

# Cosmological Signatures of Anisotropic Spatial Curvature

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**Abstract.** If one is willing to give up the cherished hypothesis of spatial isotropy, many interesting cosmological models can be developed beyond the simple anisotropically expanding scenarios. One interesting possibility is presented by *shear-free* models in which the anisotropy emerges at the level of the curvature of the homogeneous spatial sections, whereas the expansion is dictated by a single scale factor. We show that such models represent viable alternatives to describe the large-scale structure of the inflationary universe, leading to a kinematically equivalent Sachs-Wolfe effect. Through the definition of a complete set of spatial eigenfunctions we compute the two-point correlation function of scalar perturbations in these models. In addition, we show how such scenarios would modify the spectrum of the CMB assuming that the observations take place in a small patch of a universe with anisotropic curvature.

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## 1 Introduction

The combination of observational data from modern galaxy surveys with those from the Cosmic Microwave Background (CMB) radiation strongly suggests that, on scales around 100 Mpc and above, the spatial distribution of the cosmic web is homogeneous and isotropic. This highly symmetric description of the universe comprises what is known as the cosmological principle, and is mathematically encoded in the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime metric.

Besides being backed up by observational data, the use of the FLRW metric has a technical advantage. Since the only free parameter in this metric is a function of time, background dynamical equations become ordinary differential equations, which can be solved by means of known analytical or semi-analytical techniques. More important, though, is the fact that, in this description of the spacetime, “pictures” of the three-dimensional spatial section at different times are copies of one another, differing only by a global conformal factor. If one further expands cosmological observables in terms of the eigenfunctions of the spatial Laplace-Beltrami operator, the aforementioned simplification implies that the eigenmodes of physical quantities will not couple through the evolution of the universe, as far

as the perturbative regime is maintained, and then one can trace their dynamics individually by means of linear differential equations. The easiness and versatility of this program has enormously boosted the field of cosmology in the past decades, culminating with the so-called standard cosmological model (the  $\Lambda$ CDM model), in which the qualitative features of the large-scale universe can be understood by means of only six free parameters.

As comfortable as this description might seem, it is important to keep in mind that the cosmological principle constitutes a set of working hypotheses, and, in reality, a particularly critical set, given the intrinsic human limitations to perform observations at cosmological scales. In other words, one might be careful to not let arguments of simplicity prevent the discovery of new cosmological phenomena.

The necessity of scrutinizing the symmetry hypothesis of the standard cosmological model also has the important role of testing the robustness of cosmological observations, which could otherwise be biased by *a posteriori* assumptions. As a matter of fact, while the observed approximate isotropy of the CMB is in agreement with the cosmological principle, this does not mean that different symmetry principles are ruled out by it and, in fact, different principles are known to be compatible with the CMB observations. Indeed, in the past few years several authors have considered the possibility that the spectrum of CMB could be fitted by large void models in which we would be located near its center [1–4], in clear contrast to the homogeneity hypothesis. The spectrum of CMB fluctuations has also been contrasted with spatially homogeneous but anisotropic cosmological models, either as a mean of constraining the impact of spatial anisotropy on CMB data [5–7], or as an attempt to explain known statistical anomalies of the CMB [8–11]. Of these two different routes, the latter is the one which we plan to follow in this work. That is, we assume spatial homogeneity of the universe and investigate the impact of spatial anisotropies on the large-angle spectrum of CMB temperature fluctuations.

The simplest example of an anisotropic spacetime corresponds to a generalization of the FLRW metric in which each spatial direction expands according to its own scale factor. Such geometries are known as Bianchi I metrics. Given that the CMB signal points to a great degree of spatial isotropy of the early universe, Bianchi I metrics can be deployed in two different ways. For example, one could suppose that the onset of inflation has taken place in an anisotropic spacetime, and ask for its impact on primordial quantum fluctuations re-entering the horizon at the time of CMB formation [12]. On a different context, the possibility of a late-time anisotropization of the spacetime could leave the CMB spectrum essentially unchanged, while affecting both the growth rate of structures [13] as well as weak-lensing observables [14, 15]. As in the case of FLRW geometries, Bianchi I metrics also enjoy the simplicity of having unperturbed quantities that obey ordinary differential equations. However, the anisotropic expansion of the spacetime inevitably leads to a coupling of Fourier modes, which renders the use of this metric quite ingenious at the technical level.

However, once we are willing to abandon the isotropy hypothesis of the cosmological principle, much richer models can be constructed besides the Bianchi I solution [16–20]. In fact, it has been demonstrated in references [18, 21–23] that the anisotropy of the universe does not need to emerge from its dynamical expansion; instead, it could result from the curvature of spatial sections, while having the expansion controlled by one scale factor, at the cost of having an imperfect fluid sourcing the anisotropic curvature. This idea has been successfully applied in the context of late-time anisotropies in references [24–26]. From the phenomenological point of view, this possibility offers two main advantages. First, since the background metric is controlled by a single scale factor, the redshift of electromagnetic

radiation is exactly isotropic, which means that the CMB remains exactly isotropic at the background level. Second, the fact that there is a single scale factor implies that pictures of the universe at different times are also copies of each other, exactly as in the FLRW description. Then, in the perturbative regime, eigenmodes of physical quantities will also evolve independently of one another, something which greatly facilitates the computation of cosmological observables.

In this work we consider two spacetime metrics presenting isotropic expansion but anisotropic spatial curvature. These are the metrics from the Bianchi type III and Kantowski-Sachs families. Following the formalism of linear and gauge-invariant perturbations introduced in [27], we evaluate the impact of these metrics on the large-angle spectrum of CMB fluctuations, assuming that the particle horizon during CMB formation is much smaller than the curvature scale of these spaces. After introducing an explicit set of spatial eigenfunctions in Section 2, we show in Section 3 that, provided that the universe has gone an early period of exponential expansion, the dynamics of scalar perturbations will be described by one single gravitational potential  $\Phi$ . From this result, an important corollary follows: given that large-angle CMB fluctuations result from the geodesics of photons in a perturbed spacetime, the Sachs-Wolfe effect in the considered geometries will have exactly the same functional form as in the FLRW case. This is demonstrated in Section 4, where, using the explicit eigenfunctions found in Section 2, we compute the temperature two-point correlation function in these spaces, as well as their perturbative series around the isotropic two-point correlation function in the limit of large curvature radius. The effect of anisotropic curvature leads to off-diagonal multipolar correlations in the spectrum of the CMB which extends to a large multipolar range. This signature is investigated numerically in Section 5. We conclude with some remarks and perspectives of further developments in Section 6.

## 2 Geometry and spatial eigenfunctions

We start this section by introducing the main mathematical and technical tools which will be needed later on to compute the angular power spectrum of the CMB.

