Dimensional flow in discrete quantum geometries

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In various theories of quantum gravity, one observes a change in the spectral dimension from the topological spatial dimension d at large length scales to some smaller value at small, Planckian scales. While the origin of such a flow is well understood in continuum approaches, in theories built on discrete structures a firm control of the underlying mechanism is still missing. We shed some light on the issue by presenting a particular class of quantum geometries with a flow in the spectral dimension, given by superpositions of states defined on regular complexes. For particular superposition coefficients parametrized by a real number $0 < \alpha < d$, we find that the spatial spectral dimension reduces to $d_S \simeq \alpha$ at small scales. The spatial Hausdorff dimension of such class of states varies between 1 and d, while the walk dimension takes the usual value $d_W = 2$. Therefore, these quantum geometries may be considered as fractal only when $\alpha = 1$, where the "magic number" $D_S \simeq 2$ for the spectral dimension of space*time*, appearing so often in quantum gravity, is reproduced as well. These results apply, in particular, to special superpositions of spin-network states in loop quantum gravity, and they provide more solid indications of dimensional flow in this approach.

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I. INTRODUCTION

The identification of good geometric observables is a thorny issue in (quantum) gravitational physics, and it is of particular importance in nonperturbative, background-independent approaches to quantum gravity, especially where the fundamental degrees of freedom characterizing quantum states and histories of the system are nongeometric in the standard sense and characterized by intrinsic discreteness. Examples are loop quantum gravity (LQG) [1–3], spin-foam models [4,5] and group field theory (GFT) [6,7], strictly related to LQG [8,9]. Here, the major challenge is to find a relation to the continuum spacetime geometries of classical general relativity, i.e., to show that the latter emerge from the fundamental discrete quantum structures of the theory in some approximation. This emergence has to be expressed in terms of suitable geometry observables, both classical and quantum, that should indicate that the desired features of smooth spacetimes are recovered. This is, in fact, a precondition for extracting physics from such quantum-gravity formalisms.

Effective-dimension observables provide important information about the geometric properties of quantum states of space and spacetime histories in quantum gravity. In particular, the spectral dimension d_S, which depends on the spectral properties of a geometry through its definition as the scaling of the heat-kernel trace, has attracted special

attention due to the observation of a dimensional flow (i.e., the change of spacetime dimensionality across a range of scales [10–12]) in various approaches, such as causal dynamical triangulations (CDT) [13], the functional renormalization-group approach of asymptotic safety [14,15] and Hořava-Lifshitz gravity [16] among others.

In all these approaches, the spectral dimension of spacetime exhibits a scale dependence itself, flowing from the topological dimension D in the infrared (IR) to $D_S \simeq 2$ in the ultraviolet (UV) [16–21] (although new CDT calculations [22] rather hint at $D_S \simeq 3/2$). While modified dispersion relations provide an obvious reason for this behavior in smooth geometries [16,17,23–26], dimensional flow remains to be better understood in the case of discrete calculations as in the CDT approach [19–22]. Causal dynamical triangulations, in fact, aim at a definition of the continuum path integral for quantum gravity via a regularization of the same in terms of a superposition of simplicial complexes (thus a form of discrete geometries) weighted by the Regge action. While it is more difficult to identify the underlying reason for the dimensional flow in this context, the same is obtained in a very direct manner from the evaluation of the heat trace as a quantum geometric observable inside the CDT partition function.

Here we take a very similar direct approach, but in a context that is closer to the formalism of loop quantum gravity. In LQG, quantum states are defined as superpositions of spin networks, which are graphs labeled by algebraic data from the representation theory of $SU(2)$. There is thus an interplay between two types of data and their corresponding discreteness: a combinatorial

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discreteness due to the graph substratum for the quantum states, and an algebraic discreteness due to the fact that the labels are half-integers corresponding to $SU(2)$ irreducible representations. Quantum effects in the evaluation of observables are thus to be expected, in general, from both these sources and it is an important limitation to focus only on one of them, as in preliminary studies of dimensional flow in LQG [26].

In a previous work [27], we have already tackled the issue of computing the spectral dimension on LQG states based on a given graph, dealing both with coherent states and with their superpositions. There, we showed that the underlying discrete structure plays a dominant role. Here, we intend to explore the role of combinatorial discreteness and of superpositions of combinatorial structures in greater detail.

In this paper, we present a special class of superpositions of discrete quantum states characterized by a real-valued parameter α. This parameter will control the scale-dependent values taken by the spectral dimension, and therefore the dimensional flow. These superpositions are over states based on regular complexes corresponding to hypercubic lattices to which a single quantum label is assigned, uniformly to all cells of a certain dimension. Such states occur indeed in the kinematical Hilbert space of the quantum gravity formalisms we just mentioned: LQG, spin-foam models and GFT. Because of the uniform labeling, these superpositions are also similar to the discrete geometries in CDT, although we understand the former not as regularization tools for physically smooth geometries but as fundamentally discrete structures with their own physical interpretation. Contrary to the CDT setting, we interpret the combinatorial structures we superpose as defining quantum gravity states, not histories, and the coefficients in the superpositions to have no immediate dynamical content. However, we point out that this interpretation enters only minimally in the actual calculations and it could be generalized.

Perhaps surprisingly, superpositions of quantum states supported on different complexes have not been considered much in the LQG literature so far. Instead, most analyses have involved only states based on one and the same complex. A first simple example of states based on superpositions of combinatorial structures are the condensate states with a homogeneous cosmology interpretation introduced recently in the GFT context [28–32].

Using a known analytic expression for the spectral dimension of single members in the superposition [27], we compute numerically superpositions over up to 10^6 discrete geometries. On these grounds, we find strong evidence for a dimensional flow, characterized by the parameter α.

Similarly, we find analytic solutions for the walk dimension and Hausdorff dimension of lattice geometries and perform again numerical calculations of superpositions. For

these observables, however, while we recover the topological dimension at large scales, we do not find any special properties for superpositions as compared to states defined on fixed complexes.

II. A GENERAL CLASS OF SUPERPOSITION STATES

Let us now explain in detail the construction of super-position states of interest, and the calculation of their spectral, walk and Hausdorff dimension.

