On conjectures in t-norm based fuzzy logics*

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Abstract. This paper is a humble homage to Enric Trillas. Following his foundational contributions on models of ordinary reasoning in an algebraic setting, we study here elements of these models, like conjectures and hypothesis, in the logical framework of continuous t-norm based fuzzy logics. We consider notions of consistency, conjecture and hypothesis arising from two natural families of consequence operators definable in these logics, namely the ones corresponding to the so-called truth-preserving and degree-preserving consequence relations. We pay special attention to the particular cases of three prominent fuzzy logics: Gödel, Product and Łukasiewicz logics

Keywords: CHC models, consequence operators, t-norm based fuzzy logics, consistency, conjectures

1 Introduction

The paper deals with *models of ordinary reasoning* as defined by Trillas et al. [2, 11, 8], based on the notions of conjecture and hypothesis [13]. These models try to capture the main properties of some of the basic types of ordinary reasoning: deduction, induction, abduction and speculative reasoning. These types of reasoning are represented, respectively, by consequences, conjectures, hypotheses and speculations. And all of them can be defined from a given consequence operator in the sense of Tarski.

Actually, any reasoning process starts from a body of information or, in logical terms, from a set of premises. If this set is finite, to obtain its consequences is usually reduced to look for the consequences of the conjunction (or meet in algebraic terms) of all the premises. This is done for instance in [11], in the setting of preordered sets, where for a given set of premises the consequence operator $C_\wedge$ provides as consequences all those elements greater or equal (with respect to a given preorder) than the conjunction of its premises. Nevertheless, there are different ways of defining a consequence operator, even in that general setting. Still in [11], the authors also consider the operator $C_\cup$ for which the

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consequences of a set of premises is taken as the union of the consequences of each premise.

Once the set of consequences is defined, one can try to characterize the information that is not inconsistent with a set of premises P. Here consistency refers to the impossibility of deducing the negation from an already deduced element, i.e. if $q \in C(P)$, then $not\text{-}q \notin C(P)$, assuming a negation *not* is available in the framework. Then the set of *conjectures* for a set of premises P are those elements consistent with P. On the other hand, the *hypotheses* for P will be those conjectures which allow to deduce every premise in P. Finally, the set of *speculations* for P is defined as the set of those conjectures that are neither consequences nor hypotheses. In this way, one obtains a partition of the set of conjectures in terms of consequences, hypotheses and speculations.

These models of ordinary reasoning, also called CHC models (from "Consequences, Hypotheses and Conjectures" [10]) have been usually defined in a particular algebraic setting, most notably in orthomodular lattices [2], but also in residuated lattices [12], and more recently [8] very general mathematical structures, called *Basic Flexible Algebras* (BFAs), have been considered, having as particular cases Boolean algebras, ortholattices, orthomodular lattices, De Morgan algebras and standard algebras of fuzzy sets. On the other hand, in some papers CHC models have benn directly defined from a Tarski consequence operator, independently from the order of the underlying lattice structure. However, in [12] the notion of consequence is already based on the implication operation of the underlying residuated lattice, and hence compatible with the lattice ordering, and, as we have already mentioned before, in [11] the models are generalized to the setting of pre-ordered sets.

In the classical logic framework, given a set of premises or theory Γ, it is easy to identify what conjectures, hypotheses and refutations of Γ are, by resorting to the well known notions of logical consequence and consistency. Indeed, the set of possible conjectures relative to Γ consist of all those formulas φ that are consistent with Γ, while the possible hypotheses for Γ are those formulas that entail every formula of Γ. Sometimes one can single out the set of *strict* hypotheses, in the sense of not being consequences of Γ. In formal terms, they are defined respectively as:

$$Conj(\Gamma) = \{\varphi \mid \Gamma \cup \{\varphi\} \not\models_{CL} \bot\},$$
$$Hyp(\Gamma) = \{\varphi \mid \varphi \models_{CL} \psi, \text{ for each } \psi \in \Gamma\},$$
$$Hyp^*(\Gamma) = Hyp(\Gamma) \setminus \mathbb{CL}(\Gamma),$$

where $\models_{CL}$ and $\mathbb{CL}$ denote respectively the consequence relation and consequence operator of classical propositional logic. Furthermore, CHC models deal with the notions of refutations and speculations. Speculations are usually understood as those conjectures that are neither hypotheses nor consequences, while refutations of a theory are those formulas that are inconsistent with the theory, or equivalently that the theory proves their negation. That is:

$$Spec(\Gamma) = Conj(\Gamma) \setminus (Hyp(\Gamma) \cup \mathbb{CL}(\Gamma)),$$

$$Ref(\Gamma) = \{\varphi \mid \Gamma \cup \{\varphi\} \models_{CL} \bot\} = \{\varphi \mid \Gamma \models_{CL} \neg\varphi\}.$$

In this paper we study all these notions in the context of t-norm-based fuzzy logics [9, 4], in relation to two natural consequence operators that are at work in these logics, namely the ones associated to the truth-preserving and degree-preserving notions of logical consequence [1]. We show that sets of conjectures arising in these logical frameworks verify similar properties to those described in [8], and in particular that they can be partitioned into consequences, hypotheses and speculations as well.

The paper is organized as follows. After some preliminaries on t-norm-based fuzzy logics in Section 2, we describe in Section 3 a set of four different consequence operators for t-norm based fuzzy logics. These consequence operators are used in Section 4 to study the notion of consistency for sets of formulas relative to them. Finally, in Section 5, based on the notions of consequence and consistency, we characterize the sets of conjectures and hypothesis for the logics of a continuous t-norm depending on properties of the t-norm. We pay special attention to the cases of the logics corresponding to the three basic t-norms, i.e. Łukasiewicz, Gödel and Product fuzzy logics. We end up with some remarks on future work.

