On strongly standard complete fuzzy logics: $MTL_*^{\mathbb{Q}}$ and its expansions

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Abstract

Finding strongly standard complete axiomatizations for t-norm based fuzzy logics (i.e. complete for deductions with infinite sets of premises w.r.t. semantics on the real unit interval $[0,1]$) is still an open problem in general, even though results are already available for some particular cases like some infinitary logics based on a continuous t-norm or certain expansions of Monoidal t-norm based logic (MTL) with rational constant symbols. In this paper we propose a new approach towards the problem of defining strongly standard complete for logics with rational constants in a simpler way. We present a method to obtain a Hilbert-Style axiomatization of the logic associated to an arbitrary standard MTL-algebra expanded with additional connectives whose interpretations on $[0,1]$ are functions with no jump-type discontinuities.

Keywords: Fuzzy logics, Strong standard completeness, MTL logic expansions, rational expansions, Pavelka-style completeness, Infinitary logics

1. Introduction

Within the mathematical logic field, the problem of finding an adequate (complete) axiomatization of a logical consequence relation has been largely studied, both within the classical and non-classical logic frameworks. In particular, within the mathematical fuzzy logics field, much effort has been devoted to prove completeness of different axiomatizations with respect to classes of algebras defined on the real unit interval $[0,1]$ (see for instance [1] and [2]), but in general, what has been mainly achieved are axiomatizations and results concerning *finitary* completeness, that is, for deductions from *a finite number of premises*.

In this paper we are concerned with the problem of the *strong* completeness, i.e., completeness for deductions with an arbitrary number of premises. In particular, we will focus on showing strong completeness for logics of a left-continuous t-norm (extensions of the monoidal t-norm based logic, MTL) expanded with rational truth-constants and with an arbitrary set of connectives respecting some constraints.

The paper is structured as follows. In the following section we gather some preliminaries about fuzzy logics and expansions with rational truth constants. In Section 3 we focus on the main results of the paper. We analyse which kind of operations from $[0,1]$ can be naturally axiomatized and how. We then present certain rules and prove that the logic resulting from adding these rules to the well known axiomatic system of *MTL* (extended with book-keeping axioms for all the connectives) is strongly standard complete. In Section 4 we pay particular attention to the case of logics with the Monteiro-Baaz Δ operation. Finally, we present some conclusions and notes for future research.

2. Preliminaries

A logic L, in its more general definition, consists in nothing more than in an abstract consequence relation between sets of formulas in a corresponding language. There exist different formalisms that allow to define a logic in a finite or recursively enumerable way; we will focus here on two of the most well known approaches: Hilbert-style axiomatic systems and algebraic logic.

Since *MTL* is the logic of the left-continuous t-norms [3, 2], it is natural to provide a formal definition for it in terms of algebras and homomorphisms. The standard algebra induced by a left-continuous t-norm $*$ is $[\mathbf{0},\mathbf{1}]_* = \langle [0,1], *, \rightarrow_*, \min, 0, 1\rangle$, where $\rightarrow_*$ is the residuum of $*$. Concerning this paper, we will be interested in the expansions of these algebras with rational constant symbols. If $*$ is closed on the set $[0,1]_{\mathbb{Q}}$ of rational numbers of $[0,1]$, the corresponding rational standard algebra is the structure:

$$[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}} = \langle [0,1], *, \rightarrow_*, \min, \{c\}_{c\in[0,1]_{\mathbb{Q}}}\rangle.$$

Note that the language of logics having these standard algebras with rational constants as intended semantics expands the one of MTL, $\{\&, \rightarrow, \wedge, \overline{0}, \overline{1}\}$, with rational truth-constants $\{\overline{c}\}_{c\in[0,1]_{\mathbb{Q}}}$.

Definition 2.1. Let $\mathbf{A}$ be the (rational) standard algebra from a left-continuous t-norm $*$, and $\Gamma\cup\{\varphi\}$ be a set of formulas. Then, φ **is consequence of** Γ **in** $\mathbf{A}$, and we will write $\Gamma \models_{\mathbf{A}} \varphi$, whenever for any homomorphism h from Fm into $\mathbf{A}$ such that $h(\Gamma) \subseteq \{1\}$ it holds that $h(\varphi) = 1$.

If $\mathbb{K}$ is a class of such algebras (of the same type), we say φ is a consequence of Γ in $\mathbb{K}$, and write $\Gamma \models_{\mathbb{K}} \varphi$, if $\Gamma \models_{\mathbf{A}} \varphi$ for each $\mathbf{A} \in \mathbb{K}$.

It has been proved in [2] that the logic (without the extra set of constants) of the class of standard algebras defined by all left-continuous t-norms is strongly complete with respect to the the axiomatic system proposed in [3] defined by the following set of axioms and Modus Ponens as inference rule:

(MTL1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(MTL2) $(\varphi \,\&\, \psi) \to \varphi$
(MTL3) $(\varphi \,\&\, \psi) \to (\psi \,\&\, \varphi)$
(MTL4a) $(\varphi \wedge \psi) \to \varphi$
(MTL4b) $(\varphi \wedge \psi) \to (\psi \wedge \varphi)$
(MTL4c) $(\varphi \,\&\, (\varphi \to \psi)) \to (\varphi \wedge \psi)$
(MTL5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \,\&\, \psi) \to \chi)$
(MTL5b) $((\varphi \,\&\, \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(MTL6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(MTL7) $\overline{0} \to \varphi$

By MTL we will usually refer to this axiomatic system and we will denote by $\vdash_{MTL}$ the corresponding notion of (finitary) proof.

For axiomatic extensions of *MTL* more particular completeness results have been proved. For instance, Hajek's Basic logic, Gödel logic, Product logic or Łukasiewicz logic, enjoy completeness with respect to a single particular standard algebra. It is well known, however, that these completeness results are, in general, true only for deductions from a finite set of premises. While Gödel logic is truly strongly standard complete, this is not the case, for instance, for any other extension of *BL*. In [4], Montagna studied the problem of how to enforce strong standard completeness on extensions of *BL* and he arrived to an elegant solution based on one infinitary rule.

