DIFFERENT T-NORMS FOR DEA-FUZZY EFFICIENCY COMPUTATIONS

Lidia Angulo Meza (UFF)
lidia_a_meza@yahoo.com
Eliane Goncalves Gomes (Embrapa)
eliane.gomes@embrapa.br
João Carlos Correia Baptista Soares de Mello (UFF)
jcsmello@pesquisador.cnpq.br
Luiz Biondi Neto (UERJ)
luizbiondi@terra.com.br

Interval DEA frontiers are here used in situations where one input or output is subject to uncertainty in its measurement and is presented as an interval data. We built an efficient frontier without any assumption about the probability disttribution function of the imprecise variable. We take into account only the minimum and the maximum values of each imprecise variable. Two frontiers are constructed: the optimistic and the pessimistic ones. We use fuzzy relationships to introduce a new efficiency index based on a set of some Fuzzy T Norms. We will explore only the case where only one single variable presents a certain degree of uncertainty.

Palavras-chaves: Data envelopment analysis, Fuzzy sets, Interval data

1. Introduction

Classic Data Envelopment Analysis (DEA) models (COOPER *et al.*, 2000) estimate a non-parametric linear piecewise frontier determined by efficient Decision-Making Units (DMUs). Such models assume that the values involves are known with absolute precision. However, such hypothesis might not be true either due to uncertainty hidden in the measurements or because the data are given in interval format (COOPER *et al.*, 2000). In the first case, the classical solution is to use the Stochastic Frontier Analysis (COELLI *et al.*, 1998), which assumes that uncertainties follow some probability distribution. An introduction to such an approach with the use of parametric and econometric methods is found in Lovell (1993).

This paper proposes a method to evaluate efficiency in the case where the data are in interval form. The method uses a geometrical approach in order to build a fuzzy efficient frontier sets. Instead of calculating an efficiency score we will attribute to each and every DMU a membership degree to the frontier which will become a fuzzy set (ZADEH, 1965).

Alternative solutions for interval data in DEA models can be found in the literature with Fuzzy Linear Programming Problems. Yet another approach found is to present the efficiency measurements in terms of fuzzy functions. We emphasize that the approach followed in this paper uses only the concept of fuzzy sets and T Norms. The scores obtained herebellow are based mainly on geometrical considerations.

2. Literature Review in DEA Models with Uncertainties

A comprehensive literature review on methods used to deal with imprecise DEA data can be found in Zhu (2003), who classifies data uncertainties into three types: interval data, ordinal data and interval data ratios. The author run a model called Imprecise Data Envelopment Analysis (IDEA) (COOPER *et al.*, 1999), which treats the three types of data uncertainties using scale transformations. Due to the problems regarding scale transformations, the author proposes a simplified approach that converts the variables employed into exact data.

The IDEA model was used by Despotis and Smirlis (2002) to deal with uncertain data of two types: interval data and ordinal data. The use of such a linear model is carried out through a change of variable scales, turning the non-linear model into a linear programming model. As a result, upper and lower bounds are obtained for the efficiency of each DMU. According to the authors, the use of post DEA models allows a better discrimination among DMUs. The authors still proposed a post DEA model in order to determine target inputs for inefficient DMUs.

Cooper et al. (2001) proposed an extended IDEA model, which enables not only the use of imprecise data, but also the use of weight restrictions in the form of assurance regions or cone-ratios. In that case, the variable limits are changed to data adjustments. Such a model was applied to the efficiency evaluation of a Korean mobile telecommunication company.

Lertworasirikul et al. (2003) considered uncertain inputs and outputs as fuzzy sets. Efficiency computations are then carried out by means of linear fuzzy programming. As an alternative approach, the same authors proposed the use of possibility DEA models. A fuzzy variable is associated to a possibility distribution (ZADEH, 1978), where the fuzzy-DEA scores, although not unique, depend on the level of possibility used.

Entani et al. (2002) employed a DEA model to assess DMUs optimistically. These results were used to determine interval efficiencies by means of new DEA models. Consequently, the efficiency score is not represented by a number, but by an interval. On the other hand, Yamada et al. (1994) assessed each DMU pessimistically based on the Inverted DEA model and calculated interval efficiency scores. The authors still considered interval data and proposed a model to evaluate interval efficiency and inefficiency as carried out using crisp data.

