ON THE EXPLICIT EXPRESSIONS OF THE CANONICAL 8-FORM ON RIEMANNIAN MANIFOLDS WITH Spin(9) HOLONOMY

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ABSTRACT. Two explicit expressions of the canonical 8-form on a Riemannian manifold with holonomy group Spin(9) have been given: One by the present authors and another by Parton and Piccinni. The relation between these two expressions is obtained. Moreover, it is shown that they are different only from a combinatorial viewpoint.

1. INTRODUCTION AND PRELIMINARIES

The group Spin(9) belongs to Berger's list of restricted holonomy groups of locally irreducible Riemannian manifolds which are not locally symmetric. Manifolds with holonomy group Spin(9) have been studied by Alekseevsky [3], Brown and Gray [6], Brada and Pécaut-Pison [5, 4], Abe [1], Abe and Matsubara [2], Friedrich [8, 9], Lam [10], the present authors [7], Sati [15], Parton and Piccinni [12], [13], [14], and Ornea, Parton, Piccinni and Vuletescu [11], among other authors. As proved in [6], a connected, simply-connected, complete non-flat Spin(9)-manifold is isometric to either the Cayley projective plane $\mathbb{O}\mathrm{P}(2) \cong F_4/\mathrm{Spin}(9)$ or its dual symmetric space, the Cayley hyperbolic plane $\mathbb{O}\mathrm{H}(2) \cong F_{4(-20)}/\mathrm{Spin}(9)$ (see also [3]).

We recall that a Spin(9)-structure on a connected, oriented 16-dimensional Riemannian manifold (M, g) is defined as a reduction of its bundle of oriented orthonormal frames $\mathrm{SO}(M)$, via the spin representation $\rho(\mathrm{Spin}(9)) \subset \mathrm{SO}(16)$. Equivalently (Friedrich [8, 9]), a Spin(9)-structure is given by a nine-dimensional subbundle ν^9 of the bundle of endomorphisms $\mathrm{End}(TM)$, locally spanned by a set of endomorphisms $I_i \in \Gamma(\nu^9)$, $0 \leqslant i \leqslant 8$, satisfying the relations

$$I_iI_j + I_jI_i = 0, \;\; i \neq j, \quad I_i^2 = \mathrm{I}, \quad I_i^T = I_i, \quad \mathrm{tr}\, I_i = 0, \;\; i, j = 0, \dots, 8.$$

These endomorphisms define 2-forms ω_{ij}, $0 \leqslant i < j \leqslant 8$, on M locally by $\omega_{ij}(X, Y) = g(X, I_iI_jY)$. Similarly, using the skew-symmetric endomorphisms $I_iI_jI_k$, $0 \leqslant i < j < k \leqslant 8$, one can define 2-forms σ_{ijk}. The 2-forms $\{\omega_{ij}, \sigma_{ijk}\}$ are linearly independent and a local basis of the bundle $\Lambda^2 M$.

Moreover, Δ_9 being the unique irreducible 16-dimensional Spin(9)-module, the Spin(9)-module $\Lambda^8(\Delta_9^*)$ contains one and only one (up to a non-zero factor) 8-form Ω_0^8 which is Spin(9)-invariant and defines the unique parallel 8-form on $\mathbb{O}\mathrm{P}(2)$. It induces a *canonical* 8-*form,* which we denote again by Ω_0^8, on any 16-dimensional

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manifold with a fixed Spin(9)-structure. This form is said to be canonical because [6, p. 48] it yields, for the compact case, a generator of $H^8(\mathbb{O}\mathrm{P}(2),\mathbb{R})$.

Some expressions of Ω_0^8 have been given. The first one by Brown and Gray [6, p. 49], in terms of a Haar integral. An explicit expression was given in [4, pp. 150, 153] and [5], by using a vector cross-product. Unfortunately, this expression is not correct (see [7] for a more detailed explanation). Another explicit expression was then given in [2, p. 8], as a sum of 702 suitable terms (see also [1]). This expression contains some errors (see [7] for a more detailed explanation).

We gave in [7] an explicit expression of Ω_0^8, which we denote here by Ω^8, in terms of the 9×9 skew-symmetric matrix of local Kähler 2-forms, $\omega=(\omega_{ij})$. The invariance and non-triviality of Ω^8 was proved using the properties of the automorphisms of the octonion algebra.

Another explicit expression of Ω_0^8, as the fourth coefficient $\tau_4(\omega)$ of the characteristic polynomial of the matrix ω, was given by Parton and Piccinni in [12]. To prove the non-triviality of $\tau_4(\omega)$, they performed a computer computation with the help of the software `Mathematica`.