Regarding logics expanded with rational constant symbols, the main references for strong completeness approaches are [5] and [6]. While the first work focuses on the strong standard completeness for the product logic extended with rational constant symbols following the usual algebraic approach, the second work is framed in the context of logics that are Pavelka-style complete, originally introduced by Pavelka in the context of Łukasiewicz logic [7]. This is a different (infinitary) notion of completeness that we will not detail here in order not to overload the reader, but it is weaker than strong standard completeness (in the sense that if a logic is strongly standard complete, then it is Pavelka-style complete, but not the other way round). For instance, Łukasiewicz logic with rational constant symbols and extended by the so-called book-keeping axioms (we will detail these later) is Pavelka-style complete, but not strongly standard complete. The paper by Cintula [6] explores different notions of rational expansions of *MTL*, and he proposes a pair of infinitary deduction rules for each discontinuity point in the truth-functions of connectives in $[0,1]$, that must be added to the logic to be Pavelka-complete.

In what follows, we will detail an alternative way (with respect to the Pavelka-style approach) to study the strong standard completeness of rational expansions of *MTL*. The approach is based on the idea that the problems of devising an axiomatization that is strongly standard complete is not exactly linked to the discontinuity points of the connectives but rather to changes in some regularity conditions, like monotonicity and continuity.

3. Towards strong standard completeness

Our main objective within this work is to study possible axiomatizations of an (infinitary) logic strongly complete with respect to the standard algebra $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}$ of an arbitrary left-continuous t-norm $*$ expanded with a further set of operations OP, that we will denote $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)$. We will begin by treating this problem in its more abstract version (for an arbitrary set of operations), and then we will provide some particular results when slightly restricting the possible sets of operations.

The language $\mathcal{L}(OP)$ of the corresponding logic will be the language of MTL (with connectives $\&, \wedge, \to$) expanded with rational truth-constants (a constant $\overline{c}$ for each rational c in $[0,1]$) plus an n-ary operation symbol $\overline{op}$ for each n-ary function $op \in OP$.

Given a left-continuous t-norm $*$, we start by considering the initial axiomatic system $MTL_*^{\mathbb{Q}}$:

- MTL-axioms and rules
- Book-keeping axioms for & and $\to$:

  $\overline{c}\&\overline{d} \leftrightarrow \overline{c * d}$
  $\overline{c} \to \overline{d} \leftrightarrow \overline{c \to_* d}$

  for every $c, d \in [0,1]_{\mathbb{Q}}$
- A rule: $\dfrac{\overline{c} \vee \varphi}{\varphi}$, for each rational $c < 1$.

where the last rule enforces the interpretation of a truth constant $\overline{c}$ with $c < 1$ in a corresponding algebra to some element different from 1.

When we consider an additional set of connectives $\overline{OP}$ in the language and their corresponding operations OP in the standard algebra, the first axioms we have to add to the system $MTL_*^{\mathbb{Q}}$ are book-keeping axioms for each n-ary operation $\star \in OP$ of the algebra:

$$\text{(Book-}\star\text{)} \quad \overline{\star}(\overline{c_1}, ..., \overline{c_n}) \leftrightarrow \overline{\star(c_1, ..., c_n)}.$$

for every $c_1, ..., c_n \in [0,1]_{\mathbb{Q}}$. In the next sections we consider additional rules to be added.

3.1. Initial observations

An important result that is worth to be recalled is that, in most of the cases, it is not possible to provide an axiomatization like the one commented above with only finitary deduction rules. This follows as an immediate corollary of [6, Prop. 18]. Note that if a logic with rational truth-constants is strongly standard complete, it is Pavelka-style complete as well.

Proposition 3.1. *Let $*$ be a left-continuous t-norm and* $\mathbf{A}$ *an expanded rational standard $*$-algebra which has a non-continuous operation. Then there is no finitary axiomatic system that is strongly complete with respect to* $\mathbf{A}$.

This sounds natural by observing the previous works on strong standard completeness and Pavelka-style completeness, and fully justifies the use of infinitary deduction rules, and the intuition coming out from this result is that of using infinitary deduction rules to control the discontinuity points of the operations of the algebra. This is actually the idea found in [6], but following this reasoning (that is to say, adding an infinitary rule for each discontinuity point) would result, in the general case, in extremely complex axiomatic systems. For instance, the residuum of the Gödel t-norm has an uncountable number of discontinuity points (the diagonal). Then, the addition of an infinitary rule for each one of these points leads to non-enumerable axiomatic system. Our aim is to propose an alternative axiomatization that does not directly depend on the cardinality of the set of discontinuity points, but on the regularity of the function as a whole.

3.2. The density rule

The problem of proving completeness with respect to the rational standard algebra associated with a particular left-continuous t-norm and a set of functions can be approached exploiting other characteristics of the operations (different from just the discontinuity points) that can be more generally studied, instead of proposing a rule for each discontinuity point. First, from the area of the first-order non-classical logics we can consider the following deduction rule

$$\frac{(A \to p) \vee (p \to B) \vee C}{(A \to B) \vee C}$$

where p is a propositional variable not occurring in A, B or C. It is a widely studied rule, that was first presented by Takeuti and Titani in [8] to axiomatize the so-called Intuitionistic fuzzy logic, and it exploits the concept of free variable from first order logics. it is called *density rule* since its validity in a given algebra enforces its universe to be dense (in the sense that between two different elements there is always a third one in between).

In our framework, it is possible to propose a similar rule with an infinite number of premises (that depend on a “free" constant symbol) in order to enforce the density of the constants within the elements of the algebra. Indeed, we extend the axiomatic system $MTL_*^{\mathbb{Q}}$ with the following infinitary deduction rule:

$$(\vee \mathsf{R}^\infty) \quad \frac{\{\gamma \vee (\varphi \to \overline{c}) \vee (\overline{c} \to \psi)\}_{c \in [0,1]_{\mathbb{Q}}}}{\gamma \vee (\varphi \to \psi)}$$

The notion of proof when infinitary rules are present is worth to be recalled.