Lopes and Lanzer (2002) carried out a performance evaluation of University academic departments. The DEA results on teaching, research and quality are set as fuzzy numbers. A unique performance score for each department was built using a weighted ordered aggregator.

Guo and Tanaka (2001) extended the DEA CCR model with fuzzy inputs and outputs to a model named DEARA. This model uses regression analysis concepts and a Fuzzy-DEA model, in which the resulting efficiency scores are interval fuzzy evaluations.

Kao and Liu (2000) proposed a method to measure DMUs efficiencies with fuzzy variables. The fuzzy model then turns out to a family of conventional DEA models based on crisp data using the α-cut approach. According to the authors, the fuzzy efficiency scores obtained are given by interval functions yielding more information to the decision-maker. This approach uses Fuzzy Linear Programming. A similar approach has been used by Lin (2006).

To measure the technical efficiency of DMUs, Triantis and Eeckaut (2000) relaxed the concept of production frontier and proposed a pair-wise comparison, checking the dominance or non-dominance of each DMU when compared to any other. The use of fuzzy variables to take into account imprecise data yields a fuzzy pair-wise comparison. Such results are represented in matrix form that shows two-way dominance. In other words, efficiency scores are not actually obtained, but only an indication of domination among DMUs. It should be stressed that should this model be used with crisp data, it would generate a model equivalent to the Free Disposal Hull (FDH) model (DEPRINS *et al.*, 1984).

Hougaard (1999) used fuzzy intervals to combine the information given by DEA analytical efficiency scores with subjective efficiency scores. Qualitative and organizational aspects are in fuzzy intervals format. The relationship between this information is given by a fuzzy interval function. Ideally, the two sources of information related to the performance of a DMU can be joined in such a way that the objective DEA aspect is used to control the subjectivity in the expert point of view, and vice-versa. That leads to a modified score set in terms of a fuzzy interval.

Triantis and Girod (1998) suggested a three-stage approach to measure technical efficiency in a fuzzy environment. This approach uses classic DEA techniques and is built on fuzzy parametric programming concepts (CARLSSON e KORHONEN, 1986).

Sengupta (1992) used fuzzy sets theory in a DEA context. The author uses three types of fuzzy statistics (fuzzy mathematical programming, fuzzy regression and fuzzy entropy) to illustrate the types of decision and solution that can be reached when we have imprecise data and a priori information is uncertain and imprecise. The same author (SENGUPTA, 2005) generalized the nonparametric approach of DEA in both static and dynamic directions by incorporating uncertainties. He addressed an extension of the convex hull method of DEA for determining a production frontier in the presence of demand and supply uncertainty of outputs and inputs.

An approach based on randomized ranks is presented by Sant'Anna (2002).

When applications are concerned, we may cite a study on the location for the geographic situation of hydroelectric plants (SANT'ANNA e SANT'ANNA, 2008) and the study on the efficiency of Taiwan hotels (SHEN e HSIEH, 2006).

3. Fuzzy Efficient Frontier

The approach developed here makes no assumption regarding the way each input or output varies. Only maximum and minimum values for each output and each input are required. Only geometric relationships are required to obtain the membership degree of each DMU to the fuzzy frontier. The algebraic calculation of those relations uses only classic DEA models. If the variables are in interval form, the exact location of the efficient frontier is unknown. It may be placed between upper and lower bounds. That is, the frontier is not a piecewise linear surface but a region of the space. In the case of one single input and one single output, such a frontier would be a strip. In other words, this frontier is a fuzzy set (ZADEH, 1965). To such sets, instead of stating that a single element belongs or not to the set, we consider that all elements belong to it with a certain degree of membership.

