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Analytically pricing European options under the CGMY model

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Analytically pricing European options under the CGMY model

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This thesis is presented as part of the requirements for the conferral of the degree:

Master of Science-Research

The University of Wollongong

School of Mathematics and Applied Statistics

August, 2014

Declaration

I, Meiyu, Du, declare that this thesis submitted in partial fulfilment of the requirements for the conferral of the degree Master of Science-Research, from the University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. This document has not been submitted for qualifications at any other academic institution.

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Meiyu, Du

February 29, 2016

Abstract

This thesis investigates the pricing of European-style options under the CGMY model, which can fit the empirically observed data in financial market better than the B-S (Black-Scholes) model. Under this model, the price of options is governed by a FPDE (fractional partial differential equation) with two spatial-fractional derivatives defined in the Weyls sense. In comparison with the derivative of integer order, the fractional-order derivative requires the function value over the entire domain rather than its value at one particular point. This has added an additional degree of difficulty when either the analytical solution or the numerical method is attempted. Albeit difficult, we have managed to derive a closed-form analytical solution for European options under the CGMY model. Based on the solution, we further discuss its asymptotic behaviors and the put-call parity under the adopted CGMY model. Finally, we propose an efficient numerical evaluation technique for the current formula so that it can be easily used in trading practice.

Acknowledgments

Foremost, I would like to convey my sincerest gratitude and appreciation to my supervisor, Dr. Wenting Chen, for her insightful supervision and continuous encouragement throughout my study. I appreciate all her contributions of time and ideas to make my research studies productive and stimulating. Her passion in science and rigorous attitude towards research have exceptionally inspired and enriched my growth as a student, a researcher. I would give my best regards and respect to her.

My appreciation also goes to my parents and my girlfriend for their understanding and supporting, especially to my parents who have encouraged me to be disciplined and systematic in my work ever since I was young.

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Chapter 1

Introduction and background

1.1 Option derivatives

In finance, an option is a contract which gives the buyer the right, but not the obligation, to buy or sell an underlying asset or instrument at a specified price on or before a specified date. Generally speaking, the buyer pays a premium to the seller for this right. The price in the contract is known as the exercise price or the strike price, and the specified date in the contract is defined as the expiration date, exercise date or maturity date. An option which conveys the right to buy something at a specific price is referred to as a call. Similarly, an option which conveys the right to sell something at a specific price is referred to as a put [23].

According to the exercise time, options can be classified as European style or American style. A European option can be exercised only at the expiration date, while an American option can be exercised at any time before the expiration date [23].

Options can also be divided into vanilla options and exotic options. A vanilla option is a normal call or put option that has standardized terms and no special or unusual features. On the other hand, an exotic option is an option whose payoff function is more general and complicated than the vanilla call or put options. Typical examples include the barrier options, Asian options, digital options and so on [23].

Option derivatives have been known for many centuries. However, the trading

activities have increased significantly only since 1973 when the B-S (Black-Scholes) model was established [28]. From then on, options were issued with standardized terms and traded through a guaranteed clearing house at the CBOE (Chicago Board Options Exchange). Today, many options are traded through clearing houses on regulated options exchanges throughout the world [55].

1.2 Mathematical background

One of the major challenges in today's financial industry is to determine the prices of financial derivatives efficiently and accurately. Such evaluations require advanced mathematical methods. In this section, we shall briefly review the mathematical background that is employed as a basic tool for the studies in the current thesis.

1.2.1 Fractional derivatives

The fractional derivative was originated in a letter from Leibniz to L'Hôpital in 1695:" *Can the meaning of derivatives with integer orders be generalized to derivatives with non-integer orders?*" Later on, many mathematicians have contributed to this topic [30, 46, 48], and thus the fractional derivative has been developed.

Recently, many models are formulated in terms of fractional derivatives, such as control process, viscoelasticity, signal process, and anomalous diffusion. There exists a vast literature on the use of different types of fractional derivatives. The most popular ones include the Riemann-Liouville fractional derivative, the Caputo fractional derivative and the Weyls fractional derivative. We shall briefly review them in this subsection.

The Riemann-Liouville fractional derivative was introduced by Liouville in 1832 [40]. It is defined upon the Lagrange rule for differential operators. It is formally defined in [30] as follows.

Definition 1.2.1 *The left-side Riemann-Liouville derivative is*

$$_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}f(t)dt, \quad n-1 \leq \alpha < n,$$

where $\alpha \in R^+$ and n is the least integer greater than α, $f(x)$ has derivatives of at least m orders and $m \geq [\alpha] = n - 1$.

Definition 1.2.2 *The right-side Riemann-Liouville derivative is*

$$_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_x^b (t-x)^{n-\alpha-1}f(t)dt, \quad n-1 \leq \alpha < n,$$

where α, n and $f(x)$ are the same defined as Definition 1.2.1.

The Caputo fractional derivative was introduced by Caputo in [15]. In contrast to the Riemann-Liouville fractional derivative, it is not necessary to define the fractional order initial conditions when solving FPDEs (fractional partial differential equations) involving the Caputo fractional derivative [15]. The Caputo fractional derivative can also be defined from the left side and the right side. They are formally defined in [5] as follows.

Definition 1.2.3 *The left-side Caputo derivative is*

$$_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_a^x (x-t)^{n-\alpha-1}\frac{d^n f(t)}{dt^n}dt, \quad n-1 \leq \alpha < n,$$

where α, n and $f(x)$ are the same defined as Definition 1.2.1, $f^{(k)}(a) = 0, k = 0, 1, ..., n-1$.

Definition 1.2.4 *The right-side Caputo derivative is*

$$_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\int_x^b (t-x)^{n-\alpha-1}\frac{d^n f(t)}{dt^n}dt, \quad n-1 \leq \alpha < n,$$

where α, n and $f(x)$ are the same defined as Definition 1.2.1, $f^{(k)}(a) = 0, k = 0, 1, ..., n-1$.

We remark that in the above definitions, if $\alpha \geq 0$, they are fractional derivatives, if $\alpha < 0$, they become fractional integrals. It is remarked that if a, b take on $-\infty$ and $+\infty$, respectively, the Weyls fractional derivatives can be obtained in [31] as follows.

Definition 1.2.5 *The left-side Weyls derivative is*

$$_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^x f(y)(x-y)^{n-\alpha-1}dy, \qquad n-1 \leq \alpha < n.$$

Definition 1.2.6 *The right-side Weyls derivative is*

$$_xD_\infty^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_x^\infty f(y)(y-x)^{n-\alpha-1}dy, \qquad n-1 \leq \alpha < n.$$

Here, **Definition 1.2.5** is referred to as "forward Weyl's fractional derivative" whereas **Definition 1.2.6** is referred to as "backward Weyl's fractional derivative" [31]. Correspondingly, there are two fractional integral definitions.

It should be remarked that there is a relationship between the forward fractional integral and the forward fractional derivative, i.e.,

$$_{-\infty}D_x^u f(x) = D^m[_{-\infty}J_x^{-v}f(x)] = D^m\frac{1}{\Gamma(v)}\int_{-\infty}^x f(y)(x-y)^{v-1}dy,$$

$$= \frac{1}{\Gamma(m-u)}\frac{d^m}{dx^m}\int_{-\infty}^x (x-y)^{m-u-1}f(y)dy,$$

where $_{-\infty}J_x^{-v}f(x)$ is the forward fractional integral defined as

$$_{-\infty}J_x^{-v}f(x) = \frac{1}{\Gamma(v)}\int_{-\infty}^x f(y)(x-y)^{v-1}dy,$$

and $v = m - u > 0$.

Similarly, we can obtain the relationship between the backward fractional integral and the backward fractional derivative.

Now, let D_t^α be any of the α-order fractional derivative mentioned above. According to [12], it is known that D_t^α has the following properties.

1. $D_t^{\alpha+\beta}[f(t)] = D_t^\alpha D_t^\beta [f(t)];$

2. Linearity: $D_t^\alpha[af(t) + bg(t)] = aD_t^\alpha[f(t)] + bD_t^\alpha[g(t)],$ where a, b are constants;

3. Leibniz rules: $D_t^\alpha[f(t)g(t)] = \Sigma_{n=0}^\infty \binom{\alpha}{n} D_t^n[g(t)]D_t^{\alpha-n}[f(t)].$

1.2.2 Fourier transform

The Fourier transform is a useful mathematical transformation employed to transform signals between the time (or spatial) domain and the frequency domain. This transform has many applications in physics and engineering [13].

Definition 1.2.7 *The Fourier transform is defined by*

$$F[f(x)] = F(\xi) = \int_{-\infty}^{\infty} f(x)e^{ix\xi}dx,$$

where $f(x)$ is analytic on $(-\infty, +\infty)$, $F[\cdot]$ denotes the Fourier transform, and ξ is the Fourier transform parameter.

Definition 1.2.8 *The inverse Fourier transform is given by*

$$f(x) = F^{-1}[F(\xi)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\xi)e^{-ix\xi}d\xi.$$

The Fourier transform is an invertible, linear transform. It has the following important properties [13].

1. Linearity property: $F[af(x) + bg(x)] = aF[f(x)] + bF[g(x)],$ and $F^{-1}[af(x) + bg(x)] = aF^{-1}[f(x)] + bF^{-1}[g(x)];$

2. Shifting property: $F[f(x - x_0)] = e^{ix_0\xi}F[f(x)],$ and $F[f(x + x_0)] = e^{-ix_0\xi}F[f(x)];$

3. Derivative property: $F[f^{(k)}(x)] = (-i\xi)^k F[f(x)],$ where $f(x) \to 0$ for $|x| \to \infty;$

4. Exponential function property: $F[e^{ax}f(x)] = F(\xi - ia);$

5. Odd-even property: If $f(x)$ is an (odd) even function, $F(\xi)$ is an (odd) even function as well.

One of the most important properties of the Fourier transform is the so-called convolution theorem. Convolution is similar to the cross-correlation. It has applications in many areas including the probability, statistics, computer vision, image and signal processing, electrical engineering, and differential equations [27].

Theorem 1 *(Convolution Theorem)* $F[f_1(x) * f_2(x)] = F[f_1(x)] \cdot F[f_2(x)]$ *and* $F^{-1}[F(f_1(x)) \cdot F(f_2(x))] = f_1(x) * f_2(x)$, *where* $f_1(x) * f_2(x)$ *is the convolution of the function of* f_1 *and* f_2, *i.e.,* $f_1(x) * f_2(x) = f(x) = \int_{-\infty}^{\infty} f_1(t) f_2(x - t) dt$.

According to this theorem, the following corollary on the joint pdf (probability density function) can be proved.

Corollary 1.2.1 *The pdf of the sum of two independent continuous random variables* U *and* V *is the convolution of their separate density functions, i.e.,*

$$f_{U+V}(x) = \int_{-\infty}^{+\infty} f_U(y) f_V(x - y) dy = (f_U * f_V)(x),$$

where f_U *and* f_V *are the pdfs of* U *and* V, *respectively.*

Proof.   Let $f_{U,V}(x, y)$ and $F_{U,V}(x, y)$ be the joint pdf and the joint cdf (cumulative distribution function) of U and V, respectively. Therefore, for $Z = U + V$, we have

$$\begin{aligned} F_Z(z) &= P(Z \le z) = P(U + V \le z), \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{z-y} f_{U,V}(x, y) dx dy, \end{aligned}$$

where $F_Z(z)$ is the cdf of Z. Now, let $u = x + y$, and we obtain

$$\begin{aligned} F_Z(z) &= \int_{-\infty}^{+\infty} \int_{-\infty}^{z} f_{U,V}(u - y, y) du dy, \\ &= \int_{-\infty}^{z} \int_{-\infty}^{+\infty} f_{U,V}(u - y, y) dy du. \end{aligned}$$

Therefore, the pdf of z, i.e., $f_z(z)$ can be determined as

$$
\begin{aligned}
f_Z(z) &= \frac{d(F_Z(z))}{dz} = \frac{d(\int_{-\infty}^{z} \int_{-\infty}^{+\infty} f_{U,V}(u-y,y)dydu)}{dz}, \\
&= \int_{-\infty}^{+\infty} f_{U,V}(z-y,y)dy.
\end{aligned}
$$

Since U and V are independent, we have

$$
\begin{aligned}
f_Z(z) &= \int_{-\infty}^{+\infty} f_{U,V}(z-y,y)dy, \\
&= \int_{-\infty}^{+\infty} f_U(z-y)f_V(y)dy, \\
&= f_U(x) * f_V(y).
\end{aligned}
$$

This has completed the proof of this corollary.

On the other hand, it should be remarked that the fractional derivatives usually have simpler behaviors in the Fourier space [31].

Corollary 1.2.2 *The Fourier transforms of fractional integrals are given by $F\left[_{-\infty}J_x^\alpha f(x)\right] = (-i\xi)^{-\alpha}F(\xi)$, and $F\left[_x J_\infty^\alpha f(x)\right] = (i\xi)^{-\alpha}F(\xi)$, where $f(x)$ is analytic on $(-\infty,\infty)$, $\alpha < 0$, ξ is the Fourier transform parameter, and $F(\xi)$ is the Fourier transform of $f(x)$.*

Proof. According to the definition of fractional integrals, it is known that

$$
_{-\infty}J_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} f(\xi)d\xi,
$$

which can be rewritten as

$$
\begin{aligned}
_{-\infty}J_x^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty (y)^{\alpha-1} f(x-y)dy, \\
&= \frac{1}{\Gamma(\alpha)} \int_{-\infty}^\infty (y)^{\alpha-1} f(x-y)H(y)dy,
\end{aligned}
$$

with $H(\cdot)$ being the step function defined as

$$H(y) = \begin{cases} 1, & y \geq 0; \\ 0, & y < 0. \end{cases}$$

It is clear now, $_{-\infty}J_x^\alpha f(x)$ is the convolution of $x^{\alpha-1}H(x)/\Gamma(\alpha)$ and $f(x)$. Therefore, according to **Theorem 1**, we obtain

$$F\left[_{-\infty}J_x^\alpha f(x)\right] = F\left[\frac{x^{\alpha-1}H(x)}{\Gamma(\alpha)}\right]F\left[f(x)\right].$$

On the other hand, applying the Fourier transform directly to $\dfrac{x^{\alpha-1}H(x)}{\Gamma(\alpha)}$, we obtain

$$F\left[\frac{x^{\alpha-1}H(x)}{\Gamma(\alpha)}\right] = \mid \xi \mid^{-\alpha} \exp\left[(i\alpha\pi/2)sgn(\xi)\right], \tag{1.2.1}$$

where $sgn(\xi)$ is the sign function defined as

$$sgn(\xi) = \begin{cases} 1, & \xi > 0; \\ 0, & \xi = 0; \\ -1, & \xi < 0. \end{cases} \tag{1.2.2}$$

For the case of $\xi > 0$, according to Euler's theory, we have

$$\begin{aligned} \exp[(i\alpha\pi/2)sgn(\xi)] &= \exp(i\alpha\pi/2), \\ &= [\exp(-i\pi/2)]^{-\alpha}, \\ &= [\cos(-\pi/2) + i\sin(-\pi/2)]^{-\alpha}, \\ &= (-i)^{-\alpha}, \end{aligned}$$

which, combined with **(1.2.1)**, yields

$$F\left[\frac{x^{\alpha-1}H(x)}{\Gamma(\alpha)}\right] = (-i\xi)^{-\alpha}. \tag{1.2.3}$$

For the case of $\xi < 0$, we have

$$
\begin{aligned}
\exp[(i\alpha\pi/2)sgn(\xi)] &= \exp(-i\alpha\pi/2), \\
&= [\exp(i\pi/2)]^{-\alpha}, \\
&= [\cos(\pi/2) + i\sin(\pi/2)]^{-\alpha}, \\
&= (i)^{-\alpha},
\end{aligned}
$$

which, combined with $(\mathbf{1.2.1})$, yields,

$$
F\left[\frac{x^{\alpha-1}H(x)}{\Gamma(\alpha)}\right] = (-i\xi)^{-\alpha}. \tag{1.2.4}
$$

For the case of $\xi = 0$, we have

$$
F\left[\frac{x^{\alpha-1}H(x)}{\Gamma(\alpha)}\right] = (-i\xi)^{-\alpha}. \tag{1.2.5}
$$

From $(\mathbf{1.2.3})$, $(\mathbf{1.2.4})$ and $(\mathbf{1.2.5})$, one can draw the conclusion that

$$
F\left[{}_{-\infty}J_x^\alpha f(x)\right] = (-i\xi)^{-\alpha}F(\xi).
$$

The proof of this corollary is thus completed.