Definition 3.2. Let L be an axiomatic system with infinitary deduction rules. A **proof of** φ **from** Γ **in** L is a well-founded tree (with possibly infinite width but with finite depth) labelled by formulas such that

- The root is labelled by φ, and the leaves are axioms of L or elements from Γ.
- For each intermediate node ψ with Σ being its immediate successors in the tree, $\frac{\Sigma}{\psi}$ is an instance of a rule of L.

The rule $(\vee \mathsf{R}^\infty)$ will be strong enough to account for the left-continuous t-norm operation and its residuum (and in general, "very regular" operations), but if we want to expand the logic with arbitrary operations, particular rules for each function will be needed.

3.3. The general problem: representable operations

In his work [6], Cintula studies the extensions of the standard *MTL* algebras with rational constants by argument-wise monotonic operations (i.e. those which, fixed all variables except one result in a monotonic (unary) operation, either increasing or decreasing) that moreover are closed on the rationals (i.e., the application of the operation on rational numbers yields a rational). Our approach allows us to partially generalize his results to a much wider family of operations, namely those that can be decomposed in argument-wise monotonic and directionally continuous regions that can be determined in the language of the logic. However, in our approach we will lose the capacity to work with some operations that Cintula considers in his paper: the ones that have jump-type discontinuity points for which, for some argument, the value of the function coincides neither with the left nor the right limit. This is natural, since using the density rule presented before, it is not clear how to deal with functions whose limit points cannot be reached through the rationals.

For a n-ary function $\star$ that is component-wise monotonic and (left or right) continuous on $U = U^1 \times ... \times U^n \subseteq [0,1]^n$, we let

$$\eta_i^{\star U} = \begin{cases} + & \text{if } \star \text{ is increasing}^1 \text{ in } U^i \\ - & \text{otherwise (decreasing)} \end{cases}$$

$$\delta_i^{\star U} = \begin{cases} L & \text{if } \star \text{ is left-continuous in } U^i \\ R & \text{otherwise (right-continuous)} \end{cases}$$

and then we introduce the following notation:

$$impl(s, \varphi, \psi) = \begin{cases} \varphi \to \psi & \text{if } s = + \text{ or } s = L \\ \psi \to \varphi & \text{if } s = - \text{ or } s = R \end{cases}$$

We will say that an n-ary function $\star : [0,1]^n \to [0,1]$ **has a simplifiable universe** whenever there exists $I \subseteq \omega$ and $\{U_i\}_{i \in I}$, called a **simplified universe** (and we will refer to the U_i's as **regions** of this simplified universe) of $\star$ such that

1. $\bigcup_{i \in I} U_i = [0,1]^n$, and for each $i \in I$, $U_i = U_i^1 \times \cdots \times U_i^n$ with U_i^j being a closed interval of $[0,1]$. [2]
2. For each $i \in I$, $\star$ is component-wise continuous in U_i and component-wise monotonic in *the interior* of U_i;
3. For each $(x_1, ..., x_n) \in [0,1]^n$, either it is a tuple of rational numbers or there exists U_j such that for each $1 \leq i \leq n$, $x_i \in U_j^i$ and $\inf U_j^i < x_i$ if $\delta_i^{\star U_j} = L$ and $\sup U_j^i > x_i$ otherwise. [3]

Formally, the sets of functions whose addition to the logic is studied here are the following ones.

Definition 3.3. Let $\star$ be an n-ary operation on $[0,1]$. We say $\star$ is **logically representable** if the following hold:

- $\star$ is closed on the rationals
- $\star$ has a simplifiable universe

We call these operations logically representable because of two logically-oriented characteristics. First, it is clear that splitting the universe on intervals we can write, in the syntax (using the rational constants), a set of formulas that will represent logically the idea of a certain point belonging in that part. For instance, the truth of formulas like $(\overline{0.4} \to \varphi)\&(\varphi \to \overline{0.7})$ expresses that the value of φ belongs to the interval $[0.4, 0.7]$. For simplicity, we will use the symbol $\in$ in the logic to write these kind of formulas. In the previous example, the formula would be equivalently expressed as "$\varphi \in [0.4, 0.7]$". Similarly, an expression "$(\varphi, \psi) \in U$", where $U = [a,b] \times [c,d]$, will be used as a shorthand for "$\varphi \in [a,b] \wedge \psi \in [c,d]$". Second, the regularity of the function in the regions of its simplified universe is characterizable with rules, as we will see in the following sections. This will imply that the interpretations of the operations from OP in the class of algebras of the logic we are defining have the right behaviour from the point of view of logical deductions.

Figure 1 gives an intuitive idea of which kind of functions belong to this class. On the other hand, functions that are not in this class are those that have a discontinuity jump in a non-rational point. A simple example of an operation not logically representable is the Dirichlet function,

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is a rational} \\ 0 & \text{otherwise} \end{cases}$$

[2]For simplicity we will assume the extreme points of the interval to be rational numbers, but this is not necessary. In other case, some deduction rules that will be defined later would have an infinite set of premises.

[3]This last condition implies that x_i does not coincide with the edge point that is not covered by the continuity direction.

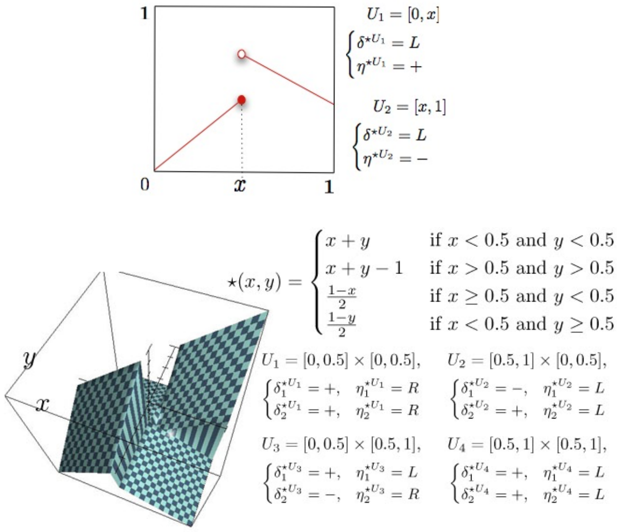

Figure 1: Examples of logically representable functions.