The expression Ω^8 of the (global) canonical 8-form on the Spin(9)-manifold (M,g,ν^9) is given [7] by

$$\Omega^8=\sum_{i,j,i',j'=0,\dots,8}\omega_{ij}\wedge\omega_{ij'}\wedge\omega_{i'j}\wedge\omega_{i'j'},$$

where $\omega_{ij}=-\omega_{ji}$ if $i>j$ and $\omega_{ij}=0$ if $i=j$.

In turn, the expression $\tau_4(\omega)$ of the canonical 8-form on (M,g,ν^9) is given [12] by

$$\tau_4(\omega)=\sum_{0\leqslant\alpha_1<\alpha_2<\alpha_3<\alpha_4\leqslant 8}(\omega_{\alpha_1\alpha_2}\wedge\omega_{\alpha_3\alpha_4}-\omega_{\alpha_1\alpha_3}\wedge\omega_{\alpha_2\alpha_4}+\omega_{\alpha_1\alpha_4}\wedge\omega_{\alpha_2\alpha_3})^2.$$

Note that the fourth coefficient $\tau_4(\omega)$ of the characteristic polynomial of the skew-symmetric matrix ω is given as the summation of the squared Pfaffians of the principal 4×4-submatrices of ω.

Let $\mathbb{R}[x_{01},\dots,x_{78}]$ be the commutative polynomial ring on the 36 variables x_{ij}, $i<j$. Put $x_{ij}=-x_{ji}$, for $i>j$, and $x_{ii}=0$ for convenience. Consider the following three polynomial functions $F,P,Q\in\mathbb{R}[x_{01},\dots,x_{78}]$,

$$F=\sum_{i,j,i',j'=0,\dots,8}x_{ij}x_{ij'}x_{i'j}x_{i'j'},\qquad P=\sum_{0\leqslant i<j\leqslant 8}x_{ij}^2, \tag{1.1}$$

and

$$Q=\sum_{0\leqslant\alpha_1<\alpha_2<\alpha_3<\alpha_4\leqslant 8}(x_{\alpha_1\alpha_2}x_{\alpha_3\alpha_4}-x_{\alpha_1\alpha_3}x_{\alpha_2\alpha_4}+x_{\alpha_1\alpha_4}x_{\alpha_2\alpha_3})^2.$$

The aim of the present paper is to prove that the (combinatorial) relation

$$F=2P^2-4Q, \tag{1.2}$$

holds in $\mathbb{R}[x_{01},\dots,x_{78}]$.

Now, as proved in [7, Corollary 7], the 4-form

$$\omega^4=\sum_{0\leqslant i<j\leqslant 8}\omega_{ij}\wedge\omega_{ij},$$

vanishes on any Spin(9)-manifold. This fact also follows easily from the results of Brown and Gray [6, Section 5]: any Spin(9)-invariant 4-form on the space Δ_9 is trivial.

Since all 2-forms commute (in particular, the forms ω_{ij} commute) and the 8-form $\omega^4 \wedge \omega^4$, corresponding to the polynomial function P^2, vanishes, it follows immediately from (1.2) the

Proposition 1.1. *The expressions Ω^8 and $\tau_4(\omega)$ of the canonical 8-form on the* $\mathrm{Spin}(9)$*-manifold (M, g, ν^9), are related by*

$$\Omega^8 = -\,4\tau_4(\omega). \tag{1.3}$$

2. Proof of the main relation

We will denote the union of two disjoint sets A and B by $A \sqcup B$.

First of all consider the function F defined by (1.1). Denote by W the set of all ordered pairs ij, where $i, j = 0, \dots, 8$. Let $D = \{ii, i = 0, \dots, 8\}$ be the diagonal in W and let $\overline{D} = W \setminus D$. Since $x_{ii} = 0$ for all $0 \leqslant i \leqslant 8$, we have that

$$F = \sum_{(ij,i'j') \in \overline{D} \times \overline{D}} x_{ij} x_{ij'} x_{i'j} x_{i'j'}.$$

Note that the sequence $ij, ij', i'j, i'j'$ is a sequence of vertices of either a rectangle or a degenerate rectangle made of entries of a square 9×9 matrix. This sequence originates an either 1– or 2– or 4–element subset of W. So it is natural to represent the product $\overline{D} \times \overline{D}$ as the union $\overline{D} \times \overline{D} = \overline{D}^{\times}_{12} \sqcup \overline{D}^{\times}_{4}$ of the two disjoint nonempty subsets,

$$\begin{aligned} \overline{D}^{\times}_{12} &= \big\{(ij, i'j') \in \overline{D} \times \overline{D} : i = i' \text{ or } j = j'\big\}, \\ \overline{D}^{\times}_{4} &= \big\{(ij, i'j') \in \overline{D} \times \overline{D} : i \neq i',\ j \neq j'\big\}. \end{aligned}$$

