\varphi}{\partial t}(s,x)\,dx \\ &\quad +\int_0^t\int_{\mathbb{R}}\left(u_{xx}(s,x)+\bar{f}(u(s,x))\right)\varphi(s,x)\,dxds \\ &\quad +\text{martingale term.} \end{aligned}$$

Integrating by parts over x allows us to move the derivatives from u onto φ, on the right-hand-side, and using the boundary conditions for φ and the bound on $\bar{f}$ we can write:

$$\begin{aligned}\int_{\mathbb{R}} (u(t,x)\varphi(t,x) - u(0,x)\varphi(0,x))\, dx \\ &\le \int_0^t \int_{\mathbb{R}} u(s,x) \left(\varphi_{xx}(s,x) + K_1\varphi(s,x) + \frac{\partial \varphi}{\partial t}(s,x) \right) dxds \\ &+\text{martingale term.} \end{aligned} \tag{4.5}$$

We now take expectations to remove the martingale term and set $\varphi(s,y) = \mathcal{H}_{t-s}(\Xi, y)$ where $\mathcal{H}_t(\Xi, y) = \int_{\mathbb{R}} \mathcal{H}_t(x-y)\Xi(x)\, dx$ for suitable functions Ξ such that the integral is well defined. It is clear $\mathcal{H}_0(\Xi, y) = \Xi(y)$ and, by integration by parts due to $\mathcal{H}_t(x)$ satisfying equation (4.4),

$$\begin{aligned}\mathcal{H}_t(\Xi, y) &= \mathcal{H}_0(\Xi, y) + \int_0^t \mathcal{H}_s(\Xi_{xx} + K_1\Xi, y)\Xi(x)\, ds \\ &= \Xi(y) + \int_0^t \mathcal{H}_s(\Xi_{xx} + K_1\Xi, y)\Xi(x)\, ds. \end{aligned} \tag{4.6}$$

By the definition of φ, it is clear $\varphi(t,y) = \Xi(y)$ and $\varphi_{xx} + K_1\varphi + \frac{\partial\varphi}{\partial t} = 0$ by equation (4.6). Thus (4.5) becomes

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbb{R}} u(t,x)\Xi(x)\, dx\right] &\le \int_{\mathbb{R}} u(0,x)\mathcal{H}_t(\Xi, x)\, dx \\ &= \int_{\mathbb{R}} u(0,x) \left(\int_{\mathbb{R}} \mathcal{H}_t(z-x)\Xi(z)\, dz \right) dx. \end{aligned}$$

Letting $\Xi(r)$ approach a delta function, that is, for fixed $l \in \mathbb{R}$, be of the form $(4\pi n)^{-\frac{1}{2}} \exp\left(-\frac{(r-l)^2}{4n}\right)$ and let $n \to \infty$ the above equation tends to $\mathbb{E}\left[u(t,l)\right] \le \int_{\mathbb{R}} u(0,x)\mathcal{H}_t(l-x)\, dx$. To see this we apply Lesbesgue's differentiation Theorem on the left-hand-side and note $\mathcal{H}_t(\Xi, x) \to \mathcal{H}_t(l-x)$ on the right-hand-side. We will now relabel the dummy variables and define $C(t,x) = \mathbb{E}[u(t,x)]$ and rewrite the above as

$$\begin{aligned}
C(t,x) &\leq \exp(K_1 t) \int_{\mathbb{R}} H(y)\Gamma(t, x-y)\, dy \text{ where } H(y) = \mathbb{I}_{[y\leq 0]} \\
&= \exp(K_1 t) \int_{-\infty}^{0} \Gamma(t, x-y)\, dy \\
&= \frac{\exp(K_1 t)}{\sqrt{4\pi t}} \int_{-\infty}^{0} \exp\left(-\frac{(x-y)^2}{4t}\right) dy.
\end{aligned}$$

Using the substitution $z = (x-y)/\sqrt{t}$ we can express the above integral as

$$\int_{-\infty}^{0} \exp\left(-\frac{(x-y)^2}{4t}\right) dy = \int_{x/\sqrt{t}}^{\infty} \exp\left(-\frac{z^2}{4}\right) \sqrt{t}\, dz.$$

Noting the similarities between the Gaussian distribution mean 0, variance 2 this can be written

$$\int_{x/\sqrt{t}}^{\infty} \exp\left(-\frac{z^2}{4}\right) \sqrt{t}\, dz = \sqrt{4t\pi}\left(1 - \Phi(x/\sqrt{t})\right).$$

Hence,

$$C(t,x) \leq \exp(K_1 t)\left(1 - \Phi(x/\sqrt{t})\right).$$

Now, given the definition of the Gaussian Φ function we can write and bound

$$\begin{aligned}
1 - \Phi(x) &= \int_{x}^{\infty} \exp\left(-\frac{z^2}{4}\right) dy \leq \int_{x}^{\infty} \frac{z}{x} \exp\left(-\frac{z^2}{2}\right) dz \\
&= \frac{1}{x} \exp\left(-\frac{x^2}{4}\right).
\end{aligned}$$

Putting all this together allows us to write

$$\begin{aligned}
C(t,x) &\leq \exp(K_1 t)\left(1 - \Phi(x/\sqrt{t})\right) \\
&= \exp(K_1 t) \frac{\exp(-\frac{x^2}{4t})}{x} \sqrt{t}. \qquad (4.7)
\end{aligned}$$

Also note that $C(t,x) \leq 1$ for all $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$ trivially. Then

$$\begin{aligned}
\mathbb{E}\left[(\gamma_t^a)^+\right] &= \int_0^{\infty} \mathbb{P}\left[\gamma_t^a \geq x\right] dx \\
&= \int_0^{\infty} \mathbb{P}\left[u(t,x) \geq a\right] dx
\end{aligned}$$

since $\{\gamma_t^a \geq x\} = \{u(t,x) \geq a\}$ almost surely for $t > 0$.

Hence,

$$\begin{aligned}
\mathbb{E}\left[(\gamma_t^a)^+\right] &= \int_0^\infty \mathbb{P}\left[u(t,x) \geq a\right] dx \\
&\leq \int_0^\infty 1 \wedge \frac{\mathbb{E}\left[u(t,x)\right]}{a} dx \text{ by Tchebychev's inequality} \\
&\leq \int_0^{2\sqrt{K_1}t} dx + \frac{\sqrt{t}}{a} \int_{2\sqrt{K_1}t}^\infty \exp\left(K_1 t\right) \frac{\exp\left(-\frac{x^2}{4t}\right)}{x} dx \\
&\leq \int_0^{2\sqrt{K_1}t} dx + \frac{\sqrt{t}}{a} \int_{2\sqrt{K_1}t}^\infty \exp\left(K_1 t\right) \frac{\exp\left(-\frac{x^2}{4t}\right)}{x} \times \left(\frac{x^2}{4K_1 t^2}\right) dx \\
&\leq 2\sqrt{K_1}t + \frac{1}{a}\frac{1}{2K_1\sqrt{t}}.
\end{aligned}$$

This allows us to write

$$\overline{\lim_{t\to\infty}} \mathbb{E}\left[\frac{(\gamma_t^a)^+}{t}\right] \leq 2\sqrt{K_1}. \tag{4.8}$$

To control $\mathbb{E}\left[(\gamma_t^a)^-\right]$ where $(z)^- = -\left(z \wedge 0\right) \geq 0$ we note that for $x \geq 0$

$$\begin{aligned}
\{(\gamma_t^a)^- \geq x\} &= \{\gamma_t^a \leq -x\} \\
&= \{u(t,-x) \leq a\} \text{ almost surely if } t > 0 \\
&= \{1 - u(t,-x) \geq 1 - a\}.
\end{aligned}$$

Defining $v(t,x) = 1 - u(t,-x)$ we note that $v(t,x)$ solves the equation $dv = v_{xx}dt + \hat{f}(v)dt + \hat{g}(v) \circ dW_t$ where $\hat{f}(v) = -f(1-v)$, $\hat{g}(v) = -g(1-v)$. This means the condition $-f(z) - 1/2g(z)g'(z) = \hat{f}(1-v) + \frac{1}{2}\hat{g}\hat{g}'(1-v) \leq K_2(1-v)$ allows the same argument as shown in the $(\gamma_t^a)^+$ case to yield

$$\overline{\lim_{t\to\infty}} \mathbb{E}\left[\frac{(\gamma_t^a)^-}{t}\right] \leq 2\sqrt{K_2}. \tag{4.9}$$

Combining both estimates we have

$$\begin{aligned}\overline{\lim_{t\to\infty}}\,\mathbb{E}\left[|\gamma_t^a|\right] &= \overline{\lim_{t\to\infty}}\,\mathbb{E}\left[(\gamma_t^a)^+\right] + \overline{\lim_{t\to\infty}}\,\mathbb{E}\left[(\gamma_t^a)^-\right] \\ &\leq 2\sqrt{K_1} + 2\sqrt{K_2}\end{aligned}$$

as required. ■

Lemma 66. *Let M_t be a continuous local martingale. If $[M]_t \xrightarrow{a.s.} 0$ then $M_t \xrightarrow{a.s.} 0$.*

Proof. By Dubins-Schwarts (see [24]) there exists a Brownian motion B such that we can write $M_t = B_{[M]_t}$. Given that $[M]_t \to 0$ almost surely we have $M_t = B_{[M]_t} \to 0$ almost surely. ■

Lemma 67. *For $t \geq t_0 > 0$,*

$$\lim_{L\to\infty}\int_{t_0}^{t}(\tilde{u}(s,-L)-1)\circ\, d\gamma_s^a = 0 \textit{ almost surely.}$$

and

$$\lim_{U\to\infty}\int_{t_0}^{t}\tilde{u}(s,U)\circ\, d\gamma_s^a = 0 \textit{ almost surely.}$$

Proof. Given the definition of γ_s^a (see Chapter 2) we may write $\int_{t_0}^{t}(\tilde{u}(s,-L)-1)\circ\, d\gamma_s^a$ in its component parts, recall γ_s^a is defined by definition 17 and Theorem 21:

$$\begin{aligned}\int_{t_0}^{t}(\tilde{u}(s,-L)-1)\circ\, d\gamma_s^a &= \int_{t_0}^{t}(\tilde{u}(s,-L)-1)\left(\frac{u_{xx}(s,\gamma_s^a)+f(a)}{u_x(s,\gamma_s^a)}\right)ds \\ &\quad+\int_{t_0}^{t}(\tilde{u}(s,-L)-1)\left(\frac{g(a)}{u_x(s,\gamma_s^a)}\right)dW_s \\ &\quad+\frac{1}{2}\int_{t_0}^{t}\left[(\tilde{u}(s,-L)-1)\left(\frac{g(a)}{u_x(s,\gamma_s^a)}\right), W\right]ds\end{aligned}$$

where the square brackets denote the quadratic covariation process in calculating the correction term when moving from Stratonovich to Itô noise. We will consider each of these terms in turn. For the first term we shall, pathwise, use the Dominated-Convergence Theorem. Also note:

(1) $u_x(s, \gamma_s^a) < 0$ (Property (**iv**) Theorem 15),

(2) Given that $s \to \gamma_s^a$ and $u_{xx}(s, x)$ are continuous so is the composite and hence, $u_{xx}(s, \gamma_s^a)$ achieves its bounds and is bounded,

(3) $(\tilde{u}(t, -L) - 1)$ converges to zero as L increases and is bounded by 2.

Using these it is clear

$$\begin{aligned}
\lim_{L\to\infty} \int_{t_0}^{t} (\tilde{u}(s, -L) - 1) \left(\frac{u_{xx}(s, \gamma_s^a) + f(a)}{u_x(s, \gamma_s^a)} \right) ds \\
&= \int_{t_0}^{t} \lim_{L\to\infty} (\tilde{u}(s, -L) - 1) \left(\frac{u_{xx}(s, \gamma_s^a) + f(a)}{u_x(s, \gamma_s^a)} \right) ds \\
&= 0 \text{ almost surely by the Dominated Convergence Theorem.}
\end{aligned}$$

For the second term we appeal to Lemma 66 and show the quadratic covariation process converges almost surely to zero:

$$\int_{t_0}^{t} (\tilde{u}(s, -L) - 1)^2 \left(\frac{g(a)}{u_x(s, \gamma_s^a)} \right)^2 ds \to 0$$

almost surely as L tends to infinity similarly to the previous term.

For the third term we calculate the quadratic covariation. The decomposition of

$$d\left((\tilde{u}(s, -L) - 1) \left(\frac{g(a)}{u_x(s, \gamma_s^a)} \right) \right)$$

can be written

$$g(a)\frac{(u_x(s,\gamma_s^a)\circ d\tilde{u}(s,-L)+(\tilde{u}(s,-L)-1)\circ du_x(s,\gamma_s^a))}{u_x(s,\gamma_s^a)^2}$$

$$\begin{aligned} &= \text{increments with bounded variation}\\ &\quad +g(a)\frac{(u_x(s,\gamma_s^a)g(\tilde{u}(s,-L))\circ dW_s+u_x(s,\gamma_s^a)\tilde{u}_x(s,-L)\circ d\gamma_s^a)}{u_x(s,\gamma_s^a)^2}\\ &\quad +g(a)\frac{((\tilde{u}(s,-L)-1)\,g'(u(s,\gamma_s^a))u_x(s,\gamma_s^a)\circ dW_s)}{u_x(s,\gamma_s^a)^2}. \end{aligned}$$

Then,

$$\begin{aligned} &d\left[(\tilde{u}(s,-L)-1)\left(\frac{g(a)}{\tilde{u}_x(s,\gamma_s^a)}\right),W\right]\\ &\quad = g(a)\frac{(\tilde{u}_x(s,\gamma_s^a)g(\tilde{u}(s,-L))-u_x(s,\gamma_s^a)\tilde{u}_x(s,-L)\,(g(a)\,(\gamma_x^a)_s))}{\tilde{u}_x(s,\gamma_s^a)^2}\\ &\qquad +\frac{((\tilde{u}(s,-L)-1)\,g'(\tilde{u}(s,\gamma_s^a))\tilde{u}_x(s,\gamma_s^a))}{\tilde{u}_x(s,\gamma_s^a)^2}. \end{aligned}$$

Note m is the inverse of u and $(\gamma_x^a)_s = m_x(s)$ from equation (2.7) and is bounded on $s\in[t_0,t]$ almost surely. By Property (**iv**) of Theorem 15 and, given that g is bounded and continuous, $g(x)\to 0$ as $x\to 1$, this term clearly tends to zero as L tends to infinity by the same properties as those used in the first two terms. ∎

Proof of Theorem 61 (Stochastic McKean bound). For $t\geq t_0>0$, we have

$$\begin{aligned} \tilde{u}(t,x)-\tilde{u}(t_0,x) &= \int_{t_0}^t \tilde{u}_{xx}(s,x)\,ds+\int_{t_0}^t \tilde{u}_x(s,x)\circ d\gamma_s^a+\int_{t_0}^t f(\tilde{u}(s,x))\,ds\\ &\quad +\int_{t_0}^t g(\tilde{u}(s,x))\circ dW_s \ \mathbb{P}\text{-almost surely.} \end{aligned}$$

We now transform from the Stratonovich noise to an Itô noise, at the expense of a correction term which we will absorb into the forcing term writing $\bar{f}(z) = f(z) +$

$\frac{1}{2}g(z)g'(z)$.

$$\tilde{u}(t,x) - \tilde{u}(t_0,x) = \int_{t_0}^{t} \tilde{u}_{xx}(s,x)\,ds + \int_{t_0}^{t} \tilde{u}_x(s,x) \circ d\gamma_s^a + \int_{t_0}^{t} \bar{f}(\tilde{u}(s,x))\,ds + \int_{t_0}^{t} g(\tilde{u}(s,x))dW_s \tag{4.10}$$

$\mathbb{P}$-almost surely.

Note that for $U > 0$ we have $\tilde{u}(t,U) \leq a$ and $\tilde{u}(t,U) \overset{U\to\infty}{\longrightarrow} 0$.

We integrate over $[0,U]$ and use Fubini's and the stochastic Fubini Theorem to get

$$\begin{aligned} \int_0^U \tilde{u}(t,x) - \tilde{u}(t_0,x)\,dx = & \int_{t_0}^{t} \left(\tilde{u}_x(s,U) - \tilde{u}_x(s,0)\right)\,ds \\ & + \int_{t_0}^{t} \left(\tilde{u}(s,U) - a\right) \circ d\gamma_s^a \\ & + \int_{t_0}^{t}\int_0^U \bar{f}(\tilde{u}(s,x))\,dxds \\ & + \int_{t_0}^{t}\int_0^U g(\tilde{u}(s,x))\,dsdW_s. \end{aligned} \tag{4.11}$$

Remark. The first and third terms on the right-hand-side only require the ordinary Fubini Theorem, path-by-path, and this is easy to justify.

Remark. A suitable stochastic Fubini Theorem can be found in [24] page 176 for integrands that are bounded. This covers the fourth term on the right-hand-side. For the second term we proceed similarly but, given the unboundedness of $\tilde{u}_x(t,x)$ near $t = 0$, we cannot directly apply Fubini's Theorem. However, if we define the stopping time $\sigma_n = \inf\{t \geq t_0 > 0 : \sup_{0\leq y\leq U} |\tilde{u}_x(t,y)| \geq n\}$ and consider instead the integral

$$\int_0^U \int_{t_0}^{t\wedge\sigma_n} \tilde{u}_x(s,x) \circ d\gamma_s^a dx = \int_0^U \int_{t_0}^{t} \tilde{u}_x(s,x)\mathbb{I}_{[s\leq\sigma_n]} \circ d\gamma_s^a dx,$$

we can use Fubini's Theorem to interchange integrals given that the integrand is now bounded. Given that the stopping time satisfies $\sigma_n \to \infty$ almost surely as $n \to \infty$ by the regularity of $\tilde{u}_x$, we have the desired result almost surely.

We aim to progress in the following steps

(1) Let $U \to \infty$ to "decouple" $\circ\, d\gamma_s^a$ term;

(2) Take expectations;

(3) Then take the $\limsup \frac{1}{t}$.