2 Preliminaries on t-norm based logics

Continuous t-norm-based fuzzy logics[3] are a family of logics whose language $\mathcal{L}$ is built from a countable set of propositional variables using different connectives $\&, \wedge, \vee, \to, \neg$ and truth constants $\overline{0}, \overline{1}$ for truth and falsity. Semantically, they correspond to logical calculi with the real interval $[0, 1]$ as set of truth-values and taking the conjunction $\&$, the implication $\to$ and the truth-constant $\overline{0}$ as primitive. Further connectives are definable as: $\varphi \wedge \psi = \varphi\&(\varphi \to \psi)$, $\varphi \vee \psi = ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$, $\neg\varphi = \varphi \to \overline{0}$ and $\overline{1} = \neg\overline{0}$.

In this framework, each continuous t-norm $\star$ uniquely determines a logic $\mathrm{L}_\star$ as a propositional calculus over formulas interpreting the conjunction & by the t-norm $\star$, the implication $\to$ by its residuum $\Rightarrow_\star$ and the truth-constant $\overline{0}$ by the value 0. More precisely, evaluations of propositional variables are mappings e assigning to each propositional variable p a truth-value $e(p) \in [0, 1]$, which extend univocally to compound formulas as follows:

$$e(\overline{0}) = 0,$$
$$e(\varphi\&\psi) = e(\varphi) \star e(\psi),$$
$$e(\varphi \to \psi) = e(\varphi) \Rightarrow_\star e(\psi).$$

From these defintions, it follows that $e(\varphi \wedge \psi) = \min(e(\varphi), e(\psi))$, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$, $e(\neg\varphi) = e(\varphi) \Rightarrow_\star 0$ and $e(\varphi \equiv \psi) = e(\varphi \to \psi) \star e(\psi \to \varphi)$.

[3] We assume readers to be familiar with the notions of t-norm, the three basic continuous t-norms, i.e, minimum, product and Łukaseewicz t-norm, and the notion of ordinal sum. We also assume familiarity with the decomposition of continuous t-norm as ordinal sums of isomorphic copies of the three basic continuous t-norms.

Two types of logics $L_\star$ are considered in the rest of the paper. The first one are logics $L_\star$ where $\star$ is a t-norm satisfying the pseudo-complementation condition $\min\{x, \neg x\} = 0$ (we will call them SBL t-norms). The second contains the rest of $L_\star$ logics, defined by so-called non-SBL t-norms. These two type of t-norms are algebraically characterized in [6] as follows:

- A t-norm $\star$ is an SBL t-norm if and only if one of the following conditions hold: $\star$ is an ordinal sum with a first component that is isomorphic either to minimum or product t-norm, or it is an ordinal sum without a first component. For any SBL t-norm $\star$, the corresponding negation ($n(x) = x \Rightarrow_\star 0$) is the Gödel negation ($n(0) = 1$ and $n(x) = 0$ otherwise). Prominent examples of SBL t-norms are minimum and product t-norms.
- A t-norm is a non-SBL t-norm if and only if it is an ordinal sum with a first component that is (isomorphic) to Łukasiewicz t-norm. From now on and for any non-SBL t-norm, we will denote by a the idempotent element that is the upper bound of its first (Łukasiewicz) component. The corresponding negation is defined by $n(0) = 1$, $n(x) = a - x$ for $x \in (0, a]$ and $n(x) = 0$ for $x \geq 0$. A prominent example of a non-SBL t-norm is therefore the Łukasiewicz t-norm itself, that corresponds to the case when $a = 1$.

For each logic $L_\star$, we consider two kinds of finitary consequence relations, $\models$ and $\models^{\leq}$, defined as follows, where $\Gamma \cup \{\varphi\}$ is a set of formulas from $\mathcal{L}$:

- $\Gamma \models \varphi$ if there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $e(\varphi) = 1$ for every evaluation $e : \mathcal{L} \to [0, 1]_*$ such that $e(\psi) = 1$ for every $\psi \in \Gamma_0$.
- $\Gamma \models^{\leq} \varphi$ if there exists a finite $\Gamma_0 \subseteq \Gamma$ such that $\min\{e(\psi) \mid \psi \in \Gamma_0\} \leq e(\varphi)$ for every evaluation $e : \mathcal{L} \to [0, 1]_*$.

The consequence relation $\models$ is usually called "truth-preserving" while $\models^{\leq}$ is called "degree-preserving", for obvious reasons [1]. Observe that $\{\psi_1, \ldots, \psi_n\} \models^{\leq} \varphi$ iff $\models (\psi_1 \wedge \ldots \wedge \psi_n) \to \varphi$, so that deductions from premises with $\models^{\leq}$ can be translated to deductions of theorems with $\models$.

Well-known axiomatic systems, like Łukasiewicz logic (Ł), Gödel logic (G) or Product logic (Π), syntactically capture the "truth-preserving" consequence relation for $L_\star$ when $\star$ is Łukasiewicz, min or product t-norm respectively [9]. Moreover, in [7] it is proved that every logic $L_\star$, for $\star$ being any continuous t-norm, is finitely axiomatizable. Logics $L_\star$ where $\star$ is a SBL t-norm are called SBL logics. All these axiomatic systems are extensions of Hájek's BL logic [9, 3]. It is worth noticing that Gödel logic is the only t-norm based fuzzy logic for which $\models$ coincides with $\models^{\leq}$.

3 Consequence operators on $L_\star$ logics

Given a logic $L_\star$, we consider the consequence operators associated to the logical consequence relations $\models$ and $\models^{\leq}$: for any subset of formulas $\Gamma \subseteq \mathcal{L}$,

- $C(\Gamma) = \{\varphi \in \mathcal{L} \mid \Gamma \models \varphi\}$ and

– $C^{\leq}(\Gamma) = \{\varphi \in \mathcal{L} \mid \Gamma \models^{\leq} \varphi\}$.