Most generally speaking, a discrete quantum state of geometry $|\{j_c\}, \mathcal{C}\rangle$ is given by an assignment of quantum numbers j_c to a certain subset of cells $c \in \mathcal{C}$ of a (combinatorial) complex $\mathcal{C}$, diagonalizing volume operators of these cells

$$\widehat{V_{c'}^{(p)}}|\{j_c\}, \mathcal{C}\rangle \propto l^p(j_{c'})|\{j_c\}, \mathcal{C}\rangle, \qquad (1)$$

where we have adopted natural units. An example of such states is spin-network states in LQG, based on the 1-skeleton of the dual complex $\mathcal{C}^\star$, with the j's identifying irreducible representations of $SU(2)$. In three spacetime dimensions, the spatial $(d = 2)$ states in the spin network basis diagonalize the length operators $\hat{L}_e$ associated with all edges $e \in \mathcal{C}$. Thus, they are labeled by spins j_e on the corresponding dual edges $e^\star \in \mathcal{C}^\star$. The form of the $\hat{L}_e$ spectra is $l(j_e) = \sqrt{j_e(j_e + 1) + C}$, with a free parameter $C \in \mathbb{R}$ due to a quantization ambiguity for the Euclidean theory (as well as for timelike edges in the Lorentzian theory, a continuous positive variable being instead assigned to spacelike edges) [33,34].

In four spacetime dimensions $(d = 3)$, spin-network states have the same spectrum for area operators $\hat{A}_f$ on faces $f \in \mathcal{C}$ such that [35,36]

$$l(j_f) = [j_f(j_f + 1) + C]^{1/4}. \qquad (2)$$

Generic quantum-geometry states are superpositions of the discrete quantum geometries $|\{j_c\}, \mathcal{C}\rangle$, which indeed form a complete spin-network basis of states of the Hilbert space in LQG. In particular, this Hilbert space can be cast in the form of a direct sum of Hilbert spaces $\mathcal{H} = \oplus_{\mathcal{C}} \mathcal{H}_{\mathcal{C}}$.

In the following, we will restrict to superpositions with nonzero coefficients only for states $|j, \mathcal{C}\rangle$ labeled by a single quantum number $j_c = j$ for all cells. Thus, one can consider the individual states $|j, \mathcal{C}\rangle$ as corresponding to equilateral lattices. Given this class of quantum states, we then consider generic superpositions of the form

$$|\psi\rangle = \sum_{j, \mathcal{C}} a_{j, \mathcal{C}}|j, \mathcal{C}\rangle. \qquad (3)$$

We also impose a constraint on the overall volume V_0 computed from such superposition states:

$$|\psi, V_0\rangle = \sum_{j,\mathcal{C}} a_{j,\mathcal{C}} \delta(\langle j, \mathcal{C}|\hat{V}|j,\mathcal{C}\rangle, V_0)|j,\mathcal{C}\rangle, \qquad (4)$$

where the delta is a Kronecker delta. We will further restrict the sum to certain regular complexes, i.e., hypercubic lattices $\mathcal{C}_N$ based on the canonical vertex set $\mathcal{C}_N^{[0]} := (\mathbb{Z}_N)^d$ of d-tuples of integers modulo N. In this case, the fixed volume condition is explicitly

$$V_0 = \langle j, \mathcal{C}_N|\hat{V}|j,\mathcal{C}_N\rangle \propto N^d l^d(j), \qquad (5)$$

which fixes the lattice size $N = N(j)$ for a given j (at least approximately).[1] In general, there are three scales involved in the superposition states

$$|V_0, j_{\min}, j_{\max}\rangle := \sum_{j=j_{\min}}^{j_{\max}} a_j |j, \mathcal{C}_{N(j)}\rangle, \qquad (6)$$

when summing over a finite range from $j_{\min}$ to $j_{\max}$: a minimal length scale $l(j_{\min})$, an intermediate scale $l(j_{\max})$ and the overall volume size $V_0^{1/d} \propto N(j_{\min})l(j_{\min}) = N(j_{\max})l(j_{\max})$. Note that a finite volume V_0 bounds also possible cutoffs $j_{\max}$ (since N is a positive integer).

One can also consider the limit of noncompact geometries $N(j_{\min}) \to \infty$, where all complexes in the superposition of fixed-volume states (6) converge to the infinite lattice $\mathcal{C}_\infty$. Thus, they are technically the same as superpositions on the fixed complex $\mathcal{C}_\infty$, although the physical interpretation is different. Due to the combinatorial simplicity, results of infinite-size calculations can be directly applied to the finite-volume case.

Having defined our superposition states, we can move on to the evaluation of the geometric observables of interest, i.e., dimension estimators.

III. EVALUATION OF DIMENSION OBSERVABLES OF SUPERPOSITION STATES

A. Spectral dimension

Let the heat kernel $K(x, x'; \tau)$ be the solution of the diffusion equation $(\partial_\tau - \Delta_x)K = 0$ on a space X, with initial condition $K(x, x'; 0) = \delta(x - x')$, where Δ is the Laplace operator on X. It is a function of the geometry of X via its assigned metric. In the resolution interpretation of [37,38], the parameter $\sqrt{\tau}$ and its inverse represent, respectively, the length scale and the resolution at which

a geometry is inspected by a pointwise probe deployed at a spatial point x'. The trace of the heat kernel over all points is denoted as $P(\tau) = \mathrm{Tr}_X K(x, x'; \tau)$ and called "return probability" from the traditional but somewhat problematic interpretation in terms of a diffusing process (see [18,38] for a discussion and resolutions of such problems).

While ordinary diffusion takes place on continuous manifolds, the whole setup, and in particular the definition of the Laplace operator, can be generalized to discrete spaces, like (combinatorial) complexes. This was indeed the subject of [39,40]. The Laplacian on $\mathcal{C}$, as a differential operator acting on a field ϕ_a on the d-cells $c_a \in \mathcal{C}$ (equivalently, on dual vertices), is then [39]

$$-(\Delta_\mathcal{C}\phi)_a = \sum_{b \sim a}(\Delta_\mathcal{C})_{ab}(\phi_a - \phi_b)$$

$$= \frac{1}{V_a^{(d)}}\sum_{b \sim a}\frac{V_{ab}^{(d-1)}}{l_{ab}^\star}(\phi_a - \phi_b), \qquad (7)$$

where the sum runs over d-cells c_b adjacent to c_a, $V_a^{(d)}$ is the volume of the cell c_a, $V_{ab}^{(d-1)}$ is the volume of the common bounding $(d-1)$-cell and $l_{ab}^\star$ is the length of its dual edge. Accordingly, the heat trace on $\mathcal{C}$ is given by a trace $\mathrm{Tr}_\mathcal{C}$ over maps of that field space.