In the absence of objective reasons to choose among one of the various classical membership functions we will use some geometric measurements on the fuzzy efficient frontier. To do so, we need to introduce some concepts:

- Upper frontier: It is the frontier obtained by a classic DEA model (CCR or BCC) that considers the maximum value of the imprecise output for each DMU. As in terms of production this is the most desirable situation for all DMUs, the frontier so obtained may also be named Optimistic Frontier.
- Lower frontier: It is the frontier obtained by a classic DEA model (CCR or BCC) that considers the minimum value of the imprecise output for each DMU. Since in terms of production this is the least favorable situation for all DMUs, the frontier so obtained may also be named Pessimistic Frontier.

The definitions hereabove are concerning to the case when the variable in interval form is an output. Moreover, these concepts are similar to those defined by Kao and Liu (2000). Those authors have used Fuzzy Linear Programming, and we will use a geometrical approach. The relations derived from our geometrical approach are a generalization of the relation obtained by Soares de Mello et al (2005) and used by Gomes et al. (2006) and by Correia and Soares de Mello (2008).

Figure 1 illustrates these concepts, considering the BCC DEA model (BANKER *et al.*, 1984). The interval data DEA frontier comprises the region between the lower and the upper frontiers. In opposition to classic DEA frontier, a DMU cannot be represented as a point in a multidimensional space. Its geometric representation must be a line segment (even in multidimensional cases). In Figure 1, the DMU under analysis is represented by the segment $\overline{P1P2}$. The point $P2$ corresponds to the lower value of the imprecise output and the point $P1$ corresponds to the upper value of the imprecise output.

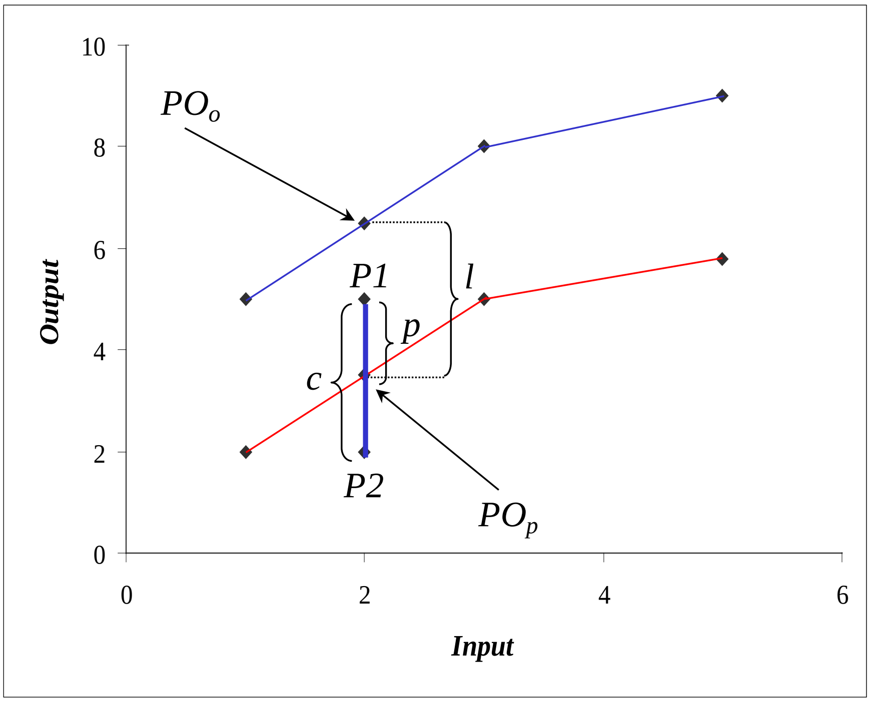


Figure 1 - Optimistic and pessimistic frontiers.

Also in Figure 1, PO_o and PO_p are the projected output on the optimistic and pessimistic frontiers; c is the DMU length, i.e., the difference between the optimistic and pessimistic values of the output; l is the width of the strip connecting the DMU projections on both frontiers; p is the difference between the optimistic output of each DMU and its projection on the pessimistic frontier.

To determine the DMU's membership degrees to the frontier we consider the following cases.

1) Figure 2 shows that DMUs A and F are totally inside the region defining the fuzzy frontier. Such DMUs must have a unitary membership degree the fuzzy frontier.

2) DMUs B and C slightly touch the frontier and so their membership degrees are zero.

