Implicator-Conjunctor Based Models of Fuzzy Rough Sets: Definitions and Properties

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Abstract. Ever since the first hybrid fuzzy rough set model was proposed in the early 1990's, many researchers have focused on the definition of the lower and upper approximation of a fuzzy set by means of a fuzzy relation. In this paper, we review those proposals which generalize the logical connectives and quantifiers present in the rough set approximations by means of corresponding fuzzy logic operations. We introduce a general model which encapsulates all of these proposals, evaluate it w.r.t. a number of desirable properties, and refine the existing axiomatic approach to characterize lower and upper approximation operators.
Keywords: fuzzy sets, rough sets, hybridization, lower and upper approximation, implication, conjunction, axiomatic approach

1 Introduction

Fuzzy sets [1] generalize classical or *crisp sets* in a sense that objects can be assigned intermediary membership degrees to a set or relation, drawn from a partially ordered set, typically $[0, 1]$. On the other hand, *rough sets* [2] characterize a set of objects by means of a lower and an upper approximation, taking into account an equivalence relation that represents indiscernibility between objects. Both theories have fostered broad research communities and have been applied in a wide range of settings. It was recognized early on that they are complementary, rather than competitive; a first hybrid fuzzy rough set model was proposed by Dubois and Prade [3] in 1990. Now, more than 20 years later, interest in fuzzy rough sets is thriving; this is mainly thanks to their proven application potential in machine learning, and in particular in feature selection [4–6] and instance selection [7].

Fuzzy-rough hybridization has been pursued in a variety of ways; in this paper, we focus on the most common approach, i.e., using fuzzy logical extensions of the Boolean implication and conjunction, along with infimum and supremum as extensions of the universal and existential quantifiers. This idea sparked the

seminal proposal in [3], and since then many papers [8–18] have focused on the refinement of this model using both *constructive approaches*, which propose new definitions of approximation operators, and *axiomatic approaches*, which set forth a set of axioms or desirable properties, and characterize the operators that satisfy them.

It was found that through a deliberate choice of fuzzy logical operators, and the use of a similarity relation (also called fuzzy equivalence relation) to model approximate indiscernibility, most properties of the original rough set model can be maintained [8, 9]; on the other hand, from a practical point of view, the use of similarity relations is not always convenient (see e.g. [6]), and as De Cock et al. [16] argued, they cause part of the hybridization potential to remain unexplored. Moreover, also in the crisp case, various types of binary relations have been considered to replace the indiscernibility equivalence relation, see e.g. [19, 20]. For all of these reasons, several other authors considered fuzzy rough set models based on general fuzzy relations [10–15, 17].

In this paper, we unify all these approaches under the umbrella of a general implicator-conjunctor based fuzzy rough set model, imposing minimal restrictions on the approximations. After recalling some preliminaries in Section 2, we present the definitions of the approximations in Section 3, and give a chronological overview of special cases studied in the literature. In Section 4, we evaluate the model w.r.t. desirable properties, while in Section 5, we refine the axiomatic approach of Wu et al. [13], weakening some of its conditions and proposing an alternative characterization that caters specifically to residual implications. Finally, in Section 6, we conclude and outline future work.

2 Preliminaries

2.1 Fuzzy Logical Connectives

A *conjunctor* is a mapping $\mathcal{C}\colon [0,1]^2 \to [0,1]$ which is increasing in both arguments and which satisfies $\mathcal{C}(0,0) = \mathcal{C}(0,1) = \mathcal{C}(1,0) = 0$ and $\mathcal{C}(1,1) = 1$. It is called a *border conjunctor* if it satisfies $\mathcal{C}(1,x) = x$ for all x in $[0,1]$. A commutative, associative border conjunctor $\mathcal{T}$ is called a *t-norm*.

A *disjunctor* is a mapping $\mathcal{D}\colon [0,1]^2 \to [0,1]$ which is increasing in both arguments and which satisfies $\mathcal{D}(1,0) = \mathcal{D}(0,1) = \mathcal{D}(1,1) = 1$ and $\mathcal{D}(0,0) = 0$. It is called a *border disjunctor* if it satisfies $\mathcal{D}(0,x) = x$ for all x in $[0,1]$. A commutative, associative border disjunctor $\mathcal{S}$ is called a *t-conorm*.

A *negator* is a decreasing mapping $\mathcal{N}\colon [0,1] \to [0,1]$ which satisfies $\mathcal{N}(0) = 1$ and $\mathcal{N}(1) = 0$. It is *involutive* if for all $x \in [0,1]$, $\mathcal{N}(\mathcal{N}(x)) = x$. The standard negator $\mathcal{N}_s$ is defined by, for x in $[0,1]$, $\mathcal{N}_s(x) = 1 - x$.

