DIFFERENTIABLE APPROXIMATIONS TO BROWNIAN MOTION ON MANIFOLDS

by

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DECLARATION

The results of Chapter Three first appeared in my M.Sc. dissertation, 'Langevin Equations on Manifolds' in 1975.

SUMMARY

The main part of this thesis is devoted to generalised Ornstein-Uhlenbeck processes. We show how to construct such processes on 2-uniformly smooth Banach spaces. We give two methods of constructing Ornstein-Uhlenbeck type processes on manifolds with sufficient structure, including on a finite dimensional Riemannian manifold where we actually construct a process on the orthonormal bundle $O(M)$ and project down to M to obtain the required process. We show that in the simplest case on a finite dimensional Riemannian manifold the two constructions give rise to the same process. We construct the infinitesimal generator of this process.

We show that, given a Hilbert space and a Banach space E with W a Brownian motion on E whose index set includes $[0,R]$, and $X: H \longrightarrow L(E;H), V: H \longrightarrow H$ satisfying sufficient boundedness and Lipschitz conditions, the solutions of the family of o.d.e.'s $dx_\beta = X(x_\beta)v_\beta dt + V(x_\beta)dt$ (where v_β is an O-U velocity process on E), indexed by $\omega \in \Omega$ where Ω is the probability space over which W is defined, converges in $\mathcal{L}^2$-norm to a solution of $dx = X(x)dW + V(x)dt$, both solutions having the same starting point. We show that the convergence is uniform over $[0,R]$ in probability, and include a proof of Elworthy, from 'Stochastic Differential Equations on Manifolds' (Warwick University preprint, 1978) to show that convergence still occurs when the processes are constructed on suitable manifolds (Elworthy's proof is for piecewise linear approximations). We extend our results to include O-U processes in 'force-fields'. We follow the method of Elworthy to show the uniform convergence of the flows of the constructed processes.

Finally we prove similar convergence theorems for piecewise-linear approximations, following the proofs of Elworthy.

INTRODUCTION

The main topic considered in this thesis is that of Ornstein-Uhlenbeck (O-U) processes which we will construct on suitable objects. We show that, under certain conditions, our O-U processes converge to similarly constructed Brownian motions. We also consider piecewise linear approximations to Brownian motion and mention, briefly, Malliavin's approximation.

The first chapter deals with the concepts we will need for the later work. Its first section extracts a few key concepts and results from the work of Eells and Elworthy (as expounded in Elworthy [4]) on stochastic dynamical systems, while the second uses one of their results (the Itô formula on a Hilbert space) to obtain an 'integration by parts' formula which generalises that of McShane [15, p.127]. The third section draws mainly from the work of Neidhardt [16] and lists some of his results for stochastic integration on 2-uniformly smooth Banach spaces (a Banach space E is 2-uniformly smooth if there exists $A_E>0$ s.t. $\frac{1}{2}\{\|x+y\|_E^2 + \|x-y\|_E^2\} \le \|x\|_E^2 + A_E\|y\|_E^2$ for all $x,y\in E$). In this case we have the inequality, $E\|\int_0^R G\,dW\|_E^2 \le A_E E\int_0^R \|G\|_\gamma^2 dt$, and if E is a Hilbert space we have corresponding $\mathcal{L}^{2m}$ inequalities. We prove the well-known fact that if E is a Hilbert space and $L : H \longrightarrow E_0$ is the inclusion of a Hilbert space into a Banach space , which generates a Brownian motion W on E_0 then $\int B(dW,dW) = \int \mathrm{tr}_L B\,dt$ for $B(t)\in L^2(E_0,E)$.

In sections 4 and 5 we again extract from Elworthy [4]. Section 4 gives a formula for the infinitesimal generator of a process on a manifold , based on a Brownian motion on a

Hilbert space. Section 5 shows how to construct a Brownian motion on a finite-dimensional Riemannian manifold, based on the canonical diffeomorphism $\theta: O(M)\times \mathbb{R}^n \longrightarrow HO(M)$, $(u,v) \longrightarrow (T\tau|H_uO(M))^{-1}u(v)$ where $\tau:O(M)\longrightarrow M$ is the orthonormal bundle over M.

Section 6 briefly describes Malliavin's C^∞ approximation to Brownian motion on manifolds.

The second chapter is concerned with the construction of O-U processes on 2-uniformly smooth Banach spaces. In the first section we state a few facts and formulae regarding O-U processes on the real line and in the second we show how these formulae generalise to 2-uniformly smooth Banach spaces. We show that neither the O-U position process nor the O-U velocity process satisfy McShane's condition 'A(q)' for $q\geq 1$, and that the O-U velocity process is a quasi-martingale. We show the boundedness of the $\mathcal{L}^2$-norm of an O-U velocity process on a Hilbert space.

In section 3 we consider the family of C^n approximations to Brownian motion on 2-uniformly smooth Banach spaces obtained by iterating the construction of the O-U process. I have not seen this treated, even on the real line.

The third chapter deals with the construction of O-U processes on manifolds. This chapter formed (embryonically) my M.Sc. dissertation. More recently Jørgensen [9] has published a construction of O-U type processes on Riemannian manifolds, which we see in section 4, by considering the infinitesimal generator, gives rise to the same process as our own. The first section shows how we can construct O-U processes on any manifold M with sufficient structure (i.e. a spray and a C^3 section of $Hom_M(E,TM)$ for some Banach space E). In the second section we show how to apply the basis of this construction to finite dimensional Riemannian manifolds, where we have a spray (the Riemannian spray which generates the geodesics) but not a section of $Hom_M(E,TM)$. Instead we lift to the orthonormal bundle and consider the section of $Hom_{\mathbb{R}^n}(O(M),HO(M))$. The construction is then similar to that of Brownian motion on a Riemannian manifold with the added complication that we construct a process on the tangent bundle. In the third section we consider another type of O-U process on a manifold based on an O-U process (rather than a Brownian motion) on E. This process is essentially different from the one constructed in the first two sections but we show that, in the simplest case on a Riemannian manifold, the processes are the same. In the fourth section we construct the infinitesimal generator of an O-U process on a Riemannian manifold.

The fourth chapter deals with the approximation of O-U processes to Brownian motions. The first two sections prove $\mathcal{L}^2$-convergence and supremum-norm convergence in probability on Hilbert spaces while the third shows convergence on manifolds, using a proof taken from Elworthy [4]. As examples we consider a paper by Arnold et al [1], and the Riemannian O-U process constructed in Chapter Three. Section 4 generalises the results by considering O-U processes in force-fields (see Nelson [17, Ch.10]. We show that equivalent convergence theorems hold for such processes. In section 5 we follow Elworthy [5] to show the uniform convergence of the flows of O-U processes on compact manifolds to the flow of an s.d.e. based on a Brownian motion.

In Chapter Five we consider the piecewise-linear approximation, proving $\mathcal{L}^2$-convergence, supremum-norm convergence (both on Hilbert spaces) and (hence) convergence on manifolds. The $\mathcal{L}^2$-convergence follows the proof in Elworthy [4] which I extracted from proofs of McShane [15] for a more general class of approximations on the real line. The proof of convergence in probability uses the martingale inequality, $E \sup_{0\le t\le R} |X(t)|^2 \le 4\, E|X(R)|^2$ and the expressions derived in the previous theorem, following the proof in Elworthy.

A few notes about notation are required. We will always consider processes x defined over $\Omega \times [0,R]$ where $(\Omega, \mathcal{F}, \mu)$ is a probability space with an increasing family of σ-subalgebras $\{\mathcal{F}_t\}_{t\in[0,R]}$ of $\mathcal{F}$ such that $x|[0,t]$ is $\mathcal{F}_t$-measurable. We will assume for convenience that the sets of measure zero in $\mathcal{F}$ are contained in each $\mathcal{F}_t$. E_t will denote expectation conditioned by $\mathcal{F}_t$.

$\mathcal{L}^2$-norm will usually be written as $\|\cdot\|$ while all other norms will carry subscripts to identify them. We use the terms 'Brownian motion' and 'Wiener process' interchangeably and do not restrict Brownian motion (in finite dimensions) to having covariance matrix I, although we do assume that the covariance matrix is diagonal.

We write $\mathrm{Hom}_M(E;TM)$ or $\mathrm{Hom}(M\times E;TM)$ for the fibre bundle over M whose elements are maps $v : M\times E \longrightarrow TM$ such that $v_m : E \longrightarrow T_mM$ is a homomorphism.

CHAPTER ONE. PRELIMINARIES.

§1. Stochastic Dynamical Systems.

The main reference for this section is Elworthy [4], but the theory is developed in finite dimensions in McShane [15].

Let z be a stochastic process with values in some separable Banach space G,

$$z : [a,b] \longrightarrow \mathcal{L}^0(\Omega, \mathfrak{F}; G),$$

where $[a,b] \subseteq T \subseteq \mathbb{R}$ and $(\Omega, \mathfrak{F}, \mu)$ is a probability space. We assume the existence of a family of σ-subalgebras of $\mathfrak{F}$, $\{\mathfrak{F}_t ; t \in T\}$ such that $t, t' \in T$ and $t < t' \Rightarrow \mathfrak{F}_t \subseteq \mathfrak{F}_{t'}$.

(1.1.1) Definition

z is said to satisfy condition A(q), where q is a positive integer, if z(t) is $\mathfrak{F}_t$-measurable $\forall\, t \in [a,b]$ and there exist constants $K>0, \delta>0$ such that if $a \leq s \leq t \leq b$ and $t-s<\delta$ then almost everywhere,

$$|E_s(z(t)-z(s))| \leq K(t-s),$$
$$\text{and } E_s(|z(t)-z(s)|^{2p}) \leq K(t-s) \text{ for } 1 \leq p \leq q$$

where E_s is expectation conditioned by $\mathfrak{F}_s$ and $|\,.\,|$ is G-norm.

In this section we will be concerned with stochastic processes z which satisfy A(q) for $q \geq 2$ and which also satisfy the condition that $z|_{[a,t]}$ has $\mathfrak{F}_t$-almost all of its sample paths continuous where $a \leq t \leq b$. Later we will only be interested in the case of z being a Brownian motion.

Let $G_1, \ldots, G_q$ be Banach spaces, H a Hilbert space and B a map $T \times \Omega \longrightarrow L(G_1, \ldots, G_q; H)$ which is $\mathfrak{F}_t$-random for each $t \in T$, where $L(G_1, \ldots, G_q; H)$ denotes the continuous q-linear maps.

(1.1.2) Definition

B is said to satisfy B(p) for $p \geq 1$, $p \in \mathbb{Z}$ if for each $t \in T$ $B(t)$ is $\mathfrak{F}_t$-random and B is continuous in $\mathcal{L}^{2p}$-norm at each point of [a,b], i.e. if $t \in [a,b]$ then $\|B(t)\|_{\mathcal{L}^{2p}} < \infty$ and as $\tau \longrightarrow t$ in T so $\|B(\tau) - B(t)\|_{\mathcal{L}^{2p}} \longrightarrow 0$.

(1.1.3) Definition

B is said to satisfy B_o if :-

(i) for each $t \in T$ $B(t)$ is $\mathfrak{F}_t$-random;

(ii) B is norm continuous in measure at each point of [a,b], i.e. for each $t \in [a,b]$

$$|B(\tau) - B(t)| \longrightarrow 0 \text{ in measure as } \tau \longrightarrow t \text{ in } T,$$

where $|\cdot|$ is norm in $L(G_1, \ldots, G_q; H)$;

and (iii) B has bounded sample paths almost surely, i.e. there is $\mathfrak{F}$-measurable $b : \Omega \longrightarrow \mathbb{R}$ with

$$|B(t,\omega)| < b(\omega) \ \forall \ t \in T \text{ a.e.}$$

Let $z^1, \ldots, z^q$ be such that $z^i : [a,b] \longrightarrow \mathcal{L}^o(\Omega, \mathfrak{F}; G_i)$ and the z^i satisfy A(q).

We take the following facts from Elworthy .

(1.1.4) <u>Theorem</u>

If B satisfies B_o or $B(1)$ then the belated integral

$\int_c^e B(s)(dz^1(s),\ldots,dz^q(s))$ exists for all $a\leq c<e\leq b$.

The integral has the following properties:-

(i) $\exists\ \beta > 0$, depending only on the K with respect to which $z^1,\ldots,z^q$ satisfy $A(q)$ and $b-a$, such that

$$\left\| \int_c^e B(dz^1,\ldots,dz^q) \right\|_2 \leq \beta\left(\int_c^e \|B(t)\|_2^2\, dt\right)^{\frac{1}{2}}$$

where $\|.\|_2$ is $\mathcal{L}^2$-norm (we will hereafter write $\mathcal{L}_2$-norm as $\|.\|$);

(ii) If $q=2$ and a.s. $|z^2(t,\omega) - z^2(s,\omega)| = o(|t-s|^{\frac{1}{2}})$ as $|t-s| \longrightarrow 0$ for $t,s \in [a,b]$ then

$$\int_a^b B(dz^1,dz^2) = 0 \text{ a.s.},$$

e.g. $z^2(t,\omega) = t\ \forall\ \omega \in \Omega$;

(iii) If for some k, $1\leq k\leq q$, z^k has a.s. continuous sample paths on $[a,b]$ then a.s.

$$\int_a^b B(dz^1,\ldots,dz^q) = 0 \text{ for } q\geq 3;$$

(iv) Let W be a Brownian motion on $\mathbb{R}^n$ with covariance matrix I and mean zero. Then,

$$\int_a^b B(t,\omega)(dW(t,\omega),dW(t,\omega)) = \int_a^b trB(t,\omega)dt,$$

where $trB(t,\omega)$ is the trace of $B(t,\omega): \mathbb{R}^n \longrightarrow \mathbb{R}^n = (\mathbb{R}^n)^*$ and the right hand side denotes the vector-val ued Riemann integral.

Let M be a separable, metrisable C^3-manifold modelled on a

Hilbert space, H. Let E be a Banach space and X a section of Hom(MxE,TM), i.e. X assigns to each $m \in M$ a linear map $X_m : E \longrightarrow T_mM$.

The pair (X,z) with z a stochastic process on E satisfying A(4) is called a <u>stochastic dynamical system</u> (S.D.S.) on M and we restrict our attention to S.D.S.s on M with X C^1 and having first derivatives (in coordinate charts) locally Lipschitz. X will normally be C^2.

(1.1.5) A regular localization of the S.D.S. (X,z) at the point $m \in M$ is a triple $\Lambda = ((U,\phi), U_o, \lambda)$ where:-

(i) (U,ϕ) is a C^2 chart about m with $\phi(U)$ a bounded open subset of H, $\phi(U) = W$;

(ii) U_o is an open neighbourhood of m in U with $\overline{\phi(U_o)} \subset W$. Let $W_o = \phi(U_o)$;

(iii) $\lambda : H \longrightarrow [0,1]$ is C^2 with $\operatorname{supp}\lambda \subset W$, $\lambda | W_o \equiv 1$;

(iv) If $X_\Lambda(h) = \lambda(h)\phi_*(X)(h)$,

then X_Λ and $\frac{1}{2}DX_\Lambda \circ X_\Lambda$ are globally Lipschitz [where $\phi_*(X)(h) = T_{\phi^{-1}(h)}\phi \circ X(\phi^{-1}(h))$ is the local representative of X in the chart (U,ϕ), and $DX_\Lambda \circ X_\Lambda : H \longrightarrow \mathbb{L}_2(E,H)$ is given by, $DX_\Lambda \circ X_\Lambda(h,e) = DX_\Lambda(h)(X_\Lambda(h)e)(e)$].

A regular localization exists at each point $m \in M$. Given a localization Λ and $a \le t_o \le t \le b$ we have a solution y of,

$$(1.1.6)\quad y(t) = y(t_o) + \int_{t_o}^t X_\Lambda(y(s))dz(s) + \tfrac{1}{2}\int_{t_o}^t DX_\Lambda(y(s))(X_\Lambda(y(s))dz(s), dz(s)),$$

(the integrals being belated integrals).

A process $x : [a,\xi') \times \Omega \longrightarrow M$, defined up to a stopping time $\xi : \Omega \longrightarrow (a,b]$ is a solution of the stochastic differential equation (s.d.e.) $dx = Xdz$ if it is non-anticipating, has continuous sample paths almost surely and there is a cover of M by regular localizations such that if $\Lambda = ((U,\phi), U_0, \lambda)$ is a member of the cover and $a \le t_0 \le b$ then $\theta \circ x$ agrees with the solution of 1.1.6 for $y_0 = \theta \circ x(t_0)$, almost surely on $\{x(t_0) \in U_0\}$ from time t_0 to the first exit time after t_0 of that solution from W_0, where $\theta : M \longrightarrow H$ is some extension of ϕ.

The following theorem can be found in Elworthy [4].

(1.1.7) Theorem

For (X,z) as above dx = Xdz has a unique maximal solution $x : [a, \xi) \times \Omega \longrightarrow M$, where $\xi : \Omega \longrightarrow (a,b]$ is the explosion time. The explosion time is b almost everywhere if M is compact.

Examples

(1.1.8) Let V be a vector field on M and define $X \in \mathrm{Hom}(M \times \mathbb{R}, TM)$ by $X_m(t) = tV(m)$. Define $z : (-\infty, \infty) \times \Omega \longrightarrow \mathbb{R}$ by $z(t,\omega) = t \ \forall \omega \in \Omega$. Then (X,z) is an ordinary dynamical system or drift, since in this case solutions are locally of the form $x(t) = x_0 + \int_0^t X(x(s))ds + 0$,

$$= x_0 + \int_0^t V(x(s))ds,$$

since $\int B(ds,ds) = 0$ by property (ii) following Theorem 1.1.4.

V need only be C^1 for a solution to exist, and in this case we write (X,z) as (V,t).

(1.1.9) Let (X_1, z_1) be an S.D.S. and (V,t) a drift, both defined on M, with X_1 C^2 and V C^1,

$$X_1 : M\times E \longrightarrow TM, \quad z_1 : [a,b]\times\Omega \longrightarrow E.$$

Using these we define $X \in \mathrm{Hom}(M\times(E\times\mathbb{R}),TM)$ by $X_m(e,t) = X_1(m,e) + tV(m)$, and we define $z:[a,b]\times\Omega \longrightarrow E\times\mathbb{R}$ by $z(t,\omega) = (z_1(t,\omega),t)$.

(X,z) is the <u>direct sum</u> of (X_1,z_1) and (V,t) and can be written $(X_1\oplus V, z_1\oplus t)$.

$dx = Xdz$ has solutions which look locally like,

$$x(t) = x_a + \int_a^t X_1(x(s))dz_1(s) + \int_a^t V(x(s))ds +$$

$$+ \tfrac{1}{2}\int_a^t DX_1(x(s))\circ X_1(x(s))(dz_1(s),dz_1(s)).$$

We write $dx = X_1 dz_1 + Vdt$.

Let $M=E=\mathbb{R}^n$ and let z_1 be a Brownian motion on $\mathbb{R}^n$ with an o.n. basis $\{e_i\}$ of $\mathbb{R}^n$ with respect to which z_1 has independent components $(z_1^1,\ldots\ldots,z_1^n)$ and mean zero such that $E(z_1^i(s)^2) = c_i|s|$. In this case the above equation becomes,

$$x(t) = x_a + \int_a^t V(x(s))dz_1(s) + \int_a^t X_1(x(s))dz_1(s) +$$

$$+ \tfrac{1}{2}\int_a^t \mathrm{tr}_{z_1}(DX_1(x(s))\circ X_1(x(s)))ds,$$

where $\mathrm{tr}_{z_1}(DX_1(x(s))\circ X_1(x(s)) = \sum c_i DX_1(x(s))\circ X_1(x(s))(e_i,e_i)$.

(1.1.10) We will construct an Ornstein-Uhlenbeck process on $\mathbb{R}^n$ using example 1.1.9.

Define $X_1 : T\mathbb{R}^n\times\mathbb{R}^n \longrightarrow TT\mathbb{R}^n$ by

$$X_1(x,v,w) = (x,v,0,\beta w)$$

and $V : T\mathbb{R}^n \longrightarrow TT\mathbb{R}^n$ by

$$V(x,v) = (x,v,v,-\beta v),$$

where $\beta>0$ is a constant.

Let z_1 be a Brownian motion on $\mathbb{R}^n$ as in example 1.1.9 and let $(x(t),v(t))$ be a solution of the s.d.e $d(x,v)=X_1dz_1+Vdt$, $(x(0),v(0)) = (x_o,v_o)$.

Thus $(dx(t),dv(t)) = (v(t)dt,-\beta v(t)dt + \beta dz_1(t))$

$$\text{giving} \quad \begin{cases} dx(t) = v(t)dt \ , \ x(0) = x_o \\ dv(t) = -\beta v(t)dt + \beta dz_1(t) \ , \ v(0) = v_o, \end{cases}$$

the equations of an Ornstein-Uhlenbeck process on $\mathbb{R}^n$ (see Nelson [17] and also Chapter Two). Note that the integral form of the above s.d.e. for v has zero second-order term.

If V was instead given by

$$V(x,v) = (x,v,v,-\beta v+b(x)),$$

then we have the equations of an Ornstein-Uhlenbeck process in a force field (see §10 of Nelson [17]) which we will consider in more detail later.

§2 Integration by Parts

We will develop a formula for integration by parts which will prove useful for Ornstein-Uhlenbeck processes on Hilbert spaces, using the following theorem from Elworthy [4].

(1.2.1) Theorem

Let G be a separable Banach space and H,F separable Hilbert spaces. Assume $z : [a,b] \times \Omega \longrightarrow G$ satisfies A(4) and has almost all sample paths continuous. Suppose

$$B : T\times\Omega \longrightarrow L(G;H)$$

and $$C : T\times\Omega \longrightarrow L(G,G;H)$$

satisfy either $B(2)$ or B_0.

Let $u : [a,b]\times\Omega \longrightarrow H$ be defined by

$$u(t) = u(a) + \int_a^t B dz + \int_a^t C(dz,dz)$$

where $u(a)$ is $\mathfrak{F}_a$-measurable.

Then, if $\theta : H \longrightarrow F$ is C^2,

$$\theta(u(t)) = \theta(u(a)) + \int_a^t D\theta(u(s)) \circ B(s)dz(s) + \\ + \int_a^t D\theta(u(s)) \circ C(s)(dz(s),dz(s)) + \\ + \tfrac{1}{2}\int_a^t D^2\theta(u(s))(B(s)dz(s),B(s)dz(s)).$$

(1.2.2) <u>Corollary</u>

Let E,G be separable Banach spaces, F,H separable Hilbert spaces and suppose $z_1 : [a,b]\times\Omega \longrightarrow G$, $z_2 : [a,b]\times\Omega \longrightarrow E$ satisfy $A(4)$ and have almost all sample paths continuous. Suppose that $L(H;F)$ is separable and Hilbertable and that

$$B : T\times\Omega \longrightarrow L(G;L(H;F)),$$

$$C : T\times\Omega \longrightarrow L(G,G;L(H;F)),$$

$$B' : T\times\Omega \longrightarrow L(E;H)$$

and $C' : T\times\Omega \longrightarrow L(E,E;H)$ satisfy either $B(2)$ or B_0.

Let $f : [a,b]\times\Omega \longrightarrow L(H;F)$, $g : [a,b]\times\Omega \longrightarrow H$ be defined by $f(t) = f(a) + \int_a^t B dz_1 + \int_a^t C(dz_1,dz_1)$,

$$g(t) = g(a) + \int_a^t B' dz_2 + \int_a^t C'(dz_2,dz_2),$$

where $f(a)$, $g(a)$ are $\mathcal{F}_a$-measurable and assume that $f(-,\omega)|_{[a,t]}$, $g(-,\omega)|_{[a,t]}$ are continuous for $\mathcal{F}_t$-almost all ω.