For each consequence operator C and $C^{\leq}$ we will also consider the consequence operators $C_{\cup}$ and $C^{\leq}_{\cup}$ (used in Trillas et al.'s paper [8]) defined as follows:

– $C_{\cup}(\Gamma) = \{\varphi \in \mathcal{L} \mid \gamma \models \varphi$ for some $\gamma \in \Gamma\}$ and
– $C^{\leq}_{\cup}(\Gamma) = \{\varphi \in \mathcal{L} \mid \gamma \models^{\leq} \varphi$ for some $\gamma \in \Gamma\}$.

Since $\models$ is a stronger notion of consequence than $\models^{\leq}$, we have the following chains of inclusions among these operators:

$$C^{\leq}_{\cup} \subseteq C^{\leq} \subseteq C \subseteq \mathbb{CL}, \tag{1}$$

$$C^{\leq}_{\cup} \subseteq C_{\cup} \subseteq C \subseteq \mathbb{CL}. \tag{2}$$

where $\mathbb{CL}$ denotes the consequence operator of classical propositional logic (CL) in the language $\mathcal{L}$ where we identify the connectives & and $\wedge$. In the particular case of Gödel logic ($\star$ = min), since $\models$ coincides with $\models^{\leq}$, it turns out that $C_{\cup} = C^{\leq}_{\cup} \subseteq C = C^{\leq}$.

Observe that, unlike C, the operators $C^{\leq}_{\cup}$ and $C_{\cup}$ are not closed by modus ponens, a fact that makes the associated notion of inference rather weak. Actually also $C^{\leq}$ is not closed by modus ponens but it is closed by a restricted version of modus ponens: if $\varphi \to \psi$ is a theorem of $L_{\star}$, from φ derive ψ.

All these operators are consequence operators in the sense of Tarski, that is, any $C^* \in \{C, C^{\leq}, C_{\cup}, C^{\leq}_{\cup}\}$ verifies the following well-known properties:

(Inc) $\forall \Gamma,\ \Gamma \subseteq C^*(\Gamma)$,
(Mon) $\forall \Gamma_1, \Gamma_2$, if $\Gamma_1 \subseteq \Gamma_2$ then $C^*(\Gamma_1) \subseteq C^*(\Gamma_2)$,
(Idem) $\forall \Gamma,\ C^*(C^*(\Gamma)) = C^*(\Gamma)$.

The following lemmas highlight several properties of the above defined consequence operators. In what follows, given a finite set of formulas Γ, we will write $\Gamma^{\wedge}$ for $\bigwedge\{\psi \mid \psi \in \Gamma\}$. Moreover, to simplify the notation, we will also write $C^*(\varphi)$, for $C^*(\{\varphi\})$.

Lemma 1. *The operators C and $C^{\leq}$ are closed by the (weak) conjunction $\wedge$. That is, for $C^* \in \{C, C^{\leq}\}$, if $\varphi \in C^*(\Gamma)$ and $\psi \in C^*(\Gamma)$, then $\varphi \wedge \psi \in C^*(\Gamma)$. Thus, in particular, it holds that $C^*(\Gamma) = C^*(\Gamma^{\wedge})$.*

Lemma 2. *The operator C is closed by the (strong) conjunction &: if $\varphi \in C(\Gamma)$ and $\psi \in C(\Gamma)$, then $\varphi \& \psi \in C(\Gamma)$.*

These results do not hold in general for the operators $C_{\cup}$ and $C^{\leq}_{\cup}$, while Lemma 2 does not even hold for $C^{\leq}$.

On the other hand, by definition, it is clear that the $C_{\cup}$ and $C^{\leq}_{\cup}$ operators satisfy the following property.

Lemma 3. *Let $C^* \in \{C_{\cup}, C^{\leq}_{\cup}\}$. Then $C^*(\Gamma) = \bigcup_{\varphi \in \Gamma} C^*(\varphi)$.*

As it has been already mentioned, it is well-known that for any logic $L_{\star}$, the operators C and $C^{\leq}$ (and also $C_{\cup}$ and $C^{\leq}_{\cup}$) coincide if and only if $\star$ is the minimum t-norm. For instance, in $L_{\star}$ with $\star \neq$ min, if p is a propositional variable then $p\&p$ belongs to $C(p)$, but it obviously does not belong to $C^{\leq}(p)$.

4 Notions of consistency relative to the consequence operators C, $C^{\leq}$, $C_{\cup}$ and $C_{\cup}^{\leq}$

In this section we study which is the proper notion of consistency relative to our four consequence operators C, $C^{\leq}$, $C_{\cup}$ and $C_{\cup}^{\leq}$, within the framework of the logics $L_{\star}$.

In connection with the consequence relation $\models$, it is customary in the framework of *Mathematical Fuzzy logic* [4] to define a set of premises Γ to be consistent whenever $\Gamma \not\models \overline{0}$. However, this notion of consistency does not make much sense in connection with the degree-preserving consequence relation $\models^{\leq}$, since $\models^{\leq}$ is paraconsistent [5], i.e. it is not always the case that $\{\varphi, \neg\varphi\} \models^{\leq} \overline{0}$. For this reason we will adopt the following general definition.

Definition 1 (C^*-consistency). *For any logic $L_{\star}$, let C^* be any of its associated four consequence operators $C, C^{\leq}, C_{\cup}$ and $C_{\cup}^{\leq}$. We say that a set of premises Γ is C^*-consistent whenever the following condition holds:*

> *(C^*-Cons) For any $\varphi \in \mathcal{L}$, if $\varphi \in C^*(\Gamma)$, then $\neg\varphi \notin C^*(\Gamma)$.*

Notice that, due to the closure condition (Idem), if Γ is C^*-consistent, then so is $C^*(\Gamma)$. In the next two subsections, we provide equivalent consistency conditions for C and $C^{\leq}$, and for $C_{\cup}$ and $C_{\cup}^{\leq}$ respectively, depending on the two types of t-norms considered and their corresponding logics.

