# Logics for belief functions on MV-algebras☆

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## Abstract

In this paper we present a generalization of belief functions over fuzzy events. In particular we focus on belief functions defined in the algebraic framework of finite MV-algebras of fuzzy sets. We introduce a fuzzy modal logic to formalize reasoning with belief functions on many-valued events. We prove, among other results, that several different notions of belief functions can be characterized in a quite uniform way, just by slightly modifying the complete axiomatization of one of the modal logics involved in the definition of our formalism.

*Keywords:* Belief functions; Łukasiewicz Logic; Modal logics; Fuzzy Events.

## 1. Introduction and motivation

Dempster-Shafer theory of evidence [8, 33] is a generalization of Bayesian probability theory in which degrees of uncertainty are evaluated by belief functions, rather than by probability measures. Belief functions [33, 34] can be regarded as a special class of measures of uncertainty on Boolean algebras of events representing an agent's degree of confidence in the occurrence of some event by taking into account different bodies of evidence that support that belief [33]. Such evidence plays a pivotal role in determining the agent's belief. Indeed, as we will recall in a while, although any belief function on the Boolean algebra $2^X$ of subsets of a finite set $X$ might be seen as a particular probability, its associated distribution (called *mass* in Dempster-Shafer theory) maps the whole algebra $2^X$ into $[0, 1]$, and not only its atoms. Every set $Y \subseteq X$ with a strictly positive mass represents a body of evidence and is called a *focal element*.

In the literature several attempts to extend belief functions on fuzzy events can be found. The first extension of Dempster-Shafer theory to the general framework of fuzzy set theory was proposed by Zadeh in the context of information granularity and possibility theory [37] in the form of an expected conditional necessity. After Zadeh, several further generalizations were proposed depending on the way a measure of *inclusion among fuzzy sets* is used to define the belief functions of fuzzy events based on fuzzy evidence. Indeed, given a mass assignment $m$ for the bodies of evidence $\{A_1, A_2, \ldots\}$, and a measure $I(A \subseteq B)$ of inclusion among fuzzy sets, the belief of a fuzzy

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set $B$ can be defined in general by the value: $Bel(B) = \sum_i I(A_i \subseteq B) \cdot m(A_i)$. We refer the reader to [24, 35] for exhaustive surveys, and [1] for another approach through fuzzy subsethood. Different definitions were also introduced by Dubois and Prade [11] and by Denœux [9, 10] to deal with belief functions ranging over intervals or fuzzy numbers.

Recently, in [26, 27] and in [15], the authors introduce a treatment of belief functions on fuzzy sets within the algebraic framework of MV-algebras. We will recall the main ideas of these approaches in Section 4, but it is worth pointing out that the choice of MV-algebras as a setting for that investigation will play a notable role in the development of the present work. In fact here we will focus our attention on the introduction of a multimodal logic for belief functions on fuzzy sets, and, since MV-algebras are the equivalent algebraic semantics for Łukasiewicz calculus, the latter can be used both as ground logic to treat fuzzy events and as setting to axiomatize belief functions over them as well.

The idea of formalising a logical system to reason with belief functions within the framework of Łukasiewicz logic is not new. In fact, a logic to reason with classical belief functions over Łukasiewicz logic was defined in [17] as a fuzzy probabilistic extension of the classical S5 modal logic. The approach is based on exploiting the fact that a belief function on classical logic formulas $\varphi$ can be interpreted as a probability on modal formulas $\Box\varphi$, and hence, in that setting, a formula of the kind $P\Box\varphi$, where $P$ is a fuzzy modality for probability, can be read as *$\varphi$ is believable* and its semantics given by belief functions.

The treatment we propose here can be considered as an extension and a generalization of [17]. In particular we will focus on representing belief functions defined over fuzzy sets of finite range, that is, fuzzy sets on a finite set $X$ and with membership values on a finite subset $S_k = \{0, 1/k, \ldots, (k-1)/k, 1\}$ of the real unit interval $[0, 1]$. As we will recall later, every finite MV-algebra can be easily represented as a subalgebra of fuzzy sets of the form $(S_k)^X = \{f \mid f : X \to S_k\}$, for some natural $k$. Then, a probabilistic modality $P$ will be introduced into a suitable modal logic $\Lambda_k$ over the $(k+1)$-valued Łukasiewicz logic $\text{Ł}_k$, and we will define the belief degree of a fuzzy event modeled by a $\text{Ł}_k$ formula $\psi$ as the probability of $\Box\psi$, i.e. as the truth degree of $P\Box\psi$.

It is worth noticing that there is not a unique way to generalize belief functions on MV-algebras. In fact, we can distinguish at least the cases in which the belief functions are such that their focal elements are (1) *crisp sets*, (2) *fuzzy sets*, and (3) *normalized fuzzy sets*. Remarkably, all these cases can be uniformly treated in our multimodal setting only by distinguishing among several axiomatic extensions of the intermediate modal logic $\Lambda_k$. We will discuss these topics in the subsections 6.1 and 6.2.

This paper is organized as follows. In Section 2 we will recall the basic notions about classical belief functions, while Section 3 is devoted to preliminaries on finitely and infinitely-valued Łukasiewicz logics, MV-algebras and states. Then in Section 4 we will introduce belief functions on MV-algebras and we will prove some basic properties. In Section 5 we consider another equivalent approach to define belief functions on MV-algebras based on a generalization of Dempster's spaces. Section 6 will be devoted to the modal expansion $\Lambda_k$ of $\text{Ł}_k^{\mathbf{c}}$, the $(k+1)$-valued Łukasiewicz logic $\text{Ł}_k$ with truth constants, proving results concerning local finiteness and completeness. Moreover, in Subsections 6.1 and 6.2, we will introduce two relevant axiomatic extensions of $\Lambda_k$ that will be used to characterize distinguished classes of belief functions. In Section 7 we finally introduce the probabilistic logic over $\Lambda_k$, $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$, a class of probabilistic-based models, and we prove completeness. Subsection 7.1 will focus on completeness of the logic $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$ with respect to the semantics defined by belief function-based models, while in Subsection 7.2 we will introduce an extension of $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$ to deal with normalized belief functions. We end with Section 8, where we discuss our future work.

## 2. Preliminaries on Belief functions on Boolean algebras

Consider a finite set $X$ whose elements can be regarded as mutually exclusive (and exhaustive) propositions of interest, and whose powerset $2^X$ represents all such propositions. The set $X$ is usually called the *frame of discernment*, and every element $x \in X$ represents the lowest level of discernible information we can deal with.

A map $m : 2^X \to [0, 1]$ is said to be a *basic belief assignment*, or a *mass assignment* whenever

$$m(\emptyset) = 0 \text{ and } \sum_{A \in 2^X} m(A) = 1.$$

Given such a mass assignment $m$ on $2^X$, for every $A \in 2^X$, the *belief of* $A$ is defined as

$$\mathbf{b}_m(A) = \sum_{B \subseteq A} m(B). \tag{1}$$

Every mass assignment $m$ on $2^X$ is in fact a probability distribution on $2^X$ that naturally induces a probability measure $P_m$ on $2^{2^X}$. Consequently, the belief function $\mathbf{b}_m$ defined from $m$ can be equivalently described as follows: for every $A \in 2^X$,

$$\mathbf{b}_m(A) = P_m(\{B \in 2^X : B \subseteq A\}). \tag{2}$$

Therefore, identifying the set $\{B \in 2^X : B \subseteq A\}$ with its characteristic function on $2^{2^X}$ defined by

$$\beta_A : B \in 2^X \mapsto \begin{cases} 1 & \text{if } B \subseteq A \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

it is easy to see that, for every $A \in 2^X$, and for every mass assignment $m : 2^X \to [0, 1]$, we have $\mathbf{b}_m(A) = P_m(\beta_A)$. This easy characterization will be important when we discuss the extensions of belief functions on MV-algebras. The following is a trivial observation about the map $\beta_A$ that can be useful to understand our generalization: for every $A \in 2^X$, $\beta_A$ can be regarded as a map evaluating the (strict) inclusion of $B$ into $A$, for every subset $B$ of $X$.

A subset $A$ of $X$ such that $m(A) > 0$ is said to be a *focal element*. Every belief function is characterized by the value that $m$ takes over its focal elements, and therefore, the focal elements of a belief function $\mathbf{b}_m$ contain the pieces of evidence that characterize $\mathbf{b}_m$ itself. For every set $X$ and for every mass assignment $m$, call $\mathfrak{F}_m$ the set of focal elements of $2^X$ with respect to $m$. It is well known that several subclasses of belief functions can be characterized just by the structure of their focal elements. In particular, when $\mathfrak{F}_m \subseteq \{\{x\} : x \in X\}$, it is clear that $\mathbf{b}_m$ is a probability measure. Moreover, if the focal elements are nested subsets of $X$, i.e. $\mathfrak{F}_m$ is a chain with respect to the inclusion relation between sets, then $\mathbf{b}_m$ is a *necessity measure* [11, 33]; this means e.g. that $\mathbf{b}_m(A_1 \cap A_2) = \min\{\mathbf{b}_m(A_1), \mathbf{b}_m(A_2)\}$.

## 3. Preliminaries on Łukasiewicz logic, MV-algebras and states

The logical setting in which we frame our study is that of (infinitely-valued) Łukasiewicz logic Ł, and its finitely-valued schematic extensions $Ł_k$. Formulas of (any finitely-valued) Łukasiewicz logic are inductively defined from a countable set $V = \{p_1, p_2, \ldots\}$ of variables, along with the binary connective $\to$ and the unary connective $\neg$. We will denote by $\mathfrak{F}(V)$ the class of formulas defined from the set of variables $V$.

Further connectives are definable from $\to$ and $\neg$ as follows:

$$\begin{array}{lll} \varphi \oplus \psi & \text{is} & \neg\varphi \to \psi \\ \varphi \odot \psi & \text{is} & \neg(\neg\varphi \oplus \neg\psi) \\ \varphi \vee \psi & \text{is} & (\varphi \to \psi) \to \psi \\ \varphi \wedge \psi & \text{is} & \neg(\neg\varphi \vee \neg\psi) \\ \varphi \leftrightarrow \psi & \text{is} & (\varphi \to \psi) \wedge (\psi \to \varphi) \end{array}$$

The truth constant $\top$ is $\varphi \to \varphi$ and the truth constant $\bot$ is $\neg\top$, and we will henceforth use the following abbreviations: for every $n \in \mathbb{N}$ and for every $\varphi \in \mathfrak{F}(V)$, $n\varphi$ will stand for $\varphi \oplus \ldots \oplus \varphi$ ($n$-times), and $\varphi^n$ will stand for $\varphi \odot \ldots \odot \varphi$ ($n$-times).

The propositional Łukasiewicz logic (Ł in symbols) is defined as the following Hilbert style system of axioms and rules (cf. [21]):

(Ł1) $\varphi \to (\psi \to \varphi)$,

(Ł2) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$,

(Ł3) $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$,

(Ł4) $(\varphi \vee \psi) \to (\psi \vee \varphi)$,

(MP) The rule of modus ponens: from $\varphi$ and $\varphi \to \psi$, deduce $\psi$.

For every $k \in \mathbb{N}$, the $(k+1)$-valued Łukasiewicz logic $Ł_k$ is the axiomatic extension of Ł defined by the following axioms (cf. [19, 21]):

(Ł5) $(k-1)\varphi \leftrightarrow k\varphi$,

(Ł6) $(l\varphi^{l-1})^k \leftrightarrow k\varphi^l$, for every $l = 2, \ldots, k-2$ that does not divide $k-1$.

The notion of *deduction* and *proof* are the usual ones (see [21]). A *theory* is any subset of $\mathfrak{F}(V)$, and for every theory $\Gamma$ and for every formula $\varphi$ we will write $\Gamma \vdash \varphi$ if $\varphi$ can be proved from $\Gamma$ in the logic $Ł_k$.