Now, for each pair $(ij, i'j') \in \overline{D}^{\times}_{4}$, the 4–element subset (that is, the rectangle $\{ij, ij', i'j, i'j'\}$) of W has either 0 or 1 or 2 common elements with the diagonal $D \subset W$. So it is natural to represent the set $\overline{D}^{\times}_{4}$ as the union of the two disjoint nonempty subsets $\overline{D}^{\times}_{4,0}$ and $\overline{D}^{\times}_{4,12}$, where $\overline{D}^{\times}_{4,12} = \overline{D}^{\times}_{4} \setminus \overline{D}^{\times}_{4,0}$ and

$$\overline{D}^{\times}_{4,0} = \big\{(ij, i'j') \in \overline{D} \times \overline{D} : i \neq i',\ j \neq j', i \neq j', i' \neq j\big\}.$$

Since $x_{ij} x_{ij'} x_{i'j} x_{i'j'} = x_{ij}^2 x_{i'j'}^2$ for $(ij, i'j') \in \overline{D}^{\times}_{12}$, and $x_{ij} x_{ij'} x_{i'j} x_{i'j'} = 0$ if the rectangle generated by the pair $(ij, i'j') \in \overline{D} \times \overline{D}$ intersects the diagonal D, we obtain that

$$F = \sum_{(ij,i'j') \in \overline{D}^{\times}_{12}} x_{ij}^2 x_{i'j'}^2 + \sum_{(ij,i'j') \in \overline{D}^{\times}_{4,0}} x_{ij} x_{ij'} x_{i'j} x_{i'j'},$$

because, by the definition of $\overline{D}$, one has $\overline{D} \times \overline{D} = \overline{D}^{\times}_{12} \sqcup \overline{D}^{\times}_{4,12} \sqcup \overline{D}^{\times}_{4,0}$ (the union of three disjoint nonempty subsets).

It is clear that $2P = \sum_{ij \in \overline{D}} x_{ij}^2$ because $x_{ij} = -x_{ji}$. Thus for the polynomial function $4P^2 = \sum_{(ij,i'j') \in \overline{D} \times \overline{D}} x_{ij}^2 x_{i'j'}^2$ we have that

$$4P^2 = \sum_{(ij,i'j') \in \overline{D}^{\times}_{12}} x_{ij}^2 x_{i'j'}^2 + \sum_{(ij,i'j') \in \overline{D}^{\times}_{4,12}} x_{ij}^2 x_{i'j'}^2 + \sum_{(ij,i'j') \in \overline{D}^{\times}_{4,0}} x_{ij}^2 x_{i'j'}^2.$$

Consider now the involution $\mu_1 \colon \overline{D} \times \overline{D} \to \overline{D} \times \overline{D}$, $(ij, i'j') \mapsto (ji, i'j')$. Since this map is a reflection with respect to the diagonal D, one has

$$\mu_1(\overline{D}^{\times}_{4,12}) = \overline{D}^{\times}_{12} \quad \text{and} \quad \mu_1(\overline{D}^{\times}_{12}) = \overline{D}^{\times}_{4,12}, \tag{2.1}$$

so, in particular, $\#(\overline{D}^{\times}_{12}) = \#(\overline{D}^{\times}_{4,12})$. Indeed, for each pair $(ij, ij') \in \overline{D} \times \overline{D}$, $j \neq j'$, generating a 2–element subset of W, the pair $\mu_1(ij, ij') = (ji, ij') \in \overline{D} \times \overline{D}$, generates a 4–element subset (the rectangle $\{ji, jj', ii, ij'\}$) having one common point with the diagonal D. For each pair $(ij, ij) \in \overline{D} \times \overline{D}$, generating an 1–element subset in W, the pair $\mu_1(ij, ij) = (ji, ij) \in \overline{D} \times \overline{D}$ generates a 4–element subset (the rectangle $\{ji, jj, ii, ij\}$) having two common points with the diagonal D. In other words, $\mu_1(\overline{D}^{\times}_{4,12}) \subset \overline{D}^{\times}_{12}$ and $\mu_1(\overline{D}^{\times}_{12}) \subset \overline{D}^{\times}_{4,12}$. Since μ_1 is an involution on $\overline{D} \times \overline{D}$ and $\overline{D}^{\times}_{12} \cap \overline{D}^{\times}_{4,12} = \emptyset$, relation (2.1) follows.