Moreover, since we are working over propositional expansions of MTL (in the sense that the new operations are functions over the standard $*$-algebra), it is natural to require that an axiomatic system for the logic induced by the $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)$ be an *implicative logic*, in the sense of Rasiowa. To ensure this, we have to add to $MTL_*^{\mathbb{Q}}$, for each new connective of the logic $\overline{\star}$, the following congruence rule (from the definition of Rasiowa implicative logic):

$$(\vee\mathsf{CONG}^\star)\frac{\gamma \vee \{\varphi_1 \leftrightarrow \psi_1, ..., \varphi_n \leftrightarrow \psi_n\}}{\gamma \vee (\overline{\star}(\varphi_1, ..., \varphi_n) \to \overline{\star}(\psi_1, ..., \psi_n))}$$

Besides this, we need two new kinds of rules in order to control the behaviour of the operation on the "non-rational" elements of the algebra (i.e. elements that do not coincide with the interpretation of any rational truth-constant). One type of rules will cope with the monotonicity of the functions, and the other will refer to the continuity.

The rules for expressing the monotonicity of the operation in each component have to control the extreme points of the regions from the simplified universe, since there exists the possibility of one of the extreme points behaving non-monotonically. Then, we just need to assume there exists a constant below (in the sense of the monotonicity) the point we are studying, and in the consequences we just control if that constant coincides with the minimum rational of the region of the universe.

Formally, the rules that are added to $MTL_*^{\mathbb{Q}}$ in order to characterize the monotonicity of an n-ary operation $\star$ are of the following form: for each region U from its simplified universe and each coordinate $1 \leq i \leq n$ we consider the following rule $(\vee \mathsf{M}_\mathsf{i}^{\star\mathsf{U}})$:

$$\frac{\gamma \vee \{(\varphi_1, ..., \varphi_i, ..., \varphi_n) \in U, \overline{e} \in U^i, \; impl(\eta_i^{\star U}, \overline{e}, \psi), impl(\eta_i^{\star U}, \psi, \varphi_i)\}}{\gamma \vee \chi \vee (\overline{\star}(\varphi_1, ..., \psi, ..., \varphi_n) \to \overline{\star}(\varphi_1, ..., \varphi_i, ..., \varphi_n))}$$

where $\chi = \overline{e} \leftrightarrow \overline{extr}$, and

$$extr = \begin{cases} \min U^i & \text{if } \delta_i^{\star U} = L \\ \max U^i & \text{if } \delta_i^{\star U} = R \end{cases}$$

Observe that the meaning of the formula χ is just to check if a certain value coincides with the edge –in the direction of the continuity of the operation in that component.

On the other hand, we need rules that determine the continuity of the function in the regions of the simplified universe. That will be done by translating some of the information on the operation to the axiomatic system. In particular, the intuitive meaning of the two rules below capture the fact that, for a given point in a continuity fragment of a function, if the value of the function is smaller/greater than a certain value in all rationals from that fragment, the so is the image of that point. Formally, for each region U of the simplified universe of $\star$ and each component $1 \leq i \leq n$, we add to $MTL_*^{\mathbb{Q}}$ the following two rules: (with χ is as above):

- If $\star$ is **left-continuous and increasing** in U^i ($\delta_i^{\star U} = L, \eta_i^{\star U} = +$) or **right-continuous and decreasing** ($\delta_i^{\star U} = R, \eta_i^{\star U} = -$):

$$(\vee \mathsf{C}_\mathsf{i}^{\star \mathsf{U}})\,\frac{\gamma \vee \{(\varphi_1, ..., \varphi_n) \in U, \overline{c} \to \overline{\star}(\varphi_1, ..., \varphi_n), \{\chi \vee impl(\delta_i^{\star U}, x_i, \overline{d}) \vee \overline{\star}(\varphi_1, ..., \overline{d}, ..., \varphi_n) \to \overline{c}\}_{d \in U^i \cap [0,1]_{\mathbb{Q}}}\}}{\gamma \vee (\overline{\star}(\varphi_1, ..., \varphi_i, ..., \varphi_n) \to \overline{c})}$$

- If $\star$ is **left-continuous and decreasing** in U^i ($\delta_i^{\star U} = L, \eta_i^{\star U} = -$) or **right-continuous and increasing** ($\delta_i^{\star U} = R, \eta_i^{\star U} = +$):

$$(\vee \mathsf{C}_\mathsf{i}^{\star \mathsf{U}})\,\frac{\gamma \vee \{(\varphi_1, ..., \varphi_n) \in U, \overline{\star}(\varphi_1, ..., \varphi_n) \to \overline{c}, \{\chi \vee impl(\delta_i^{\star U}, x_i, \overline{d}) \vee \overline{c} \to \overline{\star}(\varphi_1, ..., \overline{d}, ..., \varphi_n)\}_{d \in U^i \cap [0,1]_{\mathbb{Q}}}\}}{\gamma \vee (\overline{c} \to \overline{\star}(\varphi_1, ..., \varphi_i, ..., \varphi_n))}$$

It is an exercise to check that all the rules introduced in this section are sound. Indeed, the only case that could be somewhat not obvious is the last couple of formulas, but observe that they hold in the standard algebra with the corresponding operations $[\mathbf{0}, \mathbf{1}]_*^{\mathbb{Q}}(OP)$: if $c < \star(x_1, ..., x_n)$, there is c_i, with $c_i \leq x_i$ if $\delta_i^{\star U} = +$ or with $x_i \leq c_i$ if $\delta_i^{\star U} = -$, such that $c < \star(x_1, ..., c_i, ..., x_n)$. Figure 2 shows this for some examples.

These rules enforce that the value of a function in a point can be approached through the values on rational constants near it (in the direction in which the function is continuous).