### 2.1 Anisotropy through spatial curvature

In this work we are interested in spatially homogeneous but *anisotropic* solutions to Einstein equations which nonetheless display an *isotropic* expansion, i.e., solutions whose dynamics is described by a single scale factor. Rather, the anisotropy of these solutions arises from the spatial curvature of the sections of constant comoving time, curvature which is not the same in all directions. This can be achieved by considering spatial sections which are the Cartesian product of curved subspaces. Since one-dimensional manifolds are trivially flat, the only possibility left for a three-dimensional manifold to present anisotropic spatial curvature is to be the product of a two-dimensional and a one-dimensional manifolds. In the case of maximally symmetric two-dimensional manifolds, and again apart from trivial flat cases, we only have to consider the possibility of a two-sphere or a bi-dimensional pseudo-sphere. Therefore, the spatial topology turns out to be  $\mathbb{S}^2 \times \mathbb{R}$  or  $\mathbb{H}^2 \times \mathbb{R}$ , respectively, accepting the topology of the real line for the third spatial dimension. The metric of the spaces that we are going to consider is thus conformally (ultra-)static and can be written as

$$ds^2 = a^2(\eta) \left[ -d\eta^2 + \gamma_{ab} dx^a dx^b + dz^2 \right], \quad (2.1)$$

where  $\{a, b\}$  represent coordinates in the two-dimensional subspace and  $z$  is the coordinate of the real line. For future reference it will prove convenient to rewrite the line element in cylindrical coordinates

$$ds^2 = a^2(\eta) [-d\eta^2 + d\rho^2 + S^2(\rho)d\varphi^2 + dz^2] , \quad (2.2)$$

where the function  $S(\rho)$  is given by

$$S^2(\rho) = \begin{cases} \rho^2, & (\mathbb{R}^3) \\ \sinh^2 \rho, & (\mathbb{H}^2 \times \mathbb{R}) \\ \sin^2 \rho. & (\mathbb{S}^2 \times \mathbb{R}) \end{cases} \quad (2.3)$$

and where, to facilitate comparison with standard results, we have included the flat FLRW case ( $\mathbb{R}^3$ ) as well. Furthermore, note that while  $z \in (-\infty, +\infty)$  and  $\varphi \in [0, 2\pi)$  in all three cases, the variable  $\rho$  belongs to the interval  $[0, +\infty)$  in the  $\mathbb{R}^3$  and  $\mathbb{H}^2 \times \mathbb{R}$  cases, while  $\rho \in [0, \pi)$  for  $\mathbb{S}^2 \times \mathbb{R}$ . Moreover, note that we can always recover the flat space limit by taking  $\rho \ll 1$ , which gives

$$S^2(\rho) \sim \rho^2 .$$

In the cosmology literature, metrics described by either the  $\mathbb{H}^2 \times \mathbb{R}$  or the  $\mathbb{S}^2 \times \mathbb{R}$  spatial slices are known as Bianchi type III (BIII) and Kantowski-Sachs (KS) solutions, respectively.

## 2.2 Spatial eigenfunctions

In order to compute correlation functions between physical observables (for instance, the gravitational potential), it is very convenient to deal with their spatial dependence by expanding them in modes that decouple in the dynamical field equations. These field equations are typically of a generalized Klein-Gordon type. Hence, for the case of scalar observables, we will first need to explicitly find and normalize (with respect to the inner product given by the volume element of the spatial sections) a complete set of scalar modes which are solutions of the eigenvalue problem

$$\nabla^2 \mathcal{Q} = -q^2 \mathcal{Q}, \quad q \in \mathbb{R} \quad (2.4)$$

where  $\mathcal{Q}$  is a scalar function, and

$$\nabla^2 \equiv \frac{1}{\sqrt{h}} \partial_i \left( \sqrt{h} \partial^i \right), \quad h = \det(h_{ij}), \quad (2.5)$$

is the Laplace-Beltrami operator in a three-dimensional Riemannian space with metric  $h_{ij}$ . The eigenfunctions in the three cases considered here can be found by means of a simple separation of variables

$$\mathcal{Q}(\rho, \varphi, z) = f(\rho)g(\varphi)h(z) .$$

Below, we present these normalized eigenfunctions.

### 2.2.1 Flat FLRW

In the flat FLRW case the spatial metric is  $h_{ij} = \text{diag}(1, \rho^2, 1)$ . Using separation of variables, the normalized solutions to the eigenvalue problem (2.4) are easily found to be

$$Z_{\omega mk}(\mathbf{x}) \equiv \omega^{1/2} J_m(\omega\rho) \frac{e^{im\varphi}}{\sqrt{2\pi}} \frac{e^{ikz}}{\sqrt{2\pi}}, \quad (2.6)$$

where the eigenvalues are

$$\omega \in \mathbb{R}^+, \quad m \in \mathbb{Z}, \quad k \in \mathbb{R},$$

and obey the dispersion relation

$$q^2 = \omega^2 + k^2. \quad (2.7)$$

The normalization constant is such that

$$\int \sqrt{h} d^3 \mathbf{x} Z_{\omega m k}(\mathbf{x}) Z_{\omega' m' k'}^*(\mathbf{x}) = \delta_{m m'} \delta(\omega - \omega') \delta(k - k'). \quad (2.8)$$

Any scalar function  $\mathcal{Q}(\mathbf{x})$  can be expanded in terms of this basis as

$$\mathcal{Q}(\mathbf{x}) = \sum_{m=-\infty}^{\infty} \int_0^{\infty} d\omega \int_{-\infty}^{+\infty} dk \mathcal{Q}_m(\omega, k) Z_{\omega m k}(\mathbf{x}), \quad (2.9)$$

with inverse given by

$$\mathcal{Q}_m(\omega, k) = \int \sqrt{h} d^3 \mathbf{x} \mathcal{Q}(\mathbf{x}) Z_{\omega m k}^*(\mathbf{x}). \quad (2.10)$$

### 2.2.2 Bianchi III

In the BIII case the spatial metric in cylindrical coordinates is  $h_{ij} = \text{diag}(1, \sinh^2 \rho, 1)$ . The normalized solutions in this case are

$$Z_{\omega m k}(\mathbf{x}) \equiv N_{m\omega} P_{-\frac{1}{2}+i\omega}^m(\cosh \rho) \frac{e^{im\varphi}}{\sqrt{2\pi}} \frac{e^{ikz}}{\sqrt{2\pi}}, \quad (2.11)$$

where

$$\omega \in \mathbb{R}^+, \quad m \in \mathbb{N}, \quad k \in \mathbb{R},$$

and

$$q^2 = \omega^2 + k^2 + \frac{1}{4}. \quad (2.12)$$

Note that, owing to the curvature of the spatial sections, there is a strictly positive lower bound to the frequency of a mode:

$$q^2 \geq \frac{1}{4}. \quad (2.13)$$

Modes with  $0 \leq q^2 \leq 1/4$  do not belong to the basis of square-integrable eigenfunctions of the Laplace-Beltrami operator, and are thus excluded from the frequency spectrum. Such frequencies lead to what is known as supercurvature modes [28].

The Legendre function  $P_{-\frac{1}{2}+i\omega}^m(\cosh \rho)$  – sometimes also called conical function – is real and has the following symmetries

$$P_{-1/2+i\omega}^m(x) = P_{-1/2-i\omega}^m(x) = \frac{\Gamma(i\omega + m + 1/2)}{\Gamma(i\omega - m + 1/2)} P_{-1/2+i\omega}^{-m}(x), \quad (x \geq 1). \quad (2.14)$$

The (real) normalization constant  $N_{m\omega}$  is given by

$$N_{m\omega} = \sqrt{(-1)^m \omega \tanh \pi \omega \frac{\Gamma(i\omega - m + 1/2)}{\Gamma(i\omega + m + 1/2)}}, \quad (2.15)$$

and is such that [29]

$$\int \sqrt{h} d^3 \mathbf{x} Z_{\omega m k}(\mathbf{x}) Z_{\omega' m' k'}^*(\mathbf{x}) = \delta_{m m'} \delta(\omega - \omega') \delta(k - k'). \quad (2.16)$$

Since these eigenfunctions form a basis for the Hilbert space of square-integrable functions with the volume element corresponding to  $h_{ij}$ , any such function  $\mathcal{Q}(\mathbf{x})$  can be expanded as

$$\mathcal{Q}(\mathbf{x}) = \sum_{m=-\infty}^{+\infty} \int_0^{\infty} d\omega \int_{-\infty}^{+\infty} dk \mathcal{Q}_m(\omega, k) Z_{\omega m k}(\mathbf{x}), \quad (2.17)$$

with the inverse following directly from the orthogonality relation (2.16). The flat-space limit is reached when  $\rho \ll 1$  and the frequency is large,  $\omega \gg 1$ , such that  $\omega\rho$  is kept fixed. Using the approximations

$$(i\omega)^m P_{i\omega-1/2}^{-m}(\cosh \rho) \sim (-1)^m J_m(\omega\rho), \quad (2.18)$$

$$\frac{\Gamma(i\omega + m + 1/2)}{\Gamma(i\omega - m + 1/2)} \sim \omega^{2m}, \quad (2.19)$$

one can check that, up to an overall and unimportant phase,

$$Z_{\omega m k}(\mathbf{x}) \sim \omega^{1/2} J_m(\omega\rho) \frac{e^{im\varphi}}{\sqrt{2\pi}} \frac{e^{ikz}}{\sqrt{2\pi}}, \quad (2.20)$$

as it should be.

