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High rate serially concatenated codes with low error floors

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Keywords
codes, concatenated, serially, floors, error, rate, low, high

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High Rate Serially Concatenated Codes with Low Error Floors

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***Abstract*—Serial concatenation of Hamming codes and an accumulator has been shown to achieve near capacity performance at high code rates. However, these codes usually exhibit poor error floor performance due to their small minimum distances. To overcome this weakness, we propose to replace the outer Hamming codes by product codes constructed from Hamming codes and single-parity-check codes. In this way, the minimum distance of the outer code can be doubled, which is expected to increase the minimum distance of the serially concatenated code and thus to improve error floor performance. Three-dimensional EXIT chart is used for their convergence analysis and the derived thresholds are shown to be close to Shannon limit. Low weight distance spectrum of the proposed code is also calculated and compared with the original code. Simulation results show that the proposed codes can lower the error floor by two orders of magnitudes without waterfall performance degradation at short block length.**

1. Introduction

High rate codes with low error floors are of interests for some applications where high data rates and low error probabilities are required, e.g., magnetic recording systems, optical communications [1], and some future wireless transmission systems [2]. Recently, a class of high-rate serially concatenated codes with Hamming codes as the outer code and an accumulator as the inner code, termed as HA codes (or exHA codes for extended Hamming outer codes), has been shown to achieve near capacity performance in the waterfall region [1], [3]. However, since the outer Hamming codes have minimum distance 3 (or 4 for extended Hamming codes), the resulting serially concatenated codes usually have rather small minimum distances, thus leading to poor error floor performance. For example, the minimum distances of HA codes are typically 2 or 3 when overall code length is 992 (see the analysis in Section 4). This weakness hinders its applications in systems where low error rate is expected such as optical communication systems and data storage devices [4]. One way to mitigate the weakness is to optimize the interleaver design [5]. Another approach is to append a second accumulator to HA codes and form double serially concatenated codes, termed as HAA codes [6]. It is shown that HAA codes have minimum distance growing linearly with the block length and thus they are expected to achieve very good error floor performance [6]. However, due to the serial concatenation with two accumulators, iterative decoding of HAA codes incurs a non-negligible loss at the convergence threshold. For example, using (31, 26) Hamming code as the outer code, the convergence thresholds of HA codes and HAA codes are $E_b/N_0 = 2.77$ dB and 3.48 dB, respectively [6]. This implies that the serial concatenation of a second accumulator leads to a threshold loss of 0.71 dB. Thus, how to balance the performance in error floor region and waterfall region is a critical issue in the design of high rate codes with iterative decoding [1].

To increase the minimum distances of HA codes while maintaining their good decoding thresholds, this paper proposes to enhance the outer Hamming codes by using high-rate single-parity-check (SPC) codes. More specifically, the outer codes are replaced by product codes [7] with Hamming codes and high-rate SPC codes as the two component codes. The resulting serially concatenated codes are called HSA codes (or exHSA codes for extended Hamming codes). Using a high-rate SPC code as one component code, the product code can double the minimum distance of Hamming code and the code rate loss can be marginal.

The remainder of this paper is organized as follows. Section 2 gives a detailed description of the encoder and the associated iterative decoder for the proposed high rate codes. Three-dimensional EXIT charts are used for analysing the iterative decoding behaviour of the proposed codes and iterative decoding threshold are determined in Section 3. In Section 4, the low-weight distance spectrum of the proposed codes is calculated, and the simulation results are presented to confirm the analysis in Section 5.

2. HSA Codes: Encoder and Decoder

HSA codes is a class of serially concatenated codes with product codes (constructed from Hamming codes and SPC codes) as the outer code and an accumulator as the inner code. The encoder and decoder of HSA codes are detailed in the following two subsections, respectively.