Given an involutive negator $\mathcal{N}$, a conjunctor $\mathcal{C}$ and a disjunctor $\mathcal{D}$, the *$\mathcal{N}$-dual of $\mathcal{C}$* is a disjunctor $\mathcal{D}_{\mathcal{C},\mathcal{N}}$, defined by $\mathcal{D}_{\mathcal{C},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{C}(\mathcal{N}(x),\mathcal{N}(y)))$, and the *$\mathcal{N}$-dual of $\mathcal{D}$* is a conjunctor $\mathcal{C}_{\mathcal{D},\mathcal{N}}$, defined by $\mathcal{C}_{\mathcal{D},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{D}(\mathcal{N}(x), \mathcal{N}(y)))$, for all x, y in $[0,1]$. It can be verified that the $\mathcal{N}$-dual of a t-norm is a t-conorm, and vice versa.

An *implicator* $\mathcal{I}$ is a mapping $\mathcal{I}\colon [0,1]^2 \to [0,1]$ satisfying $\mathcal{I}(1,0) = 0$, $\mathcal{I}(1,1) = \mathcal{I}(0,1) = \mathcal{I}(0,0) = 1$ which is decreasing in the first and increasing in the second argument. If $\mathcal{I}$ satisfies $\mathcal{I}(1,x) = x$ for all x in $[0,1]$, it is called a *border implicator*, and if it satisfies the exchange principle, $\mathcal{I}(x,\mathcal{I}(y,z)) = I(y,\mathcal{I}(x,z))$ for all x, y, z in $[0,1]$, it is called an *EP implicator*.

Let $\mathcal{C}$, $\mathcal{D}$ and $\mathcal{N}$ be a border conjunctor, a disjunctor and a negator respectively. The *S-implicator* $\mathcal{I}_{\mathcal{D},\mathcal{N}}$ based on $\mathcal{D}$ and $\mathcal{N}$ is defined by, for x, y in $[0,1]$, $\mathcal{I}_{\mathcal{D},\mathcal{N}}(x,y) = \mathcal{D}(\mathcal{N}(x), y)$. The *R-implicator* $\mathcal{I}_{\mathcal{C}}$ based on $\mathcal{C}$ is defined by, for x, y in $[0,1]$, $\mathcal{I}_{\mathcal{C}}(x,y) = \sup\{\gamma \in [0,1] \mid \mathcal{C}(x,\gamma) \leq y\}$. Both S-implicators and R-implicators are particular cases of border implicators.

Given an involutive negator $\mathcal{N}$ and an implicator $\mathcal{I}$, the *induced conjunctor of $\mathcal{I}$ and $\mathcal{N}$* is a conjunctor $\mathcal{C}_{\mathcal{I},\mathcal{N}}$ defined by, for $x, y \in [0,1]$, $\mathcal{C}_{\mathcal{I},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{I}(x,\mathcal{N}(y))$. It is not necessarily a t-norm.

2.2 Fuzzy Sets and Relations

A fuzzy set A in a non-empty universe set U is a mapping $A : U \to [0,1]$. The collection of all fuzzy sets in U is denoted by $\mathcal{F}(U)$.

Given α in $[0,1]$, the *constant (fuzzy) set* $\hat{\alpha}$ is defined by, for x in U, $\hat{\alpha}(x) = \alpha$. In the crisp case, the only constant sets are $\emptyset$ and U.

Let $A, B \in \mathcal{F}(U)$ and $x \in U$. Given a negator $\mathcal{N}$, the *$\mathcal{N}$-complement of A* is defined by $(co_{\mathcal{N}}(A))(x) = \mathcal{N}(A(x))$. Given a conjunctor $\mathcal{C}$ and a disjunctor $\mathcal{D}$, the *$\mathcal{C}$-intersection and $\mathcal{D}$-union of A and B* are defined by $(A \cap_{\mathcal{C}} B)(x) = \mathcal{C}(A(x), B(x))$ and $(A \cup_{\mathcal{D}} B)(x) = \mathcal{D}(A(x), B(x))$. If $\mathcal{C} = \min$ and $\mathcal{D} = \max$, we simply write $\cap$ and $\cup$. Given an implicator $\mathcal{I}$, the *$\mathcal{I}$-implication of A and B* is defined by $(A \Rightarrow_{\mathcal{I}} B)(x) = \mathcal{I}(A(x), B(x))$.

A binary fuzzy relation R in U is a fuzzy set in $U \times U$. We define its inverse fuzzy relation R' by $R'(x,y) = R(y,x)$ for x, y in U. R is called *reflexive* if $R(x,x) = 1$, *symmetric* if $R(x,y) = R(y,x)$ and *inverse serial* if $\sup_{x \in U} R(x,y) = 1$ for all y in U. For a symmetric binary fuzzy relation R, it obviously holds that $R = R'$.