### 2.2.3 Kantowski-Sachs

Finally, for the KS case, the spatial metric in cylindrical coordinates is  $h_{ij} = (1, \sin^2 \rho, 1)$ . Since the spatial sections are just the product of the unit two-sphere with the real line, the eigenfunctions are simply the standard spherical harmonics multiplied by a one-dimensional plane wave:

$$Z_{\ell m k}(\mathbf{x}) \equiv N_{m\ell} P_{\ell}^m(\cos \rho) \frac{e^{im\varphi}}{\sqrt{2\pi}} \frac{e^{ikz}}{\sqrt{2\pi}}, \quad (2.21)$$

where

$$\ell \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad k \in \mathbb{R},$$

and we now have

$$q^2 = \ell(\ell + 1) + k^2. \quad (2.22)$$

Although the  $\ell = 0$  mode is included in the basis of square-integrable functions, this mode corresponds to an overall monopole, and is excluded from the definition of cosmological perturbations. Thus, the lower bound to the frequency of a mode is

$$q^2 \geq 2. \quad (2.23)$$

The normalization constant is defined as

$$N_{m\ell} \equiv \sqrt{\frac{(2\ell + 1)(\ell - m)!}{4\pi(\ell + m)!}} \quad (2.24)$$

and is chosen to give

$$\int \sqrt{h} d^3 \mathbf{x} Z_{\ell m k}(\mathbf{x}) Z_{\ell' m' k'}^*(\mathbf{x}) = \delta_{\ell \ell'} \delta_{m m'} \delta(k - k'). \quad (2.25)$$

Since these solutions also form a complete basis for the space of square-integrable functions, we can again expand any such function as

$$\mathcal{Q}(\mathbf{x}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int_{-\infty}^{+\infty} dk \mathcal{Q}_{\ell m}(k) Z_{\ell m k}(\mathbf{x}), \quad (2.26)$$

with the inverse following from (2.25). The flat-space limit may again be achieved by letting  $\rho \ll 1$  and  $\ell \gg 1$ , with  $\ell \rho$  held constant. Using the identity  $P_{\ell}^{-m} = (-1)^m (\ell - m)! / (\ell + m)! P_{\ell}^m$  and the approximations  $P_{\ell}^{-m}(\cos \rho) \sim \ell^{-m} J_m(\ell \rho)$  and  $(\ell + m)! / (\ell - m)! \sim \ell^{2m}$ , we find

$$Z_{\ell m k}(\mathbf{x}) \sim \ell^{1/2} J_m(\ell \rho) \frac{e^{im\varphi}}{\sqrt{2\pi}} \frac{e^{ikz}}{\sqrt{2\pi}}, \quad (2.27)$$

again, up to a phase.

### 3 Scalar perturbations

In this section we recall the basic formalism of linear perturbation theory in BIII and KS backgrounds [27]. Then, by analyzing the linearized Einstein equations of the perturbative degrees of freedom we show that, provided that the universe has gone an early period of exponential expansion, scalar perturbations can be parameterized by a single free function, exactly as in the standard FLRW case. Our main result is (3.16). Readers not interested in the details of the derivation can skip them and go directly to Section 4. Throughout this section we use the coordinates introduced in formula (2.1).

#### 3.1 General formalism

The theory of linear cosmological perturbations is built upon the idea that the large-scale universe can be described by means of small perturbations added to a background (exact) solution of Einstein equations. In general terms, this program can be implemented with the introduction of ten arbitrary functions on top of the background spacetime metric. Then, for homogeneous and isotropic spatial sections, these functions can be classified as either Scalars, Vectors or Tensors (SVT) by exploring their transformation properties under the action of the spatial symmetry group. In an FLRW universe this leads to the known SVT mode decomposition [30]. The main advantage of this procedure is that, in the linear regime, each kind of perturbation evolves independently of the others, which simplifies the analysis of their dynamical evolution. In this decomposition, non-trivial SVT modes exist in all FLRW cosmologies whose spatial slices are copies of a maximally symmetric space. In the BIII or KS spacetimes, on the other hand, spatial slices are given by the product of the real line with a two-dimensional maximally symmetric space. Therefore, in this case it seems natural to classify the perturbations modes just according to their behavior with respect to the symmetry group of that two-dimensional manifold [27]. In this sense, it is worth remarking that there is no non-trivial transverse and traceless symmetric tensor in two dimensions. More specifically,



on the two-dimensional maximally symmetric subspaces, any vector  $V_a$  and symmetric tensor  $T_{ab}$  can be decomposed as

$$V_a = D_a V + \bar{V}_a, \quad (3.1)$$

$$T_{ab} = X\gamma_{ab} + D_a D_b Y + D_{(a} \bar{Z}_{b)}, \quad (3.2)$$

where  $D_a$  is the covariant derivative compatible with the metric  $\gamma_{ab}$  [see (2.1)] and barred vectors are transverse, i.e.,  $D_a \bar{V}^a = 0 = D_a \bar{Z}^a$ . Let us also notice that, similarly, no non-trivial transverse vector can exist in the remaining one-dimensional section of the spatial slices, in which only scalar modes can be defined. Actually, this decomposition of the perturbations is an adaptation of the more general formalism of Gerlach and Sengupta [31], extended beyond spherical symmetry, where we treat the two dimensions supplementary to those of the bi-dimensional maximally symmetric manifold not in a covariant way as a Lorentz manifold on its own, but rather described in terms of the conformal time and the  $z$ -coordinate introduced above.

Since the background metric of the BIII and KS models considered here have a single scale factor, there will be no dynamical coupling between SV modes, something which further simplifies the dynamical evolution of the system. As stated in the Introduction, this situation is in sharp contrast with perturbation theory in Bianchi I universes where, although one can define uncoupled SVT modes at a given initial time, the anisotropic evolution of the background metric couples those modes, leading to a dynamical see-saw effect [12, 32]. Once clarified that perturbative mode couplings will not arise in the present case, we can focus exclusively on the evolution of scalar perturbations, which are the most important ones for the CMB large-angle perturbations. The most general metric with scalar perturbations can be written

$$ds^2 = a^2(\eta)[-(1 + 2\Phi)d\eta^2 - 2\Pi d\eta dz + (1 + 2\Psi)\gamma_{ab}dx^a dx^b + (1 + 2\Lambda)dz^2], \quad (3.3)$$

where  $\Phi$ ,  $\Pi$ ,  $\Psi$ , and  $\Lambda$  represent the four effective scalar degrees of freedom of the metric.<sup>1</sup>

Next, we need to introduce perturbations to the matter sector. Such perturbations can be parameterized in the same way as above, with the addition of extra SV degrees of freedom to the matter components. One important aspect of the formalism is that, since the considered models do not have a dynamical FLRW limit, the anisotropic spatial curvature needs to be sourced by an imperfect matter component [21]. This can be done either by introducing a stress component in the energy-momentum tensor [21] or a two-form field [24]. We can also take the simpler route provided by the choice of scalar fields, which can be chosen to be a real massless field in the BIII case [22, 23], or a complex massless field in the KS case [33]. Thus, our matter sector is composed of two fields: the inflaton, whose perturbations can be introduced exactly as in the FLRW case (see, for example, [34]), and the above anisotropic and massless scalar field  $\phi(\eta, \mathbf{x})$ . Their perturbations are as follows:

$$a^2 \delta T_{(\phi)0}^0 = \Lambda - \partial_z \delta \phi, \quad (3.4)$$

$$a^2 \delta T_{(\phi)z}^0 = -\Pi - \delta \phi', \quad (3.5)$$

$$a^2 \delta T_{(\phi)b}^a = (\Lambda - \partial_z \delta \phi) \delta_b^a, \quad (3.6)$$

$$a^2 \delta T_{(\phi)a}^z = \partial_a \delta \phi, \quad (3.7)$$

$$a^2 \delta T_{(\phi)z}^z = -\Lambda + \partial_z \delta \phi, \quad (3.8)$$

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<sup>1</sup>Seven scalar functions in the original parameterization minus three scalar gauge functions [27].

where the prime denotes the derivative with respect to  $\eta$ . As we are going to show, the perturbation  $\delta\phi$  of the anisotropic scalar field is not dynamically important in the inflationary stage, and can be safely ignored.