2.1. Encoder

Figure 1(a) depicts the encoder structure of the serial concatenation of an outer code and an inner accumulator through an interleaver π. The use of Hamming codes and extended Hamming codes as outer codes has been considered in [1], [3]. Since the minimum distances of Hamming and extended Hamming codes are very small (3 and 4, respectively), the resultant serially concatenated codes generally

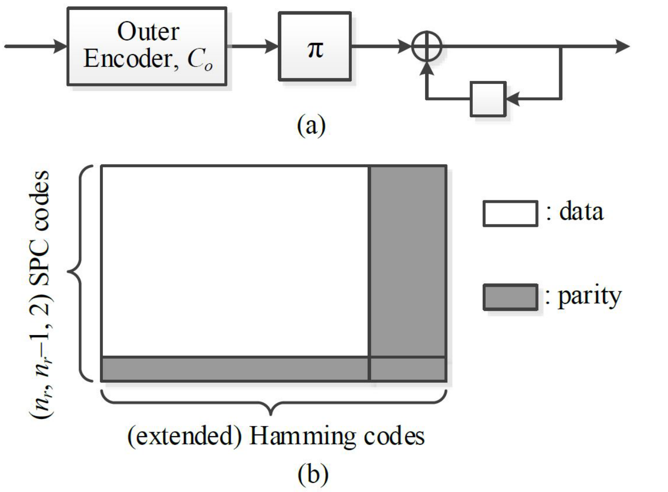

Figure 1. (a) Encoder strucuture of serially concatenated codes with an inner accumulator; (b) An outer product code with (extended) Hamming codes and SPC codes as component codes.

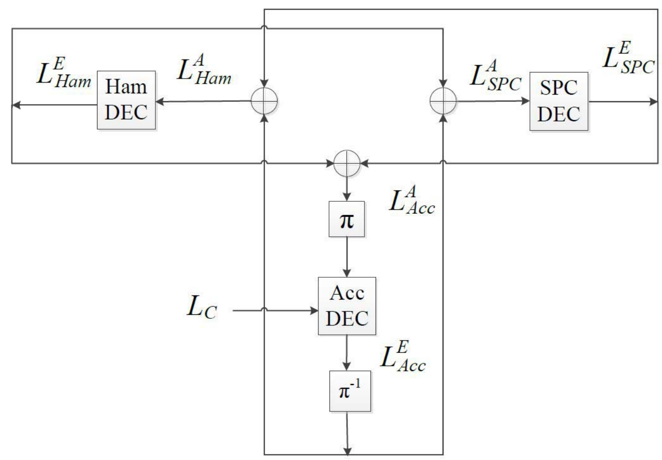

Figure 2. Iterative decoder of HSA codes with three constituent decoders. "Acc DEC", "Ham DEC" and "SPC DEC" denote the accumulator decoder, Hamming decoder and SPC decoder, respectively. π and π^{-1} denote the interleaver and deinterleaver.

exhibit small minimum distances, thus leading to high error floor performance. Here, we propose to use as the outer code the product code with Hamming and SPC component codes as shown in Fig. 1(b). The product code is depicted as an array, where each row is a Hamming code and each column is an SPC code. Compared to HA codes, the rates of HSA codes are reduced by a factor of $(n_r-1)/n_r$, which is the code rate of the SPC code. It is easy to control the rate loss by adjusting the number of rows, n_r, in the code array. It is well known that the minimum distance (d_{min}) of a product code is the product of the d_{min}'s of its two component codes [8]. Moreover, the d_{min} of SPC codes is 2. Thus, an advantage of using proposed product codes as the outer code is that the minimum distance of the outer code can be doubled. More specifically, the minimum distance of the outer code is increased from 3 to 6 for the case of Hamming codes and from 4 to 8 for the case of extended Hamming codes.