Then,

$$(1.2.3) \qquad f(t)og(t) = f(a)og(a) + \int_a^t B(s)og(s)dz_1(s) + \int_a^t f(s)oB'(s)dz_2(s) + \int_a^t C(s)og(s)(dz_1(s),dz_2(s)) + \int_a^t f(s)oC'(s)(dz_2(s),dz_2(s)) + \int_a^t B(s)oB'(s)(dz_1(s),dz_2(s)).$$

Proof

Define $u : [a,b] \times \Omega \longrightarrow (L(H;F) \times H)$ by

$$u(t,\omega) = (f(t,\omega),g(t,\omega)) = (f(a),g(a)) + \left(\int_a^t Bdz_1, \int_a^t B'dz_2\right) + \left(\int_a^t C(dz_1,dz_1), \int_a^t C'(dz_2,dz_2)\right)$$

and define $\theta : L(H;F) \times H \longrightarrow F$ by $\theta(f,g) = fog$.

Noting that $L(H;F) \times H$ is a separable Hilbert space an application of Theorem 1.2.3 yields the result. #

(1.2.4) Example

Let x_1 and x_2 be 1-dimensional processes satisfying, in the notation of McShane [15],

$$dx_1(t) = f^1(t) + \sum_{\rho=1}^{r} g_\rho^1(t)dz^\rho(t) + \sum_{\rho,\sigma=1}^{r} h^1_{\rho,\sigma}(t)dz^\rho(t)dz^\sigma(t),$$

$$dx_2(t) = f^2(t) + \sum_{\rho=1}^{r} g_\rho^2(t)\,dz^\rho(t) +$$

$$+ \sum_{\rho,\sigma=1}^{r} h^2_{\rho,\sigma}(t)\,dz^\rho(t)\,dz^\sigma(t).$$

In Corollary 1.2.2 we take $G=E=\mathbb{R}^{r+1}$ and $z_1(t) = z_2(t) = (t, z^1(t), \ldots, z^r(t))$, $H = F = \mathbb{R}$. Thus $L(H;F) = \mathbb{R}$.

$B : \Omega \times [a,b] \times \mathbb{R}^{r+1} \longrightarrow \mathbb{R}$ is given by

$$(\omega, t, y_1, \ldots, y_{r+1}) \longrightarrow f^1(t,\omega)y_1 + \sum_{\rho=1}^{r} g_\rho^1(t,\omega)y_{\rho+1} + \ldots$$

etc.

Then equation 1.2.3 gives,

$$x_1(t)x_2(t) = x_1(a)x_2(a) + \int_a^t f^1(s)x_2(s)ds + \int_a^t f^2(s)x_1(s)ds +$$

$$+\sum_{\rho=1}^{r}\int_a^t g_\rho^1(s)x_2(s)dz^\rho(s) + \sum_{\rho=1}^{r}\int_a^t g_\rho^2(s)x_1(s)dz^\rho(s) +$$

$$+\sum_{\rho,\sigma=1}^{r}\int_a^t h^1_{\rho,\sigma}(s)x_2(s)\,(dz^\rho(s), dz^\sigma(s)) +$$

$$+\sum_{\rho,\sigma=1}^{r}\int_a^t h^2_{\rho,\sigma}(s)x_1(s)\,(dz^\rho(s), dz^\sigma(s)) +$$

$$+\sum_{\rho,\sigma=1}^{r}\int_a^t g_\rho^1(s)g_\sigma^2(s)\,(dz^\rho(s), dz^\sigma(s)),$$

which is the integral form of the equation which McShane derives on page 127.

Looking at equation 1.2.3 we see that we are justified in using the mnemonic

$$(1.2.5)\quad f(t)\circ g(t) = f(a)\circ g(a) + \int_a^t f(s)dg(s) + \int_a^t g(s)df(s) + \int_a^t df(s)dg(s),$$

although neither f nor g need satisfy A(4) or even A(1).

If g is C^1, i.e. $g(t) = g(a) + \int_a^t g'(s)ds$, then 1.2.3 becomes,

$$(1.2.6) \quad f(t)\circ g(t) = f(a)\circ g(a) + \int_a^t f(s)\circ g'(s)ds + \int_a^t B(s)\circ g(s)dz_1(s) + \int_a^t C(s)\circ g(s)(dz_1(s),dz_1(s)).$$

§ 3. Stochastic Integration in 2-Uniformly Smooth Banach Spaces

This section is mainly extracted from Neidhardt [16].

Let E_o be a real separable Banach space and W a continuous Brownian motion with mean-zero increments on E_o. Let ν be the distribution of $[W(s+t)-W(s)](t)^{-\frac{1}{2}}$ on E_o, where $s,t \in [0,R]$ and we assume that $[0,R]$ is contained in the index set of W. Since ν is a mean-zero Gaussian measure on E_o it determines a Hilbert space $H = H(\nu) \subset E_o$ and we use the same letter ν to denote the canonical distribution on H.

For F a real separable Hilbert space and G a real separable Banach space we define,

$$M(F,G) = \{L \in L(F;G) \mid \|L.\|_G \text{ is ameasurable norm on } F\}.$$

$L : H \longrightarrow G$ generates a Gaussian measure ν_L on G if and only if $L \in M(H;G)$.

Let E be a real separable Banach space which is 2-uniformly smooth, i.e. $\exists\, A_E < \infty$ such that,

$$\tfrac{1}{2}\left\{ \|x-y\|_E^2 + \|x+y\|_E^2 \right\} \leq \|x\|_E^2 + A_E\|y\|_E^2 \quad \forall\, x,y \in E.$$

We define $\|L\|_\nu = \left\{ \int_E \|x\|^2 d\nu_L(x) \right\}^{\frac{1}{2}}$ for $L \in M(H;E)$. $M(H;E)$ is a separable Banach space under this norm.

Let $L \in L(E_o;E)$. Then νL^{-1} is a measure on E and hence the restriction of L to H is in $M(H;E)$. So we say that $L(E_o;E) \subset M(H;E)$, and in fact $M(H;E)$ is the completion of $L(E_o;E)$ with respect to $\|\cdot\|_\nu$.

Let S be a normed, measurable space and for $0 \leq c < d \leq R$ define the non-anticipating functions,

$$\mathcal{N}(c,d;S) = \left\{ G : \Omega \times (c,d) \longrightarrow S \;\middle|\; G|_{(c,t]\times\Omega} \text{ is } \mathcal{B}_{(c,t]} \times \mathcal{F}_t\text{-measurable } \forall\, t \in (c,d] \right\}$$

where $\mathcal{B}_{(c,t]}$ is the class of Borel subsets of $(c,t]$.

We define,

$$\mathcal{M}^2(c,d;S) = \left\{ G \in \mathcal{N}(c,d;S) \;\middle|\; E\int_c^d \|G(t)\|_S^2\, dt < \infty \right\}.$$

The following facts are taken from Neidhardt [16].

(1.3.1) Let $G,G' \in \mathcal{M}^2(c,d;M(H,E))$ and let $\tilde{E}$ be another 2-uniformly smooth separable Banach space. Then :-

(i) $\int_c^t G\,dW$ is defined a.s. for $c < t \leq d$;

(ii) $\int_c^t G\,dW$ is $\mathcal{F}_t$-measurable;

(iii) $\int_c^d G\,dW = 0$ a.s. on $\left\{ \omega \,\middle|\, \int_c^d \|G(t,\omega)\|_\nu^2 dt = 0 \right\}$;

(iv) $\int_c^d G dW = \int_c^e G dW + \int_e^d G dW$ for $c<e<d$;

(v) if $L \in \mathcal{L}_0(\Omega; L(E,\tilde{E}))$ is $\mathcal{F}_c$-measurable then

$$\int_c^d LG dW = L\int_c^d G dW \text{ a.s. };$$

(vi) $\int_c^d (G+G')dW = \int_c^d G dW + \int_c^d G' dW$ a.s. ;

(vii) $E\|\int_c^d G dW\|_E^2 \leq A_E E\int_c^d \|G(t)\|_\nu^2 dt$,
and $E\int_c^d G dW = 0$;

(viii) $E(\int_c^d G dW | \mathcal{F}_c) = 0$,
and $E(\|\int_c^d G dW\|_E^2 | \mathcal{F}_c) \leq A_E E(\int_c^d \|G(t)\|_\nu^2 dt | \mathcal{F}_c)$;

(ix) $\int_c^d G dW$ has a continuous version ;

and (x) $E \sup_{c \leq t \leq d} \|\int_c^t G dW\|_E^2 \leq 4A_E E\int_c^d \|G(t)\|_\nu^2 dt.$

Note that Neidhardt proves many of these properties for more general G.

From now on we will let L be the inclusion of H in E_o which generates $\nu \equiv \nu_L$ on E_o. For $B \in L(E_o, E_o; E)$ we define

$$tr_L(B) = \int_{E_o} B(y,y)d\nu(y),$$

and $tr(L) = \int_{E_o} \|y\|^2 d\nu(y) < \infty$ by Fernique's theorem.

Thus $tr_L(B) \leq \|B\| \; tr(L) < \infty$.

If $E, \tilde{E}$ and G are as above, $f: E \longrightarrow \tilde{E}$ is C^2 and $F \in \mathcal{L}^1(c,d;E)$ such that $\forall\, t \in [c,d]$

$$X(t) = X(c) + \int_c^t F(s)ds + \int_c^t G(s)dW(s) \text{ where } X(c) \in E,$$

then we have the Ito formula,

$$\begin{aligned}(1.3.2) \quad f(X(d)) = f(X(c)) &+ \int_c^d Df(X(t))F(t)dt + \\ &+ \int_c^d Df(X(t))G(t)dW(t) + \\ &+ \tfrac{1}{2}\int_c^d\int_E D^2 f(X(t))(y,y)\,d\nu_{G(t)}(y)dt \text{ a.s.}\end{aligned}$$

In particular, let $Y:[c,d] \longrightarrow \mathbb{R}$ be C^1. Then we have the integration by parts formula,

$$(1.3.3) \quad Y(d)X(d) = Y(c)X(c) + \int_c^d Y(t)F(t)dt + \int_c^d Y'(t)G(t)dW(t).$$

If E is a Hilbert space then we have, in addition to the above results, for $m \geq 1$,

$$(1.3.4) \quad E[\|\int_c^d GdW\|_E^{2m}] \leq \{m(2m-1)\}^m R^{m-1} \int_0^R E[\|G(s)\|_\nu^{2m}]\, ds$$

(see Baxendale [2]). This is proved using the Ito formula, but it is not clear how to extend the proof to E non-Hilbert. For this reason we will later be considering Ornstein-Uhlenbeck processes on Hilbert spaces.

(1.3.5)' Theorem (Neidhardt)
Let $a : E \longrightarrow E$ and $b : E \longrightarrow M(H,E)$ be measurable and let V be a random variable in E independent of $W(t) - W(c)$ for $t > c$.

We assume,

$$(1.3.5) \qquad \|a(x_1)-a(x_2)\|_E \leq C\|x_1-x_2\|_E$$

$$\|b(x_1)-b(x_2)\|_\nu \leq C\|x_1-x_2\|_E$$

$$\|a(x)\|_E \leq G(1+\|x\|_E)$$

$$\|b(x)\|_\nu \leq G(1+\|x\|_E)$$

$$E(\|V\|_E^2) < \infty$$

$\forall\, x, x_1, x_2 \in E.$

Then there is exactly one solution X of

$$(1.3.6) \quad X(t) = V + \int_c^t a(X(s))ds + \int_c^t b(X(s))dW(s) \quad c \leq t < d,$$

satisfying $E\int_c^d \|X(t)\|_E^2\, dt < \infty$.

Let E_o be a Hilbert space with covariance operator S_ν of W on E_o, $S_\nu : E_o \longrightarrow E_o$ given by

$$\langle S_\nu x, y\rangle = \int_{E_o} \langle x,z\rangle_{E_o} \langle y,z\rangle_{E_o} d\nu_L(z).$$

Note that L can be taken to be the identity on E_o. We may choose an o.n. basis $\{e_i\}$ of E_o with associated eigenvalues $\{\lambda_i\}$ such that S_ν satisfies

$$S_\nu(x) = \sum \lambda_i \langle x, e_i \rangle e_i$$ (see, e.g. Kuo [11]).

Then if $B \in L(E_0, E_0)$,

$$\begin{aligned} tr_L(B) &= \int_{E_0} B(y,y)\, d\nu_L(y) \\ &= \sum B(e_i, e_j) \int_{E_0} \langle y, e_i \rangle \langle y, e_j \rangle d\nu_L(y) \\ &= \sum B(e_i, e_j) \langle e_i, S_\nu e_j \rangle \text{ by definition of } S \\ &= \sum \lambda_i B(e_i, e_i) \text{ by the choice of basis.} \end{aligned}$$

From now on, in this section, we will assume that E is a Hilbert space though E_0 need not be.

Let $(\Omega, \mathfrak{F}, \mu)$ be the probability space on which W_t is defined and let $\{\mathfrak{F}_s\}_{s \geq 0}$ be , as usual, an increasing family of σ-subalgebras of $\mathfrak{F}$ such that $W_t - W_s$ is independent of $\mathfrak{F}_s$ for each $0 \leq s \leq t$. Since W is a Brownian motion it satisfies $A(q)$ for each $q \geq 1$ (definition 1.1.1).

Let $B : \Omega \times [0,R] \longrightarrow L(E_0, E_0; E)$ satisfy B_0 or $B(1)$ (definitions 1.1.2 and 1.1.3). Then $\int_a^b B(s)(dW(s), dW(s))$ exists for $0 \leq a < b \leq R$ by Theorem 1.1.4.

(1.3.6) <u>Proposition</u>

Using the notation above assume that $\|B(\omega,t)\|_{L(E_0,E_0;E)} \leq K$, where $K>0$, $\forall\, \omega \in \Omega, t \in [0,R]$.

Then,

$$\int_a^b B(s)(dW(s), dW(s)) = \int_a^b tr_L(B(s))\, ds \quad , 0 \leq a < b \leq R$$

almost surely.

Proof

We follow the 1-dimensional proof in Gihman and Skorohod [6].

We may write,

$$\int_a^b B(s)(dW(s),dW(s)) = \lim_{\text{mesh}\,\Pi\to 0} \sum B(t_i)(\Delta_i W,\Delta_i W)$$

for (Cauchy) partitions, $\Pi = \{a=t_o<\ldots<t_m=b\}$ and mesh $\Pi = \min |t_{i+1}-t_i|$, $\Delta_i W= W_{t_{i+1}}-W_{t_i}$, $\Delta_i t = t_{i+1}-t_i$.

Let $\alpha(\Pi) = \sum_i B(t_i)(\Delta_i W,\Delta_i W)$.

Let $\beta(\Pi) = \sum_i E[B(t_i)(\Delta_i W,\Delta_i W)|\mathcal{F}_{t_i}]$

$= \sum_i \mathrm{tr}_L(B(t_i))\Delta_i t$ by definition of tr_L and W,

$\longrightarrow \int_a^b \mathrm{tr}_L(B(s))ds$ as mesh $\Pi \longrightarrow 0$ (convergence is in $\mathbb{R}^1$)

Let us look at

$$E[\|\alpha(\Pi)-\beta(\Pi)\|_E^2]$$

$$= E[\|\sum_i(B(t_i)(\Delta_i W,\Delta_i W) - \mathrm{tr}_L(B(t_i))\Delta_i t)\|^2]$$

$$= E\sum_{i,j}\langle B(t_i)(\Delta_i W,\Delta_i W) - \mathrm{tr}_L(B(t_i))\Delta_i t, \\ B(t_j)(\Delta_j W,\Delta_j W) - \mathrm{tr}_L(B(t_j))\Delta_j t\rangle$$

$$= \sum_i E\|B(t_i)(\Delta_i W,\Delta_i W) - \mathrm{tr}_L(B(t_i))\Delta_i t\|^2 \\ + 2\sum_{i<j} E\langle B(t_i)(\Delta_i W,\Delta_i W) - \mathrm{tr}_L(B(t_i))\Delta_i t, \\ B(t_j)(\Delta_j W,\Delta_j W) - \mathrm{tr}_L(B(t_j))\Delta_j t\rangle .$$

The second term is

$$2\sum_{i<j} E[E\langle B(t_i)(\Delta_i W,\Delta_i W)-tr_L(B(t_i))\Delta_i t,$$
$$B(t_j)(\Delta_j W,\Delta_j W)-tr_L(B(t_j))\Delta_j t\rangle_E \mid \mathcal{F}_{t_j}]$$

$$= 2\sum_{i<j} E[\langle B(t_i)(\Delta_i W,\Delta_i W)-tr_L(B(t_i))\Delta_i t,$$
$$E(B(t_j)(\Delta_j W,\Delta_j W)-tr_L(B(t_j))\Delta_j t \mid \mathcal{F}_{t_j})\rangle_E$$

$= 0.$

Hence,

$$E[\|\alpha(\pi)\ -\beta(\pi)\|_E^2] = \sum E\|B(t_i)(\Delta_i W,\Delta_i W)-tr_L(B(t_i))\Delta_i t\|_E^2.$$

But,

$$E\|B(t_i)(\Delta_i W,\Delta_i W)-tr_L(B(t_i))\Delta_i t\|_E^2$$

$$= E\langle B(t_i)(\Delta_i W\Delta_i W)-tr_L(B(t_i))\Delta_i t,\ B(t_i)(\Delta_i W,\Delta_i W)\rangle$$
$$- E\langle B(t_i)(\Delta_i W,\Delta_i W)-tr_L(B(t_i))\Delta_i t,\ tr_L(B(t_i))\Delta_i t\rangle$$

$$= E\langle B(t_i)(\Delta_i W,\Delta_i W),\ B(t_i)(\Delta_i W,\Delta_i W)\rangle$$

$$\le K^2 E[\|\Delta_i W\|^4]$$

$= K^2 \int_{E_0}\|y\|^4(\Delta_i t)^2 d\nu(y)$ since ν is the distribution of a unit increment of W

$= K^2 m_4(\Delta_i t)^2$

where $m_4 = \int_{E_0}\|y\|^4 d\nu(y) < \infty$ by Fernique's theorem.

Hence

$$E[\|\alpha(\pi)\ -\ \beta(\pi)\|^2] \le K^2 m_4 \sum(\Delta_i t)^2$$

$\leq K^2 m_4 (\text{mesh}\Pi)(b-a).$

Thus,

$$\mu\{\omega; \|\alpha(\Pi,\omega) - E\alpha(\Pi,\omega)\|_E \geq \varepsilon\} \leq \frac{E(\|\alpha(\Pi) - E(\alpha(\Pi))\|_E^2)}{\varepsilon^2}$$

$$\longrightarrow 0 \text{ as } \text{mesh}\Pi \longrightarrow 0.$$

Hence we can see that, taking the limit as $\text{mesh}\Pi \longrightarrow 0$,

$$\int_a^b B(s)(dW(s), dW(s)) = \int_a^b \text{tr}_L(B(s))ds \quad \text{a.s.} \qquad \#$$

(1.3.7) Corollary

If E_o is also a Hilbert space with an o.n. basis as in the remarks preceding Proposition 1.3.5, then under the conditions of that proposition,

$$\int_a^b B(s)(dW(s), dW(s)) = \sum \lambda_i \int_a^b B(s)(e_i, e_i)ds.$$

§4. Infinitesimal Generators

We follow Elworthy [4, Ch.VIII,§1].

Let (X,z) be an s.d.s. over a manifold M, with drift, $(X,z) = (X_1 \oplus V, W \oplus t)$, where $X_1: M \times H \longrightarrow TM$ and W is a Brownian motion on the Hilbert space H. We assume that all of these objects satisfy the conditions imposed in §1.

We can write the corresponding s.d.e. as

$$dx = X_1 dW + V dt.$$

Let $x_m : [0,\xi^m) \times \Omega \longrightarrow M$ denote a maximal solution to this s.d.e. with $x_m(0) = m$ a.s., where $m \in M$.

If $f : M \longrightarrow \mathbb{R}$ we define $P_t f : M \longrightarrow \mathbb{R}$ by

$$P_t f(m) = \int_{\Omega_t^m} f\, x_m(t,\omega)\, d\mu(\omega)$$

(where $\Omega_t^m = \{\omega \in \Omega;\ t < \xi^m(\omega)\}$)

whenever f is measurable and the integral exists.

If $B\mathcal{L}^0(M;\mathbb{R})$ is the set of all bounded measurable functions on M then P_t exists as a map from $B\mathcal{L}^0(M;\mathbb{R})$ into itself, but $P_t f$ also exists in many cases where f is not bounded.

The infinitesimal generator of (X,z), $\mathcal{A}_X$ maps a function $f : M \longrightarrow \mathbb{R}$ to a function $\mathcal{A}_X(f) : M \longrightarrow \mathbb{R}$,

$$\mathcal{A}_X(f)(m) = \lim_{t \to 0} \frac{P_t f(m) - f(m)}{t}$$

whenever this is defined.

Let $\{e_i\}$ be an orthonormal basis of H as in §3, and define the second order differential operator $\mathcal{L}^2_{X_1}$ on M by,

$$\mathcal{L}^2_{X_1}(f) = \sum_i X_1^i (X_1^i f),$$

where X_1^i is the vector field on M defined by $X_1^i(m) = X_1(m, e_i)$. $\mathcal{L}^2_{X_1}$ is independent of the chosen orthonormal basis of H. Let $S(m,t)e$ be a solution in M of $dS(m,t)e = X(S(m,t)e)dt$. Then for $f : M \longrightarrow \mathbb{R}$,

$$\mathcal{L}^2_{X_1}(f) = \Sigma\lambda_i \frac{d^2 f\circ S(m,t)e_i}{dt^2}\bigg|_{t=0}$$

(see Elworthy [4]). In fact this is the formulation of $\mathcal{L}^2_{X_1}$ which we shall use in Chapter Three when we come to calculate the infinitesimal generator of an O-U process.

(1.4.1) Theorem (see Elworthy [4])

If $f: M \longrightarrow \mathbb{R}$ is C^2 with $df\circ X_1 : M \longrightarrow L(H;\mathbb{R})$, $V(f) : M \longrightarrow \mathbb{R}$ and $\mathcal{L}^2_{X_1}(f) : M \longrightarrow \mathbb{R}$ all bounded on M then $P_t f$ exists and

$$\mathcal{A}_X f = \tfrac{1}{2}\mathcal{L}^2_{X_1}(f) + V(f)$$

$$= \tfrac{1}{2}\Sigma \nabla df(X_1^i, X_1^i) + \tfrac{1}{2}\Sigma\lambda_i df\circ\nabla X_1^i(X_1^i) + V(f)$$

for any linear connection ∇ on M.

Proof

As in Elworthy except that we use Corollary 1.3.6 to show that

$$\int_0^{t\wedge\xi} \frac{d^2}{dt^2} f\, B(t, x(s,\omega))\Big|_{t=0} (dW(s,\omega), dW(s,\omega))$$

$$= \int_0^{t\wedge\xi} \mathrm{tr}_L \frac{d^2}{dt^2} f\, B(t, x(s,\omega))\Big|_{t=0} ds$$

$$= \int_0^{t\wedge\xi} \Sigma\lambda_i \frac{d^2}{dt^2} f\, B(t, x(s,\omega))\Big|_{t=0} (e_i, e_i) ds.$$

#

§5. Brownian Motion on Riemannian Manifolds

Let M be a Riemannian manifold modelled on R^n. The orthonormal frame bundle, O(M), is a reduction of the bundle of linear frames, with fibre $O_m(M)$ over $m \in M$ which can be thought of as the set of isometries from R^n to T_mM. O(n) acts naturally on $O_m(M)$ on the right and $O_m(M)$ is isomorphic to O(n).

The Levi-Cevita (or Riemannian) connection on M determines a splitting of TO(M),

$$TO(M) = HO(M) \oplus VO(M)$$

into 'horizontal' and 'vertical' parts. If $\tau : O(M) \longrightarrow M$ is projection then the fibre $V_uO(M)$ over $u \in O(M)$ is, by definition, $\ker(T\tau | T_uO(M))$.

Thus $T\tau | H_uO(M) : H_uO(M) \longrightarrow T_{\tau(u)}M$ is an isomorphism and there is a natural diffeomorphism,

$$\theta : O(M) \times R^n \longrightarrow HO(M)$$

given by $\theta : (u, v) \longmapsto (T\tau | H_uO(M))^{-1}(u(v))$.