By Lemma 67 $\int_{t_0}^t \tilde{u}(s,U) \circ d\gamma_s^a \to 0$ almost surely. This shows that term 2 on the right-hand-side of equation (4.11) converges almost surely to $-a\left(\gamma_t^a - \gamma_{t_0}^a\right)$. By regularity of solutions we know $\tilde{u}_x(s,U) \xrightarrow{a.s.} 0$ as $U \to \infty$. We need to justify

$$\int_{t_0}^t \tilde{u}_x(s,U)\, ds \xrightarrow{a.s.} 0 \text{ as } U \to \infty. \tag{4.12}$$

Given that

$$\mathbb{E}\left[\left(\int_{t_0}^t |\tilde{u}_x(s,U)|\; ds\right)^2\right] \leq C(T)\mathbb{E}\left[\int_{t_0}^t |\tilde{u}_x(s,U)|^2\; ds\right]$$

by the Cauchy-Schwarz inequality it is enough to show $|\tilde{u}_x(s,U)|^2$ is uniformly integrable. From Theorem 15, for solutions started from the Heaviside initial condition

$$\mathbb{E}\left[\sup_y |u_x(s,y)|^2\right] \leq C(t_0,T) < \infty$$

for $s \in (t_0, T]$.

Then,

$$\mathbb{E}\left[\int_{t_0}^t |\tilde{u}_x(s,U)|^2\; ds\right] \leq \mathbb{E}\left[\int_{t_0}^t \sup_y |u_x(s,y)|^2\; ds\right]$$

is bounded independently of U.

This shows that the variable $\int_{t_0}^t \tilde{u}_x(s,U)\,ds$ is uniformly integrable and this justifies (4.12).

As $u \in [0,1]$ given hypothesis (**H2**) we can write

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} u(s,x)(1-u(s,x))\,dxds\right] &\leq \int_0^T \int_0^\infty \mathbb{E}\left[u(s,x)\right]\,dxds \\
&\quad + \int_0^T \int_{-\infty}^0 \mathbb{E}\left[(1-u(s,x))\right]\,dxds \\
&< \infty
\end{aligned}$$

by the first moment bounds on $C(t,x) = \mathbb{E}\left[u(t,x)\right]$, as before (see equation (4.7)), and $\mathbb{E}\left[1-u(t,x)\right]$. This allows us to dominate (since $\left|\bar{f}(z)\right| \leq Kz(1-z)$) and justifies the limit

$$\int_{t_0}^t \int_0^U \bar{f}(\tilde{u}(s,x))\,dxds \to \int_{t_0}^t \int_0^\infty \bar{f}(\tilde{u}(s,x))\,dxds.$$

We will also show that, since g is bounded above, $g(z) \leq Cz(1-z)$ for some constant C, we have

$$\int_{t_0}^t \int_0^U g(\tilde{u}(s,x))\,dxdW_s \to \int_{t_0}^t \int_0^\infty g(\tilde{u}(s,x))\,dxdW_s \text{ as } U \to \infty. \tag{4.13}$$

Note that

$$\begin{aligned}
\mathbb{E}\left[\int_{t_0}^t \left(\int_{\mathbb{R}} u(1-u)(s,x)\,dx\right)^2 ds\right] &\leq 2\mathbb{E}\left[\int_0^T \left(\int_0^\infty u(s,x)\,dx\right)^2 ds\right] \qquad (4.14)\\
&\quad +2\mathbb{E}\left[\int_0^T \left(\int_{-\infty}^0 (1-u(s,x))\,dx\right)^2 ds\right]
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \mathbb{E}\left[\left(\int_0^\infty u(s,x)\,dx\right)^2 ds\right] &= \int_0^T\int_0^\infty\int_0^\infty \mathbb{E}\left[u(s,x)u(s,y)\right]\,dxdyds \\
&\leq \int_0^T\int_0^\infty\int_0^\infty \mathbb{E}\left[u(s,x)\right]\wedge\mathbb{E}\left[u(s,y)\right]\,dxdyds \\
&\leq \int_0^T\int_0^\infty\int_0^\infty C(s,x)\wedge C(s,y)\,dxdyds \\
&< \infty
\end{aligned}$$

by first moment bounds (see (4.7)). The second term in (4.14) is similar. This indicates that the right-hand-side of (4.13) is well defined. Also, this gives the required domination to show that (4.13) holds in $\mathbb{L}^2$ and hence, almost surely along a suitable sequence $U_n \to \infty$. Finally

$$\int_0^U \tilde{u}(t,x) - \tilde{u}(t_0,x)\,dx \xrightarrow{a.s.} \int_0^\infty \tilde{u}(t,x) - \tilde{u}(t_0,x)\,dx$$

where we dominate by

$$C(a)\int_{\mathbb{R}} u(1-u)(t,x) + u(1-u)(t_0,x)\,dx < \infty \text{ almost surely.}$$

This completes step 1 and we have

$$\begin{aligned}
\int_0^\infty \tilde{u}(t,x) - \tilde{u}(t_0,x)\,dx &= -\int_{t_0}^t \tilde{u}_x(s,0)\,dx - a\left(\gamma_t^a - \gamma_{t_0}^a\right) \\
&\quad + \int_{t_0}^t\int_0^\infty \bar{f}(\tilde{u}(s,x))\,dxds \qquad (4.15) \\
&\quad + \int_{t_0}^t\int_0^\infty g(\tilde{u}(s,x))dxdW_s
\end{aligned}$$

almost surely. To complete step 2 we now take expectations and rearrange.

$$\begin{aligned}\int_{t_0}^{t} \mathbb{E}\left[\int_0^\infty \bar{f}(\tilde{u}(s,x))\,dx\right] ds &= \mathbb{E}\left[\int_0^\infty \tilde{u}(t,x) - \tilde{u}(t_0,x)\,dx\right] \\ &\quad + a\mathbb{E}\left[\gamma_t^a - \gamma_{t_0}^a\right] \\ &\quad + \mathbb{E}\left[\int_{t_0}^{t} \tilde{u}_x(s,0)\,dx\right].\end{aligned} \tag{4.16}$$

For step 3 we multiply by $\frac{1}{t}$ for $t \geq t_0 > 0$ and take the limit as $t \to \infty$. The domination in equation (4.15) shows that

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\left|\mathbb{E}\left[\int_0^\infty \tilde{u}(t,x) - \tilde{u}(t_0,x)\,dx\right]\right| < \infty$$

by using first moment bounds on $u(t,x)$ and $1 - u(t,x)$. Lemma 65 shows that

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\left|\mathbb{E}\left[\gamma_t^a - \gamma_{t_0}^a\right]\right| < \infty.$$

Also, by Lemma 64

$$\begin{aligned}\overline{\lim_{t\to\infty}} -\frac{1}{t}\mathbb{E}\left[\int_{t_0}^{t} \tilde{u}_x(s,0)\,ds\right] &\leq \mathbb{E}\left[-\tilde{u}_x(t_0,0)\right] \\ &< \infty\end{aligned}$$

and $\frac{1}{t}\mathbb{E}\left[\int_{t_0}^{t} \tilde{u}_x(s,0)\,ds\right] \leq 0$. This shows that in the case that either $f(z) \geq \epsilon z(a-z)$ for $z \in [0,a]$ or $f(z) \leq -\epsilon z(a-z)$ for $z \in [0,a]$ we may use equation (4.16) to deduce that

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\mathbb{E}\left[\int_{t_0}^{t}\int_0^\infty \tilde{u}(s,x)\left(a - \tilde{u}(s,x)\right)\,dx\right] < \infty. \tag{4.17}$$

In the case $\left|\bar{f}(z)\right| \geq \epsilon(1-z)(z-a)$ for $z \in [a,1]$ we repeat the above analysis but integrate over $[-L,0]$ and let $L \to \infty$ to deduce

$$\overline{\lim_{t\to\infty}} \frac{1}{t}\mathbb{E}\left[\int_{t_0}^{t}\int_{-\infty}^{0}(1-\tilde{u}(t,x))(\tilde{u}(t,x)-a)\,dxds\right] < \infty. \tag{4.18}$$

We can also replace t_0 with 0 if necessary as, by first moment bounds on

$$\mathbb{E}\left[\int_{\mathbb{R}} u(t,x)(1-u(t,x)\,dx\right],$$

we can show that $\frac{1}{t}\mathbb{E}\left[\int_0^{t_0}\int_{\mathbb{R}} u(s,x)(1-u(s,x))\,dxds\right] \to 0$ as $t \to \infty$ and this completes the proof. ∎

Lemma 68. *Let u be a solution to equation (2.1) under hypotheses (**H1**) and (**H2**). If $\left|\bar{f}\right| \geq \epsilon z(1-z)$ on $[0,1]$ then*

$$\overline{\lim_{t\to\infty}}\,\frac{1}{t}\mathbb{E}\left[\int_0^t\int_{\mathbb{R}} u(s,x)(1-u(s,x))\,dxds\right] < \infty.$$

Proof. The proof of this follows in the same way as Theorem 61, we first integrate over $(-L,U)$ and then by splitting up the range of integration to $[-L,0]$ and $[0,U]$ the proof carries over unchanged. ∎

Remark. This lemma shows that in the case $\bar{f}(u) \leq \epsilon u(1-u)$, the limiting solution as time tends to infinity is not the constant function $u_\infty \neq a \in (0,1)$. However, in the cases $\bar{f}(u) \leq \epsilon u(1-u)(u-a)$ and $\bar{f}(u) \leq \epsilon u(1-u)(a-u)$ more work is required to remove the possibility that $u_\infty = a$. We will look at this in the next section.

4.2 Dynamics around centring point

Introduction. *In the previous section we have seen that if the solution becomes flat around the centring point then this adds little contribution to the McKean bound type estimates. For a fixed centring point $a \in (0,1)$ such that $\bar{f}(a) = 0$, the trivial solution of $\lim_{t\to\infty}\mathcal{L}\left(\tilde{u}(t,x)\right) = \delta_a$ still remains a possibility. In this section we explore the dynamics* *of the solution to equation (2.2) around the centring point a and show that such a limiting solution is impossible. We will also discuss the implications in the case of $\bar{f}$ being of the unstable form, $\bar{f} = u(1-u)(a-u)$ (see Chapter 1).*

Theorem 69. *Suppose $\bar{f}$ is either of KPP, Nagumo or Unstable type and, for the Nagumo and Unstable type, suppose further $\bar{f}(a) = 0$ and $g(a) \neq 0$. Then $\nu\left(\mathbb{B}_{dec}^{1,0}\right) = 1$.*

Proof. $\bar{f}$ **is of KPP type.**

In this case $\bar{f} \geq \epsilon z(1-z)$ for all $\epsilon > 0$. Since $\mathcal{L}(\tilde{u}(t)) \to \nu$ in $\mathcal{M}_1(\mathbb{B}_{dec})$ then for any N

$$\mathbb{E}\left[\int_{-N}^{N} \tilde{u}(t,x)(1-\tilde{u}(t,x))\,dx\right] \to \int_{\mathbb{B}_{dec}} \int_{-N}^{N} \varphi(1-\varphi)\,dx\nu(d\varphi) \text{ as } t \to \infty$$

(since the map $\varphi \to \int_{-N}^{N} \varphi(1-\varphi)\,dx$ is bounded and continuous on $\mathbb{B}_{dec}$). Hence,

$$\frac{1}{t}\mathbb{E}\left[\int_0^t \int_{-N}^{N} \tilde{u}(t,x)(1-\tilde{u}(t,x))\,dxds\right] \to \int_{\mathbb{B}_{dec}} \int_{-N}^{N} \varphi(1-\varphi)\,dx\nu(d\varphi)$$

and by Lemma 68 the right-hand-side must be bounded independently of N. Letting $N \to \infty$ we find

$$\int_{\mathbb{B}_{dec}} \int_{\mathbb{R}} \varphi(1-\varphi)\,dx\nu(d\varphi) < \infty.$$

This implies that for $d\nu$ almost all φ either $\varphi \in \mathbb{B}_{dec}^{1,0}$ or $\varphi = 0$ or $\varphi = 1$. In Theorem 51 we found variables $\hat{\tilde{u}}_{t_n}$ so that

$$\hat{\tilde{u}}_{t_n} \uparrow u(\infty,x) \text{ for } x \geq 0$$

$$\hat{\tilde{u}}_{t_n} \downarrow u(\infty,x) \text{ for } x \leq 0$$

so that $\bar{u}(\infty,x)$, the right continuous modification of $u(\infty,x)$ which has law ν, takes values in $[0,a]$ for $x > 0$ and in $[a,1]$ for $x < 0$. This eliminates the possibility that

$\varphi = 0$ and $\varphi = 1$ and shows $\nu(\mathbb{B}^{1,0}_{dec}) = 1$.

$\bar{f}$ is of the Nagumo or Unstable type (where $\bar{f}(a) = 0$, $g(a) \neq 0$).

We have $|\bar{f}| \geq \epsilon(1-z)(z-a)$ on $[a,1]$ and $|\bar{f}| \geq \epsilon z(a-z)$ on $[0,a]$. By the same reasons as in the KPP type case we find

$$\int_{\mathbb{B}_{dec}} \int_0^\infty \varphi(a-\varphi)\, dx \nu(d\varphi) < \infty$$

and

$$\int_{\mathbb{B}_{dec}} \int_{-\infty}^0 (1-\varphi)(\varphi - a)\, dx \nu(d\varphi) < \infty.$$

This implies that for $d\nu$ almost all φ either $\varphi \in \mathbb{B}^{1,0}_{dec}$ or $\varphi = a$. We need to eliminate the latter possibility. The key idea is that if $\nu(\varphi : \varphi = a) = \delta_1 > 0$ then with probability at least $\delta_1/2$ there will be an arbitrary large flat-ish patch in $\tilde{u}(t)$. However, this would lead to an arbitrary large contribution to the McKean bound when the noise $g(u) \circ dW$ perturbs it.

We will prove by contradiction. Suppose $\nu(\varphi : \varphi = a) = \delta_1 > 0$. Then, defining $z_+ = z \vee 0$, setting $\eta > 0$ and using the property that $\varphi \mapsto \left(1 - \int_{\mathbb{R}} |\varphi(x) - a|^2 \exp(-\eta |x|)\, dx\right)_+$ is bounded and continuous, we can write

$$\begin{aligned}
\tilde{\mathcal{Q}}^H_t &\left(\varphi : \int_{\mathbb{R}} |\varphi(x) - a|^2 \exp(-\eta |x|)\, dx \leq 1\right) \\
&\geq \int_{\mathbb{B}_{dec}} \left(1 - \int_{\mathbb{R}} |\varphi(x) - a|^2 \exp(-\eta |x|)\, dx\right)_+ \tilde{\mathcal{Q}}^H_t(d\varphi) \\
&\to \int_{\mathbb{B}_{dec}} \left(1 - \int_{\mathbb{R}} |\varphi(x) - a|^2 \exp(-\eta |x|)\, dx\right)_+ \nu(d\varphi) \\
&\geq \delta_1.
\end{aligned}$$

So for $t \geq T(\eta)$, $\tilde{\mathcal{Q}}^H_t\left(\varphi : \int_{\mathbb{R}} |\varphi(x) - a|^2 \exp(-\eta |x|)\, dx \leq 1\right) \geq \delta_1/2$. Let u be the solution started from the Heaviside initial condition such that $\mathcal{L}(\tilde{u}(t)) = \tilde{\mathcal{Q}}^H_t$. Suppose this is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and with a Brownian motion $(W_t : t \geq 0)$. Fix $\delta_1 > 0$ and choose $\Omega_t \in \mathcal{F}_t$ so that $\mathbb{P}(\Omega_t) = \delta_1/2$ and $\int_{\mathbb{R}} |u(t,x) - a|^2 \exp(-\eta |x|)\, dx \leq 1$ on Ω_t. Let $(Y_s : s \in [t, t+1])$ solve the SDE $dY = \bar{f}(Y)dt + g(Y)dW_t$ with initial condition $Y_t = a$. We will prove a proposition (Proposition 71) which will show that $\tilde{u}$ stays close to Y_t if the initial conditions are close and then use this to show a contradiction to the McKean Bound.

$$\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}} \tilde{u}(t+t^*, x)|1 - \tilde{u}(t+t^*, x)||\tilde{u}(t+t^*, x) - a|\, dx\right] \\
&\qquad = \mathbb{E}\left[\int_{\mathbb{R}} u(t+t^*, x)|1 - u(t+t^*, x)||u(t+t^*, x) - a|\, dx\right] \\
&\qquad \geq \mathbb{E}\left[\int_{\mathbb{R}} u(t+t^*, x)|1 - u(t+t^*, x)||u(t+t^*, x) - a|\right. \\
&\qquad\qquad \left.\times \exp(-\eta |x|)\, dx \mathbb{I}(\Omega_t)\right] \\
&\qquad \geq \int_{\mathbb{R}} \exp(-\eta |x|)\, dx \mathbb{E}\left[Y_{t+t^*}|1 - Y_{t+t^*}||Y_{t+t^*} - a|\mathbb{I}(\Omega_t)\right] \\
&\qquad\qquad - L\mathbb{E}\left[\int_{\mathbb{R}} \exp(-\eta|x|) \left|u(t+t^*, x) - Y_{t+t^*}\right| dx \mathbb{I}(\Omega_t)\right] \\
&\qquad = I - II
\end{aligned} \tag{4.19}$$

where L is the Lipschitz constant of $z|1-z||a-z|$ on $[0,1]$. Before continuing this proof, we will prove two results which will be useful in calculating a lower bound and hence a contradiction.

Lemma 70. *Suppose $\bar{f}, g$ satisfy hypotheses **(H1)** and **(H2)** of Chapter 2 and consider the solution Y_t to the SDE $dY_t = \bar{f}(Y_t)dt + g(Y_t)dW_t$ for $t \geq 0$ with initial condition $Y_0 = a$. Suppose also $g(a) \neq 0$. Then there exists $t^* \in (0,1)$, $\delta_0 > 0$ and $\epsilon_0 > 0$ satisfying $4\epsilon_0 < a \wedge (1-a)$ such that $\mathbb{P}\left[|Y_{t^*} - a| \in (\epsilon_0, 2\epsilon_0)\right] > \delta_0$.*

Proof. The proof of this follows from the fact that solutions are unique and $Y_t = a$ for $t \in [0,1]$ is not a solution for $g(a) \neq 0$. $\blacksquare$

Remark. If $|Y_{t^*} - a| \in (\epsilon_0, 2\epsilon_0)$ where $4\epsilon_0 < a \wedge (1-a)$ then $Y_{t^*} \geq \frac{a}{2}$ and $1 - Y_{t^*} \geq \frac{1-a}{2}$.