2.2. Decoder

From Fig. 1(b), each coded bit in the outer product code joins a Hamming code and an SPC code. After the outer product encoding, we can see from Fig. 1(a) that the coded bits of the outer product code are interleaved and then used as the input to the accumulator. Thus, each coded bit in the outer product code in fact joins 3 code constraints: a Hamming code, an SPC code, and the accumulator. Accordingly, the iterative decoder can be constructed by employing three soft-input/soft-output (SISO) decoders (i.e., Accumulator decoder, Hamming decoder and SPC decoder) as shown in Fig. 2. The SISO decoder for the inner accumulator can be efficiently conducted with low complexity by performing the forward-backward algorithm on its factor graph representation [11]. For SISO decoding of Hamming codes, we adopt the low complexity algorithm proposed in [14], which is based on the dual code decoding principle firstly developed by Hartmann and Rudolph in [13]. As the dual codes of extended Hamming codes are first order Reed-Muller codes whose symbol-by-symbol maximum a posteriori (MAP) decoding can be done with fast Hadamard transforms (FHTs), SISO decoding of (extended) Hamming codes can also be efficiently implemented by using FHTs. The SISO decoding of SPC codes is exactly the same as the row decoding in LDPC codes, which can be implemented by the famous sum-product algorithm (see, e.g., [8]).

Denote the overall code length as N. Let $\mathbf{c} = (c_0, c_1, ..., c_{N-1})$ be the codeword ,where c_n denotes the nth element of $\mathbf{c}$ and takes valuse from $\{0,1\}$. Assume binary phase-shift keying (BPSK) modulation (with mapping $0 \rightarrow +1$ and $1 \rightarrow -1$) is used, and the modulated signal is transmitted over an additive white Gaussian noise (AWGN) channel. The received signal can be written as

$$y_n = (1-2c_n) + \omega_n,$$

where y_n is the nth element in the received vector $\mathbf{y} = (y_0, y_1, ..., y_{N-1})$ and $\omega_n \sim N(0, \sigma^2)$ is the nth sample of the AWGN.

Before we introduce the detailed decoding algorim, it is necessarily to introduce some notations. $L^A_{Acc,n}$, $L^A_{Ham,n}$ and $L^A_{SPC,n}$ denote a priori information of nth bit in log-likelihood ratio form (L-value) [9] for accumulator decoder, Hamming decoder, and SPC decoder, respectively. Similarly, $L^E_{Acc,n}$, $L^E_{Ham,n}$ and $L^E_{SPC,n}$ denote the generated extrinsic L-values of nth bit for the three decoders. The iterative decoding of the proposed codes can be performed by serially activating the three component SISO decoders, i.e., "Acc DEC", "Ham DEC" and "SPC DEC" as shown in Fig. 2. The detailed decoding algorithm is summarized in Algorithm 1.

Algorithm 1 Iterative decoding of HSA codes

Step 1: Initialization

Calculate the L-value, $L_{C,n}$, from channel observation $L_{C,n} = \log \frac{P(y_n|x_n=0)}{P(y_n|x_n=1)} = \frac{2y_n}{\sigma^2}$,
and set all the extrinsic information to 0,
$L^E_{Acc,n} = L^E_{Ham,n} = L^E_{SPC,n} = 0, n = 0, 1, ..., N-1$.

Step 2: Iterative decoding

At each iteration, the three constituent decoders are performed in a serial fashion.

Accumulator decoder: Compute a priori information $L^A_{Acc,n} = L^E_{Ham,n} + L^E_{SPC,n}$. Then, input $\{L^A_{Acc,n}\}$ and $\{L_{C,n}\}$ to the accumulator decoder and generate extrinsic information $\{L^E_{Acc,n}\}$.

Hamming decoder: Compute a priori information $L^A_{Ham,n} = L^E_{Acc,n} + L^E_{SPC,n}$. Then, input $\{L^A_{Ham,n}\}$ into the Hamming decoder and generate extrinsic information $\{L^E_{Ham,n}\}$.

SPC decoder: Compute a priori information $L^A_{SPC,n} = L^E_{Ham,n} + L^E_{Acc,n}$. Then, input $\{L^A_{SPC,n}\}$ into the SPC decoder and generate extrinsic information $\{L^E_{SPC,n}\}$.