4.1 The case of the C and $C^{\leq}$ operators

We first show that the notion of C-consistency from the above Definition 1 coincides with the usual one in truth-preserving fuzzy logics.

Lemma 4. *For any logic $L_{\star}$, the following statements are equivalent:*

- *Γ is C-consistent,*
- *$\Gamma \not\models \overline{0}$ (i.e. there exists an evaluation e such that $e(\psi) = 1$ for all $\psi \in \Gamma$),*
- *$\Gamma^{\wedge} \not\models \overline{0}$ (i.e. there exists an evaluation e such that $e(\Gamma^{\wedge}) = 1$).*

The proof is an immediate consequence of Lemma 2 since if Γ is C-inconsistent, then $\Gamma \models \varphi \& \neg\varphi$, and $\varphi \& \neg\varphi$ is equivalent to $\overline{0}$.

The corresponding (not very elegant) general condition for $C^{\leq}$ is as follows.

Lemma 5. *For any logic $L_{\star}$, the following statements are equivalent:*

- *Γ is $C^{\leq}$-consistent,*
- *For any formula φ, if $\models \Gamma^{\wedge} \to \varphi$ then $\not\models \Gamma^{\wedge} \to \neg\varphi$.*

This lemma follows directly from the definition of $\models^{\leq}$ in terms of $\models$.

Next we show some more specific conditions for $C^{\leq}$ depending on the choices of the t-norm $\star$.

Lemma 6. *For any logic $L_\star$ with $\star$ being a SBL t-norm, the notions of C-consistency and $C^{\leq}$-consistency coincide.*

Proof. First notice that, by definition, if Γ is not $C^{\leq}$-consistent, then there exists φ such that $\varphi \wedge \neg\varphi \in C^{\leq}(\Gamma)$. But if $\star$ is a SBL t-norm, then $L_\star$ proves $(\varphi \wedge \neg\varphi) \to \overline{0}$, and hence $\Gamma \models \overline{0}$. Then the proof of this lemma easily follows from the following chain of equivalences: Γ is $C^{\leq}$-consistent iff $\Gamma \not\models^{\leq} 0$ iff there is an evaluation e such that $e(\Gamma^\wedge) > 0$ iff there is an evalutaion e' such that $e'(\Gamma^\wedge) = 1$ iff $\Gamma \not\models 0$ iff Γ is C-consistent. For the latter step, it is enough to take e' such that $e'(p) = e(\neg\neg p)$ for each propositional variable p.[4] ⊣

Proposition 1. *For any logic $L_\star$ with $\star$ being a SBL t-norm, let $C^* \in \{C, C^{\leq}\}$. Then the following conditions are equivalent:*

(i) Γ is C^*-consistent,
(ii) there exists a Boolean $L_\star$-evaluation e (i.e. only taking values 0 or 1) such that $e(\Gamma^\wedge) = 1$.

Proof. The proof is easy taking into account that the evaluation e' defined in the proof of the previous lemma is Boolean, i.e. $e'(\varphi) \in \{0, 1\}$ for any formula φ. ⊣

This last lemma amounts to say that, in a logic $L_\star$ with $\star$ being an SBL t-norm, Γ is C^*-consistent if and only if Γ is *classically* consistent (identifying the weak and strong conjunctions). However, in the case that $\star$ is a non-SBL t-norm, C-consistency is not equivalent to $C^{\leq}$-consistency, and condition (ii) is satisfied neither for C nor for $C^{\leq}$. Indeed, the formula $\varphi = (p \vee \neg p) \to (p \equiv \neg p)$, with p a propositional variable, is not consistent in classical logic but it is consistent in $L_\star$ when $\star$ is a non-SBL t-norm. To see this, if a is the smallest positive idempotent of $\star$, it is enough to take an evaluation e with $e(p) = \frac{a}{2}$ and check that $e(\varphi) = 1$, since in this case, $e(\neg p) = a - \frac{a}{2} = \frac{a}{2} = e(p)$. Recall that in such a case, $\star$ over the interval $[0, a]$ is isomorphic to the Łukasiewicz t-norm, and hence its corresponding negation $n_\star$ over $(0, a]$ is given by $n_\star(x) = a - x$.

Proposition 2. *In any logic $L_\star$ where $\star$ is a non-SBL t-norm with a being the smallest positive idempotent, the following conditions are equivalent:*

- *Γ is $C^{\leq}$-consistent,*
- *there exists an $L_\star$-evaluation e such that $e(\Gamma^\wedge) > a/2$.*

Proof. Assume there exists an evaluation e such that $e(\Gamma^\wedge) > a/2$. Then if $\Gamma \models^{\leq} \varphi$, necessarily $e(\varphi) \geq e(\Gamma^\wedge) > a/2$, and thus either $e(\neg\varphi) = 0$ or $e(\neg\varphi) = a - e(\varphi)$, and in both cases $e(\neg\varphi) < a/2$, and thus $\Gamma \not\models^{\leq} \neg\varphi$.

Conversely, assume $e(\Gamma^\wedge) \leq a/2$ for any evaluation e. Then, we would have both $\Gamma \models^{\leq} \Gamma^\wedge$ (by definition of $\models^{\leq}$), but also $\Gamma \models^{\leq} \neg\Gamma^\wedge$, since by hypothesis $e(\neg\Gamma^\wedge) \geq a/2$, and hence Γ is $C^{\leq}$-inconsistent. ⊣

[4] Recall that in a SBL-chain, both $\neg\neg 0 = 0$ and $\neg\neg x = 1$ if $x > 0$. Moreover $\neg\neg$ defines a morphism from the algebra $([0, 1], \star, \Rightarrow_\star, 0, 1)$ into itself.