3) Between those extreme situations, DMUs E and G would have intermediate membership degrees.

   a) The segment that represents DMU G covers all the length of the fuzzy frontier. However, its membership degree cannot be one, as it still has a strip outside the fuzzy frontier. This means that although this DMU totally includes the frontier, it is not totally included there. The ratio p/c is adequate to evaluate the membership degree in situations similar to that of DMU G.

   b) An inverse situation is presented by DMU E, which is fully contained in the fuzzy frontier, but does not entirely cover it. Like DMU G, this DMU cannot present a unitary membership degree to the frontier. For such situation the ratio p/l adequately represents the membership degree.

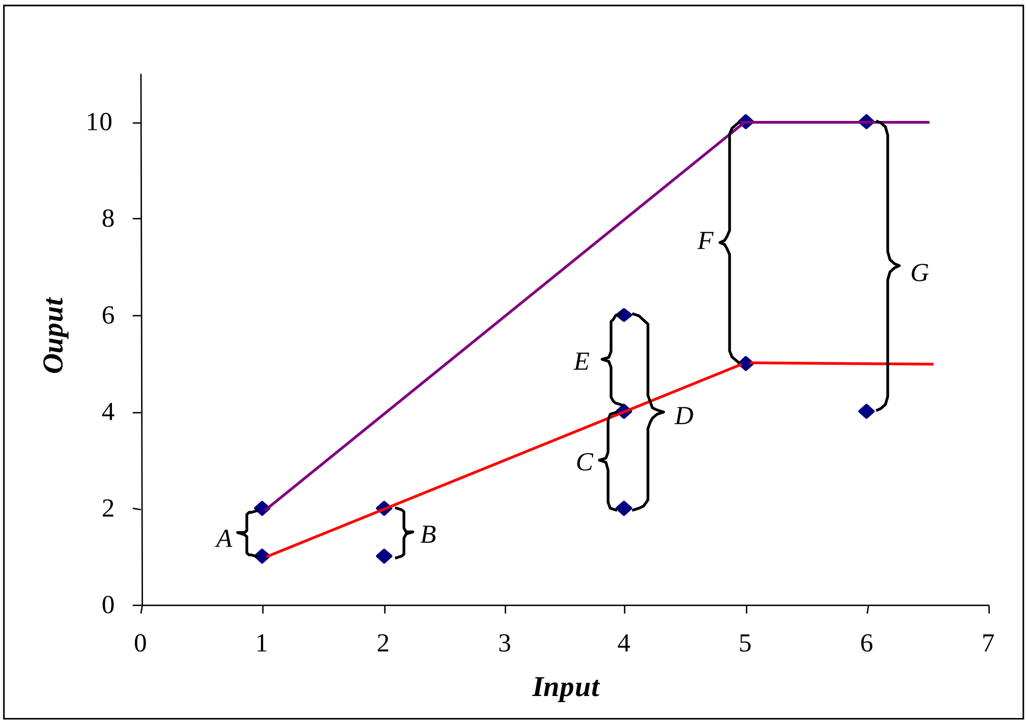


Figure 2 - Interval data DEA frontier in a BCC model.

Both ratios above are only adequate in particular situations and lead to meaningless results when used in a different situation. In order to obtain a membership function with properties required in items 1), 2) and 3) (a and b), we need to combine the two ratios.

Properties 1), 2) and 3) are satisfied if we defined a T Norm between the fuzzy set defined by the membership function p/l and the fuzzy set defined by the membership function p/c.

We will use three 'T Norm' to evaluate the membership degree of a DMU to the Fuzzy Frontier: (i) The Product; (ii) the Drastic Product; (iii) the Min. The graphic representation of these three T Norms can be seen in Figure 3.