Proposition 71. *Let u be the solution to (2.1) driven by W_t. Define Y_t as the solution to the SDE $dY_t = \bar{f}(Y_t)dt + g(Y_t)dW_t$, $t \in \mathbb{R}^+$, $Y_0 = a$ on the same probability space. Then there exists a constant $C(f,g,T)$ so that for all $t \in [0,T]$ and $\eta \in (0,1)$,*

$$\mathbb{E}\left[\int_{\mathbb{R}} \exp\left(-\eta\,|x|\right)|u(t,x) - Y_t|^2\,dx\right] \leq C(f,g,T)\mathbb{E}\left[\int_{\mathbb{R}} \exp\left(-\eta\,|x|\right)|u(0,x) - a|^2\;dx\right].$$

Remark. In the proof of the above proposition we will make use of the following bound:

$$\begin{aligned}
\int_{\mathbb{R}} \Gamma_t(x-y)\exp\left(-\eta\,|x|\right)\,dx &= \int_{\mathbb{R}} \Gamma_t(z)\exp\left(-\eta\,|y+z|\right)\,dz \\
&\leq \int_{\mathbb{R}} \Gamma_t(z)\exp\left(-\eta\,(y+z)\right)\,dz \\
&= \exp\left(-\eta y\right)\int_{\mathbb{R}} \Gamma_t(z)\exp\left(-\eta z\right)\,dz \\
&= (4\pi t)^{-\frac{1}{2}}\exp\left(-\eta y\right)\int_{\mathbb{R}} \exp\left(-\frac{1}{4t}\left(z+2t\eta\right)^2 + t\eta^2\right)\,dz \\
&= (4\pi t)^{-\frac{1}{2}}\exp\left(-\eta y + t\eta^2\right)\int_{\mathbb{R}} \exp\left(-y^2\right)2\sqrt{t}\,dy \\
&= \exp\left(-\eta y + t\eta^2\right) \\
&\leq \exp\left(-\eta y\right)\exp\left(C(T)\eta^2\right) \text{ for all } 0 \leq t \leq T.
\end{aligned}$$

Similarly, the above can also be bounded by $\exp(\eta y)\exp(C(T)\eta^2)$ and hence we may write

$$\int_{\mathbb{R}} \Gamma_t(x-y)\exp\left(-\eta\,|x|\right)\,dx \leq \exp\left(-\eta\,|y|\right)\exp\left(C(T)\eta^2\right). \tag{4.20}$$

Proof of Proposition 71. Writing the solutions to u and Y in the Green's function representation form and defining $z(t,x) = u(t,x) - Y_t$ we can write

$$\begin{aligned} z(t,x) &= \int_{\mathbb{R}} \Gamma_t(x-y) z(0,y)\, dy + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(Y_s)\right)\, dyds \\ &\quad + \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(g(u(s,y)) - g(Y_s)\right)\, dydW_s. \end{aligned}$$

We will now multiply throughout by $\exp(-\eta|x|)$ and integrate over $\mathbb{R}$ and consider each term separately. As the initial conditions are deterministic we have:

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Gamma_t(x-y) z(0,y)\, dy \right)^2 \exp(-\eta\,|x|)\, dx \;\leq\; \int_{\mathbb{R}} \int_{\mathbb{R}} \Gamma_t(x-y) z(0,y)^2 \exp(-\eta\,|x|)\, dydx$$

by the Cauchy-Schwarz inequality. Now using the bound in (4.20) we have

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \Gamma_t(x-y) z(0,y)\, dy \right)^2 \exp(-\eta\,|x|)\, dx \;\leq\; \exp\left(C(T)\eta^2\right) \int_{\mathbb{R}} z(0,y)^2 \exp(-\eta\,|y|)\, dy.$$

Also, by use of the Lipschitz property of $\bar{f}$, we have

$$\begin{aligned} &\int_{\mathbb{R}} \left(\int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(Y_s)\right)\, dyds \right)^2 \exp(-\eta\,|x|)\, dx \\ &\qquad = \int_{\mathbb{R}} \left| \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left(\bar{f}(u(s,y)) - \bar{f}(Y_s)\right)\, dyds \right|^2 \\ &\qquad\qquad \times \exp(-\eta\,|x|)\, dx \\ &\qquad \leq C(T) \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) \left|\bar{f}(u)(s,y) - \bar{f}(Y_s)\right|^2 \\ &\qquad\qquad \times \exp(-\eta\,|x|)\, dydsdx \\ &\qquad \leq C(T, \|f'\|_\infty) \int_{\mathbb{R}} \int_0^t \int_{\mathbb{R}} \Gamma_{t-s}(x-y) z(s,y)^2 \\ &\qquad\qquad \times \exp(-\eta\,|x|)\, dydsdx \\ &\qquad \leq C(T, \|f'\|_\infty) \exp\left(C(T)\eta^2\right) \int_0^t \int_{\mathbb{R}} z(s,y)^2 \\ &\qquad\qquad \times \exp(-\eta\,|y|)\, dyds \end{aligned}$$

by use of (4.20) and Fubini's Theorem. Using the Itô isometry as well as the Lipschitz property for g, a similar bound holds for the $(g(u(s,y)) - g(Y_s))$ term with constant $C(\|g'\|_\infty)\exp(C(T)\eta^2)$. Putting all of these terms together shows,

$$\begin{aligned}
\mathbb{E}\left[\int_{\mathbb{R}} \exp(-\eta|x|)\, z(t,x)^2\, dx\right] &\leq \exp\left(C(T)\eta^2\right)\int_{\mathbb{R}} \mathbb{E}\left[z(0,y)^2\right]\exp(-\eta|y|)\, dy \\
&\quad + C(T,\|f'\|_\infty,\|g'\|_\infty)\exp\left(C(T)\eta^2\right) \\
&\quad \times \int_0^t \int_{\mathbb{R}} \mathbb{E}\left[z(s,y)^2\right]\exp(-\eta|y|)\, dyds.
\end{aligned}$$

Using Gronwall's argument we have the required result. ■

In continuing Theorem 69, the proof that ν is concentrated on $\mathbb{B}^{1,0}_{dec}$ where $\bar{f}$ is of Nagumo or Unstable type, we will bound I and II in equation (4.19) separately. For the first term

$$\begin{aligned}
I &= \frac{2}{\eta}\mathbb{E}\left[Y_{t+t^*}(1-Y_{t+t^*})|a - Y_{t+t^*}|\mathbb{I}(\Omega_t)\right] \\
&= \frac{2}{\eta}\mathbb{E}\left[\mathbb{I}(\Omega_t)\mathbb{E}\left[Y_{t^*}(1-Y_{t^*})|a-Y_{t^*}|\,|\,Y_0 = a\right]\right] \text{ by the Markov property} \\
&\geq \frac{2}{\eta}\delta_0\frac{a}{2}\frac{(1-a)}{2}\epsilon_0\mathbb{P}\left[\Omega_t\right] \\
&= \frac{2}{\eta}\delta_0\frac{a}{2}\frac{(1-a)}{2}\epsilon_0\frac{\delta_1}{2}.
\end{aligned}$$

For the second term

$$
\begin{aligned}
II &\leq L\mathbb{E}\left[\mathbb{I}(\Omega_t)\mathbb{E}\left[\int_{\mathbb{R}} \exp\left(-\eta|x|\right)|u(t+t^*,x)-Y_{t+t^*}|\ dx \mid \mathcal{F}_t\right]\right] \\
&\leq L\mathbb{E}\left[\mathbb{I}(\Omega_t)\left|\mathbb{E}\left[\left|\int_{\mathbb{R}} \exp\left(-\eta|x|\right)|u(t+t^*,x)-Y_{t+t^*}|\ dx\right|^2 \mid \mathcal{F}_t\right]\right|^{1/2}\right] \\
&\qquad \text{by use of the Cauchy-Schwarz inequality} \\
&\leq L\sqrt{\frac{2}{\eta}}\mathbb{E}\left[\mathbb{I}(\Omega_t)\left|\mathbb{E}\left[\int_{\mathbb{R}} \exp\left(-\eta|x|\right)|u(t+t^*,x)-Y_{t+t^*}|^2\ dx \mid \mathcal{F}_t\right]\right|^{1/2}\right] \\
&\leq LC(f,g)\sqrt{\frac{2}{\eta}}\mathbb{E}\left[\mathbb{I}(\Omega_t)\left|\int_{\mathbb{R}} \exp\left(-\eta|x|\right)|u(t,x)-a|^2\ dx\right|^{1/2}\right]
\end{aligned}
$$

by Proposition 71. Now since $\int_{\mathbb{R}} \exp\left(-\eta|x|\right)|u(t,x)-a|^2\ dx \leq 1$ on Ω_t we can write

$$
\begin{aligned}
LC(f,g)\sqrt{\frac{2}{\eta}}\mathbb{E}\left[\mathbb{I}(\Omega_t)\left|\int_{\mathbb{R}} \exp\left(-\eta|x|\right)|u(t,x)-a|^2\ dx\right|^{1/2}\right] &\leq LC(f,g)\sqrt{\frac{2}{\eta}}\mathbb{E}\left[\mathbb{I}(\Omega_t)\right] \\
&= LC(f,g)\sqrt{\frac{2}{\eta}\frac{\delta_1}{2}}.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\mathbb{E}\left[\int_{\mathbb{R}} \tilde{u}(t+t^*,x)|1-\tilde{u}(t+t^*,x)||\tilde{u}(t+t^*,x)-a|\,dx\right] \\
&\qquad \geq \frac{2}{\eta}\delta_0\frac{a}{2}\frac{(1-a)}{2}\epsilon_0\frac{\delta_1}{2} - LC(f,g)\sqrt{\frac{2}{\eta}\frac{\delta_1}{2}}.
\end{aligned}
$$

By taking η small, the above bound can be made arbitrarily large and as, by the use of the stochastic McKean bound (Theorem 61),

$$
\begin{aligned}
&\overline{\lim_{t\to\infty}}\,\frac{1}{t}\int_0^t \mathbb{E}\left[\int_{\mathbb{R}} u(t,x)(1-u(t,x))|a-u(t,x)|\,dx\right]\,ds \\
&\qquad \leq \overline{\lim_{t\to\infty}}\,\frac{1}{t}\int_0^t \mathbb{E}\left[\int_{\mathbb{R}}(1-u(t,x))|a-u(t,x)|\,dx\right]\,ds \qquad (4.21) \\
&\qquad\quad + \overline{\lim_{t\to\infty}}\,\frac{1}{t}\int_0^t \mathbb{E}\left[\int_{\mathbb{R}} u(t,x)|a-u(t,x)|\,dx\right]\,ds \qquad (4.22) \\
&\qquad < \infty
\end{aligned}
$$

this is a contradiction and shows that the supposition $\nu(\varphi : \varphi = a) = \delta_1 > 0$ was false. ■

Remark. The bounds for equations (4.21) and (4.22) also indicate that $\nu(\varphi : \varphi = 0)$ and $\nu(\varphi : \varphi = 1)$ are, respectively, almost surely equal to 0.

4.3 Stationary Travelling Wave

Introduction. *In this section we combine the sections above to conclude that the limit law to which the wavefront converges is a stationary travelling wave. However, this result is based upon a general Markovian framework which, following a statement of the key result in this section, we explain further.*

Theorem 72. *Suppose $\bar{f}$ is of KPP, Nagumo or Unstable type. Then ν is a stationary travelling wave, that is, it is invariant for the pinned process. We can write this as $\mathcal{Q}^\nu\left[\tilde{u}(t) \in dg\right] = \nu(dg)$.*

The proof of this is at the end of the section.

4.4 Markovian framework

Let $X = (X_t, t \geq 0)$ be a continuous-time Markov process defined on a probability space $(\Omega, \mathcal{F}, (\mathbb{P}_x : x \in E))$ with values on a metric space $(E, \mathcal{E})$. (So under $\mathbb{P}_x$, X_t has starting condition x). Transition Markov kernels $P_t(x, dg) = \mathbb{P}_x\left[X_t \in dg\right]$ satisfy, for all $0 \leq s \leq t$, $x \in E$ and $dg \in \mathcal{E}$,

(1) $x \mapsto P_t(x, A)$ are measurable for all $A \in \mathcal{E}$;

(2) $P_{t+s}(x, dg) = \int_E P_s(x, dg) P_t(g, dx)$.

If $F : E \to \hat{E}$ is a measurable map from $(E, \mathcal{E})$ to another measurable space $(\hat{E}, \hat{\mathcal{E}})$ we may ask under what condition the image $Y_t = F(X_t)$ is still Markov. This is set out in the next lemma.

Lemma 73. ***Dynkin Criterion***

If for all $y \in E$ and $A \in \hat{\mathcal{E}}$ the values $P_t(y, F^{-1}(A))$ are all equal for all $y \in F^{-1}(x)$ then Y_t is a Markov Process.

Remark. The proof of this can be found in [25], Theorem 13.5.

Sketch proof. Define for $A \in \hat{\mathcal{E}}$ and $y \in \hat{E}$, $\hat{P}_t(y, A) = P_t(x, F^{-1}(A))$ for $x \in F^{-1}(y)$. It then follows by checking the two criteria defining a transition kernel above. ∎

Lemma 74. *Suppose u satisfies equation (2.1) with hypotheses (**H1**) and (**H2**). Then, defining the map $F : \mathbb{B}^{1,0}_{dec} \to \mathbb{B}^{1,0}_{dec}$ by $F(\varphi) = \tilde{\varphi}$ where $\tilde{\varphi}(x) = \varphi(x + \gamma^a_t(\varphi))$ for some fixed $a \in (0, 1)$, $\tilde{u} = F(u)$ is still a Markov process.*

Proof. Consider the Dynkin criteria. In our setting we have $E = \hat{E} = \mathbb{B}^{1,0}_{dec}$. From Theorem 15 we know u is a Markov process. To finish we need to show that the map $F(\varphi) = \tilde{\varphi}$ is measurable upon which, by the use of Lemma 73, we have that $F(\varphi)$ describes a Markov family. Now, for fixed $a \in (0, 1)$ it is clear the map $\varphi \mapsto \varphi(\cdot - a)$ is continuous in the $\mathbb{L}^1_{loc}$ metric. Consider the map $\varphi \mapsto (\varphi, \gamma^a_t(\varphi))$. By Lemma 20 we know $\varphi \mapsto \gamma^a_t(\varphi)$ is measurable and hence, so is the composite. Now in our context, we consider a pinned $f \in \mathbb{B}^{1,0}_{dec}$, a fixed $A \subseteq \mathbb{B}^{1,0}_{dec}$ and choose $g \in F^{-1}(f)$. Then for some translation τ^a we can write $g = \tau^a f$ and

$$\begin{aligned}
P_t(g, F^{-1}(A)) &= \mathbb{P}_g\left[u(t) \in F^{-1}(A)\right] \\
&= \mathbb{P}_g\left[F(u(t)) \in A\right] \text{ by inverting the pinning} \\
&= \mathbb{P}_g\left[\tilde{u}(t) \in A\right] \text{ by the definition of } F \text{ being the centring function} \\
&= \mathbb{P}_{\tau^a f}\left[\tilde{u}(t) \in A\right] \text{ by definition of } g \text{ being a translation of } f \\
&= \mathbb{P}_f\left[\tilde{u}(t) \in A\right] \text{ as } \tilde{u} \text{ is translation invariant.}
\end{aligned}$$

Then for all x, all $t \geq 0$ and all A in $\mathbb{B}^{1,0}_{dec}$ we have the transition kernels $P_t(x, F^{-1}(A))$ are equal for all $x \in F^{-1}(a)$. Hence the kernel $\hat{P}(x, B) = P_t(x, F^{-1}(B))$, being equal, shows that the map $u(t,x) \mapsto \tilde{u}(t,x)$ preserves the Markov property given that $u(t,x)$ is Markov. ■

Lemma 75. *Suppose the map $F : \mathbb{B}_{dec} \to \mathbb{R}$ if bounded and continuous then $P_t F : \mathbb{B}_{dec} \to \mathbb{R}$ is bounded and continuous.*

Proof. Suppose $\varphi^n \to \varphi$. Take u^n, u as solutions with initial conditions φ^n, φ and with respect to the same Brownian motion. Then, for any $N > 0$,

$$\mathbb{E}\left[\int_{-N}^{N} |u^n(t,x) - u(t,x)|^2 \, dx\right] \to 0 \text{ as } n \to \infty. \tag{4.23}$$

This is enough since for any subsequence n', by considering a suitable sub-subsequence, n'' say, we find that $u^{n''}(t,\cdot) \xrightarrow{a.s.} u(t,\cdot)$. Then

$$\begin{aligned}
P_t F(\varphi_{n''}) &= \mathbb{E}\left[F(u^{n''}(t))\right] \\
&\to \mathbb{E}\left[F(u(t))\right] \text{ since } F \text{ is bounded, continuous} \\
&= P_t F(\varphi).
\end{aligned}$$

This is true for any sequence n' such that $P_tF(\varphi_n) \to P_tF(\varphi)$. To prove equation (4.23) we use Proposition 71 where we set $v(t,x) = Y_t$. Then

$$\begin{aligned}
\mathbb{E}\left[\int_{-N}^{N} |u^n(t,x) - u(t,x)|^2 \, dx\right] &\leq \exp(\eta N)\,\mathbb{E}\left[\int_{-N}^{N} |u^n(t,x) - u(t,x)|^2 \exp(-\eta\,|x|)\, dx\right] \\
&\leq \exp(\eta N)\,\mathbb{E}\left[\int_{\mathbb{R}} |u^n(t,x) - u(t,x)|^2 \exp(-\eta\,|x|)\, dx\right] \\
&\leq \exp(\eta N)\, C(f,g,T) \\
&\quad \times \mathbb{E}\left[\int_{\mathbb{R}} \exp(-\eta\,|x|)\,|\varphi^n(x) - \varphi(x)|^2 \, dx\right].
\end{aligned}$$

We can then choose N sufficiently large such that

$$\mathbb{E}\left[\int_{\mathbb{R}} \exp(-\eta\,|x|)\,|\varphi^n(x) - \varphi(x)|^2 \, dx\right] \leq \mathbb{E}\left[\int_{-N}^{N} |\varphi^n(x) - \varphi(x)|^2 \, dx\right].$$

Given $\varphi^n \to \varphi$ in $\mathbb{L}^1_{loc}(\mathbb{R})$ we have $\mathbb{E}\left[\int_{-N}^{N} |\varphi^n(x) - \varphi(x)|^2 \, dx\right] \to 0$ as $n \to \infty$ and hence, the desired result. ■

Corollary 76. *Suppose the map $F : \mathbb{B}^{1,0}_{dec} \to \mathbb{R}$ is bounded, continuous and translation invariant, that is $F(\varphi) = F(\tau^a\varphi)$ for all $\varphi \in \mathbb{B}^{1,0}_{dec}$ and all $a \in \mathbb{R}$. Then $\tilde{P}_tF = P_tF$ and hence $\tilde{P}_tF$ is bounded and continuous.*

Proof. Let $\{f_n\}_{n\in N}$ and f be functions in $\mathbb{B}_{dec}$ such that $f_n \xrightarrow{\mathbb{L}^2_{loc}} f$. Then Lemma 75 shows that $P_tF(f_n) \to P_tF(f)$. However, given that F is translation invariant we can write

$$\begin{aligned}
\tilde{P}_tF(f) : &= \mathbb{E}_f\left[F(\tilde{u}_t)\right] \\
&= \mathbb{E}_f\left[F(u_t)\right] \text{ given that } F \text{ is translation invariant} \\
&= P_tF(f).
\end{aligned}$$

Hence, $\tilde{P}_sF(f_n) = P_sF(f_n) \to P_sF(f) = \tilde{P}_sF(f)$ as n tends to infinity as required. ■

Remark. It is unknown, as yet, whether $\gamma_t^a(\varphi) = 0$ for $d\nu$ almost all φ. We now centre the law ν.