In conclusion, given the above perturbations to both the geometrical and the matter sectors, linear and gauge-invariant dynamical equations can be straightforwardly computed [27].

### 3.2 Inflation and metric perturbations

We are primarily interested in computing large-angle effects ( $\gtrsim 1^\circ$ ) of the BIII and KS geometries on the temperature angular spectrum of the CMB. At these scales, temperature perturbations are mainly driven by quantum fluctuations of the early universe – a period which, we assume, was sourced by scalar fields. Then, from the  $a - b$  component of the linearized Einstein equation, and from the fact that scalar fields do not produce anisotropic stress, it follows that (see (A.5) in [27])

$$-D_{(a}D_{b)}(\Lambda + \Phi) = 0, \quad a \neq b. \quad (3.9)$$

The above equation translates into the following constraint:

$$\Lambda = -\Phi. \quad (3.10)$$

This is in close analogy to the situation of standard (FLRW) perturbation theory, where the absence of anisotropic stress is used to eliminate one scalar degree of freedom of the metric.

Given the constraint (3.10), metric scalar perturbations are still described by three free functions:  $\Phi$ ,  $\Psi$  and  $\Pi$ . Surprisingly, though, only one degree of freedom is relevant during inflation. In order to see that, let us consider the  $z - a$  component of Einstein equations:

$$-(\partial_z\Phi + \partial_z\Psi) + \mathcal{H}\Pi + \frac{1}{2}\Pi' = \delta\phi, \quad (3.11)$$

where  $\mathcal{H}$  is the Hubble parameter. If inflation is sourced by a slowly-rolling scalar field, the variable  $\delta\phi$  obeys [27]

$$w'' + \left[ q^2 + \frac{2 + 3\epsilon}{\eta^2} \right] w = 0, \quad w \equiv a\delta\phi, \quad (3.12)$$

where  $\epsilon \equiv 1 - \mathcal{H}'/\mathcal{H}^2$  is the slow-roll parameter and  $\eta \in (-\infty, 0^-)$  is the conformal time during inflation. Note that, since the “mass” term  $(2 + 3\epsilon)/\eta^2$  is strictly positive,  $w$  will never grow, which in turn implies that  $\delta\phi$  will be suppressed by the expansion. Indeed, for  $q^2 \gg \eta^{-2}$ ,  $w \sim \exp[\pm iq\eta]$  and  $\delta\phi$  decays as  $1/a$ . For  $q^2 \ll \eta^{-2}$ ,  $w \sim \exp[\pm i\sqrt{3\epsilon + 7/4} \log(-\eta)]/(-\eta)^{1/2}$ , which implies that<sup>2</sup>  $\delta\phi \sim 1/a^{3/2}$ . Since this result holds for both BIII and KS, we can set

$$\delta\phi = 0 \quad (3.13)$$

with no loss of generality.

If we now rewrite (3.11) in momentum space ( $\partial_z \rightarrow ik$ ) and use (3.13), we find that the constraint

$$ik[\Phi(q, \eta) + \Psi(q, \eta)] = \frac{[a^2\Pi(q, \eta)]'}{2a^2} \quad (3.14)$$

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<sup>2</sup>We recall that, during slow-roll inflation,  $a \sim 1/(-\eta)$  at dominant order in  $\epsilon$ .

must hold for *all values of  $k$* . Moreover, from the dispersion relations (2.12) and (2.22) we see that  $q$  and  $k$  can be treated as independent variables whose specific values fix  $\omega$  (BIII) or  $\ell$  (KS). In other words, the case  $k = 0$  in the constraint (3.14) must hold for all possible values of  $\Pi(q, \eta)$ . This is only possible if  $a^2\Pi(q, \eta)$  is just a function of  $q$ , that is, if  $\Pi(q, \eta) = f(q)/a^2(\eta)$ . We thus see that the variable  $\Pi$  is dynamically suppressed as well, and can be ignored. Since this result holds for both BIII and KS geometries, we can set

$$\Pi = 0, \quad \Psi = -\Phi. \quad (3.15)$$

We have thus arrived at the important conclusion that, provided that the universe has experienced an early period of exponential expansion, the metric describing large-scale perturbations in both BIII and KS geometries depends on a single free function, and can be parameterized as

$$ds^2 = a^2(\eta)[-(1 + 2\Phi)d\eta^2 + (1 - 2\Phi)h_{ij}dx^i dx^j]. \quad (3.16)$$

This is formally the same as the metric of adiabatic scalar perturbations in FLRW universes. However, our result is more general because  $h_{ij}(x^k)$  can represent the FLRW, the BIII, or the KS spatial metrics.

## 4 Sachs-Wolfe effect and correlation functions

The main conclusion of the previous section is encoded in (3.16), and it implies that scalar perturbations over the considered manifolds with anisotropic spatial curvature are described by a single gravitational potential, exactly like standard perturbation theory in FLRW universes. Moreover, (3.16) implies that *all* dynamical equations are formally the same as in the standard isotropic case – the only difference being that covariant derivatives are now taken with respect to the more general metric  $h_{ij}$ . A corollary of this result is that the Sachs-Wolfe (SW) effect, being purely kinematical, is given by the same known expression

$$\Delta T(\hat{\mathbf{n}}) = \frac{1}{3}\Phi(\mathbf{x}, \eta), \quad (4.1)$$

where  $\hat{\mathbf{n}}$  is the direction of detection of a photon emitted at spacetime position  $(\mathbf{x}, \eta)$  (i.e., the coordinates of the last-scattering surface) and, again,  $\mathbf{x}$  are the coordinates in the more general spacetimes under discussion. In this description, the effect of the anisotropic spatial curvature appears, in particular, through the non-trivial spectrum of the Laplace-Beltrami operator, and affects the angular power spectrum of the CMB at large scales, as we are now going to see.

### 4.1 FLRW case

We move now to the computation of the temperature two-point correlation function (2pcf) resulting from the Sachs-Wolfe effect in a flat FLRW universe. While this result is well-known, this section is intended to introduce the main formalism, and also to facilitate comparison with new results which will follow below. Moreover, here we will carry out the computation using cylindrical coordinates, that are not the ones most commonly employed.