Let $u : [a,b] \longrightarrow O(M)$ be a C^1 path.

(1.5.1) Definition

u is said to be horizontal if $u'(t) \in H_{u(t)}O(M) \ \forall t \in [a,b]$.

If $s : [a,b] \longrightarrow M$ is C^1 there exists a unique path $\tilde{s} : [a,b] \longrightarrow O(M)$ with $\tilde{s}(a) = u_a \in O_m(M)$ such that $\tau \circ \tilde{s} = s$ and $\tilde{s}$ is horizontal. $\tilde{s}$ is called the <u>horizontal lift</u> (or simply <u>lift</u>) of s starting at u_a.

If X is a vector field on M there is the related notion of the lift of X. $\tilde{X}$ is the unique vector field on O(M) such that $\tilde{X}(u) \in H_u O(M)$ and $T\tau \circ \tilde{X}(u) = X \circ \tau(u) \ \forall\, u \in O(M)$. s is an integral curve of X if and only if $\tilde{s}$ is an integral curve of $\tilde{X}$.

We will construct Brownian motion on M following Elworthy [4] (see also Jørgensen [9]). Let W be a standard Brownian motion on $\mathbb{R}^n$ (i.e. W(t) has mean zero and covariance matrix tI). Let $u : [0, \xi^{u_0}) \times \Omega \longrightarrow O(M)$ be a maximal solution of $du = \theta\, dW$ with $u(0) = u_0$. Then $\tau \circ u \equiv x$ is called Brownian motion on M. Note that W could be any process satisfying A(4) on $\mathbb{R}^n$ in order for the s.d.e. to have a solution by Theorem 1.1.7. We prove a result which will require the following fact from Elworthy [4].

(1.5.2)

Let $h : M \longrightarrow N$ be a C^3 diffeomorphism and $x : [a, \xi) \times \Omega \longrightarrow M$ a maximal solution of $dx = X dz$, $x(a) = x_a$ for an s.d.s. (X,z) on M, where all objects satisfy the conditions of §1. Then $y = h \circ x$ is a maximal solution of $dy = Y dz$, $y(a) = h(x_a)$, where $Y(u,v) = T_{h^{-1}(u)} h \circ X(h^{-1}u, v)$.

(1.5.3) <u>Proposition</u>

Let z be a process on $\mathbb{R}^n$ with orthogonally invariant distribution, and which satisfies A(4). Let u be the corresponding process on O(M) as constructed above with,

$$du = \theta\, dz \ , \ u(a) = u_a \in O(M) \ , \tau(u_a) = x_a \in M,$$

and v the process given by,

$$dv = \theta\, dz \ , \ v(a) = v_a \in \tau^{-1}(x_a).$$

Then a.s. $R_g u(t,\omega)) = v(t,p(\omega))$, where p is a measure preserving transformation of Ω (over which z is defined - we take Ω to be a space of paths on $\mathbb{R}^n$, e.g. in the case $z = W$, $\Omega = C^0([a,b]; \mathbb{R}^n)$) & R_g is right translation by $g \in O(n)$ s.t. $R_g(u_a) = v_a$.

Proof (see also next page)

If $p : \Omega \longrightarrow \Omega'$ is a transformation then by 1.5.2

$y'(t,\omega) = y(t,p^{-1}(\omega))$ is a solution of $dy' = Ydz'$ where $z'(t,\omega) = z(t,p^{-1}\omega)$.

Now, $v_a = u_a \circ g$ for some $g \in O(n)$. Define $R_g : O(M) \longrightarrow O(M)$ by $R_g(r) = r \circ g$.

Thus $R_g \circ u$ is a maximal solution of $d\nu = Ydz$ where $Y(\nu) = T_{\nu g^{-1}} R_g \circ \theta(\nu g^{-1})$.

Define $p : \Omega \longrightarrow \Omega$ by $p(\omega) = g^{-1}\omega$. p is measure preserving since g^{-1} is an isometry.

p is obviously bijective, hence

$v(t,\omega) = R_g \circ u(t,p^{-1}(\omega))$ is a maximal solution of $dv = Ydz'$ where

$$\begin{aligned} z'(t,\omega) &= z(t,p^{-1}(\omega)) \\ &= p^{-1}(\omega)(t) \\ &= g(\omega(t)). \end{aligned}$$

Hence v is a maximal solution of the s.d.e.

$$dv = Y'dz,$$

where

$$\begin{aligned} Y'(v) &= T_{v.g^{-1}} R_g \circ \theta(v.g^{-1}) \circ g \\ &= \theta(v) \text{ by the invariance of connections.} \end{aligned}$$

Addendum to 1.5.3

We have yet to show that z' satisfies A(4) and that $Y'dz = Ydz'$ in any meaningful sense.

Since $z'(t,\omega) = g(z(t,\omega))$, z' satisfies A(4) from the fact that z does and from

(i) $z'(t)$ is $\mathcal{F}_t$-measurable since g maps Borel sets to Borel sets (in $\mathbb{R}^n$);

(ii) $$\left|E_s(g(z(t,\omega)) - g(z(s,\omega)))\right| = \left|g(E_s(z(t,\omega) - z(s,\omega)))\right| = \left|E_s(z(t,\omega) - z(s,\omega))\right|$$
since g is an isometry

(iii) $\left|g(z(t,\omega)) - g(z(s,\omega))\right| = \left|z(t,\omega) - z(s,\omega)\right|$

We see, therefore, that (Y,z') is an S.D.S. with maximal solution v satisfying, in a regular localisation $=((U,\phi),U_o,\lambda)$

$$\phi\circ v(t) = \phi\circ v(t_o) + \int_{t_o}^{t} Y_\Lambda(\phi\circ v(s))d(g\circ z)(s) +$$
$$+ \tfrac{1}{2}\int_{t_o}^{t} DY_\Lambda(\phi\circ v(s))\circ(Y_\Lambda(\phi\circ v(s))d(g\circ z)(s), d(g\circ z)(s))$$

for suitable t_o, t and ω (see 1.1.5 and 1.1.6).

By looking at the belated integral as the almost sure limit of belated sums, we see that $d(g\circ z)$ is equivalent to $g\circ dz$ (i.e. a belated integral using one is almost surely equal to the equivalent integral using the other).

But $Y_\Lambda(\phi\circ v(s))\circ g = (Y\circ g)_\Lambda(\phi\circ v(s))$ and similarly ,

$$DY_\Lambda(\phi\circ v(s))\circ(Y_\Lambda(\phi\circ v(s))\circ g(-), g(-)) = D(Y\circ g)_\Lambda(\phi\circ v(s))\circ((Y\circ g)_\Lambda(\phi\circ v(s))-,-).$$

This shows that v is also a (maximal) solution of the S.D.S. $(Y',z) = (Y\circ g,z)$ and we can say that $Ydz' \equiv Y'dz$.

Thus v is the process in the statement of the proposition and $\tau(v(t,p(\omega))) = \tau(u(t,\omega).g) = \tau(u(t,\omega))$ as required. #

In particular Proposition 1.5.2 holds for z = W.

Using Theorem 1.4.1 we can show that x, the process on M defined by W, has infinitesimal generator $\frac{1}{2}\Delta$ where Δ is the Laplace-Beltrami operator on M determined by the given Levi-Cevita connection.

§6. Malliavin's Approximation

One form of differentiable approximation to Brownian motion on manifolds is that of Malliavin [13]. This starts by taking a bump function u on $\mathbb{R}$ with support in [0,1] and with unit integral over $\mathbb{R}$. With this is associated, for each $\varepsilon>0$,

$$u_\varepsilon(t) = \varepsilon^{-1}u(\varepsilon^{-1}t),$$

which also has unit integral over $\mathbb{R}$. Let W be a Brownian motion on $\mathbb{R}^n$ and define $W_\varepsilon(t) = \int W(t+\lambda)u_\varepsilon(\lambda)d\lambda$. Then $W(.,\omega)$ is a C^∞ function almost surely.

Also $\|W_\varepsilon(t)-W(t)\|$ tends to zero as $\varepsilon \longrightarrow 0$ for $t\geq 0$:-

$$\|W_\varepsilon(t)-W(t)\| = \left\|\int[W(t+\lambda)u_\varepsilon(\lambda)-W(t)u_\varepsilon(\lambda)]d\lambda\right\|$$

$$\leq \int \| W(t+\lambda) - W(t) \| u_\varepsilon(\lambda) d\lambda$$

$\leq \int \sqrt{\lambda}\, K\, u_\varepsilon(\lambda) d\lambda$ where K is a constant determined by the covariance matrix of W, in our notation $K = \sqrt{trL}$,

$\leq \sqrt{\varepsilon}\, K$ since the support of u_ε is contained in $[0,\varepsilon]$.

If $X \in Hom_M(\mathbb{R}^n, TM)$ is C^2 and V is a C^1 vector field on M then $(X\oplus V, W\oplus t)$ and $(X\oplus V, W_\varepsilon \oplus t)$ form s.d.s.'s on M since it is easy to see that $\tilde{W}_\varepsilon$ satisfies A(q) for $q\geq 1$ ($\tilde{W}_\varepsilon(t) = W_\varepsilon(t-\varepsilon)$ - note that $W_\varepsilon(t)$ is $\mathcal{F}_{t+\varepsilon}$-measurable).
Let u, u_ε be maximal solutions of

$$du = XdW + Vdt\ , \quad u(0) = m \in M;$$

$$du_\varepsilon = Xd\tilde{W}_\varepsilon + Vdt\ , \quad u_\varepsilon(0) = m.$$

Then Malliavin proves in [14] that u_ε tends almost surely to u as ε tends to zero

The Ornstein-Uhlenbeck approximation will differ in that it wi ll not satisfy A(q) and hence we will have to consider families of ordinary differential equations.

CHAPTER TWO. ORNSTEIN-UHLENBECK PROCESSES ON 2-UNIFORMLY SMOOTH BANACH SPACES

§1. Ornstein-Uhlenbeck Processes on the Real Line

The basic reference for this section is Nelson [17].

The Einstein theory of Brownian motion (mathematically formalised in the Wiener process) provides an excellent quantitative approximation to the behaviour of most Brownian particles (such as pollen in water). It is not, however, a dynamical theory, since the velocity of a Wiener particle is undefined. Thus it is not a good qualitative theory of the behaviour of Brownian particles, and in more complicated cases such as that of a particle immersed in a force field, it breaks down completely.

The Ornstein-Uhlenbeck (O-U) theory is an attempt, based on the Langevin equation, to describe the behaviour of Brownian particles in a dynamical way. We will just consider particles on the real line. The situation on $\mathbb{R}^n$ is easily developed from this, and will in any case be a special instance of the more general theory we will develop on Banach spaces.

Let $z_\beta(t)$ denote the position of aBrownian particle in $\mathbb{R}$ at time $t \geq 0$ and assume that $v_\beta(t) = dz_\beta(t)/dt$ exists and satisfies $dv_\beta(t) = -\beta v_\beta(t)dt + \beta dW(t)$,
the Langevin equation, with β a positive constant and W a Wiener process on $\mathbb{R}$ (W has mean zero and variance c).

z_β is an O-U position process and v_β an O-U velocity process. To demonstrste the relevance of the Langevin equation consider, formally, though dW/dt does not exist,

$$md^2 z_\beta(t)/dt^2 = -m\beta v_\beta(t) + m\beta dW(t)/dt,$$

where m is the mass of the Brownian particle. Using Newton's second law, F=ma, this shows that we consider two forces to be acting on the particle. One is a frictional force, proportional to the velocity, and the other is a vacillating force caused, for instance, by changing molecular pressure on all sides of the particle.

If we let the particle have position z_o and velocity v_o at time t=0 the solutions to the equations

$$\begin{aligned} dz_\beta(t) &= v_\beta(t) \ , \ z_\beta(0) = z_o \\ dv_\beta(t) &= -\beta v_\beta(t)\,dt + \beta dW(t) \end{aligned} \tag{2.1.1}$$

are $v_\beta(t) = v_o e^{-\beta t} + \beta e^{-\beta t}\int_0^t e^{\beta s} dW(s)$
$z_\beta(t) = z_o + \int_0^t v_\beta(s)\,ds,$
(see Nelson or McShane [15]). We will show in §2 that the above solutions still hold in 2-uniformly smooth Banach spaces. Both z_β and v_β are Gaussian processes.

A few formulae which are easily derived from the above solution for v_β are

$$\begin{aligned} E(v_\beta(t)) &= v_o e^{-\beta t} \\ E_s(v_\beta(t)) &= v_\beta(s) e^{-\beta(t-s)} \quad \text{for } 0 \le s \le t \\ E(v_\beta(t)^2) &= v_o^2 e^{-2\beta t} + (c\beta/2)(1-e^{-2\beta t}) \end{aligned} \tag{2.1.2}$$

$$E_s[v_\beta(t)^2] = v_\beta(s)^2 e^{-2\beta(t-s)} + \frac{c}{2}\beta(1-e^{-2\beta(t-s)})$$

for $0 \le s < t$.

v_β has infinitesimal generator $-\beta v \frac{d}{dv} + \beta^2 \frac{c}{2} \frac{d^2}{dv^2}$

(see Nelson).

As $\beta \longrightarrow \infty$ the O-U position process converges in $\mathcal{L}^2$ -norm to the Wiener process W on $\mathbb{R}$. We will prove this for O-U processes on 2-uniformly smooth Banach spaces, and a similar theorem for O-U type processes which we will construct on suitable manifolds.

§2. Ornstein-Uhlenbeck Processes on 2-Uniformly Smooth Banach Spaces

Let E be a separable real Banach space and W a Wiener process on E. We will use the notation of 1.3 with $E=E_o$ a 2-uniformly smooth Banach space with constant A_E.

We will define an O-U process on E by the equations,

(2.2.1) $\qquad dv_\beta(t) = -\beta v_\beta(t)dt + \beta dW(t),\ v_\beta(0)=v_o$;

(2.2.2) $\qquad dz_\beta(t) = v_\beta(t)dt\ ,\ z_\beta(0) = z_o$

where we use the symbol β both for the positive constant and the map $\beta I : E \longrightarrow E$.

Then solutions exist by theorem 1.3.5. Note that there are no second order integral terms in the integral form of 2.2.2.

To find solutions of 2.2.1 and 2.2.2 we will use the integration by parts formula 1.3.3. If E is a Hilbert space we can, equivalently, use the formula 1.2.6.

(2.2.3) Lemma

$$v_\beta(t) = v_0 e^{-\beta t} + \beta\int_0^t e^{-\beta(t-s)} I dW(s) \text{ a.s. for } 0 \leq t \leq R.$$

I is the identity map from E to E which can be considered as the map $L \in M(H;E)$ which generates the Wiener process.

Proof

From 1.3.3 we have that

$$\beta e^{-\beta t}\int_0^t e^{\beta s} I dW(s) =$$

$$= \beta W(t) - \beta e^{-\beta t} W(0) - \beta\int_0^t \beta e^{-\beta(t-s)} W(s)ds$$

$$= \beta W(t) - \beta^2\int_0^t e^{-\beta(t-s)} W(s)ds.$$

Define $u_\beta(t) = v_0 e^{-\beta t} + \beta\int_0^t e^{-\beta(t-s)} I dW(s)$.

Then $u_\beta(t) = v_0 e^{-\beta t} - \beta^2\int e^{-\beta(t-s)} W(s)ds + \beta W(t)$.

So

$$\int_0^t u_\beta(s)ds = \frac{v_0}{\beta}(1-e^{-\beta t}) + \int_0^t \beta e^{-\beta s}\int_0^s e^{\beta r} I dW(r)ds$$

$$= \frac{v_0}{\beta}(1-e^{-\beta t}) + \beta e^{-\beta t}\int_0^t e^{\beta s} W(s)ds$$

integrating by parts and applying the formula we worked out above.

Thus $u_\beta(t) = v_0 - \beta\int_0^t u_\beta(s)ds + \beta W(t)$,

and so $u_\beta(t)$ satisfies equation 2.2.2 and is almost surely equal to $v_\beta(t)$. #

(2.2.4) Lemma

$$z_\beta(t) = z_o + \frac{v_o}{\beta}(1-e^{-\beta t}) + W(t) - e^{-\beta t}\int_0^t e^{\beta s} dW(s) \text{ a.s.}$$

for $0 \leq t \leq R$.

Proof

By 2.2.1 $z_\beta(t) = z_o + \int_0^t v_\beta(s)ds.$

Using the proof of lemma 2.2.3 thisis

$$z_o + \frac{v_o}{\beta}(1-e^{-\beta t}) \quad -e^{-\beta t}\int_0^t e^{\beta s} W(s)ds.$$

Another application of 1.3.3 gives the desired result. #

(2.2.5) Theorem

$$\|z_\beta(t) - W(t) - z_0\| \longrightarrow 0 \text{ as } \beta \longrightarrow \infty, \text{ uniformly}$$

for $0 \leq t \leq R$.

Proof

From Lemma 2.2.4 we see that

$$\|z_\beta(t) - W(t) - z_o\|^2 \leq 2\left\|\frac{v_o}{\beta^2}\right\|^2 (1-e^{-\beta t})^2 + 2e^{-2\beta t}\left\|\int_0^t e^{\beta s} I dW(s)\right\|^2.$$

The first term on the right tends to zero as β tends to infinity, uniformly for $t \in [0,R]$. Note that v_o need only be in $\mathcal{L}^2(\Omega, \mathcal{F}; E)$ for this to hold.

Also

$$e^{-2\beta t}\left\|\int_0^t e^{\beta s} dW(s)\right\|^2 \leq A_E e^{-2\beta t}\left(\int_0^t \|e^{\beta s} I\|_\nu^2 \, ds\right)$$

by property vii of 1.3.1

$$= A_E(\mathrm{tr}L)\, e^{-2\beta t} \frac{1}{2\beta}(e^{2\beta t} - 1)$$

which certainly tends to zero as β tends to infinity, uniformly over $t \in [0,R]$, proving the theorem. #

(2.2.6) Lemma

If $v_\beta : [0,R] \times \Omega \to E$ is a solution of 2.2.2 then v_β is a quasi-martingale (see, e.g. Ito [8]).

Proof

By lemma 2.2.3,

$$\begin{aligned} v_\beta(t) &= v_o e^{-\beta t} + \beta\int_0^t e^{-\beta(t-s)} dW(s) \qquad \text{a.s.} \\ &= v_o e^{-\beta t} + \beta W(t) - \beta^2 \int_0^t e^{-\beta(t-s)} W(s)\,ds. \end{aligned}$$

Let $X_1(t,\omega) = \beta W(t,\omega)$
and $X_2(t,\omega) = v_o e^{-\beta t} - \beta^2 \int_0^t e^{-\beta(t-s)} W(s)\,ds.$

Then,

(i) X_1 is a martingale since W is;

(ii) X_2 is almost surely of bounded variation since it has almost surely C^1 sample paths (W has almost surely continuous sample paths).

Hence $X = X_1 + X_2$ is a quasi-martingale. #

Remark Since z_β has almost surely C^1 sample paths z_β is almost surely of bounded variation :-

$$z_\beta(t) = z_o + \frac{v_o}{\beta}(1-e^{-\beta t}) + \beta e^{-\beta t}\int_0^t e^{\beta s} W(s)\,ds.$$

(2.2.7) Lemma

(i) $E(v_\beta(t)) = v_o e^{-\beta t}$

(ii) $E_s(v_\beta(t)) = v_\beta(s)e^{-\beta(t-s)}$ for $0\leq s\leq t\leq R$.

Proof

(i) $v_\beta(t) = v_o e^{-\beta t} + \beta\int_0^t e^{-\beta(t-s)} I\,dW(s)$

and the second term has zero expectation by 1.3.1 (vii).

(ii) $$v_\beta(t) = v_o e^{-\beta s}(e^{-\beta(t-s)}) + \beta\int_0^s e^{-\beta(t-s)}e^{-\beta(s-r)} I\,dW(r)$$

$$+ \beta\int_s^t e^{-\beta(t-r)} I\,dW(r)$$

$$= e^{-\beta(t-s)}v_\beta(s) + \beta\int_s^t e^{-\beta(t-r)} I\,dW(r).$$

Thus $E_s(v_\beta(t)) = e^{-\beta(t-s)}v_\beta(s)$ by 1.3.1 (viii). #

(2.2.8) Lemma

Neither z_β nor v_β satisfy $A(q)$ for $q\geq 1$ (even if $E=\mathbb{R}$).

Proof

(i) In order to satisfy $A(q)$ v_β must satisfy a.s. for $0\leq s<t\leq R$, $t-s<\delta$,

$$\|E_s(v_\beta(t)-v_\beta(s))\|_E \leq K(t-s) \text{ for some } \delta>0,\ K>0.$$

But Lemma 2.2.7 shows that

$$\|E_s(v_\beta(t)-v_\beta(s))\|_E = \|v_\beta(s)\|_E(1-e^{-\beta(t-s)})$$

and while $(1-e^{-\beta(t-s)}) \leq \beta(t-s)$,

$v_\beta(\omega)$ is unbounded, i.e. given $N>0$

$$\mu\{\omega; \|v_\beta(s,\omega)\|_E > N\} > 0.$$ Thus K does not exist.

(ii) Since $v_\beta(t) = \frac{dz}{dt}\beta(t)$ a.s.,

$$\text{for } t>s \ \| E_s(z_\beta(t)-z_\beta(s))\|_E = \| \int_s^t E_s(v_\beta(r))dr \|_E$$

$$= \| \int_s^t v_\beta(s)e^{-\beta(r-s)}dr \|_E$$

$$= \| v_\beta(s)\|_E \frac{(1-e^{-\beta(t-s)})}{\beta} .$$

Again, we note that v_β is unbounded. #

We will be interested in 'stochastic differential equations' of the form $dx = Xdz_\beta$. Since z_β does not satisfy A(4) we cannot use Theorem 1.1.7 to show the existence of solutions to this equation. However, since $dz_\beta = v_\beta dt$ we rewrite this s.d.e. as the family of ordinary differential equations $dx = Xv_\beta\, dt$, indexed by ω, which can be dealt with by the theory of ordinary differential equations.

Now we assume that E is a Hilbert space with an orthonormal basis as in §1.3 and we define P_n to be projection from E onto the first n coordinates of e_i. Let W^n be the n-dimensional Brownian motion $P_n \circ W$. Then we have an O-U system on $P_n(E)$ given by

$$dv_\beta^n = -\beta v_\beta^n dt + \beta I_n dW^n , \quad v_\beta^n(0)=v_o^n$$

$$dz_\beta^n = v_\beta^n dt , \quad z_\beta^n(0) = z_o^n,$$

where I_n is the identity map on $P_n(E)$.

(2.2.9) Lemma

If $v_o^n = P_n(v_o)$ and $z_o^n = P_n(z_o)$ then

$$v_\beta^n = P_n \circ v_\beta \quad \text{and} \quad z_\beta^n = P_n \circ z_\beta \quad \text{a.s..}$$

Proof

We know that $dv_\beta = -\beta v_\beta dt + \beta dW$,
i.e $v_\beta(t) = v_o - \beta\int_0^t v_\beta(s)ds + \beta W(t)$.

Thus, since P_n is linear,

$$\begin{aligned} P_n \circ v_\beta(t) &= P_n(v_o) - \beta\int_0^t P_n \circ v_\beta(s)ds + \beta P_n \circ W(t) \\ &= v_o^n - \beta\int_0^t P_n \circ v_\beta(s)ds + \beta W^n(t) \text{ a.s.} \end{aligned}$$

Hence $P_n \circ v_\beta = v_\beta^n$ a.s..

Also, $z_\beta(t) = z_o + \int_0^t v_\beta(s)ds$

so $P_n \circ v_\beta(t) = z_o^n + \int_0^t v_\beta^n(s)ds$ a.s.

#

Remark Using this result we see that, in the case of an O-U process on a Hilbert space, we can split (z_β, v_β) into independent 1-dimensional O-U processes based on the independent 1-dimensional Wiener processes $\{W_i\}$ where each W_i is the projection of W onto the i-th coordinate of E.