4.2 The case of the $C_\cup$ and $C_\cup^\leq$ operators

In this section we carry out a similar analysis of the notion of consistency corresponding to the weak consequence operators $C_\cup$ and $C_\cup^\leq$. We start by a set of general equivalent characterizations of the notion of $C_\cup$-(in)consistency.

Lemma 7. *In any logic $L_\star$, the following are equivalent:*

- *Γ is $C_\cup$-inconsistent,*
- *there exists $\psi, \chi \in \Gamma$, such that $\psi, \chi \models \overline{0}$,*
- *there exists $\psi, \chi \in \Gamma$ and $n \in \mathbb{N}$ such that $\neg\chi^n \in C_\cup(\psi)$, where $\chi^n = \chi \& \overset{n}{\ldots} \& \chi$; or equivalently, such that $\psi \models \neg\chi^n$,*
- *there exists $\psi, \chi \in \Gamma$ and $n \in \mathbb{N}$ such that for every $L_\star$-evaluation e, if $e(\psi) = 1$, then $e(\chi) \star \overset{n}{\ldots} \star e(\chi) = 0$.*

Proof. By definition, if Γ is $C_\cup$-inconsistent, there exist $\chi, \psi \in \Gamma$ such that $\chi \models \varphi$ and $\psi \models \neg\varphi$. Therefore it holds that $\chi, \psi \models \varphi \& \neg\varphi$, but $\varphi \& \neg\varphi$ is equivalent to $\overline{0}$. Moreover, this condition is equivalent in turn to the existence of $n \in \mathbb{N}$ such that $\psi \models \neg\chi^n$.[5] In semantic terms, this exactly corresponds to the fact that for every $L_\star$-evaluation e, if $e(\psi) = 1$, then $e(\chi) \star \overset{n}{\ldots} \star e(\chi) = 0$.

Reciprocally, if $\psi, \chi \in \Gamma$ are such that there exists $n \in \mathbb{N}$ verifying $\psi \models \neg\chi^n$, then we have $\chi \models \chi^n$ and $\psi \models \neg\chi^n$. Therefore, Γ is $C_\cup$-inconsistent. ⊣

As a direct consequence of Lemmas 3 and 6, if $\star$ is any SBL t-norm, we have that $C_\cup = C_\cup^\leq$ in the logic $L_\star$, and hence both related notions of consistency coincide as well. Therefore the following result holds.

Proposition 3. *Let $L_\star$ be any logic where $\star$ is an SBL t-norm, and let $C^* \in \{C_\cup, C_\cup^\leq\}$. Then the following conditions are equivalent:*

- *Γ is C^*-consistent,*
- *for any $\psi, \chi \in \Gamma$, $\chi \not\models \neg\psi$,*
- *for any $\psi, \chi \in \Gamma$, there exists an evaluation e such that $e(\psi \wedge \chi) > 0$,*
- *for any $\psi, \chi \in \Gamma$, there exists an $\{0, 1\}$-evaluation e such that $e(\psi \wedge \chi) = 1$.*

Proof. Since consistency with respect to $C_\cup$ and $C_\cup^\leq$ coincides, we the proposition for $C^* = C_\cup$. If Γ is $C_\cup$-inconsistent, there exist $\psi, \chi \in \Gamma$ such that $\psi \models \varphi$ and $\chi \models \neg\varphi$ for some φ. Let us prove that $\chi \models \neg\psi$. The hypothesis implies that for any $L_\star$-evaluation e, if $e(\psi) = 1$, then $e(\varphi) = 1$, and if $e(\chi) = 1$, then $e(\varphi) = 0$. Therefore, $e(\chi) = 1$ implies $e(\psi) < 1$. Then we have two cases: either $e(\psi) = 0$ and we are done, or $e(\psi) > 0$. In the latter case, define a new evaluation e' by putting $e'(p) = e(\neg\neg p)$ for all propositional variables p. It can be checked (see e.g. [9]) that $e'(\varphi) = e(\neg\neg\varphi)$ for any formula φ, and hence $e'(\chi) = e'(\varphi) = e'(\psi) = 1$, contradicting the hypothesis that $\chi \models \neg\varphi$.

[5] Here we use the local deduction theorem that is valid for all t-norm based logics, namely $\Gamma \cup \{\varphi\} \models \psi$ iff there exists an $n \in \mathbb{N}$ such that $\Gamma \models \varphi \& \overset{n}{\ldots} \& \varphi \to \psi$ (see e.g. [9, 4]).

Reciprocally, assume that for any evaluation e such that $e(\chi) = 1$, we have $e(\neg\psi) = 1$. Therefore, Γ is $C_\cup$-inconsistent, since $\psi \models \psi$ and $\chi \models \neg\psi$.

The last two items are easy consequences of previous ones. Recall that for any $L_\star$-evaluation e, $e(\neg\neg\varphi) \in \{0, 1\}$ for any φ. ⊣

In fact this last proposition simply says that in $L_\star$, with $\star$ being an SBL t-norm, a set of formulas is consistent with respect to the consequence operators $C_\cup$ and $C_\cup^\leq$ iff they are *pairwise* consistent in the usual sense of the operators C and $C^\leq$ respectively, which is a very natural condition according to the definition of the consequence operators $C_\cup$ and $C_\cup^\leq$.

Note that the second item of the last proposition is not true for Łukasiewicz logic, since there is the possibility of having formulas $\chi, \psi \in \Gamma$, such that $\chi, \psi \models \overline{0}$, with $\psi \not\models \neg\chi$ and $\chi \not\models \neg\psi$. For instance, take $\chi := \neg(\neg p \& \neg q)$ and $\psi := (p \equiv \neg p) \wedge \neg q$, where p and q are propositional variables. It is clear that $\chi, \psi \models \overline{0}$, since there is no evaluation e such that $e(\chi) = e(\psi) = 1$. Indeed, if $e(\psi) = 1$, then $e(p) = 1/2$ and $e(q) = 0$, but then $e(\psi) = e(\neg(\neg p \& \neg q)) = 1/2$. On the one hand, one can check that $\psi \not\models \neg\chi$, since for $e(p) = 1/2$ and $e(q) = 0$, we have $e(\psi) = 1$, while $e(\neg\chi) = 1 - 1/2 = 1/2$. On the other hand, it holds that $\chi \not\models \neg\psi$, since for $e(p) = e(q) = 1/2$ we have $e(\chi) = 1$ and $e(\psi) = 1/2$.