Now, by (2.1) we have

$$4P^2 = 2 \sum_{(ij,i'j') \in \overline{D}^{\times}_{12}} x^2_{ij} x^2_{i'j'} + \sum_{(ij,i'j') \in \overline{D}^{\times}_{4,0}} x^2_{ij} x^2_{i'j'}$$

because $x^2_{ij} = x^2_{ji}$. So that

$$F - 2P^2 = \sum_{(ij,i'j') \in \overline{D}^{\times}_{4,0}} \left(x_{ij} x_{ij'} x_{i'j} x_{i'j'} - \frac{1}{2} x^2_{ij} x^2_{i'j'} \right).$$

Since exactly four different pairs $(ij, i'j') \in \overline{D}^{\times}_{4,0}$ determine the same 4–element subset $\{ij, ij', i'j, i'j'\} \subset \overline{D}$, we obtain that

$$F - 2P^2 = \sum_{\{ij,ij',i'j,i'j'\} \in \overline{D}_4} \left(4x_{ij} x_{ij'} x_{i'j} x_{i'j'} - x^2_{ij} x^2_{i'j'} - x^2_{ij'} x^2_{i'j} \right),$$

where $\overline{D}_4 = \big\{ \{ij, ij', i'j, i'j'\} \subset \overline{D}, i \neq i',\ j \neq j', i \neq j', i' \neq j \big\}$.

To prove the relation (1.3), note that for each pair $(ij, i'j') \in \overline{D}^{\times}_{4,0}$ and, consequently, for each subset $\{ij, ij', i'j, i'j'\} \in \overline{D}_4$, the sequence (i, j, i', j') consists of distinct elements of the set $\{0, \dots, 8\}$. For each subset $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, where $0 \leqslant \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leqslant 8$, there exist exactly 6 different subsets $\{ij, ij', i'j, i'j'\} \in \overline{D}_4$ (rectangles) such that $\{i, j, i', j'\} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ (see Remark 2.1 below for a particular case). These 4-element subsets are determined by the following pairs $(ij, i'j')$ of the set $\overline{D}^{\times}_{4,0}$,

$$\begin{aligned} &(\alpha_1\alpha_2, \alpha_3\alpha_4), \quad (\alpha_2\alpha_1, \alpha_4\alpha_3), \quad (\alpha_1\alpha_2, \alpha_4\alpha_3), \\ &(\alpha_2\alpha_1, \alpha_3\alpha_4), \quad (\alpha_1\alpha_3, \alpha_2\alpha_4), \quad (\alpha_3\alpha_1, \alpha_4\alpha_2). \end{aligned}$$

Thus $F - 2P^2 = \sum_{0 \leqslant \alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 \leqslant 8} A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$, where

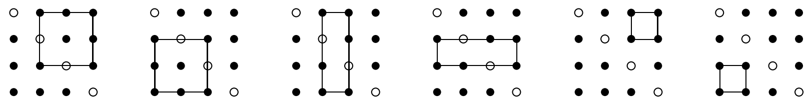
$$\begin{aligned} A(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = {} & 4x_{\alpha_1\alpha_2} x_{\alpha_1\alpha_4} x_{\alpha_3\alpha_2} x_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_2} x^2_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_4} x^2_{\alpha_3\alpha_2} \\ & + 4x_{\alpha_2\alpha_1} x_{\alpha_2\alpha_3} x_{\alpha_4\alpha_1} x_{\alpha_4\alpha_3} - x^2_{\alpha_2\alpha_1} x^2_{\alpha_4\alpha_3} - x^2_{\alpha_2\alpha_3} x^2_{\alpha_4\alpha_1} \\ & + 4x_{\alpha_1\alpha_2} x_{\alpha_1\alpha_3} x_{\alpha_4\alpha_2} x_{\alpha_4\alpha_3} - x^2_{\alpha_1\alpha_2} x^2_{\alpha_4\alpha_3} - x^2_{\alpha_1\alpha_3} x^2_{\alpha_4\alpha_2} \\ & + 4x_{\alpha_2\alpha_1} x_{\alpha_2\alpha_4} x_{\alpha_3\alpha_1} x_{\alpha_3\alpha_4} - x^2_{\alpha_2\alpha_1} x^2_{\alpha_3\alpha_4} - x^2_{\alpha_2\alpha_4} x^2_{\alpha_3\alpha_1} \\ & + 4x_{\alpha_1\alpha_3} x_{\alpha_1\alpha_4} x_{\alpha_2\alpha_3} x_{\alpha_2\alpha_4} - x^2_{\alpha_1\alpha_3} x^2_{\alpha_2\alpha_4} - x^2_{\alpha_1\alpha_4} x^2_{\alpha_2\alpha_3} \\ & + 4x_{\alpha_3\alpha_1} x_{\alpha_3\alpha_2} x_{\alpha_4\alpha_1} x_{\alpha_4\alpha_2} - x^2_{\alpha_3\alpha_1} x^2_{\alpha_4\alpha_2} - x^2_{\alpha_3\alpha_2} x^2_{\alpha_4\alpha_1}. \end{aligned}$$