Both the return probability $P(\tau)$ and the Laplacian Δ can be turned into operators $\widehat{P(\tau)}$ and $\hat{\Delta}$ acting on quantum states of geometry. Quantizing the metric-dependent coefficients $(\Delta_\mathcal{C})_{ab}$ which enter in the definition of the discrete Laplacian (7) results in an operator $\hat{\Delta}_\mathcal{C}$ acting on the Hilbert space $\mathcal{H}_\mathcal{C}$ of states on a given complex $\mathcal{C}$ which returns states together with discrete Laplacians.[2] This can be done in different ways, depending on the geometric variables that are most convenient in the specific quantum geometric context that is chosen. It has been discussed in detail in [39]. In general, the resulting expression will be a complicated function of the quantum labels assigned to the

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[1]Ratios $l(j_1)/l(j_2)$ for pairs of quantum numbers j_1, j_2 can be nonrational, so that one should take the integer value (floor function) of $N(j)$. Physically, it is certainly enough to apply Eq. (4) in such an approximative way. Note also that our results are independent of the spacing of quantum numbers in the superposition. Thus, one could as well define the states as sums over only those j's for which $N(j) \in \mathbb{N}$ strictly.

[2]Note that only the coefficients of $\hat{\Delta}_\mathcal{C}$ are quantum operators in the usual sense, i.e., maps from the Hilbert space $\mathcal{H}_\mathcal{C}$ to itself. $\hat{\Delta}_\mathcal{C}$ itself is an operator properly defined only on the coupled Hilbert space of geometry and test fields, which we do not introduce. We do not consider quantum states of test fields, since the relevant object $\widehat{P(\tau)}$ to define the spectral dimension is a functional of pure geometry and, as such, it can eventually be defined as a quantum operator in the strict sense. Let us expand this technical point for the interested reader. As a vector space over complex numbers, any state in a Hilbert space can be expanded in the elements of a complete basis with complex numerical coefficients. Elements in the image of the quantum Laplacian $\hat{\Delta}_\mathcal{C}$ are sums over such a basis, but with coefficients that are discrete Laplacians instead of complex numbers, that is, maps from a functional space on a complex to itself. These elements are obviously not states in $\mathcal{H}$. Still, we can use $\hat{\Delta}_\mathcal{C}$ to define $\widehat{P(\tau)}$ which is the quantum operator acting on pure-geometry states that we are interested in here.

complex, which is however both well-defined and explicitly computable [27].

The operators $\widehat{P(\tau)}$ and $\hat{\Delta}$ on the full Hilbert space $\mathcal{H} = \oplus_{\mathcal{C}} \mathcal{H}_{\mathcal{C}}$ are then defined in terms of the family of orthogonal projections $\pi_{\mathcal{C}} : \mathcal{H} \to \mathcal{H}_{\mathcal{C}}$. In this way, the Laplacian acting on generic quantum states of geometry is the formal sum

$$\hat{\Delta} := \sum_{\mathcal{C}} \pi_{\mathcal{C}} \hat{\Delta}_{\mathcal{C}} \pi_{\mathcal{C}}. \tag{8}$$

With the appropriate notion of a trace $\overline{\mathrm{Tr}} := \sum_{\mathcal{C}} \mathrm{Tr}_{\mathcal{C}} \pi_{\mathcal{C}}$, based on the trace $\mathrm{Tr}_{\mathcal{C}}$ over discrete field space on $\mathcal{C}$ introduced above, the heat trace is then defined as

$$\widehat{P(\tau)} := \overline{\mathrm{Tr}} e^{\tau \hat{\Delta}}. \tag{9}$$

$\widehat{P(\tau)}$ is a map from $\mathcal{H}$ on itself, and thus a quantum operator in the strict sense. Then, the spectral dimension $d_{\mathrm{S}}^{\psi}(\tau)$ of a quantum state of geometry $|\psi\rangle \in \mathcal{H}$ is the scaling of the expectation value of $\widehat{P(\tau)}$ [27]:

$$d_{\mathrm{S}}^{\psi}(\tau) := -2 \frac{\partial}{\partial \ln \tau} \ln \langle \widehat{P(\tau)} \rangle_{\psi}. \tag{10}$$

Note that it depends only on pure geometry, since the relevant operators are acting on pure-geometry states.

For the discrete quantum geometries $|\{j_c\}, \mathcal{C}\rangle$ it is reasonable to assume that they are eigenvectors of $\hat{\Delta}_{\mathcal{C}}$, based on the definition of these labels (1) and on our previous work [27]. On the states (3) that we are interested in here, the heat-trace expectation value is thus

$$\langle \widehat{P(\tau)} \rangle_{\psi} = \sum_{\mathcal{C}} \left(\sum_{j} a_{j,\mathcal{C}}^* \langle j, \mathcal{C} | \right) \left(\sum_{j'} a_{j',\mathcal{C}} \mathrm{Tr}_{\mathcal{C}} e^{\tau \hat{\Delta}_{\mathcal{C}}} | j', \mathcal{C} \rangle \right)$$
$$= \sum_{j,\mathcal{C}} |a_{j,\mathcal{C}}|^2 \mathrm{Tr}_{\mathcal{C}} e^{\tau \langle j, \mathcal{C} | \hat{\Delta}_{\mathcal{C}} | j, \mathcal{C} \rangle}. \tag{11}$$

Some simplifying assumptions are however needed in order to proceed with systematic computations on extended complexes. In the following, we assume that the expectation value of the Laplacian $\hat{\Delta}_{\mathcal{C}}$ scales as

$$\langle j, \mathcal{C} | \hat{\Delta}_{\mathcal{C}} | j, \mathcal{C} \rangle_{ab} \propto l^{-2}(j) (\Delta_{\mathcal{C}})_{ab}, \tag{12}$$

where $\Delta_{\mathcal{C}}$ is the combinatorial Laplacian (7) on the complex $\mathcal{C}$. This assumption is sensible if the Laplacian can be expressed as a function of the volumes (1). A similar ansatz is, in fact, made in [26], although in that work this is not justified on the basis of a detailed analysis of the underlying graph and on the complete expression for the Laplacian, such as the one presented in [39].