We finish this subsection with the following characterization of $C_\cup^\leq$-consistency when $\star$ is a non-SBL t-norm.

Proposition 4. *Let $L_\star$ be any logic where $\star$ is a non-SBL t-norm. The following conditions are equivalent:*

- *Γ is $C_\cup^\leq$-consistent,*
- *for all $\psi, \chi \in \Gamma$, $\neg\chi \notin C^\leq(\psi)$,*
- *for all $\psi, \chi \in \Gamma$, there exists an $L_\star$-evaluation e such that $e(\psi) > e(\neg\chi)$, i.e. such that $e(\psi \& \chi) > 0$.*

Proof. We prove the equivalence between the first two conditions, the third condition being only a rewriting of the second condition. By definition, if Γ is not $C_\cup^\leq$-consistent, there is a formula φ such that both $\varphi \in C_\cup^\leq(\Gamma)$ and $\neg\varphi \in C_\cup^\leq(\Gamma)$, or in other words, such that there exist $\psi, \chi \in \Gamma$ such that $\varphi \in C^\leq(\psi)$ and $\neg\varphi \in C^\leq(\chi)$. This means that, for any $L_\star$-evaluation e, $e(\psi) \leq e(\varphi)$ and $e(\chi) \leq e(\neg\varphi)$, and from the latter, it follows that $e(\neg\chi) \geq e(\neg\neg\varphi)$ as well. Therefore we have that, for any evaluation e, $e(\psi) \leq e(\varphi) \leq e(\neg\neg\varphi) \leq e(\neg\chi)$, hence $\psi \models^\leq \neg\chi$, i.e. $\neg\chi \in C^\leq(\psi)$.

Conversely, if there exist $\psi, \chi \in \Gamma$ such that $\neg\chi \in C^\leq(\psi)$, then Γ is $C_\cup^*$-inconsistent since we obviously have $\chi \in C^\leq(\chi)$ as well. ⊣

5 Conjectures and Hypothesis

In this section we study how the notions of conjecture, hypothesis and speculation can be characterized under the different notions of consequence operators and logics we have considered in the previous sections. We start by adapting the usual definitions of these notions to our framework of t-norm based fuzzy logics of the form $L_\star$, where $\star$ is a continuous t-norm.

Definition 2. *For any logic $L_\star$, let C^* be any of its associated four consequence operators (C, $C^{\leq}$, $C_\cup$, $C_\cup^{\leq}$), and let $\Gamma \subseteq \mathcal{L}$ be a set of premises. We respectively define the set of conjectures, hypotheses, strict hypotheses and speculations of Γ with respect to C^* as follows:*

- $Conj_{C^*}(\Gamma) = \{\varphi \in \mathcal{L} \mid \Gamma \cup \{\varphi\}$ *is C^*-consistent*$\}$,
- $Hyp^+_{C^*}(\Gamma) = \{\varphi \in \mathcal{L} \mid \varphi$ *is C^*-consistent and* $\Gamma \subseteq C^*(\varphi)\}$,
- $Hyp_{C^*}(\Gamma) = Hyp^+_{C^*}(\Gamma) \setminus C^*(\Gamma)$,
- $Spec_{C^*}(\Gamma) = Conj_{C^*}(\Gamma) \setminus (C^*(\Gamma) \cup Hyp_{C^*}(\Gamma))$.

From this definition, it readily follows that consequences, (strict) hypotheses and speculations form a partition of the set of conjectures. In partiicula, for any set of premises Γ, it holds that

$$C^*(\Gamma) \cup Hyp_{C^*}(\Gamma) \cup Spec_{C^*}(\Gamma) = Conj_{C^*}(\Gamma).$$

Moreover, the following general properties also hold:

1. $Conj_{C^*}(\Gamma)$ may not be C^*- consistent
2. $Conj_{C^*}(\Gamma) = \bigcup\{T \subseteq \mathcal{L} \mid \Gamma \subseteq T$ and T is maximally C^*-consistent$\}$
3. If $\varphi, \psi \in Conj_{C^*}(\Gamma)$, then $\varphi \vee \psi \in Conj_{C^*}(\Gamma)$

   Proof. The case $C^* = C$ follows from the fact that the following condition holds true in any logic $L_\star$: $\Gamma, \varphi, \models \chi$ and $\Gamma, \psi \models \chi$ iff $\Gamma, \varphi \vee \psi \models \chi$. The remaining cases are easy consequences of this. ⊣

4. For $C^* \in \{C, C^{\leq}\}$, if $\varphi, \psi \in Hyp_{C^*}(\Gamma)$, then $\varphi \wedge \psi \in Hyp_{C^*}(\Gamma)$

   Proof. First of all note that if φ and ψ are C^*-consistent, so is $\varphi \wedge \psi$ (see Lemma 1). Now, if $\Gamma \subseteq C^*(\varphi) \cap C^*(\psi)$, it is clear that $\Gamma \subseteq C^*(\varphi \wedge \psi)$ as well. ⊣

5. If $\Gamma_1 \subseteq \Gamma_2 \subseteq \mathcal{L}$, then it holds:
   $Conj_{C^*}(\Gamma_2) \subseteq Conj_{C^*}(\Gamma_1)$,
   $Hyp_{C^*}(\Gamma_2) \subseteq Hyp_{C^*}(\Gamma_1)$.
6. If $C_1, C_2 \in \{C, C^{\leq}, C_\cup, C_\cup^{\leq}\}$ are such that $C_1 \subseteq C_2$, then it holds:
   $Conj_{C_2}(\Gamma) \subseteq Conj_{C_1}(\Gamma)$,
   $Hyp_{C_1}(\Gamma) \subseteq Hyp_{C_2}(\Gamma)$,
   $Spec_{C_2}(\Gamma) \subseteq Spec_{C_1}(\Gamma)$.