The temperature correlation function between two CMB photons arriving from the last scattering surface, considering only the SW effect, is defined by

$$\langle \Delta T(\hat{\mathbf{n}})\Delta T(\hat{\mathbf{n}}') \rangle \equiv C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \frac{1}{9}\langle \Phi(\mathbf{x})\Phi(\mathbf{x}') \rangle, \quad (4.2)$$

where, for simplicity, we have dropped the instants of emission, since these should be clear from the context. We now use (2.6) to decompose the gravitational potential  $\Phi(\mathbf{x})$  and use the fact that, in a Gaussian and statistically isotropic universe, the correlation between modes takes the form

$$\langle \Phi_m(\omega, k) \Phi_{m'}^*(\omega', k') \rangle = \mathcal{P}(q) \delta_{mm'} \delta(\omega - \omega') \delta(k - k'). \quad (4.3)$$

Here  $\mathcal{P}(q)$  is the primordial power spectrum and  $q$  is defined as in (2.7). Employing the summation theorem for Bessel functions (A.1), we then arrive at

$$C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \frac{1}{(6\pi)^2} \int_0^\infty \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(q) J_0(\omega \Delta \rho_0) e^{ik\Delta z}, \quad (4.4)$$

where we have introduced the definitions

$$(\Delta \rho_0)^2 \equiv \rho^2 + \rho'^2 - 2\rho\rho' \cos \Delta \varphi, \quad \Delta \varphi \equiv \varphi - \varphi', \quad \Delta z \equiv z - z'. \quad (4.5)$$

As we will see, expressions (4.4) and (4.5) play a central role in the derivation of results for geometries with anisotropic curvature. Just for completeness, though, let us note that the right hand side of (4.4) can be further simplified by the change of variables

$$\omega = q \sin \psi \quad \text{and} \quad k = q \cos \psi, \quad (4.6)$$

and the subsequent integration over the range<sup>3</sup>  $0 \leq \psi \leq \pi$ . The fact that the 2pcf depends only on the angle between  $\hat{\mathbf{n}}$  and  $\hat{\mathbf{n}}'$  can be seen by introducing spherical coordinates through

$$\rho^{(\prime)} = \Delta \eta \sin \theta^{(\prime)} \quad \text{and} \quad z^{(\prime)} = \Delta \eta \cos \theta^{(\prime)}, \quad (4.7)$$

where  $\Delta \eta$  is the conformal distance to the last scattering surface. This leads to

$$C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \equiv C(\vartheta) = \frac{1}{18\pi^2} \int_0^\infty q^2 dq \mathcal{P}(q) \frac{\sin [q\Delta\eta\sqrt{2-2\cos\vartheta}]}{q\Delta\eta\sqrt{2-2\cos\vartheta}}, \quad (4.8)$$

where  $\cos \vartheta \equiv \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos \Delta \varphi$ . From here, the angular power spectrum  $C_\ell$  can be directly computed by means of a Legendre transform over  $\vartheta$ :

$$C_\ell = 2\pi \int_{-1}^1 d(\cos \vartheta) P_\ell(\cos \vartheta) C(\vartheta), \quad (4.9)$$

which, after performing the integral in  $\cos \vartheta$  by means of the change of variable  $2y^2 = 1 - \cos \vartheta$ , leads to the well known expression [30]

$$C_\ell = \frac{2}{9\pi} \int_0^\infty q^3 \mathcal{P}(q) j_\ell^2(q\Delta\eta) d \ln q. \quad (4.10)$$

## 4.2 Bianchi III case

In the BIII geometry the temperature 2pcf is still defined by (4.2). However, the gravitational potential is now decomposed using the eigenfunctions (2.11). The perturbations in this case are supposed to be Gaussian but statistically anisotropic, so that the most general correlation function between modes will have the form

$$\langle \Phi_m(\omega, k) \Phi_{m'}^*(\omega', k') \rangle = \bar{\mathcal{P}}(\mathbf{q}) \delta_{mm'} \delta(\omega - \omega') \delta(k - k'). \quad (4.11)$$

---

<sup>3</sup>The range of  $\psi$  is conditioned by the restriction  $\omega \geq 0$ .

Since we are working with homogeneous spaces, both Kronecker and Dirac deltas are needed to ensure translational invariance [35]. The power spectrum, on the other hand, is a general function of the vector  $\mathbf{q} \equiv (\omega, m, k)$ , since the underlying geometries are anisotropic. In principle, a rigorous quantization procedure is required to fix and normalize  $\bar{\mathcal{P}}(\mathbf{q})$  properly. Nonetheless, note that invariance under rotations in the maximally symmetric bi-dimensional subspace requires a power spectrum independent of  $m$ . Moreover, since we are interested in departures from the CMB features typical of an FLRW universe resulting from the BIII geometry, it suffices to fix  $\bar{\mathcal{P}}(\mathbf{q})$  by demanding that the two-point correlation functions in these geometries agree in the limit in which the two points coincide (limit in which the spatial curvature is ignorable). This condition leads to

$$\bar{\mathcal{P}}(\mathbf{q}) = \frac{\mathcal{P}(\tilde{q})}{\tanh \pi\omega}, \quad (4.12)$$

where  $\mathcal{P}$  is the same function appearing in (4.3) and where we have defined  $\tilde{q}^2 \equiv \omega^2 + k^2 = q^2 - 1/4$  in order to account for the discrepancy between the dispersion relation in the FLRW case and (2.12). The factor  $\tanh \pi\omega$  is needed to ensure the correct functional dependence of the 2pcf when  $\hat{\mathbf{n}} \rightarrow \hat{\mathbf{n}}'$ , according to our arguments. Using the above expression and the summation theorem (A.3) for Legendre polynomials, we arrive at

$$C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \frac{1}{(6\pi)^2} \int_0^\infty \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(\tilde{q}) P_{-\frac{1}{2}+i\omega}(\cosh \Delta\rho) e^{ik\Delta z}, \quad (4.13)$$

where

$$\cosh \Delta\rho \equiv \cosh \rho \cosh \rho' - \sinh \rho \sinh \rho' \cos \Delta\varphi. \quad (4.14)$$

Before continuing our analysis, it is interesting to compare these formulas with their counterparts in the FLRW geometry. First, since  $P_{-1/2+i\omega}(1) = 1 = J_0(0)$ , we can see that (4.13) agrees with (4.4) when  $\Delta\rho = \Delta\rho_0 = 0 = \Delta z$ , as they should. The main differences between these expressions are: (i) the law of cosines, equation (4.5), which now becomes the corresponding law in a two-dimensional hyperbolic space, namely (4.14), and (ii) the Bessel function in the kernel of the integral (4.4), which is now replaced by a Legendre function. As we will see, all expressions agree at first order in the limit where  $\Delta\eta \ll 1$  in units of curvature scale (and hence, strictly speaking, even beyond the limit of coincident points that we commented above).

Apart from our physical requirements of coincident points, expression (4.13) is still quite general, inasmuch as it provides the form of the correlation between any two scalar functions in a space with  $\mathbb{H}^2 \times \mathbb{R}$  topology. However, we are only interested in situations in which this 2pcf does not lead to large modifications in the CMB data from the results of the flat FLRW case, since we know from observations that our universe is well fitted by those data. With this motivation, we can think of the CMB observations as taking place in a region described just by a small patch of a larger BIII region. If that is the case, small curvature corrections from the global BIII metric might be lurking beyond the particle horizon scale, and their signatures should be encoded in small corrections to the large-angle CMB spectrum. Such corrections come from two different terms. The first of them is the function  $P_{-\frac{1}{2}+i\omega}(\cosh \Delta\rho)$ , which can be written as a powers series in  $\Delta\rho$  (see Appendix A.2):

$$P_{-\frac{1}{2}+i\omega}(\cosh \Delta\rho) = J_0(\omega\Delta\rho) - \frac{(\Delta\rho)^2}{12} \left[ J_0(\omega\Delta\rho) - \frac{J_1(\omega\Delta\rho)}{2\omega\Delta\rho} \right] + \dots \quad (4.15)$$