We now evaluate the spectral dimension on our superposition states. Under the assumption (12), the expression

for the expectation value of the return probability further simplifies to

$$\langle \widehat{P(\tau)} \rangle_{\psi} \propto \sum_{j,\mathcal{C}} |a_{j,\mathcal{C}}|^2 \mathrm{Tr}_{\mathcal{C}} e^{\tau l^{-2}(j) \Delta_{\mathcal{C}}}. \tag{13}$$

The above expression can be computed most efficiently considering the limit of infinite lattices, for which an analytic expression for the heat trace is available. In [27], we showed that the heat trace on $\mathcal{C}_{\infty} = \mathbb{Z}^d$ is

$$P^{\mathcal{C}_{\infty}}(\tau) = [e^{\tau} I_0(\tau)]^d, \tag{14}$$

where I_0 is the modified Bessel function of the first kind. In the same limit, one can give precise formulas for the contribution to the spectral dimension coming from individual lattices, so that we are in the ideal position to investigate the effect of superpositions of the same. The spectral dimension $d_{\mathrm{S}}^{j,\mathcal{C}_{\infty}}$ on a single state $|j, \mathcal{C}_{\infty}\rangle$ equals d for $\tau \gg l^2(j)$ and vanishes for $\tau \ll l^2(j)$:

$$d_{\mathrm{S}}^{j,\mathcal{C}_{\infty}} \simeq \begin{cases} d, & \tau \gg l^2(j) \\ 0, & \tau \ll l^2(j) \end{cases}. \tag{15}$$

Around the scale $\tau \approx l^2(j)$, there is a peak of approximate height $1.22d$ [27]. We consider these features as discretization artifacts, and we conclude that no real dimensional flow is seen for individual states in the superposition [27].

Therefore, we are prompted to extend the search for quantum geometry states that would show true signs of dimensional flow to superposition states, to which we now move. Using the above solution, the spectral dimension of $|V_0, j_{\min}, j_{\max}\rangle$, Eq. (6) in the limit $N(j_{\min}) \to \infty$, is given by the scaling of

$$\langle \widehat{P(\tau)} \rangle_{V_0, j_{\min}, j_{\max}} \propto \sum_{j=j_{\min}}^{j_{\max}} |a_j|^2 \{ e^{l^{-2}(j)\tau} I_0[l^{-2}(j)\tau] \}^d. \tag{16}$$

For asymptotic power-law spectra

$$l(j) \simeq j^{\beta}, \tag{17}$$

where $\beta > 0$ as usual in LQG [see Eq. (2)], we have done numerical calculations for various classes of coefficient functions a_j and various values of spatial dimension d and cutoffs $j_{\max}$. In all the examples presented here, we use $j_{\min} = 1$; calculations with lower cutoffs of the same order (e.g., $j_{\min} = 1/2$) give similar results. Notice also that the same finite minimal value for the geometric spectra could be obtained in correspondence with a quantum label $j = 0$, for choices of quantization map that give a nonzero value for C in Eq. (2).

The first general class of coefficients to be considered is of power-law functions,

$$a_j \propto j^\gamma. \tag{18}$$

Defining the parameter

$$\alpha := -\frac{2\gamma + 1}{\beta}, \tag{19}$$

the spectral dimension of the state under consideration has the following behavior depending on the range of values of α.

(1) For $0 < \alpha < d$:

(a) In the IR, i.e., for large length scales $\tau \gg l^2(j_{\max})$, $d_S(\tau) = d$ (Fig. 1). This is of course a consistency check for the validity of the formalism, since at large scales we recover the topological dimension of the space the quantum states are supposed to represent. It is however already a nontrivial test, as identifying quantum states with the right semi-classical continuum properties at large scales is no small task in background-independent quantum gravity.

(b) Below the smallest lattice scale, i.e., for $\tau \ll l^2(j_{\min})$, $d_S(\tau) = 0$. This is the usual discreteness effect which we find also for individual lattice-based states (15), which remains at the Planck scale for discrete spectra induced by holonomies valued in compact groups [33,35,36]. For noncompact groups, spectra are typically continuous and no volume discreteness effect at Planck scale occurs, as $j_{\min} \to 0$ [34].

(c) Between these scales, there is a plateau with value $d_S(\tau) = \alpha$ (Fig. 2). This plateau indicates a regime in which the effective dimension is physically smaller than the topological one, and thus a proper dimensional flow. In the light of our previous results [27], which, as discussed, were performed on the same type of quantum states and in the same formalism, but without considering large superpositions of lattice structures, we regard this as a truly quantum effect stemming from the superposition of states $|j, \mathcal{C}\rangle$ with geometric spectra on different scales and based on complexes of different size. It is interesting that, at such intermediate scales, the effective dimension is independent of the topological one (again, provided $d > \alpha$) and depends instead only on the specific choice of quantum states.

(d) In particular, for infinite superpositions $(j_{\max} \to \infty)$ this plateau takes the value α and extends indefinitely (Fig. 3). Formally, one can express this behavior by

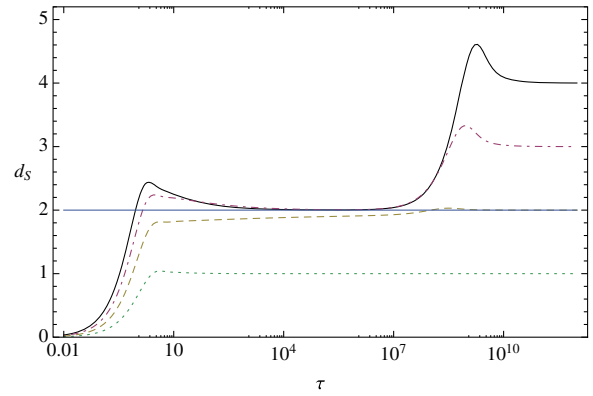$$\Delta\tau|_{d_S = \alpha} \xrightarrow[j_{\max} \to \infty]{} \infty. \tag{20}$$

FIG. 1 (color online). Spectral dimension of a superposition with $\alpha = 2$ in $d = 1, 2, 3, 4$ (dotted, dashed, dot-dashed, solid curve) with cutoff $j_{\max} = 10^4 d$.
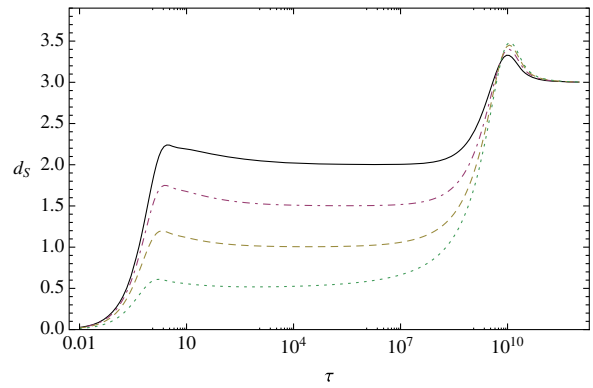

FIG. 2 (color online). Spectral dimension of superposition states with $\alpha = 1/2, 1, 3/2, 2$ (dotted, dashed, dot-dashed, solid curve) in $d = 3$ with cutoff $j_{\max} = 10^5$.
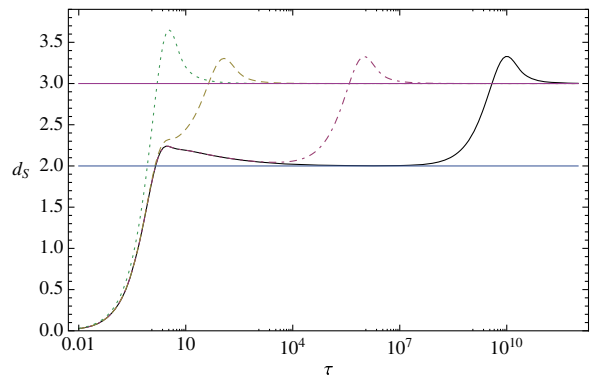

FIG. 3 (color online). Spectral dimension of superpositions with $\alpha = 2$ in $d = 3$ for cutoffs $j_{\max} = 1, 10, 10^3, 10^5$ (dotted, dashed, dot-dashed, solid curve).