The algebraic counterpart of (finitely-valued) Łukasiewicz calculus is the class of (finitely-valued) MV-algebras. An MV-algebra (cf. [7, 21, 30]) is a system $M = (M, \oplus, \neg, 0^M)$ of type $(2,1,0)$ such that the reduct $(M, \oplus, 0^M)$ is a commutative monoid, and the following equations hold:

(MV1) $x \oplus \neg 0^M = \neg 0^M$,

(MV2) $\neg\neg x = x$,

(MV3) $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

For every $k \in \mathbb{N}$, an $\mathrm{MV}_k$-algebra is any MV-algebra that also satisfies:

(MV4) $kx = (k-1)x$,

(MV5) $(lx^{l-1})^k = kx^l$, for every $l = 2, \ldots, k-2$ that does not divide $k-1$,

where, in (MV4) and (MV5), $1^M$ stands for $\neg 0^M$, and for every $n \in \mathbb{N}$, $nx = x \oplus \ldots \oplus x$ ($n$-times), and $x^n = x \odot \ldots \odot x$ ($n$-times). As in the case of the logical language, here other operations can also be defined, among them $x \to y$ is $\neg x \oplus y$ and $x \odot y$ is $\neg(\neg x \oplus \neg y)$.

In every MV-algebra $M$ we can define an order relation by the following stipulation: for every $x, y \in M$,

$$x \leq y \text{ iff } \neg x \oplus y = 1.$$

An MV-algebra is said to be linearly ordered, or an MV-chain, provided that the order $\leq$ is linear.

An *evaluation* $e$ of formulas of $\mathfrak{F}(V)$ into an MV-algebra ($\mathrm{MV}_k$-algebra) $M$ is any map $e : V \to M$ that extends to compound formulas by truth functionality using the operations in $M$. We say that $e$ is a model of (or satisfies) a formula $\varphi \in \mathfrak{F}(V)$ when $e(\varphi) = 1^M$. The class of MV-algebras constitutes a variety (i.e. an equational class [3]), and MV-algebras are the equivalent algebraic semantics for Łukasiewicz logic. Similarly, for every $k$, $\mathrm{MV}_k$-algebras form a variety that is the equivalent algebraic semantics for $Ł_k$. Therefore Łukasiewicz logic is complete with respect to the class of MV-algebras, and $Ł_k$ is complete with respect to class of $\mathrm{MV}_k$-algebras.

**Example 3.1** (Standard Algebras). *(1) Equip the real unit interval* $[0,1]$ *with the operations of*

- *truncated sum: for all* $x, y \in [0,1]$, $x \oplus y = \min(1, x+y)$,
- *standard negation: for all* $x \in [0,1]$, $\neg x = 1 - x$.

*Then the algebra* $[0,1]_{\mathrm{MV}} = ([0,1], \oplus, \neg, 0)$ *is an MV-algebra called the* standard MV-algebra. *The variety of MV-algebras is generated, as a variety and as a quasi-variety, by* $[0,1]_{\mathrm{MV}}$ *(cf. [4, 7]). This means that, in order to show that a given equality, or quasi-equality, written in the algebraic language of MV-algebras, holds in every MV-algebra, it is sufficient to check whether it holds in* $[0,1]_{\mathrm{MV}}$.

*(2) For every* $k \in \mathbb{N}$ *let* $S_k = \{0, 1/k, \ldots, (k-1)/k, 1\}$. *Equip* $S_k$ *with the restrictions to* $S_k$ *of the above defined truncated sum and standard negation. We will henceforth denote by* $\mathbf{S}_k$ *the obtained structure, that is usually called the* standard $\mathrm{MV}_k$-algebra. *The variety of* $MV_k$*-algebras is generated by* $\mathbf{S}_k$ *(cf. [7]).*

Clearly, the above examples (and the results cited therein) show a stronger version of completeness for Ł and $Ł_k$ that we are going to make clear as follows.

**Theorem 3.2.** (1) *Łukasiewicz logic has the* finite strong real completeness *(*FSRC *for short), i.e.: for every finite theory* $\Gamma \subseteq \mathfrak{F}(V)$, *and for every formula* $\varphi$, $\Gamma \vdash \varphi$ *in Ł iff every evaluation into the MV-algebra* $[0,1]_{\mathrm{MV}}$ *that satisfies* $\Gamma$, *satisfies* $\varphi$ *as well.*

(2) *For every* $k \in \mathbb{N}$, $Ł_k$ *has the* strong real completeness *(*SRC *for short), i.e.: for every theory* $\Gamma \subseteq \mathfrak{F}(V)$, *and for every formula* $\varphi$, $\Gamma \vdash \varphi$ *in* $Ł_k$ *iff every evaluation into the* $MV_k$*-algebra* $\mathbf{S}_k$ *that satisfies* $\Gamma$, *satisfies* $\varphi$ *as well.*

Every MV-algebra $M$ contains a largest Boolean algebra $B(M)$ called the *Boolean skeleton* of $M$, which is constituted by all the idempotent elements of $M$. Indeed, the universe of $B(M)$ coincides with the set $\{x \in M : x \odot x = x\}$.

**Remark 3.3.** *It is worth noticing that every finite MV-algebra $M$ can be represented as a finite direct product of finite MV-chains. In other words, for every finite MV-algebra $M$, there exists a finite MV-chain $\mathbf{S}_k$, and a finite index set $X$ such that $M$ embeds into the direct product $\mathbf{S}_k)^X$. This means that every finite MV-algebra can be seen as a MV-subalgebra of functions from $X$ into $S_k$, i.e. as a MV-algebra of $S_k$-valued fuzzy sets of $X$. Therefore, without loss of generality, we will henceforth concentrate on finite MV-algebras of fuzzy sets of this form.*

*3.1. Expanding Łukasiewicz logic with rational truth constants*

Let $\mathcal{L}$ denote either Ł or Ł$_k$, and let $\mathcal{Q}(\mathcal{L})$ denote the set of all the rational numbers included into the standard algebra of $\mathcal{L}$ (recall Example 3.1). Therefore, if $\mathcal{L}$ stands for Ł then $\mathcal{Q}(\text{Ł})$ stands for $[0,1]\cap\mathbb{Q}$, while if $\mathcal{L}$ stands for any $(k+1)$-valued Łukasiewicz logic Ł$_k$, then clearly $\mathcal{Q}(\text{Ł}_k) = S_k$.

The logic $\mathcal{L}^{\mathbf{c}}$ is obtained by expanding the language of Łukasiewicz logic by means of symbols $\overline{r}$ for each $r \in \mathcal{Q}(\mathcal{L})$,[1] and adding the following *bookkeeping axiom schemes*:

(Q1) $(\overline{r_1} \to \overline{r_2}) \leftrightarrow \overline{\min\{1, 1 - r_1 + r_2\}}$;

(Q2) $\neg\overline{r} \leftrightarrow \overline{1 - r}$.

The algebraic counterpart of Ł$^{\mathbf{c}}$, are structures $(M, \{\overline{r}^M\}_{r\in\mathcal{Q}(\text{Ł})})$ where $M$ is an MV-algebra, the $\overline{r}^M$'s are nullary operations in $M$, and for every $r, r_1, r_2 \in \mathcal{Q}(\text{Ł})$ the following hold:

$$\begin{aligned} \overline{r_1}^M \to \overline{r_2}^M &= \overline{\min(1, 1 - r_1 + r_2)}^M \\ \neg\overline{r}^M &= \overline{1 - r}^M \end{aligned}$$

We will henceforth omit the superscript $^M$ whenever it will be superfluous.

The standard Ł$^{\mathbf{c}}$-chain is the structure $[0,1]_{\text{Ł}^c} = ([0,1]_{MV}, \{r\}_{r\in\mathbb{Q}})$, i.e. the standard MV-chain together with the rational truth constants $\overline{r}$ interpreted as themselves. For every $k \in \mathbb{N}$, Ł$_k^{\mathbf{c}}$-algebras and the standard Ł$_k^{\mathbf{c}}$-chain are defined in analogous way.

The notion of evaluation of $\mathfrak{F}(V)^{\mathbf{c}}$-formulas into expanded MV-structures with truth constants is defined in the natural way. In particular, an $\mathcal{L}^{\mathbf{c}}$-evaluation on the standard $\mathcal{L}^{\mathbf{c}}$-chain is such that $e(\overline{r}) = r$ for every $r \in \mathcal{Q}(\mathcal{L})$.

**Theorem 3.4** ([13]). (1) *The logic $L^{\mathbf{c}}$ logic is finitely strong real complete, i.e. for every finite theory $\Gamma \subseteq \mathfrak{F}(V)^{\mathbf{c}}$ and for every formula $\varphi$ in $\mathfrak{F}(V)^{\mathbf{c}}$, $\Gamma \vdash \varphi$ in $L^{\mathbf{c}}$ iff for every evaluation $e$ into the standard $L^{\mathbf{c}}$-chain such that $e(\gamma) = 1$ for every $\gamma \in \Gamma$, then $e(\varphi) = 1$ as well.*

(2) *For every $k \in \mathbb{N}$, the logic $L_k^{\mathbf{c}}$ is strong real complete, i.e. for every theory $\Gamma \subseteq \mathfrak{F}(V)^{\mathbf{c}}$, and for every formula $\varphi$ in $\mathfrak{F}(V)^{\mathbf{c}}$, $\Gamma \vdash \varphi$ in $L_k^{\mathbf{c}}$ iff for every evaluation $e$ into the standard $L_k^{\mathbf{c}}$-chain such that $e(\gamma) = 1$ for every $\gamma \in \Gamma$ it holds that $e(\varphi) = 1$ as well.*

[1] We will henceforth denote by $\mathfrak{F}(V)^{\mathbf{c}}$ the class of formulas obtained from this expanded language.

*3.2. States on MV-algebras*

The notion of *state* on an MV-algebra generalizes that of a finitely additive probability on a Boolean algebra. More specifically, by a *state* on an MV-algebra $M$ (cf. [29]) we mean a map from $M$ into the real unit interval $[0, 1]$, $\mathbf{s} : M \to [0, 1]$, satisfying:

(S1) $\mathbf{s}(1^M) = 1$,

(S2) For every $x, y \in M$ such that $x \odot y = 0^M$, $\mathbf{s}(x \oplus y) = \mathbf{s}(x) + \mathbf{s}(y)$.

It can be easily shown that every state $\mathbf{s}$ on $M$ satisfies $\mathbf{s}(\neg x) = 1 - \mathbf{s}(x)$, and hence in particular $\mathbf{s}(0^M) = 0$.

**Remark 3.5.** *The notion of state easily extends to expanded MV-algebras with truth constants, just by requiring the same two properties (S1) and (S2). Namely, if $M^c = (M, \{\overline{r}\}_{r\in\mathcal{Q}(\mathcal{L})})$ is any $\mathcal{L}^{\mathbf{c}}$-algebra, then (S1) and (S2) enforce every state $\mathbf{s}$ on $M^c$ to satisfy $\mathbf{s}(\overline{r}) = r$ for every rational $r \in \mathcal{Q}(\mathcal{L})$, and hence states on MV-algebras with truth constants are homogeneous. Therefore, this enables us to concentrate on states on MV-algebras, regardless of the fact that the languages are enriched by rational truth constants.*

A state $\mathbf{s}$ on $M$ is said to be *faithful* provided that $\mathbf{s}(x) = 0$, implies $x = 0^M$. In other words, a state of $M$ is faithful if the unique element of $M$ sent to 0 is the bottom element of $M$.

**Example 3.6.** *Consider any MV-algebra $M$. Then, every homomorphism $h : M \to [0, 1]_{\mathrm{MV}}$ is a state. In addition, since the class $St(M)$ of all the states of $M$ is a convex subset of $[0, 1]^M$ (cf. [29]), the homomorphisms of $M$ into $[0, 1]_{\mathrm{MV}}$ coincide with the extremal points of $St(M)$, and therefore by Krein-Mil'man theorem [18, Theorem 5.17] every state can be represented as a limit of convex combinations of homomorphisms from $M$ into $[0, 1]_{\mathrm{MV}}$ (cf. [29, Theorem 2.5]).*

Given a state $\mathbf{s} : M \to [0, 1]$, we denote by $Supp(\mathbf{s})$ its support, i.e. $Supp(\mathbf{s}) = \{x \in M : \mathbf{s}(x) > 0\}$. The following theorem is an immediate consequence of [25, Corollary 29].