Similarly, we can show that $F\left[{}_xJ_\infty^\alpha f(x)\right] = (i\xi)^{-\alpha}F(\xi)$.

On the other hand, when the Fourier transform is applied to fractional derivatives, we have the following corollary.

Corollary 1.2.3 *The Fourier transforms of the Weyls fractional derivatives are given by $F\left[{}_{-\infty}D_x^\alpha f(x)\right] = (-i\xi)^\alpha F(\xi)$, and $F\left[{}_xD_\infty^\alpha f(x)\right] = (i\xi)^\alpha F(\xi)$, where $f(x)$ is analytic on $(-\infty, \infty)$, $\alpha > 0$, ξ is the Fourier transform parameter, $\xi \neq 0$ and $F(\xi)$ is the Fourier transform of $f(x)$.*

Proof.   From **Definition 1.2.5**, it is known that

$$
{}_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^x f(\xi)(x-\xi)^{n-\alpha-1}d\xi.
$$

Now, let $y = x - \xi, n - \alpha = -p \quad (-1 < p < 0)$. We have

$$
\begin{aligned}
_{-\infty}D_x^\alpha f(x) &= {}_{-\infty}D_x^{n+p} f(x), \\
&= \frac{1}{\Gamma(-p)} \frac{\partial^n}{\partial x^n} \int_\infty^0 f(x-y)(y)^{-p-1}(-dy), \\
&= \frac{1}{\Gamma(-p)} \frac{\partial^n}{\partial x^n} \int_0^\infty f(x-y)(y)^{-p-1}dy.
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
_{-\infty}D_x^p f(x) &= \frac{1}{\Gamma(-p)} \int_0^\infty f(x-y)(y)^{-p-1}dy, \\
&= \frac{1}{\Gamma(-p)} \int_{-\infty}^\infty f(x-y)(y)^{-p-1}H(y)dy, \quad (1.2.6)
\end{aligned}
$$

where $H(\cdot)$ is the Heviside function. From **(1.2.6)**, it is clear that $_{-\infty}D_x^p f(x)$ is the convolution of $x^{-p-1}H(x)/\Gamma(-p)$ and $f(x)$.

According to **Theorem 1**, we have

$$
\begin{aligned}
F[_{-\infty}D_x^p f(x)] &= F\left[\frac{x^{-p-1}H(x)}{\Gamma(-p)}\right] F[f(x)], \\
&= |\xi|^p \exp[(-ip\pi/2)sgn(\xi)]F(\xi), \quad (1.2.7)
\end{aligned}
$$

where $sgn(\xi)$ is the sign function defined in **(1.2.2)**. In fact, **(1.2.7)** can be further simplified according to the following two different choices of ξ. Here, $\xi \neq 0$ because there would be meaningless if $\xi = 0$ since $p < 0$.

For the case of $\xi > 0$, according to Euler's theory, it is known that

$$
\exp[(-ip\pi/2)sgn(\xi)] = (-i)^p,
$$

and thus, we have

$$
F\left[\frac{x^{-p-1}H(x)}{\Gamma(-p)}\right] = (-i\xi)^p.
$$

On the other hand, for $\xi < 0$, we have

$$\exp[(-ip\pi/2)sgn(\xi)] = \exp(ip\pi/2) = [\exp(i\pi/2)]^p = [\cos(\pi/2) + i\sin(\pi/2)]^p = (i)^p.$$

Combining the results of the two cases, we have

$$F\left[\frac{x^{-p-1}H(x)}{\Gamma(-p)}\right] = (-i\xi)^p,$$

and thus

$$F[_{-\infty}D_x^p f(x)] = (-i\xi)^p F(\xi).$$

It is clear at this stage that

$$
\begin{aligned}
F[_{-\infty}D_x^\alpha f(x)] &= F\left[D^n[_{-\infty}D_x^p f(x)]\right], \\
&= (-i\xi)^{n+p}F(\xi), \\
&= (-i\xi)^\alpha F(\xi).
\end{aligned}
$$

This has completed the proof of this corollary.

Similarly, we can show that $F[_xD_\infty^\alpha f(x)] = (i\xi)^\alpha F(\xi)$. The details are omitted here.

1.2.3 Laplace transform

The Laplace transform is a widely used integral transform technique with many applications in the physics and engineering area. It is a linear operator of a function $f(t)$ with a real argument $t(t \geq 0)$ that transforms $f(t)$ to a function $F(s)$ with complex argument s, i.e.,

$$F(s) = \mathcal{L}_s\{f(t)\} = \int_0^\infty e^{-st}f(t)dt.$$

The Laplace transform can be formally defined in [39] as follows.

Definition 1.2.9 *The Laplace transform of a function $f(t)$ is given by*

$$\mathcal{L}_s\{f(t)\} = \int_{-\infty}^{\infty} e^{-st} f(t)dt,$$

where $\mathcal{L}_s\{f(t)\}$ is the Laplace transform of $f(t)$, s is the Laplace transform parameter, and is a complex number.

The inverse Laplace transform is known as the "Bromwich integral", which is defined as follows.

Definition 1.2.10 *The inverse Laplace transform of $\mathcal{L}_s\{f(t)\}$ is given by*

$$f(t) = \mathcal{L}_t^{-1}\{\mathcal{L}_s\{f(t)\}\} = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\gamma-iT}^{\gamma+iT} e^{st}\mathcal{L}_s\{f(t)\}\,ds.$$

The Laplace transform has a number of properties that make it useful for analyzing linear dynamical systems [39], i.e.,

1. Linearity property: $\mathcal{L}_s\{af(t) + bg(t)\} = a\mathcal{L}_s\{f(t)\} + b\mathcal{L}_s\{g(t)\}$;

2. Frequency domain differentiation property: $\mathcal{L}_s\{t^n f(t)\} = (-1)^n\left(\mathcal{L}_s\{f(t)\}\right)^{(n)}$;

3. General differentiation property: $\mathcal{L}_s\{f^{(n)}(t)\} = s^n\mathcal{L}_s\{f(t)\} - \sum_{k=1}^{n} s^{k-1} f^{(n-k)}(0)$;

4. Time scaling property: $\mathcal{L}_s\{f(at)\} = \frac{1}{|a|}\mathcal{L}_{\frac{s}{a}}\{f(t)\}$;

5. Convolution property: $\mathcal{L}_s\{f(t) * g(t)\} = \mathcal{L}_s\{f(t)\}\cdot\mathcal{L}_s\{g(t)\}$, here
   $f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ for $f, g \in [0, +\infty) \to R$.

We remark that in the above properties, $\mathcal{L}_s\{f(t)\}$ and $\mathcal{L}_s\{g(t)\}$ are the Laplace transform of $f(t)$ and $g(t)$, respectively.

1.2.4 Mellin transform

The Mellin transform is an integral transform that can be regarded as the multiplicative version of two-sided Laplace transform. This integral transform is closely related to the theory of Dirichlet series, and is often used in the number theory and the theory of asymptotic expansions [49]. The Mellin transform is formally defined in [32] as follows.

Definition 1.2.11 *The Mellin transform of a function $f(x)$ is given by*

$$M\{f(x)\}_s = \varphi(s) = \int_0^\infty x^{s-1} f(x) dx,$$

where $\varphi(s)$ is the Mellin transform of $f(x)$, and s is the Mellin transform parameter.

Definition 1.2.12 *The inverse Mellin transform of $\varphi(s)$ is given by*

$$M^{-1}\{\varphi(s)\}_x = f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-s} \varphi(s) ds.$$

The Mellin transform is also very useful in the applied mathematics area because of the following properties [35].

1. Scaling property: $M\{f(at)\}_s = \int_0^{+\infty} f(at) t^{s-1} dt = a^{-s} \int_0^{+\infty} f(x) x^{s-1} dx = a^{-s} \varphi(s)$;

2. Multiplication by t^a: $M\{t^a f(t)\}_s = \int_0^{+\infty} f(t) t^{(s+a)-1} dt = \varphi(s+a)$;

3. Raising the independent variable to a real power: $M\{f(t^a)\}_s = \int_0^{+\infty} f(t^a) t^{s-1} dt = \int_0^{+\infty} f(x) x^{\frac{s}{a}-\frac{1}{a}} \left(\frac{1}{a} x^{\frac{1}{a}-1} dx\right) = a^{-1} \varphi\left(\frac{s}{a}\right)$, where $a > 0$;

4. Inverse of independent variable: $M\{t^{-1} f(t^{-1})\}_s = \varphi(1-s)$;

5. Multiplication by $\ln t$: $M\{\ln t f(t)\}_s = \frac{d}{ds} \varphi(s)$;

6. Multiplication by a power of $\ln t$: $M\{(\ln t)^k f(t)\}_s = \frac{d^k}{ds^k} \varphi(s)$.

1.2.5 The Lévy process

In probability theory, the Lévy process, named after the French mathematician Paul Lévy, is a stochastic process with independent, stationary increments. It represents the motion of a point whose successive displacements are random and independent, and statistically identical over different time intervals of the same length [50]. A Lévy process can thus be viewed as the continuous time analog of a random walk. The most well known examples of Lévy processes are the Brownian motion and the Poisson process [50]. The Lévy process is formally defined in [2] as follows.

Definition 1.2.13 *A stochastic process $X = \{X_t : t \geq 0\}$ is a Lévy process if it has the following properties:*

1. *$X_0 = 0$, almost surely;*

2. *Independence of increments: For any $0 \leq t_1 < t_2 < \cdots < t_n < \infty, X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent;*

3. *Stationary increments: For any $s < t, X_t - X_s$ has the same distribution as X_{t-s};*

4. *Continuity in probability: For any $\epsilon > 0$ and $t \geq 0$ it holds that $\lim_{h \to 0} P(|X_{t+h} - X_t| > \epsilon) = 0$.*

We remark that if X is a Wiener process, the probability distribution of $X_s - X_t$ is normally distributed with mean 0 and variance $s - t$. If X is the Poisson process, the probability distribution of $X_s - X_t$ is a Poisson distribution with mean $\lambda(s-t)$, where $\lambda > 0$ is the "intensity" or "rate" of the process.

It should also be noticed that in any Lévy process with finite moments, the nth moment $\mu_n(t) = E(X_t^n)$ is a polynomial function of t, and it satisfies

$$\mu_n(t + s) = \sum_{k=0}^{n} \binom{n}{k} \mu_k(t) \mu_{n-k}(s).$$

On the other hand, it should be pointed out that the distribution of a Lévy process is characterized by its characteristic function, which can be determined by the Lévy Khinchine formula [8]. In specific, if $(X_t)_{t\geq0}$ is a Lévy process, then its characteristic function is given by

$$\mathbb{E}\left[e^{iuX_t}\right] = \exp\left(bitu - t\frac{1}{2}\sigma^2 u^2 + t \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbf{I}_{|x|<1}\right) \Pi(dx)\right),$$

where $b \in \mathbb{R}$, $\sigma^2 \geq 0$, $\mathbf{I}$ is the indicator function and Π is a sigma-finite measure called the Lévy measure of X, satisfying $\int_{\mathbb{R}\setminus\{0\}} \min\{x^2, 1\}\Pi(dx) < \infty$. $\Pi(dx)$ here represents the rate of arrival (intensity) of the Poisson process with jump size x.

We further remark that the Lévy process usually includes three independent components: a linear drift, a Brownian motion and a superposition of independent

(centered) Poisson processes with different jump sizes. These three components can all be determined by the Lévy?Khintchine triplet (b, σ^2, Π) [20].

1.2.6 Special functions

Gamma function

The Gamma function was introduced into mathematics by Euler [1]. For complex numbers with a positive real part, it is defined via a convergent improper integral [24] as follows.

Definition 1.2.14 *The Gamma function is defined as*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt,$$

where $z = \sigma + i\tau$ and $\sigma > 0$.

We remark that the Gamma function is known as the Euler integral of the second kind. Using the integration by part technique, it can be shown that the Gamma function satisfies the following identity, i.e.,

$$\Gamma(z+1) = z\Gamma(z).$$

The Gamma function also satisfies the Euler reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)},$$

which implies

$$\Gamma(\varepsilon - n) = (-1)^{n-1} \frac{\Gamma(-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(n+1-\varepsilon)},$$

and

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

The detailed treatments of the Gamma function can be found in [24].

H-function

The H-function is an important function, because it includes most of the special functions occurring in applied mathematics and statistics as special cases. In 1961, Fox defined the H-function in his paper [36] as follows.

Definition 1.2.15 *The H-function $H(\cdot)$ is given by*

$$
\begin{aligned}
H(cz) = H_{p,q}^{m,n}(cz) &= H_{p,q}^{m,n}\left[cz \,\middle|\, \begin{array}{l} (a_i, A_i)_{i=1,\ldots,p} \\ (b_j, B_j)_{j=1,\ldots,q} \end{array} \right], \\
&= \frac{1}{2\pi i} \int_C \chi(s)(cz)^{-s} ds,
\end{aligned}
$$

where C is a certain contour separating the poles of the two factors in the numerator and $\chi(s)$ is the integral density defined by

$$
\chi(s) = \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{i=1}^n \Gamma(1 - a_i - A_i s)}{\prod_{i=n+1}^p \Gamma(a_i + A_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - B_j s)},
$$

z, c, a_i and b_j are real or complex numbers, all A_i and B_j are positive real numbers, and m, n, p and q are integers satisfying $0 \le m \le q$ and $0 \le n \le p$.

From the above definition, it can be noticed that the H-function is in fact the inverse Meillin transform of $\chi(s)$.

The H-function has the following useful properties [42].

1. Reciprocal property:

$$
H_{p,q}^{m,n}\left[\frac{1}{z} \,\middle|\, \begin{array}{l} (a_i, A_i)_{i=1,\ldots,p} \\ (b_j, B_j)_{j=1,\ldots,q} \end{array} \right] = H_{q,p}^{n,m}\left[z \,\middle|\, \begin{array}{l} (1 - b_j, B_j)_{j=1,\ldots,q} \\ (1 - a_i, A_i)_{i=1,\ldots,p} \end{array} \right];
$$

2. Power property:

$$
H_{p,q}^{m,n}\left[z^c \,\middle|\, \begin{array}{l} (a_i, A_i)_{i=1,\ldots,p} \\ (b_j, B_j)_{j=1,\ldots,q} \end{array} \right] = \frac{1}{c} H_{p,q}^{m,n}\left[z \,\middle|\, \begin{array}{l} a_i, \frac{A_i}{c})_{i=1,\ldots,p} \\ (b_j, \frac{B_j}{c})_{j=1,\ldots,q} \end{array} \right],
$$

where $c > 0$;

3.

$$z^c H_{p,q}^{m,n} \left[z^c \middle| \begin{array}{c} (a_i, A_i)_{i=1,\ldots,p} \\ (b_j, B_j)_{j=1,\ldots,q} \end{array} \right] = H_{p,q}^{m,n} \left[z \middle| \begin{array}{c} (a_i + A_i c, A_i)_{i=1,\ldots,p} \\ (b_j + B_j c, B_j)_{j=1,\ldots,q} \end{array} \right];$$

4. Reduction property:

$$H_{p,q}^{m,n} \left[z \middle| \begin{array}{cc} (a_i, A_i)_{i=1,\ldots,p} \\ (b_j, B_j)_{j=1,\ldots,q-1} & (a_i, A_i) \end{array} \right] = H_{p-1,q-1}^{m,n-1} \left[z \middle| \begin{array}{c} a_i, A_i)_{i=2,\ldots,p} \\ (b_j, B_j)_{j=1,\ldots,q-1} \end{array} \right].$$

It should be remarked that the H-function will exhibit simpler behaviors after some integral transforms are applied [11]. We have the following lemmas.