Notice that this only means that the topological dimension d is obtained further away at large τ. Physically, one never takes the infinite limit in practice: for large spin labels, the plateau is long but has finite extension $\Delta\tau$.
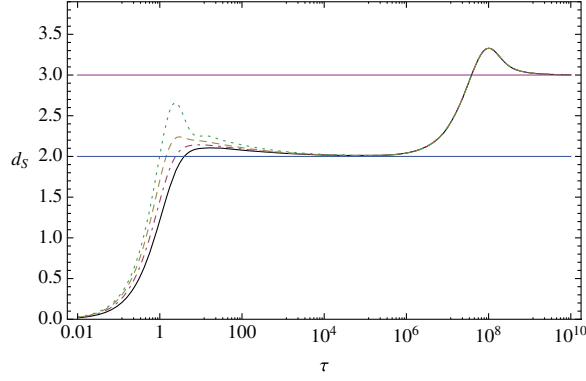
FIG. 4 (color online). Spectral dimension of a superposition with $\alpha = 2$ in $d = 3$ summing over positive $j \in \frac{1}{q}\mathbb{N}$ up to $j_{\max} = 10^4$ for $q = 1/2, 1, 2, 10$ (dotted, dashed, dot-dashed, solid curve).

(e) Moreover, these results are independent of the spacing of the quantum labels j. Summing over $j \in \frac{1}{q}\mathbb{N}$ for some $q \in \mathbb{Q}$ slightly changes the results only at the scale $l(j_{\min})$ (Fig. 4). Therefore, neither the IR nor the UV regime depends on the spacing of the state label j. The numerical calculations show, in particular, that this should also be true in the limit $q \to \infty$, i.e., for positive real j.

(2) For $\alpha < 0$, no superposition effect occurs and the profile of the spectral dimension equals approximately the one of the single state $|j_{\max}, \mathcal{C}_\infty\rangle$, Eq. (15):

$$d_{\mathrm{S}}^{V_0, j_{\min}, j_{\max}}(\tau) \approx d_{\mathrm{S}}^{j_{\max}, \mathcal{C}_\infty}(\tau). \qquad (21)$$

This is a numerical result, for which we lack, at present, a complete analytical or physical understanding. Nevertheless, we can offer an intuitive explanation. We saw that, in the range $0 < \alpha < d$, α is the spectral dimension of the state at sufficiently small scales. On a continuous medium, the case $\alpha < 0$ would correspond to an unphysical one with negative dimension. This situation is meaningless both in the conventional diffusion interpretation of the spectral dimension (where the probe would do "less than not propagating") and in the resolution interpretation of [37,38]. In the latter, the return probability $P(\tau) \sim (\sqrt{\tau})^{-d_{\mathrm{S}}} \sim \ell^{-d_{\mathrm{S}}} \sim (\mathrm{res})^{d_{\mathrm{S}}}$ is the probability to find the probe anywhere when the geometry is probed at scales ℓ, i.e., with resolution $1/\ell$. For positive d_{S}, this probability decreases with the resolution: if $1/\ell$ is too small, there is a chance that we do not see the probe at all. On the other hand, a negative d_{S} implies that the coarser the probe, the greater the chance to find it somewhere. In our case, this pathological behavior is screened by discreteness effects and d_{S} is saturated by the lattice with

labels $j_{\max}$. The resolution interpretation coupled with the LQG interpretation of the spin labels helps in explaining Eq. (15): coarser resolutions can effectively probe only large volumes and the largest volume available for the states (16) is at the upper cutoff $j_{\max}$. Under such conditions, it is natural to expect that the lattice structure completely dominates the profile of the spectral dimension.

(3) For $\alpha > d$, no superposition effect occurs and the profile of the spectral dimension equals approximately the one of the single state $|j_{\min}, \mathcal{C}_\infty\rangle$,

$$d_{\mathrm{S}}^{V_0, j_{\min}, j_{\max}}(\tau) \approx d_{\mathrm{S}}^{j_{\min}, \mathcal{C}_\infty}(\tau). \qquad (22)$$

In the continuum limit, $\alpha > d$ would imply a spectral dimension larger than the ambient space. Similarly to the previous case, one has both the diffusion and the resolution interpretation at hand. In the conventional diffusion interpretation of the spectral dimension, the case $d_{\mathrm{S}} > d$ may be regarded as physical: the probe effectively sees more than d dimensions and tends to superdiffuse. In the resolution interpretation, the probability of finding the probe somewhere grows more steeply than for the normal case (Brownian motion) and probes with large resolution (small scales ℓ) become even more effective. However, in the present quantum framework there is a limit to which one can probe the microscopic structure of geometry: volume spectra are discrete with minimum eigenvalue determined by $j_{\min}$. Again, the variation of the spectral dimension is dominated by lattice effects, this time governed by the lower cutoff in the spin labels.

A partial understanding of the results with $0 < \alpha < d$, in particular concerning the dependence of the UV value of d_{S} on the powers β and γ in (19), is provided by the following rewriting of the heat trace (16). A redefinition of variables $k(j) := l^{-\alpha}(j)$ demands a change of summation-integration measure by

$$\frac{dk}{dj} = \frac{d}{dj} l^{-\alpha}(j) = -\alpha \frac{d \ln l(j)}{dj} l^{-\alpha-1}(j). \qquad (23)$$

In particular, for the power-law spectra (17) and the definition of α (19)

$$\frac{dk}{dj} = -\alpha\beta j^{-\alpha\beta-1} \overset{(19)}{=} (2\gamma + 1) j^{2\gamma} \qquad (24)$$

which is proportional to $|a_j|^2$ for the power-law coefficients (18). Thus, the heat trace on these superpositions is a uniformly weighted sum in the k-variable [over the range corresponding to (16)]:

$$\langle\hat{P}(\tau)\rangle_{V_0,j_{\min},j_{\max}} \propto \sum_k [e^{-k^{2/\alpha}\tau}I_0(k^{2/\alpha}\tau)]^d. \qquad (25)$$

Therefore, genuine dimensional flow comes from a subtle balancing of d and α in this expression, while a negative α yields just a dominant $k_{\max} = k(j_{\max})$ contribution in the sum. Indeed, we have also calculated the spectral dimension directly from (25) for various values of d, α and summing ranges of integer k's, obtaining qualitatively similar results as discussed above for (16). As a consequence, dimensional flow has some dependence on the form of the spectrum (17) but only in combination with appropriate superposition coefficients.