At this point we can provide a formal definition of our logic expanding $MTL_*^{\mathbb{Q}}$.

Definition 3.4. Let $*$ be a left continuos t-norm and let OP a set of logically representable operations. Then the axiomatic system $MTL_*^{\mathbb{Q}}(OP)$ is

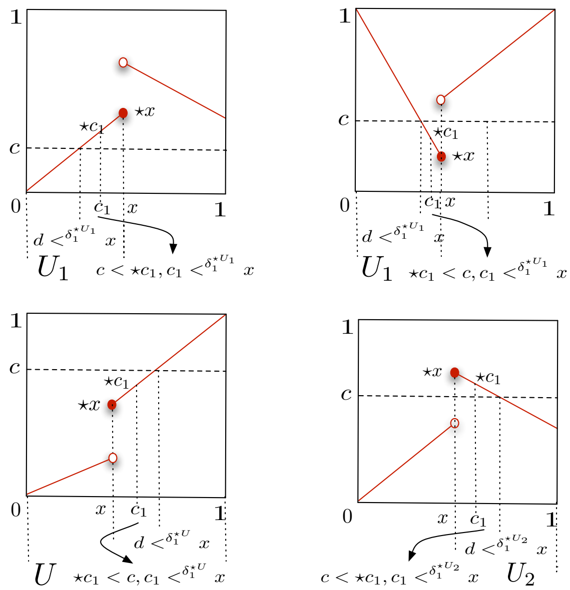

Figure 2: Examples for the rule $\mathsf{C}_\mathsf{i}^{\star \mathsf{U}}$.

defined as the expansion of $MTL_*^{\mathbb{Q}}$ with the following axioms and rules:

- book-kepping axioms (Book-$\star$), for each $\star \in OP$
- density rule ($\vee \mathsf{R}^\infty$)
- congruence rule ($\vee \mathsf{CONG}^\star$), for each $\star \in OP$
- monotonicity rules ($\vee \mathsf{M}_\mathsf{i}^{\star \mathsf{U}}$), for each $\star \in OP$ and region U of its universe
- continuity rules ($\vee \mathsf{C}_\mathsf{i}^{\star \mathsf{U}}$), for each $\star \in OP$ and region U of its universe

The associated notion of *infinitary* proof (according to Def. 3.2) will be denoted $\vdash_{MTL_*^{\mathbb{Q}}(OP)}$.

It is remarkable to notice that the original axiomatization of $MTL_*^{\mathbb{Q}}$ already allows to prove all the previous deductions concerning the left-continuous t-norm operation $*$ and its residuum $\Rightarrow_*$. Then, in the case where no extra operation is added (when $OP = \emptyset$), no extra rules are needed (apart from $\vee \mathsf{R}^\infty$). In fact, it is natural to think in the rules presented in this section ($\vee \mathsf{CONG}^\star$, $\vee \mathsf{M}_\mathsf{i}^{\star \mathsf{U}}$ and $\vee \mathsf{C}_\mathsf{i}^{\star \mathsf{U}}$) as a way of emulating a "usual axiomatization" using only book-keeping axioms.

3.4. The semilinearity issue

Before continuing, the reader may wonder about the necessity of having closed by $\vee$ each new inference rule introduced in the previous sections. The idea is of simplifying the study of the logic: even if the general results concerning semilinearity are mostly limited to finitary logic [9], in our case we can prove that the logics $MTL_*^{\mathbb{Q}}(OP)$ (with OP a set of logically representable functions) is semilinear. It is clear that $MTL_*^{\mathbb{Q}}(OP)$ is Rasiowa-implicative, and thus, algebraizable in the sense of Blok and Pigozzi [10]. Its algebraic companion is the proper **generalised quasi-variety**

$\mathbb{MTL}_*^{\mathbb{Q}}(OP)$ of MTL-algebras expanded with rational truth-constants and operations from OP, further satisfying the axioms, equations and generalised quasi-equations naturally obtained respectively from the additional axioms, finitary rules and infinitary rules of $MTL_*^{\mathbb{Q}}(OP)$.[4] Also, this implies that there is an isomorphism between filters and congruences on algebras in the class $\mathbb{MTL}_*^{\mathbb{Q}}(OP)$.

On the other hand, it is possible to prove that, as it happens in the finitary case, if the inference rules of an implicative logic L are closed under the $\vee$ operation (that is to say, if for any rule $\Gamma \vdash \varphi$ of the logic, the rule $\{\chi \vee \gamma\}_{\gamma \in \Gamma} \vdash \chi \vee \varphi$ is derivable in the logic) then the logic is semilinear, that is, it is strongly complete with respect to the linearly ordered L-algebras. This is the motivation for which all the new rules proposed are directly formulated as their $\vee$-closure.[5]

Actually, it is possible to prove that a (deductive) filter of an algebra in $\mathbb{MTL}_*^{\mathbb{Q}}(OP)$ is the intersection of the prime filters that contain it, understanding as prime filters those for which, for two arbitrary elements a, b of the algebra, either $a \to b \in F$ or $b \to a \in F$. We will not detail this result here but the proof is very similar to the one found in [11, Cor. 2.5.4.], it is only necessary to adapt the definition of semantical proof to the infinitary case (following 3.2). The semilinearity of $MTL_*^{\mathbb{Q}}(OP)$ is then a corollary of this characterization.