**Theorem 3.7.** *Let $M = (S_k)^X$ be a finite MV-algebra. Then for every state $\mathbf{s} : M \to [0, 1]$ there exists a finitely additive probability measure $P$ on $B(M) = 2^X$ such that for every $f \in M$,*

$$\mathbf{s}(f) = \sum_{x\in X} f(x) \cdot P(\{x\}).$$

## 4. Belief functions on MV-algebras

In [26, 27], Kroupa provides a generalization of belief functions that can be easily adapted to the framework of finite MV-algebras. Recalling Remark 3.3, we can assume that the finite MV-algebra we are going to work with is $M = (S_k)^X$ for a suitable MV-chain $S_k$, and a finite set $X$. Denote by $2^X$ the powerset of $X$, and consider, for every $a : X \to S_k$ the map $\hat{\rho}_a : 2^X \to S_k$ defined as follows: for every $B \subseteq X$,

$$\hat{\rho}_a(B) = \min\{a(x) : x \in B\}. \tag{4}$$

**Remark 4.1.** *(1) $\hat{\rho}_a$ generalizes $\beta_A$ in the following sense: whenever $A \in B(M) = 2^X$, then $\hat{\rho}_A = \beta_A$. Namely, for every $A \in B(M)$, $\hat{\rho}_A(B) = 1$ if $B \subseteq A$, and $\hat{\rho}_A(B) = 0$ otherwise.*

*(2) If the finite MV-algebra we are dealing with coincides with the free $n$-generated $MV_k$-algebra $\mathbf{L}_{V_0}(Ł_k)$ (i.e. the Lindenbaum-Tarski $MV_k$-algebra generated from a language with propositional variables in $V_0 = \{p_1, \ldots, p_n\}$[2]), then for every $a \in \mathbf{L}_{V_0}(Ł_k)$, we also have that $\hat{\rho}_a \in \mathbf{L}_{V_0}(Ł_k)$. In other words we can consider $\hat{\rho}$ as a map $\hat{\rho} : \mathbf{L}_{V_0}(Ł_k) \to \mathbf{L}_{V_0}(Ł_k)$.*

**Definition 4.2.** *We call a map $\hat{\mathbf{b}} : (S_k)^X \to [0,1]$ a* Kroupa belief function *whenever there exists a state $\hat{\mathbf{s}} : (S_k)^{2^X} \to [0,1]$ such that for every $a \in M$, $\hat{\mathbf{b}}(a) = \hat{\mathbf{s}}(\hat{\rho}_a)$.*

The state $\hat{\mathbf{s}}$ needed in the definition of $\hat{\mathbf{b}}$ is called the *state assignment* in [26]. Although $\hat{\mathbf{b}}$ has been directly introduced as a combination of $\hat{\rho}$ with the state assignment $\hat{\mathbf{s}}$, a notion of *mass assignment* can also be introduced even for this generalized case. Indeed, since $X$ is finite, it turns out that one can equivalently define

$$\hat{\mathbf{b}}(a) = \sum_{B \subseteq X} \hat{\rho}_a(B) \cdot \hat{\mathbf{s}}(B).$$

In particular, since $1 = \hat{\mathbf{b}}(X) = \sum_{B \subseteq X} \hat{\mathbf{s}}(B)$, the restriction of the state $\hat{\mathbf{s}}$ to $2^X$ (call it $\hat{m}$) is a classical mass assignment. Now, we are allowed to speak about *focal elements* of $\hat{\mathbf{b}}$ as those elements in $2^X$ that the mass assignment $\hat{m}$ maps into a non-zero value.

Notice that, although the arguments in Kroupa's definition of belief function are fuzzy sets, the mass assignments that characterize each of these belief functions are defined on crisp (i.e. Boolean) sets, and therefore, the focal elements associated to every Kroupa belief function are crisp sets. In other words, every Kroupa belief function $\hat{\mathbf{b}}$ is defined from crisp, and not fuzzy, pieces of evidence.

Kroupa's definition of belief function makes use (with the necessary modification in using a state instead of a probability measure) of the maps $\hat{\rho}_a$, for every $a \in M$, which evaluate the degree of inclusion $\hat{\rho}_a(B)$ of each classical (i.e. crisp, Boolean) subset $B$ of $X$ into the fuzzy set $a$. The definition that we introduce below generalizes Kroupa's definition by introducing, for every $a \in M$, a map $\rho_a$ assigning to every fuzzy set $b \in M$ its degree of inclusion into $a$ (cf. [1]). To be more precise, let $M = (S_k)^X$, and consider, for every $a \in M$ a map $\rho_a : M \to [0,1]$ defined as follows: for every $b \in M$,

$$\rho_a(b) = \min\{b(x) \Rightarrow a(x) : x \in X\} \tag{5}$$

where $\Rightarrow$ denotes the Łukasiewicz implication function ($x \Rightarrow y = \min(1, 1 - x + y)$). [3]

**Remark 4.3.** *(1) In a sense, for every $a \in M$, $\rho_a$ can be identified as the membership function of the fuzzy set of elements of $M$ (and hence the fuzzy subsets of $X$) that are* included *in $a$. In particular one has $\rho_a(b) = 1$ whenever $b \leq a$ (for each point). Also notice that the Boolean skeleton $B(M)$ of any finite MV-algebra $M = (S_k)^X$ coincides with $2^X$ and hence, as also shown by the following result, for every $a \in M$ the map $\rho_a$ extends $\hat{\rho}_a$ in the domain.*

*(2) What we have already noticed in Remark 4.1 (2) with respect to $\hat{\rho}$, can be similarly shown for the map $\rho : a \in \mathbf{L}_{V_0}(Ł_k) \mapsto \rho_a \in \mathbf{L}_{V_0}(Ł_k)$.*

[2] We remind the reader that, whenever we fix a language $\mathscr{LC}$, a set of variables $V$, and a logic $\mathcal{L}$ together with its consequence relation $\vdash_{\mathcal{L}}$, the Lindenbaum-Tarski algebra $\mathbf{L}_V(\mathcal{L})$ is the quotient algebra of formulas modulo the equiprovability relation. We invite the reader to consult [5] for further details.

[3] Here the choice of $\Rightarrow$ is due to the MV-algebraic setting, but other choices could be made in other algebraic frameworks (see e.g. [1]).

**Proposition 4.4.** (i) *For all* $a, a' \in M$, $\rho_{a\wedge a'} = \min\{\rho_a, \rho_{a'}\}$, *and* $\rho_{a\vee a'} \geq \max\{\rho_a, \rho_{a'}\}$.

(ii) *For every* $a \in M$, *the restriction of* $\rho_a$ *to* $B(M)$ *coincides with the transformation* $\hat{\rho}_a$ *defined in equation* (4).

(iii) *For every* $A \in B(M)$, *the restriction of* $\rho_A$ *to* $B(M)$ *coincides with the transformation* $\beta_A$ *defined in equation* (3).

*Proof.* (i) In every MV-chain, and in particular in the standard chain $[0,1]_{\mathrm{MV}}$ the equation $\neg\gamma \oplus (\alpha \wedge \beta) = (\neg\gamma \oplus \alpha) \wedge (\neg\gamma \oplus \beta)$ holds:, i.e. $(\gamma \Rightarrow (\alpha \wedge \beta)) = (\gamma \Rightarrow \alpha) \wedge (\gamma \Rightarrow \beta)$. Therefore, for every $a, a', b \in M$,

$$\begin{aligned}\rho_{a\wedge a'}(b) &= \min\{b(x) \Rightarrow (a \wedge a')(x) : x \in X\}\\ &= \min\{b(x) \Rightarrow (a(x) \wedge a'(x)) : x \in X\}\\ &= \min\{(b(x) \Rightarrow a(x)) \wedge (b(x) \Rightarrow a'(x)) : x \in X\}\\ &= \min\{\rho_a(b), \rho_{a'}(b)\}.\end{aligned}$$

An easy computation shows that $\rho_{a\vee a'} \geq \max\{\rho_a, \rho_{a'}\}$.

(ii) For every $B \in B(M)$, $\rho_a(B) = \min\{B(x) \Rightarrow a(x) : x \in X\}$. Whenever $x \notin B$, $B(x) = 0$, and hence $B(x) \Rightarrow a(x) = 1$ for all those $x \notin B$. On the other hand for all $x \in B$, $B(x) = 1$, and so $B(x) \Rightarrow a(x) = 1 \Rightarrow a(x) = a(x)$ for all $x \in B$. Consequently, $\rho_a(B) = \min\{a(x) : x \in B\}$.

(iii) It trivially follows from (ii) and Remark 4.1. □

Now we introduce our definition of belief functions on MV-algebras of fuzzy sets.

**Definition 4.5.** *Let* $X$ *be finite, and let* $M = (S_k)^X$ *be the finite MV-algebra of fuzzy sets of* $X$ *with values in* $S_k$. *A map* $\mathbf{b} : M \to [0,1]$ *is called a* belief function *if there exists a state* $\mathbf{s} : (S_k)^M \to [0,1]$ *such that for every* $a \in M$,

$$\mathbf{b}(a) = \mathbf{s}(\rho_a). \tag{6}$$

*We denote the class of all belief functions over* $M$ *by* $Bel(M)$.

Notice that, as we already observed in Remark 4.3 (2), if $a \in M = (S_k)^X$ then $\rho_a \in (S_k)^M$ and hence $\mathbf{s}(\rho_a)$ is defined for every $a \in (S_k)^X$.

It is clear from the definition that $Bel(M)$ is a convex set, since states are closed by convex combinations (recall Example 3.6).

**Proposition 4.6.** *For every finite MV-algebra* $M$, *and for every* $\mathbf{b} \in Bel(M)$, $\mathbf{b}$ *is totally monotone, i.e.* $\mathbf{b}$ *is monotone, and it satisfies: for all* $a_1, \ldots, a_n \in M$,

$$\mathbf{b}\left(\bigvee_{i=1}^{n} a_i\right) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot \mathbf{b}\left(\bigwedge_{k\in I} a_k\right).$$

*Proof.* Since for every $a \in M$, $\rho_a$ is monotone, and every state $\mathbf{s}$ is monotone, $\mathbf{b}$ is monotone as well. Moreover, for every $n$ and for every $a_1, \ldots, a_n \in M$, from (6) and Proposition 4.4 (i) we have the following chain of inequalities:

$$\begin{aligned}\mathbf{b}\left(\bigvee_{i=1}^{n} a_i\right) &= \mathbf{s}(\rho_{a_1\vee\ldots\vee a_n})\\ &\geq \mathbf{s}(\rho_{a_1} \vee \ldots \vee \rho_{a_n})\\ &= \textstyle\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot \mathbf{s}\left(\bigwedge_{k\in I} \rho_{a_k}\right)\\ &= \textstyle\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot \mathbf{s}\left(\rho_{(\bigwedge_{k\in I} a_k)}\right)\\ &= \textstyle\sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} \cdot \mathbf{b}\left(\bigwedge_{k\in I} a_k\right).\end{aligned}$$

□

On Boolean algebras, total monotonicity is a property that fully characterizes belief functions. It is an open problem whether the same holds for belief functions on MV-algebras, even in our restricted setting.