Definition 77. $\tilde{\nu}$

Define $\tilde{\nu}$ to be the image of ν under the map $J : \mathbb{B}^{1,0}_{dec} \to \mathbb{B}^{1,0}_{dec}$ defined by $\varphi \mapsto \tilde{\varphi}$, that is for all bounded measurable F,

$$\int_{\mathbb{B}^{1,0}_{dec}} F(\varphi)\, d\tilde{\nu}(\varphi) = \int_{\mathbb{B}^{1,0}_{dec}} F(\tilde{\varphi})\, d\nu(\varphi). \tag{4.24}$$

Theorem 78. *$\tilde{\nu}$ is a stationary travelling wave in the sense of definition 60.*

Proof. Take $F : \mathbb{B}^{1,0}_{dec} \to \mathbb{R}$ bounded, continuous and translation invariant. Note $\tilde{P}_t F$ is still bounded, continuous and translation invariant.

$$\begin{aligned}
\int_{\mathbb{B}^{1,0}_{dec}} \tilde{P}_s F\, d\tilde{\nu} &= \int_{\mathbb{B}^{1,0}_{dec}} \tilde{P}_s F\, d\nu \text{ by equation (4.24)} \\
&= \lim_{t\to\infty} \int_{\mathbb{B}^{1,0}_{dec}} \tilde{P}_s F\, d\tilde{\mathcal{Q}}^H_t \\
&= \lim_{t\to\infty} \int_{\mathbb{B}^{1,0}_{dec}} F\, d\tilde{\mathcal{Q}}^H_{t+s} \text{ by the Markov property} \\
&= \int_{\mathbb{B}^{1,0}_{dec}} F\, d\nu \\
&= \int_{\mathbb{B}^{1,0}_{dec}} F\, d\tilde{\nu}.
\end{aligned}$$

■

Remark. A definition of a stationary travelling wave as in the deterministic case where, for a solution $(u(t,x) : t \geq 0, x \in \mathbb{R})$, $(u(x+ct) : x \in \mathbb{R},\, t \geq 0)$ describes the dynamics of the stationary travelling wave isn't possible in the stochastic case given the noise. To overcome this we may consider the solution in expectation or investigate the laws of the solution, as in this thesis.

Let $(\tilde{P}_t^*)$ be the dual semigroup acting on measures $\mathcal{M}_1(\mathbb{B}_{dec}^{1,0})$. We have shown

$$\int_{\mathbb{B}_{dec}^{1,0}} F\, d\left(\tilde{P}_s^* \tilde{\nu}\right) = \int_{\mathbb{B}_{dec}^{1,0}} F\, d\tilde{\nu} \tag{4.25}$$

for all bounded, continuous translation invariant $F : \mathbb{B}_{dec}^{1,0} \to \mathbb{R}$. This may then be extended to all bounded measurable translation invariant functions F by considering measures on quotient spaces. Hence, using

$$\tilde{\nu}(\varphi : \gamma_t^a(\varphi) = 0) = \tilde{P}_t^* \tilde{\nu}(\varphi : \gamma_t^a(\varphi) = 0) = 1, \tag{4.26}$$

for all measurable $A \subseteq \mathbb{B}_{dec}^{1,0}$

$$\begin{aligned} \tilde{P}_t^* \tilde{\nu}(\varphi : \varphi \in A) &= \tilde{P}_t^* \tilde{\nu}(\varphi : \tilde{\varphi} \in A) \text{ by equation (4.26)} \\ &= \tilde{\nu}(\varphi : \tilde{\varphi} \in A) \text{ by equation (4.25)} \\ &= \tilde{\nu}(\varphi : \varphi \in A) \text{ again by equation (4.26)} \end{aligned}$$

and $\tilde{\nu}$ is a stationary travelling wave as required.

Remark. In fact, as we now show, we didn't need to centre the law ν as the limiting law is already centred.

Lemma 79. $\nu = \tilde{\nu}$.

Proof. By regularity $\varphi \in \mathcal{C}^1$, $\varphi_x < 0$ for $\tilde{P}_t^* \tilde{\nu}$ almost all φ if $t > 0$. Hence, $\varphi \in \mathcal{C}^1$, $\varphi_x < 0$ for $\tilde{\nu}$ almost all φ. Hence, $\varphi \in \mathcal{C}^1$, $\varphi_x < 0$ ν almost all φ. We already know that $\varphi(x) \leq a$ for $x > 0$ ν-almost all φ and $\varphi(x) \geq a$ for $x < 0$ ν-almost all φ. Given that φ is $\mathcal{C}^1$, $\gamma_t^a(\varphi) = 0$ for $\nu-$almost all φ. Hence, $\nu = \tilde{\nu}$. ■

Finally we argue that ν depends in the obvious way upon the choice of the centring point a. For this lemma only we denote by ν^a the limit law centred at level $a \in (0, 1)$.

Lemma 80. *The image of ν^a under the map $\varphi(\cdot) \mapsto \varphi(\cdot + \Gamma^b(\varphi))$ is ν^b.*

Proof. Note that $\nu^a \overset{s}{\succ} \delta_H$. By Corollary 33, and the fact that ν^a is invariant for the pinned process, we find $\nu^a \overset{s}{\succ} \mathcal{Q}_t^{H,b}$. Letting $t \to \infty$ and using Lemma 45, we find $\nu^a \overset{s}{\succ} \nu^b$. Similarly, $\nu^b \overset{s}{\succ} \nu^a$. Using Theorem 47, as $\mathbb{B}_{dec}$ is Polish, we may find variables $p, q, r \in \mathbb{B}_{dec}$ with $p \overset{D}{=} \nu^a \overset{D}{=} r$ and $q \overset{D}{=} \nu^b$ so that $p \overset{s}{\succ} q \overset{s}{\succ} r$ almost surely. Note that all three functions are continuous almost surely and hence, centring is well defined. Let $\tilde{p}(\cdot) = p(\cdot + \Gamma^b(p))$, $\tilde{r}(\cdot) = r(\cdot + \Gamma^b(r))$. Then $\tilde{p} \overset{s}{\succ} q \overset{s}{\succ} \tilde{r}$ almost surely. Fix an $\omega \in \Omega$ such that this is true then,

$$\tilde{p}(x) \geq q(x) \geq \tilde{r}(x) \text{ for } x \geq 0$$

$$\tilde{p}(x) \leq q(x) \leq \tilde{r}(x) \text{ for } x \leq 0.$$

However, $\tilde{p}(x) \overset{D}{=} \tilde{r}(x)$ for all x and hence $\tilde{p}(x) = q(x) = \tilde{r}(x)$ for all x almost surely. ■

Chapter 5

Wave speed formula and domains of attraction

Introduction. *In this chapter we will describe the domains of* ν *and in section 4.1 we will develop an abstract result about such domains. We will then extend this result, via an implicit wave speed formula, to show for an initial condition trapped between two Heaviside initial conditions, the "trapped" solution converges to the same law as to that for the Heaviside initial condition (up to translation).*

5.1 Domains for trapped law

Introduction. *If it were easy to describe the law* $\mathcal{L}(u_0)$ *then, following from the work in the previous section, we could also conclude that the domain of attraction contains all random* u_0 *satisfying*

$$\mathcal{L}(\mathbb{I}_{[x\leq 0]}) \overset{s}{\prec} \mathcal{L}(u_0) \overset{s}{\prec} \nu,$$

since the law of $u(t)$ is trapped between and must converge to $\mathcal{L}(\tilde{u}(\infty)) = \nu$.

Before proving this result we will prove a lemma which will be of use.

Lemma 81. *Suppose $\varphi_n \to \varphi$ in $\mathbb{B}^{1,0}_{dec}$ and $\varphi \in \mathcal{C}^1$, $\varphi_x < 0$. Then $\Gamma(\varphi_n) \to \Gamma(\varphi)$.*

Proof. We first recall the definition of the wavemarker (definition 16):

For a function $\varphi : \mathbb{R} \to (0,1)$ and $a \in [0,1]$, define $\Gamma^a(\varphi) = \inf\{x : \varphi(x) \leq a\}$.

Then, if $\Gamma^a = A$ we have $\varphi(x) > a$ for $x < A$. However, $\varphi_n \to \varphi$ almost everywhere, that is $\varphi_n \to \varphi$ for all x in a dense set $\mathbb{D} \subset \mathbb{R}$. Fix $x < A$, such that $x \in \mathbb{D}$ and $\varphi(x) > a$. Then $\varphi_n(x) > a$ for large n and hence, $\Gamma(\varphi_n) \geq x$ for large n, which yields $\underline{\lim}\, \Gamma(\varphi_n) \geq A$. Since $\varphi_x(A) < 0$, $\varphi(y) < a$ for all $y > A$. Take $y > A$, $y \in \mathbb{D}$ then $\varphi(y) < a$. Hence, for large n $\varphi_n(y) < a$. So $\Gamma(\varphi_n) \leq y$ for large n and $\overline{\lim}\, \Gamma(\varphi_n) \leq A$. Hence $\lim \Gamma(\varphi_n) = \Gamma(\varphi) = A$ as required. ■

Remark. Note that $\underline{\lim}\, \Gamma(\varphi_n) \geq \Gamma(\varphi)$ always.

Proposition 82. *Suppose $\delta_H \overset{s}{\prec} \mathcal{L}(u_0) \overset{s}{\prec} \nu$ then $\mathcal{L}(\tilde{u}(t)) \overset{D}{\to} \nu$ as $t \to \infty$.*

Proof. Since $\mathcal{M}(\mathbb{B}_{dec})$ is compact by Proposition 11, given a sequence $t_n \to \infty$, there exists a subsequence $t_{n'}$ such that $\mathcal{L}(\tilde{u}_{t'_n}) \to \mu$ for some $\mu \in \mathcal{M}(\mathbb{B}_{dec})$. By Corollary 33 we have $\tilde{\mathcal{Q}}^H_t \overset{s}{\prec} \mathcal{L}(\tilde{u}_t) \overset{s}{\prec} \nu$ for all t. Using this subsequence and the result of Lemma 45 (see remark post the stated Lemma) $\nu \overset{s}{\prec} \mu \overset{s}{\prec} \nu$. We need laws that charge only the continuous paths. Let ν^ϵ and μ^ϵ denote the image of ν and μ under the map $\varphi \mapsto \varphi^\epsilon$ defined by $\varphi^\epsilon(x) = \Gamma_\epsilon * \varphi(x)$. Then $\nu^\epsilon \overset{s}{\prec} \mu^\epsilon \overset{s}{\prec} \nu^\epsilon$ by Corollary 33 in the special case $f = g = 0$. Arguing as in Lemma 80 we find $\widetilde{\nu^\epsilon} = \widetilde{\mu^\epsilon}$. Letting $\epsilon \to 0$, using the fact that

$\varphi \mapsto \varphi^\epsilon$ is continuous, we find

$$\int_{\mathbb{B}^{1,0}_{dec}} F(x)d\nu = \int_{\mathbb{B}^{1,0}_{dec}} F(x)d\mu$$

for all bounded translation invariant $F : \mathbb{B}^{1,0}_{dec} \to \mathbb{R}$. From this we conclude that $\nu = \tilde{\mu}$. Note that $\tilde{\mu}$, and hence ν, charge only φ for which $\varphi \in \mathcal{C}^1$ and $\varphi_x < 0$. Lemma 81 shows that Γ^a is continuous for such a φ. Since $\mathcal{L}(\tilde{u}_{t_n}) \to \mu$ and μ is continuous on the discontinuous set of Γ^a, $\Gamma^a(\varphi) = 0$ for μ-almost all φ (to see this we consider the function $(\Gamma^a(u(t_n)) \wedge 1) \vee (-1)$ and note that this is bounded and continuous) and $\tilde{\mu} = \mu = \nu$. This completes the proof. ■

Remark. Despite Proposition 82 suggesting a very strong result on the domains of ν, the conditions on $\mathcal{L}(u_0)$ are difficult to check.

5.2 Additional assumptions

In this chapter we make the following assumption on $\bar{f}$ as well as hypotheses (**H1**) and (**H2**) as detailed in Chapter 2:

(**H3**) $\bar{f}$ is of KPP type, in particular there exists a constant K such that $\left|\bar{f}(z)\right| \leq Kz(1-z)$.

As we concentrate on solutions with initial conditions other than the Heaviside initial condition in this chapter, the following condition will prove useful as detailed in the forthcoming results.

Definition 83. ***Trapped criteria***

*The initial condition $u_0 \in \mathbb{B}^{1,0}_{dec}$ is said to be **Trapped** if, for some $A < B$ both in $\mathbb{R}$,*

$$\mathbb{I}_{(-\infty,A)}(x) \leq u(0,x) \leq \mathbb{I}_{(-\infty,B)}(x) \text{ for all } x \in \mathbb{R}. \tag{5.1}$$

5.3 Wave speed formula

Introduction. *In this section we prove a wave speed formula which shows the connection between the gradient of the wave front and the wave speed. Intuitively, the flatter the wavefront the faster it will travel.*

Lemma 84. *Consider equation (2.1) satisfying hypotheses (**H1**) - (**H2**). Suppose also that the deterministic initial condition u_0 is **Trapped**. Then*

$$u(t,x,\mathbb{I}_{(-\infty,A)}) \leq u(t,x,u_0) \leq u(t,x,\mathbb{I}_{(-\infty,B)}) \textit{ for all } t \geq 0$$

almost surely. Note $u(t,x,G)$ is the solution to equation (2.1) on one fixed filtered probability space with a fixed Brownian motion (W) started at G. Moreover,

$$\gamma_t^a(\mathbb{I}_{(-\infty,A)}) \leq \gamma_t^a(u_0) \leq \gamma_t^a(\mathbb{I}_{(-\infty,B)}) = \gamma_t^a(\mathbb{I}_{(-\infty,A)}) + (B-A) \tag{5.2}$$

for all $t \geq 0$ almost surely where $\gamma_t^a(G)$ is the wave marker $\Gamma^a(u(t,\cdot,G))$.

Proof. The first part of the lemma follows directly from the coupling result in Property (**ii**) Theorem 15. For the second part we note that from the **Trapped** property, and the result of the first part of the lemma, it is easy to deduce that $\gamma_t^a(u_0)$ must satisfy

$$\gamma_t^a(\mathbb{I}_{(-\infty,A)}) \leq \gamma_t^a(u_0) \leq \gamma^a(\mathbb{I}_{(-\infty,B)}) \tag{5.3}$$

for all subsequent times $t > 0$. The last equality in equation (5.2) is proved noting $u(t,x,\mathbb{I}_{(-\infty,B)}) = u(t,x-(B-A),\mathbb{I}_{(-\infty,A)})$ for all $t \geq 0$, all $x \in \mathbb{R}$ $\mathbb{P}$-almost surely by the uniqueness of solution result from Theorem 15. ■

Remark. For solutions u started from the Heaviside initial condition we will write $u(t,\cdot,H)$ as shorthand when the exact position of the Heaviside function is unimportant. We will also write $\gamma_t^a(H)$ as shorthand for the wave marker $\Gamma^a(u(t,\cdot,H))$.

Lemma 85. *Let u be the solution to equation (2.1) started from the Heaviside initial condition. Then*

$$\mathbb{E}\left[\int_{\mathbb{R}} u(t,x)(1-u(t,x))\,dx\right] \uparrow \int_{\mathbb{B}^{1,0}_{dec}} \left(\int_{\mathbb{R}} \varphi(1-\varphi)\,dx\right)\nu(d\varphi)$$

and this limit is finite. Let $\tilde{u}^{(1/2)}$ be the solution centred at $a=\frac{1}{2}$ and let $\nu^{\frac{1}{2}}$ be the corresponding limiting law. Then for all $\epsilon>0$ there exists $M(\epsilon)$ such that for all $N > M(\epsilon)$, defining $I=[-N,N]$,

$$\begin{aligned}\mathbb{E}\left[\int_{I^c} \tilde{u}^{(1/2)}(t,x)(1-\tilde{u}^{(1/2)}(t,x))\,dx\right] &\leq \int_{\mathbb{B}^{1,0}_{dec}}\int_{I^c} \varphi(x)(1-\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi)\\ &\leq \epsilon\end{aligned}$$

for all $t\geq 0$.