Step 3: Decision

Compute the L-value $L_{H,n} = L^E_{Acc,n} + L^E_{Ham,n} + L^E_{SPC,n}$. and make hard decision $\hat{x}_n$ for the nth bit as follows,

$$\hat{x}_n = \begin{cases} 0 & L_{H,n} > 0 \\ 1 & L_{H,n} \le 0 \end{cases}.$$

If the maximum iteration number reached, stop decoding. Otherwise, go back to Step 2.

3. Threshold Analysis Via Three-Dimensional EXIT Chart

The convergence behavior of iteratively decoded systems can be accurately analyzed by using the density evolution (DE) algorithm [16]. However, as DE tracks the evolution of probability density functions (pdfs) of soft information, its computational complexity is very high. A simplified version of DE, referred to extrinsic information transfer (EXIT) chart, is proposed in [9], which uses mutual information as the surrogate of pdfs. The input-output relations of constituent decoders are depicted by EXIT functions which characterizes how a priori information transfer into extrinsic information at the SISO decoder. A decoding trajectory for the exchange of extrinsic information between constituent decoders can be visualized in an EXIT chart.

For iterative decoding systems with two component decoders, each decoder can be characterized by an EXIT function, which is usually obtained via simulation with the assumption that the a priori decoder input follows the symmetric Gaussian distribution [9]. Graphically, an EXIT function can be visualized as a curve in the EXIT chart. Notice that as the extrinsic information from one decoder is used as the priori information for the other decoder, the EXIT curve for the second decoder can be drawn in the same chart for the first decoder by swapping the axes. In this way, the convergence behavior of the iteratively decoded system with two component decoders can be visualized by the decoding trajectory between the two EXIT curves [9].

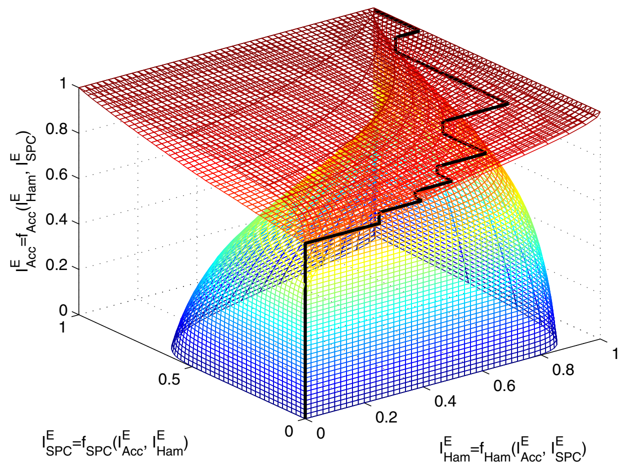

Figure 3. Three-dimensional EXIT chart for Hamming(31,26)-SPC(32,31)-Accumulate code at $E_b/N_0 = 3.19$ dB

Later, the EXIT chart tool is further extended for the analysis of three-dimensional parallelly concatenated system by Ten Brink in [10] and three-dimensional serially concatenated system by Tüchler in [15]. From the encoding perspective, the proposed HSA codes can be viewed as a Hybrid concatenation scheme, the product code and the accumulator are serially concatenated, while the product code itself can be viewed as a parallel concatenation. However, as mentioned in Subsection 2.2 each coded bit in the outer product code joins 3 code constraints and then an HSA code can be treated as a parallelly concatenated code by viewing the product codeword as the "input". In fact, the proposed decoder as shown in Fig. 2 has the same structure as that for parallelly concatenated code (see Fig. 2 in [9]). Hence, the three-dimensional EXIT chart developed for parallelly concatenated codes in [10] can be adopted for the analysis of the proposed codes. As seen from Fig. 2, for a three-dimensional parallelly concatenated code, each constituent decoder has two inputs and one output, which means the associated EXIT function is a two-input and one-output function, and is visualized as a surface rather than a curve in the case of two-dimensional EXIT chart. Now we use the Hamming decoder as an example to explain how to generate the EXIT surface for a constituent decoder. The EXIT function denoted as $I^E_{Ham} = f_{Ham}(I^E_{Acc}, I^E_{SPC})$, where I^E_{Acc}, I^E_{Ham} and I^E_{SPC} denote the mutual information which are related to the extrinsic information generated by accumulator decoder, Hamming decoder and SPC decoder, respectively. To approximate the function, we only need a fine grid of the (I^E_{Acc}, I^E_{SPC}) over the area $[0, 1]^2$, and for each point (I^E_{Acc}, I^E_{SPC}) in the grid simulation is required to find the associated I^E_{Ham}. The detailed procedure is similar to that in [9].