For every belief function $\mathbf{b} : M \to [0,1]$ defined by a state $\mathbf{s}$ on the finite MV$_k$-algebra $(S_k)^M$ we know from Theorem 3.7 that there exists a unique finitely additive probability measure $P$ on $2^M$, the Boolean skeleton of $(S_k)^M$, such that, for every $a \in (S_k)^M$

$$\mathbf{s}(a) = \sum_{f \in (S_k)^X} a(f) \cdot P(\{f\}). \tag{7}$$

Let $m_{\mathbf{b}} : (S_k)^X \to [0,1]$ be the probability distribution associated to the probability measure $P$ of (7), i.e. defined as $m_{\mathbf{b}}(f) = P(\{f\})$, for every $f \in (S_k)^X$. In this case we get, for every $f \in M$,

$$\mathbf{b}(a) = \mathbf{s}(\rho_a) = \sum_{f \in (S_k)^X} \rho_a(f) \cdot m_{\mathbf{b}}(f). \tag{8}$$

Then, for obvious reasons, we call $m_{\mathbf{b}}$ the *mass assignment associated to* $\mathbf{b}$.

Given a belief function $\mathbf{b}$ on $M$, in analogy with the classical case, an element $f \in M$ is said to be a *focal element*, provided that $m_{\mathbf{b}}(f) > 0$. Notice that the focal elements, are elements of the MV-algebra $M = (S_k)^X$, and hence they are not crisp sets in general. This supports the interpretation that the belief functions defined as in (6) differ from Kroupa definition by offering a more general setting for evidence theory.

Let us denote by $\perp$ the bottom element of $M$, i.e. the function $\perp : X \to S_k$ such that $\perp(x) = 0$ for all $x \in X$. However, in general, $\rho_\perp$ does not coincide with the bottom element of $(S_k)^M$. In fact, if $a \in M$ is a function such that for no $x \in X$, $f(x) = 1$, then it immediately follows that $\rho_\perp(f) > 0$. Therefore, $\mathbf{b}(\perp) = 0$ does not hold in general (and in particular, whenever $\mathbf{s}$ is a faithful state). We call a belief function $\mathbf{b}$ on $M$ *normalized* provided that all the focal elements of $\mathbf{b}$ are normalized fuzzy sets, i.e. for every focal element $f \in M$ for $\mathbf{b}$ there exists a $x \in X$ such that $f(x) = 1$.

For every $r \in S_k$, let $\overline{r} : X \to S_k$ be the function constantly equal to $r$. Then for every normalized fuzzy set $f \in M$, $\rho_{\overline{r}}(f) = \inf\{f(x) \Rightarrow r : x \in X\} = r$. Hence, if $\mathbf{b}$ is a normalized belief function, $\mathbf{b}(\overline{r}) = \sum_{f \in (S_k)^X} \rho_{\overline{r}}(f) \cdot m(f) = r$. In other words the following holds.

**Proposition 4.7.** *Let $\mathbf{b} \in Bel(M)$ be a normalized belief function. Then $\mathbf{b}$ is homogeneous, i.e. for every $r \in S_k$, $\mathbf{b}(\overline{r}) = r$.*

**Example 4.8.** *Let us revisit Smets' well-known story of the murder of Mrs. Jones [34]. There are 3 suspects of being her murderer: Peter, Paul and Mary. Consider the information provided by the janitor of the building where Mrs. Jones lives. He heard the victim yelling and saw* a small man *running. It turns out that Paul and Mary are not tall while Peter is taller ((Paul is 1.65 m. tall, Mary is 1.60 m tall and Peter is 1.85 m.). So, actually, the subset of small suspects of $X = \{Peter, Paul, Mary\}$ can be considered as a fuzzy set, with membership function, say,*

$$\mu_{small}(Peter) = 0, \ \mu_{small}(Paul) = 0.7, \ \mu_{small}(Mary) = 0.9.$$

*On the other hand, Mary has short hair, so she may be mistaken as a man at first sight, and hence, the subset of suspects looking like a man can be considered fuzzy as well, with membership function:*

$$\mu_{\textit{man-like}}(Peter) = 1,\ \mu_{\textit{man-like}}(Paul) = 1,\ \mu_{\textit{man-like}}(Mary) = 0.5.$$

*The evidence supplied by the janitor may be represented by a mass assignment* $m : [0,1]^X \to [0,1]$ *such that* $m(\textit{small} \wedge \textit{man-like}) = \alpha > 0$, $m(X) = 1 - \alpha$ *and* $m(f) = 0$ *for any other* $f \in [0,1]^X$. *Here we interpret* $\wedge$ *by the* min *operator, so we have*

$$\mu_{\textit{small}\wedge\textit{man-like}}(Peter) = 0, \mu_{\textit{small}\wedge\textit{man-like}}(Paul) = 0.7, \mu_{\textit{small}\wedge\textit{man-like}}(Mary) = 0.5.$$

*Suppose we are interested in computing the belief that the suspect is Paul. We then need to compute*

$$\begin{aligned}\rho_{\{Paul\}}(\textit{small} \wedge \textit{man-like}) &= \min_{x\in X}\{\mu_{\textit{small}\wedge\textit{man-like}}(x) \Rightarrow \mu_{Paul}(x)\}\\ &= \min\{0 \Rightarrow 0, 1 \Rightarrow 1, 0.5 \Rightarrow 0\}\\ &= \min\{1, 0.5\} = 0.5\end{aligned}$$

*and* $\rho_{\{Paul\}}(X) = 0$. *Finally, we have*

$$\begin{aligned}\mathbf{b}(\{Paul\}) &= \sum_{f\in Supp(m)} \rho_{\{Paul\}}(f)\cdot m(\{f\})\\ &= \rho_{\{Paul\}}(\textit{small} \wedge \textit{man-like})\cdot m(\textit{small} \wedge \textit{man-like})\\ &= 0.5\cdot\alpha > 0.\end{aligned}$$

*Hence, we get a positive belief degree of Paul being the murderer. This is in contrast with the results we would obtain with both the classical and Kroupa's models, where focal elements are only allowed to be classical subsets of* $X$, *in case we assume Mary can be mistaken as a man. Indeed, in that case, we would be forced to take as focal element, besides* $X$ *itself, the set* $\textit{small} \wedge \textit{man-like} = \{Paul, Mary\}$, *and since there would be no focal element included in* $\{Paul\}$, *we would get* $\mathbf{b}(\{Paul\}) = 0$.

*One can analogously compute the belief of other (fuzzy) events of interest:*

$\mathbf{b}(\bot) = 0.3\cdot\alpha$
$\mathbf{b}(\{Mary\}) = 0.3\cdot\alpha$
$\mathbf{b}(\{Peter\}) = 0.3\cdot\alpha$
$\mathbf{b}(\textit{small} \wedge \textit{man-like}) = \alpha$
$\mathbf{b}(\textit{small}) = \alpha$
$\mathbf{b}(\textit{man-like}) = \alpha + 0.5\cdot(1-\alpha)$
$\mathbf{b}(X) = 1$

*Due to the fact that the focal element "small* $\wedge$ *man-like" is a non-normalized fuzzy set, the belief of the of bottom element of* $\mathbf{b}(\bot)$ *is strictly positive.*

## 5. An alternative definition of belief functions based on Dempster spaces

The definition of a belief function on a MV-algebra functions $M = (S_k)^X$ we have proposed in Definition 4.5 cannot be done by only working inside the MV-algebra $M$ where the belief function is defined. In fact the definition also involves a state on the bigger algebra $(S_k)^M$.

A possibility to overcome this, so to say, peculiar situation is to resort to the original Dempster model of defining a belief function as a lower probability induced by a multivalued mapping [8]. Indeed, given a probability $\mu$ on the power set of a finite set $E$ and a multivalued mapping $\Gamma : E \to 2^X$, one can consider an induced lower probability on $2^X$ defined as $bel(A) = \mu(\{v \in E \mid \Gamma(v) \subseteq A\})$, for every $A \subseteq X$. This is in fact a belief function, and moreover, every belief function on $X$ comes defined in this way. The 4-tuple $D = (W, E, \Gamma, \mu)$ is called a Dempster space.

In this section we show how to define belief functions on MV-algebras of functions $M = (S_k)^X$ based on a natural generalization of Dempster spaces and we will show, as in the classical case, that both approaches turn out to be equivalent. The approach based on generalized Dempster spaces will have some advantages regarding the logical approach to belief functions developed in Section 7.

**Definition 5.1 (Generalized Dempster space).** *A generalized Demptser space is a 4-tuple* $D = (W, E, \Gamma, \mu)$ *where*

- $W$ *and* $E$ *are non-empty sets*
- $\mu : (S_k)^E \to [0, 1]$ *is a state*
- $\Gamma : E \to (S_k)^W$ *is a fuzzy set-valued mapping*

For simplicity, generalized Dempster spaces will be simply called Dempster spaces from now on.

For each $f \in (S_k)^W$ define $\varrho_f : E \to S_k$ by $\varrho_f(v) = \inf_{w\in W} \Gamma(v)(w) \Rightarrow f(w)$.

**Definition 5.2 (Belief function given by a Dempster space).** *Given a Dempster space* $D = (W, E, \Gamma, \mu)$*, the induced belief function* $bel_D : (S_k)^W \to [0, 1]$ *is defined as*

$$bel_D(f) = \mu(\varrho_f).$$

In order to distinguish the two notions of belief functions that we have introduced so far (namely those from Definition 4.5 that we will denote by $\mathbf{b}$, and the ones introduced above in Definition 5.2 that we will denote by $bel_D$), we will henceforth call *Dempster belief funcions* those induced by a Dempster space as in Definition 5.2.

**Lemma 5.3.** *For any Dempster space* $D = (W, E, \Gamma, \mu)$*, there is a mapping* $m : (S_k)^W \to [0, 1]$ *such that*

$$\sum_{g\in(S_k)^W} m(g) = 1$$

*and the Dempster belief function* $bel_D$ *is defined as follows: for any* $f \in (S_k)^W$

$$bel_D(f) = \sum_{g\in(S_k)^W} \rho_f(g) \cdot m(g).$$

*Proof.* For any $f \in (S_k)^W$, we have $bel_D(f) = \mu(\varrho_f)$, where $\varrho_f : E \to S_k$ is defined by $\varrho_f(v) = \inf_{u\in W} \Gamma(v)(u) \Rightarrow f(u) = \rho_f(\Gamma(v))$. But $\mu(\varrho_f) = \sum_{v\in E} \varrho_f(v) \cdot \mu(\{v\}) = \sum_{v\in E} \rho_f(\Gamma(v)) \cdot \mu(\{v\})$. Define now the mapping $m : (S_k)^W \to [0, 1]$ by

$$m(g) = \mu(\{v \in E \mid \Gamma(v) = g\}). \tag{9}$$

Then it is clear that $\sum_{v\in E} \rho_f(\Gamma(v)) \cdot \mu(\{v\}) = \sum_{g\in(S_k)^W} \rho_f(g) \cdot m(g)$. □

Finally, as it happens in the classical case, one can show that the two notions of belief functions given in Definitions 4.5 and 5.2 are equivalent.

**Proposition 5.4.** *Let $W$ be finite. A mapping $\mathbf{b} : (S_k)^W \to [0,1]$ is a belief function in the sense of Definition 4.5 iff there is a Dempster space $D = (W, E, \Gamma, \mu)$ such that $\mathbf{b} = bel_D$.*

*Proof.* Let $\mathbf{b}$ be a belief function defined by a state $\mathbf{s} : (S_k)^M \to [0,1]$ as in Definition 4.5. We define the Dempster space $D = (W, E, \Gamma, \mu)$ where $E = (S_k)^W$, $\Gamma : E \to (S_k)^W$ is the identity function, and $\mu = \mathbf{s}$. Then we have

$$\varrho_f(g) = \inf_{g \in (S_k)^W} \Gamma(g)(w) \Rightarrow f(w) = \inf_w g(w) \Rightarrow f(w) = \rho_f(g),$$

hence $\mathbf{b}(f) = \mathbf{s}(\rho_f) = \mu(\varrho_f) = bel_D(f)$.