Expression (1): $\wp_P$ is the membership degree using the Product.

$$\wp_P = \frac{p^2}{lc} \tag{1}$$

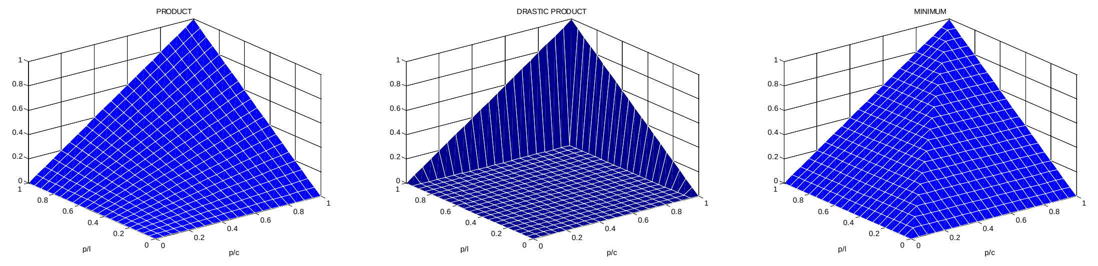
Expression (2): $\wp_D$ is the membership degree using the Drastic Product.

$$\wp_D = \begin{cases} 0, \text{ if } p<c \text{ and } p<l \\ p/c, \text{ if } p=l \\ p/l, \text{ if } p=c \end{cases} \tag{2}$$

Expression (3): $\wp_M$ is the membership degree using the Min.

$$\wp_M = Min\left(p/c, p/l\right) \tag{3}$$

These expressions may be used only if the uncertainty of the output is not null, to avoid divisions by zero. In other words, expressions (1), (2) and (3) are not valid if a DMU has no uncertainty in its output.



(a) (b) (c)

Figure 3 - Representations of the T Norms: (a) the Product; (b) the Drastic Product; (c) the Min.

Table 1 shows the results of membership degrees calculations for the DMUs of Figure 2, where O_p and O_o are the output values for the pessimistic and optimistic frontiers, respectively, and I is the input value.

DMU	I	O_p	O_o	c	l	p	$\wp_P$	$\wp_D$	$\wp_M$
A	1	1	2	1	1	1	1,00	1,00	1,00
B	2	1	2	1	2	0	0,00	0,00	0,00
C	4	2	4	2	4	0	0,00	0,00	0,00
D	4	2	6	4	4	2	0,25	0,00	0,50
E	4	4	6	2	4	2	0,50	0,50	0,50
F	5	5	10	5	5	5	1,00	1,00	1,00
G	6	4	10	6	5	5	0,83	0,83	0,83

Table 1 - Membership degrees regarding the fuzzy DEA frontier.

From the algebraic properties of the T Norms follows that $\wp_D \leq \wp_P \leq \wp_M$.
If we want to choose only one of these norms, we should prefer the T Norm “Product”. The reason for this choice is that the $\wp_P$ value is neither the largest, nor the smallest value of the membership degree. That is, it is not too much benevolent, or to much aggressive.

4. Algebraic Calculation of the Membership Degree

The previous calculations are based on a geometrical definition, which is feasible only for very simple models. In order to obtain an expression that might be used for multidimensional general models, in which only one output is imprecise, it is essential to change the geometric terms in equation (1), (2) and (3) into variables that might be derived from the classic DEA models.

For the case of one imprecise output, considering the classic DEA definitions for output oriented models, and also remembering that for that case the efficiencies are greater than one (BCC DEA model), equations (4) and (5) can be rewritten for O_p and O_o, that are the output values for the pessimistic and optimistic frontiers, where

– PO_p and PO_o are the output targets on the pessimistic and optimistic frontiers, i.e., the projected output on the optimistic and pessimistic frontiers;

– Eff_p is the efficiency calculated using the lower output values, i.e., the efficiency related to the pessimistic frontier;

– Eff_o is the efficiency calculated using the upper output values, i.e., the efficiency related to the optimistic frontier.

$$Eff_p = \frac{PO_p}{O_p} \tag{4}$$

$$Eff_o = \frac{PO_o}{O_o} \tag{5}$$

With the purpose of avoiding misunderstandings, Eff_o and Eff_p should not be named optimistic and pessimistic efficiencies, as there is no guarantee that $Eff_o \geq Eff_p$.