Still maintaining the power-law spectrum (17) (which is the most reasonable assumption, consistent with known results in LQG and related approaches), we have calculated the spectral dimension for various other classes of coefficient functions. In most cases, there are no surprising results.

(a) For example, exponential coefficients $a_j \propto e^{aj}$ let either the maximal state $j_{\max}$ dominate when $a > 0$, or the minimal one $j_{\min}$ when $a < 0$. (b) Gaussian coefficients, on the other hand, result in a dominance of the j_0 at which they are peaked. (c) Trigonometric functions add some oscillations to $d_S^{j_{\max},\mathcal{C}_\infty}$ in the intermediate regime, depending on the relation of the periods to the spacing of j in the sum. In all these cases, therefore, the overall behavior of the spectral dimension is the same as that found for coefficients given by simple powers.

More interesting is the case of coefficients which are linear combinations of power functions in j. Then one finds, for their asymptotic behavior $a_j \sim j^\gamma$, the same effect as for power-law coefficients. In particular, if there are several regimes with different approximate scaling $\gamma_1, \gamma_2, \ldots$, one obtains plateaux in the spectral dimension plot of different values $\alpha_1, \alpha_2, \ldots$ accordingly. An example is shown in Fig. 5. This effect coincides, both in its qualitative shape and origin, to the one obtained in the multiscale generalization of the diffusion equation with different powers of the Laplacian [41]. In general, all coefficient functions with an approximate power-law behavior in some regime give rise to dimensional flow at those scales. Details such as the value of $j_{\min}$ and the spacing in j are not relevant for the value of the spectral dimension in these intermediate regimes, in agreement with the discussion in [41] on the role of regularization parameters in the profile of d_S. The details of regularization schemes are nonphysical and affect only transient regimes in $d_S(\tau)$, not the value of the plateaux.

B. Walk dimension of superpositions

The spectral dimension is only one of the possible dimensional observables. Our strategy is well suited to analyze other observables as well, and it is interesting to do
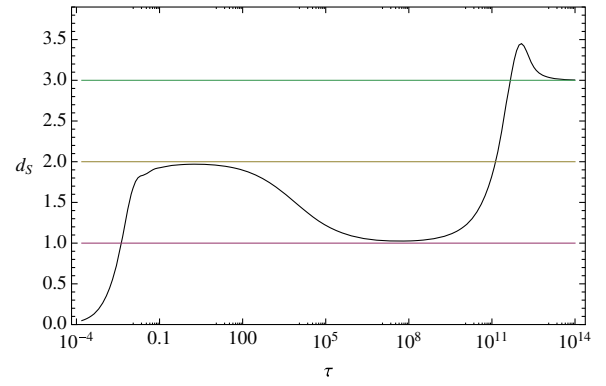

FIG. 5 (color online). Spectral dimension of superpositions with coefficients $|c_j|^2 = j^{-4} + 200j^{-7}$ summing from $j_{\min} = 1/2$ to $j_{\max} = 200$ for $d = 3$ and $\beta = 3$ (to be able to numerically cover enough scales with a feasible number of states in the sum). According to (19), two different UV regimes with dimension $d_S \approx 2$ and then $d_S \approx 1$ can be observed.

so because there exist several relations among them, in classical and continuum spaces. Only a detailed analysis of their combined behavior can give solid indications on the nature of the quantum geometries corresponding to quantum gravity states.

A closely related observable is the walk dimension d_W. It is defined via the scaling of the mean square displacement

$$\langle X^2 \rangle_y(\tau) = \int dx |x-y|^2 K(x,y;\tau) \propto \tau^{2/d_W}, \qquad (26)$$

that is

$$d_W(\tau) := 2\left(\frac{\partial \ln\langle X^2\rangle_y}{\partial \ln\tau}\right)^{-1}. \qquad (27)$$

In the case of the d-dimensional hypercubic lattice $\mathcal{C}_\infty$, we can choose the origin $y = 0$, so that

$$\langle X^2\rangle_0^{\mathcal{C}_\infty}(\tau) = \sum_{\vec{n}\in\mathbb{Z}^d} |\vec{n}|^2 K(\vec{n},0;\tau) \qquad (28)$$

$$= \sum_{\vec{n}\in\mathbb{Z}^d}\left(\sum_{j=1}^d n_j^2\right)e^{-\tau}\prod_{k=1}^d I_{n_k}(\tau). \qquad (29)$$

This can be evaluated using standard relations of the Bessel functions I_n,

$$\langle X^2\rangle_0^{\mathcal{C}_\infty}(\tau) \propto e^{-d\tau}\sum_{j=1}^d\sum_{\vec{n}\in\mathbb{Z}^d} n_j^2\prod_{k=1}^d I_{n_k}(\tau) \qquad (30)$$

$$= e^{-d\tau} \sum_{j=1}^{d} \left[\sum_{n_j \in \mathbb{Z}^d} n_j^2 I_{n_j}(\tau) \right]$$

$$\times \prod_{k \neq j}^{d} \left[\sum_{n_k \in \mathbb{Z}} I_{n_k}(\tau) \right]$$

$$= e^{-d\tau} d \left[\frac{\tau}{2} \sum_{n \in \mathbb{Z}} I_{n-1}(\tau) + I_{n+1}(\tau) \right] (e^{\tau})^{d-1}$$

$$= d\tau. \tag{31}$$

Thus, the walk dimension on the lattice is

$$d_{\mathrm{W}}^{\mathcal{C}_\infty}(\tau) = 2, \tag{32}$$

as in the continuum.

Quantum superpositions $|V_0, j_{\min}, j_{\max}\rangle$ are characterized by the Laplacian (12), so that along the same lines as (16) one has

$$\langle\langle X^2 \rangle_0(\tau)\rangle_{V_0, j_{\min}, j_{\max}} = \sum_{j=j_{\min}}^{j_{\max}} |a_j|^2 \langle X^2 \rangle_0^{\mathcal{C}_\infty}[l^{-2}(j)\tau]$$

$$= d \sum_{j=j_{\min}}^{j_{\max}} |a_j|^2 l^{-2}(j)\tau \tag{33}$$

$$\propto \tau. \tag{34}$$

Therefore, also for quantum superpositions the scaling of the mean square displacement yields the standard result

$$d_{\mathrm{W}}^{V_0, j_{\min}, j_{\max}} = 2, \tag{35}$$

independent of the form of the coefficients a_j. Notice that the dependence on the topological dimension in Eq. (33) is only through a proportionality coefficient. Therefore, Eq. (35) is valid both for space and spacetime.