Conversely, let $bel_D$ be the belief function given by a Dempster space $D = (W, E, \Gamma, \mu)$ with $W$ finite. According to the preceeding lemma, there is a mass $m$ on $(S_k)^W$ such that $bel_D(f) = \sum_{g \in (S_k)^W} \rho_f(g) \cdot m(g)$. Define the state $\mathbf{s} : (S_k)^M \to [0,1]$ such that, for every $h \in (S_k)^M$,

$$\mathbf{s}(h) = \sum_{f \in M} h(f) \cdot m(f).$$

Then, the belief function on $(S_k)^W$ defined by $\mathbf{s}$ does the job, since $\mathbf{b}(f) = \mathbf{s}(\rho_f) = \sum_{g \in M} \rho_f(g) \cdot m(g) = bel_D(f)$. □

## 6. The minimal modal extension of $Ł_k^{\mathbf{c}}$ without nested modalities

In [17] the authors introduce a probabilistic fuzzy modal logic defined over the classical modal logic S5 to axiomatize reasoning with classical belief functions. Roughly speaking, the intuition behind that approach is that the two modalities $P$ for *probably*, and the classical modality $\Box$ of S5, can be used to define a modality $B$ by the combination $P\Box$, which behaves as a belief function over classical events. Although there are no particular requirements for choosing S5, this modal logic has the advantage of being locally finite. This requirement is crucial to prove completeness of the resulting probabilistic logic with respect to a Kripke style semantics.

As mentioned in the introduction, in this paper we introduce a similar approach for belief functions on fuzzy sets of $(S_k)^X$ and, following the definition we introduced in Section 4, we will define a probabilistic logic over a suitable fuzzy modal logic $\Lambda_k$. In fact, in order to keep the defined logic sufficiently expressive and locally finite, we will take $\Lambda_k$ as the non-nested fragment of $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$, the minimal modal logic over the standard MV$_k$-chain $\mathbf{S}_k$ defined and studied in [2]. We will devote this section to describe these modal logics and to show completeness of $\Lambda_k$.

The language of $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$ is obtained by enlarging the language of $Ł_k^{\mathbf{c}}$ by a unary modality $\Box$, and defining well formed formulas in the usual inductive manner: (1) every formula of $Ł_k^{\mathbf{c}}$ is a formula; (2) if $\varphi$ and $\psi$ are formulas, then $\Box\varphi$, $\varphi \odot \psi$, and $\varphi \to \psi$, are formulas.

A $Ł_k^{\mathbf{c}}$-*Kripke frame* is a tuple $(W, R)$ where $W$ is a non-empty set of possible worlds and $R : W \times W \to S_k$ is an many-valued accessibility relation. We denote by $\mathbf{Fr}$ the class of all $Ł_k^{\mathbf{c}}$-Kripke frames. A $Ł_k^{\mathbf{c}}$-*Kripke model* is a triple $(W, e, R)$ where $(W, R)$ is a $Ł_k^{\mathbf{c}}$-Kripke frame, and for every possible world $w$, $e(\cdot, w)$ is a truth evaluation of $Ł_k^{\mathbf{c}}$-formulas into $S_k$.

Given a formula $\phi$, and a $Ł_k^{\mathbf{c}}$-Kripke model $K = (W, e, R)$, for every $w \in W$, we define the truth value of $\phi$ in $w$, $\|\phi\|_w$, as follows:

- If $\phi$ is a formula of $Ł_k^{\mathbf{c}}$, then $\|\phi\|_{K,w} = e(\phi, w)$,
- If $\phi = \Box\psi$, then $\|\Box\psi\|_{K,w} = \bigwedge_{w' \in W} \left(R(w, w') \Rightarrow \|\psi\|_{K,w'}\right)$,
- If $\phi$ is a compound formula, its truth value is computed truth functionally by means of $Ł_k^{\mathbf{c}}$ truth functions.

The truth value of a formula $\phi$ in $K$ is then defined as $\|\phi\|_K = \inf\{\|\phi\|_{K,w} \mid w \in W\}$. As usual, the notion of (local) logical consequence in **Fr** is defined as follows: given a set of formulas $\Gamma \cup \{\varphi\}$, $\varphi$ follows from $\Gamma$, written $\Gamma \models_{\mathbf{Fr}} \varphi$, iff for every Kripke model $K = (W, e, R)$ such that $(W, R) \in \mathbf{Fr}$ and every $w \in W$, if $\|\psi\|_{K,w} = 1$ for every $\psi \in \Gamma$, then $\|\varphi\|_{K,w} = 1$ as well.

The axioms of $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$ are the following:

- All the axioms for $Ł_k^{\mathbf{c}}$
- $(\Box 1)$ $\Box\overline{1}$
- $(\Box 2)$ $(\Box\varphi \wedge \Box\psi) \to \Box(\varphi \wedge \psi)$
- $(\Box 3)$ $\Box(\overline{r} \to \varphi) \leftrightarrow (\overline{r} \to \Box\varphi)$, for each $r \in S_k$

The rules of $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$ are Modus Ponens (from $\varphi$ and $\varphi \to \psi$ infer $\psi$) and Monotonicity for $\Box$ (from $\varphi \to \psi$ infer $\Box\varphi \to \Box\psi$).

The notion of proof in $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$, denoted $\vdash_{\Lambda(Fr, Ł_k^{\mathbf{c}})}$, is defined as usual from the above axioms and rules. In [2] the authors show that $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$ is sound and complete with respect to the class **Fr** of $Ł_k^{\mathbf{c}}$-Kripke frames: for every set of formulas $\Gamma \cup \{\varphi\}$, $\Gamma \models_{\mathbf{Fr}} \varphi$ iff $\Gamma \vdash_{\Lambda(Fr, Ł_k^{\mathbf{c}})} \varphi$.

**Remark 6.1.** *In [6] it is shown that the classical modal logic $K$ is not locally finite. This means that the Lindenbaum-Tarski algebra of $K$ generated by any finite set of propositional variables is infinite in general. In particular there is an infinite class of modal formulas $\phi_1, \phi_2, \dots$ such that for every $i \neq j$, $\phi_i \leftrightarrow \phi_j$ is not valid in some Kripke frame. Since every Kripke frame for $K$ belongs to* **Fr** *as well, this means that $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$ is not locally finite either.*

Now we define $\Lambda_k$ as the fragment of $\Lambda(Fr, Ł_k^{\mathbf{c}})$ obtained by restricting the language to formulas without nested modalities. Namely, the set $\mathfrak{F}(V)^{\Box}$ of formulas of $\Lambda_k$ is defined as follows:

(1) formulas of $Ł_k^{\mathbf{c}}$ are formulas of $\Lambda_k$, i.e. $\mathfrak{F}(V)^c \subseteq \mathfrak{F}(V)^{\Box}$;

(2) for every formula $\varphi \in \mathfrak{F}(V)$, $\Box\varphi \in \mathfrak{F}(V)^{\Box}$;

(3) $\mathfrak{F}(V)^{\Box}$ is taken closed under the connectives of $Ł_k^{\mathbf{c}}$.

Notice that, in this restricted case, nested modalities are not allowed, and hence, if for instance $\varphi$ and $\psi$ are non-modal formulas, then $(\Box\varphi) \odot \psi$ is a formula of $\mathfrak{F}(V)^{\Box}$, but $\Box(\Box\varphi \odot \psi)$ is not. In particular notice that the above axioms $(\Box 1)$-$(\Box 3)$ are formulas in $\mathfrak{F}(V)^{\Box}$.

The axioms of the logic $\Lambda_k$ are those of $\Lambda(Fr, Ł_k^{\mathbf{c}})$, and its inference rules are Modus Ponens, and the Monotonicity rule for $\Box$, the latter being restricted in the premises to formulas in $\mathfrak{F}(V)^{\mathbf{c}}$. We will denote by $\vdash_{\Lambda_k}$ the provability relation in $\Lambda_k$.

**Lemma 6.2.** *The logic $\Lambda_k$ is locally finite.*

*Proof.* The claim can be easily proved recalling that $Ł_k^{\mathbf{c}}$ is locally finite for every $k$, and then, fixing a finite set of variables $V_0$, there are, up to logical equivalence, only finitely many formulas in $\mathfrak{F}(V_0)^{\mathbf{c}}$, i.e. the Lindenbaum-Tarski algebra $\mathbf{L}_{V_0}(Ł_k^{\mathbf{c}})$ is finite. Therefore the Lindenbaum-Tarski algebra $\mathbf{L}_{V_0}(\Lambda_k)$ is finite as well, since its domain is contained in the finite set $\{[\Box\varphi] : [\varphi] \in \mathbf{L}_{V_0}(Ł_k^{\mathbf{c}})\}$. Therefore $\Lambda_k$ is locally finite. □

Now we are going to show that $\Lambda_k$ is sound and complete with respect to the class $\mathbf{Fr}$ of $Ł_k^{\mathbf{c}}$-Kripke frames. This clearly shows that in fact $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$ can be seen as a conservative expansion of $\Lambda_k$.

The formulas in $\mathfrak{F}(V)^{\Box}$ can be translated into the language of $Ł_k^{\mathbf{c}}$ as follows. For every atomic modal formula $\Box\varphi$, we expand the language with a fresh new variable $q_\varphi$, and let $V^q$ be this new set of variables. Then we define the translation $^{\#}$ from $\mathfrak{F}(V)^{\Box}$ to $\mathfrak{F}(V \cup V^q)^{\mathbf{c}}$ by stipulating: $p^{\#} = p$ if $p \in V$, $(\Box\varphi)^{\#} = q_\varphi$, $(\overline{r})^{\#} = \overline{r}$, $(\neg\varphi)^{\#} = \neg\varphi^{\#}$, and $(\varphi \to \psi)^{\#} = \varphi^{\#} \to \psi^{\#}$. Moreover we define the set $K^{\#}(V)$ to be the subset of all $Ł_k^{\mathbf{c}}$-formulas from $\mathfrak{F}(V \cup V^q)^{\mathbf{c}}$ obtained by translating all the instances of the axioms of $\Lambda_k$ plus the set of formulas of the form $q_\varphi \to q_\psi$, for each $\varphi \to \psi \in \mathfrak{F}(V)^{\mathbf{c}}$ such that $Ł_k^{\mathbf{c}} \vdash \varphi \to \psi$.

**Lemma 6.3.** *Let $\Gamma \cup \varphi$ be a subset of $\mathfrak{F}(V)^{\Box}$. Then $\Gamma \vdash_{\Lambda_k} \varphi$ iff $\Gamma^{\#} \cup K^{\#}(V) \vdash_{Ł_k^{\mathbf{c}}} \varphi^{\#}$.*

*Proof.* This is semantically proved in [2, Theorem 3.26]. From a syntactic point of view, it is easy to see that any proof $\Phi_1, \ldots, \Phi_l = \varphi$ of $\varphi$ from $\Gamma$ in $\Lambda_k$ can be translated into a proof $\Phi_1^{\#}, \ldots, \Phi_{l'}^{\#} = \varphi^{\#}$ of $\varphi^{\#}$ from $\Gamma^{\#} \cup K^{\#}(V)$ in $Ł_k^{\mathbf{c}}$. □

**Theorem 6.4.** *The logic $\Lambda_k$ is strongly complete with respect to the class* $\mathbf{Fr}$ *of $Ł_k^{\mathbf{c}}$-Kripke frames.*

*Proof.* Let $\Gamma \cup \{\varphi\}$ be a set of formulas from $\mathfrak{F}(V)^{\Box}$, and assume that $\Gamma \not\vdash_{\Lambda_k} \varphi$. From Lemma 6.3, $\Gamma^{\#} \cup K^{\#}(V) \not\vdash_{L_k} \varphi^{\#}$ and hence, from the strong real completeness of $Ł_k^{\mathbf{c}}$, there is a evaluation $e$ of $V \cup V^q$ into $S_k$ such that $e(\psi) = 1$ for every $\psi \in \Gamma^{\#} \cup K^{\#}(V_0)$, and $e(\varphi^{\#}) < 1$.