Lemma 1.2.1 *If all real b_j satisfy that $\dfrac{-b_j}{B_j} < 1$ for $j = 1, \ldots, m$, then the Fourier transform applied to $H_{p,q}^{m,n}(cz)$ admits*

$$\begin{aligned} \mathcal{F}\{H_{p,q}^{m,n}(cz)\} &= \int_0^{+\infty} e^{itz} H_{p,q}^{m,n}(cz)dz, \\ &= \frac{1}{c} H_{q,p+1}^{n+1,m} \left[-\frac{i}{c}t \middle| \begin{array}{cc} (1-b_1-B_1, B_1) & (1-b_j-B_j, B_j)_{j=2,\ldots,q} \\ (0,1) & (1-a_i-A_i, A_i)_{i=1,\ldots,p} \end{array} \right]. \end{aligned}$$

However, if all real b_j satisfy $\dfrac{-b_j}{B_j} \geq 1$ for $j = 1, \ldots, m$, then the Fourier transform becomes

$$\mathcal{F}\{H_{p,q}^{m,n}(cz)\} = \frac{(-1)^I}{c} H_{q+1,p+1}^{n+1,m+1} \left[-\frac{i}{c}t \middle| \begin{array}{ccc} (I,1) & (1-b_j-B_j, B_j)_{j=1,\ldots,q} \\ (I,1) & (1-a_i-A_i, A_i)_{i=1,\ldots,p} & (0,1) \end{array} \right],$$

where $I = max\left\{0, \lfloor \dfrac{-b_j}{B_j} \rfloor \right\}$.

From **Lemma 1.2.1**, it is clear that the Fourier transform of the H-function is another H-function.

Lemma 1.2.2 *The Mellin transform of the H-function admits*

$$\begin{aligned} M\{H_{p,q}^{m,n}(cz)\} &= \int_0^{+\infty} z^{s-1} H_{p,q}^{m,n}(cz)dz, \\ &= \frac{1}{c^s} \frac{\prod_{j=1}^m \Gamma(b_j + B_j s) \prod_{i=1}^n \Gamma(1 - a_j - A_j s)}{\prod_{i=n+1}^p \Gamma(a_i + A_i s) \prod_{j=m+1}^q \Gamma(1 - b_j - B_j s)}. \end{aligned}$$

Lemma 1.2.3 *If all real b_j satisfy $\dfrac{-b_j}{B_j} < 1$ for $j = 1, ..., m$, then the Laplace transform applied to $H_{p,q}^{m,n}(cz)$ admits*

$$
\begin{aligned}
\mathcal{L}\{H_{p,q}^{m,n}(cz)\} &= \int_0^{+\infty} e^{-rz} H_{p,q}^{m,n}(cz)dz, \\
&= \frac{1}{c} H_{q,p+1}^{n+1,m}\left[\frac{1}{c}r \middle| \begin{array}{cc} (1-b_1-B_1, B_1) & (1-b_j-B_j, B_j)_{j=2,...,q} \\ (0,1) & (1-a_i-A_i, A_i)_{i=1,...,p} \end{array}\right].
\end{aligned}
$$

However, if all real b_j satisfy $\dfrac{-b_j}{B_j} \geq 1$ for $j = 1, ..., m$, the Laplace transform becomes

$$
\mathcal{L}\{H_{p,q}^{m,n}(cz)\} = \frac{(-1)^I}{c} H_{q+1,p+1}^{n+1,m+1}\left[\frac{1}{c}r \middle| \begin{array}{ccc} (I,1) & (1-b_j-B_j, B_j)_{j=1,...,q} & \\ (I,1) & (1-a_i-A_i, A_i)_{i=1,...,p} & (0,1) \end{array}\right],
$$

*where I is the same as defined in **Lemma 1.2.1**.*

From **Lemma 1.2.3**, it is clear that the Laplace transform of the H-function is another H-function.

1.3 The option pricing models

1.3.1 The Black-Scholes model

The B-S (Black-Scholes) model was established in 1973 by Black and Scholes [10]. They derived a PDE (partial differential equation), now called the B-S equation, governing the price of option derivatives.

The B-S equation is given by

$$
\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,
$$

where V is the price of the option, S is the price of the stock, t is the current time, and σ and r are the volatility and the annualized risk-free interest rate, respectively.

With appropriate boundary conditions, Black and Scholes also derived the closed-

form analytical solution for European call options as

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)},$$

where

$$d_1 = \frac{1}{\sigma\sqrt{T-t}} \left[\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t) \right],$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t},$$

where $C(S,t)$ is the price of a European call option, and K is the strike price.

By using the put-call parity, the price of the corresponding European put option can be obtained as [38]

$$P(S,t) = Ke^{-r(T-t)} - S + C(S,t),$$
$$= N(-d_2)Ke^{-r(T-t)} - N(-d_1)S.$$

1.3.2 The FMLS model

The FMLS (finite moment log stable) model was introduced in [18]. This model can not only capture the high-frequency empirical probability distribution of the S&P 500 data, but also fit simultaneously volatility smirks at different maturities [25].

Let $V(x,t;\alpha)$ be the price of European-style options, with x being the log underlying price defined as $x = \ln S$ and α being the tail index. Cartea and del-Castillo-Negrete [21] have shown that under the FMLS model, $V(x,t;\alpha)$ satisfies

the following FPDE (fractional partial differential equation):

$$\begin{cases} \dfrac{\partial V}{\partial t} + (r + \dfrac{1}{2}\sigma^\alpha \sec \dfrac{\alpha\pi}{2})\dfrac{\partial V}{\partial x} - \dfrac{1}{2}\sigma^\alpha \sec \dfrac{\alpha\pi}{2} {}_{-\infty}D_x^\alpha V - rV = 0, \\ V(x, T; \alpha) = \Pi(x), \end{cases}$$

where $\Pi(x)$ is the payoff function, defined as $\max(e^x - K, 0)$ and $\max(K - e^x, 0)$ for European calls and puts, respectively, with K being the strike price. ${}_{-\infty}D_x^\alpha(\cdot)$ here is the one-dimensional Weyls factional derivative defined as

$$ {}_{-\infty}D_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial x^n}\int_{-\infty}^{x}\frac{f(y)}{(x-y)^{n-\alpha-1}}dy, \qquad n - 1 \le \Re(\alpha) < n. $$

Recently, Chen et al. [25] derived the closed-form analytical solution for European put options under the FMLS model as

$$ V_p(x, \tau; \alpha) = Ke^{-\gamma\tau}\int_{d_1}^{+\infty} f_{\alpha,0}(|m|)dm - e^x \int_{d_1}^{+\infty} e^{-\tau - \tau^{\frac{1}{\alpha}}m} f_{\alpha,0}(|m|)dm, $$

where $d_1 = \dfrac{x - \ln K - (1 - \gamma)\tau}{\tau^{\frac{1}{\alpha}}}$, and $f_{\alpha,0}(|m|)$ is the Lévy stable density defined as

$$ f_{\alpha,0}(|m|) = \frac{1}{\alpha} H_{2,2}^{1,1}\left[|m| \,\middle|\, \begin{matrix} (1 - \frac{1}{\alpha}, \frac{1}{\alpha}) & (\frac{1}{2}, \frac{1}{2}) \\ (0, 1) & (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right]. $$

1.3.3 The CGMY model

The CGMY model was introduced by Carr et al. in [16]. The CGMY model is in fact a continuous time model that has a unique advantage. This model can have either finite or infinite activities according to the number of price jumps in any time intervals. Besides, the CGMY model further allows the jump component to have finite or infinite variation. In other words, the CGMY model could synthesize the features of other continuous time models. Because of its unique advantages, the CGMY model can be employed to study both the statistical process needed to assess risk and allocate investments and the risk-neutral process used in pricing and

hedging derivatives [16].

Under the risk neutral measure Q, the CGMY model assumes that the log value of the underlying i.e., $x_t = \ln S_t$ follows a stochastic differential equation of a geometric Lévy process

$$d(\ln S_t) = (r - v)dt + dL_t,$$

with solution

$$S_T = S_t e^{(r-v)(T-t)+\int_t^T dL_u}, \tag{1.3.8}$$

where r is the risk-free rate, t is the current time, v is a convexity adjustment so that $E^Q[S_T] = e^{r(T-t)S_t}$, and dL_t is the increment of a Lévy process under the EMM (Equivalent Martingale Measure).

Let $V(x,t)$ be the price of European-style options, with x being the log underlying price and t being the current time. Cartea and del-Castillo-Negrete [21] have shown that under the CGMY model, $V(x,t)$ satisfies the following FPDE system:

$$\begin{cases} \dfrac{\partial V}{\partial t} + (r - v)\dfrac{\partial V}{\partial x} + C\Gamma(-Y)e^{Mx}{}_xD_\infty^Y \left(e^{-Mx}V\right) \\ +C\Gamma(-Y)e^{-Gx}{}_{-\infty}D_x^Y \left(e^{Gx}V\right) = \left[r + C\Gamma(-Y)\left(M^Y + G^Y\right)\right]V, \\ V(x,T) = \Pi(x), \end{cases}$$

where $v = C\Gamma(-Y)\left[(M-1)^Y - M^Y + (G+1)^Y - G^Y\right]$ and $\Pi(x)$ is the payoff function. ${}_{-\infty}D_x^Y(\cdot)$ and ${}_xD_\infty^Y(\cdot)$ here are the one-dimensional Weyls factional derivatives defined in **Subsection 1.2.1**. The parameters C, G, M and Y will be explained in detail in Section 2.2. Due to the complicity of this model, the pricing of option derivatives under this model has not been extensively explored. A closed-form analytical solution for European-style options is the main topic of the current thesis.

1.3.4 The KoBol model

The KoBol model is also a pure jump Lévy process. In comparison to the CGMY model, this process additionally introduces a damping effect in the tails of the LS (Lévy-α-stable) distribution to ensure finite moments and to gain mathematical tractability [21].

The Lévy density of this model is given in [43] by

$$\omega_{KoBol}(x) = \begin{cases} Dq|x|^{-1-\alpha}e^{-\lambda|x|}, x < 0; \\ Dqx^{-1-\alpha}e^{-\lambda x}, x > 0, \end{cases}$$

where $D > 0$, $\lambda > 0$, $p, q \in [-1, 1]$, $p + q = 1$, and $0 < \alpha \leq 2$. It should be remarked that the parameter λ controls the decay of the exponent, p and q control the skewness, and D is a measure of the overall activity level. The FPDE system governing the price of European-style options is also established in [21] as

$$\begin{cases} \dfrac{\partial V}{\partial t} + \left(r - v - \lambda^\alpha(q-p)\dfrac{\partial V}{\partial t}\right) + \dfrac{1}{2}\sigma^\alpha[pe^{\lambda x}{}_x D_\infty^\alpha(e^{-\lambda x}V) \\ + qe^{-\lambda x}{}_{-\infty}D_x^\alpha(e^{\lambda x}V)] = \left(r + \frac{1}{2}\sigma^\alpha\lambda^\alpha\right)V, \\ V(x, T) = \Pi(x), \end{cases}$$

where $v = \dfrac{1}{2}\sigma^\alpha[p(\lambda-1)^\alpha + q(\lambda+1)^\alpha - \lambda^\alpha - \alpha\lambda^{\alpha-1}(q-p)]$.

1.4 Various representations of European-style option price in the Fourier space

As mentioned earlier, the Fourier transform has been widely used in the financial engineering area [13]. European-style options under various different models can be expressed in terms of Fourier integrals [51]. In this section, we shall briefly review some of these representations.

1.4.1 Attari's approach

In [3], Attari mentioned that the price of a European call at time zero with spot price S_0, strike price K and maturity T is given by

$$C(S_0, K, T) = S_0\Pi_1 - e^{-rT}K\Pi_2,$$

where

$$\Pi_1 = 1 + \frac{e^k}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-iuk}\phi_T(u)}{i(u+i)}du,$$

$$\Pi_2 = \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-iuk}\phi_T(u)}{iu}du,$$

$k = \ln\dfrac{Ke^{-rT}}{S_0}$, S is a general Lévy process with risk neutral density $q_T(x)$, $q_T(x)$ is the risk neutral density of $x = \ln S$ relative to the martingale measure Q, and $\phi_T(u)$ is the characteristic function of $q_T(x)$ defined as $\phi_T(u) = \displaystyle\int_{-\infty}^{\infty}e^{iux}q_T(x)dx$.

1.4.2 Bate's formula

A similar approach to that of Attari (2004) is outlined in [7]. Here, the value of a European call option is determined from

$$C(S_0, K, T) = S_0 - e^{-rT}K\left(\frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-iu\ln\frac{K}{S_0}}\phi_T(u)}{iu(1-iu)}du\right),$$

where S_0, K and T are the same as defined in **Subsection 1.4.1**. Using the fact that option prices are real valued, we obtain

$$C(S_0, K, T) = S_0 - e^{-rT}K\left(\frac{1}{2} + \frac{1}{\pi}\int_{0}^{+\infty}\frac{e^{-iu\ln\frac{K}{S_0}}\phi_T(u)}{iu(1-iu)}du\right). \tag{1.4.9}$$

In the literature, **(1.4.9)** is referred to as the Bate formula.

1.4.3 The B-M formula

Bakshi and Madan [4] reduced the European option price valuation problem to the estimation of the price of the ArrowDebreu securities under appropriately modified equivalent probability measures. They pointed out that the price of a European call at time zero with spot price S_0, strike price K and maturity T is given by,

$$
\begin{aligned}
C(S_0, K, T) &= e^{-rT} \int_k^{+\infty} e^x q(x) dx - e^{-rT} K \int_k^{+\infty} q(x) dx, \\
&= S\Pi_1 - e^{-rT} K \Pi_2,
\end{aligned}
$$

where $k = \ln K$, and Π_1 and Π_2 are defined as

$$
\Pi_1 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \Re \left[\frac{e^{-iuk} \phi_T(u-i)}{iu\phi_T(-i)} \right] du,
$$

and

$$
\Pi_2 = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \Re \left[\frac{e^{-iuk} \phi_T(u)}{iu} \right] du.
$$

Hereafter, we name the current formula as "The B-M formula". This formula will be further used in **Chapter 4** for comparison purpose. Interested readers can refer to [51] for further study.

Before leaving this section, it should be remarked that the option price expressed in terms of Fourier integral is not truly "explicit", although it is also of closed form. A clear advantage of our formula that will be derived in later chapters against these representations is that ours has no need to work out the expression of the characteristic function in advance, which is however, an essential part of the latter.