(2.2.10) Lemma

$$E(\|v_\beta(t)\|_E^2) \leq 2e^{-2\beta t}\|v_o\|_E^2 + A_E\beta(1-e^{-2\beta t})(\mathrm{tr}L).$$

If E is a Hilbert space the result becomes

$$E(\|v_\beta(t)\|_E^2) = e^{-2\beta t}\|v_0\|_E^2 + \frac{\beta}{2}(1-e^{-2\beta t})(\mathrm{tr}L).$$

<u>Proof</u>

$$v_\beta(t) = v_0 e^{-\beta t} + \beta e^{-\beta t}\int_0^t e^{\beta s} I\,dW(s).$$

Hence

$$E(\|v_\beta(t)\|_E^2) \leq 2e^{-2\beta t}\|v_0\|_E^2 + 2\beta^2 e^{-2\beta t}\|\int_0^t e^{\beta s} I\,dW(s)\|^2$$

$$\leq 2e^{-2\beta t}\|v_0\|_E^2 + 2\beta^2 e^{-2\beta t} A_E \int_0^t (\mathrm{tr}L)e^{2\beta s}ds$$

by 1.3.1 (vii)

$$= 2e^{-2\beta t}\|v_0\|_E^2 + \beta A_E(\mathrm{tr}L)(1-e^{-2\beta t}).$$

If E is Hilbert then we write

$$\|v_\beta(t)\|_E^2 = e^{-2\beta t}\|v_0\|_E^2 + $$

$$+2\beta e^{-2\beta t}\left\langle v_0 , \int_0^t e^{\beta s} I\,dW(s)\right\rangle_E$$

$$+ \beta^2 e^{-2\beta t}\|\int_0^t e^{\beta s} I\,dW(s)\|_E^2$$

and the middle term has zero expectation by 1.3.1(vii) (note that this is true even if v_0 is a member of $\mathcal{L}^2(\Omega,\mathcal{F};E)$ which is $\mathcal{F}_0$-measurable). #

If E is a Hilbert space an alternative way to see the above result is to split v_β into its 1-dimensional component processes and use the result in 2.1.2 for 1-dimensional O-U processes :

$$E(\|v_\beta(t)\|_E^2) = E(\Sigma\|v_\beta^i(t)\|_E)^2 = E(\Sigma\|v_\beta^i(t)\|_E^2)$$

$$= \Sigma E(\|v_\beta^i(t)\|_E^2) = \Sigma(\|v_0^i\|_E^2 e^{-2\beta t} + \frac{\lambda_i}{2}\beta(1-e^{-2\beta t}))$$

and we note that $\sum \lambda_i = \mathrm{tr}L$.

(2.2.11) Lemma

If E is a Hilbert space then,

$$E(\|v_\beta(t)\|^{2m}) \leq \sum_{k=0}^{m} \frac{(2m)!(\mathrm{tr}L)^k}{(2(m-k))!\, 2^{2k}.k!} \|v\|_E^{2(m-k)} \beta^k e^{-2(m-k)\beta t}(1-e^{-2\beta t})^k$$

for $m \geq 1$.

Proof

We will use induction. Note that from Lemma 2.2.10 we see that the result holds for m=1. We assume the result for m-1.

Let $f : E \longrightarrow \mathbb{R}$ be defined by $f(x) = \|x\|_E^{2m}$.
Then f is C^2 and

$$Df(x) = 2m\|x\|_E^{2(m-1)}\langle x,\cdot\rangle_E \quad \in L(E;\mathbb{R})$$

$$D^2f(x)y = 2m\|x\|_E^{2(m-1)}\langle y,\cdot\rangle_E + 4m(m-1)\|x\|_E^{2(m-2)}\langle x,y\rangle_E\langle x,\cdot\rangle_E$$

$\in L(E;\mathbb{R})$ for $x,y \in E$.

Also.

$$v_\beta(t) = v_0 + \beta\int_0^t I\,dW(s) - \beta\int_0^t v_\beta(s)\,ds.$$

Thus the Itô formula, 1.3.2 gives us that

$$f(v_\beta(t)) = f(v_0) + 2m\beta\int_0^t \|v_\beta(s)\|_E^{2(m-1)}\langle v_\beta(s), dW(s)\rangle_E$$

$$- 2m\beta\int_0^t \|v_\beta(s)\|^{2m}\,ds$$

$$+ \frac{\beta^2}{2}\int_0^t \mathrm{tr}_L F(s)\,ds + \frac{\beta^2}{2}\int_0^t \mathrm{tr}_L G(s)\,ds$$

where $F(s)$ and $G(s)$ are the first and second terms of $D^2f(v_\beta(s))$.

Now,

$$\int_0^t \mathrm{tr}_L F(s)\,ds = \int_0^t\int_E 2m\|v_\beta(s)\|_E^{2(m-1)}\langle y,y\rangle_E d\nu(y)\,ds$$

$$= \int_0^t 2m\|v_\beta(s)\|_E^{2(m-1)}(\mathrm{tr}L)\,ds,$$

and

$$\int_0^t \mathrm{tr}_L G(s)\,ds = 4m(m-1)\int_0^t\int_E\|v_\beta(s)\|_E^{2(m-2)}\langle v_\beta(s),y\rangle_E\langle v_\beta(s),y\rangle_E\, d\nu(y)\,ds$$

$$= 4m(m-1)\int_0^t\|v_\beta(s)\|_E^{2(m-2)}\sum\lambda_i\langle v_\beta(s),e_i\rangle_E^2 ds.$$

Hence,

$$E(\|v_\beta(t)\|_E^{2m}) = \|v_0\|_E^{2m} + m\beta^2(\mathrm{tr}L)\int_0^t E(\|v_\beta(s)\|_E^{2(m-1)})\,ds$$

$$+2m(m-1)\beta^2\int_0^t E(\|v_\beta(s)\|_E^{2(m-2)}\sum\lambda_i\langle v_\beta(s),e_i\rangle_E^2)\,ds$$

$$-\ 2m\beta\int_0^t E\|v_\beta(s)\|_E^{2m}ds$$

$$= a(t) - 2m\beta\int_0^t E\|v_\beta(s)\|_E^{2m}ds$$

where $a(t)$ is the sum of the first three terms on the r.h.s. We use lemma 2.2.13, which we prove shortly, to obtain

$$E(\|v_\beta(t)\|_E^2) = a(t) - 2m\beta\int_0^t e^{-2m\beta(t-s)}a(s)\,ds$$

which, integrating by parts, is

$$a(t) - 2m\beta\left[\frac{e^{-2m\beta(t-s)}}{2m\beta}a(s)\right]_0^t - \int_0^t\frac{e^{-2m\beta(t-s)}}{2m\beta}a'(s)\,ds$$

$$= a(0)e^{-2m\beta t} + \int_0^t e^{-2m\beta(t-s)}a'(s)\,ds.$$

Now, $a(0) = \|v_0\|_E^{2m}$

and $a'(s) = m\beta^2(\mathrm{tr}L)E(\|v_\beta(s)\|_E^{2(m-1)})$

$$+\ 2m(m-1)\beta^2E(\|v_\beta(s)\|_E^{2m(m-2)}\sum\lambda_i\langle v_\beta(s),e_i\rangle_E^2).$$

But $\langle v_\beta(s), e_i\rangle_E^2 \le \|v_\beta(s)\|_E^2$

so

$$a'(s) \le m\beta^2(\mathrm{tr}L)E(\|v_\beta(s)\|_E^{2(m-1)})$$

$$+ 2m(m-1)\beta^2(\mathrm{tr}L)E(\|v_\beta(s)\|_E^{2(m-1)})$$

$$= m(2m-1)\beta^2(\mathrm{tr}L)E(\|v_\beta(s)\|_E^{2(m-1)})$$

giving by induction,

$$E(\|v_\beta(t)\|_E^{2m}) \le e^{-2m\beta t}\|v_0\|_E^{2m} +$$

$$+ \int_0^t e^{-2m\beta(t-s)} m(2m-1)\beta^2(\mathrm{tr}L)E(\|v_\beta(s)\|_E^{2(m-1)})ds$$

$$= e^{-2m\beta t}\|v_0\|_E^{2m} +$$

$$+ m(2m-1)\beta^2(\mathrm{tr}L)e^{-2m\beta t}\int_0^t e^{2m\beta s}E(\|v_\beta(s)\|_E^{2(m-1)})ds$$

$$\le e^{-2m\beta t}\|v_0\|_E^{2m} +$$

$$+ m(2m-1)\beta^2(\mathrm{tr}L)e^{-2m\beta t}\sum_{k=0}^{m-1}\frac{(2(m-1))!\|v_0\|_E^{2(m-k-1)}}{(2(m-k-1))!k!2^{2k}}(\mathrm{tr}L)^k\beta^k.$$

$$\cdot\int_0^t e^{-2(k+1)\beta s}(1-e^{-2\beta s})^k ds$$

$$= e^{-2m\beta t}\|v_0\|_E^{2m} +$$

$$+ \sum_{k=1}^{m}\frac{(2m)!e^{-2m\beta t}(\mathrm{tr}L)^k}{2(2(m-k))!2^{2(k-1)}(k-1)!}\|v_0\|_E^{2(m-k)}\beta^{k+1}\int_0^t e^{2k\beta s}(1-e^{2\beta s})^{k-1}ds$$

$$= e^{-2m\beta t}\|v_0\|_E^{2m} +$$

$$+ \sum_{k=1}^{m}\frac{(2m)!e^{-2\beta kt}(\mathrm{tr}L)^k}{2(2(m-k))!2^{2(k-1)}(k-1)!}\|v_0\|_E^{2(m-k)}\beta^{k+1}\left(\frac{e^{2\beta kt}}{2\beta k}(1-e^{-2\beta t})^k\right)$$

$$= \sum_{k=0}^{m}\frac{(2m)!(\mathrm{tr}L)^k}{(2(m-k))!2^{2k}k!}\|v_0\|_E^{2(m-k)}e^{-2\beta(m-k)t}\beta^k(1-e^{-2\beta t})^k.$$

#

(2.2.12) Remark

In particular, if E is a Hilbert space and $\beta>1$,

$$E\left(\frac{\|v_\beta(t)\|_E^{2m}}{\beta^m}\right) \leq \sum_{k=0}^{m} \frac{(2m)!}{(2(m-k))!\,2^{2k}.k!} \|v_o\|_E^{2(m-k)} (trL)^k$$

and hence is bounded by a constant depending only on m, v_o and W.

We say $\quad E\left(\frac{\|v_\beta(t)\|_E^{2m}}{\beta^m}\right) \leq P_m^2 \qquad m\geq 1.$

(2.2.13) Gronwall's Lemma

Let ϕ, α be real-valued Lebesgue-integrable functions on the interval $[0,R]$ such that for some $L>0$

$(*)$...... $\phi(t) \leq \alpha(t) + L\int_0^t \phi(s)ds \qquad t \in [0,R].$

Then $\phi(t) \leq \alpha(t) + L\int_0^t e^{L(t-s)}\alpha(s)ds$ a.a. $t \in [0,R]$.

If we have equality in $(*)$ then L need not be positive, and the result becomes equality.

Proof

We will just prove the result for inequality taking the proof from Elworthy [4].

Set $\psi(t) = \int_0^t \phi(s)ds$. Then ψ is differentiable a.e. and

$$\psi'(t) \leq \alpha(t) + L\ (t)$$

giving $\frac{d}{dt}(e^{-Lt}\psi(t)) \leq e^{-Lt}\alpha(t)$

whence $\quad \psi(t) \leq e^{Lt}\int_0^t e^{-Ls}\alpha(s)ds.$

Substitution gives the result. #

§3. Iterated Ornstein-Uhlenbeck Processes

We will construct a sequence of processes $\{z_\beta^n\}$ on E for $n \geq 1$ such that each z_β^n has almost surely C^n sample paths.

Consider the pair (v_β^2, z_β^2) of stochastic processes on E given by the equations,

$$\begin{cases} dz_\beta^2(t) = v_\beta^2(t)\,dt \quad , \quad z_\beta^2(0) = z_o \\ dv_\beta^2(t) = -\beta^2 v_\beta^2(t)\,dt + \beta^2 v_\beta(t)\,dt. \end{cases}$$

By the theory of ordinary differential equations solutions exist for all $t \in [0,R]$ for almost all $\omega \in \Omega$.

In general we define z_β^n, v_β^n by

$$(2.3.1) \quad \begin{cases} dz_\beta^n(t) = v_\beta^n(t)\,dt \quad , \quad z_\beta^n(0) = z_o. \\ dv_\beta^n(t) = -\beta^{2^{(n-1)}} v_\beta^n(t)\,dt + \beta^{2^{(n-1)}} v_\beta^{n-1}(t)\,dt \;, v_\beta^n(0) = v_o. \end{cases}$$

where z_β^1 and v_β^1 are z_β and v_β respectively and

$$'v_\beta^0 dt' = 'dz_\beta^0(t)' = dW(t).$$

(2.3.2) <u>Theorem</u>

For each $n \geq 1$ $\|z_\beta^n(t) - W(t) - z_o\| \longrightarrow 0$ as $\beta \longrightarrow \infty$, uniformly for $t \in [0,R]$.

<u>Proof</u>

We use induction on n , noting that the result is true for n=1 by theorem 2.2.5.

Assume the result is true for n.

Then we use Gronwall's lemma to obtain (noting that in the case of equality in Lemma 2.2.13 ϕ and α can map into any Banach space),

$$v_\beta^{n+1}(t) = v_o e^{-\beta^{2^n}t} + \beta^{2^n}\int_0^t v_\beta^n(s)\,ds - \beta^{2^{(n+1)}}\int_0^t e^{-\beta^{2^n}(t-s)} \cdot \int_0^s v_\beta^n(r)\,dr\,ds$$

$$= v_o e^{-\beta^{2^n}t} + \beta^{2^n}\int_0^t v_\beta^n(s)\,ds - \beta^{2^{(n+1)}}\frac{1}{\beta^{2^n}}\int_0^t v_\beta^n(s)\,ds$$

$$+ \frac{\beta^{2^{(n+1)}}}{\beta^{2^n}} e^{-\beta^{2^n}t}\int_0^t e^{\beta^{2^n}s} v_\beta^n(s)\,ds$$

(integrating by parts)

$$= e^{-\beta^{2^n}t} v_o + \beta^{2^n} e^{-\beta^{2^n}t}\int_0^t e^{\beta^{2^n}s} v_\beta^n(s)\,ds.$$

Hence

$$z_\beta^{n+1}(t) = z_o + \beta^{2^n}\int_0^t e^{-\beta^{2^n}s}\int_0^s e^{\beta^{2^n}r} v_\beta^n(r)\,dr\,ds + v_o\int_0^t e^{-\beta^{2^n}s}\,ds$$

$$= z_o - e^{-\beta^{2^n}t}\int_0^t e^{\beta^{2^n}s} v_\beta^n(s)\,ds + v_o\beta^{-2^n}\left[1 - e^{-\beta^{2^n}t}\right]$$

$$+ \frac{\beta^{2^n}}{\beta^{2^n}}\int_0^t e^{-\beta^{2^n}s} e^{\beta^{2^n}s} v_\beta^n(s)\,ds$$

$$= z_o - e^{-\beta^{2^n}t}\int_0^t e^{\beta^{2^n}s} v_\beta^n(s)\,ds + v_o\beta^{-2^n}\left[1 - e^{-\beta^{2^n}t}\right]$$

$$+ \int_0^t v_\beta^n(s)\,ds$$

$$= z_\beta^n(t) - e^{-\beta^{2^n}t}\int_0^t e^{\beta^{2^n}s} v_\beta^n(s)\,ds + v_o\beta^{-2^n}\left[1 - e^{-\beta^{2^n}t}\right].$$

Thus

$$\|z_\beta^{n+1}(t) - W(t) - z_o\| \le \|z_\beta^n(t) - W(t) - z_o\| + \|v_o\|\beta^{-2^n}$$

$$+ e^{-\beta^{2^n}t}\int_0^t e^{\beta^{2^n}s}\|v_\beta^n(s)\|\,ds.$$

We can easily show by induction that $\|v_\beta^n(s)\| \le n\|v_o\| + \beta/2(t+L) = K(n,\beta)$ for all $t \in [0,R]$

and hence

$$\|z_\beta^{n+1}(t)-W(t)-z_0\|^2 \le 3\|z_\beta^n(t)-W(t)-z_0\|^2 + 3\|v_0\|^2\beta^{-2^{(n+1)}}$$

$$+ 3e^{-2\beta^{2^n}t}R\int_0^t e^{2\beta^{2^n}s}\|v_\beta^n(s)\|^2 ds$$

by th Cauchy-Schwarz inequality

$$\le 3Re^{-2\beta^{2^n}t}K(n,\beta)^2\int_0^t e^{2\beta^{2^n}s}ds \quad +3\|v_0\|^2\beta^{-2^{(n+1)}}$$

$$+ 3\|z_\beta^n(t)-W(t)-z_0\|^2$$

$$\le 3R(2\beta^{2^n})^{-1}K(n,\beta)^2 + 3\|z_\beta^n(t)-W(t)-z_0\|^2 + 3\|v_0\|^2\beta^{-2^{(n+1)}}$$

which tends to zero as β tends to infinity (uniformly for $t \in [0,R]$) by induction.

#

CHAPTER THREE. ORNSTEIN-UHLENBECK PROCESSES ON MANIFOLDS

§1. General Ornstein-Uhlenbeck Processes

In this section we will show how it is possible to construct O-U type processes on any manifold with sufficient structure, before particularising to Riemannian manifolds in §2.

Let M be a separable, metrisable C^4 manifold modelled on a Hilbert space. We assume the existence of Q, a C^3 section of $\mathrm{Hom}_M(E,TM)$ for some Banach space E and of a spray over M (i.e. a vector field ξ over TM satisfying, for $s \in \mathbb{R}$ identified with the map $TTM \to TTM$ given by scalar multiplication by s on fibres, $\xi(sv) = s_* s\xi(v)$ for all $v \in TM$).

As in example 1.1.10 we will construct an S.D.S. (X,z) on TM, $(X,z) = (X_1 \oplus Y, W \oplus t)$ where X_1 is a section of $\mathrm{Hom}_M(E,TM)$, W is a Brownian motion on E and (Y,t) is a drift.

We will construct X_1 and Y in turn.

X_1 arises from Q in the following manner. We have a map $\tilde{Q} : E \times TM \to TTM$,

$$\begin{tikzcd} E \times TM \arrow[r, "\simeq"] \arrow[rrrr, bend right, "\tilde{Q}"'] & \pi^*(M \times E) \arrow[r, "\pi^* Q"] & \pi^*(TM) \arrow[r, "\alpha"] & VT(TM) \arrow[r] & TTM \end{tikzcd}$$

Here π^* is the 'pull-back' by π, the natural projection from TM to M :- in a chart (U,ϕ) modelled on a Hilbertspace H, $\pi^*(M \times E)$ has the representation $\phi(U) \times H \times E$ and $\pi^*(TM)$ has the representation $\phi(U) \times H \times H$ (if B is a fibre bundle over M then $\pi^* B$ is the fibre-bundle product $TM \times_M B$). If in the chart (U,ϕ)

Ω has the local representation,

$$\Omega_U : \phi(U)\times E \longrightarrow \phi(U)\times H$$
$$(x,v) \longmapsto (x,\Omega'_U(x,v))$$

(we will frequently use the same symbol for representations and their principal parts)
then $\pi^*(\Omega)$ has the representation

$$\pi^*(\Omega)_U : \phi(U)\times H\times E \longrightarrow \phi(U)\times H\times H$$
$$(x,e,v) \longmapsto (x,e,\Omega'_U(x,v)).$$

We will construct α for more general vector bundles - this will prove useful in §2. Let $\tau: B \longrightarrow V$ be a vector bundle over a C^1 manifold V. Then there is an exact sequence

$$0 \longrightarrow \tau^*B \xrightarrow{\alpha} TB \xrightarrow{T^*\tau} TV \longrightarrow 0 \qquad \text{(see Lang [12])}.$$

To see this , let (U,ϕ) be a chart of V modelled on a Banach space E, in which $\tau^{-1}(U)$ has representation $U\times F$ for some Banach space F. In this chart,

τ^*B has representation $\phi(U)\times F\times F$,

TB has representation $\phi(U)\times E\times F\times F$,

$\tau^*(TV)$ has representation $\phi(U)\times E\times F$.

α has the local representation,

$$\alpha_U : \phi(U)\times F\times F \longrightarrow \phi(U)\times E\times F\times F$$
$$(x,v,w) \longmapsto (x,0,v,w).$$

We will show,

(3.1.1) <u>Lemma</u>

$\alpha : \tau^*B \longrightarrow TB$ is well-defined by the above local representation.

<u>Proof</u>

Let $(\tilde{U},\psi)$ be another chart of V with $U\cap\tilde{U} \neq \phi$. We must show that the above local definition of α is invariant under the change of coordinates in B,

$$\gamma : \phi(U\cap\tilde{U})\times F \longrightarrow \psi(U\cap\tilde{U}) \times \tilde{F}$$
$$(x,v) \longmapsto (\psi\circ\phi^{-1}(x), L(x)v)$$

where $\tilde{F}$ is a Banach space and $L \in C^{0}(U\cap\tilde{U}, L(F;\tilde{F}))$

$$\text{and} \quad T\gamma : \phi(U\cap\tilde{U})\times E\times F\times F \longrightarrow (U\cap\tilde{U})\times E\times\tilde{F}\times\tilde{F}$$
$$(x,e,v,w) \longmapsto (\psi\circ\phi^{-1}(x), D(\psi\circ\phi^{-1})(x)e, L(x)v, L(x)w + T_xL(e)v).$$

In other words we require the following diagram to be commutative,

$$\begin{array}{ccc} U\times F\times F & \xrightarrow{\alpha_U} & U\times E\times F\times F \\ \Big\downarrow{\scriptstyle \tau^*(\gamma)_U} & & \Big\downarrow{\scriptstyle T\gamma} \\ \tilde{U}\times\tilde{F}\times\tilde{F} & \xrightarrow{\alpha_{\tilde{U}}} & \tilde{U}\times E\times\tilde{F}\times\tilde{F} \end{array}$$

where $\tau^*(\gamma)_U(x,v,w) = (\psi\circ\phi^{-1}x, L(x)v, L(x)w)$.

This is true if and only if

$$(\psi\circ\phi^{-1}(x), 0, L(x)v, L(x)w) =$$
$$(\psi\circ\phi^{-1}(x), D(\psi\circ\phi^{-1})(x)(0), L(x)v, L(x)w+T_xL(0)v)$$

which is clearly true. #

Returning to the case $\tau : B \longrightarrow V$ is $\pi : TM \longrightarrow M$ we see that the local representation of $\alpha : \pi^*TM \longrightarrow TTM$ in the chart (U,ϕ) is

$$\alpha_U : \phi(U)\times H\times H \longrightarrow (\phi(U)\times H)\times(H\times H)$$
$$(x,v,w) \longmapsto (x,v,0,w).$$

Hence $\tilde{Q}$ is represented locally by,

$$\tilde{Q}_U : \phi(U)\times H\times E \longrightarrow \phi(U)\times H\times H\times H$$
$$(x,v,w) \longmapsto (x,v,0,Q'_U(x,w)).$$

We define q_β : $E\times TM \longrightarrow E\times TM$
by $(w,v_x) \longmapsto (\beta w, v_x)$.

Then $\tilde{Q}\circ q_\beta$ defines X_1. Locally,

$$X_{1U} : \phi(U)\times H\times E \longrightarrow \phi(U)\times H\times H\times H$$
$$(x,v,w) \longmapsto (x,v,0,\beta Q_U(x,w)).$$

We will now construct Y. This will consist of three parts:-

(i) the analogue of the s.d.e. '$dz_\beta = v_\beta dt$'

(ii) the analogue of '$-\beta v_\beta dt$' in equation 2.2.1;

(ii) a C^1 vector field F on M (which may be zero) which is the analogue of the force field in example 1.1.10.