Proof. Set $a=\frac{1}{2}$, that is centre the solution such that $\tilde{u}^{(1/2)}(t,0)=u^{(1/2)}(t,\gamma_t^{1/2})=\frac{1}{2}$. Now, from Theorem 51, ordering and taking a realisation, denoted $\hat{\cdot}$, we may write,

$$\hat{\tilde{u}}^{(1/2)}(t,x,H)\geq \hat{\tilde{u}}^{(1/2)}(s,x,H) \text{ for all } x\geq 0$$

$$\hat{\tilde{u}}^{(1/2)}(t,x,H)\leq \hat{\tilde{u}}^{(1/2)}(s,x,H) \text{ for all } x\leq 0$$

almost surely, for a fixed $s\leq t$. Hence, for all $x\in\mathbb{R}$,

$$\hat{\tilde{u}}^{(1/2)}(t,x,H)(1-\hat{\tilde{u}}^{(1/2)}(t,x,H))\geq \hat{\tilde{u}}^{(1/2)}(s,x,H)(1-\hat{\tilde{u}}^{(1/2)}(s,x,H)).$$

This shows that $\mathbb{E}\left[\tilde{u}^{1/2}(t,x,H)(1-\tilde{u}^{1/2}(t,x,H))\right]$ is increasing in t for all $x\in\mathbb{R}$. We now note that the map $\varphi\mapsto\int_I \varphi(x)(1-\varphi(x))\,dx$ is bounded and continuous on $\mathbb{B}^{1,0}_{dec}$ and by the Monotone Convergence Theorem,

$$\mathbb{E}\left[\int_I \tilde{u}^{(1/2)}(t,x,H)(1-\tilde{u}^{(1/2)}(t,x,H))\,dx\right] \tag{5.4}$$

$$= \int_{\mathbb{B}^{1,0}_{dec}} \int_I \varphi(x)(1-\varphi(x)\,dx\,\tilde{\mathfrak{Q}}^H_t(d\varphi)$$

$$\uparrow \int_{\mathbb{B}^{1,0}_{dec}} \int_I \varphi(x)(1-\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi) \tag{5.5}$$

as $t \to \infty$. Lemma 68 shows that the left-hand-side of the above equation can be bounded by $4\sqrt{K}$, a bound independent of t and N. This bound, therefore, also applies to the limit on the right-hand-side. Now $\int_{\mathbb{B}^{1,0}_{dec}} \int_{I^c} \varphi(x)(1-\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi) \to 0$ as $N \to \infty$ by the Monotone Convergence Theorem. The lemma follows. ■

Lemma 86. *Suppose $u_0 \in \mathbb{B}^{1,0}_{dec}$ is **Trapped** then*

$$\mathbb{E}\left[\int_{\mathbb{R}} u(t,x,u_0)(1-u(t,x,u_0))\,dx\right] \le \left(\frac{4}{a}+\frac{4}{1-a}\right)\sqrt{K} + 2\,(B-A) \;\; \textit{for all } t \ge 0$$

and given $\epsilon > 0$ there exists $M(\epsilon)$ such that for all $N > M(\epsilon)$, defining $I = [-N,N]$,

$$\mathbb{E}\left[\int_{I^c} \tilde{u}^{(1/2)}(t,x,u_0)(1-\tilde{u}^{(1/2)}(t,x,u_0))\,dx\right] \le \epsilon.$$

Proof. Recalling the notation $\gamma^a_t(u_0)$ we breakup the integral into two bits and bound each one separately.

$$\begin{aligned}
\int_{\mathbb{R}} u(t,x,u_0)(1-u(t,x,u_0))\,dx &= \int_{-\infty}^{\gamma^a_t(u_0)} u(t,x,u_0)(1-u(t,x,u_0))\,dx \\
&\qquad + \int_{\gamma^a_t(u_0)}^{\infty} u(t,x,u_0)(1-u(t,x,u_0))\,dx \\
&\le \int_{-\infty}^{\gamma^a_t(u_0)} (1-u(t,x,\mathbb{I}_{(-\infty,A)}))\,dx \\
&\qquad + \int_{\gamma^a_t(u_0)}^{\infty} u(t,x,\mathbb{I}_{(-\infty,B)})\,dx
\end{aligned}$$

by use of the bounds $u(t,x) \in [0,1]$ and Lemma 84. Now, using the McKean bound, Lemma 68, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_{-\infty}^{\gamma_t^a(u_0)} (1-u(t,x,\mathbb{I}_{(-\infty,A)}))\,dx\right] \\
&\qquad = \mathbb{E}\left[\int_{-\infty}^{\gamma_t^a(\mathbb{I}_{(-\infty,A)})} (1-u(t,x,\mathbb{I}_{(-\infty,A)}))\,dx\right] \\
&\qquad\quad +\mathbb{E}\left[\int_{\gamma_t^a(\mathbb{I}_{(-\infty,A)})}^{\gamma_t^a(u_0)} (1-u(t,x,\mathbb{I}_{(-\infty,A)}))\,dx\right] \\
&\qquad \leq \frac{1}{a}\mathbb{E}\left[\int_{\mathbb{R}} u(t,x,\mathbb{I}_{(-\infty,A)})(1-u(t,x,\mathbb{I}_{(-\infty,A)}))\,dx\right] \\
&\qquad\quad +\mathbb{E}\left[\int_{\gamma_t^a(\mathbb{I}_{(-\infty,A)})}^{\gamma_t^a(u_0)} 1\,dx\right] \\
&\qquad \leq \frac{1}{a}\mathbb{E}\left[\int_{\mathbb{R}} u(t,x,\mathbb{I}_{(-\infty,A)})(1-u(t,x,\mathbb{I}_{(-\infty,A)}))\,dx\right] \\
&\qquad\quad + (B-A) \\
&\qquad \leq \frac{4}{a}\sqrt{K} + (B-A)\,.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}&\left[\int_{\gamma_t^a(u_0)}^{\infty} u(t,x,\mathbb{I}_{(-\infty,B)})\,dx\right] \\
&= \mathbb{E}\left[\int_{\gamma_t^a(\mathbb{I}_{(-\infty,B)})}^{\infty} u(t,x,\mathbb{I}_{(-\infty,B)})\,dx\right] \\
&\quad + \mathbb{E}\left[\int_{\gamma_t^a(u_0)}^{\gamma_t^a(\mathbb{I}_{(-\infty,B)})} u(t,x,\mathbb{I}_{(-\infty,B)})\,dx\right] \\
&\leq \frac{1}{1-a}\mathbb{E}\left[\int_{\mathbb{R}} u(s,x,\mathbb{I}_{(-\infty,B)})(1-u(t,x,\mathbb{I}_{(-\infty,B)}))\,dx\right] \\
&\quad + \mathbb{E}\left[\int_{\gamma_t^a(u_0)}^{\gamma_t^a(\mathbb{I}_{(-\infty,B)})} 1\,dx\right] \\
&\leq \frac{1}{1-a}\mathbb{E}\left[\int_{\mathbb{R}} u(t,x,\mathbb{I}_{(-\infty,B)})(1-u(t,x,\mathbb{I}_{(-\infty,B)}))\,dx\right] \\
&\quad + (B-A) \\
&\leq \frac{4}{1-a}\sqrt{K} + (B-A)\,.
\end{aligned}$$

Putting this all together we can write

$$\begin{aligned}
\mathbb{E}&\left[\int_{\mathbb{R}} u(t,x,u_0)(1-u(t,x,u_0))\,dx\right] \\
&\leq \frac{1}{a}\mathbb{E}\left[\int_{\mathbb{R}} u(t,x,\mathbb{I}_{(-\infty,A)})(1-u(t,x,\mathbb{I}_{(-\infty,A)}))\,dx\right] \\
&\quad + (B-A) \\
&\quad + \frac{1}{(1-a)}\mathbb{E}\left[\int_{\mathbb{R}} u(s,x,\mathbb{I}_{(-\infty,B)})(1-u(t,x,\mathbb{I}_{(-\infty,B)}))\,dx\right] \\
&\quad + (B-A)
\end{aligned}$$

and

$$\mathbb{E}\left[\int_{\mathbb{R}} u(t,x)(1-u(t,x))\,dxds\right] \leq 4\left(\frac{1}{a}+\frac{1}{1-a}\right)\sqrt{K} + 2\,(B-A)$$

as required. For the second part of the proof we note, defining $N = N(\epsilon)$ and $\bar{N} =$

$N+(B-A)$,

$$\begin{aligned}
\int_{\bar{N}}^{\infty} \tilde{u}^{(1/2)}(t,x,u_0)(1-\tilde{u}^{(1/2)}(t,x,u_0))\,dx \\
&= \int_{\gamma_t^{1/2}(u_0)+\bar{N}}^{\infty} u^{(1/2)}(t,x,u_0)(1-u^{(1/2)}(t,x,u_0))\,dx \\
&\leq \int_{\gamma_t^{1/2}(u_0)+\bar{N}}^{\infty} u^{(1/2)}(t,x,\mathbb{I}_{(-\infty,B)})\,dx \\
&\leq \int_{\gamma_t^{1/2}(\mathbb{I}_{(-\infty,A)})+\bar{N}}^{\infty} u^{(1/2)}(t,x,\mathbb{I}_{(-\infty,B)})\,dx \\
&= \int_{\gamma_t^{1/2}(\mathbb{I}_{(-\infty,B)})+N}^{\infty} u^{(1/2)}(t,x,\mathbb{I}_{(-\infty,B)})\,dx \\
\text{given that } \gamma_t^{1/2}(\mathbb{I}_{(-\infty,A)})+(B-A)=\gamma_t^{1/2}(\mathbb{I}_{(-\infty,B)}) \\
&= \int_{\gamma_t^{1/2}(H)+N}^{\infty} u^{(1/2)}(t,x,H)\,dx \\
&= \int_{N}^{\infty} \tilde{u}^{(1/2)}(t,x,H)\,dx \\
&\leq 2\int_{N}^{\infty} \tilde{u}^{(1/2)}(t,x,H)(1-\tilde{u}^{(1/2)}(t,x,H))\,dx.
\end{aligned}$$

Similarly for the other tail, we calculate

$$
\begin{aligned}
\int_{-\infty}^{-\bar{N}} & \tilde{u}^{(1/2)}(t,x,u_0)(1-\tilde{u}^{(1/2)}(t,x,u_0))\,dx \\
&= \int_{-\infty}^{\gamma_t^{1/2}(u_0)-\bar{N}} u^{(1/2)}(t,x,u_0)(1-u^{(1/2)}(t,x,u_0))\,dx \\
&\leq \int_{-\infty}^{\gamma_t^{1/2}(u_0)-\bar{N}} (1-u^{(1/2)}(t,x,\mathbb{I}_{(-\infty,A)}))\,dx \\
&\leq \int_{-\infty}^{\gamma_t^{1/2}(\mathbb{I}_{(-\infty,B)})-\bar{N}} (1-u^{(1/2)}(t,x,\mathbb{I}_{(-\infty,A)}))\,dx \\
&= \int_{-\infty}^{\gamma_t^{1/2}(\mathbb{I}_{(-\infty,A)})-N} (1-u^{(1/2)}(t,x,\mathbb{I}_{(-\infty,A)}))\,dx \\
&\qquad \text{given that } \gamma_t^{1/2}(\mathbb{I}_{(-\infty,B)})-(B-A)=\gamma_t^{1/2}(\mathbb{I}_{(-\infty,A)}) \\
&= \int_{-\infty}^{\gamma_t^{1/2}(H)-N} (1-u^{(1/2)}(t,x,H))\,dx \\
&= \int_{-\infty}^{-N} (1-\tilde{u}^{(1/2)}(t,x,H))\,dx \\
&\leq 2\int_{-\infty}^{-N} \tilde{u}^{(1/2)}(t,x,H)(1-\tilde{u}^{(1/2)}(t,x,H))\,dx.
\end{aligned}
$$

Putting both of these bounds together we have

$$
\begin{aligned}
\mathbb{E}&\left[\int_I \tilde{u}^{(1/2)}(t,x,u_0)(1-\tilde{u}^{(1/2)}(t,x,u_0))\,dx\right] \\
&\leq 2\mathbb{E}\left[\int_N^{\infty} \tilde{u}^{(1/2)}(t,x,H)(1-\tilde{u}^{(1/2)}(t,x,H))\,dx\right] \\
&\qquad +2\mathbb{E}\left[\int_{-\infty}^{-N} \tilde{u}^{(1/2)}(t,x,H)(1-\tilde{u}^{(1/2)}(t,x,H))\,dx\right] \\
&\leq 4\epsilon
\end{aligned}
$$

by Lemma 85 which we can make as small as required. ■

We now prove an implicit formula for the wave speed. The aim is to find a formula for $\lim_{t\to\infty}\mathbb{E}\left[\frac{\gamma_t^a(H)}{t}\right]$ and compare this to the analogous formula for $\lim_{t\to\infty}\mathbb{E}\left[\frac{\gamma_t^a(u_0)}{t}\right]$.

Theorem 87. *Consider equation (2.1) satisfying hypotheses (**H1**) - (**H3**). Let u be the solution started from $u_0 \in \mathbb{B}^{1,0}_{dec}$ and suppose u_0 is **Trapped**. Then,*

$$\lim_{t\to\infty} \mathbb{E}\left(\frac{\gamma_t^a}{t} - \frac{1}{t}\int_{t_0}^{t}\int_{\mathbb{R}} \bar{f}(u(s,x))dxds\right) = 0.$$

Proof. We proceed in the same way as the proof of the stochastic McKean bound using both Fubini's theorem and the stochastic Fubini Theorem to interchange integrals as required. First, we integrate over the interval $(-L, U)$.

$$\begin{aligned}\int_{-L}^{U} [\tilde{u}(t,x) - \tilde{u}(t_0,x)]\,dx &= \int_{t_0}^{t} [\tilde{u}_x(s,U) - \tilde{u}_x(s,-L)]\,ds \\ &\quad + \int_{t_0}^{t} (\tilde{u}(s,U) - \tilde{u}(s,-L)) \circ d\gamma_s^a \\ &\quad + \int_{t_0}^{t}\int_{-L}^{U} \bar{f}(\tilde{u}(s,x))dxds \\ &\quad + \int_{t_0}^{t}\int_{-L}^{U} g(\tilde{u}(s,x))\,dxdW_s.\end{aligned} \tag{5.6}$$

To expand the range of integration, we again have to check that the limit of each of the above terms is well defined. To do this we use that the first moment of u started from the **Trapped** initial condition u_0 can be bounded by the first moment of the solution u started from the Heaviside initial condition which we have already shown is bounded:

$$\mathbb{E}[u(t,x,u_0)] \leq \mathbb{E}\left[u(t,x,\mathbb{I}_{(-\infty,B)})\right] = \mathbb{E}[u(t,x-B,H)]$$

and we may use the bounds given in equation (4.7). Similarly for $\mathbb{E}[1 - u(t,x,u_0)]$.

Using these bounds repeatedly we can again show

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \int_{\mathbb{R}} u(s,x,u_0)(1-u(s,x,u_0))\,dxds\right]& \\
&\le \int_0^T \int_0^\infty \mathbb{E}\left[u(s,x,u_0)\right]\,dxds \\
&\quad + \int_0^T \int_{-\infty}^0 \mathbb{E}\left[(1-u(s,x,u_0))\right]\,dxds \\
&\le \int_0^T \int_0^\infty \mathbb{E}\left[u(s,x,\mathbb{I}_{(-\infty,B)})\right]\,dxds \\
&\quad + \int_0^T \int_{-\infty}^0 \mathbb{E}\left[(1-u(s,x,\mathbb{I}_{(-\infty,A)}))\right]\,dxds \\
&< \infty.
\end{aligned}$$

From hypothesis (**H3**), the limit

$$\int_{t_0}^t \int_{-L}^U \bar{f}(\tilde{u}(s,x))\,dxds \to \int_{t_0}^t \int_{-\infty}^\infty \bar{f}(\tilde{u}(s,x))\,dxds$$

is justified. Similarly for the dW term since g is bounded above. Also, the limit for the term on the left-hand-side of (5.6) is justified and exists by use of Lemma 86. For the $\int_{t_0}^t (\tilde{u}(s,U)-\tilde{u}(s,-L))\circ\, d\gamma_s^a$ term we write

$$\begin{aligned}
\int_{t_0}^t (\tilde{u}(s,U)-\tilde{u}(s,-L))\circ\, d\gamma_s^a &= \int_{t_0}^t \tilde{u}(s,U)\circ\, d\gamma_s^a \\
&\quad + \int_{t_0}^t (1-\tilde{u}(s,-L))\circ\, d\gamma_s^a \\
&\quad - \left(\gamma_t^a - \gamma_{t_0}^a\right)
\end{aligned}$$

and the argument reduces to the same argument as that shown in Theorem 61 by the use of Lemma 67. This shows that this term converges almost surely to $\left(\gamma_{t_0}^a - \gamma_t^a\right)$. By regularity of solutions we know $\tilde{u}_x(s,R) \xrightarrow{a.s.} 0$ as $|R| \to \infty$. We again need to justify

$$\int_{t_0}^t \tilde{u}_x(s,R)\,ds \xrightarrow{a.s.} 0 \text{ as } |R| \to \infty. \tag{5.7}$$

Note, from Property (**vii**) of Theorem 15, for solutions u with **Trapped** initial conditions that

$$\mathbb{E}\left[\sup_y |u_x(s,y)|^2\right] \le C(t_0,T) < \infty$$

for $s \in (t_0, T]$. Then,

$$\mathbb{E}\left[\int_{t_0}^t |\tilde{u}_x(s,R)|^2 \, ds\right] \le \mathbb{E}\left[\int_{t_0}^t \sup_y |u_x(s,y)|^2 \, ds\right]$$

is bounded independently of R and so the variable $\tilde{u}_x(s,R)$ is uniformly integrable as a function of s and ω. This justifies taking the limit in each of the terms in equation (5.6). Dividing by $t > 0$ and taking the limits as $U \to \infty$ and $L \to -\infty$ we will get:

$$\begin{aligned} \frac{\gamma_t^a}{t} - \frac{\gamma_{t_0}^a}{t} - \frac{1}{t}\int_{t_0}^t \int_{\mathbb{R}} \bar{f}(\tilde{u}(s,x))dxds &= \frac{1}{t}\int_{t_0}^t \int_{\mathbb{R}} g(\tilde{u}(s,x))\, dxdW_s \\ &\quad -\frac{1}{t}\int_{\mathbb{R}} \left[\tilde{u}(t,x) - \tilde{u}(t_0,x)\right] dx. \end{aligned} \tag{5.8}$$

Upon taking expectations the Itô noise term vanishes and the last term on the right-hand-side, by use of Lemma 86, tends to zero as t tends to infinity, this completes the proof. ■

Remark. To our knowledge there are no concrete examples of applications of wave speed formula. What we present in this thesis is the use of an implicit rather than explicit wave speed formula as the only convergence we required is convergence in expectation. We did feel that this may be relaxed to almost sure convergence but this was beyond our requirements and, as such, not pursued avidly. Wave speed formula are an open area of future research.