1.5 Literature review

Although the B-S framework has led to a boom in the trading of option derivatives around the world, it is known that several assumptions used in the B-S model are unrealistic [53]. For example, the geometric Brownian motion used to model the

underlying returns implies that the series of the first difference of the log prices are uncorrelated [52]. Also, the assumption that the log-increments of the underlying returns are Gaussian has underestimated the probability of underlying price moving significantly over small time steps [18]. To overcome these drawbacks, some other financial models with non-Gaussian log-increments were introduced over the last few decades. One of the most important models being used is to assume that under an equivalent martingale measure, the underlying price stays within a family of Lévy processes [25]. For example, Barndorff-Nielsen introduced the NIG (normal inverse Gaussian) model by modeling key stylized features of observational series from finance [6]. Then, Madan and Seneta introduced the VG (variance gamma) model to analyze the underlying uncertainty driving stock market returns [41]. Besides, the maximally skewed LS (Lévy stable) process introduced in [18] has been studied by a number of researchers [25, 37], and the FMLS (finite moment log stable) model is introduced. It is shown that this model can not only successfully capture the high-frequency empirical probability distribution of the S&P 500 data, but also fit simultaneously volatility smirks at different maturities [25]. Other different Lévy processes used to extend the FMLS model include the CGMY model introduced in [16] and the KoBol model introduced in [43]. It should be pointed out that the CGMY model further generalizes the VG model. In specific, it could synthesize the features of other continuous time models. Therefore, the CGMY model can be used effectively to study both the statistical process and the risk-neutral process [16].

There are mainly two types of fractional derivatives documented in the quantitative finance area, a time-fractional derivative and a spatial-fractional derivative. A closed-form solution for European vanilla options under the modified B-S equation with a time-fractional derivative was derived by Wyss [29]. However, he did not provide plausible financial reason why a time-fractional derivative should be adopted. Then, a model involved information on the waiting-time between trades was proposed by Cartea and Meyer-Brandis [22]. This model was further analyzed by Cartea in [19], where he established the FPDE system with the Caputo fractional derivative

involved. In terms of option pricing models with a spatial-fractional derivative, Carr and Wu [18] formally described the FMLS model under the risk-neutral measure. A substantial progress has then been made by Cartea and del-Castillo-Negrete [21]. They have successfully connected the pricing of options under the FMLS model, the KoBol model and the CGMY model to the solving of different FPDE systems. Recently, Chen et al. [25] derived a closed-form analytical solution for European put options under the FMLS model. By "closed-form", it is meant that one can write the solution in terms of generally accepted mathematical functions and operations [56]. In this thesis, we shall consider the pricing of European options under the CGMY model purely analytically.

1.6 Structure of the thesis

This thesis is divided into five chapters. In the first chapter, we briefly review some mathematical and financial background of the CGMY model. In **Chapter 2**, we revisit the CGMY model in detail. The implied volatility of the CGMY model is also considered in this chapter. In **Chapter 3**, we derive the closed-form analytical solution for European-style options by solving the corresponding FPDE system using the Fourier transform technique. We also analyze the asymptotic behaviors of the current solution. In **Chapter 4**, some numerical examples and discussions are provided to illustrate the validity of the current solution and the proposed implementation technique. We also conduct some quantitative analysis on the impacts of different parameters. A brief conclusion is finally given in **Chapter 5**.

Chapter 2

CGMY model

To obtain a clear overview of the CGMY model, we shall discuss this model in a reasonable detail in this chapter. This chapter is further divided into five sections, according to five important issues to be addressed. In the first section, we compare the CGMY model with the VG (Variance Gamma) model. In the second section, we discuss various parameters appeared in the CGMY model, whereas in the third section, we discuss the relationship between the Y value and the pdf (probability density function) of the CGMY model. The implied volatility of the CGMY model is then provided in **Section 4**. Finally, the characteristic exponent and the characteristic function of the CGMY model are discussed.

2.1   The revisit of the VG model

The CGMY model is closely related to the VG model, which is also an infinite-activity pure jump process. To better explain the CGMY model, we shall briefly review the VG model in this section. Let $G(t; 1, \nu)$ be a Gamma process with unit mean and variance ν. According to [16], it is known that the corresponding pdf is given by

$$f(g) = \frac{g^{t/\nu - 1} \exp(-g/\nu)}{\nu^{t/\nu} \Gamma(t/\nu)},$$

and the characteristic function is given by

$$\Phi_G(u,t) = E\{\exp[iuG(t)]\} = (\frac{1}{1-i\nu u})^{t/\nu}.$$

The VG process $X_{VG}(t;\sigma,\nu,\theta)$ is denoted by

$$X_{VG}(t;\sigma,\nu,\theta) = \theta G(t;\nu) + \sigma W[G(t;\nu)],$$

where σ, ν, and θ are real constants and $W(\cdot)$ is the standard Brownian motion. According to [16], The characteristic function of the VG model can be calculated as

$$
\begin{aligned}
\Phi_{VG}(u,t) = E\{\exp[iuX_{VG}(u,t)]\} &= (\frac{1}{1-i\theta\nu u + \sigma^2\nu u^2/2})^{t/\nu}, \\
&= (\frac{1}{1-i\eta_p u})^{t/\nu}(\frac{1}{1+i\eta_n u})^{t/\nu},
\end{aligned}
$$

where $\eta_p = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} + \frac{\theta\nu}{2}$ and $\eta_n = \sqrt{\frac{\theta^2\nu^2}{4} + \frac{\sigma^2\nu}{2}} - \frac{\theta\nu}{2}$.

We can also use two independent Gamma processes, namely, $G_p(t;\mu_p,\nu_p)$ and $G_n(t;\mu_n,\nu_n)$ to define the VG process [16]. We assume that the ratio of the variance to the square of the mean is the same for both gamma processes, and is equal to ν. We obtain $\nu = \frac{\eta_p}{\mu_p}$ and $\nu = \frac{\eta_n}{\mu_n}$, $\nu_p = \mu_p^2\nu$ and $\nu_n = \mu_n^2\nu$. Accordingly, the VG model can also be written as

$$X_{VG}(t;\sigma,\nu,\theta) = G_p(t;\mu_p,\nu_p) - G_n(t;\mu_n,\nu_n).$$

From the above representation, the Lévy density of the VG model can be obtained as

$$
\omega_{VG}(x) = \begin{cases} \dfrac{\mu_n^2}{\nu_n}\dfrac{e^{-\frac{\mu_n}{\nu_n}|x|}}{|x|}, & x < 0; \\[2ex] \dfrac{\mu_p^2}{\nu_p}\dfrac{e^{-\frac{\mu_p}{\nu_p}|x|}}{|x|}, & x > 0. \end{cases}
\tag{2.1.1}
$$

Please refer to [16] for more details.

We can now generalize the VG model to the CGMY model with four new pa-

rameters C, G, M and Y. The CGMY process is also a pure jump Lévy process with Lévy measure $W(dx) = w_{CGMY}(x)dx$ given by,

$$\omega_{CGMY}(x) = \begin{cases} C\dfrac{e^{-G|x|}}{|x|^{1+Y}}, & x < 0; \\ C\dfrac{e^{-M|x|}}{|x|^{1+Y}}, & x > 0, \end{cases} \tag{2.1.2}$$

where $C > 0, G \geq 0, M \geq 0$, and $Y < 2$.

From the two density functions **(2.1.1)** and **(2.1.2)**, it can be observed that the VG model is a special case of the CGMY model if $Y = 0$, $C = \dfrac{1}{\nu}$, $G = \dfrac{1}{\eta_n}$ and $M = \dfrac{1}{\eta_p}$. In fact, C, G, M and Y are four key parameters capturing the characteristics of the CGMY model. We shall discuss them in detail in the next section.

2.2   C, G, M, Y of the CGMY model
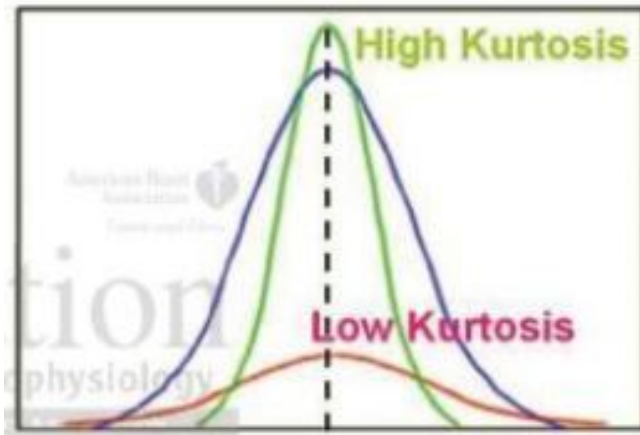
Figure 2.1: High kurtosis and low kurtosis have different effects on the probability density function

Madan et al. mentioned in [17] that the parameter C can be viewed as a measure of the overall level of activity. In the special case of $G = M$, the CGMY model becomes a symmetric process. The parameter C then controls the overall kurtosis of the distribution. It can be observed from **Fig 2.1** that the pdf of the CGMY model would become flatter as the kurtosis becomes smaller.

The parameters G and M control the rate of exponential decay on the right and left of the Lévy density, respectively. When they are equal to each other, the distribution of the CGMY model becomes symmetric, as mentioned earlier. In the case of $G \neq M$, it leads to a skewed distribution. For example, if $G < M$, the left tail of the distribution is heavier than the right one, whereas $G > M$, the right tail is heavier than the left one [16].

On the other hand, as mentioned in [54], the parameter Y is used to characterize the fine structure of the stochastic process. In specific, the parameter Y determines whether the CGMY model has a complete monotone Lévy density, and whether the process has finite or infinite activity, or variation.

According to different values of Y, the CGMY has the following different properties [16].

The CGMY model:

(a) does not have a complete monotone Lévy density, but has a finite activity for $Y < -1$;

(b) has a complete monotone Lévy density with a finite activity for $-1 < Y < 0$;

(c) has a complete monotone Lévy density with an infinite activity and finite variation for $0 < Y < 1$;

(d) has a complete monotone Lévy density with an infinite activity and infinite variation for $1 < Y < 2$;

(e) degenerates to the classical B-S model for $Y \to 2$.

The explanations of CM (complete monotone Lévy densities), FV (finite variation) and FA (finite activity) are left in **Appendix B** for interested readers.

2.3   The pdf of the CGMY model

As mentioned earlier, the CGMY process is in fact a Lévy process $\{X(t)\}_{t \geq 0}$ such that $X(1)$ is CGMY distributed with parameters C, G, $M > 0$ and $Y < 2$.

According to [14], a CGMY-distributed random variable will have the mean value $C(M^{Y-1} - G^{Y-1})\Gamma(1-Y)$, and variance value $C(M^{Y-2} + G^{Y-2})\Gamma(2-Y)$. Furthermore, its skewness and kurtosis are given by $\dfrac{C(M^{Y-3} + G^{Y-3})\Gamma(3-Y)}{[C(M^{Y-2} + G^{Y-2})\Gamma(2-Y)]^{3/2}}$ and $3 + \dfrac{C(M^{Y-4} + G^{Y-4})\Gamma(4-Y)}{[C(M^{Y-2} + G^{Y-2})\Gamma(2-Y)]^2}$, respectively.
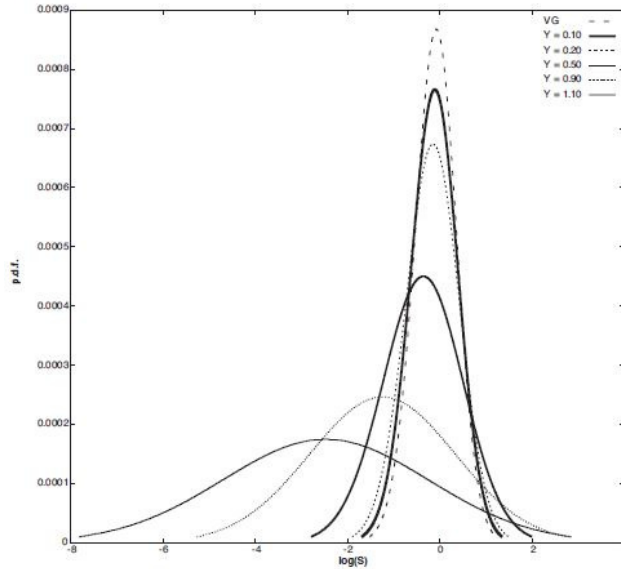
Figure 2.1 The p.d.f. for the CGMY model. Several values of Y

Figure 2.2: The pdf of the CGMY model for different values of Y with $G=M$.

Displayed in **Fig 2.2** is the pdf of the CGMY model for different values of Y with $G = M$. This figure is a typical example shown in [26]. From the figure, it can be observed that both tails of the density function are fatter and the center of the distribution shifts to the left when Y becomes larger.

2.4 The implied volatility of the CGMY model

Now, we turn to investigate the implied volatility of the CGMY model. For comparison purpose, we set $G = M = 1$. In this case, the CGMY distribution is symmetric. Furthermore, the parameters in the CGMY model are chosen such that the CGMY distribution has the same quartiles as the B-S distribution with a volatility σ_{BS}. Specifically, $X_{BS}(t) = (\mu - \dfrac{\sigma_{BS}^2}{2})t + \sigma_{BS}W(t)$, where $\sigma_{BS} > 0$ and $\mu \in R$ are parameters, W(t) is a standard Brownian motion. From the view of risk-neutral valuation, the expected return μ on the option is equal to r. Therefore, the secondary moment

of the B-S model can be obtained as

$$E[X_{BS}^2(t)] = \{E[X_{BS}(t)]\}^2 + V[X_{BS}(t)] = [(\mu - \frac{\sigma_{BS}^2}{2})t]^2 + \sigma_{BS}^2 t.$$

On the other hand, we have known the mean and variance of $X_{CGMY}(t)$. The secondary moment of the CGMY model can be written as

$$
\begin{aligned}
E[X_{CGMY}^2(t)] &= \{E[X_{CGMY}(t)]\}^2 + V[X_{CGMY}(t)], \\
&= [(r-v)t]^2 + tC(M^{Y-2} + G^{Y-2})\Gamma(2-Y),
\end{aligned}
$$

where $v = C\Gamma(-Y)[(M-1)^Y - M^Y + (G+1)^Y - G^Y]$. Next, substituting $\mu = r$ and $G = M = 1$, we have the result $C = \dfrac{r^2 t + \sigma_{BS}^2}{2\Gamma(2-Y)}$ with the secondary moments $E[X_{BS}^2(t)]$ and $E[X_{CGMY}^2(t)]$ being equal to each other. The implied volatility curves
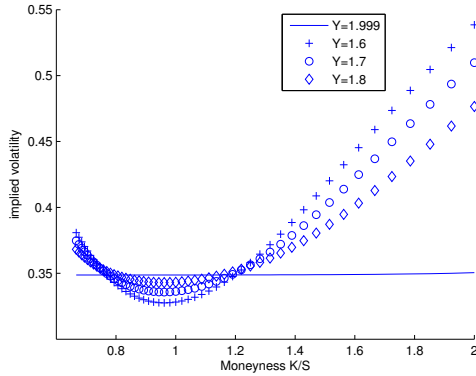
(a) implied volatility for different Y. (b) Implied volatility surfaces with $Y = 1.8$.
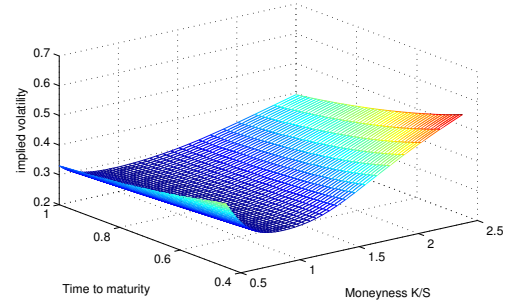
Figure 2.3: The implied volatility for the CGMY model. Model parameters are $K = 10$, $G = M = 1$, $r = 0.1$, $t = 0.55$(year), $C = \frac{0.064}{\Gamma(2-Y)}$ and $\sigma_{BS} = 0.35$.

as a function of the moneyness $\dfrac{K}{S}$ at different levels of Y are shown in **Fig 2.3(a)**. The implied volatility surface of a particular value of Y, i.e., $Y = 1.8$ is further shown in **Fig 2.3(b)**. From this figure, one can clearly observe the volatility smile for all maturities. However, the smile is weakened as the time to maturity becomes larger. This agrees with the observation in [47].