The first part is given by the spray which we have assumed exists over M - we have a vector field on TM satisfying, locally,

$$\phi(U)\times H \longrightarrow (\phi(U)\times H)\times(H\times H)$$
$$(x,v) \longmapsto (x,v,v,f_U(x,v)),$$

where for $s \in \mathbb{R}$, $f_U(x,sv) = s^2 f_U(x,v)$.

Note that in example 1.1.10 we took f_U to be identically zero.

The second ('frictional') part of the vector field, like X_1, arises from α. Define the section p_β : $TM \longrightarrow \pi^*(TM)$ by the local sections,

$$(p_\beta)_U : \phi(U)\times H \longrightarrow \phi(U)\times H\times H$$
$$(x,v) \longmapsto (x,v,-\beta v).$$

This is easily seen to be invariant under a change of coordinates and hence is well defined globally.

$\alpha\circ p_\beta : TM \longrightarrow TTM$ becomes the second part of the vector field. Locally

$$(\alpha\circ p_\beta)_U : \phi(U)\times H \longrightarrow (\phi(U)\times H)\times(H\times H)$$
$$(x,v) \longmapsto (x,v,0,-\beta v).$$

For the third part of the vector field (the 'force field') we assume that we are given a vector field F over M with local representation

$$F_U : \phi(U) \longrightarrow \phi(U)\times H$$
$$x \longmapsto (x,F_U(x)) .$$

In the same way that Q generates $\tilde{Q}$ we get $\tilde{F} : TM \longrightarrow TTM$, locally

$$\tilde{F}_U : \phi(U)\times H \longrightarrow \phi(U)\times H\times H\times H$$
$$(x,v) \longmapsto (x,v,0,F'_U(x)).$$

We put the three parts of the vector field together to give $Y : TM \longrightarrow TTM$ defined by the local representations,

$$Y_U : \phi(U)\times H \longrightarrow \phi(U)\times H\times H\times H$$
$$(x,v) \longmapsto (x,v,v,-\beta v + f_U(x,v) + F'_U(x))$$

which gives the section (also called Y) of $\mathrm{Hom}_{TM}(\mathbb{R},TTM)$ represented locally by

$$Y_U : \phi(U)\times H\times \mathbb{R} \longrightarrow \phi(U)\times H\times H\times H$$
$$(x,v,t) \longmapsto (x,v,vt,-\beta vt + f_U(x,v)t+F_U(x)t)$$

Thus $X = X_1 \oplus Y : TM \times E \times \mathbb{R} \longrightarrow TTM$ is represented locally by

$$X_U : \phi(U) \times H \times E \times \mathbb{R} \longrightarrow \phi(U) \times H \times H \times H$$

(3.1.2) $$(x,v,w,t) \longmapsto (x,v,vt,-\beta vt + f_U(x,v)t + F_U(x)t + \beta Q_U(x,w)).$$

As we stated at the beginning of this section, we consider the S.D.S. (X,z) where $z = W \oplus t$. Then Theorem 1.1.7 tells us that the s.d.e. $dx = Xdz$, $x(0) = x_0 \in M$ has maximal solutions. Such a solution $x:[0,\zeta) \times \Omega \longrightarrow TM$ may be called an 'O-U process' on TM. We will show in §3 how we can construct another O-U process on TM given an O-U process on E.

(3.1.3) Suppose that we are given another C^2 vector field V on M. Then we create an 'O-U process plus drift' on TM as follows. Let V be represented locally by

$$V_U : \phi(U) \longrightarrow \phi(U) \times H$$
$$x \longmapsto (x, V_U(x)).$$

TV is a C^1 vector field on TM represented locally by

$$TV_U : \phi(U) \times H \longrightarrow \phi(U) \times H \times H \times H$$
$$(x,v) \longmapsto (x,v,V_U(x), TV_U(x)(v)).$$

We add this term to the drift Y to give

$$X : TM \times E \times \mathbb{R} \longrightarrow TTM \text{ represented locally by}$$

$$X_U : \phi(U) \times E \times H \times \mathbb{R} \longrightarrow \phi(U) \times H \times H \times H$$

$$(x,v,w,t) \longmapsto (x,v,vt+V_U(x)t, -\beta vt + f_U(x,v)t + F_U(x)t + \beta Q_U(x,w) + TV_U(x)(v)t).$$

(3.1.4) Example

Let M=H=E. Then we have Q globally defined by

$$Q : H\times H \longrightarrow TH \cong H\times H$$
$$(x,v) \longmapsto (x,v).$$

We take the zero spray over H,

$$H\times H \longrightarrow H\times H\times H\times H$$
$$(x,v) \longmapsto (x,v,v,0).$$

Let F be the vector field $F : H \longrightarrow H\times H$, $x \longmapsto (x,g(x))$

for some continuous map g from H to H. We have $\tilde{F}: H\times H \longrightarrow H\times H\times H\times H$ given by $(x,v) \longmapsto (x,v,0,g(x))$.

V is any C^2 vector field over H. Then $X=X_1 \oplus Y$ is the section,

$$X : H\times H\times H\times \mathbb{R} \longrightarrow H\times H\times H\times H$$
$$(x,v,w,t) \longmapsto (x,v,vt+V(x)t,-\beta vt+g(x)t+\beta w+TV(x)vt).$$

We may rewrite the s.d.e. $d\nu = Xdz$ as ,

$$dx(t) = v(t)dt + V(x(t))dt, x(0)=x_0$$
$$dv(t) = -\beta v(t)dt + g(x(t))dt + \beta dW(t) + TV(x(t))\dot{x}dt$$

Thus if V is not identi cally zero v is no longer the'velocity' of z. If V is zero we have the equation of an O-U process in a force-field on H, and if g is also zero we have the situation of Chapter Two...

§2. Ornstein-Uhlenbeck Processes on Riemannian Manifolds

Let M be a finite-dimensional Riemannian manifold modelled on $\mathbb{R}^n$, with orthonormal frame bundle O(M).

We do not, in general, have a section Q of $\mathrm{Hom}_M(\mathbb{R}^n, TM)$ as we assumed existed in §1, but we do have a spray - the Riemannian spray which generates the geodesics in M.

However, as we saw in §1.5 there is a section θ of $\mathrm{Hom}_{O(M)}(\mathbb{R}^n, TO(M))$, $\theta : O(M) \times \mathbb{R}^n \longrightarrow HO(M) \hookrightarrow TO(M)$.

We will use the method of §1 to construct an O-U process on HO(M). Projection to TM by $T\tau$ (where $\tau : O(M) \longrightarrow M$ is projection) will give us an O-U process on TM.

We take the construction in four parts - spray, 'friction', 'force field' (generated by some C^2 vector field over M), and 'noise'.

Spray

We require a vector field over TO(M). In fact as we intend to construct, on O(M), the horizontal lift of a process on M (i.e. the process on O(M) will consist of the lifts of the C^1 sample paths of the process on M) we are only interested in obtaining a vector field over HO(M). We will use the map,

$$\begin{array}{ccccccc} HO(M) & \longrightarrow & O(M) \times \mathbb{R}^n & \longrightarrow & TO(M) \times \mathbb{R}^n \times \mathbb{R}^n & \longrightarrow & THO(M) \\ \nu_u & \longmapsto & \theta^{-1}(\nu_u) = (u,v) & \longmapsto & (\theta(u,v), v, 0) & \longmapsto & T\theta(\theta(u,v), v, 0). \end{array}$$

Let v be a solution curve in HO(M) of this vector field with initial point $\theta(u_o, v_o)$. Then v projects onto $O(M)$ as a solution curve of $du = \theta(u, v_o)dt$ with $u(0) = u_o$.

A proof of the following proposition is contained in Kobayashi and Nomizu [10] where the θ^{v_o} are 'standard horizontal vector fields'.

(3.2.1) <u>Proposition</u>

Let $v_o \in \mathbb{R}^n$ with $|v_o| = 1$ and define $\theta^{v_o} : O(M) \longrightarrow HO(M)$ by $\theta^{v_o}(u) = \theta(u, v_o)$. The solution curves of θ^{v_o} project by $\tau : O(M) \longrightarrow M$ to precisely the geodesics of M.

<u>Friction</u>

We require a vector field on $HO(M)$ and we will obtain this from the map α constructed in §1. The vector bundle is $\rho : HO(M) \longrightarrow O(M)$ and $\alpha : \rho^*(HO(M)) \longrightarrow THO(M)$ is given (globally) by

$$\rho^*(HO(M)) \xrightarrow[\simeq]{\rho^*(\theta^{-1})} \rho^*(O(M) \times \mathbb{R}^n) \xrightarrow{\simeq} O(M) \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\tilde{\alpha}} TO(M) \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow[\simeq]{T\theta} THO(M)$$

$$(u, v, w) \longmapsto (\theta(u, 0), v, w).$$

Thus the frictional vector field is, again globally,

$$HO(M) \xrightarrow{\simeq} O(M) \times \mathbb{R}^n \longrightarrow O(M) \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow{\tilde{\alpha}} TO(M) \times \mathbb{R}^n \times \mathbb{R}^n \xrightarrow[\simeq]{T\theta} THO(M)$$

$$\nu_u \longmapsto (u, v) \longmapsto (u, v, -\beta v) \longmapsto (\theta(u, 0), v, -\beta v)$$

<u>Force Field</u>

We suppose that we are given a vector field (C^2) F on M. The horizontal lift F_1 of F on $O(M)$ is given by

$$F_1(u_x) = (T\tau|H_{u_x}O(M))^{-1}(F(x)),$$

i.e. $F_1 : O(M) \longrightarrow O(M) \times R^n \longrightarrow HO(M)$

$$u_x \longmapsto (u_x, u_x^{-1}(F(x)).$$

This, in turn, determines $\tilde{F}$ on HO(M),

$$HO(M) \longrightarrow O(M) \times R^n \longrightarrow TO(M) \times R^n \times R^n \xrightarrow{T\theta} THO(M)$$

$$\nu_u \longmapsto (u_x, v) \longmapsto (\theta(u_x, 0), v, u_x^{-1}(F(x))).$$

<u>Noise</u>

The noise is determined in precisely the same way as it was in §1, though on HO(M) rather than on TO(M) :-

$\tilde{\theta} : HO(M) \times R^n \longrightarrow THO(M)$ is given by

$$HO(M) \times R^n \xrightarrow{\simeq} \rho^*(O(M) \times R^n) \xrightarrow{\rho^*(\theta)} \rho^*(HO(M)) \xrightarrow{\alpha} THO(M)$$

$$(\nu_u, w) \longmapsto T\theta(\theta(u,0), v, w)$$

where $(u,v) = \nu_u$ and $\rho : HO(M) \longrightarrow O(M)$ is projection.

Define $q_\beta : HO(M) \times R^n \longrightarrow HO(M) \times R^n$ by

$$(\nu_u, w) \longmapsto (\nu_u, \beta w).$$

Then $\tilde{\theta} \circ q_\beta$ is the section X_1 of $\mathrm{Hom}_{HO(M)}(R^n, THO(M))$ determining the noise.

We now combine the four parts of the S.D.S. constructed above on HO(M). The total drift, given by 'spray', 'friction', and 'force field' is

$$Y : HO(M) \longrightarrow O(M) \times R^n \longrightarrow TO(M) \times R^n \times R^n \longrightarrow THO(M)$$

$$\nu_u \longmapsto T\theta(\theta(u,v), v, -\beta v + u^{-1}(F(\tau(u))).$$

Adding the noise we get the section X of $\mathrm{Hom}_{HO(M)}(\mathbb{R}^n \times \mathbb{R}, THO(M))$,

$$HO(M)\times \mathbb{R}^n \times \mathbb{R} \xrightarrow{\simeq} O(M)\times \mathbb{R}^n\times \mathbb{R}^n \times \mathbb{R} \longrightarrow TO(M)\times \mathbb{R}^n\times \mathbb{R}^n \xrightarrow{\simeq} THO(M)$$
$$(\nu_u, w, t) \longmapsto (u,v,w,t) \longmapsto (\theta(u,vt), v, -\beta vt+\beta w+u^{-1}F(\tau(u))).$$

Then, again letting $z = W\oplus t$ on $\mathbb{R}^n\times\mathbb{R}$, (X,z) is a canonically defined S.D.S. on $HO(M)$ and the s.d.e. $dx=Xdz$, $x(0)=u_o$ possesses a maximal solution on $HO(M)$ by Theorem 1.1.7.

Let $u : [0,\xi) \times \Omega \longrightarrow HO(M)$ be such a solution. Then we have a process $v : [0,\xi)\times\Omega \longrightarrow TM$ defined by $v= T\tau\circ u$. v is an 'Ornstein-Uhlenbeck' process on TM. Note that $\theta^{-1}\circ u$ is the process on $O(M)\times \mathbb{R}^n$,

$$[0,\xi)\times\Omega \longrightarrow O(M)\times \mathbb{R}^n$$
$$(t,\omega) \longrightarrow (\rho(u(t,\omega)), (\rho(u(t,\omega)))^{-1}v(t,\omega)).$$

As in §1 we can use a C^2 vector field on M to create an extra drift term in the s.d.e. determining the O-U process.

§3. The Rolled Ornstein-Uhlenbeck Process

Let M be a manifold, E a Banach space and Q a section of $\mathrm{Hom}_M(E,TM)$ all satisfying the conditions imposed on them in §1. Also we let V be a C^2 vector field on M as in3.1.3.

In §1 we constructed an 'O-U' process on M by analogy with the construction of such processes on Banach spaces. We now assume that we are given an O-U process (z_β, v_β) on E and that E is 2-uniformly smooth, hence permitting ourselves to apply the results of Chapter 2.

(z_β, v_β) may be an O-U process in a force field on E, in which case we get an extra term in the s.d.e. 2.2.1.

Consider the family of o.d.e.s,

$$dx(t)/dt = Q(x(t))v_\beta(t) + V(x(t)), \quad x(0)=x_o \in M$$

(cf. remark following Lemma 2.2.8). Then maximal solutions exist for almost all $\omega \in \Omega$, giving a process on M if we take $x(t,\omega) = x_o$ for all ω such that $z_\beta(.,\omega)$ is not C^1. $x : [0,\xi)\times\Omega \longrightarrow M$ has almost surely C^1 sample paths.

Let (U,ϕ) be a chart of M such that $x(t,\omega) \in U$ for some $t>0$ and $\omega\in\Omega$ s.t. $x(-,\omega)$ is C^1. Locally we have $x_U(t)\in\phi(U)$, $x'_U(t)\in\phi(U)\times H$,

$$x'_U(t) = (x_U(t), Q_U(x_U(t))v_\beta(t) + V_U(x_U(t))).$$

In what follows we will drop the subscript 'U'. We form from this a local (Itô) s.d.e. on M. The significance of (Itô) throughout this paper is that there will be no second order terms in the corresponding integral equations.

Locally,

$$d(x(t),v(t)) = (Q(x(t))v_\beta(t)dt + V(x(t))dt, DV(x(t))v(t)dt + D_1Q(x(t),v_\beta(t))v(t)dt + Q(x(t))dv_\beta(t)).$$

As in example 3.1.4 we have,

$$dv_\beta(t) = -\beta v_\beta(t)dt + g(z_\beta(t))dt + \beta dW(t).$$

Thus we have the connected pair of equations,

$$\text{(3.3.1)} \quad \begin{aligned} dx(t) &= Q(x(t))v_\beta(t)dt + V(x(t))dt \\ \text{(Itô)}\ dv(t) &= DV(x(t))v(t)dt + D_1Q(x(t),v_\beta(t))v(t)dt - \\ &\quad - \beta Q(x(t))v_\beta(t)dt + Q(x(t))g(z_\beta(t))dt + \\ &\quad + \beta Q(x(t))dW(t). \end{aligned}$$

We can compare the O-U process on TM which is a maximal solution of 3.3.1 starting from a given point, with a solution of the s.d.e. corresponding to the S.D.S. of 3.1.3:-

(3.3.2)

$$dx_1(t) = v_1(t)dt + V(x_1(t))dt \quad , \; x_1(0) = x_o \in M$$

$$dv_1(t) = -\beta v_1(t)dt + f(x_1(t), v_1(t))dt + F(x_1(t))dt + \\ + \beta Q(x_1(t))dW(t) + DV(x_1(t))v_1(t)dt,$$

$$v_1(0) = Q(x_0)v_o + V(x_o) = v(0).$$

Now let us assume that M=H is the Hilbert space on which v_β and z_β are defined and that in §1 we have the zero spray, so that f=0, and that F=g on H. Assume also that $Q(x)v=v$ for all $x,v \in H$. Then 3.3.1 becomes

$$dx(t) = v_\beta(t)dt + V(x(t))dt$$

$$dv(t) = DV(x(t))v(t)dt - \beta v_\beta(t)dt + g(z_\beta(t))dt \\ + \beta dW(t),$$

and 3.3.2 becomes

$$dx_1(t) = v_1(t)dt + V(x_1(t))dt$$

$$dv_1(t) = -\beta v_1(t)dt + g(x_1(t))dt + \beta dW(t) + \\ DV(x_1(t))v_1(t)dt.$$

Thus even in this simple case the two constructions give different processes unless V is identically zero.

In particular, let M be a finite dimensional Riemannian manifold. In §1.5 we saw how to construct a Brownian motion on M given one on $\mathbb{R}^n$. We use a similar construction to obtain an Ornstein-Uhlenbeck process on M. Let z be a stochastic process on $T_{x_o}M$ starting at 0_{x_o} and choose $u_o \in \tau^{-1}(x_o)$. Let us assume that z has almost surely C^1 sample paths. $u_o^{-1} \circ z$ is a process on $\mathbb{R}^n$ with almost surely C^1 sample paths starting at 0. If θ is the canonical diffeomorphism from $O(M) \times \mathbb{R}^n$ to $HO(M)$ then by the theory of ordinary differential equations $du/dt = \theta(u(t), u(t)^{-1}\frac{dz}{dt}(t))$, $u(0) = u_o$ has solutions for almost all $\omega \in \Omega$. Let $u : [0, \xi^{u_o}) \times \Omega \longrightarrow O(M)$ be such a family of maximal solutions and define $\mathcal{D}(z)(u_o) \equiv \tau \circ u : [0, \xi^{u_o}) \times \Omega \longrightarrow M$. $\mathcal{D}(z)(u_o)$ is a process on M starting at x_o and is independent of the u_o chosen in $\tau^{-1}(x_o)$. Hence we write $\mathcal{D}(z)$. $\mathcal{D}(z)$ is the composition of z with the Cartan development and the process by which we obtain $\mathcal{D}(z)$ is called <u>rolling</u>.

In fact we wish to construct an O-U process on M based on an O-U process on $\mathbb{R}^n$. In this case the development of z does depend on u_o. However, similarly to Proposition 1.5.3 we can show that

(3.3.3) <u>Lemma</u>

Let (z_β, v_β) be an O-U process on $\mathbb{R}^n$ and let u,v be maximal solutions of the family of o.d.e.'s $du = \theta(u)v_\beta dt$ with initial points u_o, v_o respectively, where $u_o, v_o \in \tau^{-1}(x_o)$. Then $R_g \circ u(t,\omega) = v(t, p(\omega))$ a.s. , where p is a measure-preserving transformation of $\Omega = C^1([0,R]; \mathbb{R}^n)$ & $R_g(u_o) = v_o$.

There is an inverse map to $\mathcal{D}$, $\mathcal{D}^{-1}$ constructed as follows. Let $x : [0, \xi) \times \Omega \longrightarrow M$ be a process with almost surely C^1 sample paths starting at x_o and choose $\omega \in \Omega$ such that $x_\omega : [0, \xi(\omega)) \longrightarrow M$

is C^1. Let $\tilde{x}_{u_o}(-,\omega)$ be the horizontal lift of $x(-,\omega)$ to $O(M)$ with $\tilde{x}_{u_o}(0) = u_o \in \tau^{-1}(x_o)$.

Then $\tilde{x}'_{u_o}(t,\omega) = \nu_{u_o}(t,\omega) = \theta(\tilde{x}_{u_o}(t,\omega), v_{u_o}(t,\omega))$ for v_{u_o} some process on $\mathbb{R}^n$. $\mathcal{D}^{-1}(x) = u_o \circ v_{u_o}$ is a process on $T_{x_o}M$ which can easily be seen to roll onto x on M.

We have.

(3.3.4) Theorem

Let (z_β, v_β) be an O-U process on $\mathbb{R}^n$ and let us roll this onto M to obtain $x: [0,\xi) \times \Omega \longrightarrow M$ starting at x_o (choosing $z_0 = 0$, $v_0 \in \mathbb{R}^n$ & $u_0 \in \tau^{-1}(x_0)$).

Let $y : [0,\xi) \times \Omega \longrightarrow M$ be another O-U process on M starting at x_0 which is the projection of an O-U process on TM obtained by the method of §2 with a constant of β and V and F both identically zero & starting point $\theta(u_0, v_0)$. We assume also that (z_β, v_β) is an O-U process without a force-field so that its components satisfy 2.2.1 and 2.2.2.

Then $y(t,\omega) = x(t,\omega)$ almost surely.

Proof

We will use $\mathcal{D}^{-1}$ to 'unroll' y. Choose $\omega \in \Omega$ such that $y(-,\omega) : [0,\xi(\omega)) \longrightarrow M$ is C^1. Then the lift of $y(-,\omega)$, $\tilde{y}(-,\omega) : [0,\xi(\omega)) \longrightarrow O(M)$ starting at $u_o \in \tau^{-1}(x_o)$ is the projection of a solution curve of the s.d.e. $dx = Xdz$ constructed in §2. Recall that $X : HO(M) \times \mathbb{R}^n \times \mathbb{R} \longrightarrow THO(M)$ is defined by

$$HO(M) \times \mathbb{R}^n \times \mathbb{R} \longrightarrow O(M) \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \longrightarrow TO(M) \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow THO(M)$$

$$(u,v,w,t) \longrightarrow (\theta(u,vt), v, -\beta vt + \beta w).$$

This gives that

$$\tilde{y}'(t) = \theta(\tilde{y}(t), v(t))$$

where v is a process on $\mathbb{R}^n$ such that $dv(t) = -\beta v(t)dt + \beta dW(t)$.

Hence $v(t) = v_\beta(t)$ a.s.

$\#$

§4 The Infinitesimal Generator

We will use the construction of §2 to determine the infinitesimal generator of an O-U process on a finite dimensional Riemannian manifold.

Let $f : TM \xrightarrow{b} \mathbb{R}$ be C^2 and define $\tilde{f} : HO(M) \longrightarrow \mathbb{R}$ by $\tilde{f}(\sigma) = f \circ T\tau(\sigma)$. Let $\tilde{P}_t$ be the transition operator of the O-U process $\tilde{y}$ on HO(M) and P_t of $T\tau \circ \tilde{y} = y$ on TM.

$$\begin{aligned}
\tilde{P}_t(\tilde{f})(\sigma) &= \int_{\Omega_t^\sigma} \tilde{f}(\tilde{y}(\sigma, t, \omega))\, d\mu(\omega) \\
&= \int_{\Omega_t^\sigma} f \circ T\tau(\tilde{y}(\sigma, t, \omega))\, d\mu(\omega) \\
&= \int_{\Omega_t^\sigma} f(y(T\tau(\sigma), t, \omega))\, d\mu(\omega) \\
&= P_t f(T\tau(\sigma)).
\end{aligned}$$

Let $\tilde{\mathcal{A}}$ be the infinitesimal generator of $\tilde{y}$ on HO(M) and $\mathcal{A}$ the infinitesimal generator of y on TM. Then from the above we have that $\mathcal{A}(f)(T\tau(\sigma)) = \tilde{\mathcal{A}}(\tilde{f})(\sigma)$.

From §1.4 we have that, given an S.D.S. $(X, z) = (X_1 \oplus Y, W \oplus t)$, the infinitesimal generator of (X, z) is

$$\tfrac{1}{2}\mathcal{L}_{X_1}^2 + Y \text{ where } \mathcal{L}_{X_1}^2 f(x) = \sum_i \frac{d^2}{dt^2} f \circ S(x, t) e_i \Big|_{t=0} \quad (3.4.1)$$

where $S(x,t)e_i$ is a solution of $dS(x,t)e_i = X_1(S(x,t)e_i)e_i$, f is a function on M, $x \in M$, and $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^n$.