As an example, Fig. 3 shows the three-dimensional EXIT chart at $E_b/N_0 = 3.19$ dB for the HSA code with (31, 26) Hamming codes and (32, 31) SPC component codes . There are three surfaces in the three dimensional EXIT chart; each surface corresponds to the extrinsic mutual information transfer characteristic of a constituent decoder which accepts the priori knowledge from other two decoders. To insure that successful decoding, it is necessary to guarantee that the trajectory can go up to (1, 1, 1). Equivalently, a tunel from (0, 0, 0) to (1, 1, 1) is required. Otherwise, the trajectory will get stuck and decoding cannot converge to the correct codeword. The threshold is the minimum E_b/N_0-value at which a tunel from (0,0,0) to (1,1,1) is possible. With the help of the three dimensional EXIT chart, we can easily identify the thredholds of HSA codes by graduately tunning the value of E_b/N_0. As an example, the iterative decoding threshold of HSA codes with the outer product code constructed from (31,26) Hamming code and (32,31) SPC code is found to be $E_b/N_0 = 2.80$ dB, which is better than the threshold, $E_b/N_0 = 3.48$ dB, of the HAA codes. Notice that the corresponding Shannon limit is $E_b/N_0 = 2.2$ dB. Hence, the proposed HSA code is about 0.6 dB away from the Shannon limit.

4. Low Weight Profile Analysis

As the error floor performance is largely determined by low weight codewords, this section examines and compares the low weight distance spectra of the proposed codes and the existing ones. For comparison purpose, we compute the low weight profiles for 3 length-992 code ensembles, i.e., HA code ensemble with (31,26) Hamming outer code, exHA code ensemble with (32, 26) extended Hamming outer code, and HSA code ensemble with the product code constructed from (31,26) Hamming code and (32,31) SPC code as the outer code. Note that the HSA code ensemble has the same code rate as the exHA code ensemble. Using the uniform interleaver concept [11], the ensemble-average weight enumerator (WE) of the code ensemble can be computed as:

$$A_h = \sum_w \frac{A_w^{C_0} A_{w,h}^{Acc}}{\binom{N}{w}}, \qquad (1)$$

where A_h denotes the ensemble-average number of codewords of weight h, $A_w^{C_0}$ is the WE of the outer code C_0 and $A_{w,h}^{ACC}$, the input-output WE of the inner accumulator, can be written in closed form as [8]:

$$A_{w,h}^{ACC} = \binom{N-h}{\lfloor w/2 \rfloor}\binom{h-1}{\lceil w/2 \rceil - 1} \qquad (2)$$

The ensemble-average low-weight profiles are summarized in Table 1. From Table 1, we can see that the ensemble-average number of weight-4 codewords for HA codes is 8.5271, which implies that a randomly generated HA code has a high probability of having a minimum distance no greater than 4. In fact, 20 length-992 HA codes are constructed by randomly generating 20 interleavers and the triple impulse method in [6] was used to determine their minimum distances (d_{min}), among which 9 codes were found to have $d_{min} = 2$, 10 were found to have $d_{min} = 3$, and only 1 has $d_{min} = 4$. Similar results are observed for exHA codes. However, when the outer code is replaced by the product code with (31,26) Hamming code and (32,31) SPC code as component codes, the values of the low-weight profile of HSA codes are more than two orders smaller than those of HA codes and exHA codes. In this case, the triple impulse method fails to find dmin with reasonable values. This implies that HSA codes could have larger d_{min}'s and much better error floor performance could be expected. For comparison, we also include in Table 1 the low-weight profile for the length-992 HAA code ensemble with (31,26) Hamming outer code. Although the HAA code ensemble has even smaller values at the low-weight profile, it incurs a non-negligible performance in the waterfall region as stated above. In fact, at short block lengths, we find HSA codes could provide similar error floor performance as HAA codes, as will be shown in Fig. 4.