2.5   The characteristic exponent and the characteristic function of the CGMY model

In this section, we shall discuss the characteristic exponent and characteristic function of the CGMY model in detail. According to the Lévy Khintchine formula [8], the characteristic exponent of the Lévy process is given by

$$\Psi_t(u) = \left(bitu - t\frac{1}{2}\sigma^2 u^2 + t \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbf{I}_{|x|<1} \right) \Pi(dx) \right).$$

Based on the relationship between $\Phi_t(u)$ and $\Psi_t(u)$, we have $\Phi_t(u) = \exp(\Psi_t(u))$, where $\Phi_t(u)$ is the characteristic function of the CGMY model. The analytical expressions of the characteristic exponent and the characteristic function of the CGMY model can be obtained through the following lemma [34].

Lemma 2.5.1 *The characteristic exponent of the CGMY model is given by*

$$\Psi_{CGMY}(u,t) = tC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y],$$

and the characteristic function of the CGMY model is

$$\Phi_{CGMY}(u,t) = \mathbb{E}\left[e^{iuX_t} \right] = \exp\{tC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]\}.$$

Proof.   For convenience, we set $\sigma = 0, b = 0$ in the characteristic exponent of the Lévy process. For $\sigma \neq 0$ and $b \neq 0$, it is too complicated to obtain a simpler behavior of the characteristic exponent and function of the CGMY model, which is omitted. We first determine $\Psi_1(u)$, where $\Psi_1(u)$ is the characteristic exponent $\Psi_t(u)$ at $t = 1$. We have

$$\Psi_1(u) = \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1 - iux\mathbf{I}_{|x|<1} \right) \Pi(dx),$$

which can be simplified into

$$\Psi_1(u) = \int_{\mathbb{R}\setminus\{0\}} \left(e^{iux} - 1\right)\omega_{CGMY}dx. \tag{2.5.3}$$

Then, since ω_{CGMY} has different expressions for $x > 0$ and $x < 0$, the integral in **(2.5.3)** can be divided into the sum of two integrals $\int_0^{+\infty} (e^{iux} - 1)\omega_{CGMY}dx$ and $\int_{-\infty}^0 (e^{iux} - 1)\omega_{CGMY}dx$, where ω_{CGMY} is the same as defined in **Section 2.1**.

For $\int_0^{+\infty} (e^{iux} - 1)\omega_{CGMY}dx$, we have

$$
\begin{aligned}
\int_0^{+\infty} (e^{iux} - 1)\omega_{CGMY}dx &= \int_0^{+\infty} (e^{iux} - 1)C\frac{e^{-Mx}}{x^{Y+1}}dx, \\
&= C\int_0^{+\infty} x^{-Y-1}\exp\{[-(M - iu)x] - \exp(-Mx)\}dx, \\
&= C(M - iu)^Y\Gamma(-Y) - CM^Y\Gamma(-Y), \\
&= C\Gamma(-Y)[(M - iu)^Y - M^Y].
\end{aligned}
$$

Similarly, the integral $\int_{-\infty}^0 (e^{iux} - 1)\omega_{CGMY}dx$ can be simplified as

$$\int_{-\infty}^0 (e^{iux} - 1)\omega_{CGMY}dx = C\Gamma(-Y)[(G + iu)^Y - M^Y].$$

Therefore, the characteristic exponent at $t = 1$ of the CGMY model is equal to

$$\Psi_{CGMY}(u, 1) = C\Gamma(-Y)[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y],$$

which, after the time being added on, becomes

$$\Psi_{CGMY}(u, t) = t\Psi_{CGMY}(u, 1) = tC\Gamma(-Y)[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y].$$

Therefore, the characteristic function of the CGMY model can be obtained as

$$
\begin{aligned}
\Phi_{CGMY}(u,t) &= \mathbb{E}\left[e^{iuX_t}\right] = e^{\Psi_{CGMY}(u,t)}, \\
&= \exp\{tC\Gamma(-Y)[(M-iu)^Y - M^Y + (G+iu)^Y - G^Y]\}.
\end{aligned}
$$

This has completed the proof.

After the details of the CGMY model are explored, we turn to investigate the pricing of European-style option derivatives under this model. This issue will be the main topic of the next chapters.

Chapter 3

Analytical solution approach

In this chapter, we consider the pricing of European-style options under the CGMY model purely analytically. This chapter is further divided into three sections, according to three important issues to be addressed. In the first section, we shall briefly revisit the derivation of the FPDE system governing the European-style options under the CGMY model. In the second section, a closed-form analytical solution is derived from the established FPDE system. In the third section, we investigate some asymptotic behaviors of our solution, the put-call parity and the Greek Letters.

3.1   FPDE system for European-style options

In this section, we shall briefly introduce the FPDE system that the price of European-style options must satisfy under the CGMY model. For more details, please refer to [21].

Let $V(x, t)$ be the price of European-style options, with x being the log underlying price and t being the current time. It is known that under the risk-neutral measure Q, the option price is the discounted expectation of the payoff values, i.e.,

$$V(x, t) = e^{-r(T-t)} E^Q \left[\Pi(x_T) \right].$$

With the Fourier transform of the payoff available, i.e.,

$$\tilde{\Pi}(\xi) = F\left[\Pi(x)\right] = \int_{-\infty}^{+\infty} e^{i\xi x_T} \Pi(x_T) dx_T,$$

$V(x,t)$ can be rewritten as

$$V(x,t) = \frac{e^{-r(T-t)}}{2\pi} E^Q \left[\int_{-\infty}^{+\infty} e^{-i\xi x_T} \tilde{\Pi}(\xi) d\xi\right].$$

Now, changing the order of the expectation and the integration, we obtain

$$
\begin{aligned}
V(x,t) &= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty}^{+\infty} E^Q \left[e^{-i\xi x_T}\right] \tilde{\Pi}(\xi) d\xi, \\
&= \frac{e^{-r(T-t)}}{2\pi} \int_{-\infty}^{+\infty} e^{-i\xi x_t - i\xi(r-v)(T-t) + (T-t)\Psi(-\xi)} \tilde{\Pi}(\xi) d\xi, \\
&= F^{-1}\left[e^{[-r-i\xi(r-v)+\Psi(-\xi)](T-t)} \tilde{\Pi}(\xi)\right],
\end{aligned}
\tag{3.1.1}
$$

where $\Psi(\xi)$ is the characteristic exponent of the Lévy process. Specifically, $E^Q[e^{-i\xi x_T}] = e^{-r\xi x_t - i\xi(r-v)(T-t) + (T-t)\Psi(-\xi)}$ can be proved straightforwardly through the fact the characteristic function of $\ln S_T$, using **(1.3.8)**, is given by

$$E^Q[e^{i\xi \ln S_T}] = e^{i\xi \ln S_t + i\xi(r-v)(T-t) + (T-t)\Psi(\xi)}.$$

Taking the Fourier transform on both sides of **(3.1.1)**, we obtain

$$
\begin{aligned}
\tilde{V}(t;\xi) &= F[V(x,t)] = F\left\{F^{-1}[e^{[-r-i\xi(r-v)+\Psi(-\xi)](T-t)} \tilde{\Pi}(\xi)]\right\}, \\
&= e^{[-r-i\xi(r-v)+\Psi(-\xi)](T-t)} \tilde{\Pi}(\xi).
\end{aligned}
$$

Therefore, in the Fourier space, $\tilde{V}(t;\xi)$ satisfies the following ODE:

$$
\begin{cases}
\dfrac{d\tilde{V}}{dt} = [r + i\xi(r-v) - \Psi(-\xi)] \tilde{V}, \\
\tilde{V}(T) = \tilde{\Pi}(\xi).
\end{cases}
\tag{3.1.2}
$$

Applying the inverse Fourier transform to **(3.1.2)**, the FPDE system governing the

European options under the CGMY model can be obtained as

$$
\begin{cases}
\dfrac{\partial V}{\partial t} + (r - v)\dfrac{\partial V}{\partial x} + C\Gamma(-Y)e^{Mx}{}_xD_\infty^Y\left(e^{-Mx}V\right) \\[2mm]
+ C\Gamma(-Y)e^{-Gx}{}_{-\infty}D_x^Y\left(e^{Gx}V\right) = \left[r + C\Gamma(-Y)\left(M^Y + G^Y\right)\right]V, \qquad (3.1.3) \\[2mm]
V(x, T) = \Pi(x),
\end{cases}
$$

where

$$
v = C\Gamma(-Y)[(M - 1)^Y - M^Y + (G + 1)^Y - G^Y].
$$

It should be remarked that for $Y < 0$, $_{-\infty}D_x^Y(\cdot)$ and $_xD_\infty^Y(\cdot)$ are Weyls fractional integrals defined by

$$
_{-\infty}D_x^Y f(x) = \frac{1}{\Gamma(-Y)} \int_{-\infty}^x f(y)(x - y)^{-Y-1}dy,
$$

and

$$
_xD_\infty^Y f(x) = \frac{1}{\Gamma(-Y)} \int_x^\infty f(y)(x - y)^{-Y-1}dy,
$$

respectively.   For $Y > 0$, $_{-\infty}D_x^Y(\cdot)$ and $_xD_\infty^Y(\cdot)$ are Weyls fractional derivatives defined by

$$
_{-\infty}D_x^Y f(x) = \frac{1}{\Gamma(n - Y)} \frac{\partial^n}{\partial x^n} \int_{-\infty}^x f(y)(x - y)^{n-Y-1}dy,
$$

and

$$
_xD_\infty^Y f(x) = \frac{1}{\Gamma(n - Y)} \frac{\partial^n}{\partial x^n} \int_x^\infty f(y)(y - x)^{n-Y-1}dy,
$$

respectively.

It can be observed that **(3.1.3)** is quite complicated in comparison to the classical B-S system or even the FPDE governing the option price under the FMLS model [21]. The current FPDE system **(3.1.3)** involves both left-side and right-side fractional derivatives, which are difficult to deal with either numerically or analyti-

cally. In the literature, the analytical solution of the CGMY model has never been achieved, which has greatly hindered further quantitative analysis of the CGMY model. It should also be noticed that **(3.1.3)** is fundamentally different from the case where the fractional derivative appears in the time direction and can be eliminated by using the Laplace transform [29]. In our case, the Laplace transform would not work. Despite those difficulties, we still have managed to derive a closed-form analytical solution for the CGMY model. This issue will be further discussed in the next section.

3.2 Closed-form analytical solution

After the FPDE system governing the price of European-style options being successfully established in the previous section, we shall now concentrate on deriving the closed-form analytical solution. Here, we consider the case for $1 < Y < 2$, because the CGMY model has a complete monotone density with an infinite activity and infinite variation.

To solve for **(3.1.3)** analytically, we shall start from the expression of V in the Fourier space. As mentioned in **Section 3.1**, in the Fourier space, $\tilde{V}(t; \xi)$ satisfies

$$\begin{cases} \dfrac{d\tilde{V}}{dt} = [r + i\xi(r - v) - C\Gamma(-Y)((M + i\xi)^Y - M^Y + (G - i\xi)^Y - G^Y)]\tilde{V}, \\ \tilde{V}(T) = \tilde{\Pi}(\xi), \end{cases}$$

$$(3.2.4)$$

where $\tilde{V}(t; \xi) = F[V(x, t)]$ and $\tilde{\Pi}(\xi) = F[\Pi(x)]$. More details are left in **Appendix A** for interested readers.

Upon solving **(3.2.4)**, the option price in the Fourier space can be obtained as

$$\tilde{V}(t; \xi) = \exp\{-[r + i\xi(r - v) - C\Gamma(-Y)((M + i\xi)^Y - M^Y + (G - i\xi)^Y - G^Y)](T - t)\}\tilde{\Pi}(\xi).$$

$$(3.2.5)$$

To obtain an analytical formula for the option price in the original space, the Fourier inversion needs to be carried out. After the Fourier inversion is applied to **(3.2.5)**,

we obtain

$$
\begin{aligned}
V(x,t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ix\xi}\cdot e^{-[r+i\xi(r-v)-C\Gamma(-Y)((M+i\xi)^{Y}-M^{Y}+(G-i\xi)^{Y}-G^{Y})](T-t)}\tilde{\Pi}(\xi)d\xi, \\
&= \frac{k_0}{2\pi}\int_{-\infty}^{\infty}e^{-ix\xi-(T-t)\{i\xi(r-v)-C\Gamma(-Y)[(M+i\xi)^{Y}+(G-i\xi)^{Y}]\}}\tilde{\Pi}(\xi)d\xi, \\
&= \frac{k_0}{2\pi}\int_{-\infty}^{\infty}e^{-ix\xi}\cdot e^{-i\xi k_1}\tilde{\Pi}(\xi)\cdot e^{k_2(M+i\xi)^{Y}}e^{k_2(G-i\xi)^{Y}}d\xi, \\
&= k_0 F^{-1}\left[e^{-i\xi k_1}\Pi(\tilde{\xi})\cdot e^{k_2(M+i\xi)^{Y}}e^{k_2(G-i\xi)^{Y}}\right], \\
&= k_0 F^{-1}\left[e^{-i\xi k_1}\Pi(\tilde{\xi})\right]*F^{-1}\left[e^{k_2(M+i\xi)^{Y}}e^{k_2(G-i\xi)^{Y}}\right], \\
&= k_0\Pi(x+k_1)*F^{-1}\left[e^{k_2(M+i\xi)^{Y}}\right]*F^{-1}\left[e^{k_2(G-i\xi)^{Y}}\right], \qquad (3.2.6)
\end{aligned}
$$

where $k_0 = \exp\{-\left[r+C\Gamma(-Y)(M^{Y}+G^{Y})\right](T-t)\}$, $k_1 = (r-v)(T-t)$, and $k_2 = C\Gamma(-Y)(T-t)$.

To obtain the purely analytical formula, the inversions of $\exp\{k_2(M+i\xi)^{Y}\}$ and $\exp\{k_2(G-i\xi)^{Y}\}$ contained in **(3.2.6)** need to be considered. We have the following lemmas for them.

Lemma 3.2.1 *The Fourier inversion of $e^{-k_2|\xi|^{Y}}$ is given by*

$$
\begin{aligned}
F^{-1}\left[e^{-k_2|\xi|^{Y}}\right] &= P(x;Y) = \frac{1}{k_2^{1/Y}}f_{Y,0}\left(\frac{|x|}{k_2^{1/Y}}\right), \\
&= \frac{1}{Yk_2^{1/Y}}H_{2,2}^{1,1}\left[\frac{|x|}{k_2^{1/Y}}\,\middle|\, \begin{array}{cc} (1-\frac{1}{Y},\frac{1}{Y}) & (\frac{1}{2},\frac{1}{2}) \\ (0,1) & (\frac{1}{2},\frac{1}{2}) \end{array}\right],
\end{aligned}
$$

where $H(\cdot)$ is the H-function.

Proof.   Upon realizing that $e^{-|\xi|^{Y}}$ is nothing but the characteristic function of a centered and symmetric Lévy distribution, as well as the relationship between the Fourier transform and the characteristic function of a probability density function, it can be deduced that the Fourier inversion of $e^{-k_2|\xi|^{Y}}$ is equal to multiples of the closed-form representation of the Lévy stable density $f_{Y,0}(x)$, which is usually

expressed in terms of the H-function in [45] as follows.

$$f_{Y,0}(x) = \frac{1}{Y} H_{2,2}^{1,1} \left[x \middle| \begin{array}{cc} (1 - \frac{1}{Y}, \frac{1}{Y}) & (\frac{1}{2}, \frac{1}{2}) \\ (0, 1) & (\frac{1}{2}, \frac{1}{2}) \end{array} \right].$$

This has completed the proof.