The second correction comes from the law of cosines (4.14), which can be expanded in powers of  $\rho$  and  $\rho'$  as

$$(\Delta\rho)^2 = (\Delta\rho_0)^2 + \frac{\sin^2 \Delta\varphi}{3}(\rho\rho')^2 + \dots \quad (4.16)$$

where  $\Delta\rho_0$  was defined in equation (4.5). Both expansions above are exact, and can be carried at any desired order in their perturbation parameters. In our case, this parameter is the distance to the last scattering surface *in units of the curvature scale*,  $\Delta\eta$ , which relates to  $\Delta\rho$ ,  $\rho$  and  $\rho'$  through equations (4.7), (4.5) and (4.16). Notice that this is the only free parameter appearing in our equations. If we thus combine equations (4.15) and (4.16), retaining only corrections of order  $(\Delta\eta)^2$ , we arrive at

$$P_{-\frac{1}{2}+i\omega}(\cosh \Delta\rho) \approx J_0(x) + \frac{(\rho\rho')^2 \sin^2 \Delta\varphi}{6(\Delta\rho_0)^2} \left[ x \frac{dJ_0(x)}{dx} \right] - \frac{(\Delta\rho_0)^2}{12} \left[ J_0(x) - \frac{J_1(x)}{2x} \right], \quad (4.17)$$

where  $x = \omega\Delta\rho_0$ . Inserting this result in equation (4.13) we finally have that

$$\begin{aligned} C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \approx & \frac{1}{(6\pi)^2} \int_0^\infty \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(\tilde{q}) J_0(\omega\Delta\rho_0) e^{ik\Delta z} \\ & + \frac{1}{(6\pi)^2} \left[ \frac{(\rho\rho')^2 \sin^2 \Delta\varphi}{6(\Delta\rho_0)^2} \int_0^\infty \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(\tilde{q}) (\omega\Delta\rho_0) \frac{dJ_0(\omega\Delta\rho_0)}{d(\omega\Delta\rho_0)} e^{ik\Delta z} \right. \\ & \left. - \frac{(\Delta\rho_0)^2}{12} \int_0^\infty \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(\tilde{q}) \left[ J_0(\omega\Delta\rho_0) - \frac{J_1(\omega\Delta\rho_0)}{2\omega\Delta\rho_0} \right] e^{ik\Delta z} \right]. \quad (4.18) \end{aligned}$$

Expression (4.18) is one of our main results. Note that the effect of the anisotropic curvature will appear both at the isotropic ( $\hat{\mathbf{n}} = \hat{\mathbf{n}}'$ ) and anisotropic ( $\hat{\mathbf{n}} \neq \hat{\mathbf{n}}'$ ) parts of the CMB spectrum. We will investigate these signatures with more details in Section 5.

### 4.3 Kantowski-Sachs case

The computation of the 2pcf in the KS case follows the same lines as for BIII. Starting from (4.2), we decompose the gravitational potential using the eigenfunctions (2.21). Since we are considering a homogeneous but spatially anisotropic geometry, the correlation between modes must take the form

$$\langle \Phi_{\ell m}(k) \Phi_{\ell' m'}^*(k') \rangle = \bar{\mathcal{P}}(\mathbf{q}) \delta_{\ell\ell'} \delta_{mm'} \delta(k - k'), \quad (4.19)$$

where  $\mathbf{q} \equiv (\ell, m, k)$ . Using arguments similar to those given in the previous section, one can show that

$$\bar{\mathcal{P}}(\mathbf{q}) = 2\pi \mathcal{P}(\tilde{q}). \quad (4.20)$$

Here we have defined  $\tilde{q}^2 \equiv k^2 + (\ell + 1/2)^2 = q^2 + 1/4$  to account for the difference between the FLRW dispersion relation and (2.22), and we have chosen the numerical factor in the power spectrum so as to formally reproduce the expression for the 2pcf of the flat FLRW case in the limit of coincident points. Using the above power spectrum and the summation theorem (A.5) we find

$$C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = \frac{1}{(6\pi)^2} \sum_{\ell=1}^{\infty} \left( \ell + \frac{1}{2} \right) \int_{-\infty}^{+\infty} dk \mathcal{P}(\tilde{q}) P_\ell(\cos \Delta\rho) e^{ik\Delta z}, \quad (4.21)$$

where

$$\cos \Delta\rho \equiv \cos \rho \cos \rho' + \sin \rho \sin \rho' \cos(\varphi - \varphi'). \quad (4.22)$$

A remark is in order, concerning the derivation of (4.20) and (4.21). In the limit of coincident points, equation (4.4) becomes  $(6\pi)^{-2} \int \omega d\omega \int dk \mathcal{P}(q)$ . Clearly, the integral over  $\omega$  in this expression can be split as a sum of all the integrals over two consecutive positive integers. Assuming that the function  $\mathcal{P}$  does not vary much with  $\omega$  in any such interval, we can evaluate it at its middle point and then, in the considered case of coincident points, we are simply left with the integrals of the remaining factor  $\omega d\omega$ . In the interval  $[\ell, \ell + 1]$ , e.g., this integral is equal to  $(\ell + 1/2)$ . Hence, with our assumption, the isotropic 2pcf becomes  $(6\pi)^{-2} \sum_{\ell} (\ell + 1/2) \int dk \mathcal{P}(\bar{q})$ . The requirement that the 2pcf in the KS space should give an equivalent result in the limit of coincident points then justify (4.20). As a final comment, note that the main difference of the 2pcf in the KS space, apart from the modifications in the dispersion relation and the eigenfunctions, is the appearance of a sum over  $\ell$  instead of an integral over  $\omega$ . This is reminiscent of the fact that the KS geometries have closed spatial subspaces in their spatial sections, leading to discrete eigenvalues of the Laplace-Beltrami operator.

Moving forward, we follow the same line of reasoning as in the previous section and consider the relevant region for the CMB observations as a small patch of a larger KS space. Consequently, we expand  $P_{\ell}(\cos \Delta\rho)$  as a power series in  $\Delta\rho$  [see (A.9) in the Appendix] to find

$$P_{\ell}(\cos \Delta\rho) = J_0(\tilde{\ell}\Delta\rho) + \frac{(\Delta\rho)^2}{12} \left[ J_0(\tilde{\ell}\Delta\rho) - \frac{J_1(\tilde{\ell}\Delta\rho)}{2\tilde{\ell}\Delta\rho} \right] + \dots \quad (4.23)$$

where we have used the notation  $\tilde{\ell} = \ell + 1/2$ . Note that, in contrast to (4.15), the first non-trivial correction now appears with a positive sign. Likewise, we expand  $(\Delta\rho)^2$  defined in (4.22) as a power series, which gives

$$(\Delta\rho)^2 = (\Delta\rho_0)^2 - \frac{\sin^2 \Delta\varphi}{3} (\rho\rho')^2 + \dots \quad (4.24)$$

Combining these two expressions and retaining powers up to  $(\Delta\eta)^2$ , as before, we find

$$P_{\ell}(\cos \Delta\rho) \approx J_0(x) - \frac{(\rho\rho')^2 \sin^2 \Delta\varphi}{6(\Delta\rho_0)^2} \left[ x \frac{dJ_0(x)}{dx} \right] + \frac{(\Delta\rho_0)^2}{12} \left[ J_0(x) - \frac{J_1(x)}{2x} \right], \quad (4.25)$$

where  $x = (\ell + 1/2)\Delta\rho_0$ . It is worth noticing that, since we are assuming that  $\Delta\rho_0 \propto \Delta\eta$  is small, the Bessel functions in these expressions do not vary much in each interval  $[\ell, \ell + 1]$ . Thanks to this fact, our discussion below (4.22) still applies and allows us to regard the sum over  $\ell$  as a discretized approximation of the integral over  $d\omega$ . Combining all these results, we finally find

$$\begin{aligned} C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') \approx & \frac{1}{(6\pi)^2} \int_0^{\infty} \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(\bar{q}) J_0(\omega\Delta\rho_0) e^{ik\Delta z} \\ & - \frac{1}{(6\pi)^2} \left[ \frac{(\rho\rho')^2 \sin^2 \Delta\varphi}{6(\Delta\rho_0)^2} \int_0^{\infty} \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(\bar{q}) (\omega\Delta\rho_0) \frac{dJ_0(\omega\Delta\rho_0)}{d(\omega\Delta\rho_0)} e^{ik\Delta z} \right. \\ & \left. - \frac{(\Delta\rho_0)^2}{12} \int_0^{\infty} \omega d\omega \int_{-\infty}^{+\infty} dk \mathcal{P}(\bar{q}) \left[ J_0(\omega\Delta\rho_0) - \frac{J_1(\omega\Delta\rho_0)}{2\omega\Delta\rho_0} \right] e^{ik\Delta z} \right], \quad (4.26) \end{aligned}$$

with  $\bar{q}^2$  evaluated at  $k^2 + \omega^2$  in these integrals. This is another of our main results.