Corollary 88. *The wave speed* $\lim_{t\to\infty}\mathbb{E}\left[\frac{\gamma_t^a(H)}{t}\right]$ *exists and equals* $\int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi)\,dx\nu(d\varphi)$. *Moreover, suppose* $u(0)\in\mathbb{B}^{1,0}_{dec}$ *is* ***Trapped*** *then* $\lim_{t\to\infty}\mathbb{E}\left[\frac{\gamma_t^a(u_0)}{t}\right]$ *exists and equals* $\lim_{t\to\infty}\mathbb{E}\left[\frac{\gamma_t^a(H)}{t}\right]$.

Proof. Set the pinning point $a=\frac{1}{2}$ and define $I=[-N,N]$. Then

$$\begin{aligned}\mathbb{E}\left[\int_{\mathbb{R}}\bar{f}(u^{(1/2)}(t,x,H))\,dx\right] &= \mathbb{E}\left[\int_{I}\bar{f}(u^{(1/2)}(t,x,H))\,dx\right]\\ &\quad+\mathbb{E}\left[\int_{I^c}\bar{f}(u^{(1/2)}(t,x,H))\,dx\right].\end{aligned}$$

It is clear

$$\begin{aligned}\mathbb{E}\left[\int_{I}\bar{f}(u^{(1/2)}(t,x,H))\,dx\right] &\to \int_{\mathbb{B}^{1,0}_{dec}}\int_{I}\bar{f}(\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi)\\ &= \int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi)\\ &\quad-\int_{\mathbb{B}^{1,0}_{dec}}\int_{I^c}\bar{f}(\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi).\end{aligned}$$

By Lemmas 85 and 86, for all $\epsilon>0$ there exists $N=N(\epsilon)$ such that both

$$\mathbb{E}\left[\int_{I^c}\bar{f}(u^{(1/2)}(t,x,H))\,dx\right]\le\epsilon$$

and

$$\int_{\mathbb{B}^{1,0}_{dec}}\int_{I^c}\bar{f}(\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi)\le\epsilon$$

and hence we are done. Note that $\int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi(x))\,dx\nu^{\frac{1}{2}}(d\varphi)=\int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi(x))\,dx\nu(d\varphi)$, that is the limit is independent of the centring level $\frac{1}{2}$. We combine this with the wave speed formula, shown in Theorem 87, to show, for $t\ge t_0>0$,

$$\frac{1}{t}\int_{t_0}^{t}\mathbb{E}\left[\int_{\mathbb{R}}\bar{f}(u(s,x,H))\,dx\right]ds\to\int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi)\,dx\nu(d\varphi).$$

Then

$$\mathbb{E}\left[\frac{\gamma_t^a(u_0)}{t}\right] \to \int_{\mathbb{B}_{dec}^{1,0}} \int_{\mathbb{R}} \bar{f}(\varphi)\,dx\nu(d\varphi)$$

as required.

For the second part, we note that equation (5.2) holds for all $t \geq 0$. Dividing this equation throughout by $t > 0$ and taking expectations gives

$$\mathbb{E}\left[\frac{\gamma_t^a(\mathbb{I}_{(-\infty,A)})}{t}\right] \leq \mathbb{E}\left[\frac{\gamma_t^a(u_0)}{t}\right] \leq \mathbb{E}\left[\frac{\gamma_t^a(\mathbb{I}_{(-\infty,B)})}{t}\right].$$

Taking the limit at t tends to infinity allows us to use standard analysis results to reveal that $\lim_{t\to\infty} \mathbb{E}\left[\frac{\gamma_t^a(H)}{t}\right] = \lim_{t\to\infty} \mathbb{E}\left[\frac{\gamma_t^a(u_0)}{t}\right]$ as required. ∎

5.4 Domains of attraction

Definition 89. ***Domain of*** ν

The domain of attraction of ν *is defined by the set*

$$\{\mu \in \mathcal{M}(\mathbb{B}_{dec}) : \mathcal{Q}_t^\mu(A) = \mathcal{P}\left[u(t) \in A | u_0 \stackrel{D}{=} \mu\right] \to \nu(A)\}.$$

The following lemma is the key component in proving that the solution started from a trapped initial condition has the same limit law as that of the solution started from the Heaviside initial condition.

Lemma 90. *Suppose* $\varphi, \psi \in \mathbb{B}_{dec}^{1,0}$ *satisfying* $\varphi, \psi \in \mathcal{C}^1$ *and* $\varphi_x, \psi_x < 0$. *Suppose also* $h : (0,1) \to (0,\infty)$ *is measurable. If* $\varphi \stackrel{s}{\succ} \psi$ *and* $\int_{\mathbb{R}} h(\varphi)\,dx = \int_{\mathbb{R}} h(\psi)\,dx < \infty$ *then* $\tilde{\varphi} = \tilde{\psi}$.

Proof. Let $\bar{\varphi}$ and $\bar{\psi} : (0,1) \to \mathbb{R}$ denote the inverse functions to φ and ψ respectively.

Then, it is clear

$$\int_{\mathbb{R}} h(\varphi(x))\,dx = -\int_0^1 h(y)\bar{\varphi}_y(y)\,dy.$$

Hence

$$\int_0^1 h(y)\bar{\varphi}_y(y)\,dy = \int_0^1 h(y)\bar{\psi}_y(y)\,dy. \tag{5.9}$$

We know $\varphi \overset{s}{\succ} \psi$ and hence, by Lemma 35, $\varphi_y(\bar{\varphi}(y)) \geq \psi_y(\bar{\psi}(y))$ since $\varphi(\bar{\varphi}(y)) = \psi(\bar{\psi}(y)) = y$. As $\varphi_y(\bar{\varphi}(y))\bar{\varphi}_y(y) = 1$, and similarly for ψ, we have $\bar{\varphi}_y(y) \leq \bar{\psi}_y(y)$. Combining this with equation (5.9) we can conclude $\bar{\varphi}_y \equiv \bar{\psi}_y$ and so, $\bar{\varphi} = \bar{\psi} + C$ for some constant C. Hence $\varphi = \tau^a\psi$ for some a, that is $\tilde{\varphi} = \tilde{\psi}$. ∎

Remark. We can compare this with the result in the case that X, Y are real random variables which states that if X and Y are stochastically ordered, $X \overset{s}{\succ} Y$, and $\mathbb{E}[h(X)] = \mathbb{E}[h(Y)]$ then $X \overset{D}{=} Y$.

The following theorem discusses necessary and sufficient conditions for two probability measures to be more stretched. This result is a restatement and extension of a Corollary to Theorem 11 in [29] (see also Chapter 4 of [17]) which is proved in [16] where the partial-order requirement is relaxed to a pre-order.

Theorem 91. ***Lindvall-Strassen***

Let $\mathcal{P}$ and $\mathcal{Q}$ denote two probability measures on a general Polish space E with sigma algebra $\mathcal{E}$. Let $\prec$ define a pre-order on E. Define $M = \{(p,q) : p \prec q\}$ and let us assume that this set is closed in the product topology on $E \times E$. If for any bounded, non-decreasing measurable function F,

$$\int_E F d\mathcal{P} \leq \int_E F d\mathcal{Q}$$

then there exists a probability measure $\mathcal{X}$ on $(E \times E, \mathcal{E} \times \mathcal{E})$ with marginals $\mathcal{P}$ and $\mathcal{Q}$ such that $\mathcal{X}(M) = 1$.

Proof. The proof of this can be found in [16]. ■

Remark. It is clear that the converse of the above Theorem is also true and hence we have the following corollary.

Corollary 92. *For two probability measures $\mathcal{P}$ and $\mathcal{Q}$ on $\mathbb{B}^{1,0}_{dec}$, then $\mathcal{P} \overset{s}{\succ} \mathcal{Q}$ if-and-only-if $\int_{\mathbb{B}^{1,0}_{dec}} F d\mathcal{Q} \leq \int_{\mathbb{B}^{1,0}_{dec}} F d\mathcal{P}$ for all bounded, non-decreasing F.*

The proof of this is clear.

Lemma 93. *Suppose $u_0 \in \mathbb{B}_{dec}$. Define $\tilde{\mathcal{P}}^{u_0}_t := \frac{1}{t}\int_0^t \tilde{\mathcal{Q}}^{u_0}_s \, ds$. Then*

$$\tilde{\mathcal{P}}^{u_0}_t \overset{s}{\succ} \tilde{\mathcal{P}}^{H}_t.$$

Remark. The map $s \to \mathcal{Q}^{u_0}_s$ is continuous since solution paths $s \to u(s)$ are continuous. Then the map $s \to \tilde{\mathcal{Q}}^H_s$ is also measurable as the image of a measurable map.

Proof. Note $\tilde{\mathcal{Q}}^{u_0}_t \overset{s}{\succ} \tilde{\mathcal{Q}}^H_t$ trivially. Let $F : \mathbb{B}^{1,0}_{dec} \to \mathbb{R}$ be any bounded, non-decreasing function. Then

$$\begin{aligned}
\int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\tilde{\mathcal{P}}^{u_0}_s(\varphi) &= \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\left(\frac{1}{t}\int_0^t \tilde{\mathcal{Q}}^{u_0}_s(\varphi)\, ds\right) \\
&= \frac{1}{t}\int_0^t \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\tilde{\mathcal{Q}}^{u_0}_s(\varphi)\, ds \text{ by the use of Fubini's Theorem} \\
&\geq \frac{1}{t}\int_0^t \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\tilde{\mathcal{Q}}^{H}_s(\varphi)\, ds \text{ as } \tilde{\mathcal{Q}}^{u_0}_t \overset{s}{\succ} \tilde{\mathcal{Q}}^H_t \text{ and by Corollary 92} \\
&= \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\left(\frac{1}{t}\int_0^t \tilde{\mathcal{Q}}^{H}_s(\varphi)\, ds\right) \\
&= \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\tilde{P}^{H}_s(\varphi).
\end{aligned}$$

As $\int_{\mathbb{B}^{1,0}_{dec}} F(\varphi)d\tilde{P}^{u_0}_s(\varphi) \geq \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi)d\tilde{P}^{H}_s(\varphi)$ for all bounded, non-decreasing F we can conclude

$$\tilde{\mathcal{P}}^{u_0}_t \overset{s}{\succ} \tilde{\mathcal{P}}^{H}_t$$

as required by Corollary 92. ■

We will now prove the key result of Chapter 5.

Theorem 94. *If u_0 is **Trapped** and satisfies hypotheses **(H1)** - **(H3)** then*

$$\tilde{\mathcal{P}}^{u_0}_t := \frac{1}{t}\int_0^t \tilde{\mathcal{Q}}^{u_0}_s \, ds \to \nu \text{ in } \mathcal{M}_1(\mathbb{B}^{1,0}_{dec}).$$

Proof. We start by considering the centred value $a = \frac{1}{2}$. We know $\mathbb{B}_{dec}$ is compact and Polish from earlier results and remarks. This indicates that $\mathcal{M}(\mathbb{B}_{dec})$ is also compact and Polish (see [2] or [4]). Consider $\tilde{\mathcal{P}}^{u_0}_t := \frac{1}{t}\int_0^t \tilde{\mathcal{Q}}^{u_0}_s \, ds$. Given any sequence $t_n \to \infty$ there exists a subsequence $t_{n'}$ such that $\tilde{\mathcal{P}}^{u_0}_{t_n}$ converges. Call this limit μ. We aim to show $\mu = \nu$. The subsequence principle, that is the result being true for all subsequences, will then imply $\tilde{\mathcal{P}}^{u_0}_t \to \nu$. We first show that μ is a stationary travelling wave. This is very similar to the proof that ν is a stationary travelling wave. Note $\mu(\mathbb{B}^{1,0}_{dec}) = 1$ by the McKean bound (Lemma 68). First centre μ ($\tilde{\mu}$) as before. Take F bounded, continuous and translation invariant as a map $F : \mathbb{B}^{1,0}_{dec} \to \mathbb{R}$. Then

$$\begin{aligned}
\int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\left(\tilde{\mathcal{Q}}^*_t \mu(\varphi)\right) &= \int_{\mathbb{B}^{1,0}_{dec}} \tilde{\mathcal{Q}}_t F(\varphi)\, d\mu(\varphi) \\
&= \lim_{n\to\infty} \int_{\mathbb{B}^{1,0}_{dec}} \tilde{\mathcal{Q}}_t F(\varphi) \tilde{\mathcal{P}}^{u_0}_{t_n}(d\varphi) \\
&= \lim_{n'\to\infty} \frac{1}{t_{n'}} \int_0^{t_{n'}} \int_{\mathbb{B}^{1,0}_{dec}} \tilde{\mathcal{Q}}_t F(\varphi) d\tilde{\mathcal{Q}}^{u_0}_s(\varphi)\, ds \\
&= \lim_{n'\to\infty} \frac{1}{t_{n'}} \int_0^{t_{n'}} \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\tilde{\mathcal{Q}}^{u_0}_{t+s}(\varphi)\, ds \\
&= \lim_{n'\to\infty} \frac{1}{t_{n'}} \int_t^{t+t_{n'}} \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\tilde{\mathcal{Q}}^{u_0}_r(\varphi)\, dr \\
&= \lim_{n'\to\infty} \frac{1}{t_{n'}} \int_0^{t_{n'}} \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\tilde{\mathcal{Q}}^{u_0}_r(\varphi)\, dr \\
&= \int_{\mathbb{B}^{1,0}_{dec}} F(\varphi) d\mu(\varphi).
\end{aligned}$$

As before μ is a stationary travelling wave and $\tilde{\mu} = \mu$. This implies that $\varphi \in \mathcal{C}^1$, $\varphi_x < 0$ for μ-almost all φ.

Now recalling, for $N = N(\epsilon)$, $\bar{N} = N + (B - A)$ and defining $I_M = [-M, M]$ for $M = N, \bar{N}$, we have

$$\begin{aligned}
\frac{1}{t_n}\mathbb{E}\left[\int_0^{t_n} \int_{\mathbb{R}} \bar{f}(u(s,x,u_0))\, dxds\right] &= \int_{\mathbb{B}^{1,0}_{dec}} \int_{\mathbb{R}} \bar{f}(\varphi) dx d\tilde{\mathcal{P}}^{u_0}_{t_n}(\varphi) \\
&= \int_{\mathbb{B}^{1,0}_{dec}} \int_{-\bar{N}}^{\bar{N}} \bar{f}(\varphi) dx d\tilde{\mathcal{P}}^{u_0}_{t_n}(\varphi) \\
&\quad + \int_{\mathbb{B}^{1,0}_{dec}} \int_{I^c_{\bar{N}}} \bar{f}(\varphi) dx d\tilde{\mathcal{P}}^{u_0}_{t_n}(\varphi).
\end{aligned}$$

Now

$$\begin{aligned}
\int_{\mathbb{B}^{1,0}_{dec}} \int_{-\bar{N}}^{\bar{N}} \bar{f}(\varphi) dx d\tilde{\mathcal{P}}^{u_0}_{t_n}(\varphi) &\to \int_{\mathbb{B}^{1,0}_{dec}} \int_{-\bar{N}}^{\bar{N}} \bar{f}(\varphi) dx d\mu(\varphi) \\
&= \int_{\mathbb{B}^{1,0}_{dec}} \int_{\mathbb{R}} \bar{f}(\varphi) dx d\mu(\varphi) \\
&\quad - \int_{\mathbb{B}^{1,0}_{dec}} \int_{I^c_{\bar{N}}} \bar{f}(\varphi) dx d\mu(\varphi).
\end{aligned}$$

We know, as $\bar{f}$ is of KPP type,

$$\int_{\mathbb{B}^{1,0}_{dec}}\int_{I^c_{\bar{N}}}\bar{f}(\varphi)dxd\tilde{\mathcal{P}}^{u_0}_{t_n}(\varphi)\leq\epsilon \text{ by Lemma 86.}$$

Now we claim that for all $\epsilon>0$ there exists an $\bar{N}=\bar{N}(\epsilon)$ such that $\left|\int_{\mathbb{B}^{1,0}_{dec}}\int_{I^c_{\bar{N}}}\bar{f}(\varphi)dxd\mu(\varphi)\right|\leq\epsilon$. For all $\infty>R>0$, we have for all $\epsilon>0$ there exists an $\bar{N}$ such that

$$\begin{aligned}\epsilon &\geq \int_{\mathbb{B}^{1,0}_{dec}}\int_{\bar{N}}^{\bar{N}+R}\varphi(1-\varphi)\,dxd\tilde{\mathcal{P}}^{\mu}_{t_n}\\ &\to \int_{\mathbb{B}^{1,0}_{dec}}\int_{\bar{N}}^{\bar{N}+R}\varphi(1-\varphi)\,dx\mu(d\varphi)\end{aligned}$$

where $\bar{N}=N+(B-A)$. It is clear for the same $\bar{N}$, ϵ as above, we have

$$\left|\int_{\mathbb{B}^{1,0}_{dec}}\int_{I^c_{\bar{N}}}\bar{f}(\varphi)dxd\mu\right|\leq\epsilon.$$

Hence,

$$\begin{aligned}\frac{1}{t_n}\mathbb{E}\left[\int_0^{t_n}\int_{\mathbb{R}}\bar{f}(u(s,x,u_0))\,dxds\right] &= \int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi)dxd\tilde{\mathcal{P}}^{u_0}_{t_n}\\ &\to \int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi)dxd\mu\end{aligned}$$

as required.