Now, we consider the inversion of $e^{k_2(M+i\xi)^Y}$. We have the following lemma.

Lemma 3.2.2 *The Fourier inversion of* $e^{k_2(M+i\xi)^Y}$ *is given by*

$$\mathcal{F}^{-1}\left[e^{k_2(M+i\xi)^Y}\right] = e^{Mx} P(x; Y),$$

where $P(x; Y)$ is the same as defined in **Lemma 3.2.1**.

Proof.   According to the exponential function property of the Fourier transform discussed in **Chapter 1**, we have

$$\mathcal{F}\left[e^{Mx} P(x; Y)\right] = e^{k_2(i\lambda)^Y} \mid_{\lambda=\xi-iM} = e^{k_2(i\xi+M)^Y}.$$

Therefore, we have

$$\mathcal{F}^{-1}\left[e^{k_2(i\xi+M)^Y}\right] = e^{Mx} P(x; Y).$$

This has completed the proof of this lemma.

With the help of the odd-even property of the Fourier transform, as well as the result of **Lemma 3.2.2**, the following lemma can be obtained straightforwardly.

Lemma 3.2.3 *The Fourier inversion of* $e^{k_2(G-i\xi)^Y}$ *is given by*

$$\mathcal{F}^{-1}\left[e^{k_2(G-i\xi)^Y}\right] = e^{-Gx} P(x; Y),$$

where $P(x; Y)$ is the same as defined in **Lemma 3.2.1**.

Now, combining the results obtained in **Lemma 3.2.2** and **Lemma 3.2.3**, the analytical expression for the European options under the CGMY model can be

found as

$$
\begin{aligned}
V(x,t) &= k_0 \Pi(x+k_1) * \left[e^{Mx} P(x;Y) \right] * \left[e^{-Gx} P(x;Y) \right], \\
&= k_0 \int_{-\infty}^{+\infty} \left[e^{M\tau} P(\tau;Y) * e^{-G\tau} P(\tau;Y) \right] \Pi(x - \tau + k_1) d\tau.
\end{aligned}
\tag{3.2.7}
$$

For the case of European call options, we have

$$
\begin{aligned}
V_c(x,t) &= k_0 \int_{-\infty}^{d_0} \left(\frac{1}{k_2^{1/Y}} e^{M\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) * \frac{1}{k_2^{1/Y}} e^{-G\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) \right) \left(e^{x-\tau+k_1} - E \right) d\tau, \\
&= \frac{k_0 e^{x+k_1}}{k_2^{2/Y}} \int_{-\infty}^{d_0} \left(e^{M\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) * e^{-G\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) \right) e^{-\tau} d\tau \\
&\quad - \frac{k_0 E}{k_2^{2/Y}} \int_{-\infty}^{d_0} e^{M\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) * e^{-G\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) d\tau,
\end{aligned}
\tag{3.2.8}
$$

where $d_0 = x - \ln E + k_1$.

For European puts, we obtain

$$
\begin{aligned}
V_p(x,t) &= k_0 \int_{d_0}^{+\infty} \left(\frac{1}{k_2^{1/Y}} e^{M\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) * \frac{1}{k_2^{1/Y}} e^{-G\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) \right) \left(E - e^{x-\tau+k_1} \right) d\tau, \\
&= \frac{k_0 E}{k_2^{2/Y}} \int_{d_0}^{+\infty} e^{M\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) * e^{-G\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) d\tau \\
&\quad - \frac{k_0 e^{x+k_1}}{k_2^{2/Y}} \int_{d_0}^{+\infty} \left(e^{M\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) * e^{-G\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) \right) e^{-\tau} d\tau,
\end{aligned}
\tag{3.2.9}
$$

where d_0 is the same as defined in **(3.2.8)**.

We believe that our formula **(3.2.8)** and **(3.2.9)** are already in the simplest form.

3.3   Asymptotic behaviors of the closed-form solution

One of the most efficient ways to check the validity of our closed-form solution **(3.2.8)** is to investigate its asymptotic behaviors with parameters involved taken on some extreme values [25]. Whether or not the observed asymptotic behaviors

coincide with the financial terms set for the corresponding option model could be a necessary condition to verity the solution. On the other hand, analyzing the asymptotic behaviors could help readers understand the properties of the CGMY model. In view of these, we shall conduct some asymptotic analysis in this section.

Theorem 2 *With $C = \dfrac{\sigma^2}{4\Gamma(-Y)}$, the formula **(3.2.8)** and **(3.2.9)** degenerate to the B-S formula with volatility σ for European calls and puts, respectively, by further taking the limit as $Y \to 2$.*

Proof. When $Y \to 2$, $C = \dfrac{\sigma^2}{4\Gamma(-Y)}$, we have

$$v|_{Y \to 2} = \frac{\sigma^2}{4}[(M-1)^Y - M^Y + (G+1)^Y - G^Y]|_{Y=2} = \frac{\sigma^2}{4}(2G + 2 - 2M).$$

Substituting it into the FPDE of CGMY model, we can obtain

$$\frac{\partial V(x,t)}{\partial t} + \left(r - G\frac{\sigma^2}{2} - \frac{\sigma^2}{2} + M\frac{\sigma^2}{2}\right)\frac{\partial V(x,t)}{\partial x} + \frac{\sigma^2}{4}e^{Mx}\left[e^{-Mx}M^2V(x,t) + e^{-Mx}\frac{\partial^2 V(x,t)}{\partial x^2}\right.$$
$$\left. + 2e^{-Mx}(-M)\frac{\partial V(x,t)}{\partial x}\right] + \frac{\sigma^2}{4}e^{-Gx}\left[e^{Gx}G^2V(x,t) + e^{Gx}\frac{\partial^2 V(x,t)}{\partial x^2} + 2e^{Gx}G\frac{\partial V(x,t)}{\partial x}\right]$$
$$= \left(r + \frac{\sigma^2}{4}M^2 + \frac{\sigma^2}{4}G^2\right)V(x,t).$$

Simplifying this equation, we have

$$\frac{\partial V(x,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V(x,t)}{\partial x^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial V(x,t)}{\partial x} - rV(x,t) = 0. \qquad (3.3.10)$$

On the other hand, the standard B-S model equation is

$$\frac{\partial V(S,t)}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0.$$

Then, we make the parameter transform $x = \ln S$ and obtain the same equation as **(3.3.10)**. This has completed the proof from the view of FPDE.

Next, we shall prove this result from the solution of the CGMY model. For simplicity, we shall concentrate on the puts only. The extension to the calls is rather straightforward.

According to the definition of $f_{\alpha,0}$, it is known that

$$\lim_{\alpha \to 2} f_{\alpha,0}(|m|) = \frac{1}{2} H_{2,2}^{1,1}\left[|m| \;\middle|\; \begin{matrix} (\frac{1}{2}, \frac{1}{2}) & (\frac{1}{2}, \frac{1}{2}) \\ (0, 1) & (\frac{1}{2}, \frac{1}{2}) \end{matrix} \right],$$

which can be simplified as

$$f_{2,0}(|m|) = \frac{1}{2} H_{1,1}^{1,0}\left[|m| \;\middle|\; \begin{matrix} (\frac{1}{2}, \frac{1}{2}) \\ (0, 1) \end{matrix} \right], \tag{3.3.11}$$

whose Mellin transform admits $\mathcal{M}[f_{2,0}(|m|)] = \dfrac{1}{2}\dfrac{\Gamma(s)}{\Gamma(\frac{1}{2} + \frac{1}{2}s)}$. On the other hand, according to the property of the Gamma function discussed in **Section 1.2**, it is known that

$$\Gamma\left(\frac{1}{2}s\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}s\right) = 2^{1-s}\sqrt{\pi}\Gamma(s),$$

and thus

$$\frac{\Gamma(s)}{\Gamma(\frac{1}{2} + \frac{1}{2}s)} = \frac{(\frac{1}{2})^{-s}\Gamma(\frac{1}{2}s)}{2\sqrt{\pi}}. \tag{3.3.12}$$

Now, taking the inverse Mellin transform on both sides of **(3.3.12)**, we obtain

$$\mathcal{M}^{-1}\left[\frac{\Gamma(s)}{\Gamma(\frac{1}{2} + \frac{1}{2}s)}\right] = \mathcal{M}^{-1}\left[\frac{(\frac{1}{2})^{-s}\Gamma(\frac{1}{2}s)}{2\sqrt{\pi}}\right] = \frac{e^{-m^2/4}}{\sqrt{\pi}}, \tag{3.3.13}$$

which, combined with **(3.3.11)**, yields $f_{2,0}(|m|) = \dfrac{e^{-m^2/4}}{2\sqrt{\pi}}$, a function identical to the standard Gaussian density [45].

Now, we turn to calculate the convolution involved in our formula with the simplified Lévy density. We have

$$
\begin{aligned}
f_1(x) * f_2(x) &= \int_{-\infty}^{+\infty} \frac{1}{4\pi} \frac{1}{k_2} e^{M\tau} f_{2,0}\left(\frac{|\tau|}{\sqrt{k_2}}\right) e^{-G(x-\tau)} f_{2,0}\left(\frac{|x-\tau|}{\sqrt{k_2}}\right) d\tau, \\
&= e^{-Gx} \frac{1}{k_2} \frac{1}{4\pi} \int_{-\infty}^{+\infty} e^{(M+G)\tau} e^{-\frac{\tau^2+(x-\tau)^2}{4k_2}} d\tau, \\
&= e^{-Gx} \frac{1}{k_2} \frac{1}{4\pi} e^{-\frac{x^2}{4k_2}} \int_{-\infty}^{+\infty} e^{-\frac{\tau^2-x\tau-2k_2(M+G)\tau}{2k_2}} d\tau, \\
&= e^{-Gx} \frac{1}{k_2} \frac{1}{4\pi} e^{-\frac{x^2}{4k_2}} e^{\frac{2[\frac{x}{2}+(M+G)k_2]^2}{4k_2}} \int_{-\infty}^{+\infty} e^{-\frac{\{\tau-[\frac{x}{2}+(M+G)k_2]\}^2}{2k_2}} d\tau, \\
&= \frac{1}{k_2} \frac{1}{4\pi} e^{\frac{-4k_2Gx-x^2+\frac{x^2}{2}+2(M+G)^2k_2^2+2k_2(M+G)x}{4k_2}} \int_{-\infty}^{+\infty} e^{-\frac{\{\tau-[\frac{x}{2}+(M+G)k_2]\}^2}{2k_2}} d\tau, \\
&= \frac{1}{k_2} \frac{1}{4\pi} e^{\frac{-4k_2Gx-x^2+\frac{x^2}{2}+2(M+G)^2k_2^2+2k_2(M+G)x}{4k_2}} \sqrt{2\pi k_2}, \\
&= \frac{1}{2\sqrt{2\pi k_2}} e^{\frac{(M+G)^2 k_2}{2}} e^{\frac{-\frac{1}{2}x^2+2k_2(M-G)x}{4k_2}}.
\end{aligned}
\tag{3.3.14}
$$

Substituting **(3.3.14)** into **(3.2.7)**, we obtain

$$
V(x,t) = k_0 \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{2\pi k_2}} e^{\frac{(M+G)^2 k_2}{2}} e^{\frac{-\frac{1}{2}\tau^2+2k_2(M-G)\tau}{4k_2}} \Pi(x-\tau+k_1) d\tau.
$$

For European put options, it is known that

$$
\Pi(x-\tau+k_1) =
\begin{cases}
E - e^{x-\tau+k_1}, & x-\tau+k_1 < \ln E; \\
0, & otherwise,
\end{cases}
$$

where $d_0 = \tau = x - \ln E + k_1$. Therefore, we have

$$
\begin{aligned}
V_p(x,t) &= \frac{k_0}{2\sqrt{2\pi k_2}} e^{\frac{(M+G)^2 k_2}{2}} \int_{d_0}^{+\infty} e^{\frac{-\frac{1}{2}\tau^2 + 2k_2(M-G)\tau}{4k_2}} (E - e^{x-\tau+k_1}) d\tau, \\
&= \frac{k_0}{\sqrt{2\pi}} e^{\frac{(M+G)^2 k_2}{2}} \int_{d_2}^{+\infty} e^{\frac{-2k_2 m^2 + 2k_2^2(M-G)^2}{4k_2}} (E - e^{x-[2\sqrt{k_2}m+2(M-G)k_2]+k_1}) dm, \\
&= \frac{k_0}{\sqrt{2\pi}} e^{\frac{(M+G)^2 k_2}{2}} e^{\frac{(M-G)^2 k_2}{2}} \int_{d_2}^{+\infty} e^{-\frac{m^2}{2}} (E - e^{x-[2\sqrt{k_2}m+2(M-G)k_2]+k_1}) dm, \\
&= \frac{k_0 E}{\sqrt{2\pi}} e^{(M^2+G^2)k_2} \int_{d_2}^{+\infty} e^{-\frac{m^2}{2}} dm \\
&\quad - \frac{k_0}{\sqrt{2\pi}} e^{(M^2+G^2)k_2} e^{x+k_1} \int_{d_2}^{+\infty} e^{-\frac{m^2}{2} - 2\sqrt{k_2}m - 2(M-G)k_2} dm, \\
&= \underbrace{\frac{k_0 E}{\sqrt{2\pi}} e^{(M^2+G^2)k_2} \int_{d_2}^{+\infty} e^{-\frac{m^2}{2}} dm}_{I} \\
&\quad - \underbrace{\frac{k_0}{\sqrt{2\pi}} e^{(M^2+G^2)k_2} e^{x+k_1} e^{-2(M-G)k_2} \int_{d_2}^{+\infty} e^{-\frac{m^2}{2} - 2\sqrt{k_2}m} dm}_{II},
\end{aligned}
$$

where $\tau = 2\sqrt{k_2}m + 2(M-G)k_2$, and $d_2 = \dfrac{d_0 - 2(M-G)k_2}{2\sqrt{k_2}}$. Further calculation shows that

$$
\begin{aligned}
I &= \frac{k_0 E}{\sqrt{2\pi}} e^{(M^2+G^2)k_2} \int_{d_2}^{+\infty} e^{-\frac{m^2}{2}} dm, \\
&= k_0 E e^{(M^2+G^2)k_2} (1 - N(d_2)), \\
&= E e^{-r(T-t)} (1 - N(d_2)),
\end{aligned}
$$

and

$$
\begin{aligned}
II &= \frac{k_0}{\sqrt{2\pi}} e^{(M^2+G^2)k_2} e^{x+k_1} e^{-2(M-G)k_2} \int_{d_2}^{+\infty} e^{-\frac{m^2}{2} - 2\sqrt{k_2}m} dm, \\
&= \frac{k_0}{\sqrt{2\pi}} e^{(M^2+G^2)k_2} e^{x+k_1} e^{-2(M-G)k_2} e^{2k_2} \int_{d_1}^{+\infty} e^{-\frac{n^2}{2}} dn, \\
&= k_0 e^{(M^2+G^2)k_2} e^{x+k_1} e^{-2(M-G)k_2} e^{2k_2} (1 - N(d_1)), \\
&= e^x (1 - N(d_1)), \\
&= S (1 - N(d_1)),
\end{aligned}
$$

where $m + 2\sqrt{k_2} = n$, $d_1 = d_2 + 2\sqrt{k_2}$ and $N(\cdot)$ is the cdf (cumulative distribution function) of the standard normal distribution defined as $N(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$.