Given a t-norm $\mathcal{T}$, R is called *$\mathcal{T}$-transitive* if for all x, y and z in U, $\mathcal{T}(R(x,y), R(y,z)) \leq R(x,z)$. If R is reflexive, symmetric and $\mathcal{T}$-transitive, it is called a *$\mathcal{T}$-similarity relation*. When $\mathcal{T} = \min$, we shortly speak about a *similarity relation*. Because the minimum operator is the largest t-norm, a similarity relation is a $\mathcal{T}$-similarity relation for every t-norm $\mathcal{T}$.

2.3 Lower and Upper Approximations in Rough Set Theory

A *classical* or *Pawlak approximation space* is a couple (U, R) consisting of a non-empty set U and an equivalence relation R in U. The rough approximation of a crisp set A in U by R is the pair of sets $(R{\downarrow}A, R{\uparrow}A)$ defined by, for $x \in U$,

$$x \in R{\downarrow}A \Leftrightarrow (\forall y \in U)((y,x) \in R \Rightarrow y \in A) \tag{1}$$

$$x \in R{\uparrow}A \Leftrightarrow (\exists y \in U)((y,x) \in R \wedge y \in A). \tag{2}$$

A pair (A_1, A_2) of sets in U is called a *rough set* in (U, R) if there is a set A in U such that $A_1 = R{\downarrow}A$ and $A_2 = R{\uparrow}A$. Some of the most important properties of lower and upper approximation in a Pawlak approximation space are listed in the left hand side of Table 2. Note that we denote the complement of a crisp set A by A^c.

3 Implicator-Conjunctor Based Model

Many definitions of fuzzy rough sets emerge by faithfully extending Eqs. (1) and (2) to the $[0, 1]$-valued case. In particular, Dubois and Prade worked with a similarity relation R, and replaced the Boolean implication and conjunction by the S-implicator $\mathcal{I}_{\max,\mathcal{N}_s}$ (Kleene-Dienes implicator) and the minimum t-norm, respectively. In this section, we consider a *fuzzy approximation space*, i.e., a couple (U, R) consisting of a non-empty set U and a binary fuzzy relation R in U, and define a general format for the approximations using implicators and conjunctors.

Definition 1. *Let (U, R) be a fuzzy approximation space, A a fuzzy set in U, $\mathcal{I}$ an implicator and $\mathcal{C}$ a conjunctor. The* $(\mathcal{I}, \mathcal{C})$-fuzzy rough approximation *of A by R is the pair of fuzzy sets $(R{\downarrow_{\mathcal{I}}}A, R{\uparrow_{\mathcal{C}}}A)$ defined by, for $x \in U$,*

$$(R{\downarrow_{\mathcal{I}}}A)(x) = \inf_{y \in U} \mathcal{I}(R(y, x), A(y)) \tag{3}$$

$$(R{\uparrow_{\mathcal{C}}}A)(x) = \sup_{y \in U} \mathcal{C}(R(y, x), A(y)). \tag{4}$$

A pair (A_1, A_2) of fuzzy sets in U is called a fuzzy rough set *in (U, R) if there is a fuzzy set A in U such that $A_1 = R{\downarrow_{\mathcal{I}}}A$ and $A_2 = R{\uparrow_{\mathcal{C}}}A$.*

In Table 1 we give a chronological overview of special cases of the general model. Some authors [8, 15, 18] actually require lower semicontintuity of $\mathcal{T}$ instead of left-continuity, but by a result from [21] these two notions are equivalent for t-norms. Also, some papers [10, 11, 13, 17] consider fuzzy relations from U to W, with both U and W non-empty, finite universes, but here we restrict ourselves to the case $U = W$. As can be seen, Wu et al. [10] were the first to consider general binary fuzzy relations, while Mi and Zhang [11] initiated the use of conjunctors that are not necessarily t-norms. Also note that the t-norm $\mathcal{T}_{cos}$ used in [18] is defined, for x, y in $[0, 1]$, by $\mathcal{T}_{\cos}(x, y) = \max(xy - \sqrt{(1 - x^2)(1 - y^2)}, 0)$. Its use is inspired by the fact that some commonly used kernel functions in machine learning are in fact $\mathcal{T}_{\cos}$-similarity relations.