In our case X_1 is a map $TM \times \mathbb{R}^n \to TTM$ and comes by projection from the map,

$$\begin{tikzcd} HO(M) \times \mathbb{R}^n \arrow[r] \arrow[rrr, bend left, "\hat{X}"] & O(M) \times \mathbb{R}^n \times \mathbb{R}^n \arrow[r] & TO(M) \times \mathbb{R}^n \times \mathbb{R}^n \arrow[r] & THO(M) \end{tikzcd}$$

$$(\nu_u \,, w) \longmapsto (u \,, v \,, w) \longmapsto (\theta(u,0), v, \beta w) \longrightarrow$$

where $\theta(u,v) = \nu_u$.

We will call the middle map $\tilde{X}_1$. Then $\tilde{X}_1(w)$ is a vector field over $O(M) \times \mathbb{R}^n$ for each $w \in \mathbb{R}^n$. $\tilde{X}_1(w)$ has solution curves, starting at (u_o, v_o) at time $t = 0$ which are given by,

$$(u(t) \,, v(t)) = (u_o \,, v_o + \beta tw).$$

Thus $\hat{X}(w)$ has solution curves on $HO(M)$ given by,

$$\nu_u(t) = \theta(u_o, v_o + \beta tw) \text{ with initial point } \nu_u(0) = \theta(u_o, v_o).$$

These curves project to TM by $T\tau$ to give the curves,

$$u_o(v_o + \beta tw) = u_o(v_o) + \beta t u_o(w). \qquad (3.4.2)$$

where addition is in $T_{\tau(u_o)}M$.

These curves are vertical straight lines in TM and are the solutions of $X_1(w)$ with initial point $v = u_o(v_o)$.
Note that the solutions depend on u_o to the extent of an orthogonal transformation of $T_{\tau(u_o)}M$ but it is easy to see from what follows that the infinitesimal generator is independent of u_o. We therefore take u_o to be fixed.

3.4.1 gives that $S(v,t)e_i = v + \beta t u_o(e_i)$ for $v \in T_xM$ and $u_o \in \tau^{-1}(x)$. Let us choose the o.n. basis $\tilde{e}_i = u_o(e_i)$ of T_xM and coordinates y_i relative to this basis (let $v = (v_1, \ldots, v_n)$ in this coordinate system). From 3.1.1 and 3.1.2 we have that,

$$S(v,t)e_i = v + \beta t u_o(e_i) = v + \beta t\tilde{e}_i$$

and

$$\mathcal{L}^2_{X_1}(f)(v) = \Sigma \frac{d^2}{dt^2} f(v + \beta t\tilde{e}_i)\Big|_{t=0}$$

$$= \sum_i \frac{d}{dt}\Big[\sum_j \frac{d(v_j+\beta t\delta_{ij})}{dt} \cdot \frac{df(v+ t\tilde{e}_i)}{dy_j}\Big]\Big|_{t=0}$$

$$= \beta \Sigma \frac{d}{dt} \frac{df(v+\beta t\tilde{e}_i)}{dy_i}\Big|_{t=0}$$

$$= \beta^2 \Sigma \frac{d^2 f}{dy_i^2}(v + \beta t\tilde{e}_i)\Big|_{t=0}$$

$$= \beta^2 \Sigma \frac{d^2 f(v)}{dy_i^2} .$$

Thus $\mathcal{L}^2_{X_1} = \beta^2 \Delta_v$ where Δ_v is the vertical Laplacian over TM.

We are left with the non-stochastic bit of the S.D.S. over TM. This is a projection down of the vector field over HO(M) which may be written as

$$HO(M) \longrightarrow O(M) \times \mathbb{R}^n \longrightarrow TO(M) \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow THO(M)$$

$$\nu_u \longmapsto (u, v) \longmapsto (\theta(u,v), v, -\beta v + u^{-1}(F(\tau(u))) \longmapsto$$

where F is a vector field over M (see 'Force Field', p.51).

We will again look at the middle term, which we split into three parts. Firstly we have

$$Y_1 : (u,v) \longrightarrow (\theta(u,v), v, 0).$$

This vector field has solution curves on $O(M) \times \mathbb{R}^n$ which are given by

$$(u(t), v(t)) = (u(t), v_o)$$

where $du(t) = \theta(u(t), v_o)dt$, and the solutions of this equation are the horizontal lifts of the geodesics of M (see Proposition 3.2.1). Thus projecting Y_1 to a vector field over TM gives the Riemannian spray $\mathcal{S}$ over M.

Secondly we have

$$Y_2 : (u,v) \longrightarrow (\theta(u,0), v, -\beta v).$$

The projection of this vector field to TM has already been dealt with in §1 and is simply $\alpha \circ p_\beta$ (see p.47).

We are left with

$$(u,v) \longrightarrow (\theta(u,0), v, u^{-1}(F(\tau(u))).$$ This projects to the vector field, F_v , over TM which is the vertical lift

of F to TM (remember that F is a vector field over M).

Thus we have shown that the infinitesimal generator of the O-U process on TM is given by

$$\mathcal{A}(f) = \tfrac{1}{2}\mathcal{L}^2_{X_1}(f) + Y(f)$$

$$= \frac{\beta^2}{2}\Delta_v(f) + \mathcal{S}(f) + \alpha\, op_\beta(f) + F_v(f). \qquad (3.4.3)$$

Example

When $M = \mathbb{R}^n$ the vector field Y is given by

$(x,v) \longrightarrow (x,v,v,-\beta v+g(x))$ for some map g from $\mathbb{R}^n$ to itself. Let $(x,v) = (x_1,\ldots,x_n,v_1,\ldots,v_n)$ relative to an o.n. basis of $\mathbb{R}^n$ and let $f : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}$ be C^2.

Then $\Delta_v = \Sigma \dfrac{\partial^2}{\partial v_i^2}$

$$\mathcal{S}(x,v) = (x,v,v,0) \ : \ \mathcal{S} = \Sigma v_i \frac{\partial}{\partial x_i}$$

$$\alpha\, op_\beta(x,v) = (x,v,0,-\beta v) \ : \alpha\, op_\beta = -\beta \Sigma v_i \frac{\partial}{\partial v_i}$$

$$F^v(x,v) = (x,v,0,g(x)) \ : \ F^v = \Sigma (g(x))_i \frac{\partial}{\partial v_i} .$$

Putting these together we get that

$$\mathcal{A}(f)(x,v) = \frac{\beta^2}{2}\Sigma\frac{\partial^2}{\partial v_i^2} f(x,v) + \Sigma v_i \frac{\partial}{\partial x_i} f(x,v)$$
$$- \beta\, \Sigma v_i \frac{\partial}{\partial v_i} f(x,v) + \Sigma (g(x))_i \frac{\partial}{\partial v_i} f(x,v)$$

(cf. equation 29 on page 77 of Nelson [17] and also equation 31 of Jørgansen [9]).

CHAPTER FOUR. APPROXIMATION THEOREMS FOR ORNSTEIN-UHLENBECK PROCESSES

§1. $\mathcal{L}^2$ Approximation

In this chapter we will be considering O-U processes on manifolds, of the type $dx = Xv_\beta dt + Vdt$ as discussed in §3.3. v_β will be an O-U velocity process without force-field on a Hilbert space E.

Let E,H be real separable Hilbert spaces with (z_β, v_β) an O-U process (without force-field) on E, generated by the Brownian motion W on E, so z_β and v_β are of the type discussed in Chapter Two.

Let $X : H \xrightarrow{\mathcal{C}} L(E;H)$ be a C^2 map such that X and DX are globally Lipschitz in H and X,DX and D^2X are bounded as linear maps :-

(i) $\|X(m,v)-X(n,v)\|_H \leq F\|m-n\|_H\|v\|_E$;

(ii) $\|X(m)v\|_H \leq K\|v\|_E$;

(iii) $\|DX(m)p-DX(n)p\|_{L(E;H)} \leq J\|m-n\|_H\|p\|_H$;

(iv) $\|DX(m)p\|_{L(E;H)} \leq N\|p\|_H$;

(v) $\|D^2X(m)(n)(p)\|_{L(E;H)} \leq T\|n\|_H\|p\|_H$,

$\forall$ m,n,p $\in$ H, v $\in$ E.

Note that (i) $\Leftrightarrow$ (iv) and (iii) $\Leftrightarrow$ (v) (we could in fact take $J = T$ and $F = N$).

Note also that the above inequalities imply

(vi) $\|DX(m)\circ X(m)(v)\|_{L(E;H)} \leq NK\|v\|_E$;

(vii) $\|DX(m)\circ DX(m)\circ X(m)(v,w)\|_{L(E;H)} \leq N^2K\|v\|_E\|w\|_E$;

(viii) $\|D^2X(m)\circ X(m)(v)\circ X(m)(w)\|_{L(E;H)} \leq TK^2\|v\|_E\|w\|_E$;

(ix) $\|DX(m)\circ X(m)v - DX(n)\circ X(n)v\|_{L(E;H)} \leq (JK+NF)\|m-n\|_E\|v\|_H$

for all $w \in E$.

Let $V : H \longrightarrow H$ be C^1, Lipschitz and globally bounded :-

$\|V(m)\|_H \leq G$ and $\|V(m)-V(n)\|_H \leq S\|m-n\|_H$ $\forall$ $m,n \in H$.

(4.1.1) Theorem

Let V and X be as above and $0<R<\infty$. $z_\beta : [0,R]\times\Omega \longrightarrow E$ and $v_\beta : [0,R]\times\Omega \longrightarrow E$ with $z_\beta(0)=z_o \in E$, $v_\beta(0)=v_o \in E$ are to be O-U position and velocity processes as in Chapter Two.

Let x be a maximal solution of $dx=X(x)dW + V(x)dt$ with initial condition $x(0)=x_o \in H$,

$$x : [0,R] \times\Omega \longrightarrow H.$$

Let x_β be maximal solutions of the family of o.d.e.'s

(4.1.2) $\quad x_\beta = X(x_\beta)v_\beta dt + V(x_\beta)dt$, $x_\beta(0)=x_o$.

$x_\beta(\omega) : [0,R] \longrightarrow H$ is defined for a.a.$\omega \in \Omega$ and $x_\beta(\omega)$ is C^1 where it is defined.

Then,

$x_\beta \longrightarrow x$ in $\mathcal{L}^2$-norm, uniformly for $t \in [0,R]$, i.e. given $\varepsilon>0$ $\exists$ $N>0$ such that $\beta>N \Rightarrow$

$$\| x_\beta(t)-x(t) \| < \varepsilon \;\; \forall \; t \in [0,R].$$

Proof

For x we have the stochastic integral equation,

$$(4.1.3) \qquad x(t) = x_o + \int_0^t V(x(s))ds + \int_0^t X(x(s))dW(s) + \\ + 1/2 \int_0^t \mathrm{tr}_{L}(DX(x(s))\circ X(x(s)))ds,$$

using Proposition 1.3.6.

From 4.1.2 we have the integral equation,

$$(4.1.4) \qquad x_\beta(t) = x_o + \int_0^t V(x_\beta(s))ds + \int_0^t X(x_\beta(s))v_\beta(s)ds.$$

Remembering that $v_\beta(t) = v_o + \beta\int_0^t I dW(s) - \beta\int_0^t v_\beta(s)ds$,

and noting that $g(t) = X(x_\beta(t))$ is C^1,

$$g'(t) = DX(x_\beta(t))\circ(X(x_\beta(t))v_\beta(t) + V(x_\beta(t))),$$

we can use the integration by parts formula 1.2.6 to give

$$(4.1.5) \; x_\beta(t) = x_o + \int_0^t V(x_\beta(s))ds + \int_0^t X(x_\beta(s))dW(s) + \\ + \frac{1}{\beta} \int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(v_\beta(s),v_\beta(s))ds \\ + \frac{1}{\beta} \int_0^t DX(x_\beta(s))\circ V(x_\beta(s)) \; (v_\beta(s))ds \\ -\frac{1}{\beta}X(x_\beta(t))v_\beta(t) + \frac{1}{\beta} X(x_o)v_o.$$

But

$$\frac{1}{\beta}\|X(x_\beta(t))v_\beta(t)\| \leq \frac{K}{\beta}\|v_\beta(t)\| \leq \frac{K}{\sqrt{\beta}}P_1$$

by remark 2.2.12,

and $\quad \frac{1}{\beta}\|X(x_o)v_o\| \leq \frac{K}{\beta}\|v_o\|_E.$

Hence both of these terms have $\mathcal{L}^2$-norms which tend to zero, uniformly over $[0,R]$ as β tends to infinity. We call terms satisfying this condition $\mathcal{E}(\beta)$ (we will use $\mathcal{E}(\beta)$ both for the condition and for terms which satisfy it).

Also,

$$\frac{1}{\beta}\left\|\int_0^t DX(x_\beta(s))\circ V(x_\beta(s))(v_\beta(s))ds\right\| \leq \frac{1}{\beta}\int_0^t NG\|v_\beta(s)\| ds$$

$$\leq \frac{NGP_1R}{\sqrt{\beta}}.$$

Hence this term is $\mathcal{E}(\beta)$. From now on we will assume that V is identically zero, in order to simplify the expressions which we will derive. The conditions which we have imposed upon V are sufficient to ensure that any terms containing V can be dealt with easily.

From the above we have that

$$(4.1.6)\quad x_\beta(t) = x_o + \mathcal{E}(\beta) + \int_0^t X(x_\beta(s))dW(s) +$$

$$+ \frac{1}{\beta}\int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(v_\beta(s),v_\beta(s))ds.$$

Writing $v_\beta(s) = v_o e^{-\beta s} + \beta e^{-\beta s}\int_0^s e^{\beta r}dW(r)$ we look at the last term of 4.1.6.

This is,

$$(4.1.7)\quad \frac{1}{\beta}\int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(v_o e^{-\beta s}, v_o e^{-\beta s})ds\ +$$

$$+\int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(v_o e^{-\beta s}, e^{-\beta s}\int_0^s e^{\beta r} IdW(r))ds$$

$$+\int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(e^{-\beta s}\int_0^s e^{\beta r} IdW(r), v_o e^{-\beta s})ds$$

$$+\beta\int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(e^{-\beta s}\int_0^s e^{\beta r} IdW(r), e^{-\beta s}\int_0^s e^{\beta r} IdW(r))ds.$$

The first three terms of 4.1.7 can easily be shown to be $\mathcal{E}(\beta)$. We integrate the last term by parts using 1.2.6 to give,

$$(4.1.9)\quad -\frac{e^{-2\beta t}}{2}DX(x_\beta(t))\circ X(x_\beta(t))(\int_0^t e^{\beta r} IdW(r), \int_0^t e^{\beta u} IdW(u))$$

$$+\int_0^t \frac{e^{-2\beta s}}{2}DX(x_\beta(s))\circ X(x_\beta(s))(e^{\beta s} dW(s), \int_0^s e^{\beta r} IdW(r))$$

$$+\int_0^t \frac{e^{-2\beta s}}{2}DX(x_\beta(s))\circ X(x_\beta(s))(\int_0^s e^{\beta r} IdW(r), e^{\beta s} dW(s))$$

$$+\int_0^t \frac{e^{-2\beta s}}{2}DX(x_\beta(s))\circ X(x_\beta(s))(e^{\beta s} dW(s), e^{\beta s} dW(s))$$

$$+\int_0^t \frac{e^{-2\beta s}}{2}DX(x_\beta(s))\circ DX(x_\beta(s))\ X(x_\beta(s))(v_\beta(s)).\ (\int_0^s e^{\beta r} IdW(r), \int_0^s e^{\beta r} IdW(r))ds$$

$$+\int_0^t \frac{e^{-2\beta s}}{\beta}D^2X(x_\beta(s))\circ X(x_\beta(s))(v_\beta(s))\ X(x_\beta(s)).\ (\int_0^s e^{\beta r} IdW(r), \int_0^s e^{\beta r} IdW(r))ds.$$

$$= ⓐ + ⓑ + ⓒ + ⓓ + ⓔ + ⓕ$$

$$\|ⓐ\| \le \frac{e^{-2\beta t}}{2}NK\left\{E(\|\int_0^t e^{\beta s} IdW(s)\|_E^4)\right\}^{\frac{1}{2}}$$

$$\le \frac{e^{-2\beta t}}{2}NK36\sqrt{R}\left\{\int_0^t E(\|e^{\beta s} I\|_\gamma^4)ds\right\}^{\frac{1}{2}} \quad \text{by 1.3.4}$$

$$\le 3\ e^{-2\beta t}NK(m_4)^{\frac{1}{2}}\sqrt{R}(\int_0^t e^{4\beta s}ds)^{\frac{1}{2}}$$

$$\le \frac{3NK\sqrt{R}}{2\sqrt{\beta}}(m_4)^{1/2}$$

where $m_4 = \int_E \|y\|_E^4 \, d\nu(y) < \infty$ by Fernique's theorem.

Thus ⓐ is $\mathcal{E}(\beta)$.

$$\|ⓑ\|^2 \leq A_E \int_0^t \frac{e^{-2\beta s}}{4} E(\| DX(x_\beta(s))\; X(x_\beta(s))(\int_0^s e^{\beta r} I dW(r))\|_\nu^2) ds$$

by (vii) of 1.3.1

$$\leq A_E \int_0^t \frac{e^{-2\beta s}}{4} E(\textstyle\sum \lambda_i (NK)^2 \| \int_0^s e^{\beta r} I dW(r)\|_E^2) ds$$

$$\leq A_E \int_0^t \frac{e^{-2\beta s}}{4} tr(L)(NK)^2 E(\int_0^s \|e^{\beta r} I\|_\nu^2 dr) A_E \, ds$$

$$\leq (A_E)^2 \int_0^t \frac{e^{-2\beta s}}{4} (trL)^2 (NK)^2 \int_0^s e^{2\beta r} dr ds$$

$$= (A_E)^2 \frac{(NK)^2}{4} (trL)^2 \int_0^t \frac{e^{-2\beta s}}{2\beta} (e^{2\beta s} - 1) ds$$

$$\leq (A_E)^2 \frac{(NK)^2}{8\beta} (trL)^2 R \,.$$

Thus ⓑ is $\mathcal{E}(\beta)$.

Similarly, ⓒ is $\mathcal{E}(\beta)$.

$$\|ⓔ\| \leq \int_0^t \frac{N^2 K}{2\beta^2} (E \,\| \beta e^{-\beta s} \int_0^s e^{\beta r} I dW(r)\|_E^6)^{\frac{1}{2}} ds \,.$$

$$\leq \frac{N^2}{2\beta^2} \int_0^t (P_3) \beta^{3/2} ds \quad \text{by 2.2.12}$$

$$\leq \frac{N^2 K P_3 R'}{2\sqrt{\beta}} \,.$$

Thus ⓔ is $\mathcal{E}(\beta)$; ⓕ is dealt with similarly.

Thus we have shown that (considering V to be non-zero once more),

(4.1. 9) $x_\beta(t) = x_o + \mathcal{E}(\beta) + \int_0^t V(x_\beta(s)) ds + \int_0^t X(x_\beta(s)) dW(s) +$

$$+ \tfrac{1}{2} \int_0^t DX(x_\beta(s)) \circ X(x_\beta(s))(dW(s), dW(s)).$$

Combining this with equation 4.1.3 we have,

$$(4.1.10) \quad x_\beta(t) - x(t) = \mathcal{E}(\beta) + \int_0^t [V(x_\beta(s)) - V(x(s))]ds + \\ + \int_0^t [X(x_\beta(s)) - X(x(s))]dW(s) + \\ + \tfrac{1}{2}\int_0^t [tr_L(DX(x_\beta(s))\circ X(x_\beta(s)) - DX(x(s))\circ \\ X(x(s))]ds$$

using Proposition 1.3.6.

Hence, taking norms

$$\|x_\beta(t)-x(t)\|^2 \le 4\|\mathcal{E}(\beta)\|^2 + 4R\int_0^t S\|x_\beta(s)-x(s)\|^2 ds + \\ + 4A_E(trL)\int_0^t F^2\|x_\beta(s)-x(s)\|^2 ds \\ + 2m_4 R\int_0^t (JK+NF)^2\|x_\beta(s)-x(s)\|^2 ds.$$

We have

$$\|x_\beta(t)-x(t)\|^2 \le 4\|\mathcal{E}(\beta)\|^2 + C\int_0^t \|x_\beta(s)-x(s)\|^2 ds$$

where $C>0$ depends only on X, V, W and R.

Then Gronwall's lemma (2.2.13) gives that

$$\|x_\beta(t)-x(t)\|^2 \le 4\|\mathcal{E}(\beta)\|^2 + C\int_0^t e^{C(t-s)}4\|\mathcal{E}(\beta)\|^2 ds \\ \le 4\|\mathcal{E}(\beta)\|^2 e^{CR}$$

which tends to zero as β tends to infinity, the convergence being uniform over $t \in [0,R]$.

#

Note that while we have used the terminology of Neidhardt in the proof, since we have assumed that E is a Hilbert space we have that $A_E = 1$ and se veral of the inequalities we derived which included this term are actually equalities.

§2. Convergence in Probability on Hilbert Spaces

When we come to talk of the convergence of O-U processes on manifolds to Wiener processes we will need a different concept from that of convergence in $\mathcal{L}^2$. Note that Theorem 4.1.1 implies that, for each $t \in [0,R]$ and $\varepsilon>0$, $\mu\{\omega : \|x_\beta(t,\omega)-x(t,\omega)\|_H > \varepsilon\}$ tends to zero as β tends to infinity.

We need

(4.2.1) Lemma

Let $z : [0,R] \longrightarrow E$ be C^0 s.t. $z(0) = 0$.

Then $\sup\limits_{0\le t\le R} \| z(t) - \beta e^{-\beta t}\int_0^t e^{\beta s} z(s)ds\|_E$ tends to zero as β tends to infinity, for any Banach space E.

Proof

$$z(t) - \beta e^{-\beta t}\int_0^t e^{\beta s} z(s)ds = z(t)e^{-\beta t} + \int_0^t \beta e^{-\beta(t-s)}(z(t)-z(s))ds.$$

Let $\varepsilon>0$ be given. Since z is uniformly continuous on $[0,R]$ $\exists\, \delta>0$ s.t. $s,t \in [0,R]$ and $|s-t| < \delta \implies$
$\|z(s)-z(t)\|_E < \varepsilon/3$.

Now,

$$\sup_{0\le t\le R} \|e^{-\beta t} z(t)\|_E = \max(\sup_{0\le t\le\delta} \|z(t)\|_E e^{-\beta t}, \sup_{\delta\le t\le R} \|z(t)\|_E e^{-\beta t})$$
$$\le \max(\frac{\varepsilon}{3}, \sup_{\delta\le t\le R} \|z(t)\|_E\, e^{-\beta\delta}).$$

Since $z(t)$ is bounded for $t \in [0,R]$ the latter term tends to zero as β tends to infinity.