From the geometrical representation we easily obtain $l = PO_o - PO_p = O_oEff_o - O_pEff_p$ and $c = O_o - O_p$. In a situation where the DMU is partially contained by the fuzzy frontier, p is the difference between the optimistic output and the output target on the pessimistic frontier, which is a positive number. If the DMU is totally outside the fuzzy frontier (except by a possible single point), the difference above is negative or zero. In this situation the membership degree must be zero, and p must also equal zero to obtain this result. Expression (6) formalizes the equation for p.

$$p = \begin{cases} O_o - O_pEff_p, & \text{if } O_o - O_pEff_p \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{6}$$

From the previous relationships, it is possible to derive the expression that represents algebraically the membership degree $\wp_P$, $\wp_D$ and $\wp_M$ which are shown in (7), (8) and (9), respectively.

$$\wp_P = \begin{cases} \dfrac{\left(O_o - O_pEff_p\right)^2}{\left(O_oEff_o - O_pEff_p\right)\left(O_o - O_p\right)}, & \text{if } O_o - O_pEff_p \geq 0 \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

$$\wp_D = \begin{cases} 0, \text{ if } \left(\left(Eff_p > 1 \text{ and } Eff_o > 1\right) \text{ or } \left(O_o - O_pEff_p < 0\right)\right) \\ \dfrac{O_o - O_pEff_p}{O_o - O_p}, & \text{if } Eff_o = 1 \\ \dfrac{O_o - O_pEff_p}{O_oEff_o - O_pEff_p}, & \text{if } Eff_p = 1 \end{cases} \tag{8}$$

$$\wp_M = \begin{cases} Min\left(\dfrac{O_o - O_pEff_p}{O_o - O_p}, \dfrac{O_o - O_pEff}{O_oEff_o - O_pEff_{p_o}}\right), \text{if } O_o - O_pEff_p \geq 0 \\ 0, \text{ otherwise} \end{cases} \tag{9}$$

Table 2 details the algebraic calculation of $\wp_P, \wp_D$ and $\wp_M$. It should be noticed that, due to the output orientation model, the inefficient DMUs produce an efficient score greater than one.

DMU	I	O_p	O_o	Eff_p	Eff_o	c	l	p	$\wp_P$	$\wp_D$	$\wp_M$
A	1	1	2	1,00	1,00	1	1	1	1,00	1,00	1,00
B	2	1	2	2,00	2,00	1	2	0	0,00	0,00	0,00
C	4	2	4	2,00	2,00	2	4	0	0,00	0,00	0,00
D	4	2	6	2,00	1,33	4	4	2	0,25	0,00	0,50
E	4	4	6	1,00	1,33	2	4	2	0,50	0,50	0,50
F	5	5	10	1,00	1,00	5	5	5	1,00	1,00	1,00
G	6	4	10	1,25	1,00	6	5	5	0,83	0,83	0,83

Table 2 - Computed values based on expressions (4) to (9).

5. Fuzzy Frontier with One Imprecise Input

The case of one imprecise input may be analyzed in a way similar to that of one imprecise output. In that case, the optimistic input, I_o, is the smallest value of the input, and the pessimistic input, I_p is the largest one. An optimistic frontier is obtained when optimistic inputs are considered for all DMUs and, conversely, a pessimistic frontier is characterized when pessimistic inputs are assumed for all DMUs.

Figure 4 depicts the optimistic and the pessimistic frontiers for the case of one imprecise input. In this figure I_o, I_p, PI_o and PI_p represent, respectively, the optimistic and pessimistic values of the imprecise input, and the input target values on the optimistic and pessimistic frontiers. Conversely for the output-oriented situation, now the line segment representing the DMU with imprecise input value lies in the horizontal position.