4 Properties

In the following, we assume that (U, R), (U, R_1) and (U, R_2) are fuzzy approximation spaces, A and B are fuzzy sets in U, $\mathcal{I}$ is an implicator, $\mathcal{C}$ a conjunctor and $\mathcal{N}$ an involutive negator. In the right hand side of Table 2, we show the

Model	Conjunctor	Implicator	Relation
[3] Dubois & Prade, 1990	min	$\mathcal{I}_{\max,\mathcal{N}_s}$	similarity
[8] Morsi & Yakout, 1998	left-cont. t-norm $\mathcal{T}$	$\mathcal{I}_{\mathcal{T}}$	$\mathcal{T}$-similarity
[9] Radzikowska & Kerre, 2002	t-norm $\mathcal{T}$	border implicator $\mathcal{I}$	similarity
[10] Wu et al., 2003	min	$\mathcal{I}_{\max,\mathcal{N}_s}$	general
[11] Mi & Zhang, 2004	$\mathcal{C}_{\mathcal{I}_{\mathcal{T}},\mathcal{N}_s}$; left-cont. t-norm $\mathcal{T}$	$\mathcal{I}_{\mathcal{T}}$	general
[13] Wu et al., 2005	cont. t-norm $\mathcal{T}$	implicator $\mathcal{I}$	general
[14] Pei, 2005	min	$\mathcal{I}_{\max,\mathcal{N}_s}$	general
[15] Yeung et al., 2005	left-cont. t-norm $\mathcal{T}$	$\mathcal{I}_{\mathcal{S}_{T,\mathcal{N}},\mathcal{N}}$, $\mathcal{N}$ involutive	general
[15] Yeung et al., 2005	$\mathcal{C}_{\mathcal{I}_{\mathcal{T}},\mathcal{N}_s}$; left-cont. t-norm $\mathcal{T}$	$\mathcal{I}_{\mathcal{T}}$	general
[16] De Cock et al., 2007	t-norm $\mathcal{T}$	border implicator $\mathcal{I}$	general
[17] Mi et al., 2008	cont. t-norm $\mathcal{T}$	$\mathcal{I}_{\mathcal{S}_{\mathcal{T}},\mathcal{N}_s},\mathcal{N}_s$	general
[18] Hu et al., 2010	left-cont. t-norm $\mathcal{T}$ eft-cont. t-norm $\mathcal{T}$	$\mathcal{I}_{\mathcal{S}_{\mathcal{T}},\mathcal{N}_s},\mathcal{N}_s$	$\mathcal{T}_{\cos}$-similarity
[18] Hu et al., 2010	$\mathcal{C}_{\mathcal{I}_{\mathcal{T}},\mathcal{N}_s}$; left-cont. t-norm $\mathcal{T}$	$\mathcal{I}_{\mathcal{T}}$	$\mathcal{T}_{\cos}$-similarity

Table 1. Overview of special cases of the general fuzzy rough set model

extensions of the classical rough set properties to a fuzzy approximation space. We can prove the following propositions, which mainly generalize known results obtained in a restricted setting, see e.g. [9].

Proposition 1. *If $\mathcal{C}$ is the induced conjunctor of $\mathcal{I}$ and $\mathcal{N}$, i.e., $\mathcal{C} = \mathcal{C}_{\mathcal{I},\mathcal{N}}$, then the duality property holds.*

Corollary 1. *Let $\mathcal{D}$ be the $\mathcal{N}$-dual disjunctor of $\mathcal{C}$. If the pair $(\mathcal{I},\mathcal{C})$ consists of the S-implicator $\mathcal{I}_{\mathcal{D},\mathcal{N}}$ and the conjunctor $\mathcal{C}$, then the duality property holds.*

Corollary 2. *Let $\mathcal{T}$ be a left-continuous t-norm and $\mathcal{N} = \mathcal{N}_{\mathcal{I}_{\mathcal{T}}}$. If the pair $(\mathcal{I},\mathcal{C})$ consists of the R-implicator $\mathcal{I}_{\mathcal{T}}$ and the t-norm $\mathcal{T}$, then the duality property holds.*

To see this corollary, note that $\mathcal{C}_{\mathcal{I}_{\mathcal{T}},\mathcal{N}} = \mathcal{T}$ indeed holds: for x, y in $[0,1]$, $\mathcal{C}_{\mathcal{I}_{\mathcal{T}},\mathcal{N}}(x,y) = \mathcal{N}(\mathcal{I}_{\mathcal{T}}(x,\mathcal{N}(y))) = \mathcal{N}(\mathcal{I}_{\mathcal{T}}(x,\mathcal{I}_{\mathcal{T}}(y,0))) = \mathcal{N}(\mathcal{I}_{\mathcal{T}}(\mathcal{T}(x,y),0)) = \mathcal{N}(\mathcal{N}(\mathcal{T}(x,y))) = \mathcal{T}(x,y)$.

Proposition 2. *If the pair $(\mathcal{I},\mathcal{C})$ consists of the R-implicator $\mathcal{I}_{\mathcal{T}}$ and the left-continuous t-norm $\mathcal{T}$, then the adjointness property holds.*

Note that in generalizing the adjointness condition to a fuzzy approximation space, we have replaced R in the right hand side of the equivalence by its inverse fuzzy relation R'. Clearly, if R is symmetric (which is the case for a Pawlak approximation space), this modification is redundant.