In particular there exists $N_1>0$ s.t. $\beta> N_1 \Rightarrow$

$$\sup_{0\le t\le R} \|e^{-\beta t}z(t)\|_E < \varepsilon/3.$$

$$\sup_{0\le t\le R} \left(\left\|\int_0^t \beta e^{-\beta(t-s)}(z(t)-z(s))ds\right\|_E\right)$$

$$\le \max\Big(\sup_{0\le t\le \delta} \beta e^{-\beta t}\left\|\int_0^t e^{\beta s}(z(t)-z(s))ds\right\|_E ,$$

$$\sup_{\delta\le t\le R} \beta e^{-\beta t}\left\|\int_0^{t-\delta} e^{\beta s}(z(t)-z(s))ds\right\|_E +$$

$$\sup_{\delta\le t\le R} \beta e^{-\beta t}\left\|\int_{t-\delta}^t e^{\beta s}(z(t)-z(s))ds\right\|_E\Big)$$

$$\le \max\Big(\frac{\varepsilon}{3} e^{-\beta t}(e^{\beta t}-1),\ 2\sup_{0\le t\le R}\|z(t)\|_E \cdot \sup_{\delta\le t\le R}\left(\beta e^{-\beta t}\int_0^{t-\delta} e^{\beta s}ds\right)$$

$$+ \sup_{\delta\le t\le R}\left(\frac{\varepsilon}{3}\beta e^{-\beta t}\int_{t-\delta}^t e^{\beta s}ds\right)\Big)$$

$$\le \max\Big(\frac{\varepsilon}{3},\ 2\sup_{0\le t\le R}(\|z(t)\|_E)e^{-\beta\delta} + \frac{\varepsilon}{3}\Big)$$

$$< \frac{2\varepsilon}{3} \quad \text{for } \beta>N_2,\ \text{some } N_2>0.$$

Thus, given $\varepsilon>0$, there exists $N(\varepsilon) > 0$ s.t. $\beta>N(\varepsilon) \Rightarrow$

$$\sup_{0\le t\le R}\left\|z(t) - \beta e^{-\beta t}\int_0^t e^{\beta s}z(s)ds\right\|_E < \varepsilon.$$

#

(4.2.2) Corollary

If v_β is an O-U velocity process on a 2-uniformly smooth Banach space as in Chapter Two, then

$$\sup_{0\le t\le R}\left\|\frac{v_\beta(t)}{\beta}\right\|_E \longrightarrow 0 \text{ almost surely as } \beta \text{ tends to}$$

infinity. In particular $\sup_{0\le t\le R}\left\|\frac{v_\beta(t)}{\beta}\right\|_E$ tends to zero in probability as β tends to infinity.

Proof

$$v_\beta(t) = v_0 e^{-\beta t} + \beta e^{-\beta t}\int_0^t e^{\beta s} I dW(s)$$

$$= v_0 e^{-\beta t} + \beta W(t) - \beta^2 e^{-\beta t}\int_0^t e^{\beta s} W(s)ds.$$

Hence,

$$\frac{v}{\beta}_\beta(t) = \frac{v}{\beta}_0 e^{-\beta t} + W(t) - \beta e^{-\beta t}\int_0^t e^{\beta s} W(s)ds.$$

Since W has almost surely continuous sample paths the result follows. #

We can now prove,

(4.2.3) Theorem

Under the conditions of Theorem 4.2.1, x_β converges to x in probability, uniformly on [0,R] in the sense that for each $\varepsilon>0$,

$$\mu\{\omega; \sup_{0\le t\le R} \|x_\beta(t) - x(t)\|_H > \varepsilon\} \longrightarrow 0 \text{ as } \beta \longrightarrow \infty.$$

Proof

We will split $x_\beta(t) - x(t)$ into several terms using the expressions 4.1.3, 4.1.5, 4.1.7 and 4.1.9. These give,

(4.2.4)

$$x_\beta(t) = A(t) + B(t) + C(t) + D(t) + E(t) + F(t) + G(t) + H(t) + I(t) + J(t) + K(t) + L(t),$$

where we assume that V is identically zero, merely to keep down the number of terms in 4.2.4, and

$$A(t) = 1/\beta\ X(x_o)v_o\ ;$$

$$B(t) = -1/\beta\ X(x_\beta(t))v_\beta(t)\ ;$$

$$C(t) = \int_0^t (X(x_\beta(s)) - X(x(s)))dW(s)\ ;$$

$$D(t) = 1/\beta \int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(v_o e^{-\beta s}, v_o e^{-\beta s})ds\ ;$$

$$E(t) = \int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(v_o e^{-\beta s}, 1/\beta\ v_\beta(s))ds\ ;$$

$$F(t) = \int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(1/\beta\ v_\beta(s), v_o e^{-\beta s})ds\ ;$$

$$G(t) = -\frac{e}{2}^{-2\beta t} DX(x_\beta(t))\circ X(x_\beta(t))\left(\int_0^t e^{\beta r} I dW(r), \int_0^t e^{\beta r} I dW(r)\right);$$

$$H(t) = \int_0^t (1/2\ e^{\beta s}) DX(x_\beta(s))\circ X(x_\beta(s))\left(\int_0^s e^{\beta r} I dW(r), dW(s)\right)\ ;$$

$$I(t) = \int_0^t \frac{e}{2}^{-\beta s} DX(x_\beta(s))\circ X(x_\beta(s)\left(dW(s), \int_0^s e^{\beta r} I dW(r)\right)\ ;$$

$$J(t) = \int_0^t \frac{e}{2}^{-2\beta s} DX(x_\beta(s))\circ DX(x_\beta(s))\circ X(x_\beta(s))v_\beta(s). \left(\int_0^s e^{\beta r} I dW(r), \int_0^s e^{\beta r} I dW(r)\right)ds\ ;$$

$$K(t) = \int_0^t \frac{e}{2}^{-2\beta s} D^2X(x_\beta(s))\circ X(x_\beta(s))v_\beta(s)\circ X(x_\beta(s)). \left(\int_0^s e^{\beta r} I dW(r), \int_0^s e^{\beta r} I dW(r)\right)ds\ ;$$

$$L(t) = \int_0^t \mathrm{tr}_L\ (DX(x_\beta(s))\circ X(x_\beta(s)) - DX(x(S))\circ X(x(s)))ds\ .$$

(Several terms involving v_o are left out but cause no problems.) We will show that each of these terms has norm whose supremum tends in probability to zero as β tends to infinity. Instead of "$\sup\limits_{0\le t\le R}$" we will simply write "sup".

(i) $\sup\|A(t)\|_H = 1/\beta\ \|X(x_o)v_o\|_H \le K/\beta\ \|v_o\|_E$

$$\longrightarrow 0 \text{ as } \beta \longrightarrow \infty.$$

(ii) $\sup \|B(t)\|_H = 1/\beta \|X(x_\beta(t))v_\beta(t)\|_H \leq K/\beta \|v_\beta(t)\|_E$

$\longrightarrow 0$ in probability as $\beta \longrightarrow \infty$ by Corollary 4.2.2.

(iii) $\sup \|C(t)\|_H = \sup \| \int_0^t (X(x_\beta(s)) - X(x(s))dW(s)\|_H$.

Hence

$$\mu\{\omega \; ; \; \sup\|C(t)\|_H > \varepsilon\} \leq \frac{1}{\varepsilon^2} 4A_E \int_0^R E\|X(x_\beta(s)) - X(x(s))\|_\nu^2 ds$$

by 1.3.1 (x)

$$= \frac{4A_E}{\varepsilon^2} \int_0^R E \int_E \|X(x_\beta(s) - X(x(s)))(y)\|_H^2 d\nu(y) ds$$

$$\leq \frac{4A_E}{\varepsilon^2} \int_0^R F^2(\mathrm{tr} L) \; \|x_\beta(s) - x(s)\|^2 \; ds$$

$\longrightarrow 0$ as $\beta \longrightarrow \infty$ by Theorem 4.1.1.

(iv) $\sup \|D(t)\|_H \leq \sup 1/\beta \int_0^t NK \; e^{-2\beta s} \|v_\beta(s)\|_E^2 ds$

$\longrightarrow 0$.

(v) $\sup \|E(t)\|_H \leq \sup \int_0^t NK \; e^{-\beta s} \|v_0\|_E \|\frac{v}{\beta}_\beta(s)\|_E \; ds$

$$\leq \frac{NK}{\beta} \|v_0\|_E \; \sup \|\frac{v}{\beta}_\beta\|(s) \;_E$$

$\longrightarrow 0$ in probability by Corollary 4.2.2.

(vi) F is done similarly to E.

(vii) $G(t) = 1/2\beta^2 \; DX(x_\beta(t)) \circ X(x_\beta(t))(v_\beta(t), v_\beta(t))$ (up to v_0 terms).

Thus,

$\sup \|G(t)\|_H \leq 1/2\beta^2 \; NK \; \|v_\beta(t)\|_E^2$.

Since, by corollary 4.2.2, $\sup\|\frac{v_\beta}{\beta}(t)\|_E$ tends to zero in probability this is also true of $\sup\|\frac{v_\beta}{\beta^2}(t)\|_E^2$.

(viii) $$H(t) = \int_0^t 1/2\, e^{-\beta s} DX(x_\beta(s))\circ X(x_\beta(s))\left(\int_0^s e^{\beta r} I dW(r), dW(s)\right)$$

Hence, by 1.3.1 (x),

$$\mu\{\omega\,;\ \sup\|H(t,\omega)\|_H > \varepsilon\} \le \frac{4A_E}{\varepsilon^2}\int_0^R \frac{e^{-2\beta s}}{4} E\|[DX(x_\beta(s))\circ X(x_\beta(s)) \left(\int_0^s e^{\beta r} I dW(r)\right)]\|_\gamma^2\, ds$$

$$= \frac{4A_E}{\varepsilon^2}\int_0^R \frac{e^{-2\beta s}}{4} E\int_E \|DX(x_\beta(s))\circ X(x_\beta(s))\left(\int_0^s e^{\beta r} I dW(r)\right)(y)\|_H^2\, d\gamma(y).ds.$$

$$\le \frac{4A_E}{\varepsilon^2} E\int_0^R \frac{e^{-2\beta s}}{4}\ \mathrm{tr}L\ (NK)^2\ \|\int_0^s e^{\beta r} I dW(r)\|^2\, ds$$

$$\le \frac{4A_E^2}{\varepsilon^2}\int_0^R \frac{e^{-2\beta s}}{4}\ \mathrm{tr}L\ (NK)^2 \int_0^s e^{2\beta r}(\mathrm{tr}L)\, dr\, ds$$

$$= \frac{(\mathrm{tr}L)^2}{4\beta\varepsilon^2}(NK)^2(A_E)^2\int_0^R e^{-2\beta s}(e^{2\beta s}-1)\, ds$$

$$< \frac{R(\mathrm{tr}L.N.K.A_E)^2}{4\beta\varepsilon^2}$$

$$\longrightarrow 0.$$

(ix) I is similar to H.

(x) $$J(t) = \frac{1}{2\beta^2}\int_0^t DX(x_\beta(s))\circ DX(x_\beta(s))\circ X(x_\beta(s))(v_\beta(s), v_\beta(s), v_\beta(s))\, ds$$

up to v_0 terms.

Thus,

$$\mu\{\omega\,;\ \sup\|J(t)\|_H > \varepsilon\} \le$$

$$\frac{R}{4\beta^4\varepsilon^2}\int_0^R \|DX(x_\beta(s))\circ DX(x_\beta(s))\circ X(x_\beta(s))(v_\beta(s), v_\beta(s), v_\beta(s))\|^2 d$$

$$\leq \frac{R}{4\beta^4\varepsilon^2}\int_0^R N^4K^2E\|v_\beta(s)\|_E^6\,ds$$

$$\leq \frac{N^4K^2R}{4\beta^4\varepsilon^2}\int_0^R P_3^{\,2}\,\beta^3 ds \quad \text{by remark 2.2.12}$$

$$\longrightarrow 0.$$

(xi) K is similar to J.

(xii) $\sup \|L(t)\|_H \leq \int_0^R (JK+NL)\|x_\beta(s)-x(s)\|_H \,(m_4)^{\frac{1}{2}}\, ds$

Hence,

$$\mu\{\omega \,;\, \sup\|L(t)\|_H > \varepsilon\} \leq \frac{1}{\varepsilon^2}(JK+NL)^2 m_4 \int_0^R R\|x_\beta(s) - x(s)\|^2\, ds$$

$$\longrightarrow 0.$$

#

(4.2.5) <u>Example</u>

Arnold, Horsthemke and Lefever [1] consider the influence of fluctuations on the concentra tions of catalysts in, e.g., chemical reactions. The 'white noise' stochastic differential equation which they obtain in 2.8 is,

(4.2.6)

$$dx(t) = (\tfrac{1}{2}-x(t))dt + x(t)(1-x(t))dW(t), \quad \text{(W is a Brownian motion on } \mathbb{R})$$

which they consider as an Ito - equation.

The corresponding 'coloured noise ' differential equation is described in 3.5 and 3.6 of [1]. Let v_β be an O-U velocity process on $\mathbb{R}$,

$$dv_\beta(t) = -\beta v_\beta(t) + \beta dW(t),$$

and consider,

(4.2.7) $dx_\beta(t) = [(\frac{1}{2}-x_\beta(t)) - v_\beta(t).x_\beta(t)(1-x_\beta(t))]dt$,

i.e. $dx_\beta(t) = x_\beta(t)(1-x_\beta(t))v_\beta(t)dt + (\frac{1}{2}-x_\beta(t))dt$

$$= X(x_\beta(t))v_\beta(t) + V(x_\beta(t))dt.$$

Let x_β be a solution of 4.2.7 with $x_\beta(0)= x_o \in \mathbb{R}$, and let x be a solution of,

(4.2.8) $dx(t) = X(x(t))dW(t) + V(x(t))dt$. $x(0) = x_o$.

Note that since the coefficients are not globally Lipschitz or bounded we cannot apply 4.1.1 or 4.2.3 but we use Theorem 4.3.4 (after which this example logically belongs) to show that x_β converges to x in the sense of that theorem.

Note that if c is the variance of the Brownian motion W we can rewrite 4.2.8 as the Itô s.d.e.,

(4.2.9) $dx(t) = (\frac{1}{2}-x(t))dt + x(t)(1-x(t))dW(t)$

$$+ \frac{c}{2}(1-2x(t))x(t)(1-x(t))dt,$$

which would have corresponded to 4.2.6 had Arnold et al. considered that equation in the sense of Stratonovic.

§3. Convergence on Manifolds

This section is taken from Elworthy [4] where the result is proved for piece-wise linear approximations.

We will embed a suitable manifold, M, in a Hilbert space and use the result of §2. Firstly we need the following results whose proofs may be found in Elworthy [4].

(4.3.1) Lemma

For a complete separable metric space M and a finite measure space $(\Omega, \mathcal{F}, \mu)$ suppose $y \in \mathcal{L}^0(\Omega, \mathcal{F}; C([a,b];M))$. Then for each $\varepsilon>0$ there is a compact subset K_ε of M and $\Omega_\varepsilon \in \mathcal{F}$ such that

(i) $\mu(\Omega_\varepsilon) > \mu(\Omega) - \varepsilon$;

(ii) $\forall \omega \in \Omega_\varepsilon$ $y(\omega,t) \in K_\varepsilon$ $\forall$ $t \in [a,b]$.

(4.3.2) Lemma

Let $i : N \longrightarrow M$ be a closed C^3 embedding of a manifold N into a manifold M. Suppose the S.D.S. (X,z) on M has $X|i(N)$ tangent to $i(N)$ thus inducing an S.D.S. (Y,z) on N such that $Ti \circ Y(n,e) = X(i(n),e)$.

Then, any locally regular solution,

$$x : [a,\xi) \times \Omega \longrightarrow M \text{ of } dx = Xdz,\ x(a) \in i(N),$$

is equivalent to one of the form $i \circ y$ where $y : [a,\xi) \times \Omega \longrightarrow N$ is a locally regular solution of $dy = Ydz$ with $i \circ y(a) = x(a)$.

(4.3.3) Lemma

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and M a separable metrizable space which admits a complete metric. Then the topology of convergence in measure on $L^0(\Omega, \mathcal{F}; M)$ is independent of the choice of metric on M.

We can now prove,

(4.3.4) Theorem

For $\beta>0$ let (z_β, v_β) be an O-U process on a Hilbert space E, based on the Brownian motion W on E.

Let M be a separable, metrizable C^3 manifold, and suppose that we are given a C^2 section X of $\mathrm{Hom}_M(E,TM)$ and a C^1 vector field V on M.

Let $x : [0,\xi) \times \Omega \longrightarrow M$ be a maximal solution of
$dx(t) = V(x(t))dt + X(x(t))dW(t)$ with $x(0) = x_o \in M$.

Let $x_\beta : [0,\xi_\beta) \times \Omega \longrightarrow M$ be maximal solutions to the family of ordinary differential equations,

$$d(x_\beta(t)) = V(x_\beta(t))dt + X(x_\beta(t))v_\beta(t)dt,$$

with $x_\beta(0) = x_o$.

Define $\Omega_t = \{\omega \in \Omega;\ t<\xi(\omega)\}$.

Then,

$\{x_\beta\}$ converges to x in measure as β tends to infinity , in the sense that for $t \in [0,R]$, $\varepsilon>0$ and any metric d on M, ,

$$\mu\{\omega \in \Omega_t : \sup_{0\le s\le t} d(x_\beta(s,\omega), x(s,\omega))>\varepsilon\} \longrightarrow 0 \text{ as } \beta \longrightarrow \infty.$$

In particular $t\wedge\xi_\beta \longrightarrow t$ in measure on Ω_t as $\beta \longrightarrow \infty$.

Proof

Let $i : M \longrightarrow H$ be a closed C^3 embedding of M into a Hilbert space H. We extend X onto H as follows. Let $\{U_\alpha, p_\alpha\}$ be a partition of unity on H such that $\{U_\alpha \cap i(M)\}$ is subordinate to an atlas $\{V_\alpha, \phi_\alpha\}$ of M in the sense that $U_\alpha \cap i(M) \subset i(V_\alpha)$.

We define $X_\alpha : U_\alpha \times E \longrightarrow TH$ to be identically zero if $U_\alpha \cap i(M) = \emptyset$. Otherwise, $U_\alpha \cap i(M) \overset{j}{\cong} \phi_\alpha(V_\alpha)\times G$ where G is a Banach space and $j(u) = (j_1(u), j_2(u)) = (\phi_\alpha(m), j_2(u))$ if $u = i(m)$. We define, on $U_\alpha \cap i(M)$,

$$X_\alpha(u) = Ti \circ X(\phi_\alpha^{-1} \circ j_1(u)) = Ti \circ X(m).$$

We can extend X_α smoothly to U_α.

Then $X_H = \sum p_\alpha \cdot X_\alpha : H\times E \longrightarrow TH$ has the property that for $m \in M$, $X_H(i(m)) = Ti \circ X(m)$.

We construct the vector field V_H on H similarly.

Define $x_H : [0,\eta) \times \Omega \longrightarrow H$ to be a maximal solution of $dx_H = X_H(x_H)dW + V_H(x_H)dt$, $x_H(0) = i(x_o)$,

and $x_\beta^H : [0,\eta_\beta) \times \Omega \longrightarrow H$ to be a family of maximal solutions to the family of o.d.e.'s $dx_\beta^H = X_H(x_\beta^H)v_\beta dt + V_H(x_\beta^H)dt$, $x_\beta^H(0) = i(x_o)$.

Then, for $t \in [0,R)$ we define $y^t \in \mathcal{L}^0(\Omega_t, \mathcal{F}_t; C([0,t];H))$ by $y^t(\omega)(s) = x_H(s,\omega)$.

By Lemma4.3.1 , given $\varepsilon > 0$ there exists a compact $K^t \subset H$ and $\Omega_\varepsilon^t \subset \Omega_t$ such that

$$\text{(i) } \mu(\Omega_\varepsilon^t) > \mu(\Omega_t) - \varepsilon/2$$
$$\text{(ii) } \forall \omega \in \Omega_\varepsilon^t,\ x_H(s,\omega) \in K_\varepsilon^t (\forall s \in [0,t])$$

Let $\Lambda = ((W,id), W_o, \lambda)$ be a regular localization for X_H with $K_\varepsilon^t \subset W_o$ and W bounded. Then $X_{H\Lambda}$ and $DX_{H\Lambda}$ are bounded and globally Lipschitz (and hence $D^2X_{H\Lambda}$ is bounded).

Also, $\bar{V}_{H\Lambda}$ is bounded and globally Lipschitz.
Let $\tilde{x}$, $\tilde{x}_\beta$ correspond to x_H and x_β^H with X_H and V_H replaced by $X_{H\Lambda}$ and $V_{H\Lambda}$.

Let $\tau_\beta^\Lambda : \Omega \longrightarrow [0,R)$ be the first exit time of $\tilde{x}_\beta$ from W_o and suppose $0<\delta<d(K^t, H\setminus W_o)$ where d is the natural metric on H. From Theorem 4.2.3, letting $\Omega(\beta,\delta) = \{\omega : \sup_{0\le s\le t} d(\tilde{x}_\beta(s), \tilde{x}(s)) > \delta\}$,

there exists N>0 s.t. $\beta > N \Rightarrow \mu(\Omega(\beta,\delta)) < \varepsilon/2$.

Hence for $\beta > N$ and $\omega \in \Omega_\varepsilon^t \setminus \Omega(\beta,\delta)$,
we have that $\tau_\beta^\Lambda(\omega) > t$ which gives that,

$$\tilde{x}_\beta(s,\omega) = x_\beta^H(s,\omega) \quad 0\le s\le t \text{ a.a. } \omega\in\Omega_\varepsilon^t\setminus\Omega(\beta,\delta).$$

Thus if $\beta > N$,

$$\mu\{\omega\in\Omega_t : \sup_{0\le s\le t} d(x_\beta^H(s), x_H(s)) > \delta\} \le$$

$$\mu\{\Omega(\beta,\delta) \cup (\Omega_t \setminus \Omega_\varepsilon^t)\}$$

$$< \varepsilon.$$

Lemma 4.3.2 shows that $x_H = i\circ x$ a.s. and the corresponding result for o.d.e.s shows that $x_\beta^H = i\circ x_\beta$ almost surely.
Also, $d : H\times H \longrightarrow \mathbb{R}$ restricts to a metric on $i(M)$.
Thus using the metric $\partial(n,m) = d(i(n), i(m))$ on M we have

$$\mu\{\omega\in\Omega_t : \sup_{0\le s\le t} \partial(x_\beta(s,\omega), x(s,\omega)) > \delta\} < \varepsilon.$$

Hence Lemma 4.3.3 shows that the result holds for any metric on M.

#

Note that this theorem applies not only to the manifold case, but also to non-globally Lipschitz X (e.g. example 4.2.5).

(4.3.5) Example

Let M be a Riemannian manifold modelled on $\mathbb{R}^n$ and x_β an O-U process on M starting at x_o formed by 'rolling' a Euclidean O-U process as in §3.3.

The lift $\tilde{x}_\beta$ of x_β to $O(M)$ satisfies,

$$d\tilde{x}_\beta = \theta(\tilde{x}_\beta)v_\beta dt + \tilde{V}(\tilde{x}_\beta)dt \;,\; \tilde{x}_\beta(0) = u_o \in \tau^{-1}(x_o)$$

where $\tilde{V}$ is the lift of a vector field V on M to $O(M)$.

Let x be the Brownian motion on M whose lift $\tilde{x}$ to $O(M)$ satisfies,

$$d\tilde{x} = \theta(\tilde{x})dW + \tilde{V}(\tilde{x})dt \;,\; \tilde{x}(0) = u_o \in \tau^{-1}(x_o),$$
$$\tilde{x}(t) : \Omega_t \longrightarrow O(M).$$

Then, by Theorem 4.3.4 , for any metric $\tilde{d}$ on $O(M)$ and $\varepsilon > 0$,

$$\mu\{\omega\in\Omega_t : \sup_{0\leq s\leq t} \tilde{d}(\tilde{x}(s,\omega),\tilde{x}_\beta(s,\omega)) > \varepsilon\} \longrightarrow 0 \text{ as } \beta \text{ tends}$$

to infinity , for all $t \in [0,R]$.

We choose a Riemannian metric on $O(M)$ such that,

(i) $H_uO(M) \perp V_uO(M)$ $\forall$ $u \in O(M)$;

(ii) $T_u\tau$ is an isometry from $H_uO(M)$ to $T_{\tau(u)}M$;

(iii) the metric is invariant under $O(n)$.

This Riemannian metric on $O(M)$ generates a distance function $\tilde{d}$ on $O(M)$ which makes $O(M)$ into a metric space. Let d be the metric on M generated by the original Riemannian metric on M. Then we certainly have $\tilde{d}(u,v) \geq d(\tau(u),\tau(v))$ for $u,v\in O(M)$.

In particular, remembering that $x = \tau \circ \tilde{x}$ and $x_\beta = \tau \circ \tilde{x}_\beta$ we have shown,

$$\mu\{\omega \in \Omega_t : \sup_{0 \le s \le t} d(x(s,\omega), x_\beta(s,\omega)) > \varepsilon\} \longrightarrow 0$$

as $\beta \longrightarrow \infty$ for all $t \in [0,R]$.