Using this bound in the wave speed formula allows us to write

$$\mathbb{E}\left[\frac{\gamma^{1/2}_{t_n}(u_0)}{t_n}\right]\to\int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi)dxd\mu.$$

By the wave speed formula (as shown in Theorem 87)

$$\int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi)dx\nu(d\varphi)=\int_{\mathbb{B}^{1,0}_{dec}}\int_{\mathbb{R}}\bar{f}(\varphi)dx\mu(d\varphi). \tag{5.10}$$

In addition we know $\mathcal{L}(u(t,\cdot,u_0)) \overset{s}{\succ} \mathcal{L}(u(t,\cdot,H))$, that is $\tilde{\mathcal{Q}}_t^{u_0} \overset{s}{\succ} \tilde{\mathcal{Q}}_t^H$ for all $t \geq 0$. Now given Lemma 93 we have $\tilde{\mathcal{P}}_{t_n}^{u_0} \overset{s}{\succ} \tilde{\mathcal{P}}_{t_n}^H$. Let $t_n \to \infty$ to achieve $\mu \overset{s}{\succ} \nu$. Choose random variables u, v in $\mathbb{B}_{dec}^{1,0}$ such that $u \overset{D}{=} \nu$ and $v \overset{D}{=} \mu$, Note that $u, v \in \mathcal{C}^1$ and $u_x, v_x < 0$ almost surely. Then, we can write, using $\bar{\varphi}$ and $\bar{\psi}$ to denote the inverse functions of u and v respectively,

$$\begin{aligned}
\int_{\mathbb{R}} \bar{f}(u(x))\,dx &= -\int_0^1 \bar{f}(y)\bar{\varphi}_y(y)\,dy \\
&\geq -\int_0^1 \bar{f}(y)\bar{\psi}_y(y)\,dy \\
&= \int_{\mathbb{R}} \bar{f}(v(x))\,dx
\end{aligned}$$

as $u \overset{s}{\succ} v$. However, from equation (5.10)

$$\mathbb{E}\left[\bar{f}(u(x))\right] = \mathbb{E}\left[\bar{f}(v(x))\right]$$

and so $\int_{\mathbb{R}} \bar{f}(u(x))\,dx = \int_{\mathbb{R}} \bar{f}(v(x))\,dx$ almost surely. We now fix ω such that $\int_{\mathbb{R}} \bar{f}(u(x))\,dx = \int_{\mathbb{R}} \bar{f}(v(x))\,dx$ and $u \overset{s}{\succ} v$. By Lemma 90 this can only be true if $\tilde{u} \equiv \tilde{v}$ on this ω and so, $\tilde{u} \equiv \tilde{v}$ $\mathbb{P}$-almost surely which completes the proof. ■

Chapter 6

Fife-McLeod dynamics for the stochastic equation

6.1 Coupling argument for solutions of the Fife-McLeod equation

Introduction. *The work in this chapter is motivated by the deterministic work of Fife and McLeod (see [7],[8] and [9]) and we attempt to expand this to the stochastic setting. The formulation in the previous chapters promotes the use of the phase-plane to conduct further analysis upon travelling waves. Indeed, in the phase-plane, the concept of a stationary travelling wave starting from an Heaviside initial condition is described as movements within the* $[0,1] \times \mathbb{R}^+$ *plane collapsing from* $+\infty$ *at time* 0 *towards that of the stationary travelling wave, see figure 6.1. In the phase-plane framework, transformation to a non-moving frame of reference is not important and the concept of stretching is*

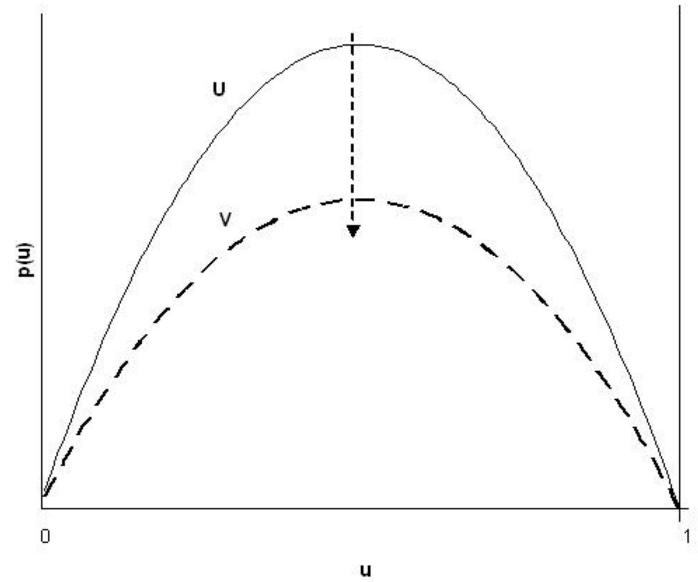

Figure 6.1: U is more stretched than V in the phase-plane.

described as one wave always lies above the other. In this way, it is clear that in the phase-plane stretching reduces to standard comparison arguments as any translate of a wavefront in the $x-t$ plane is mapped to the same wavefront in the phase plane. Although not extending the results of Chapters 3, 4 and 5 we explore the alternative definition of stretching, as presented above, and prove the equivalence of definitions in the two planes.

6.2 Stretching

Definition 95. *Define* $\mathbb{B}_{dec}^{nice} = \{\varphi \in \mathbb{B}_{dec}^{1,0} : \varphi \in \mathcal{C}^\infty$ *and* $\varphi_x(x) < 0$ *all* $x\}$.

Lemma 96. *Define the map* $p^\varphi : (0,1) \to [0,\infty)$ *for* $\varphi \in \mathbb{B}_{dec}^{nice}$ *by* $p(x) = -\varphi_x(t,x) = -\frac{1}{m_x^\varphi(x)}$ *where* m^φ *is the inverse of* φ*. Then for* $\varphi, \psi \in \mathbb{B}_{dec}^{nice}$ *if* $\varphi \overset{s}{\succ} \psi$ *then* $p^\varphi \leq p^\psi$.

Proof. We mimic the argument of Lemma 35. Suppose $\varphi(a) = \psi(a) = \theta_0 \in (0,1)$ for some $a \in \mathbb{R}$, that is $m^\varphi(\theta_0) = m^\psi(\theta_0) = a$ where m^φ, m^ψ are the inverse functions of φ and ψ respectively. The inverses exist given $\varphi, \psi \in \mathbb{B}_{dec}^{nice}$. Again note that this does not mean φ and ψ cross at a, only touch. We can then write

$$m^{\varphi}(\theta) \leq m^{\psi}(\theta) \text{ for all } \theta \geq \theta_0 \tag{6.1}$$

$$m^{\varphi}(\theta) \geq m^{\psi}(\theta) \text{ for all } \theta \leq \theta_0. \tag{6.2}$$

As the inverse m^{φ} has the same number of derivatives as φ, $m^{\varphi} \in \mathcal{C}^{\infty}$. The same is true of m^{ψ}. From equations (6.1) and (6.2) we can write $m^{\varphi}_x \leq m^{\psi}_x$ on $(0,1)$ and hence, $p^{\varphi}(x) \leq p^{\psi}(x)$ as required. ■

Remark. We believe that we can extend the results of Lemma 96 such that for any φ, $\psi \in \mathbb{B}^{1,0}_{dec}$ if $\varphi \overset{s}{\succ} \psi$ then $p^{\varphi} \leq p^{\psi}$. We do not offer a proof of this result.

6.3 Phase-plane analysis

In this chapter we will explore the phase-plane representation further through the dynamics of the SPDE. However, for such a system, there are difficulties in expanding this analysis into the stochastic case given the blow up of derivatives at the end points.

The main motivation for the use of this formulation is that the concept of stretching is easily transferred over to p space and reduces to standard comparison theory arguments (assuming the solutions are such that the map $x \to p$ makes sense). The formal map $\mathbb{I}_{[x\leq 0]} \to \infty\mathbb{I}_{(0,1)}$ makes the transformation awkward but this formulation may be best considered as an entrance law for the p process. Our main problem with using this formulation was that for solutions when $u(t,x) \sim \exp(-ax^2)$ at ∞ lead to p functions that are not in $\mathcal{C}^{0,2}$. These are needed however. Solutions with $u(t,x) \sim \exp(-ax)$ at ∞ lead to better images in p space and this should be the case for tails under a stationary distribution.

Example 7. The key example in this section that we use as primary motivation is for $u = \exp(-ax^2)$ for some $a \in \mathbb{R}^+$. Then $u_x = -2ax\exp(-ax^2) = -2u\sqrt{a\ln\left(\frac{1}{u}\right)}$. That is $p(u) = Cu\sqrt{\ln\left(\frac{1}{u}\right)} \to \infty$ and $p'(u) \uparrow \infty$ logarithmically as $u \downarrow 0$ which is undesirable.

6.4 Coupling result

Introduction. *In this section we consider the SPDE satisfying $p(t,x) = -u_x(t,x)$ (see Lemma 22) and prove a comparison theorem for solutions to the equation*

$$dp = p^2 p_{uu} dt + (f_u p - f p_u)\, dt + (g_u p - g p_u) \circ dW_t \tag{6.3}$$

satisfying a suitable Dirichlet boundary condition which we define. This comparison result is equivalent to the stretching proof put forward in Chapter 3.

Definition 97. *Suppose $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ is a filtered probability space and $(W_t : t \geq 0)$ is a one-dimensional $(\mathcal{F}_t)$ adapted Brownian motion. A function $(p(t,u) : t \geq 0, u \in [0,1])$ is a solution to the equation (6.3) up to time T if $u \in \mathbb{C}^{0,2}([0,T] \times (0,1))$ and, for all $u \in (0,1)$ and $t > 0$,*

$$\begin{aligned} p(t,u) &= p(0,u) + \int_0^t p^2(s,u) p_{uu}(s,u)\, ds \\ &\quad + \int_0^t \left(f'(u) p(s,u) - f(u) p_u(s,u)\right) ds \\ &\quad + \int_0^t \left(g'(u) p(s,u) - g(u) p_u(s,u)\right) \circ dW_s. \end{aligned}$$

It has Dirichlet boundary conditions if it satisfies $p(t,0) = p(t,1) = 0$ for all $t \geq 0$.

Theorem 98. *Suppose r, q are two solutions to (6.3) with paths in $\mathbb{C}^{0,2}([0,T] \times [0,1])$ having Dirichlet boundary conditions. If $\mathcal{P}[r(0,u) \leq q(0,u)] = 1$ then*

$$\mathcal{P}\left[r(t,u) \leq q(t,u) \text{ for all } t \in [0,T]\right] = 1.$$

Proof. Define $\tau_N = \min\{\tau_N(r), \tau_N(q)\}$ where

$$\tau_N(p) = inf\left\{t : \tfrac{d^k p(t,u)}{du^k} \geq N \text{ for some } u \in [0,1] \text{ and some } k \in \{0,1,2\}\right\}.$$

Note, since $r, q \in \mathcal{C}^{0,2}([0,T] \times [0,1])$, we have $\tau_N \to \infty$ almost surely as $N \to \infty$. The idea of the proof is to obtain a Gronwall inequality for $\mathbb{E}\left[(r(t \wedge \tau_N) - q(t \wedge \tau_N))_+\right]$ where z_+ denotes the positive part $\max\{z, 0\}$. In order to apply Itô's formula we take an approximation $J_\epsilon(x)$ to x_+ as follows: let $J_\epsilon(0) = J'_\epsilon(0) = J''_\epsilon(0) = 0$ and

$$J'''_\epsilon(x) = \begin{cases} \frac{2\pi}{\epsilon^2}\sin(2\pi x/\epsilon) & x \in [0,\epsilon], \\ 0 & x \notin [0,\epsilon]. \end{cases}$$

By integrating up we find $0 \leq J_\epsilon(x) \uparrow x_+$ and $0 \leq J'_\epsilon(x) \uparrow I(x > 0)$ as $\epsilon \to 0$. Moreover $xJ''_\epsilon(x)$ and $x^2 J'''_\epsilon(x)$ are supported in $[0,\epsilon]$ and bounded uniformly in ϵ. Also $J''_\epsilon \geq 0$.

We will then do the proof in the following five stages:

1. Use the identity $p^2 p_{uu} = \frac{1}{3}(p^3)_{uu} - 2p(p_u)^2$ and rewrite (6.3) as the equation:

$$dp = \frac{1}{3}(p^3)_{uu}dt - 2p(p_u)^2 dt + (f_u p - f p_u)\,dt + (g'p - gp_u) \circ dW_t. \tag{6.4}$$

2. Develop $J_\epsilon(r-q)$ using Itô's formula.

3. Integrate with respect to the spatial variable u: $\int_0^1 du$. This will allow us to perform integration by parts in the u variable.

4. Take expectations $\mathbb{E}$. This will allow us to neglect the Itô integrals by standard properties.

5. Then we take the limit as epsilon tends to zero $\epsilon \to 0$. this will allow us to us the limiting properties of J and its derivatives (and cancel a large number of terms).

Remark. Equation (6.4) is still non-linear but f and g no longer contribute to the non-linearity.

For the interested reader, [11] is a good introduction into Porous Media type equations and is the main inspiration for the proof that follows.

Applying Stratonovich calculus to the function $J_\epsilon(r-q)$ for fixed $x \in (0,1)$ over the range 0 to $t \wedge \tau_N$ we have

$$J_\epsilon(r-q)(t \wedge \tau_N, u) = J_\epsilon(r-q)(0,u) + \int_0^{t\wedge\tau_N} J'_\epsilon(r-q)(s,u)\, d(r-q)$$

Integrating with respect to u, noting that we can apply Fubini's Theorem to swap integrals and expanding gives

$$\begin{aligned}
\int_0^1 J_\epsilon(r-q)(t \wedge \tau_N, u)\, du &= \int_0^1 J_\epsilon(r-q)(0,u)\, du + \int_0^1 \int_0^{t\wedge\tau_N} J'_\epsilon(r-q)\, d(r-q) du \\
&= \int_0^1 J_\epsilon(r-q)(0,u)\, du \\
&\quad + \frac{1}{3}\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(\Delta(r^3)-\Delta(q^3)) duds \\
&\quad -2\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(rr_u^2 - qq_u^2) duds \\
&\quad +\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(r-q) f_u duds \\
&\quad -\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q) f (r_u - q_u) duds \\
&\quad +\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(r-q) g_u du \circ dW \\
&\quad -\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q) g (r_u - q_u) du \circ dW \\
&=: \sum_{i=1}^{7} R_i.
\end{aligned}$$

We will consider each of these terms in turn. Note that for brevity, we are writing solutions $r(s,u), q(s,u)$ as r, q respectively.

Remark. Given the assumption $r \leq q$ at $t = 0$ we trivially have $R_1 = \int_0^1 J_\epsilon(r - q)(0,u)\,du = 0$ given the definition of $J_\epsilon(z)$.

Let us now consider R_4 and R_5 together as we will for the noise terms. Let us define $C = C(N, \|f\|_\infty, \|f'\|_\infty, \|g\|_\infty)$ which may change from line to line without further note.

$$\begin{aligned}
\mathbb{E}[R_4 + R_5] &= \mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(r-q)f_u - J'_\epsilon(r-q)f(r-q)_u\,duds \\
&= \mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 2J'_\epsilon(r-q)(r-q)f_u + J''_\epsilon(r-q)f(r-q)_u(r-q)\,duds \\
&\leq \mathbb{E}\left[C\int_0^t\int_0^1 (r-q)_+\mathbb{I}_{[s<\tau_N]}duds\right] \\
&\qquad +\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J''_\epsilon(r-q)f(r-q)_u(r-q)duds \\
&\leq C\int_0^t\int_0^1 \mathbb{E}((r-q)_+\mathbb{I}_{[s<\tau_N]})duds \\
&\qquad +\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J''_\epsilon(r-q)f(r-q)_u(r-q)duds
\end{aligned}$$

where in the second equality we have used integration by parts and noted, as $J'_\epsilon(0) = 0$, that the cross terms vanish.

Note the last integral tends to zero with decreasing epsilon by use of the Dominated Convergence Theorem on $(r-q)J''_\epsilon(r-q)$. The first integral is less than $C\int_0^t\int_0^1 \mathbb{E}((r - q)_+\mathbb{I}_{[s<\tau_N]})duds$ which we can use as part of Gronwall's inequality.

Let us now consider R_2.

$$\begin{aligned}
\mathbb{E}[R_2] &= \frac{1}{3}\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(r^3-q^3)_{uu}duds \\
&= \frac{1}{3}\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r^3-q^3)(r^3-q^3)_{uu}duds \\
&\qquad +\frac{1}{3}\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 \left(J'_\epsilon(r-q)-J'_\epsilon(r^3-q^3)\right)(r^3-q^3)_{uu}duds \\
&\leq -\frac{1}{3}\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J''_\epsilon(r^3-q^3)((r^3-q^3)_u)^2duds \\
&\qquad \text{by Integration by parts noting the cross terms again vanish} \\
&\qquad +C\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 \left|J'_\epsilon(r-q)-J'_\epsilon(r^3-q^3)\right| duds \\
&\leq C\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 \mathbb{I}_{[r-q\in(0,\epsilon)\,\mathrm{or}\,r^3-q^3\in(0,\epsilon)]}\,duds \to 0
\end{aligned}$$

as epsilon tends to zero where we have used the property that the first integral is negative given $J''_\epsilon(r^3-q^3)\geq 0$ by definition.

$$\begin{aligned}
\mathbb{E}[R_3] &= -2\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(rr_u^2-qq_u^2)\,duds \\
&= -2\mathbb{E}\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)\left\{r(r_u^2-q_u^2)+(r-q)q_u^2\right\}\,duds \\
&= -2\mathbb{E}\left(\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)r(r_u-q_u)(r_u+q_u)\,duds\right. \\
&\qquad \left.+\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(r-q)q_u^2\,duds\right) \\
&\leq 2\mathbb{E}\left(\int_0^{t\wedge\tau_N}\int_0^1 J'_\epsilon(r-q)(r-q)\left\{r(r+q)_u\right\}_u\,duds\right) \\
&\qquad +2\mathbb{E}\left(\int_0^{t\wedge\tau_N}\int_0^1 J''_\epsilon(r-q)(r-q)(r-q)_u(r(r+q)_u)\,duds\right) \\
&\leq C\int_0^t\int_0^1 \mathbb{E}((r-q)_+\mathbb{I}_{[s<\tau_N]})duds \\
&\qquad +C\mathbb{E}\left(\int_0^{t\wedge\tau_N}\int_0^1 J''_\epsilon(r-q)(r-q)(r-q)_u(r(r+q)_u)\,duds\right)
\end{aligned}$$

where in the first inequality we have used integration by parts in the first term (again the cross terms vanish given $J'_\epsilon(0)=0$) and the fact that we can ignore the second term as it is negative. The second term in the last line tends to zero as epsilon tends to zero by use of the Dominated Convergence Theorem given $(r-q)_u$, $(r(r+q))_u$ are bounded by definition of the stopping time τ_N and $J''_\epsilon(z)z$ is bounded and has compact support outside of the region $(0,\epsilon)$.