Proposition 3. *If R is reflexive, $\mathcal{I}$ is a border implicator and $\mathcal{C}$ is a border conjunctor, then the inclusion property holds.*

Name	Pawlak approximation space	Fuzzy approximation space
Duality	$R{\downarrow}A = (R{\uparrow}A^{\mathrm{c}})^{\mathrm{c}}$ $R{\uparrow}A = (R{\downarrow}A^{\mathrm{c}})^{\mathrm{c}}$	$R{\downarrow_{\mathcal{I}}}A = \mathrm{co}_{\mathcal{N}}(R{\uparrow_{\mathcal{C}}}(\mathrm{co}_{\mathcal{N}}(A)))$ $R{\uparrow_{\mathcal{C}}}A = \mathrm{co}_{\mathcal{N}}(R{\downarrow_{\mathcal{I}}}(\mathrm{co}_{\mathcal{N}}(A)))$
Adjointness	$R{\uparrow}A \subseteq B \Leftrightarrow A \subseteq R{\downarrow}B$	$R{\uparrow_{\mathcal{C}}}A \subseteq B \Leftrightarrow A \subseteq R'{\downarrow_{\mathcal{I}}}B$
Inclusion	$R{\downarrow}A \subseteq A$ $A \subseteq R{\uparrow}A$	$R{\downarrow_{\mathcal{I}}}A \subseteq A$ $A \subseteq R{\uparrow_{\mathcal{C}}}A$
Set monotonicity	$A \subseteq B \Rightarrow R{\downarrow}A \subseteq R{\downarrow}B$ $A \subseteq B \Rightarrow R{\uparrow}A \subseteq R{\uparrow}B$	$A \subseteq B \Rightarrow R{\downarrow_{\mathcal{I}}}A \subseteq R{\downarrow_{\mathcal{I}}}B$ $A \subseteq B \Rightarrow R{\uparrow_{\mathcal{C}}}A \subseteq R{\uparrow_{\mathcal{C}}}B$
Relation monotonicity	$R_1 \subseteq R_2 \Rightarrow R_2{\downarrow}A \subseteq R_1{\downarrow}A$ $R_1 \subseteq R_2 \Rightarrow R_1{\uparrow}A \subseteq R_2{\uparrow}A$	$R_1 \subseteq R_2 \Rightarrow R_2{\downarrow_{\mathcal{I}}}A \subseteq R_1{\downarrow_{\mathcal{I}}}A$ $R_1 \subseteq R_2 \Rightarrow R_1{\uparrow_{\mathcal{C}}}A \subseteq R_2{\uparrow_{\mathcal{C}}}A$
Intersection	$R{\downarrow}(A \cap B) = R{\downarrow}A \cap R{\downarrow}B$ $R{\uparrow}(A \cap B) \subseteq R{\uparrow}A \cap R{\uparrow}B$	$R{\downarrow_{\mathcal{I}}}(A \cap B) = R{\downarrow_{\mathcal{I}}}A \cap R{\downarrow_{\mathcal{I}}}B$ $R{\uparrow_{\mathcal{C}}}(A \cap B) \subseteq R{\uparrow_{\mathcal{C}}}A \cap R{\uparrow_{\mathcal{C}}}B$
Union	$R{\downarrow}(A \cup B) \supseteq R{\downarrow}A \cup R{\downarrow}B$ $R{\uparrow}(A \cup B) = R{\uparrow}A \cup R{\uparrow}B$	$R{\downarrow_{\mathcal{I}}}(A \cup B) \supseteq R{\downarrow_{\mathcal{I}}}A \cup R{\downarrow_{\mathcal{I}}}B$ $R{\uparrow_{\mathcal{C}}}(A \cup B) = R{\uparrow_{\mathcal{C}}}A \cup R{\uparrow_{\mathcal{C}}}B$
Idempotence	$R{\downarrow}(R{\downarrow}A) = R{\downarrow}A$ $R{\uparrow}(R{\uparrow}A) = R{\uparrow}A$	$R{\downarrow_{\mathcal{I}}}(R{\downarrow_{\mathcal{I}}}A) = R{\downarrow_{\mathcal{I}}}A$ $R{\uparrow_{\mathcal{C}}}(R{\uparrow_{\mathcal{C}}}A) = R{\uparrow_{\mathcal{C}}}A$
Constant sets	$R{\downarrow}\emptyset = \emptyset = R{\uparrow}\emptyset$ $R{\downarrow}U = U = R{\uparrow}U$	$R{\downarrow_{\mathcal{I}}}\hat{\alpha} = \hat{\alpha}$ $R{\uparrow_{\mathcal{C}}}\hat{\alpha} = \hat{\alpha}$

Table 2. Properties in a Pawlak approximation space and their corresponding extensions to a fuzzy approximation space.