Theorem 3.5. *For any set of formulas $\Gamma \cup \{\varphi\} \subseteq Fm$, the following are equivalent:*

1. $\Gamma \vdash_{MTL_*^{\mathbb{Q}}(OP)} \varphi$
2. $\Gamma \models_{\mathbf{C}} \varphi$ *for all* $\mathbf{C} \in \mathbb{MTL}_*^{\mathbb{Q}}(OP)$ *such that* $\mathbf{C}$ *is linearly ordered.*

Proof. We only check $2 \Rightarrow 1$, the other direction being soundness. The general completeness result states that $\Gamma \nvdash_{MTL_*^{\mathbb{Q}}(OP)} \varphi$ implies that there exist $\mathbf{A} \in \mathbb{MTL}_*^{\mathbb{Q}}(OP)$, a filter F of $\mathbf{A}$, and an $\mathbf{A}$-evaluation h such that $h([\Gamma]) \subseteq F$ and $h(\varphi) \notin F$. Then one can prove there is a prime filter P of $\mathbf{A}$ that contains F and such that $h([\Gamma]) \subseteq P$ and $h(\varphi) \notin P$. It is an exercise then to see that the quotient algebra $\mathbf{A}/P$ is a linearly ordered algebra in the class $\mathbb{MTL}_*^{\mathbb{Q}}(OP)$. To conclude, recall that $\overline{h} = \pi_P \circ h$ is an evaluation on the quotient algebra $\mathbf{A}/P$, where $\pi_P : \mathbf{A} \to \mathbf{A}/P$ is the projection on the quotient algebra. Since for any $\psi \in P$ it holds that $\pi_P(\psi) = 1$ and for any $\psi \notin P$ $\pi_P(\psi) < 1$ it follows that $\pi_P \circ h([\Gamma]) \subseteq \{1\}$ and $\pi_P \circ h(\varphi) < 1$. □

Having proved strong completeness of $MTL_*^{\mathbb{Q}}(OP)$ wrt its class of linearly ordered algebras, it remains to study their relationship to the one defined in the real unit interval.

[4]Observe that in a lot of the usual many-value logics, its algebraic companion is a variety, while in this case we can prove this is not true.

[5]It is not in the scope of this work to study, in general, the problem of the semilinearity of our axiomatic systems.

3.5. Strong Standard Completeness

To show that, for an arbitrary set OP of logically representable operations, $MTL_*^{\mathbb{Q}}(OP)$ enjoys the strong standard completeness we will resort to a simple method: constructing an embedding from any (countable) linearly ordered $MTL_*^{\mathbb{Q}}$-algebra into the standard $MTL_*^{\mathbb{Q}}$-algebra.

With that aim in mind, the reason behind the addition of the rule ($\vee$R$^\infty$) is now clear: over the linearly ordered algebras, the constants are dense in the algebra, which will very helpful for the proof of standard completeness.

Lemma 3.6. *Let $\mathbf{A} \in \mathbb{MTL}_*^{\mathbb{Q}}(OP)$ be linearly ordered, and $a < b$ in A. Then there is $c \in [0,1]_{\mathbb{Q}}$ such that $a < \overline{c}^{\mathbf{A}} < b$.*

Proof. Towards a contradiction, suppose there is no such c. Then, since $\mathbf{A}$ is linearly ordered we have that for all $c \in [0,1]_{\mathbb{Q}}$, either $b \leq \overline{c}^{\mathbf{A}}$ or $\overline{c}^{\mathbf{A}} \leq a$. Then, the premises of the generalised quasi-equation associated to rule $\vee$R$^\infty$ hold, which leads to have that $b \leq a$, which contradicts the assumptions of the lemma. □

Knowing this, it is natural to construct an embedding from any countable linearly ordered $MTL_*^{\mathbb{Q}}(OP)$-algebra $\mathbf{A}$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(\mathbf{OP})$ by means of the mapping $\theta : A \to [0,1]$ defined as:

$$\theta(a) = \inf\{c \in [0,1]_{\mathbb{Q}} : \overline{c}^{\mathbf{A}} \geq a\} = \sup\{c \in [0,1]_{\mathbb{Q}} : \overline{c}^{\mathbf{A}} \leq a\}$$

We can prove that it is an injective homomorphism. We first observe that the crucial characteristics of the operations (as given in $[0,1]$) are properly translated to their correspondent symbols in the logic.

Lemma 3.7. *Let OP be a set of logically representable operations in $[0,1]$ and let $\mathbf{A}$ be a linearly ordered $\mathbb{MTL}_*^{\mathbb{Q}}(OP)$-algebra. Let $\star \in OP$ be any n-ary operation with simplified universe $U = \bigcup_{i \in I} U_i \subseteq [0,1]^n$, and for some $k \in I$, let $x_1, ..., x_n \in U_k$ such that for some $1 \leq i \leq n$, $x_i \neq \overline{c}^{\mathbf{A}}$ for any $c \in [0,1]_{\mathbb{Q}}$. Then*

$$\overline{\star}^{\mathbf{A}}(x_1, ..., x_n) = \Sigma_1\{...\Sigma_n\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}... : c_1 \in C_1\}$$

where

$$\Sigma_i = \begin{cases} \sup & \textit{if } \eta_i^{\star U_k} = +, \delta_i^{\star U_k} = L \\ & \textit{or } \eta_i^{\star U_k} = -, \delta_i^{\star U_k} = R \\ \inf & \textit{otherwise} \end{cases}$$

$$C_i = \begin{cases} \{a \in U_k^i \cap [0,1]_{\mathbb{Q}} : \overline{a}^{\mathbf{A}} \leq x_i\} & \textit{if } \delta_i^{\star U} = L \\ \{a \in U_k^i \cap [0,1]_{\mathbb{Q}} : \overline{a}^{\mathbf{A}} \geq x_i\} & \textit{otherwise} \end{cases}$$

Proof. For the sake of readability, we will write the proof assuming $\star$ is a particular binary operation, the general case can be proved similarly. Assume a simplified universe for $\star$ is given by $U_1 \cup U_2$ where $\{U_1 = [0,1] \times [0,b], U_2 = [0,1] \times [b,1]\}$ with

$$\begin{cases} \delta_1^{\star U_1} = L, & \eta_1^{\star U_1} = + \\ \delta_2^{\star U_1} = L, & \eta_2^{\star U_1} = - \end{cases} \text{ and } \begin{cases} \delta_1^{\star U_2} = L, & \eta_1^{\star U_2} = - \\ \delta_2^{\star U_2} = R, & \eta_2^{\star U_2} = - \end{cases}$$

To check the $\leq$ direction of the lemma, let $c \in [0,1]$ such that $\overline{c}^{\mathbf{A}} < \overline{\star}^{\mathbf{A}}(x_1, x_2)$. We will do the case of $(x_1, x_2) \in U_1$, the other one is analogous.