Taking into account that $x_{\alpha_a\alpha_b} = -x_{\alpha_b\alpha_a}$ and replacing each $x_{\alpha_a\alpha_b}$ by $-x_{\alpha_b\alpha_a}$ if $a > b$, we obtain

$$\begin{aligned}
A(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = &-4x_{\alpha_1\alpha_2}x_{\alpha_1\alpha_4}x_{\alpha_2\alpha_3}x_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_2}x^2_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_4}x^2_{\alpha_2\alpha_3}\\
&-4x_{\alpha_1\alpha_2}x_{\alpha_2\alpha_3}x_{\alpha_1\alpha_4}x_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_2}x^2_{\alpha_3\alpha_4} - x^2_{\alpha_2\alpha_3}x^2_{\alpha_1\alpha_4}\\
&+4x_{\alpha_1\alpha_2}x_{\alpha_1\alpha_3}x_{\alpha_2\alpha_4}x_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_2}x^2_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_3}x^2_{\alpha_2\alpha_4}\\
&+4x_{\alpha_1\alpha_2}x_{\alpha_2\alpha_4}x_{\alpha_1\alpha_3}x_{\alpha_3\alpha_4} - x^2_{\alpha_1\alpha_2}x^2_{\alpha_3\alpha_4} - x^2_{\alpha_2\alpha_4}x^2_{\alpha_1\alpha_3}\\
&+4x_{\alpha_1\alpha_3}x_{\alpha_1\alpha_4}x_{\alpha_2\alpha_3}x_{\alpha_2\alpha_4} - x^2_{\alpha_1\alpha_3}x^2_{\alpha_2\alpha_4} - x^2_{\alpha_1\alpha_4}x^2_{\alpha_2\alpha_3}\\
&+4x_{\alpha_1\alpha_3}x_{\alpha_2\alpha_3}x_{\alpha_1\alpha_4}x_{\alpha_2\alpha_4} - x^2_{\alpha_1\alpha_3}x^2_{\alpha_2\alpha_4} - x^2_{\alpha_2\alpha_3}x^2_{\alpha_1\alpha_4}\\
= &-8x_{\alpha_1\alpha_2}x_{\alpha_1\alpha_4}x_{\alpha_2\alpha_3}x_{\alpha_3\alpha_4} + 8x_{\alpha_1\alpha_2}x_{\alpha_1\alpha_3}x_{\alpha_2\alpha_4}x_{\alpha_3\alpha_4}\\
&+8x_{\alpha_1\alpha_3}x_{\alpha_1\alpha_4}x_{\alpha_2\alpha_3}x_{\alpha_2\alpha_4} - 4x^2_{\alpha_1\alpha_2}x^2_{\alpha_3\alpha_4}\\
&-4x^2_{\alpha_1\alpha_4}x^2_{\alpha_2\alpha_3} - 4x^2_{\alpha_1\alpha_3}x^2_{\alpha_2\alpha_4}\\
= &-4(x_{\alpha_1\alpha_2}x_{\alpha_3\alpha_4} - x_{\alpha_1\alpha_3}x_{\alpha_2\alpha_4} + x_{\alpha_1\alpha_4}x_{\alpha_2\alpha_3})^2.
\end{aligned}$$

Consequently, $F - 2P^2 = -4Q$, and relation (1.2) is proved.

Remark 2.1. As one may see in the pictures below, in a 4×4 rectangle there exist exactly 6 rectangles with different sets of vertices $\{ij, ij', i'j, i'j'\}$, for $\{i, j, i', j'\} = \{1, 2, 3, 4\}$, not containing the diagonal vertices $\{11, 22, 33, 44\}$.



12 34   21 43   12 43   21 34   13 24   31 42

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