Corollary 3. *Let $\mathcal{T}$ and $\mathcal{S}$ be a t-norm and its $\mathcal{N}$-dual t-conorm. If R is reflexive, and $(\mathcal{I},\mathcal{C}) = (\mathcal{I}_{\mathcal{S},\mathcal{N}},\mathcal{T})$ or $(\mathcal{I},\mathcal{C}) = (\mathcal{I}_{\mathcal{T}},\mathcal{T})$, then the inclusion property holds.*

Proposition 4. *The properties of set and relation monotonicity, intersection and union always hold.*

Proposition 5. *If R is a reflexive and $\mathcal{T}$-transitive relation, where $\mathcal{T}$ is a left-continuous t-norm and the pair $(\mathcal{I},\mathcal{C})$ consists of the R-implicator $\mathcal{I}_{\mathcal{T}}$ and the t-norm $\mathcal{T}$, then the idempotence property holds.*

Proposition 6. *If R is a reflexive relation, $\mathcal{I}$ a border implicator and $\mathcal{C}$ a border conjunctor, then the constant sets property holds.*

Summing up, in order to satisfy all properties in Table 2, $\mathcal{C}$ should be a left-continuous t-norm $\mathcal{T}$ and $\mathcal{I}$ its R-implicator, while R needs to be at least reflexive and $\mathcal{T}$-transitive. Propositions 2 and 5 do not hold in general for S-implicators, for instance, Dubois and Prade's model [3] does not satisfy them.

5 Axiomatic Approach

In the axiomatic approach, we work with unary operators on $\mathcal{F}(U)$ and some axioms to obtain a fuzzy relation R such that the operators behave as approxi-

mation operators with respect to R. Such an approach is useful to get insight in the logical structure of fuzzy rough sets.

As our starting point, we use the axiomatic approach developed by Wu et al. [13], who propose axioms to characterise lower and upper approximations, which are generalized here for an implicator-conjunctor pair.

Definition 2. *Let $H, L\colon \mathcal{F}(U) \to \mathcal{F}(U)$, $\mathcal{C}$ a conjunctor and $\mathcal{I}$ an implicator. H is a* $\mathcal{C}$-upper approximation *if it satisfies, for all $A, A_j \in \mathcal{F}(U)$, $\alpha \in [0,1]$,*

(H1) $H(\hat{\alpha} \cap_{\mathcal{C}} A) = \hat{\alpha} \cap_{\mathcal{C}} H(A)$

(H2) $H\left(\bigcup_{j\in J} A_j\right) = \bigcup_{j\in J} H(A_j)$

L is an $\mathcal{I}$-lower approximation *if it satisfies, for all $A, A_j \in \mathcal{F}(U)$, $\alpha \in [0,1]$,*

$(L1)$ $L(\hat{\alpha} \Rightarrow_{\mathcal{I}} A) = \hat{\alpha} \Rightarrow_{\mathcal{I}} L(A)$

$(L2)$ $L\left(\bigcap_{j\in J} A_j\right) = \bigcap_{j\in J} L(A_j)$

Wu et al. required $\mathcal{C}$ and $\mathcal{I}$ to be a continuous t-norm and implicator, resp., but these conditions can be slightly weakened. For this, we can use e.g. results from [22] obtained in the framework of fuzzy modal logics that can be easily adapted to approximation operators.

Proposition 7. *Let $H\colon \mathcal{F}(U) \to \mathcal{F}(U)$ and $\mathcal{T}$ a left-continuous t-norm. H is a $\mathcal{T}$-upper approximation if and only if for all $A \in \mathcal{F}(U)$, $H(A) = R{\uparrow_{\mathcal{T}}}A$, where $R(x,y) = H(\{x\})(y)$, for x, y in U.*

Proposition 8. *Let $L\colon \mathcal{F}(U) \to \mathcal{F}(U)$ and $\mathcal{I}$ an EP implicator that is left-continuous in its first argument and such that $\mathcal{N}_{\mathcal{I}}$ is continuous. L is an $\mathcal{I}$-lower approximation if and only if for all $A \in \mathcal{F}(U)$, $L(A) = R{\downarrow_{\mathcal{I}}}A$, where $R(x,y) = \mathcal{N}_{\mathcal{I}}(L(U \setminus \{x\})(y))$, for x, y in U.*

Adding more axioms to Definition 2, we can characterize specific properties of the fuzzy relation R, as the following propositions show.