Considering the last two integrals, that is R_6 and R_7:

$$\begin{aligned}
R_6 + R_7 &= \int_0^{t\wedge\tau_N} \int_0^1 J'_\epsilon(r-q)(r-q)g_u du \circ dW \\
&\qquad - \int_0^{t\wedge\tau_N} \int_0^1 J'_\epsilon(r-q)g(r_u-q_u)du \circ dW \\
&= 2\int_0^{t\wedge\tau_N} \int_0^1 J'_\epsilon(r-q)(r-q)g_u du \circ dW \\
&\qquad + \int_0^{t\wedge\tau_N} \int_0^1 J''_\epsilon(r-q)g(r-q)(r-q)_u du \circ dW \\
&= I_1 + I_2
\end{aligned}$$

by integration by parts in the u variable in the second equality. The second integral tends to zero as epsilon tends to zero due to the following lemma which is proved in the next section.

Lemma 99.

$$\int_0^{t\wedge\tau_N} \int_0^1 J''_\epsilon(r-q)g(r-q)(r-q)_u du \circ dW \to 0$$

as $\epsilon \to 0$.

Returning back to the Lemma, we rewrite I_1 in its Itô form:

$$\begin{aligned}
I_1 &= 2\int_0^{t\wedge\tau_N} \int_0^1 J'_\epsilon(r-q)(r-q)g_u du \circ dW \\
&= 2\int_0^{t\wedge\tau_N} \int_0^1 J'_\epsilon(r-q)(r-q)g_u du dW + \frac{1}{2}\left[\int_0^1 2J'_\epsilon(r-q)(r-q)g_u\, du, W\right]_t
\end{aligned}$$

where $[\cdot,\cdot]$ is the quadratic covariation. Note now that the Itô integral will vanish upon taking expectations. This leaves the covariation term which, using the following lemma, the proof of which can be found in the next section, is used in the Gronwall argument.

Lemma 100.

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^t\int_0^1 2zJ'_\epsilon(z)g_u\,du, W\right]_t \leq C(\|g\|_\infty)\int_0^t \mathbb{E}\left[\int_0^1 (r-q)_+\,du\right]\,ds$$

Hence, combining these arguments we can write $\mathbb{E}(I_1) \leq C\int_0^1 z_+du$ as epsilon tends to zero. Collecting all the terms together from the above analysis and letting epsilon tend to zero we obtain

$$\int_0^1 \mathbb{E}\left(r_{t\wedge\tau_N} - q_{t\wedge\tau_N}\right)_+\,du \leq C\int_0^t\int_0^1 \mathbb{E}\left(r_{s\wedge\tau_N} - q_{s\wedge\tau_N}\right)_+\,duds.$$

Using Gronwall's inequality shows that $\int_0^1 \mathbb{E}\left[\left(r_{t\wedge\tau_N} - q_{t\wedge\tau_N}\right)_+\,du\right] = 0$ for all $0\leq t\leq \tau_N$ and since $r_{t\wedge\tau_N}$ and $q_{t\wedge\tau_N}$ have continuous paths, so does $(r_t - q_t)\mathbb{I}_{[t\leq\tau_N]}$. This implies that $r(t,u)\leq q(t,u)$ for all $0\leq t\leq\tau_N$ almost surely. ∎

Corollary 101. *If $r_0 = q_0$ then $r_t = q_t$ for all t almost surely.*

Proof. By the above theorem this immediately follows. ∎

The future aim would be to reprove Theorem 98 under weaker regularity assumptions on derivatives at 0 and 1, that is they allow slow blow up, for example logarithmically, of p_u and p_{uu}.

6.4.1 Proof of technical lemmas.

Proof of Lemma 99. Define $z(t,u) = r(t,u) - q(t,u)$. We need to consider the quadratic covariation between $gzJ''_\epsilon(z)z_u$ and W. To calculate this we consider the decomposition of, first, $zJ''_\epsilon(z)$ and then $zz_uJ''_\epsilon(z)$.

$$\begin{aligned} d\left(zJ''_\epsilon(z)\right) &= J''_\epsilon(z)\circ dz + zJ'''_\epsilon(z)\circ dz \\ &= \text{terms of bounded variation} + \left(J''_\epsilon(z) + zJ'''_\epsilon(z)\right)\left(g_u z - g z_u\right)\circ dW_t. \end{aligned}$$

Hence, $d\left[zJ''_\epsilon(z), W\right] = \left(J''_\epsilon(z) + zJ'''_\epsilon(z)\right)\left(g_u z - g z_u\right)$. Also,

$$d\left(z_u zJ''_\epsilon(z)\right) = z_u\left(J''_\epsilon(z)\circ dz + zJ'''_\epsilon(z)\circ dz\right) + zJ''_\epsilon(z)\circ dz_u$$

and

$$dp_u = 2pp_u p_{uu}dt + p^2 p_{uuu}dt + \left(f_{uu}p - fp_{uu}\right)dt + \left(g_{uu}p - gp_{uu}\right)\circ dW_t.$$

Hence,

$$d\left[z_u zJ''_\epsilon(z), W\right] = z_u\left(J''_\epsilon(z) + zJ'''_\epsilon(z)\right)\left(g_u z - gz_u\right) + zJ''_\epsilon(z)\left(g_{uu}z - gz_{uu}\right).$$

Then,

$$\begin{aligned} \left[\int_0^{\cdot\wedge\tau_N}\int_0^1 J''_\epsilon(z)zz_u, W\right]_t &= \int_0^{t\wedge\tau_N}\int_0^1 \left[zJ''_\epsilon(z)\left(g_{uu}z - gz_{uu}\right) + z_u\left(J''_\epsilon(z) + zJ'''_\epsilon(z)\right)\right. \\ &\qquad \left.\times\left(gz_u - g_u z\right)\right] du ds \end{aligned}$$

and concentrating on the second term on the right hand side we can write, by expanding brackets,

$$
\begin{aligned}
&\int_0^{t\wedge\tau_N}\int_0^1 \left[z_u\left(J''_\epsilon(z)+zJ'''_\epsilon(z)\right)\left(gz_u-g_uz\right)\right]duds\\
&\quad=\int_0^{t\wedge\tau_N}\int_0^1 \left[g(z_u)^2J''_\epsilon(z)+g(z_u)^2zJ'''_\epsilon(z)-g_uzz_uJ''_\epsilon(z)-g_uz^2z_uJ'''_\epsilon(z))\right]ds\\
&\quad=\int_0^{t\wedge\tau_N}\int_0^1 \left[g(z_u)^2J''_\epsilon(z)+gzz_u\frac{dJ''_\epsilon(z)}{du}-g_uzz_uJ''_\epsilon(z)-g_uz^2z_uJ'''_\epsilon(z)\right]ds\\
&\quad=\int_0^{t\wedge\tau_N}\int_0^1 \left[g(z_u)^2J''_\epsilon(z)-\left\{zg_uz_u+gzz_{uu}+g(z_u)^2\right\}J''_\epsilon(z)\right.\\
&\qquad\quad\left.-g_uzz_uJ''_\epsilon(z)-g_uz^2z_uJ'''_\epsilon(z)\right]ds\\
&\quad=-\int_0^{t\wedge\tau_N}\int_0^1 \left[\left(zg_uz_u+zgz_{uu}\right)J''_\epsilon(z)+g_uzz_uJ''_\epsilon(z)+g_uz^2z_uJ'''_\epsilon(z)\right]ds.
\end{aligned}
$$

Putting this all together gives

$$
\begin{aligned}
\left[\int_0^{\cdot\wedge\tau_N}\int_0^1 J''_\epsilon(z)zz_u,W\right]_t &= \int_0^{t\wedge\tau_N}\int_0^1 \left[zJ''_\epsilon(z)\left(g_{uu}z+gz_{uu}\right)-\left(g_u(z_u)z+g(z_{uu})z\right)J''_\epsilon(z)\right.\\
&\qquad\left.-\,g_uzz_uJ''_\epsilon(z)-g_uz^2z_uJ'''_\epsilon(z)\right]ds.
\end{aligned}
$$

By the Dominated Convergence Theorem, it is clear that all the terms tend to zero as epsilon tends to zero. ■

Proof of Lemma 100. To prove this we will consider the covariation term more closely. Let z denote the difference between the two solutions r and q then we can write the decomposition

$$
\begin{aligned}
d(zJ'_\epsilon(z)) &= J'_\epsilon(z)\circ dz+zJ''_\epsilon(z)\circ dz\\
&= \text{terms of bounded variation}+\left(J'_\epsilon(z)+zJ''_\epsilon(z)\right)\left(g_uz-gz_u\right).
\end{aligned}
$$

Then

$$\left[\int_0^{\cdot}\int_0^1 2J_\epsilon'(z)g_u z\, duds, W_\cdot\right]_{t\wedge\tau_N} = \int_0^{t\wedge\tau_N}\int_0^1 2g_u\left(J_\epsilon'(z)g_u z - J_\epsilon'(z)gz_u\right) duds$$
$$+\int_0^{t\wedge\tau_N}\int_0^1 2g_u\left(z^2J_\epsilon''(z)g_u - gz_u zJ_\epsilon''(z)\right) duds$$

Now by the dominated convergence theorem, again due to the properties of J'', we can say that for any term containing $zJ_\epsilon''(z)$ tends to zero as ϵ tends to zero. Also any term $zJ_\epsilon'(z)$ is a copy of z_+ as ϵ tends to zero. Hence

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^{\cdot}\int_0^1 2zJ_\epsilon'(z)g_u\, du, W\right]_{t\wedge\tau_N} = \lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^{t\wedge\tau_N}\int_0^1 2(g_u)^2 z_+ - 2g_uJ_\epsilon'(z)gz_u\right.$$
$$\left. - 2gg_u zJ_\epsilon''(z)z_u\, duds\right].$$

Integrating the second term above by parts gives

$$\lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^{\cdot}\int_0^1 2zJ_\epsilon'(z)g_u\, du, W >_{t\wedge\tau_N}\right] = \lim_{\epsilon\to 0}\mathbb{E}\left[\int_0^{t\wedge\tau_N}\int_0^1 2(g_u)^2 z_+ + 2(gg_u)_uJ_\epsilon'(z)z\right.$$
$$\left. + (gg_u)J_\epsilon''(z)z_u z - gg_u zJ_\epsilon''(z)z_u\, duds\right]$$
$$= \mathbb{E}\left[\int_0^{t\wedge\tau_N}\int_0^1 2((g_u)^2 + (gg_u)_u)z_+\, duds\right]$$

as required. ■

6.5 Future research interests

In summary, our main results have shown that provided solutions start from the Heaviside initial condition, whether $\bar{f}$ be of KPP type, Nagumo type or Unstable type, there exists a unique stationary travelling wave (up to translation). We then extended this argument in the case where $\bar{f}$ is of KPP type and showed that this conclusion is also true for trapped initial conditions.

There are clear areas of further investigation:

- [1] Relaxing the trapped criteria into more explicit decay rates in the tails of the travelling wave.

- [2] Exploring whether we can extend our wave-speed argument in Chapter 5 for trapped initial conditions to the KPP type and Unstable type and the reasons why this fails.

- [3] Investigating how the conditions set out for $\bar{f}$ can be rewritten in terms of the f and g.

- [4] Conduct computer simulation on the dynamics of the solution and how these change for changing f and g, if at all.

- [5] Investigate if our results can be extended to the case of more general noises for example, a finite sum of independent Brownian motions, a white in time, spatially homogenous noise and even the space-time white noise case.

We hope to explore some of these open questions in future papers.

Appendix: Properties of the Heat Kernel $\Gamma_t(x)$

Introduction. *In the calculation of the solution to the heat equation* $u_t = u_{xx}$ *one finds what is termed either the heat kernel or the fundamental solution.*

Let $\Gamma(t, z)$ *be the one-dimensional heat kernel,*

$$\Gamma(t,z) = \begin{cases} \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{z^2}{4t}\right) & \text{for } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

for $z \in \mathbb{R}$.

In this section we record several properties of the heat kernel which we will use repeatedly later.

We define C as a generic constant which may change value from line to line but whose dependence will be shown.

Lemma 102. *For* $0 < t \leq t'$ *and all* $x, y \in \mathbb{R}$

$$\int_{\mathbb{R}} |\Gamma_t(x-w) - \Gamma_{t'}(y-w)|\, dw \leq Ct^{-1/2}|x-y| + Ct^{-1}|t'-t|.$$

Remark. The statement of the above inequality and similar varieties can be found in many papers concerning SPDEs (See [27] and [31]). To ensure this thesis is self contained, the proof of the inequality is shown in full below.

Remark. Before starting the proof we note an important relation which will be of immediate use. By the Fundamental Theorem of Calculus

$$\left|\exp\left(-\frac{(w+x-y)^2}{4t}\right)-\exp\left(-\frac{w^2}{4t}\right)\right|=\left|\int_0^{x-y}\frac{w+r}{2t}\exp\left(-\frac{(w+r)^2}{4t}\right)dr\right|. \quad (6.5)$$

The following lemma will also prove of use, the proof of which can be found at the end of this section.

Lemma 103. *$\int_{\mathbb{R}}\frac{|w+r|}{t}\exp\left(-\frac{|w+r|^2}{4t}\right)dw=\int_{\mathbb{R}}|w|\exp\left(-\frac{|w|^2}{4}\right)dw$ for all r and $t>0$. That is, the integral is independent of r and t.*

Proof of Lemma 102. By use of the triangle inequality we can split up the bound into two components, the first independent of t' and the second, independent of x.

$$\begin{aligned}\int_{\mathbb{R}}|\Gamma_t(x-w)-\Gamma_{t'}(y-w)|\,dw &\leq \int_{\mathbb{R}}|\Gamma_t(x-w)-\Gamma_t(y-w)|\,dw\\ &\quad+\int_{\mathbb{R}}|\Gamma_t(y-w)-\Gamma_{t'}(y-w)|\,dw\\ &= I_1+I_2 \text{ say.}\end{aligned}$$

Let us consider each of these terms in turn.

$$\begin{aligned}I_1 &= \int_{\mathbb{R}}|\Gamma_t(x-w)-\Gamma_t(y-w)|\,dw\\ &\leq Ct^{-1/2}\int_0^{|x-y|}\left[\int_{\mathbb{R}}\frac{|w+r|}{t}\exp\left(-\frac{|w+r|^2}{4t}\right)dw\right]dr\\ &= Ct^{-1/2}\int_0^{|x-y|}\int_{\mathbb{R}}|w|\exp\left(-\frac{|w|^2}{4}\right)dwdr\\ &\leq Ct^{-1/2}\int_0^{|x-y|}dr\\ &\leq Ct^{-1/2}|x-y|\end{aligned}$$

where in the first inequality we have used Lemma 103 and Fubini's theorem to swap the order of integration. Before we progress onto the I_2 term we note that through a power series expansion, the bound $\frac{w^2}{4s^2} \leq \frac{2}{s}\exp\left(\frac{w^2}{8s}\right)$ is clear. Then,

$$\begin{aligned}
|\Gamma_t(w) - \Gamma_{t'}(w)| &\leq C\left|t^{-1/2} - t'^{-1/2}\right|\exp\left(-\frac{w^2}{4t}\right) \\
&\quad +Ct'^{-1/2}\left|\exp\left(-\frac{w^2}{4t}\right) - \exp\left(-\frac{w^2}{4t'}\right)\right| \\
&\leq C|t'-t|t^{-3/2}\exp\left(-\frac{w^2}{4t}\right) \\
&\quad +Ct'^{-1/2}\int_t^{t'}\frac{w^2}{4s^2}\exp\left(-\frac{w^2}{4s}\right)\,ds \\
&\leq C|t'-t|t^{-3/2}\exp\left(-\frac{w^2}{4t}\right) \\
&\quad +Ct'^{-1/2}\int_t^{t'}\frac{2}{s}\exp\left(\frac{w^2}{8s}\right)\exp\left(-\frac{w^2}{4s}\right)\,ds \\
&= C|t'-t|t^{-3/2}\exp\left(-\frac{w^2}{4t}\right) + Ct'^{-1/2}\int_t^{t'}\frac{2}{s}\exp\left(-\frac{w^2}{8s}\right)\,ds \\
&\leq C|t'-t|t^{-3/2}\exp\left(-\frac{w^2}{4t}\right) \\
&\quad +Ct'^{-1/2}\sup_{s\in[t,t']}\left[\frac{2}{s}\right]\sup_{s\in[t,t']}\left[\exp\left(-\frac{w^2}{8s}\right)\right]\int_t^{t'}ds \\
&= C|t'-t|t^{-3/2}\exp\left(-\frac{w^2}{4t}\right) \\
&\quad +Ct'^{-1/2}\left[\frac{2}{t}\right]\left[\exp\left(-\frac{w^2}{8t'}\right)\right]\int_t^{t'}ds \text{ as } 0<t\leq t' \\
&\leq C|t'-t|t^{-1}\left[\Gamma_t(w) + \Gamma_{2t'}(w)\right].
\end{aligned}$$

Hence, integrating over $\mathbb{R}$ we find:

$$\begin{aligned}
I_2 &= \int_{\mathbb{R}}|\Gamma_t(y-w) - \Gamma_{t'}(y-w)|\,dw \\
&\leq \int_{\mathbb{R}} C|t'-t|t^{-1}\left[\Gamma_t(w) + \Gamma_{2t}(w)\right]\,dw \\
&= C|t'-t|t^{-1}
\end{aligned}$$

as $\int_R \Gamma_t(w)\,dw = 1$ and $\int_R \Gamma_{2t}(w)\,dw = 2$. Combining the above two inequalities yields the result:

$$\int_{\mathbb{R}} |\Gamma_t(x-w) - \Gamma_{t'}(y-w)|\, dw \leq C\frac{|x-y|}{t^{1/2}} + C\frac{|t'-t|}{t}$$

which gives us our required bound. ■

Proof of Lemma 103. The proof of this lemma is a simple change of variable which we specify at each step

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|w+r|}{t} \exp\left(-\frac{|w+r|^2}{4t}\right) dw \quad &= \quad \int_{\mathbb{R}} |\hat{w}+\hat{r}| \exp\left(-\frac{|\hat{w}+\hat{r}|^2}{4}\right) d\hat{w} \\
&\qquad \text{using the substitutions } w = \sqrt{t}\hat{w} \text{ and } r = \sqrt{t}\hat{r} \\
&= \quad \int_{\mathbb{R}} |\bar{w}| \exp\left(-\frac{|\bar{w}|^2}{4}\right) d\bar{w} \\
&\qquad \text{using the substitution } \bar{w} = \hat{w} + \hat{r}.
\end{aligned}$$

This is clearly a finite integral. ■

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