Then we define an $Ł_k^{\mathbf{c}}$-Kripke model $K = (W, v, R)$ as follows:

- $W$ is the set of all $Ł_k^{\mathbf{c}}$-evaluations $w : V \cup V^q \to S_k$ that are models of $\Gamma^{\#} \cup K^{\#}(V)$, and for every $\varphi \in \mathfrak{F}(V)$ and every $w \in W$, define $v(w, \varphi) = w(\varphi)$. Notice that $e \in W$, hence $W$ is not empty.
- For every $w_1, w_2 \in W$, define $R : W \times W \to S_k$ by

$$R(w_1, w_2) = \min\{w_2(\varphi) \mid \varphi \in \mathfrak{F}(V)^{\mathbf{c}}, w_1(q_\varphi) = 1\}. \tag{10}$$

Clearly $K$ is a $Ł_k^{\mathbf{c}}$-Kripke model and $(W, R) \in \mathbf{Fr}$. Following the proof of [2, Lemma 4.8] we can show by induction on the complexity of the considered formula that, for every $\psi \in \mathfrak{F}(V_0)^{\Box}$ and for every $w \in W$, $v(w, \psi) = \|\psi\|_{K,w}$. Therefore, since the evaluation $e$ belongs to $W$, we obtain that $\|\psi\|_{K,e} = 1$ for every $\psi \in \Gamma^{\#} \cup K^{\#}(V)$, and $\|\varphi\|_{K,e} < 1$. Consequently $\Gamma \not\models_{\mathbf{Fr}} \varphi$.

□

*6.1. The case of $Ł_k^{\mathbf{c}}$-frames with crisp accessibility relations*

In the same paper [2], the authors also study the subclass $\mathbf{CFr}$ of $Ł_k^{\mathbf{c}}$-Kripke frames $(W, R)$ where the accessibility relation $R$ is crisp (two-valued). The corresponding logic, $\Lambda(\mathbf{CFr}, Ł_k^{\mathbf{c}})$, is shown to be axiomatizable by extending $\Lambda(\mathbf{Fr}, Ł_k^{\mathbf{c}})$ with the well-known axiom K:

(K) $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$.

In a similar way to what we have shown in the above section, one can consider the logic $C\Lambda_k$ defined as the nested modality-free fragment of $\Lambda(\mathbf{CFr}, Ł_k^{\mathbf{c}})$. The same techniques used in the above section show that $C\Lambda_k$ is locally finite, and using [2, Lemma 4.20], one can also prove strong completeness of $C\Lambda_k$ with respect to the class **CFr** of *crisp* $Ł_k^{\mathbf{c}}$-Kripke frames.

*6.2. The case of $Ł_k^{\mathbf{c}}$-frames with reflexive accessibility relations*

Consider the logics $\Lambda_k^r$ and $C\Lambda_k^r$ obtained by adding the axiom

(T) $\Box\varphi \to \varphi$

to $\Lambda_k$ and $C\Lambda_k$ respectively. We will show that these logics are also complete with respect to the corresponding subclasses of $Ł_k^{\mathbf{c}}$-frames $(W, R)$ where $R$ is reflexive fuzzy relation, i.e. that for all $w \in W$, $R(w, w) = 1$ holds. This case is not considered in [2] so, for the sake of to be self contained, we provide a simple proof.

**Theorem 6.5.** *The logic $\Lambda_k^r$ (resp. $C\Lambda_k^r$) is sound and strongly complete with respect to the subclass of $Ł_k^{\mathbf{c}}$-Kripke frames $(W, R)$ from* **Fr** *(resp.* **CFr***) where the relation $R$ is reflexive.*

*Proof.* In order to prove soundness we need to show that $\Box\varphi \to \varphi$ holds true in every $Ł_k^{\mathbf{c}}$-Kripke model $K = (W, e, R)$ with $R$ being reflexive. For every $w \in W$, $\|\Box\varphi \to \varphi\|_{K,w} = 1$ iff

$$\bigwedge_{w' \in W} R(w, w') \to w'(\varphi) \leq w(\varphi).$$

but we have that $\bigwedge_{w' \in W} R(w, w') \to w'(\varphi) \leq R(w, w) \to w(\varphi) = 1 \to w(\varphi) = w(\varphi)$.

The completeness proof is an adaptation of the proof of Theorem 6.4. Indeed, it is sufficient to prove that, whenever $K^{\#}(V)$ contains the instances of the translations $q_\varphi \to \varphi$ of the axiom $T$, the Kripke model $K = (W, v, R)$ built in the proof of Theorem 6.4 is such that $R$ is reflexive. Indeed, from (10) it follows that for every $w \in W$, $R(w, w) = \min\{w(\varphi) \mid \varphi \in \mathfrak{F}(V)^{\mathbf{c}}, w(q_\varphi) = 1\}$. Now, since every $w \in W$ is a model of $\Gamma^{\#} \cup K^{\#}(V)$, in particular we have $w(q_\varphi \to \varphi) = 1$, that is $w(q_\varphi) \leq w(\varphi)$. Therefore $w(\varphi) = 1$ whenever $w(q_\varphi) = 1$, and hence it follows that $R(w, w) = 1$. $\Box$

## 7. Logics for belief functions on fuzzy events

In this section we are going to introduce a probabilistic modal extension (cf. [14, 16, 20, 21]) of $\Lambda_k$ (and its extensions $C\Lambda_k$, $\Lambda_k^r$ and $C\Lambda_k^r$) that we will denote $FP(\Lambda_k, Ł^{\mathbf{c}})$ ($FP(C\Lambda_k, Ł^{\mathbf{c}})$, $FP(\Lambda_k^k, Ł^{\mathbf{c}})$, $FP(C\Lambda_k^r, Ł^{\mathbf{c}})$ respectively), to deal with the two definitions of belief functions on MV-algebras of fuzzy sets we discussed in Section 4, namely Kroupa belief functions and the new equivalent definitions we have introduced there and in Section 5, together with their normalized versions.

As already mentioned before, we extend to fuzzy events the fuzzy modal approach of [17] to define a logic to reason about uncertainty on classical events modeled by belief functions. Namely, the approach is based on:

- to consider fuzzy events modeled as propositions of (finitely-valued) Łukasiewicz logic together with modality $B$, for belief, in such a way that, informally speaking, the truth degree of $B\varphi$ corresponds to the belief degree (in the sense of belief functions) of $\varphi$.

- to get a complete axiomatization of the modality $B$ by relying on the fact that any belief function on Łukasiewicz formulas[4] $\varphi$ can be obtained as a probability (or state) on formulas $\Box\varphi$ of the minimal modal extension of Łukasiewicz logic $\Lambda_k$, and hence by defining $B\varphi$ as the combination of two other modalities $P\Box\varphi$, where $P$ is a probabilistic modality like in [14].

The language of the logic $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$ is obtained by expanding the language of $\Lambda_k$ by a unary modality $P$. The class $\mathfrak{F}(V)^P$ of formulas is defined as follows:

(i) $\mathfrak{F}(V)^\Box \subseteq \mathfrak{F}(V)^P$;

(ii) for every $\psi \in \mathfrak{F}(V)^\Box$, $P\psi$ is an *atomic P-formula*, for every rational number $r \in [0,1]$, $\overline{r}$ is an atomic $P$-formula as well, and they belong to $\mathfrak{F}(V)^P$; and

(iii) $\mathfrak{F}(V)^P$ is obtained by closing the class of atomic $P$-formulas under the connectives of Łukasiewicz logic Ł.

Formulas of $\mathfrak{F}(V)^P$ which are not from $\mathfrak{F}(V)^\Box$ (i.e. propositional combinations of formulas $P\psi$) will be called *P-formulas*. For every $\varphi \in \mathfrak{F}(V)^{\mathbf{c}}$, we henceforth use the abbreviation $B(\varphi)$ for $P(\Box\varphi)$. These formulas will be formally introduced in the next section.

Notice that in $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$ we are allowing neither formulas that contain nested occurrences of $P$ nor compound formulas mixing formulas from $\mathfrak{F}(V)^\Box$ and $\mathfrak{F}(V)^P$.

Axioms and rules of $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$ are as follows:

- Axioms and rules of $\Lambda_k$ for formulas of $\mathfrak{F}(V)^\Box$
- Axioms and rules of $\text{Ł}^{\mathbf{c}}$ for formulas in $\mathfrak{F}(V)^P$
- The following probabilistic axioms for $P$-formulas (cf. [14]):

  (PAX0) $P\overline{r} \leftrightarrow \overline{r}$, for $r \in S_k$
  (PAX1) $P(\neg\varphi) \leftrightarrow \neg P\varphi$
  (PAX2) $P(\varphi \to \psi) \to (P\varphi \to P\psi)$
  (PAX3) $P(\varphi \oplus \psi) \leftrightarrow [(P\varphi \to P(\psi \odot \varphi)) \to P\psi]$

- The rule of necessitation for $P$: from $\varphi$ derive $P(\varphi)$, for $\varphi \in \mathfrak{F}(V)^\Box$

In the above definition, we could consider adding to $\Lambda_k$ the axioms $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$ and $\Box\varphi \to \varphi$ (one or both) as we did in Sections 6.1 and 6.2. This would result in similar logics $FP(C\Lambda_k, \text{Ł}^{\mathbf{c}})$, $FP(\Lambda^r_k, \text{Ł}^{\mathbf{c}})$ and $FP(C\Lambda^r_k, \text{Ł}^{\mathbf{c}})$.

**Remark 7.1.** *It is worth noticing that both $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$ and $FP(C\Lambda_k, \text{Ł}^{\mathbf{c}})$ do not prove $B(\overline{r}) \leftrightarrow \overline{r}$ for $r \in S_k \setminus \{0\}$. In fact, although $P(\overline{r}) \leftrightarrow \overline{r}$ holds (it is an instance of the axiom* (PAX0)*), $\Lambda_k \not\vdash \Box\overline{r} \leftrightarrow \overline{r}$, indeed $\Lambda_k$ only proves one direction, $\overline{r} \to \Box\overline{r}$. Then, it is clear that the extension $\Lambda^r_k$, which contains the reflexivity axiom $\Box\varphi \to \varphi$, does prove the equivalence $\overline{r} \leftrightarrow \Box\overline{r}$, and hence both $FP(\Lambda^r_k, \text{Ł}^{\mathbf{c}})$ and $FP(C\Lambda^r_k, \text{Ł}^{\mathbf{c}})$ prove $B(\overline{r}) \leftrightarrow \overline{r}$.*

The first kind of semantics we introduce for $FP(\Lambda_k, \text{Ł}^{\mathbf{c}})$ and $FP(C\Lambda_k, \text{Ł}^{\mathbf{c}})$ is given by the classes of *probabilistic* $\text{Ł}^{\mathbf{c}}_k$*-Kripke models*, and *probabilistic crisp Kripke models* respectively.

[4] According to the notions of belief functions introduced in Sections 4 and 5.

**Definition 7.2.** *A* probabilistic $Ł_k^{\mathbf{c}}$-Kripke model *is a system*

$$M = (W, e, R, \mathbf{s})$$

*such that* $(W, e, R)$ *is a* $Ł_k^{\mathbf{c}}$*-Kripke model, and* $\mathbf{s} : \mathfrak{F}_M^{\Box} \to [0,1]$ *is a state on the MV-algebra of functions* $\mathfrak{F}_M^{\Box} = \{f_\varphi^M \mid \varphi \in \mathfrak{F}(V)^{\Box}, f_\varphi^M : W \to S_k, \text{with } f_\varphi^M(w) = \|\varphi\|_{M,w}\}$.

*If* $M$ *is such that* $(W, R)$ *is a classical Kripke frame, then* $M$ *is called a* probabilistic classical $Ł_k^{\mathbf{c}}$-Kripke model.

Let $M = (W, e, R, \mathbf{s})$ be a probabilistic (classical) $Ł_k^{\mathbf{c}}$-Kripke model. For every $\Phi \in \mathfrak{F}(V)^P$, and for every $w \in W$, we define the truth value of $\Phi$ in $M$ at $w$ inductively as follows:

- If $\Phi \in \mathfrak{F}(V)^{\Box}$, then its truth value $\|\Phi\|_{M,w}$ is evaluated in $(W, e, R)$ as defined in the previous section.
- If $\Phi = P\psi$, then $\|P\psi\|_{M,w} = \mathbf{s}(f_\psi^M)$.
- If $\Phi$ is a compound formula, its truth value is computed by truth functionality.