Proposition 9. *Let $\mathcal{T}$ be a left-continuous t-norm and H a $\mathcal{T}$-upper approximation. There exists a fuzzy relation R in U such that $H = R{\uparrow_{\mathcal{T}}}$ that is*

1. *inverse serial* $\Leftrightarrow \forall \alpha \in [0,1] : H(\hat{\alpha}) = \hat{\alpha} \Leftrightarrow H(U) = U$
2. *reflexive* $\Leftrightarrow \forall A \in \mathcal{F}(U) : A \subseteq H(A)$
3. *symmetric* $\Leftrightarrow \forall x, y \in U : H(\{x\})(y) = H(\{y\})(x)$
4. *$\mathcal{T}$-transitive* $\Leftrightarrow \forall A \in \mathcal{F}(U) : H(H(A)) \subseteq H(A)$

Proposition 10. *Let $\mathcal{I}$ be a border and EP implicator that is left-continuous in its first argument such that $\mathcal{N}_{\mathcal{I}}$ is continuous, and L an $\mathcal{I}$-lower approximation. There exists a fuzzy relation R in U such that $L = R{\downarrow_{\mathcal{I}}}$ that is*

1. *inverse serial* $\Leftrightarrow \forall \alpha \in [0,1] : L(\hat{\alpha}) = \hat{\alpha}$ *and* $\mathcal{I}$ *satisfies* $x \leq y \Leftrightarrow \forall z \in [0,1] : \mathcal{I}(x,z) \geq \mathcal{I}(y,z)$
2. *reflexive* $\Leftrightarrow \forall A \in \mathcal{F}(U) : L(A) \subseteq A$
3. *symmetric* $\Leftrightarrow \forall x, y \in U, \alpha \in [0,1] : L(\{x\} \Rightarrow_{\mathcal{I}} \hat{\alpha})(y) = L(\{y\} \Rightarrow_{\mathcal{I}} \hat{\alpha})(x)$
4. $\mathcal{T}$*-transitive* $\Leftrightarrow \forall A \in \mathcal{F}(U) : L(L(A)) \subseteq L(A)$ *for all* A *in* $\mathcal{F}(U)$ *and* $\mathcal{I}$ *satisfies* $\mathcal{I}(x, \mathcal{I}(y,z)) = \mathcal{I}(\mathcal{T}(x,y), z)$ *for all* x, y, z *in* $[0,1]$

The above propositions characterize lower and upper approximations separately. If these operators are dual, we can link them together.

Proposition 11. *Let* $\mathcal{T}$ *be a left-continuous t-norm,* $\mathcal{I}$ *an EP implicator that is left-continuous in its first argument and such that* $\mathcal{N}_{\mathcal{I}}$ *is involutive,* H *a* $\mathcal{T}$*-upper approximation and* L *an* $\mathcal{I}$*-lower approximation. If* H *and* L *satisfy duality w.r.t.* $\mathcal{N}_{\mathcal{I}}$*, then there exists a binary fuzzy relation* R *in* U *such that* $H = R{\uparrow_{\mathcal{T}}}$ *and* $L = R{\downarrow_{\mathcal{I}}}$.

A drawback of the above approach is that it excludes some important operators. For instance, it can be verified that the R-implicator $\mathcal{I}_{\min}$ does not satisfy the conditions of Proposition 8, because $\mathcal{N}_{\mathcal{I}_{\min}}$ is not involutive. However, it satisfies all properties from Table 2. For this reason, below we introduce and characterize the alternative notion of a $\mathcal{T}$-coupled pair of approximations.

Definition 3. *Let* $\mathcal{T}$ *be a left-continuous t-norm,* $H, L : \mathcal{F}(U) \to \mathcal{F}(U)$*. We call* (H, L) *a* $\mathcal{T}$-coupled pair *of upper and lower approximations if the following conditions hold:*

(H1,H2) H *is a* $\mathcal{T}$*-upper fuzzy approximation operator*
(L2) $L\left(\bigcap_{j \in J} A_j\right) = \bigcap_{j \in J} L(A_j)$
(HL) $L(A \Rightarrow_{\mathcal{I}_{\mathcal{T}}} \hat{\alpha}) = H(A) \Rightarrow_{\mathcal{I}_{\mathcal{T}}} \hat{\alpha}$

Proposition 12. *Let* $\mathcal{T}$ *be a left-continuous t-norm,* $H, L : \mathcal{F}(U) \to \mathcal{F}(U)$*.* (H, L) *is a* $\mathcal{T}$*-coupled pair of upper and lower approximations if and only if there exists a binary fuzzy relation* R *in* U *such that* $H = R{\uparrow_{\mathcal{T}}}$ *and* $L = R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}$.