We begin by the second component. In our case, the monotonicity rule implies that $\overline{\star}^{\mathbf{A}}(x_1, x_2) \leq \overline{\star}^{\mathbf{A}}(c_1, x_2)$ for any $\overline{c_1}^{\mathbf{A}} \leq x_1$ (and note this set coincides with C_2). Then, in particular, it is possible to take the infimum to get $\overline{\star}^{\mathbf{A}}(x_1, x_2) \leq \inf\{\overline{\star}^{\mathbf{A}}(x_1, \overline{c_2}^{\mathbf{A}}) : c_2 \in C_2\}$. By the assumption, we have now that $\overline{c}^{\mathbf{A}} < \inf\{\overline{\star}^{\mathbf{A}}(x_1, \overline{c_2}^{\mathbf{A}}) : c_2 \in C_2\}$.

On the other hand, for the first component, we can check that the elements over which the infimum is taken can be put through another bound.

Pick an arbitrary $d \in [0,1]_{\mathbb{Q}}$ such that $d < \overline{\star}^{\mathbf{A}}(x_1, \overline{c_2}^{\mathbf{A}})$. Then, the rule $\mathsf{C}_1^{\star \mathsf{U}_1}$ can be applied: since the consequence does not hold, there must exist $c_1 \in U_1 \cap [0,1]_{\mathbb{Q}}$ such that $c_1 \neq 0$, $\overline{c_1}^{\mathbf{A}} < x_1$ and $\overline{d}^{\mathbf{A}} < \overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}})$. Then, $\overline{\star}^{\mathbf{A}}(x_1, \overline{c_2}^{\mathbf{A}}) \leq \overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}})$ for some c_1 as above.

Observe that, in particular, $c_1 \in C_1$ from the formulation of the Lemma, and so, the supremum over the elements in this set will be greater or equal to the value on that point. Then, $\overline{\star}^{\mathbf{A}}(x_1, \overline{c_2}^{\mathbf{A}}) \leq \sup\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_1 \in C_1\}$.

Applying these to the previous computation for the second component we get that $\overline{c}^{\mathbf{A}} < \inf\{\sup\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, \overline{c_2}^{\mathbf{A}}) : c_1 \in C_1\} : c_2 \in C_2\}$.

For the other direction, the reasoning is similar but using the rule $\mathsf{C}_i^{\star \mathsf{U}}$ with the form it takes for left-continuous decreasing and right-continuous increasing regions, when it is necessary (i.e. for the cases where the monotonicity is not enough). □

With this, it is not difficult to prove that the mapping θ is in fact an embedding from a countable linearly ordered $MTL_*^{\mathbb{Q}}(OP)$-algebra $\mathbf{A}$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)$.

Lemma 3.8. *Let $\mathbf{A} \in \mathbb{MTL}_*^{\mathbb{Q}}(OP)$ be a countable chain. Then, the function θ defined above is an embedding from $\mathbf{A}$ into $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)$.*

Proof. First note that for any constant $\overline{d}$, $d = \min\{c \in [0,1]_{\mathbb{Q}} : \overline{c}^{\mathbf{A}} \geq \overline{d}^{\mathbf{A}}\} = \max\{c \in [0,1]_{\mathbb{Q}} : \overline{c}^{\mathbf{A}} \leq \overline{d}^{\mathbf{A}}\}$, and so $\theta(\overline{d}^{\mathbf{A}}) = d = \overline{d}^{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)}$. On the other hand, it is immediate to see that it is strictly order preserving: if $a < b \in A$, there exists $c \in [0,1]_{\mathbb{Q}}$ such that $a < \overline{c}^{\mathbf{A}} < b$, and thus, $\theta(a) < c < \theta(b)$. This shows that θ is one-to-one.

Regarding the homomorphic conditions for the operations, we will exploit the density of the constants in $\mathbf{A}$ and in $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)$. Observe that the left-continuous t-norm and its residuum are, by definition, logically representable operations, so the proof can be done in general for any logically representable operation $\star$.[6]

As for the $\leq$ direction, let $c \in [0,1]_{\mathbb{Q}}$ such that $c < \theta(\overline{\star}^{\mathbf{A}}(x_1, ..., x_n))$. By definition, and given that θ preserves the order, $\overline{c}^{\mathbf{A}} \leq \overline{\star}^{\mathbf{A}}(x_1, ..., x_n)$. By the previous lemma, it follows that $\overline{c}^{\mathbf{A}} \leq \Sigma_1\{...\Sigma_n\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}...x_1 \in C_1\}$. Then, $\overline{c}^{\mathbf{A}} \leq \overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}})$ for some $c_i \in C_i$ if $\Sigma_i = \sup$, and for all $c_i \in C_i$ if $\Sigma_i = \inf$ (for each $1 \leq i \leq n$).

We can use now the book-keeping axioms to get that $c \leq \star(c_1, ..., c_n)$ for c_i as above. Now, we can use the properties of $\star$ in $[0,1]$ (monotonicity and left-right continuity), take limits and conclude that $c \leq \star(\theta x_1, ..., \theta x_n)$.

In order to prove the $\geq$ inequality, let $c \in [0,1]_{\mathbb{Q}}$ be such that $\star(\theta x_1, ..., \theta x_n) < c$. Then, as before (since $[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)$ is linearly ordered), from the previous lemma we get $\Sigma_1\{...\Sigma_n\{\star(c_1, ..., c_n) : c_n \in C_n\}...x_1 \in C_1\} < c$. Then, $\star(c_1, ..., c_n) < c$ for the families of c_i as above.