## 5 Qualitative analysis

We will now turn to a numerical analysis in order to extract some qualitative information from formulas (4.18) and (4.26). We start by noticing that these expressions can be summarized in a single formula as

$$C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') = C(\vartheta) \pm \mathcal{F}(\hat{\mathbf{n}}, \hat{\mathbf{n}}'), \quad (5.1)$$

where  $C(\vartheta)$ , with  $\vartheta = \arccos(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}')$ , is the isotropic 2pcf and where the anisotropic function  $\mathcal{F}(\hat{\mathbf{n}}, \hat{\mathbf{n}}')$  is defined by the additional terms in the last two lines of equation (4.18) or equation (4.26). Here, the plus and minus signs correspond to the BIII and KS cases, respectively. Alternatively, our results can be written in terms of the temperature *covariance matrix*, defined in terms of the temperature harmonic coefficients,  $a_{\ell m} = \int d^2\hat{\mathbf{n}} \Delta T(\hat{\mathbf{n}}) Y_{\ell m}^*(\hat{\mathbf{n}})$ , as

$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = \int d^2\hat{\mathbf{n}} \int d^2\hat{\mathbf{n}}' C(\hat{\mathbf{n}}, \hat{\mathbf{n}}') Y_{\ell m}^*(\hat{\mathbf{n}}) Y_{\ell' m'}(\hat{\mathbf{n}}'). \quad (5.2)$$

Decomposing each term in  $C(\hat{\mathbf{n}}, \hat{\mathbf{n}}')$  into spherical harmonics, we find

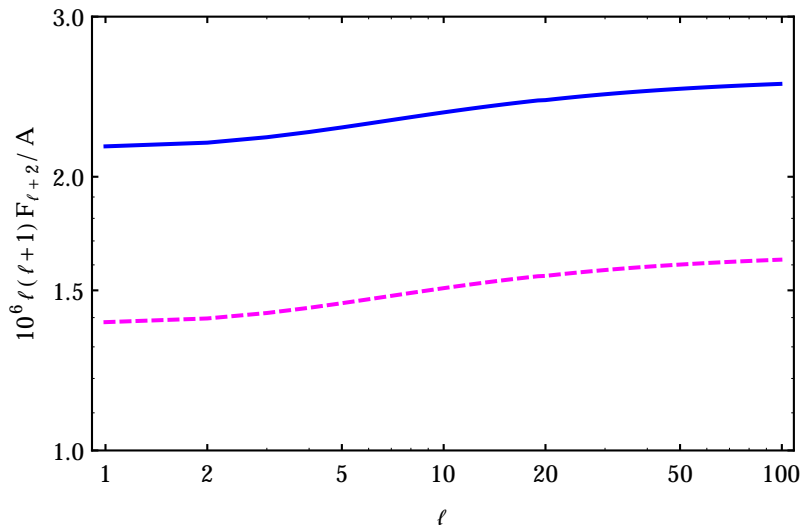
$$\langle a_{\ell m} a_{\ell' m'}^* \rangle = C_\ell \delta_{\ell\ell'} \delta_{mm'} \pm \langle \mathcal{F}_{\ell m \ell' m'} \rangle. \quad (5.3)$$

There are two types of corrections coming from the anisotropy of the spatial curvature. The first of them arises from the diagonal part of the matrix  $\langle \mathcal{F}_{\ell m \ell' m'} \rangle$ . Such correction leads to an effective temperature power spectrum which differs from  $C_\ell$  mainly at small multipoles. Unfortunately this effect does not offer a strong constraint on the model since, besides being strongly limited by cosmic variance, it is degenerate with isotropic physical mechanisms which also alter the low- $\ell$  tail of the spectrum (e.g. the Integrated Sachs-Wolfe effect). The second type of correction, on the other hand, arises from off-diagonal terms in the covariance matrix, and is a genuine signature of statistical anisotropy. Moreover, this correction is present at all multipolar scales, a feature which alleviates the problem of cosmic variance. The simplest off-diagonal term we could expect to find is of the form  $\langle a_{\ell m} a_{(\ell+1)m}^* \rangle = \langle F_{\ell m (\ell+1)m} \rangle$ . Since this term couples even and odd multipoles, it is a signature of parity breaking. However, these terms are exactly zero in our case, since the metrics we are considering are invariant under the parity transformation  $(\rho, \varphi, z) \rightarrow (\rho, \varphi \pm \pi, -z)$ . The next off-diagonal term we can calculate is  $\langle a_{\ell m} a_{(\ell+2)m}^* \rangle$ , where  $-\ell \leq m \leq \ell$ . So, for each  $\ell$  there will  $2\ell + 1$  terms of this type. Recall also that  $a_{\ell m} = (-1)^m a_{\ell, -m}^*$ , as required by the reality of  $\Delta T(\hat{\mathbf{n}})$ . Then, to further reduce the effect of cosmic variance, we will work with an azimuthally averaged measure of anisotropy defined as a sum over the  $2\ell + 1$  possible realizations of the label  $m$ . Therefore, we define the following (real) quantity:

$$\mathcal{F}_{\ell+2} \equiv \frac{1}{2\ell + 1} \sum_{m=-\ell}^{\ell} |\langle a_{\ell m} a_{(\ell+2)m}^* \rangle|. \quad (5.4)$$

We have evaluated this quantity numerically using a power spectrum per logarithmic band of the Harrison-Zel'dovich form:  $q^3 \mathcal{P}(q) = A = \text{constant}$ , and two different choices of the parameter  $\Delta\eta$  compatible with our hypothesis that  $\Delta\eta \ll 1$ . We show in figure (1) a plot of  $\ell(\ell+1)\mathcal{F}_{\ell+2}$  for a wide range of the multipole  $\ell$ . Our analysis shows that the spectrum grows smoothly with  $\ell$  up to  $\ell = 100$ , where the use of the Sachs-Wolfe effect alone is justified.





**Figure 1.** Log-log plot of the quantity  $\ell(\ell + 1)\mathcal{F}_{\ell+2}$  as a function of the angular multipole  $\ell$ , and in units of the power spectrum amplitude  $A$ . The curves show the results of our numerical computations assuming  $\Delta\eta$  equal to 0.04 (dashed, magenta) and 0.05 (continuous, blue), in units of the curvature scale.

## 6 Conclusion

The hypothesis that we live in a spatially infinite, homogeneous, and isotropic universe is one of the central pillars of modern cosmology. Besides being strongly supported by CMB measurements, the simplifications introduced by this hypothesis in the framework of general relativity has allowed cosmologists to build a robust description of the large-scale universe with only six free parameters. Nonetheless, the fact that computations are simpler in the framework of an FLRW metric should not prevent us from considering more general geometries, especially when one realizes that the observed isotropy of the CMB does not rule out less symmetric models of the universe. Indeed, it is known that the anisotropies of the background metric can manifest themselves through other channels different from the simple anisotropic expansion of the scale factor.