Proof. Assume (H, L) is a $\mathcal{T}$-coupled pair and $A \in \mathcal{F}(U)$. By (H1, H2), H is a $\mathcal{T}$-upper approximation, so by Proposition 7, $H(A) = R{\uparrow_{\mathcal{T}}}A$, where $R(x,y) = H(\{x\})(y)$, for x, y in U. On the other hand, it can be verified that $A = \bigcap_{y \in U}(\{y\} \Rightarrow_{\mathcal{I}_{\mathcal{T}}} \widehat{A(y)})$, so by (L2) and (HL), we have $L(A) = \bigcap_{y \in U} L(\{y\} \Rightarrow_{\mathcal{I}_{\mathcal{T}}} \widehat{A(y)}) = \bigcap_{y \in U} H(\{y\}) \Rightarrow_{\mathcal{I}_{\mathcal{T}}} \widehat{A(y)} = R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}A$.
Conversely, it is clear that $R{\uparrow_{\mathcal{T}}}$ and $R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}$ are an upper and a lower approximation satisfying (H1, H2) and (L2), respectively. To see (HL), let $x \in U$, $\alpha \in [0,1]$, then $\left(R{\downarrow_{\mathcal{I}_{\mathcal{T}}}}(A \Rightarrow_{\mathcal{I}_{\mathcal{T}}} \hat{\alpha})\right)(x) = \inf_{y \in U} \mathcal{I}_{\mathcal{T}}(R(y,x), \mathcal{I}_{\mathcal{T}}(A(y), \alpha)) = \inf_{y \in U} \mathcal{I}_{\mathcal{T}}(\mathcal{T}(R(y,x), A(y)), \alpha) = \mathcal{I}_{\mathcal{T}}(\sup_{y \in U} \mathcal{T}(R(y,x), A(y)), \alpha) = \mathcal{I}_{\mathcal{T}}((R{\uparrow_{\mathcal{T}}}A)(x), \alpha) = (R{\uparrow_{\mathcal{T}}}A \Rightarrow_{\mathcal{I}_{\mathcal{T}}} \hat{\alpha})(x)$.

Proposition 13. *Let $\mathcal{T}$ be a left-continuous t-norm and let (H, L) be a $\mathcal{T}$-coupled pair of upper and lower fuzzy approximation operators. There exists a binary fuzzy relation R in $U \times U$ such that $H = R{\uparrow}_{\mathcal{T}}$ and $L = R{\downarrow}_{\mathcal{I}_{\mathcal{T}}}$ that is:*

1. *inverse serial* $\Leftrightarrow H(U) = U \Leftrightarrow \forall A \in \mathcal{F}(U)\colon L(A) \subseteq H(A)$
2. *reflexive* $\Leftrightarrow \forall A \in \mathcal{F}(U)\colon L(A) \subseteq A \Leftrightarrow \forall A \in \mathcal{F}(U)\colon A \subseteq H(A)$
3. *symmetric* $\Leftrightarrow \forall x, y \in U\colon H(\{x\})(y) = H(\{y\})(x) \Leftrightarrow \forall A \in \mathcal{F}(U)\colon H(L(A)) \subseteq A \Leftrightarrow \forall A \in \mathcal{F}(U)\colon A \subseteq L(H(A))$
4. *$\mathcal{T}$-transitive* $\Leftrightarrow \forall A \in \mathcal{F}(U)\colon L(A) \subseteq L(L(A)) \Leftrightarrow \forall A \in \mathcal{F}(U)\colon H(H(A)) \subseteq H(A)$

Proof. By Proposition 12, we know that there exists a relation R such that $H = R{\uparrow}_{\mathcal{T}}$ and $L = R{\downarrow}_{\mathcal{I}_{\mathcal{T}}}$.

1. The equivalence between inverse seriality and $H(U) = U$ can be proved as follows: $H(U)(x) = \sup_{y \in U} \mathcal{T}(R(y, x), U(y)) = \sup_{y \in U} \mathcal{T}(R(y, x), 1) = \sup_{y \in U} R(y, x)$. Hence, $U = H(U)$ iff $H(U)(x) = 1$ for all $x \in U$, iff $\sup_{y \in U} R(y, x) = 1$ for all $x \in U$. The equivalence with $L(A) \subseteq H(A)$ for all $A \in \mathcal{F}(U)$ corresponds to [22, Proposition 4].
2. This corresponds to [22, Proposition 5].
3. The first equivalence is proved as in Proposition 9, item 3. The second and third one correspond to [22, Proposition 9].
4. This corresponds to [22, Proposition 13].

6 Conclusion and future work

In this paper, we have studied a general implicator-conjunctor based model for the lower and upper approximation of a fuzzy set under a binary fuzzy relation. We reviewed models from the literature that can be seen as special cases, and enriched the existing axiomatic approach with a new notion of $\mathcal{T}$-coupled pairs of approximations, which characterize the operations satisfying all relevant properties of classical rough sets, i.e., left-continuous t-norms and their R-implicators.

An important challenge is to extend the formal treatment to noise-tolerant fuzzy rough set models, such as those studied in [23–29]. Observing that the implicator-conjunctor based approximations are sensitive to small changes in the arguments (for instance, because of their reliance on inf and sup operations), many authors have proposed models that are more robust against data perturbation. However, this normally goes at the expense of the properties the corresponding fuzzy rough set model satisfies.

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