From the book-keeping axioms we have that $\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) < \overline{c}^{\mathbf{A}}$ for c_i as above. We can now clearly take suprema and infima to get $\Sigma_1\{...\Sigma_n\{\overline{\star}^{\mathbf{A}}(\overline{c_1}^{\mathbf{A}}, ..., \overline{c_n}^{\mathbf{A}}) : c_n \in C_n\}...x_1 \in C_1\} \leq \overline{c}^{\mathbf{A}}$. Again from the previous lemma, it follows that $\overline{\star}^{\mathbf{A}} x_1, ..., x_n \leq \overline{c}^{\mathbf{A}}$. Since θ is order preserving, we finally have $\theta(\overline{\star}^{\mathbf{A}} x_1, ..., x_n) \leq \theta(\overline{c}^{\mathbf{A}}) = c$. □

From here, the strong standard completeness of $MTL_*^{\mathbb{Q}}(OP)$ follows straightforwardly.

Theorem 3.9 (Strong Standard Completeness of $MTL_*^{\mathbb{Q}}(OP)$)**.** *For any set of formulas $\Gamma \cup \{\varphi\}$ the following are equivalent:*

1. $\Gamma \vdash_{MTL_*^{\mathbb{Q}}(OP)} \varphi$
2. $\Gamma \models_{[\mathbf{0},\mathbf{1}]_*^{\mathbb{Q}}(OP)} \varphi$.

Proof. One direction (from 1 to 2) is soundness, that is easy to prove. As for the other implication, suppose that $\Gamma \nvdash_{MTL_*^{\mathbb{Q}}(OP)} \varphi$. Then, by Theorem 3.5 there is a linearly ordered $MTL_*^{\mathbb{Q}}(OP)$-algebra $\mathbf{A}$ and an $\mathbf{A}$-evaluation h such that $h([\Gamma]) \subseteq \{1\}$ and $h(\varphi) < 1$. It is immediate that $h([Fm])$ is a countable subalgebra of $\mathbf{A}$ (thus linearly ordered), and so it can be embedded into the standard algebra $[0,1]_*^{\mathbb{Q}}(OP)$ by the embedding θ from the previous lemma. Then, it is clear that $\theta \circ h$ is an $[0,1]_*^{\mathbb{Q}}(OP)$-evaluation such that $\theta \circ h(\Gamma) \subseteq \{1\}$ and $\theta \circ h(\varphi) < 1$. This concludes the proof. □

4. Further issues: logics with Δ

In the approach developed in the previous sections, the resulting strongly standard complete logic $MTL_*^{\mathbb{Q}}(OP)$ depends very much on the shape of the

[6] Nevertheless, the case of the left-continuous t-norm operation has a more direct approach, that does not need any of the $\vee\mathsf{CONG}^\star$, $\vee\mathsf{M}_i^{\star \mathsf{U}}$ nor $\vee\mathsf{C}_i^{\star \mathsf{U}}$ rules and that relies on the MTL-axiomatization of a residuated operation.

operations in OP. For instance, the logic will have a finite number of rules if the number of regions of the simplified universes of all the operations is finite, and the rules will be finitary whenever the edges of these regions are rationals.

However, if we assume some given features of the set of operations we can obtain stronger and clearer results. In this section we focus on the study of logics $MTL_*^{\mathbb{Q}}(OP)$ when OP already contains an extra unary operator largely studied in the field of fuzzy logic systems: the Monteiro-Baaz Δ operator. This operator behaves over MTL chains sending the top element to itself and any other element to the bottom of the algebra. It has been axiomatized for instance in [1], and so we will consider now the basic axiomatic system $MTL_{\Delta,*}^{\mathbb{Q}}$ consisting of:

- Axioms and rules of MTL_Δ (expansion of MTL with Δ connective (see e.g. [3])
- Book-keeping axioms for $*$, its residuum and Δ: $\Delta\overline{1}$, $\neg\Delta\overline{c}$ for each $c < 1$

The power of having Δ in the logic is remarkable for instance in the approach for proving the semilinearity of the logic. Indeed, some of the new rules are not required any longer to be closed under $\vee$ forms and some other rules can be expressed as axioms. For instance the density rule can be simplified to:

$$(\mathsf{R}^\infty)\ \frac{\{(\varphi \to \overline{c}) \vee (\overline{c} \to \psi)\}_{c\in[0,1]_{\mathbb{Q}}}}{\varphi \to \psi}$$

As for the rest of the rules introduced in the previous section coping with additional operations in OP, it is now possible to transform those rules involving a finite number of premises (e.g. those related to operations such that the number of formulas that define each region of their simplified universes is finite. This is immediate, since MTL_Δ enjoys the Δ-deduction theorem ($\varphi \vdash_{MTL_\Delta} \psi$ iff $\vdash_{MTL_\Delta} \Delta\varphi \to \psi$) and so, a finitary rule can be turned into an axiom.

5. Conclusions and Future Work

In this paper we have been concerned in obtaining strongly standard complete axiomatizations for the logic of an arbitrary left-continuous t-norm, expanded with rational truth-constants and possibly with a set of additional connectives whose interpretation as operations in $[0, 1]$ satisfies some regularity conditions. The price we have to pay is that the resulting logics is not finitary any longer.

As for future research, it seems the issue has been studied to the point that nothing more but simplifying and optimizing the solutions found can be done. It is mandatory for the axiomatic system to be infinitary (Lemma 3.1), and so, an axiomatic system that in a large number of cases (for instance, MTL, BL and their extensions with Δ), has only one infinitary rule seems the best that can be achieved. For what respects operations that involve a larger number of infinitary/finitary rules, it is not clear whether a better solution can be found (both in Cintula's work, and here, the number of infinitary rules clearly depends on the regularity of the function). In this paper, we have limited the number of infinitary (and finitary) rules in terms of the regularity of the operations, but still we can come up with logics with an infinite number of them, even though the amount of such cases seems to be much smaller than in previous approaches in the literature. Nevertheless, we think some particular operations may have a good enough behaviour to allow a more specific characterization of the axioms and rules related with them, as it happens for instance in the case of the left continuous t-norm operation or the Δ operation.

Acknowledgements. Authors are grateful to anonymous reviewers and acknowledge partial support of the Mineco project TIN2012-39348-C02-01.

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