**Theorem 7.3** (Probabilistic completeness)**.** (1) *The logic* $FP(\Lambda_k, L^{\mathbf{c}})$ *is sound and finitely strong complete with respect to the class of probabilistic* $Ł_k^{\mathbf{c}}$*-Kripke models.*

(2) *The logic* $FP(C\Lambda_k, L^{\mathbf{c}})$ *is sound and finitely strong complete with respect to the class of probabilistic classical* $Ł_k^{\mathbf{c}}$*-Kripke models.*

*Proof.* The proof is an easy adaptation of the proof of [16, Theorem 25] reminding that both $\Lambda_k$ and $C\Lambda_k$ are locally finite (Lemma 6.2), and that $Ł^{\mathbf{c}}$ has the canonical FSRC (Theorem 3.4). In order to keep the paper as self-contained as possible, we sketch the main steps of the proof.

Let $\Gamma \cup \{\Phi\}$ be a finite subset of $\mathfrak{F}(V)^P$, and assume that $\Gamma \not\vdash \Phi$. Consider the translation map $^\circ$ from $\mathfrak{F}(V)^P$ to $\mathfrak{F}(V)^{\mathbf{c}}$ that, similar to what we did in Section 6, works as follows:

- for every $\psi \in \mathfrak{F}(V)^{\mathbf{c}}$, $\psi^\circ = \psi$ and $(\Box\psi)^\circ = (\Box\psi)^{\#} = q_\psi$,
- for every $\phi \in \mathfrak{F}(V)^{\Box}$, $(P\phi)^\circ = u_\phi$,
- for every $\Phi$ and $\Psi$ in $\mathfrak{F}(V)^P$, $(\Phi \to \Psi)^\circ = \Phi^\circ \to \Psi^\circ$ and $(\neg\Phi)^\circ = \neg(\Phi^\circ)$.

where the variables $q_y$ and $u_x$ are fresh for the language of $Ł_k^{\mathbf{c}}$. Since $\Lambda_k$ is locally finite, letting $V_0$ be the finite set of variables appearing in $\Gamma$ and $\Phi$, we can choose finitely many representative $\Xi_1, \ldots, \Xi_m$, one for each equivalent class in $\mathbf{L}_{V_0}(\Lambda_k)$, and use them to instantiate the probabilistic axioms (PAX1)-(PAX3). Therefore, if $\mathfrak{P}$ denotes the finite set of formulas obtained instantiating the axioms of (PAX1)-(PAX3) over $\Xi_1, \ldots, \Xi_m$ together with all the finitely many instances of the axiom (PAX0), we define

$$FP^\circ = \{\Phi^\circ : \Phi \in \mathfrak{P}\} \cup \{(P\phi)^\circ : \Lambda_k \vdash \phi\}.$$

It is easy to show that, since $\Gamma \not\vdash \Phi$ in $FP(\Lambda_k, Ł^{\mathbf{c}})$, then also $\Gamma^\circ \cup FP^\circ \cup K^{\#}(V_0) \not\vdash \Phi^\circ$ in $Ł^{\mathbf{c}}$. Therefore, since $\Gamma^\circ \cup FP^\circ \cup K^{\#}(V_0)$ is a finite theory in the language of $Ł^{\mathbf{c}}$, and since $Ł^{\mathbf{c}}$ has the canonical FSRC, from Theorem 3.4 (1) there exists a canonical evaluation $e$ into $[0,1]_{\mathrm{MV}}$ that satisfies $\Gamma^\circ \cup FP^\circ \cup K^{\#}(V_0)$, and $e(\Phi^\circ) < 1$.

Following the proof of Theorem 6.4, and since the translation $^\circ$ behaves as the translation $^{\#}$ we presented in Section 6 when restricted to formulas of $\mathfrak{F}(V_0)^{\Box}$, we can define $W$ as the set of all $\L_k^{\mathbf{c}}$-evaluations $w$ from $V_0 \cup \{q_\psi : \Box\psi \in \mathfrak{F}(V_0)^\Box\} \cup \{u_{\phi^\circ} : P(\phi) \in \mathfrak{F}(V_0)^P\}$ that are models for $\Gamma^\circ \cup FP^\circ \cup K^\#(V_0)$. The binary relation $R$ is defined as in (10): for every $w_1, w_2 \in W$,

$$R(w_1, w_2) = \min\{w_2(\varphi) : \varphi \in \mathfrak{F}(V_0)^{\mathbf{c}}, w_1(q_\varphi) = 1\}.$$

Notice that, although the evaluations in $W$ are defined on a wider class of propositional variables, the evaluation of $R$ only takes care of those variables that arise from the formulas in $\mathfrak{F}(V_0)^\Box$, nevertheless it is easy to see that $R$ is well defined. Hence $M = (W, v, R)$ is a $\L_k^{\mathbf{c}}$-Kripke model. Finally, for every $\|\varphi\|_M \in \mathfrak{F}_M^\Box$, let $\mathbf{s}(\|\varphi\|_M) = e(u_\varphi)$. The same proof of [16, Theorem 25] shows that $\mathbf{s}$ is a state, and, consequently, $N = (W, v, R, \mathbf{s})$ is a probabilistic $\L_k^{\mathbf{c}}$-Kripke model that satisfies $\Gamma$ and does not satisfy $\Phi$. This proves (1).

A similar proof can be easily adapted to the case of $FP(C\Lambda_k, \L^{\mathbf{c}})$. □

Now we can further consider the probabilistic logics $FP(\Lambda_k^r, \L^{\mathbf{c}})$ and $FP(C\Lambda_k^r, \L^{\mathbf{c}})$ built over the modal logics $\Lambda_k^r$ and $C\Lambda_k^r$ we have introduced in Section 6.2. Adapting the proof of the above Theorem 7.3, it is fairly easy to see that these logics are sound and finitely strongly complete with respect to the classes of probabilistic $\L_k^{\mathbf{c}}$-Kripke models $(W, e, R, \mathbf{s})$ in which $R$ is a reflexive relation and the class in which $R$ is a crisp reflexive relation respectively. In the next section we will show the importance of these logics to deal with normalized belief functions.

### *7.1. Belief function semantics for belief formulas*

Now, we introduce a class of models that are more closely related to belief functions on MV-algebras as we discussed in Section 4. As we have already observed in Proposition 4.4 (ii), Kroupa belief functions are particular cases of those we introduced in Definition 4.5. We will then focus on this latter generalization.

As for the formulas in $\mathfrak{F}(V)^P$ that well behave with respect to this semantics, let us consider the following class.

**Definition 7.4.** *The set of* belief formulas *(or* B-formulas*) is the subclass of $\mathfrak{F}(V)^P$ defined as follows:* atomic belief formulas *are those of the form $P(\Box\psi)$ (where of course $\psi$ is a formula in $\L_k^{\mathbf{c}}$), that will be henceforth denoted by $B(\psi)$;* compound belief formulas *are defined from atomic ones using the connectives of $L^{\mathbf{c}}$. The set of belief formulas will be denoted by $\mathfrak{F}(V)^B$.*

The class of models that we are about to introduce are based on belief functions rather than states. The idea is to use an extension of Dempster spaces that allows to evaluate formulas of $\mathfrak{F}(V)^{\mathbf{c}}$.

An *evaluated Dempster space* is a pair $(D, e)$ where $D$ is a Dempster space (Definition 5.1) and $e$ is a $\L_k^{\mathbf{c}}$-evaluation.

**Definition 7.5.** *Given an evaluated Dempster space $(D, e)$, the induced belief function on formulas of $\mathfrak{F}(V)^{\mathbf{c}}$ is defined as*

$$bel_{D,e}(\varphi) = bel_D(f_\varphi) \ (= \mu(\varrho_{f_\varphi}))$$

*where $f_\varphi \in (S_k)^W$ is the mapping defined by $f_\varphi(w) = e(w, \varphi)$.*

**Definition 7.6 (Belief function on formulas).** *A mapping $bel : \mathfrak{F}(V)^{\mathbf{c}} \to [0, 1]$ is a belief function on formulas if there is an evaluated Dempster-space $(D, e)$ such that $bel = bel_{D,e}$.*

Consider a probabilistic $Ł_k^{\mathbf{c}}$-Kripke model $K = (W, R, e, \mathbf{s})$, and define the evaluated Dempster space $(D_K, e)$, where $D_K = (W, W, \Gamma, \mu)$ where $\Gamma : W \to (S_k)^W$ is defined as $\Gamma(w) = R(w, \cdot)$, and $\mu = \mathbf{s}$. Therefore, following Definition 7.5, we can say that every probabilistic Kripke model induces (or defines) a belief function as follows:

**Definition 7.7.** *Given $K = (W, R, e, \mu)$, the induced belief function on formulas of $\mathfrak{F}(V)^{\mathbf{c}}$ is defined as*

$$bel_K(\varphi) = bel_{D_K,e}(\varphi).$$

**Lemma 7.8.** *$bel_K(\varphi) = \mu(f_{\Box\varphi})$, where $f_{\Box\varphi} : W \to S_k$ is defined as $f_{\Box\varphi}(w) = e(w, \Box\varphi)$.*

*Proof.* First observe that $bel_{D_K,e}(\varphi) = bel_D(f_\varphi) = \mu(\varrho_{f_\varphi})$. Now let us compute $\mu(\varrho_{f_\varphi})$, we have: $\varrho_{f_\varphi} : W \to S_k$ is defined by

$$\varrho_{f_\varphi}(w) = \inf_{w'} \Gamma(w)(w') \Rightarrow f_\varphi(w') = \inf_{w'} R(w, w') \Rightarrow e(w', \varphi) = e(w, \Box\varphi) = f_{\Box\varphi}(w), \qquad (11)$$

i.e. $\varrho_{f_\varphi} = f_{\Box\varphi}$. □

Therefore, the truth evaluation of belief formulas given by each probabilistic $Ł_k^{\mathbf{c}}$-Kripke model defines a belief function on non-modal formulas.

**Remark 7.9.** *It is worth noticing that, whenever $K = (W, e, R, \mathbf{s})$ is a probabilistic classical $Ł_k^{\mathbf{c}}$-Kripke model (i.e. $R$ is a crisp relation), for every $w \in W$, the mapping $\Gamma(w)$ of $D_K$ maps $w' \in W \mapsto R(w, w') \in 2^W$, and hence $\varrho_{f_\varphi}$ defined through (11) induces a $bel_K$ in the sense of [26]. In fact, recalling the proof of Lemma 5.3, the mass associated to $bel_K$ defined as in (9), provides Boolean focal elements, since $\Gamma(w)$ is a Boolean function.*

The next theorem provides the converse direction, and hence both semantics are proved to be equivalent for belief formulas.

**Theorem 7.10.** *Every belief function on formulas (defined by an evaluated Dempster space) is given by a probabilistic $Ł_k^{\mathbf{c}}$-Kripke model.*

*Proof.* Let $(D, e)$ be an evaluated Dempster space, where $D = (W, E, \Gamma, \mu)$ is Dempster space, and let $bel_{D,e}$ be its corresponding belief function defined as in Definition 7.5. We have to show that there is a probabilistic $Ł_k^{\mathbf{c}}$-Kripke model $K = (W', R, e', \mu')$ such that, for each formula $\varphi \in \mathfrak{F}(V)^{\mathbf{c}}$, $bel_{D,e}(\varphi) = bel_K(\varphi)$.

By Lemma 5.3, we know there is mass function $m : (S_k)^W \to [0, 1]$ such that, for every $f \in (S_k)^W$, $bel_D(f) = \sum_{g \in (S_k)^W} \rho_f(g) \cdot m(g)$.