In this work we have explored the ideas introduced in references [21, 23, 24], in which the anisotropy of the universe does not result from its dynamical expansion, but rather from the curvature of its spatial sections. In the context of three-dimensional spaces, this feature can be implemented naturally only in two different ways: either by considering the Bianchi type III or the Kantowski-Sachs metric. In both models the expansion of the universe is controlled by a single scale factor, and hence the resulting CMB field is exactly isotropic at the background level [23]. Moreover, the fact that the dynamics are dictated by a single scale factor introduces simplifications which render the model tractable from a computational point of view. Based on this property, and starting from the full theory of linear and gauge-invariant perturbations introduced in [27], we have established several important results. First, by means of a simple parameterization of the considered metrics in cylindrical coordinates, we have given explicit expressions for a complete set of spatial eigenfunctions possessing the symmetries of these spaces. This step is crucial for the comparison of the theory with observations and, to the best of our knowledge, has not appeared before in the context of anisotropic cosmologies. Second,

we have shown that, provided that the early universe has expanded in accordance with the inflationary paradigm, large-scale and adiabatic scalar perturbations will be described by one single gravitational potential,  $\Phi$ , just as in standard perturbation theory in isotropic universes. Since the main large-angle CMB effects depend only on the photon geodesics in the perturbed spacetime, our result shows that the temperature fluctuations at these scales will have exactly the same functional dependence on  $\Phi$  as the one found in the isotropic setup. In particular, the Sachs-Wolfe effect in both BIII and KS spaces is given by  $\Delta T = \Phi/3$ .

Focusing on the Sachs-Wolfe effect, which is the most important effect for CMB at large angles, and with the help of the basis eigenfunctions that we have found, we have explicitly computed the spatial two-point correlation function in the BIII and KS geometries in the limit of small  $\Delta\eta$ , that is, a small ratio of the distance to the last scattering surface to the curvature radius. There are two different corrections to the isotropic correlation function, one resulting from the expansion of the spatial eigenfunctions around their isotropic counterpart, and another resulting from the law of cosines in curved spaces. The general effect of these corrections is two-fold: first, it changes the low- $\ell$  tail of the isotropic spectrum, and second, it induces off-diagonal temperature correlations which ought not to be present if the universe is isotropic. We have evaluated the latter numerically, and we have found that the effect goes beyond the low multipole region of the spectrum, corresponding to small  $\ell$ , region where, at least in principle, these correlations could be detected. It is important to note that we cannot predict the ratio of the last scattering surface to the curvature radius (i.e.,  $\Delta\eta$ ), which in this work was treated as a small free parameter. Nonetheless, current bounds on superhorizon features are usually of the order of a few percent [36] or even less [37], which means that our choices for  $\Delta\eta$  are based on educated guesses.

Finally, it is worth noticing that the formalism that we have developed here is quite general and can be directly applied to situations which include other effects, such as the integrated Sachs-Wolfe and Doppler effects, as well as in scenarios involving more general matter fields. This work also paves the way to more challenging investigations, such as the effect of anisotropic curvature in the spectrum of gravitational waves and the polarization of the CMB. Such analyses, together with a thorough assessment of the detectability of these effects with actual and upcoming experimental probes, are currently in progress.

**Note Added** After this work was submitted to publication we became aware of references [38–40], in which the spatial eigenfunctions of B3 and KS spacetimes were found independently. We thank Julian Adamek for bringing these references to our attention.

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## A Mathematical complements

In this appendix we succinctly review the main mathematical tools used in this work.

## A.1 Summation theorems

The derivation of the 2pcf in flat (FLRW) geometries makes use of the following identity

$$\sum_{m=-\infty}^{+\infty} J_m(\omega\rho)J_m(\omega\rho')e^{im(\varphi-\varphi')} = J_0(\omega\Delta\rho_0), \quad (\text{A.1})$$

where  $\Delta\rho_0$  is defined in (4.5). The derivation of the 2pcf in the open (BIII) case, on the other hand, relies on the identities

$$P_\nu^{-m}(z) = \frac{\Gamma(\nu - m + 1)}{\Gamma(\nu + m + 1)} P_\nu^m(z), \quad (\text{A.2})$$

$$P_{-\frac{1}{2}+i\omega}(\cosh \Delta\rho) = \sum_{m=-\infty}^{+\infty} (-1)^m P_{-\frac{1}{2}+i\omega}^{-m}(\cosh \rho) P_{-\frac{1}{2}+i\omega}^m(\cosh \rho') e^{im(\varphi-\varphi')}, \quad (\text{A.3})$$

where  $\Delta\rho$  is defined in (4.14). Analogously, the derivation of the 2pcf in the closed case uses the following formulas:

$$P_\ell^{-m}(x) = (-1)^m \frac{(\ell - m)!}{(\ell + m)!} P_\ell^m(x), \quad (\text{A.4})$$

$$P_\ell(\cos \Delta\rho) = \sum_{m=-\infty}^{+\infty} (-1)^m P_\ell^{-m}(\cos \rho) P_\ell^m(\cos \rho') e^{im(\varphi-\varphi')}, \quad (\text{A.5})$$

where  $\Delta\rho$  is defined now in (4.22).

## A.2 Approximating Legendre polynomials by Bessel functions

In order to find an approximation to  $P_{-1/2+i\omega}(\cosh y)$  in terms of Bessel functions we start with the following integral representations [41]:

$$P_{-\frac{1}{2}+i\omega}(\cosh y) = \frac{\sqrt{2}}{\pi} \int_0^y \frac{\cos \omega t}{\sqrt{\cosh y - \cosh t}} dt, \quad y \geq 0, \quad (\text{A.6})$$

$$J_n(x) = \frac{(x/2)^n}{\Gamma(n + 1/2)\Gamma(1/2)} \int_{-1}^1 (1 - t^2)^{n-1/2} \cos xt dt. \quad (\text{A.7})$$

We now define  $t = uy$ ,  $x = \omega y$ , and write the argument of integral (A.6) as a power series in  $y$ . This gives

$$\begin{aligned} P_{-\frac{1}{2}+i\omega}(\cosh y) &= \frac{\sqrt{2}}{\pi} \int_0^1 \frac{y \cos xu}{\sqrt{\cosh y - \cosh uy}} du \\ &= \frac{1}{\pi} \int_0^1 \cos xu \left[ \frac{2}{\sqrt{1 - u^2}} - \frac{y^2(1 + u^2)}{12\sqrt{1 - u^2}} + \dots \right] du \\ &= J_0(x) - \frac{y^2}{12} \left[ J_0(x) - \frac{J_1(x)}{2x} \right] + \dots \end{aligned} \quad (\text{A.8})$$

Note that this series expansion is exact. In particular, since the resulting series is controlled by  $y$ , we can truncate it at any desired order in this variable if we regard  $x$  as a fixed number.

Following the same procedure as above, but now using

$$P_\ell(\cos \theta) = \frac{\sqrt{2}}{\pi} \int_0^\theta \frac{\cos(\ell + 1/2)t}{\sqrt{\cos t - \cos \theta}} dt, \quad 0 \leq \theta \leq \pi,$$

one can show that, defining  $x = (\ell + 1/2)\theta$ , one has [41]

$$P_\ell(\cos \theta) = J_0(x) + \frac{\theta^2}{12} \left[ J_0(x) - \frac{J_1(x)}{2x} \right] + \dots \quad (\text{A.9})$$

Again, note that the series is controlled by  $\theta$ , for constant  $x$ .

The reader familiar with the so-called flat sky approximation (usually employed in CMB analysis) will recognize expansion (A.9). It says that, for small patches of the CMB sky, the eigenfunctions of the sphere, i.e., the Legendre functions, converge towards eigenfunctions of the plane, i.e, Bessel functions [42]. For the same reason, expression (A.8) gives the eigenfunctions of the pseudo-sphere in terms of eigenfunctions of the plane.

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