C. Hausdorff dimension of superpositions

The Hausdorff dimension of a quantum state is defined in terms of the scaling of the expectation value of the volume $V(R)$ of a ball with radius R:

$$d_{\mathrm{H}}^{\psi}(R) := \frac{\partial \ln\langle V(R)\rangle_\psi}{\partial \ln R}, \tag{36}$$

which can be further expanded like the spectral dimension (11). Using the graph distance and measuring R in units of the lattice spacing, the volume on the lattice $\mathcal{C}_\infty$ is

$$V^{\mathcal{C}_\infty}(R) = 2^d \binom{R+d-1}{d} = \frac{2^d}{d!} \prod_{n=0}^{d-1}(R+n), \tag{37}$$

yielding the Hausdorff dimension

$$d_{\mathrm{H}}^{\mathcal{C}_\infty}(R) = R \sum_{n=0}^{d-1} \frac{1}{R+n} = R[\psi(R+d) - \psi(R)], \tag{38}$$

where ψ is the digamma function. At large scales, d_{H} approaches the topological dimension d, while at small scales it tends to 1:

$$d_{\mathrm{H}}^{\mathcal{C}_\infty} \simeq \begin{cases} d, & R \gg 1 \\ 1, & R \ll 1 \end{cases}. \tag{39}$$

On discrete quantum geometries $|\{j_c\}, \mathcal{C}\rangle$, we define the quantum analogue of $V(R)$ as follows. Let $v_0 \in \mathcal{C}$ be a given vertex in the complex and consider the subcomplex $\mathcal{C}_{v_0} \subset \mathcal{C}$ of all vertices v which have an expectation value of their distance to v_0 no larger than the radius,

$$\langle\{j_c\}, \mathcal{C}|\hat{L}_{vv_0}|\{j_c\}, \mathcal{C}\rangle \leq R, \tag{40}$$

where the expectation value of $\hat{L}_{vv_0}$ is the minimum of lengths derived from the sum of edge lengths of possible (combinatorial) paths between v and v_0.

The expectation value of the volume of this subcomplex $|\{j_c\}, \mathcal{C}\rangle$ is $\sum_{v \in \mathcal{C}_{v_0}} \langle V_v \rangle_{\{j_c\}, \mathcal{C}}$. To obtain the desired observable, one must average over all possible centers v_0:

$$\langle\{j_c\}, \mathcal{C}|V(R)|\{j_c\}, \mathcal{C}\rangle = \sum_{v_0 \in \mathcal{C}} \sum_{v \in \mathcal{C}_{v_0}} \langle\{j_c\}, \mathcal{C}|V_v|\{j_c\}, \mathcal{C}\rangle. \tag{41}$$

On the uniform hypercubic lattice states $|j, \mathcal{C}\rangle$, however, the sum over center vertices v_0 is not necessary due to translation invariance and because of the local volumes being all equal, $\langle j, \mathcal{C}|V_v|j, \mathcal{C}\rangle \propto l^d(j)$ for all $v \in \mathcal{C}_\infty$.

Similarly, on $|j, \mathcal{C}\rangle$ the condition (40) simplifies to

$$\langle j, \mathcal{C}|\hat{L}_{vv_0}|j, \mathcal{C}\rangle \propto l(j)N_{vv_0} \tag{42}$$

where now N_{vv_0} is the minimal number of edges in a path from v to v_0.

Therefore, the evaluation of $V(R)$ on $|j, \mathcal{C}\rangle$ can be expressed in terms of $V^{\mathcal{C}_\infty}(R)$ as

$$\langle j, \mathcal{C}|V(R)|j, \mathcal{C}\rangle \propto l^d(j)V^{\mathcal{C}_\infty}[R/l(j)]$$

$$\propto l^d(j) \prod_{n=0}^{d-1}[R/l(j) + n]. \tag{43}$$

As for the spectral dimension (16), this gives a nontrivial expectation value for generic superposition states:

$$\langle V(R)\rangle_{V_0, j_{\min}, j_{\max}} \propto \sum_{j=j_{\min}}^{j_{\max}} |a_j|^2 l^d(j) \prod_{n=0}^{d-1}[R/l(j) + n]. \tag{44}$$
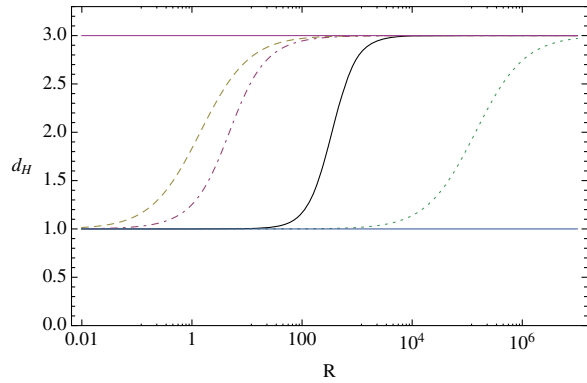
FIG. 6 (color online). Hausdorff dimension d_H of a superposition with $\alpha = 1, 2$ (solid and dash-dotted curve) in $d = 3$ summing up to $j_{max} = 10^5$, compared to d_H on single states $|1, \mathcal{C}_\infty\rangle$ (dashed curve) and $|j_{max}, \mathcal{C}_\infty\rangle$ (dotted curve).

Nevertheless, numerical calculations on the same classes of states as investigated above for the spectral dimension show qualitatively similar results to the Hausdorff dimension $d_H^{j,\mathcal{C}_\infty}$ on single states $|j, \mathcal{C}\rangle$ (Fig. 6). That is, in all instances there are plateaux as in the pure lattice case (37). Only the scale and steepness of the flow between these plateaux is modified. For example, for power-law coefficients (18) the falloff is much steeper and occurs, as α increases, closer to the scale as in the case of the single state $|j_{min}, \mathcal{C}_\infty\rangle$.

IV. DISCUSSION

Our calculations have shown that a flow in the spectral dimension occurs in quantum gravity, at least for a specific class of superpositions of regular (both from the combinatorial perspective and for what concerns the assignment of additional quantum labels) quantum states of geometry. These quantum states, although restricted by the regularity assumption, are exactly of the type appearing in the related quantum-gravity formalisms of loop quantum gravity, spinfoam models and group field theory, but can also simply be seen as quantum states of lattice quantum gravity, in the spirit of quantum Regge calculus.