Define the probabilistic $Ł_k^{\mathbf{c}}$-Kripke model $K = (W', R, e', \mathbf{s})$ such that:

- $W' = \{(f, w) \mid f \in (S_k)^W, w \in W\}$
- for every $(f, w), (g, w') \in W'$, $R((f, w), (g, w')) = f(w')$
- $e'((f, w), \varphi) = e(w, \varphi)$, for each $\varphi \in \mathfrak{F}(V)^{\mathbf{c}}$
- $\mathbf{s}$ is a state on $(S_k)^{W'}$ such that for every $f \in (S_k)^W$,

$$\sum_{w \in W} \mathbf{s}(\{(f, w)\}) = m(f).$$

Observe that, if $\psi$ is non-modal,

$$\begin{array}{rcl} e'((f,w),\Box\psi) & = & \bigwedge_{(g,w')\in W'} R((f,w),(g,w')) \Rightarrow e'((g,w'),\psi) \\ & = & \bigwedge_{(g,w')\in W'} f(w') \Rightarrow e(w',\psi) \\ & = & \bigwedge_{w'\in W} f(w') \Rightarrow f_\psi(w') \\ & = & \rho_{f_\psi}(f). \end{array}$$

Now consider a belief formula $B(\psi) = P(\Box\psi)$, and let us evaluate it in $K = (W', e', R, \mathbf{s})$:

$$\begin{array}{rcl} bel_K(\psi) = \|P\Box\psi\|_K & = & \mathbf{s}(e'(\cdot,\Box\psi)) \\ & = & \sum_{(f,w)\in W'} e'((f,w),\Box\psi)\cdot \mathbf{s}(\{(f,w)\}) \\ & = & \sum_{(f,w)\in W'} \rho_{f_\psi}(f)\cdot \mathbf{s}(\{(f,w)\}) \\ & = & \sum_{f\in(S_k)^W} \rho_{f_\psi}(f)\cdot \left(\sum_{w\in W} \mathbf{s}(\{(f,w)\})\right) \\ & = & \sum_{f\in(S_k)^W} \rho_{f_\psi}(f)\cdot m(f) \\ & = & bel_D(f_\psi) \\ & = & bel_{D,e}(\psi). \end{array}$$

□

Therefore, alternatively to the probabilistic $Ł_k^{\mathbf{c}}$-Kripke model semantics for belief formulas, we can simply define a semantics based on belief functions on formulas. This is formally done in the next two definitions.

**Definition 7.11.** *Let $\Phi$ a belief formula and let bel a belief function on formulas of $\mathfrak{F}(V)^{\mathbf{c}}$. The truth evaluation of $\Phi$ by bel is defined by induction as follows:*

- *if $\Phi$ is an atomic belief formulas $P\Box\varphi$, then $\|\Phi\|_{bel} = bel(\varphi)$;*
- *$\|\cdot\|_{bel}$ is then extended to compound belief formulas using $Ł_k^{\mathbf{c}}$ connectives.*

*If $\|\Phi\|_{bel} = 1$ we say that bel is a model of $\Psi$. Moreover, we say bel is a model of a set of belief formulas (belief theory) $T$ if bel is a model of each formula of $T$.*

**Definition 7.12.** *Let $T$ be a belief theory and let $\Phi$ be belief formula. $T \models_{BF} \Phi$ iff for every belief function bel on formulas of $\mathfrak{F}(V)^{\mathbf{c}}$, $\|\Psi\|_{bel} = 1$ for every $\Psi \in T$ implies $\|\Phi\|_{bel} = 1$ as well.*

Analogously, one can define logical consequence relations $\models_{BF_{Kroupa}}$, $\models_{BF_n}$ and $\models_{BF_{Kroupa,n}}$ corresponding to the classes of Kroupa belief functions, normalized belief functions and normalized Kroupa belief functions, respectively.

Due to Theorem 7.10, $T \models_{BF} \Phi$ can be equivalently given by probabilistic $Ł_k^{\mathbf{c}}$-Kripke models.

**Lemma 7.13.** *$T \models_{BF} \Phi$ iff for every probabilistic $Ł_k^{\mathbf{c}}$-Kripke model $K = (W, R, e, \mu)$, $\|\Psi\|_K = 1$ for every $\Psi \in T$ implies $\|\Phi\|_K = 1$ as well.*

Finally we can formulate the following completeness result.

**Theorem 7.14 (Completeness).** *Let $T$ be a finite belief theory and let $\Phi$ be belief formula. Then it holds that*

$$T \vdash_{FP(\Lambda_k, Ł^{\mathbf{c}})} \Phi \quad \textit{iff} \quad T \models_{BF} \Phi,$$

*i.e.* $\Phi$ *is derivable from* $T$ *in the logic* $FP(\Lambda_k, Ł^{\mathbf{c}})$ *if, and only if, every belief function on formulas that is a model of* $T$ *also is a model of* $\Phi$.

*Proof.* This is simply a direct consequence of the probabilistic completeness of $FP(\Lambda_k, Ł^{\mathbf{c}})$ (see Theorem 7.3) and the above Lemma 7.13. □

As a direct corollary we have the following completeness result for Kroupa belief functions.

**Corollary 7.15.** *For any finite belief theory* $R$ *and belief formula* $\Phi$ *be belief formula, it holds that* $T \vdash_{FP(C\Lambda_k, Ł^{\mathbf{c}})} \Phi$ *iff* $T \models_{BF_{Kroupa}} \Phi$.

*7.2. Dealing with normalized belief functions*

In Section 4 we called *normalized* those belief functions $\mathbf{b} : (S_k)^X \to [0,1]$ whose focal elements are normalized fuzzy sets. A belief model $(\Omega, m)$ hence is said to be *normalized* provided that every focal element $f \in (S_k)^\Omega$ (i.e. every $f \in (S_k)^\Omega$ such that $m(f) > 0$) is a normalized fuzzy set.

Consider a probabilistic $Ł_k^{\mathbf{c}}$-Kripke model $K = (W, e, R, \mathbf{s})$ for $FP(\Lambda_k^r, Ł^{\mathbf{c}})$. In other words, let $K = (W, e, R, \mathbf{s})$ be a probabilistic $Ł_k^{\mathbf{c}}$-Kripke model, whose accessibility relation $R$ is reflexive, and define from $K$ the evaluated Dempster space $D_K = (W, W, \Gamma, \mu)$ defined as in the previous section. Recall that $\Gamma(w) = R(w, \cdot)$, and hence the mass assignment associated to $bel_K$ defined as in (9) induces focal elements $g \in (S_k)^W$ such that for some $w' \in W$, $g = \Gamma(w') = R(w', \cdot)$. Therefore, if $g = \Gamma(w')$ is a focal element of $bel_K$, $g(w') = \Gamma(w')(w') = R(w', w') = 1$, and hence $g$ is normalized.

**Proposition 7.16.** *For every probabilistic* $Ł_k^{\mathbf{c}}$*-Kripke model* $K = (W, e, R, \mathbf{s})$ *with* $R$ *reflexive, there exists a normalized belief function on formulas bel such that, for every belief formula* $\Phi$, $\|\Phi\|_K = \|\Phi\|_{bel}$.

Conversely, let $(D, e) = (W, E, \Gamma, \mu, e)$ be an evaluated Dempster space inducing a normalized belief function on formulas $bel = bel_{D,e}$, and let

- $W' = \{(f, w) : f \text{ is normalized, and } f(w) = 1\}$,
- $R$ and $e'(\cdot) = \| \cdot \|_{(f,w)}$ are defined as in the proof of Theorem 7.10,
- $\mathbf{s}$ is a state on $(S_k)^{W'}$ such that for every $f \in (S_k)^W$,

$$\sum_{w \in W : f(w) = 1} \mathbf{s}(\{(f, w)\}) = m(f),$$

  where $m$ is the mass associated to $bel$ through Lemma 5.3.

Then $M = (W', e', R, \mathbf{s})$ is a probabilistic $Ł_k^{\mathbf{c}}$-Kripke model with $R$ reflexive. In fact for every $(f, w) \in W'$, $R((f, w), (f, w)) = f(w) = \max_{w' \in W} f(w') = 1$ because $f$ is a focal element for $m$, and $bel$ is normalized. Moreover, since for every $w_0 \in W$, the map $g : W \to S_k$ such that $g(w) = 1$ if $w = w_0$, and $g(w) = 0$ otherwise is a normalized fuzzy subset of $W$, it follows that

$$W = \{w \in W : (g, w) \in W'\}.$$

Therefore, taking this into account, if $\psi$ is non-modal then, following the lines of the proof of Theorem 7.10, we have $\|\psi\|_{(f,w)} = \rho_{\|\psi\|}(f)$. If $\Phi$ is any belief formula, then $\|\Phi\|_{bel} = \|\Phi\|_M$, in other words the following holds.

**Proposition 7.17.** *For each normalized belief function on formulas bel there exists a probabilistic* $Ł_k^{\mathbf{c}}$*-Kripke* $M = (W, e, R, \mathbf{s})$ *with* $R$ *reflexive, such that, for every belief formula* $\Phi$, $\|\Phi\|_M = \|\Phi\|_{bel}$.

Therefore from Proposition 7.16 and Proposition 7.17 we immediately get the following.

**Theorem 7.18.** *The logic* $FP(\Lambda_k^r, Ł^{\mathbf{c}})$ *is sound and finitely strong complete with respect to normalized belief functions on formulas.*

The following result, that we state in order to clarify what we discussed in Remark 7.1, is hence a direct consequence of Theorem 7.18 and Proposition 4.7, showing that $FP(\Lambda_k^r, Ł^{\mathbf{c}})$ proves that the belief modality $B$ is homogeneous.

**Corollary 7.19.** *For every* $k \in \mathbb{N}$, *and for every* $r \in S_k$, $FP(\Lambda_k^r, Ł^{\mathbf{c}})$ *proves* $B(\overline{r}) \leftrightarrow \overline{r}$.

**Corollary 7.20.** *For any finite belief theory* $R$ *and belief formula* $\Phi$ *be belief formula, it holds that* $T \vdash_{FP(C\Lambda_k^r, Ł^{\mathbf{c}})} \Phi$ *iff* $T \models_{BF_{Kroupa,n}} \Phi$.

## 8. Conclusion

In this paper we presented a logical approach to belief functions on MV-algebras. We have followed the idea developed in [17] where the authors defined a logic for belief functions on Boolean algebras by combining a probabilistic modality $P$ with the classical S5 modality $\Box$. Actually in [17], the choice of S5 as the modal logic for events is motivated by the need of a locally finite logical system (remember also our proof of Theorem 7.3, and the proof of [16, Theorem 25] where locally finiteness is a crucial requirement for the logic of events), and in fact S5 is the weaker classical modal logic that fulfills that requirement (see [6]). In this paper we started from a non-locally finite modal logic as logic for events, and we recovered local finiteness by working on the syntactical level of modal formulas, and specifically not allowing a nested use of $\Box$. This remark shows that, in fact, the same results the authors proved in [17] can be equivalently obtained considering, as logic for events, a variant of the weaker classical modal logic K, without nested modalities. Indeed a nested use of $\Box$ is useless when we define belief formulas as we did in Section 7.1, and as they are defined in [17, §4].

In our future work we plan to define an extension of the logics for belief functions over infinite-valued events. In order to achieve this goal, we will follow the idea of considering a modal extension of the infinitely-valued Łukasiewicz calculus as logic for events. Indeed this problem is not trivial, and although there are some papers that go in that direction (cf. [22, 23]), to adopt such kind of formalisms to treat events would not keep the logic locally finite, even in the case of a language with unnested modalities. This means that the same strategy we used in order to prove completeness for our logics (Theorem 7.3) cannot be applied in this setting. On the other hand we might exploit different kinds of completeness results like Pavelka-style completeness.

Not secondarily, we also plan to study the problem of establishing whether a partial assignment on a countable set of fuzzy events, is extendible to a belief function defined over the algebra spanned by them. This problem, which is related to the well known de Finetti coherence criterion for probability measures, can be characterized in several ways. We will focus on the development of a logico-algebraic and geometrical approach.

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