Based on the results in Section 4 on consistency for the different consequence operators, in the remaining of this section we provide some finer characterizations of conjectures and hypotheses.

5.1 The case of C and $C^{\leq}$ operators

Recall that, from Lemma 1, it follows that in any logic $L_\star$, for $C^* \in \{C, C^{\leq}\}$, it holds that $Conj_{C^*}(\Gamma) = Conj_{C^*}(\Gamma^\wedge)$ and $Hyp_{C^*}(\Gamma) = Hyp_{C^*}(\Gamma^\wedge)$. Moreover the following results hold.

Proposition 5. *Let $L_\star$ be a logic such that $\star$ is a SBL t-norm and let $C^* \in \{C, C^{\leq}\}$. Then the following statements hold:*

- *$\varphi \in Conj_{C^*}(\Gamma)$ iff there exists an $L_\star$-evaluation e such that $e(\Gamma^\wedge \wedge \varphi) = 1$,*
- *$Conj_{C^*}(\Gamma) = \alpha^{[-1]}(Conj_{\mathbb{CL}}(\alpha(\Gamma)))$,*
- *$Conj_{C^*}(\Gamma) = \{\varphi \mid \neg\varphi \notin C(\Gamma)\}$,*

where α is the mapping from formulas of $L_\star$ to formulas of classical logic obtained by identifying the weak and strong conjunctions.

Proof. The first two items follow directly from the results about consistency for C and $C^{\leq}$ in Lemma 6 and Proposition 1. The third item is an easy consequence of the second item and the fact that the property holds true for classical logic. $\dashv$

Since the mapping α is the identity for Gödel Logic, in this logic the set of conjectures $Con_{C^*}(\Gamma)$ is the same we would obtain in classical logic $Conj_{\mathbb{CL}}(\Gamma)$.

In the case of SBL logics, there is a strong relationship between conjectures and hypotheses.

Proposition 6. *Let $L_\star$ be the logic of an SBL t-norm $\star$ and let $C^* \in \{C, C^{\leq}\}$. Then $Conj_{C^*}(\Gamma) = \mathcal{L} \setminus Hyp^{+}_{C^*}(\neg\Gamma^\wedge)$.*

Proof. A formula φ does not belong to $Conj_{C^*}(\Gamma)$ iff $\{\Gamma^\wedge, \varphi\}$ is not C^*-consistent, i.e. iff $\{\Gamma^\wedge, \varphi\} \models \overline{0}$, and by the local deduction theorem, iff there is a natural n such that $\varphi \models (\Gamma^\wedge)^n \to \overline{0}$, i.e. iff $\varphi \models \neg(\Gamma^\wedge)^n$, and since $L_\star$ is a SBL logic, this happens iff $\varphi \models \neg\Gamma^\wedge$, hence iff $\varphi \in Hyp^{+}_{C^*}(\neg(\Gamma^\wedge))$. $\dashv$

For the logics $L_\star$, where $\star$ is a non-SBL t-norm, we need to distinguish the cases of the consequence operators C and $C^{\leq}$, since their notions of consistency are different.

Proposition 7. *Let $L_\star$ be the logic of a non-SBL t-norm $\star$ with $a > 0$ being its smallest positive idempotent. Then the following statements hold:*

- *$\chi \in Conj_C(\Gamma)$ iff there exists an $L_\star$-evaluation e such that $e(\Gamma^\wedge \wedge \chi) = 1$, i.e. iff $\neg\chi \notin C(\Gamma)$,*
- *$\chi \in Conj_{C^{\leq}}(\Gamma)$ iff there exists an $L_\star$-evaluation e such that $e(\Gamma^\wedge \wedge \chi) > \frac{a}{2}$.*

Proof. The two statements directly follow from the results about consistency for C and $C^{\leq}$ given in Lemma 4 and Proposition 2. Indeed, by definition, $\chi \in Conj_C(\Gamma)$ iff $\Gamma \cup \{\chi\}$ is C-consistent, iff, by Lemma 4, there exists an $L_\star$-evaluation e such that $e(\Gamma^\wedge \wedge \chi) = 1$. On the other hand, by definition as well, $\chi \in Conj_{C^{\leq}}(\Gamma)$ iff $\Gamma \cup \{\chi\}$ is $C^{\leq}$-consistent, and by Proposition 2, iff there exists an $L_\star$-evaluation e such that $e(\Gamma^\wedge \wedge \chi) > a/2$. $\dashv$

5.2 The case of $C_\cup$ and $C^{\leq}_{\cup}$ operators

In any logic $L_\star$, using the notion of $C_\cup$-consistency (see Lemma 7), it follows that the set of conjectures of a set of formulas Γ is

$$Conj_{C_\cup}(\Gamma) = \{\varphi \in \mathcal{L} \mid \forall\psi \in \Gamma,\ \{\varphi, \psi\} \not\models \overline{0}\} = \{\varphi \in \mathcal{L} \mid \forall\psi \in \Gamma,\ \forall n,\ \varphi \not\models \neg\psi^n\}.$$

In the particular case where $\star$ is an SBL t-norm, it is enough to take $n = 1$, and then the second expression can be simplified to:

$$Conj_{C_\cup}(\Gamma) = \{\varphi \in \mathcal{L} \mid \forall\psi \in \Gamma, \varphi \not\models \neg\psi\}. \quad (3)$$

Moreover, still in the case of $\star$ being an SBL t-norm, it turns out that $Conj_{C_\cup}(\Gamma) = Conj_{C^{\leq}_{\cup}}(\Gamma)$, since in such a case, $C^{\leq}_{\cup}$-consistency coincides with $C_\cup$-consistency (see Section 4.2). However, (3) is not valid for Łukasiewicz logic, as we have shown in Section 4.2.