On the other hand, we see no dimensional flow due to quantum effects in the Hausdorff and walk dimension. This conclusion is based on the interpretation, which we maintained throughout the paper, that the flow of a geometric indicator is an artifact of discretization effects whenever it approximately coincides with the flow for lattices. We will come back to this point.

Let us comment a bit further about our results from the point of view of loop quantum gravity.

Under the assumptions made for the action of the quantum Laplacian on the states (a very simple scaling behavior), an important example of states of the type we have studied are kinematical states in LQG where length ($d + 1 = 3$) or area and volume operators ($d + 1 = 4$) are diagonalized by spin-network states. In this sense, we have

identified a class of LQG states with a dimensional flow. More precisely, for any $0 < \alpha < d$ there is a class of states in the kinematical Hilbert space with a dimensional flow from the spatial topological dimension d in the IR to a smaller value α in the UV. The UV value depends on the exact superposition considered but not on the topological dimension.

This result is in contrast with earlier arguments in LQG [26]. There, the author argues for evidence of dimensional flow for individual spin-network states (thus, for a given graph or complex), and the same result is claimed in [42,43] for simple spin-network states with additional weights given by a 1-vertex spin foam (thus, not yet in a truly dynamical context). The starting point in [26] is an assumption about the scaling of the expectation value of the Laplacian, very similar to (12). The essential part of the argument is then the further assumption that the momenta p of the scalar field defining the spectral dimension are directly related to a length scale set by the quantum numbers as $p \propto 1/\sqrt{j}$. The scaling of the Laplacian in j is then translated into a modified dispersion relation in p and the result depends on the precise form of the spectrum (2) with $C = 0$.

In our case, no further assumption beyond (12) is made. Calculations are based on the momenta of the scalar field on the lattice-based geometry, that is, the spectrum of the Laplacian, but the spectral dimension is computed directly as a quantum geometric observable evaluated on quantum states. As recalled already above, in a previous work using this more direct approach [27] we have found no effects on the spectral dimension for individual quantum-geometry states of LQG based on given graphs or complexes. On the other hand, the genuine dimensional flow that we have encountered here for the states $|V_0, j_{min}, j_{max}\rangle$ is crucially related to the superposition of spin-network states also with respect to the underlying combinatorial structures, and it is not solely the result of the discreteness of geometric spectra. In this deeper sense, dimensional flow can indeed be seen as an effect of quantum discreteness of geometry.

We are also in a position to characterize the change of dimensionality more precisely than a generic "flow" of geometry. Quite often in the literature of quantum gravity, dimensional flow has been advertised as spacetime being "fractal." However, strictly speaking not all sets with varying dimensionality are fractals. Although no unique operational and rigorous definition of fractal exists, one property all fractals generally possess is a special relation among the spectral dimension d_S, the Hausdorff dimension d_H and the walk dimension d_W:

$$d_H = \frac{d_W}{2} d_S. \tag{45}$$

On the hypercubic lattice superpositions that we have considered, $d_W = 2$ and the above relation simplifies to

$d_H = d_S$. This is trivially obtained in the IR regime, where both dimensions take the value of the topological dimension. In the UV regime above the lattice scale (recall that below such scale any scaling effect is arguably unphysical), the Hausdorff dimension takes the classical value $d_H^{\mathcal{C}_\infty} \simeq 1$. Thus, (45) is only obeyed in the case of a scaling $\alpha = 1$ such that also the spectral dimension takes this value. Only then can one call the quantum superposition $|V_0, j_{\min}, j_{\max}\rangle$ an effective one-dimensional fractal. This is indeed a perfectly allowed choice of quantum states and we can conclude that we have identified a particular class of quantum geometries that corresponds, by all appearances, to a fractal quantum space.

However, we should mention a caveat here. For these geometries to be safely regarded as fractals, the origin of the dimensional flow should be the same in the left- and right-hand side of Eq. (45), which may not be the case for us: the left-hand side flows due to discreteness effects, while the right-hand side flows due to physical quantum effects. This situation might suggest either that we should not place particular significance in the fulfillment of Eq. (45) or that our discrimination between discreteness artifacts and physical effects is somewhat too strong and should be revised. We do not attempt to solve this mild conceptual issue here, which is harmless for our main results. Still, it will deserve further attention.

Interestingly, the geometry with $\alpha = 1$ is also the only one where the spectral dimension of spacetime reaches the value $D_S = d_S + 1 \simeq 2$ so often commented upon in the literature of quantum gravity. Its appearance across independent approaches such as causal dynamical triangulations, asymptotic safety, Hořava-Lifshitz gravity and others [10–12] triggered the suspicion that this "magic number" was a universal characteristic of frameworks with good ultraviolet properties or, in other words, that a two-dimensional limit of the spectral dimension was tightly related to the renormalizability or finiteness of quantum gravity. By now, it has become clear that this is not the case in general, as there exist counterexamples of nonlocal field theories with good renormalization properties [44] with $D_S \neq 2$ in the UV [45], as well as of local theories whose renormalization properties are not at all improved by dimensional flow [46]. Here we provide another instance pointing towards the same conclusion: the value of d_S is governed by a choice of states which, by itself, is not (sufficiently) connected with the dynamical UV properties of the underlying full theory.

V. CONCLUSIONS

We have investigated the effective structure of quantum superpositions of regular (hypercubic and homogeneous in label assignment) states of quantum geometry.

It is possible to identify states with a flow of the spectral dimension to a dimension α in the UV, provided the superposition includes fine enough combinatorial structures and a large enough number of (kinematical) degrees of freedom of quantum geometry, and a particular set of expansion coefficients (18) related to α (19).

For the Hausdorff and walk dimension, no physical quantum effects are observed, although discreteness effects do alter the value of the Hausdorff dimension across scales. A fractal structure in the strict sense, i.e., where (45) relating the three dimensions is fulfilled also in the UV, is realized in the case $\alpha = 1$.

In particular, these results provide evidence for a dimensional flow in a certain class of kinematical LQG states, also available in the spin-foam and group field theory context.

The results at hand can be further generalized in various directions as well as refined within individual theories of quantum gravity. In parallel, it becomes feasible to explore the phenomenological consequences of the discovered dimensional flow and (when applicable) fractal nature of quantum space as a direct effect of the full theory. This possibility is especially interesting in a quantum cosmological context, where a change of dimensionality can bear its imprint in the early stage of cosmic evolution [47–49].

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