TABLE 1. Ensemble-average low-weight profiles of 4 length-992 code ensembles with (31,26) Hamming code or (32,26) extended Hamming code as component codes. A_h is the average number of codewords with Hamming weight h.

	A_1	A_2	A_3	A_4
HA code	0	0.4541	2.4750	8.5271
exHA code	0	0.4692	1.0427	3.0685
HSA code	0	0	0.0095	0.0293
HAA code	0.0009	0.0010	0.0012	0.0013

TABLE 2. The parameters of 5 serially concatenated codes.

	Code Len.	Info. Len	(Ex)Ham Code	SPC code
HA code	992	832	(31,26)	–
exHA code	992	806	(32,26)	–
HAA code	992	832	(31,26)	–
HSA code	992	806	(31,26)	(32,31)
exHSA code	992	780	(32,26)	(31,30)

5. Numerical Results

To verify the above analysis, 5 length-992 serially concatenated codes are constructed by using random interleavers. The code parameters of the constructed 5 serially concatenated codes are listed in table 2. Figure 4 compares their frame error rate (FER) performance on the AWGN channel with BPSK modulation. The maximum iteration number is set to 30. For each simulation point at least 50 frame errors are collected. As expected, due to their small minimum distances, the HA and exHA codes exhibit poor FER performance; the FER error floors appear around FER of 10^{-3}. Although the HAA code outperforms the HA and exHA codes at the error floor region, a degradation of about 0.5 dB in the waterfall region is observed. The HSA code achieves much better error floor performance compared to HA and exHA codes and its FER curve tends to show a similar slope as that of the HAA code at the high SNR

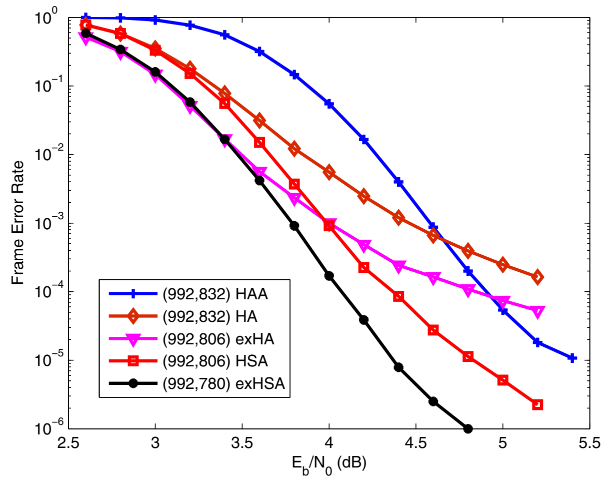

Figure 4. Frame error rate performance of length-992 high-rate serially concatenated codes on the AWGN channel with BPSK modulation.

region. The exHSA code achieves the best performance, which is achieved at the cost of a slight code rate reduction.

6. Conclusion

Low-weight profile analysis has revealed that randomly generated HA and exHA codes have high probabilities of producing low weight codewords, which are responsible for their poor error floor performance. To overcome this weakness, we have proposed to replace the outer codes in HA and exHA codes with product codes from (extended) Hamming codes and high-rate SPC codes. Such a replacement maintains the good waterfall performance of HA and exHA codes, while the minimum distance of the outer code is doubled, thus leading to much better error floor performance.

Acknowledgments

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