By setting $C = \dfrac{\sigma^2}{4\Gamma(-Y)}$, we have $d_2 = \dfrac{\ln S/E + (r - 1/2\sigma^2)(T - t)}{\sigma\sqrt{T - t}}$, and $d_1 = d_2 + \sigma\sqrt{T - t}$.

Therefore, it is clear at this stage that

$$\lim_{Y \to 2} V_p(x, t) = Ee^{-r(T-t)} N(-d_2) - SN(-d_1).$$

Therefore, it can be concluded that as $Y \to 2$, the European put option price under the CGMY model degenerates to the corresponding B-S formula with volatility σ. This has completed the proof.

After investigating the degeneration of our formula, we turn to examine the asymptotic behaviors of the European put option price for extreme underlying values. For the corresponding call option, its value can be obtained by using the put-call parity that will be established in **Section 3.4**.

Theorem 3 *(i)* $\lim\limits_{x \to -\infty} V_p(x, t) = e^{-r(T-t)} E$; *(ii)* $\lim\limits_{x \to +\infty} V_p(x, t) = 0$.

Proof. Firstly, we shall prove that $\lim\limits_{x \to -\infty} V_p(x, t) = e^{-r(T-t)} E$. We remark that $V_p(x, t)$ can also be rewritten as

$$V_p(x, t) = k_0 E \int_{d_0}^{+\infty} \rho(\tau) d\tau - k_0 e^{x + k_1} \int_{d_0}^{+\infty} \rho(\tau) e^{-\tau} d\tau,$$

where $\rho(\tau) = \dfrac{1}{k_2^{2/Y}} e^{M\tau} f_{Y,0}\left(\dfrac{|\tau|}{k_2^{1/Y}}\right) * e^{-G\tau} f_{Y,0}\left(\dfrac{|\tau|}{k_2^{1/Y}}\right)$.

According to the definition of d_0, it is not difficult to show that $d_0 \to -\infty$ as $x \to -\infty$. Therefore, it is straightforward that

$$\lim_{x \to -\infty} V_p(x, t) = k_0 E \int_{-\infty}^{+\infty} \rho(\tau) d\tau - k_0 e^{x + k_1} \int_{-\infty}^{+\infty} \rho(\tau) e^{-\tau} d\tau.$$

Firstly, let us calculate $\int_{-\infty}^{+\infty} \rho(\tau)e^{-\tau}d\tau$. According to the fact that

$$\int_{-\infty}^{+\infty} \rho(\tau)e^{-\tau}d\tau = \int_{-\infty}^{+\infty} \rho(\tau)e^{i(i)\tau}d\tau,$$

we can determine the above integral by taking the Fourier transform on $\rho(\tau)$ with respect to τ first, and then setting the Fourier transform parameter ξ to i. As a result, we obtain

$$\int_{-\infty}^{+\infty} \rho(\tau)e^{-\tau}d\tau = F\left[\rho(\tau)\right]\big|_{\xi=i} = e^{k_2(M-1)^Y + k_2(G+1)^Y},$$

and thus

$$k_0 e^{x+k_1} \int_{-\infty}^{+\infty} \rho(\tau)e^{-\tau}d\tau = e^x k_0 e^{k_1} e^{k_2(M-1)^Y + k_2(G+1)^Y} = e^x.$$

Then, the remaining task is to determine $\int_{-\infty}^{+\infty} \rho(\tau)d\tau$.

$$\int_{-\infty}^{+\infty} \rho(\tau)d\tau = \int_{-\infty}^{+\infty} \frac{1}{k_2^{1/Y}} e^{M\tau} f_{Y,0}\left(\frac{|\tau|}{k_2^{1/Y}}\right) * \frac{1}{k_2^{1/Y}} e^{-G\tau} f_{Y,0}\left(\frac{|\tau|}{k_2^{1/Y}}\right) d\tau,$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{k_2^{1/Y}} e^{M\eta} f_{Y,0}\left(\frac{|\eta|}{k_2^{1/Y}}\right) \cdot \frac{1}{k_2^{1/Y}} e^{-G(\tau-\eta)} f_{Y,0}\left(\frac{|\tau-\eta|}{k_2^{1/Y}}\right) d\eta d\tau,$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{k_2^{1/Y}} e^{M\eta} f_{Y,0}\left(\frac{|\eta|}{k_2^{1/Y}}\right) \cdot \frac{1}{k_2^{1/Y}} e^{-G(\tau-\eta)} f_{Y,0}\left(\frac{|\tau-\eta|}{k_2^{1/Y}}\right) d\tau d\eta,$$

$$= \int_{-\infty}^{+\infty} \frac{1}{k_2^{1/Y}} e^{M\eta} f_{Y,0}\left(\frac{|\eta|}{k_2^{1/Y}}\right) \int_{-\infty}^{+\infty} \frac{1}{k_2^{1/Y}} e^{-G(\tau-\eta)} f_{Y,0}\left(\frac{|\tau-\eta|}{k_2^{1/Y}}\right) d(\tau-\eta) d\eta,$$

$$= e^{k_2 G^Y} \int_{-\infty}^{+\infty} \frac{1}{k_2^{1/Y}} e^{M\eta} f_{Y,0}\left(\frac{|\eta|}{k_2^{1/Y}}\right) d\eta,$$

$$= e^{k_2(M^Y + G^Y)}, \tag{3.3.15}$$

where we use the fact that

$$\int_{-\infty}^{+\infty} \frac{1}{k_2^{1/Y}} f_{Y,0}\left(\frac{|\tau|}{k_2^{1/Y}}\right) e^{-\tau} d\tau = F\left[\frac{1}{k_2^{1/Y}} f_{Y,0}\left(\frac{|\tau|}{k_2^{1/Y}}\right)\right]\Big|_{\xi=i},$$

and the odd-even property of Fourier transform. Thus, we have

$$k_0 E \int_{-\infty}^{+\infty} \rho(\tau) d\tau = k_0 E e^{k_2(M^Y + G^Y)} = e^{-r(T-t)} E.$$

Therefore, we obtain

$$\lim_{x \to -\infty} V_p(x,t) = e^{-r(T-t)} E,$$

where S can be ignored when $x \to -\infty$.

Now, we turn to show that $\lim_{x \to +\infty} V_p(x,t) = 0$. To prove (ii), we notice that the first integral of V_p will definitely vanish as $x \to +\infty$. Therefore, we shall concentrate on showing the second integral will also vanish as $x \to +\infty$. For the second integral of V_p, we have

$$\lim_{x \to +\infty} k_0 e^{x+k_1} \int_{d_0}^{+\infty} \rho(\tau) e^{-\tau} d\tau = k_0 e^{k_1} \lim_{x \to +\infty} \frac{\int_{d_0}^{+\infty} \rho(\tau) e^{-\tau} d\tau}{e^{-x}},$$

which is equal to $\lim_{x \to +\infty} k_0 E \rho(d_0)$, after the L'hospital rule is applied. According to the fact that $\int_{-\infty}^{+\infty} \rho(\tau) d\tau$ is bounded as shown in **(3.3.15)**, it is clear $\lim_{x \to +\infty} k_0 E \rho(d_0) = 0$, because $d_0 \to +\infty$ as $x \to +\infty$. Consequently, we obtain

$$\lim_{x \to +\infty} k_0 e^{x+k_1} \int_{d_0}^{+\infty} \rho(\tau) e^{-\tau} d\tau = 0,$$

which shows that the second integral of V_p will vanish as $x \to +\infty$. Therefore, we have $\lim_{x \to +\infty} V_p(x,t) = 0$. This has completed the proof.

3.4   The put-call parity

One of the most important concepts in the quantitative finance area is the so-called put-call parity. It is a relationship between the price of European vanilla options with the same parameter settings. By using the put-call parity, the price of a European put or call can be deduced directly from its European counterpart. Considering the

importance of the put-call parity, in this section, we shall derive the put-call parity for the CGMY model.

It should be pointed out that the existence of the risk-neutral measure for the CGMY model implies that the "no arbitrage opportunity" assumption still holds under the current model [44], and thus the put-call parity should hold. Mathematically, we have the following lemma for the put-call parity under the current model.

Lemma 3.4.1 *For any given $Y \in (1, 2)$, the prices of a European call option V_c and its corresponding European put V_p satisfy the put-call parity, assuming that they have the same parameter settings, i.e.,*

$$V_c(x, t) - V_p(x, t) = S - e^{-r(T-t)} E.$$

The proof of this lemma can be easily achieved by using our closed-form analytical solutions, and is thus omitted.

Using the put-call parity, the following relationships of the Greeks can be easily achieved [38].

Lemma 3.4.2 *(i) $\Delta_c = \Delta_p + 1$; (ii) $\Gamma_c = \Gamma_p$;*
(iii) $\rho_c = \rho_p + Ee^{-r(T-t)}(T - t)$;
(iv) $Vega_c = Vega_p$;
(v) $\Theta_c = \Theta_p + Ere^{-r(T-t)}$.

It should be remarked that the put-call parity and the Greeks of CGMY model are identical with the standard Black-Scholes model, which verifies **Theorem 2**.

By using the put-call parity, the trading of European vanilla options under the CGMY model can be greatly facilitated, in the sense that the price of either a European call or put can be deduced straightforwardly from the parity once its European counterpart is determined accurately from our solution. The implementation of our formula will be illustrated in detail in the next chapter, where some numerical examples and useful discussions are also to be provided. We will explore the functions

of all the parameters in the CGMY model.

Chapter 4

Numerical examples and discussions

As shown in the previous chapter, the analytical solution of the CGMY model is derived rigorously. Therefore, there is no need to further address the "accuracy" of the solution and present any calculated results. However, from the view point that a comparison with previously published results may give the readers a sense of the verification of the newly found formula, several numerical examples are still given in this chapter. With the issues to be addressed, this chapter is further divided into three sections. In the first section, the implementation details of our solution are illustrated. In the second section, the comparison of our results with previously published ones is provided. In the last section, the impacts of different parameters on the option prices will be analyzed and discussed.

4.1 Numerical implementation of our analytical solution

Although our solution is written in a similar form as the classical B-S formula and the formula derived for the FMLS model [25], it is, however, not so straightforward as the latter, as far as the numerical implementation is concerned. From **(3.2.7)**,

it can be observed that our final solution is written in terms of a double integral. Besides, it involves the product of exponential function and H-function, as well as the convolution of two H-functions.

Hereafter, we shall consider the put case. The corresponding call values can be obtained straightforwardly by using the put-call parity established in **Section 3.4**.

To determine the Lévy density $f_{Y,\beta}$, we shall use the series representation, i.e.,

$$f_{Y,\beta}(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(1 + n\xi)}{n!} \sin(\pi n \gamma)(-x)^{n-1},$$

where $\xi = 1/Y$ and $\gamma = (Y - \beta)/2Y$. In this thesis, we set $\beta = 0$ because the distribution used in the current model is a symmetric Lévy distribution. Therefore, $\gamma = 1/2$, and the series representation can be simplified to

$$f_{Y,0} = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(1 + n/Y)}{n!} \sin(\frac{\pi n}{2})(-x)^{n-1}. \qquad (4.1.1)$$

However, from the numerical experiment, it can be observed that the series representation **(4.1.1)** converges rather slowly when x becomes very large. Alternatively, from some critical value x and onwards, we use the large asymptotic of $f_{\alpha,0}$ introduced in [33] instead of **(4.1.1)**, i.e.,

$$f_{Y,0} \sim \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\Gamma(1 + n/Y)}{n!} \sin(\frac{\pi n Y}{2})|x|^{-1-nY}. \qquad (4.1.2)$$

Numerical experiments suggested that $x \approx 3$ is the most appropriate critical value for our examples shown below.

On the other hand, from the expression of V_p, one can clearly observe that the calculation of V_p involves the determination of integrals over semi-infinite domain. One of the efficient ways to evaluate the integrals over semi-infinite domain is to use the generalized LG (Laguerre-Gauss) quadrature [9]. To apply the LG quadrature,

we transform the integration domain into semi-infinite domain as

$$
e^{M\tau} \quad f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}}) * e^{-G\tau} f_{Y,0}(\frac{|\tau|}{k_2^{1/Y}})
$$

$$
= \int_{-\infty}^{+\infty} e^{M\eta} f_{Y,0}(\frac{|\eta|}{k_2^{1/Y}}) \cdot e^{-G(\tau-\eta)} f_{Y,0}(\frac{|\tau-\eta|}{k_2^{1/Y}}) d\eta,
$$

$$
= \underbrace{\int_{0}^{+\infty} e^{M\eta} f_{Y,0}(\frac{|\eta|}{k_2^{1/Y}}) \cdot e^{-G(\tau-\eta)} f_{Y,0}(\frac{|\tau-\eta|}{k_2^{1/Y}}) d\eta}_{I}
$$

$$
+ \underbrace{\int_{0}^{+\infty} e^{-M\eta} f_{Y,0}(\frac{|\eta|}{k_2^{1/Y}}) \cdot e^{-G(\tau+\eta)} f_{Y,0}(\frac{|\tau+\eta|}{k_2^{1/Y}}) d\eta}_{II} .
$$

However, it should be pointed out that the LG method still fails to compute (I) because the decay rate of the H-function is slower than the growth rate of the exponential function.

To overcome this computational difficulty, we introduce a new scaling parameter m to control the growth rate of the exponential function. Specifically, we use the following two new integration variables, $\eta' = \dfrac{m\eta}{k_2^{1/Y}}$, and $\tau' = \dfrac{m\tau}{k_2^{1/Y}}$. As a result, $(\mathbf{3.2.9})$ becomes

$$
V_p(x,t) = \frac{k_0}{m^2} \int_{\frac{md_0}{k_2^{1/Y}}}^{+\infty} \int_{-\infty}^{+\infty} e^{\frac{k_2^{1/Y}(G+M)\eta'}{m}} f_{Y,0}(\frac{|\eta'|}{m}) f_{Y,0}(\frac{|\tau'-\eta'|}{m}) d\eta'
$$

$$
\cdot \ e^{\frac{-Gk_2^{1/Y}\tau'}{m}} (E - e^{x-\frac{-Gk_2^{1/Y}\tau'}{m}+k_1}) d\tau'. \tag{4.1.3}
$$

It should be pointed out that the introduction of the new scaling parameter m could effectively control the growth rate of the exponential function so that a balance can be obtained between the growth rate of the exponential function and the decay rate of the H-function. Our numerical experiments suggest that $m = 11$ is an appropriate value.

Now we use the generalized LG quadrature to evaluate $(\mathbf{4.1.3})$ again. We split

the inner integral of **(4.1.3)** into two integrals defined on semi-infinite domain as

$$
\begin{aligned}
V_p(x,t) \;=\; & \frac{k_0}{m^2} \int_{\frac{md_0}{k_2^{1/Y}}}^{+\infty} e^{\frac{-Gk_2^{1/Y}\tau'}{m}} \left(\int_0^{+\infty} e^{\frac{k_2^{1/Y}(G+M)\eta'}{m}} f_{Y,0}\!\left(\frac{|\eta'|}{m}\right) f_{Y,0}\!\left(\frac{|\tau'-\eta'|}{m}\right) d\eta' \right. \\
& \left. + \int_0^{+\infty} e^{\frac{-k_2^{1/Y}(G+M)\eta'}{m}} f_{Y,0}\!\left(\frac{|\eta'|}{m}\right) f_{Y,0}\!\left(\frac{|\tau'+\eta'|}{m}\right) d\eta' \right) \left(E - e^{x - \frac{-Gk_2^{1/Y}\tau'}{m} + k_1} \right) d\tau',
\end{aligned}
$$

which can now both be calculated accurately by using the LG quadrature.