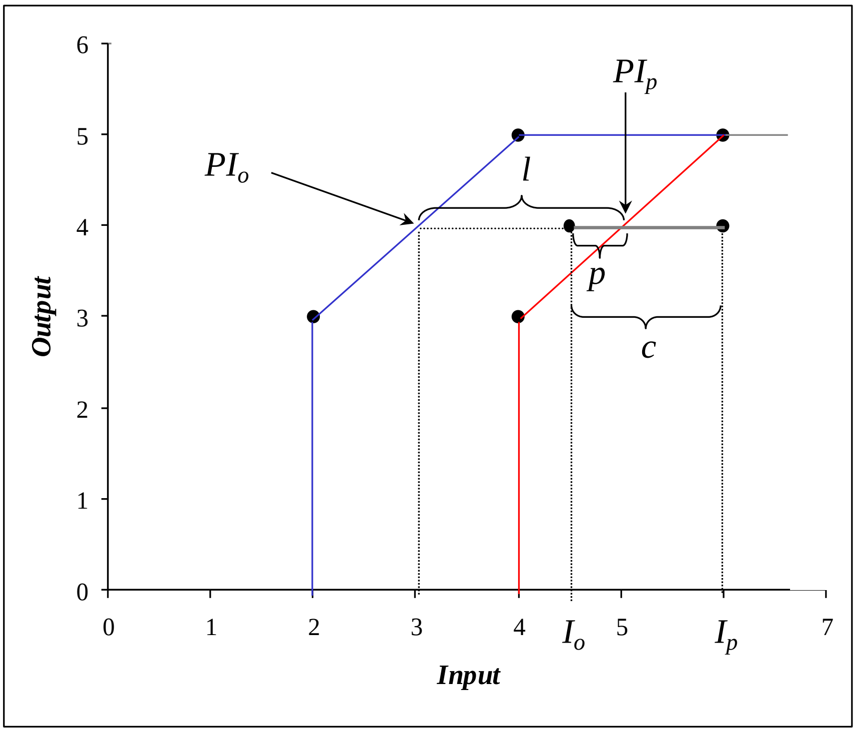


Figure 4 - Optimistic and pessimistic frontiers for the input oriented BCC model.

In a similar way to the imprecise output membership degree, we show following the membership degree based on the T Norms for the imprecise input case. Expressions (10), (11) and (12) present the membership degrees for the situations $\wp_P, \wp_D$ and $\wp_M$.

$$\wp_P = \begin{cases} \dfrac{\left(I_p Eff_p - I_o\right)^2}{\left(I_p Eff_p - I_o Eff_o\right)\left(I_p - I_o\right)}, & if\, I_p Eff_p - I_o \geq 0 \\ 0, & otherwise \end{cases} \tag{10}$$

$$\wp_D = \begin{cases} 0,\ if\ \left(\left(Eff_p < 1\ and\ \ Eff_o < 1\right)\ or\ \left(I_p Eff_p - I_o < 0\right)\right) \\ \dfrac{I_p Eff_p - I_o}{I_p - I_o},\ if\ Eff_o = 1 \\ \dfrac{I_p Eff_p - I_o}{I_p Eff_p - I_o Eff_o},,\ if\ \ Eff_p = 1 \end{cases} \tag{11}$$

$$\wp_M = \begin{cases} Min\left(\dfrac{I_p Eff_p - I_o}{I_p - I}, \dfrac{I_p Eff_p - I_o}{I_p Eff_p - I_o Eff_o}\right),\ if\ \ I_p Eff_p - I_o \geq 0 \\ 0,\ otherwise \end{cases} \tag{12}$$

6. Conclusions

The approach proposed in this paper, in order to incorporate uncertainties in classic DEA models, has the advantage of neither using any particular probability distribution for the variable uncertainties, nor a fuzzy function for them. Besides, it is at the same time mathematically simple, since the results are obtained by simple algebraic calculations (after calculating DEA classic frontiers), in opposition to the change of variable used in Despotis and Smirlis (2002), and without the need of using Inference Fuzzy System.

The location of the interval DEA frontier allows the geometrically building of a membership function and, consequently, obtaining a fuzzy result that uses the membership concept without the need of using the classical membership functions. As a matter of fact, the geometrical considerations used to define the membership index implicitly employed uniform membership functions. Those functions have constant values. One of them is equal to the inverse of the length of the DMU representative segment. The other one has his value inverse to the length of the segment determined by the pessimistic and optimistic targets. A generalization of this approach would consist on admitting other forms of membership functions, like triangular membership functions, and replace the length of segments by integrals of the membership functions along these segments.

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