The results of Propositions 5 and 6 translate into the next characterizations, where we denote by $\mathbb{CL}_\cup$ the following consequence operator related to classical logic: $\mathbb{CL}_\cup(\Gamma) = \{\psi \in \mathcal{L} \mid \psi \in \mathbb{CL}(\alpha(\varphi))$ for some $\varphi \in \Gamma\} = \bigcup_{\varphi\in\Gamma} \mathbb{CL}(\alpha(\varphi))$, where α is the map defined in Proposition 5.

Lemma 8. *Let $C^* \in \{C_\cup, C^{\leq}_{\cup}\}$ in a logic $L_\star$, where $\star$ is an SBL t-norm. Then,*

- $Conj_{C^*}(\Gamma) = \alpha^{[-1]}(Conj_{\mathbb{CL}_\cup}(\alpha(\Gamma)))$,
- $Conj_{C^*}(\Gamma) = \mathcal{L} \setminus \bigcup_{\psi\in\Gamma} Hyp^{+}_{C^*}(\neg\psi)$.

Proof. The first item basically follows by the same reasoning used in the first property in Proposition 5. As for the second one, we have the following equivalences: $\varphi \notin Conj_{C^*}(\Gamma)$ iff there exists $\psi \in \Gamma$ such that $\varphi \models \neg\psi$, iff there exists $\psi \in \Gamma$ such that $\varphi \in Hyp^{+}_{C^*}(\neg\psi)$, i.e. such that $\varphi \in \bigcup_{\psi\in\Gamma} Hyp^{+}_{C^*}(\neg\psi)$. $\dashv$

5.3 A brief summary

Finally, we present a very brief summary of the inclusion relationships that hold for consequences, conjectures and hypotheses with respect to the different consequence operators, organized by logics.

For Gödel logic it holds that:

(i) $C^{\leq} = C \subsetneq \mathbb{CL}$,
(ii) $Conj_{C^{\leq}} = Conj_C = Conj_{\mathbb{CL}}$,
(iii) $Hyp_{C^{\leq}} = Hyp_C \subsetneq Hyp_{\mathbb{CL}}$.

For any SBL logic $L_\star$ different from Gödel, it holds that, for any Γ:

(i') $C^{\leq}(\Gamma) \subsetneq C(\Gamma) \subsetneq \alpha^{-1}(\mathbb{CL}(\alpha[\Gamma]))$,
(ii') $Conj_{C^{\leq}}(\Gamma) = Conj_C(\Gamma) = \alpha^{-1}(Conj_{\mathbb{CL}}(\alpha[\Gamma]))$,
(iii') $Hyp_{C^{\leq}} \subsetneq Hyp_C \subsetneq Hyp_{\mathbb{CL}}$.

Finally, for any non-SBL logic $L_\star$ all the inclusions are strict: for any Γ,

(i") $C^{\leq}(\Gamma) \subsetneq C(\Gamma) \subsetneq \alpha^{-1}(\mathbb{CL}(\alpha[\Gamma]))$,
(ii") $Conj_{C^{\leq}}(\Gamma) \subsetneq Conj_C(\Gamma) \subsetneq \alpha^{-1}(Conj_{\mathbb{CL}}(\alpha[\Gamma]))$,
(iii") $Hyp_{C^{\leq}} \subsetneq Hyp_C \subsetneq Hyp_{\mathbb{CL}}$.

In each of the above items, the same inclusions hold when replacing C, $C^{\leq}$ and $\mathbb{CL}$ by $C_{\cup}$, $C^{\leq}_{\cup}$ and $\mathbb{CL}_{\cup}$, respectively.

6 Concluding remarks

In this paper we have presented a preliminary study towards modelling some aspects of ordinary reasoning, in the setting of t-norm based fuzzy logics, and based on a notion of consistency which depends on which consequence operator (one out of four) is chosen. The paper shows, among other interesting things, that in any logic of a SBL t-norm, the set of conjectures coincides with the set of classical conjectures (modulo identifying the strong and weak conjunctions). The proof only depends on the fact that the associated negation in all these logics is the Gödel negation.

To go further in this research, the notions of refutation and speculation have to be studied. For instance, as a first step, it seems reasonable to define the set of refutations for a set of premises Γ with respect to a consequence operator C^* as:

$$Re_{C^*}(\Gamma) = \{\chi \in \mathcal{L} \mid \Gamma \cup \{\chi\} \text{ is } C^*\text{-inconsistent}\}.$$

For SBL logics and for $C^* \in \{C, C^{\leq}\}$, this definition is equivalent to the following:

$$Re'_{C^*}(\Gamma) = \{\chi \mid \neg\chi \in C^*(\Gamma)\}.$$

However, this is no longer true for logics of non-SBL t-norms. As an example consider Łukasiewicz logic Ł, and $\Gamma = \{p \leftrightarrow \neg p\}$. Then p belongs to $Re_C(\Gamma)$, because $\Gamma \cup \{p\}$ is clearly C-inconsistent, indeed $\Gamma \cup \{p\} \models \overline{0}$. But p does not belong to $Re'_{C^*}(\Gamma)$. Indeed, $\Gamma \not\models \neg p$ since the unique Ł-evaluation e for which $e(\Gamma) \subseteq \{1\}$ is the one such that $e(p) = \frac{1}{2}$, and for this evaluation we have $e(\neg p) = \frac{1}{2}$.

This simple example shows that this study needs a deeper insigth and this is what we plan to do in the near future.

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