4.2 Validity of our closed-form analytical solution

One of the best ways to test the reliability of the proposed numerical evaluation technique for our solution is to calculate the solution as $Y \to 2$, and compare it with the standard B-S formula with the same parameter settings. Theoretically, when $Y \to 2$, the solution is identical to the B-S formula with $\sigma = 2\sqrt{\lim_{Y \to 2} C\Gamma(-Y)}$, if all the other parameters are the same, as has been shown in **Theorem 2** already. Provided in **Fig 4.1(a)** are two sets of European put prices as a function of the underlying price from a given time to maturity determined respectively from **(4.1.3)** at $Y = 1.999$ and the B-S formula with $\sigma = 2\sqrt{C\Gamma(-1.999)}$. The absolute differences between the two sets of prices are further shown in **Fig 4.1(b)**.
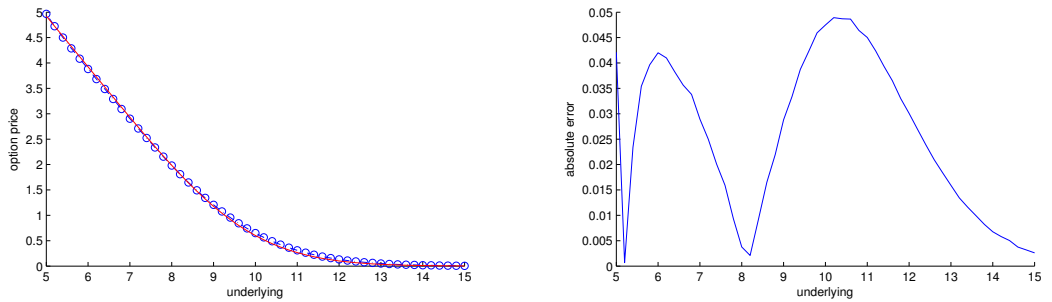
(a) B-S price VS our price at $Y = 1.999$
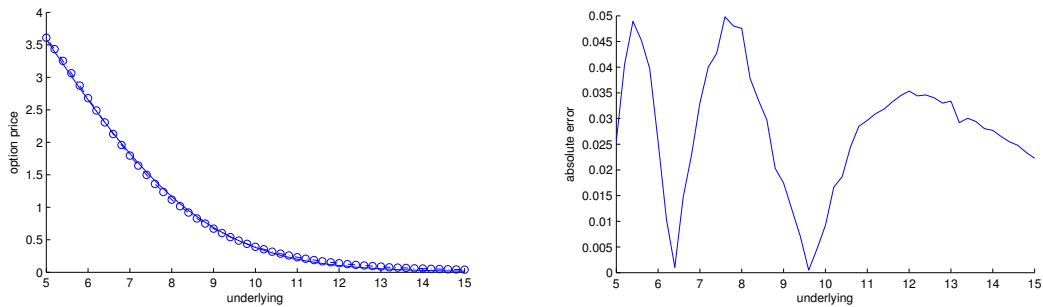

(b) Option price differences

Figure 4.1: Comparison of our solution at $Y = 1.999$ with the B-S formula. Model parameters are $K = \$10$, $r = 0.017$, $\sigma = 0.24$, $M = G = 1$, $C = \dfrac{\sigma^2}{4\Gamma(-Y)}$, and $T - t = 0.55$(year).

From these two figures, one can clearly observe that the two sets of option prices agree perfectly well with each other with the maximum absolute error between the

two being no more than 5%. This indicates that our solution is indeed approaching to the B-S formula as $Y \to 2$.

For $Y \neq 2$, we further compare our solution with the B-M formula derived in [4] for the CGMY model. This formula is also provided in **Section 1.4**. We remark that the fundamental difference between our solution and the B-M formula is that the latter is still written in terms of the inverse Fourier transform without the inversion being carried out analytically.

We also use the generalized LG quadrature to evaluate the B-M formula. In **Fig 4.2(a)**, the comparison between our formula **(4.1.3)** and the B-M formula at $Y = 1.8$ is provided. The absolute errors between the two are further shown in **Fig 4.2(b)**. From these two figures, it is clear that our option price agrees well

(a) The B-M formula price VS our price at $Y = 1.8$ (b) Option price differences

Figure 4.2: Comparison of our solution at $Y = 1.8$ with the B-M formula. Model parameters are $K = \$10$, $r = 0.35$, $C = 0.02$, $M = G = 1$, and $T - t = 0.55$(year).

with the existing formula for the CGMY model, with the maximum absolute error between the two being no more than 5%. This level of accuracy is indeed acceptable in practice. This has again confirmed the reliability of our closed-form solution as well as the proposed numerical implementation technique.

4.3 Impacts of different parameters

With confidence in our analytical solution as well as the proposed numerical implementation technique, we shall now turn to investigate the impacts of different parameters on the prices of European puts.

4.3.1 The impact of Y

Depicted in **Fig 4.3** is the comparison among several sets of European put option prices at different Y values, while all the other parameters are set to be the same. From this figure, one can observe that the option price is a monotonic increasing function of Y. This is indeed reasonable, and can be explained as follows. Firstly, we
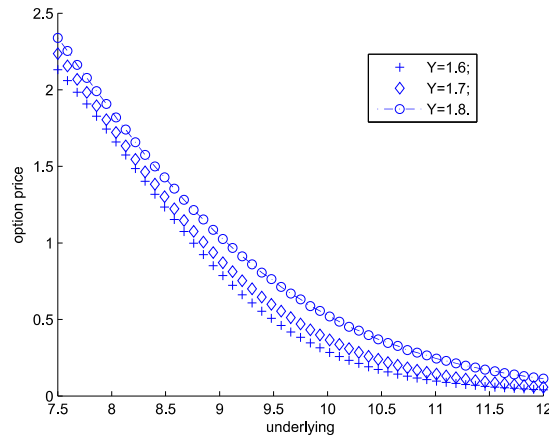

Figure 4.3: European puts at different values of Y. Model parameters are $K = 10$, $G = M = 1$, $r = 0.1$, and $C = 0.01$.

shall focus on analyzing an out-of-the-money European put option. It is known that this option will have a positive value only if there is a large decrease in the underlying price. Therefore, the option price only relies on the left tail of the distribution of the asset, and the fatter the left tail is, the more valuable the option would be. On the other hand, as mentioned earlier in **Section 2.2**, the parameter Y controls both tails of the underlying return distribution. Moreover, the left tail will become fatter as Y becomes larger. Therefore, the option would become more valuable if Y becomes larger.

Now, we turn to investigate those in-the-money European puts. To obtain the pricing biases, the put-call parity derived in **Section 3.4** will be utilized. From the put-call parity, it is clear that if a European put is in the money, the corresponding European call is out of the money, and vise versa. Therefore, analyzing those in-the-money European puts is equivalent to analyzing the corresponding out-of-the-money European calls. For an out-of-the-money European call, its intrinsic value depends

only on the right tail of the distribution of the asset, because this option will become valuable only when there is a large increase in the underlying price. Similarly, the fatter the right tail is, the more valuable the option will become. According to **Fig 2.2**, it is clear that the right tail will also become fatter as Y becomes larger. Therefore, the out-of-the-money European call option price would increase as Y becomes larger, and the corresponding in-the-money European put will do so as well.

Taking all the above points into consideration, it is clear that for a European put under the CGMY model, its price would become higher as Y becomes larger. Similarly, it can be shown that the European call option price is also a monotonic increasing function with respect to Y under the current model.

4.3.2   The impact of C

Depicted in **Fig 4.4** is the comparison among several sets of European put option prices at different C values, while all the other parameters are set to be the same. It can be observed from **Fig 4.4** that the option prices become higher as C becomes larger. From the figure, one can also observe that the option prices become flatter as
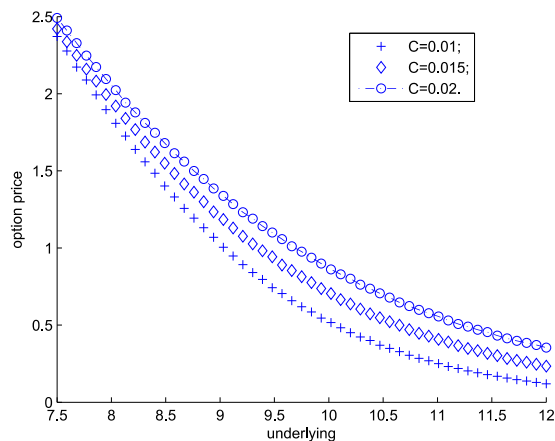

Figure 4.4: European puts at different values of C. Model parameters are $K = 10$, $G = M = 1$, $r = 0.1$, and $Y = 1.8$.

C increases. This is indeed reasonable. As mentioned in **Section 2.2**, the parameter

C controls the overall kurtosis of the distribution as

$$Kurtosis_{CGMY} = 3 + \frac{C(M^{Y-4} + G^{Y-4})\Gamma(4-Y)}{[C(M^{Y-2} + G^{Y-2})\Gamma(2-Y)]^2}.$$

From the above formula, it is clear that the kurtosis function is a monotonic decreasing function of C. Therefore, both tails would become fatter as C increases. According to the discussion in the last subsection, we can conclude that the option price shall move upwards as C becomes larger.
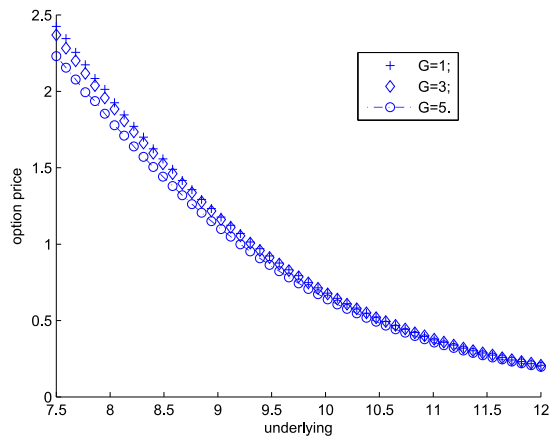
4.3.3 The impacts of G and M

Figure 4.5: European puts at different values of G. Model parameters are $K = 10$, $M = 1$, $r = 0.1$, $Y = 1.8$, and $C = 0.01$.
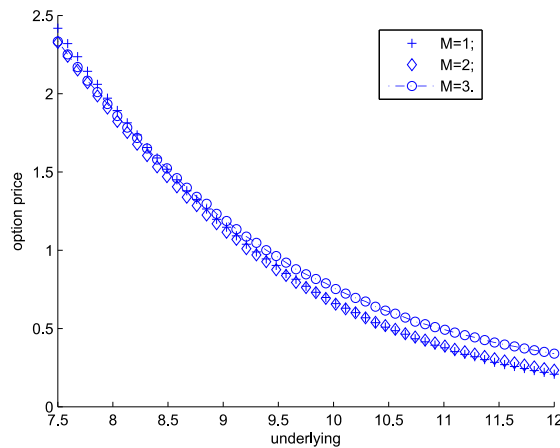

Figure 4.6: European puts at different values of M. Model parameters are $K = 10$, $G = 1$, $r = 0.1$, $Y = 1.8$, and $C = 0.01$.

In this subsection, we shall investigate the impacts of G and M on the European option price. As mentioned in **Section 2.2**, G and M control the rate of exponential decay on the right and left of the Lévy density, respectively.

Depicted in **Fig 4.5** and **Fig 4.6** are the option price of the CGMY model for different values of G and M, respectively. We can observe from **Fig 4.5** that the option price shifts downwards as the increase of the value of G, whereas in **Fig 4.6**, the option price goes upwards as the increase of the value of M. This phenomenon might also be explained from the relationship between the tail of the underlying return distribution and the price of the option. For simplicity, we omit the details here.

Chapter 5

Conclusion

In this thesis, we consider the pricing of European options under the CGMY model. Through Fourier transform, a closed-form analytical solution for European-style options under the CGMY model is successfully obtained for the first time. The asymptotic behaviors of the solution are then examined, which confirms the reliability of the CGMY model. On the other hand, for practical purpose, we propose an efficient numerical method to implement the analytical solution. Numerical experiments confirm the validity of our closed-form analytical solution as well as the reliability of the propose numerical implementation technique. Finally, we discuss quantitatively the influences of different parameters on the option prices. It is also very promising to extend the current approach to price other European-style options under different complicated models with fractional-order derivatives.

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Appendix A

Lemma A.0.1

$$\mathcal{F}\left[e^{-Gx}{}_{-\infty}D_x^Y e^{Gx}f(x)\right] = (G - i\xi)^Y F(\xi),$$

and

$$\mathcal{F}\left[e^{Mx}{}_x D_\infty^Y(e^{-Mx}f(x))\right] = (M + i\xi)^Y F(\xi).$$

Proof. According to the exponential function property of the Fourier transform discussed in **Section 1.2**, we have

$$\mathcal{F}\left[e^{-Gx}{}_{-\infty}D_x^Y(e^{Gx}f(x))\right] = \mathcal{F}\left[{}_{-\infty}D_x^Y(e^{Gx}f(x))\right]\big|_{\lambda = \xi + iG}. \tag{A.0.1}$$

On the other hand, according to **Corollary 1.2.3** contained in **Chapter 1**, we have

$$\mathcal{F}\left[{}_{-\infty}D_x^Y(e^{Gx}f(x))\right] = (-ik)^Y \mathcal{F}[e^{Gx}f(x)], \tag{A.0.2}$$

where $\mathcal{F}\left[e^{Gx}f(x)\right]$ can be further simplified as

$$\begin{aligned}
\mathcal{F}\left[e^{Gx}f(x)\right] &= F(\lambda)\big|_{(\lambda = k - iG)}, \\
&= F(k - iG). \tag{A.0.3}
\end{aligned}$$

Now, combining **(A.0.1)**-**(A.0.3)**, we obtain

$$
\begin{aligned}
F\left[e^{-Gx}{}_{-\infty}D_x^Y(e^{Gx}f(x))\right] &= (-ik)^Y F(k - iG)\,|_{k=\xi+iG}, \\
&= (G - i\xi)^Y F(\xi).
\end{aligned}
$$

Similar approach can also be applied to evaluate $F\left[e^{Mx}{}_xD_\infty^Y(e^{-Mx}f(x))\right]$, and we obtain

$$
\begin{aligned}
F\left[e^{Mx}{}_xD_\infty^Y(e^{-Mx}f(x))\right] &= (ik)^Y F(k + iM)|_{k=\xi-iM}, \\
&= (M + i\xi)^Y F(\xi).
\end{aligned}
$$

.

Appendix B

The definition of CM, FV and FA

The definitions of CM (complete monotone Lévy densities), FV (finite variation) and FA (finite activity) can all be found in [16] as follows.

Theorem 4 *By Bernstein's theorem all CM (complete monotone Lévy densities) are given by the Laplace transforms of positive measures on the positive half line, or that there exists a measure $\rho(da)$ such that*

$$k(y) = \int_0^\infty e^{-a\nu} \rho(da),$$

which means that all such densities can be written in this form for some positive measure $\rho(da)$.

Theorem 5 *Let $X = (X_t)_{t>0}$ be a Lévy process with triplet (b, σ^2, F).*
(a) If $F(\Re) < \infty$, then almost all paths of $X = (X_t)_{t>0}$ have a finite number of jumps on every compact interval. In that case, the Lévy process has finite activity.
(b) If $F(\Re) = \infty$, then almost all paths of $X = (X_t)_{t>0}$ have a infinite number of jumps on every compact interval. In that case, the Lévy process has infinite activity.

Theorem 6 *Let $X = (X_t)_{t>0}$ be a Lévy process with triplet (b, σ^2, F).*
(a) Almost all paths of X have finite variation if $\sigma = 0$ and $\int_{|x|\leq 1} |x| F(dx) < \infty$.
(b) Almost all paths of X have infinite variation if $\sigma \neq 0$ or $\int_{|x|\leq 1} |x| F(dx) = \infty$.