§4. Ornstein-Uhlenbeck Processes in Force Fields

Let $\beta > 0$ and $b : E \longrightarrow E$ be C^1 and globally Lipschitz, and globally bounded

$$\|b(e)\|_E \le K\|e\|_E \quad \forall e \in E.$$

Define (z_β^F, v_β^F) to be the O-U process in a force field on E,

$$(4.4.1) \quad \begin{cases} dz_\beta^F = v_\beta^F dt \ , \ z_\beta^F(0) = z_o \in E \\ dv_\beta^F = -\beta v_\beta^F dt + \beta b(z_\beta^F) dt + \beta I dW \ , \ v_\beta^F(0) = v_o \in E \end{cases}$$

(cf. Nelson [17]).

Consider,

$$(4.4.2) \quad dx_\beta = b(x_\beta)dt + v_\beta dt \ , \ x_\beta(0) = z_o,$$

$$(= V(x_\beta)dt + X(x_\beta)v_\beta dt)$$

and,

$$(4.4.3) \quad dx = b(x)dt + I dW \qquad , \ x(0) = z_o$$

Throughout this section we assume that E is a Hilbert space. We can apply Theorems 4.1.1 and 4.2.3 to x and x_β to show that x_β tends to x as β tends to infinity , in the senses of both of those theorems. We will assume this done.

(4.4.4) Theorem

z^F_β tends to x as β tends to infinity in the senses of both Theorems 4.1.1 and 4.2.3.

Proof

We will prove the result for $\mathcal{L}^2$ convergence. The other result follows by considering each of the terms we derive individually as in the proof of Theorem 4.2.3.

We assume that $v_o = z_o = 0$. This is simply to reduce the number of expressions we derive - any terms involving either of these can easily be seen to have small $\mathcal{L}^2$ norm for large β.

From 4.4.1 we have

$$v^F_\beta(t) = \beta W(t) - \beta\int_0^t v^F_\beta(s)ds + \beta\int_0^t b(z^F_\beta(s))ds.$$

We apply Lemma 2.2.13 to give

$$v^F_\beta(t) = \beta W(t) + \beta\int_0^t b(z^F_\beta(s))ds - \beta\int_0^t e^{-\beta(t-s)}. \\ .[\beta W(s)+\beta\int_0^s b(z^F_\beta(r))dr]ds.$$

Integrating this by parts using 1.2.6 we get,

$$v^F_\beta(t) = \beta e^{-\beta t}\int_0^t e^{\beta s} IdW(s) + \beta e^{-\beta t}\int_0^t e^{\beta s} b(z^F_\beta(s))ds$$

$$(4.4.5) \quad = v_\beta(t) + \beta e^{-\beta t}\int_0^t e^{\beta s} b(z^F_\beta(s))ds.$$

Thus, $z^F_\beta(t) = \int_0^t v^F_\beta(s)ds$

$$= \int_0^t v_\beta(s)ds + \beta\int_0^t e^{-\beta s}\int_0^s e^{\beta r} b(z^F_\beta(r))drds$$

$$(4.4.6) \quad = z_\beta(t) - e^{-\beta t}\int_0^t e^{\beta s} b(z_\beta^F(s))ds + \\ + \int_0^t b(z_\beta^F(s))ds.$$

Thus , noting that we may rewrite 4.4.2 as

$$x_\beta(t) = z_\beta(t) + \int_0^t b(x_\beta(s))ds, \quad \text{we have}$$

$$(4.4.7) \quad z_\beta^F(t) - x_\beta(t) = \int_0^t [b(x_\beta^F(s)) - b(x_\beta(s))]ds - \\ - e^{-\beta t}\int_0^t e^{\beta s} b(z_\beta^F(s))ds.$$

Now,

$$\|z_\beta^F(t)\| \leq \|z_\beta(t)\| + \int_0^t K\|z_\beta^F(s)\| ds + \\ + e^{-\beta t}\int_0^t e^{\beta s} K\|z_\beta^F(s)\| ds$$

$$\leq \|z_\beta(t)\| + 2K \int_0^t \|z_\beta^F(s)\| ds.$$

It is easily seen from Lemma 2.2.4 that $\|z_\beta(t)\|$ is bounded for all $t \in [0,R]$ and $\beta > 0$,

$$\|z_\beta(t)\| \leq M \text{ , say.}$$

Thus $\|z_\beta^F(t)\| \leq M + 2K \int_0^t \|z_\beta{}^F(s)\| ds$

$\leq M(1 + e^{2KR})$ by applying Lemma 2.2.13.

We have shown that $\|z_\beta^F(t)\|$ is bounded for all $\beta>0$ and $t\in[0,R]$.

$$(4.4.8) \quad \|z_\beta^F(t)\| \leq N \text{ , say.}$$

From 4.4.7 and 4.4.8 we have,

$$\|z_\beta^F(t) - x_\beta(t)\| \leq \int_0^t K\|z_\beta^F(s) - x_\beta(s)\| ds + e^{-\beta t}\int_0^t e^{\beta s} K\|z_\beta^F(s)\| ds$$

$$\leq \quad KN/\beta + K\int_0^t \| z_\beta^F(s) - x_\beta(s) \| \, ds \quad \text{for } t \in [0,R].$$

An application of Gronwall's lemma yields that

$\| z_\beta^F(t) - x_\beta(t) \|$ tends to zero as β tends to infinity, unformly for $t \in [0,R]$.

This , together with the result for $\| x_\beta(t) - x(t) \|$ obtained by applying Theorem 4.1.1, gives the result. #

Next we will show that Theorem 4.1.1 will still hold if we replace z_β and v_β by z_β^F and v_β^F respectively, and x by the "obvious" expression given below.

Let X and V satisfy the conditions of that theorem. We define x_β^F , x_β and x to be the maximal solutions of the differential equations (the first 2 are families of o.d.e.s and the third is an s.d.e.),

$$(4.4.9) \quad \left\{ \begin{aligned} dx_\beta^F(t) &= X(x_\beta^F(t))v_\beta^F(t)dt + V(x_\beta^F(t))dt \;, x_\beta^F(0) = x_o \in H; \\ dx_\beta(t) &= X(x_\beta(t))v_\beta(t)dt + X(x_\beta(t))b(z_\beta(t))dt + \\ &\qquad + V(x_\beta(t))dt, \; x_\beta(0) = x_o; \\ dx(t) &= X(x(t))dW(t) + X(x(t))b(W(t))dt + \\ &\qquad + V(x(t))dt \;, \; x(0) = x_o. \end{aligned} \right.$$

Note that x_β and x do not satisfy the conditions of Theorem 4.1.1 but,

(4.4.10) <u>Lemma</u>

$\| x_\beta(t) - x(t) \| \longrightarrow 0$ as β tends to infinity, uniformly for $t \in [0,R]$.

Proof

The proof is nearly the same as that of Theorem 4.1.1. The only diffemnce is that in 4.1.10 we have the extra term,

$$\int_0^t (X(x_\beta(s))b(z_\beta(s)) - X(x(s))b(W(s)))ds$$

$$= \int_0^t X(x_\beta(s))(b(z_\beta(s)) - b(W(s)))ds +$$

$$+ \int_0^t (X(x_\beta(s)) - X(x(s)))b(W(s))ds.$$

The first of these terms has $\mathcal{L}^2$ norm which tends to zero as β tends to infinity (uniformly over $t \in [0,R]$) since X is globally bounded and $z_\beta(t)$ tends to $W(t)$ in $\mathcal{L}^2$-norm by Theorem 2.2.5.

The second term has $\mathcal{L}^2$-norm

$$\leq \text{Const.} \int_0^t \| x_\beta(s) - x(s) \| ds$$

since we have assumed that b is globally bounded. Hence, we can, as in the proof of Theorem 4.1.1 , apply Gronwall's Lemma, as usual, to obtain the result. #

(4.4.11) Theorem

With the above notation and assumptions,

$\| x^F_\beta(t) - x(t) \| \longrightarrow 0$, uniformly for $t \in [0,R]$ as β tends to infinity.

Proof

Using Lemma 4.4.10 we may restrict our attention to $x^F_\beta(t) - x_\beta(t)$.

We will assume that V is identically zero as in the proof of Theorem 4.1.1. We also assume that $x_0=0$, $z_0=v_0=0$. None of these assumptions is necessary, but they simplify the expressions which we will derive.

From equations 4.4.9 we have,

$$x_\beta^F(t) = \int_0^t X(x_\beta^F(s))v_\beta^F(s)ds$$

$$= \int_0^t X(x_\beta^F(s))v_\beta(s)ds + \beta\int_0^t e^{-\beta s}X(x_\beta^F(s))\int_0^s e^{\beta r}b(z_\beta^F(r))drds$$

using equation 4.4.5.

We integrate this by parts to give,

$$x_\beta^F(t) = \int_0^t X(x_\beta^F(s))v_\beta(s)ds - e^{-\beta t}X(x_\beta^F(t))\int_0^t b(x_\beta^F(s))e^{\beta s}ds +$$

$$+ \int_0^t X(x_\beta^F(s))b(z_\beta(s))ds +$$

$$+ \int_0^t DX(x_\beta^F(s))\ X(x_\beta^F(s))v_\beta^F(s)(e^{-\beta s}.$$

$$\int_0^s e^{\beta r}b(z_\beta^F(r))drds).$$

Thus $x_\beta^F(t) - x_\beta(t) = \int_0^t (X(x_\beta^F(s))-X(x_\beta(s)))v_\beta(s)ds$

(4.4.12)
$$+ \int_0^t X(x_\beta^F(s))(b(z_\beta^F(s))-b(z_\beta(s)))ds$$

$$+ \int_0^t (X(x_\beta^F(s))-X(x_\beta(s)))b(z_\beta(s))ds$$

$$- e^{-\beta t}\ X(x_\beta(t))\int_0^t e^{\beta s}b(z_\beta^F(s))ds$$

$$+ \int_0^t DX(x_\beta^F(s))\ X(x_\beta^F(s))v_\beta^F(s)(e^{-\beta s}\int_0^s e^{\beta r}b(z_\beta^F(r))dr)ds.$$

As at the beginning of the proof of Theorem 4.1.1 we rewrite the first term on the R.H.S. of 4.4.12 as

$$\int_0^t (X(x_\beta^F(s)) - X(x_\beta(s)))dW(s) - (X(x_\beta^F(t))-X(x_\beta(t)))\frac{v_\beta}{\beta}(t)$$
$$+ \frac{1}{\beta}\int_0^t (DX(x_\beta^F(s))\circ X(x_\beta^F(s))v_\beta^F(s) -$$
$$DX(x_\beta(s))\circ X(x_\beta(s))(v_\beta(s)+b(z_\beta(s))))$$
$$v_\beta(s)ds.$$

The first two of these terms are easy to deal with, and the last term becomes, on application of equation 4.4.6,

$$\frac{1}{\beta}\int_0^t (DX(x_\beta^F(s))\circ X(x_\beta^F(s))-DX(x_\beta(s))\ X(x_\beta(s)))(v_\beta(s),v_\beta(s))ds$$
$$+ \int_0^t (DX(x_\beta^F(s))\circ X(x_\beta^F(s)))(e^{-\beta s}\int_0^s e^{\beta r}b(z_\beta^F(r))dr)(v_\beta(s))ds$$
$$- \frac{1}{\beta}\int_0^t DX(x_\beta(s))\circ X(x_\beta(s))(b(z_\beta(s)))v_\beta(s)ds.$$

The last two of these terms are no problem since b is globally bounded and $\|\frac{v}{\beta}\beta(s)\|$ tends to zero as β tends to infinity (uniformly for s e [0,R]) by Lemma 2.2.10.

The first term is dealt with similarly to the last term of 4.1.6 and will become a number of terms of type $\mathcal{E}(\beta)$ plus

$$\tfrac{1}{2}\int_0^t [DX(x_\beta^F(s))\circ X(x_\beta^F(s))-DX(x_\beta(s))\circ X(x_\beta(s))](dW(s),dW(s)).$$

The rest of the proof follows the last part of the proof of Theorem 4.1.1.

#

(4.4.13) Remark

There is no difficulty(despite the lengths of the expressions and the tediousness of the calculations) in extending the results of Lemma 4.4.10 and Theorem 4.4.11 to show convergence in the sense of Theorem 4.2.3. Thus we have,

(4.4.14) Theorem

If in Theorem 4.3.4 we replace (z_β, v_β) by (z_β^F, v_β^F), x_β by x_β^F and x by the x of 4.4.10 the results of that theorem still hold.

Proof

The proof of Theorem 4.3.4 relies on Theorem 4.2.3 for the convergence of x_β to x. By remark 4.4.13 this theorem still holds for x_β^F and the x we consider above. #

§5. Uniform Approximation

The method of this section is taken from Elworthy [5].

Let M be a compact C^∞ finite dimensional manifold of dimension m. For $s>m/2$ we can consider the space $H^s(M;M)$ of maps from M to M of Sobolev class $H^s = W^{2,s}$. $H^s(M;M)$ is a Hilbert manifold. For $s>m/2 + 1$ we let $\mathcal{D}^s$ denote the open subset of $H^s(M;M)$ consisting of diffeomorphisms. We need the following facts which may be found in Ebin and Marsden [18]:-

(i) $\mathcal{D}^s$ is a C^∞ manifold and is a topological group under the action of composition;

(ii) define, for $h \in \mathcal{D}^s$, $R_h : \mathcal{D}^s \longrightarrow \mathcal{D}^s$ by $R_h(\alpha) = \alpha \circ h$. then R_h is C^∞;

(iii) for a C^∞ manifold N the map $\phi_r : H^{s+r}(M;N) \times \mathcal{D}^s(M) \to H^s(M;N)$ defined by $\phi_r(f,h) = f \circ h$ is C^r, $r \in \mathbb{N}$;

(iv) the tangent space $T_h\mathcal{D}^s$ to $\mathcal{D}^s$ at $h \in \mathcal{D}^s$ may be identified with the Hilbertable space of H^s maps $f : M \longrightarrow TM$ with $\pi \circ f(m) = h(m)$ $\forall$ $m \in M$. In particular $T\mathcal{D}^s$ may be identified with $H^s(M;TM)$.

Let X be a section of $\mathrm{Hom}_M(E,TM)$ (for a Hilbert space E with a Brownian motion defined onit as §1.3) and assume that the vector fields $X(\ )e$ are of class H^{s+2} $\forall$ $e \in E$.

By (iv) $X(\ )e$ lies in $T_{id}\mathcal{D}^{s+2}$ where $id : M \longrightarrow M$ is the identity.

We define $\tilde{X} : \mathcal{D}^s \times E \longrightarrow T\mathcal{D}^s$ by

(4.5.1) $\tilde{X}(h,e)(y) = X(h(y))e$, or , equivalently, defining X^i to be the vector field $X(\ )e_i$ (for an o.n. basis e_i of E),

(4.5.2) $\tilde{X}^i(h) = (R_h)_* X^i = \phi_2(X^i,h)$ taking N=TM in (iii).

Since ϕ_2 is C^2 and X is C^2 , $\tilde{X}$ is C^2 and is also right invariant. For a vector field V on M of class H^{s+1} we define a right invariant vector field $\tilde{V}$ on $\mathcal{D}^s$ in the same way,

(4.5.3) $\tilde{V}(h) = (R_h)_* V = \phi_2(V,h)$.

Then (ii) together with the right invariance of $\tilde{X},\tilde{V}$ and the existence of inverses in $\mathcal{D}^s$ can be used to show that there exists a 'uniform cover' for $(\tilde{X}\oplus\tilde{V}, W\oplus t)$ on $\mathcal{D}^s$ (see Elworthy [4]).

Thus there is a solution of $d\tilde{F} = \tilde{X}dW + \tilde{V}dt, \tilde{F}(0)=id:M \longrightarrow M$, with infinite explosion time,

$\tilde{F} : [0,\infty) \times \Omega \longrightarrow \mathcal{D}^s$,, $\tilde{F}(0) = id$.

For $x_o \in M$ we define the evaluation map $ev_{x_o} : \mathcal{D}^s \longrightarrow M$ by $ev_{x_o}(h) = h(x_o)$. ev_{x_o} is C^∞.

Let $\Lambda = ((U,\phi),U_o,\lambda)$ be a localisation of $(X\oplus V, W\oplus t)$ on M and extend ϕ to a C^2 map $\psi : M \longrightarrow \mathbb{R}^m$.

Applying the Ito formula (see Elworthy [4] to $\psi \circ ev_{x_o}(\tilde{F}(t))$ we get,

$$\psi \circ ev_{x_0}(\tilde{F}(t)) = \psi(x_0) + \int_0^t X_\Lambda(\psi \circ ev_{x_0}\tilde{F}(s))dW(s)$$
$$+ \int_0^t V_\Lambda(\psi \circ ev_{x_0}\tilde{F}(s))ds$$
$$+ \tfrac{1}{2}\int_0^t tr_L(DX_\Lambda(\psi \circ ev_{x_0}(\tilde{F}(s)))(X_\Lambda(\psi \circ ev_{x_0})(\tilde{F}(s))))ds$$

and hence $ev_{x_0} \circ \tilde{F}$ is a solution of

$$dx = X\,dW + V\,dt \quad , \quad x(0) = x_0.$$

We will apply the approximation of Theorem 4.3.4 to $d\tilde{F} = \tilde{X}dW + \tilde{V}dt$ on $\mathcal{D}^s$. We define $\tilde{F}_\beta$ on $\mathcal{D}^s$ to be the family of solutions to the o.d.e.s on $\mathcal{D}^s$,

$$d\tilde{F}_\beta = \tilde{X}(\tilde{F}_\beta)v_\beta dt + \tilde{V}(\tilde{F}_\beta)dt,$$

where v_β is our usual O-U process on E. Then $\tilde{F}_\beta$ tends to $\tilde{F}$ as β tends to infinity, in the sense of Theorem 4.3.4.

In particular,

(4.5.4) <u>Theorem</u>

If M is compact and X,V are C^∞ (in Theorem 4.3.4) we define $\tilde{F}_\beta : [0,\infty) \times \Omega \longrightarrow C^\infty(M;M)$ to be a representation of the flow of the family of o.d.e.s on M,

$$dx_\beta = X(x_\beta)v_\beta dt + V(x_\beta)dt.$$

We give $C^\infty(M;M)$ the topology of uniform convergence of all derivatives. Then, as β tends to infinity, $\tilde{F}_\beta$ converges uniformly on $[0,R]$ in probability , to a map $\tilde{F} : [0,\infty) \times \Omega \longrightarrow C^\infty(M;M)$ which is a version of the flow of the s.d.e. $dx = X(x)dW + V(x)dt$

CHAPTER FIVE. PIECEWISE LINEAR APPROXIMATION

Let E, E_o, H and W be the objects of Neidhardt's stochastic integral as in §1.3. We assume that E is a Hilbert space in order to be able to apply Proposition 1.3.6. We take, as usual, $[0,R]$ to be contained in the index set of W. Let π be a (Cauchy) partition of $[0,R]$,

$$\pi = (t_1, \ldots, t_{m+1}), \quad 0=t_1<\ldots<t_{m+1} = R.$$

We define the piecewise linear approximation to W on E_o, $W_\pi : [0,R] \times \Omega \longrightarrow E_o$ by,

$$W_\pi(s,\omega) = (t_{j+1}-t_j)^{-1}[(t_{j+1}-s)W(t_j,\omega)-(s-t_j)(W(t_{j+1},\omega))]$$

for $t_j \leq s \leq t_{j+1}$.

Let X and V be objects satisfying the conditions at the start of §4.1 (although we do not need the global boundedness of D^2X), $X : E \longrightarrow L(E_o;E)$ and $V : E \longrightarrow E$ (note that the objects in §4.1 are defined over a Hilbert space H and the E of that section is the E_o of this section).

Let $x : [0,R] \times\Omega \longrightarrow E$ be the solution of the s.d.e.,

$$dx(t) = X(x(t))dW(t) + V(x(t))dt, \quad x(0) = x_o \in \mathcal{L}^2(\Omega, \mathcal{F}; E),$$

which exists by Theorem 1.1.4 (or, writing the equation in its Itô form, by Theorem 1.3.5) and satisfies,

$$\text{(5.1)} \quad x(t) = x_o + \int_0^t V(x(s))ds + \int_0^t X(x(s))dW(s) + \frac{1}{2}\int_0^t tr_L(DX(x(s))\circ X(x(s)))ds.$$

For $0 \leq s \leq t \leq R$,

$$\|x(t)-x(s)\|^2 \leq 3\left\|\int_s^t V(x(r))dr\right\|^2 + 3\left\|\int_s^t X(x(r))dW(r)\right\|^2 +$$

$$+ \frac{3}{4}\left\|\int_s^t tr_L(DX(x(r))\circ X(x(r)))dr\right\|^2$$

$$\leq 3\ (t-s)^2 G^2 + 3A_E K^2(trL)(t-s) +$$

$$+ \frac{3}{4} m_4\ (NK)^2\ (t-s)^2,$$

where the constants are those of §4.1. This gives that x is $\frac{1}{2}$-Holder continuous as a map from $[0,R]$ into $\mathcal{L}^2(\Omega,\mathcal{F};E)$.

In particular, if π_n is a sequence of partitions of $[0,R]$ such that mesh(π_n) tends to zero as n tends to infinity then

$$\sum X(x(t_i))(W(t_{i+1})-W(t_i)) - \int_0^R X(x(s))dW(s)$$

$$= \sum \int_{t_i}^{t_{i+1}} (X(x(t_i)) - X(x(t)))dW(t)$$

tends to zero in $\mathcal{L}^2$-norm as n tends to infinity. The same is true, from the proof of Proposition 1.3.7, of

$$\sum DX(x(t_i))\circ X(x(t_i))\ (W(t_{i+1})-W(t_i),W(t_{i+1})-W(t_i))$$

$$- \int_0^R tr_L(DX(x(t))\circ X(x(t)))dt.$$

(5.2) Theorem

For each partition π of $[0,R]$ let $x_\pi: [0,R]\times\Omega \longrightarrow E$ be the solutions of the family of o.d.e.s (indexed by ω),

$$(5.3)\qquad \frac{dx_\pi}{dt}(t) = X(x_\pi(t))\frac{dW_\pi}{dt}(t) + V(x_\pi(t)) \quad , 0\leq t\leq R \ , \ x_\pi(0)=x_0.$$

Then x_π converges to x in $\mathcal{L}^2$-norm, uniformly for $t \in [0,R]$, as mesh π tends to zero.

Remark A more general class of piecewise smooth approximations has been considered , in finite dimensions,by Ikeda ,Nakao and Yamato [ ] .

Proof

The line of proof will follow that in Elworthy [4], which was extracted from that of McShane [18] for a more general class of approximations on the real line.

For simplicity, as in much of Chapter Four , we consider V to be identically zero.

For a fixed partition π of $[0,R]$ we set

$y = x_\pi$, $x_j = x(t_j)$, $y_j = y(t_j)$, $\Delta_j t = t_{j+1}-t_j$ and $\Delta_j W = W(t_{j+1}) - W(t_j)$.

Then, Taylor's theorem gives,

$$y_{j+1} - y_j = Dy(t_j)\Delta_j t + \int_0^1 (1-s)D^2y(t_j+s\Delta_j t)(\Delta_j t,\Delta_j t)ds$$
$$= X(y_j)\Delta_j W + \int_0^1 (1-s)DX(y(t_j+s\Delta_j t))X(y_j+s\Delta_j t).(\Delta_j W,\Delta_j W)ds$$

on application of 5.3.

Hence,

$$y_k - y_1 = \sum_{j=1}^{k-1} (y_{j+1} - y_j)$$
$$= \sum_{j=1}^{k-1} X(y_j)\Delta_j W + \tfrac{1}{2}\sum_{j=1}^{k-1} DX(y_j)\circ X(y_j)(\Delta_j W, \Delta_j W) + A_k$$

where $A_k = \sum_{j=1}^{k-1} \int_0^1 (1-s)[DX(y(t_j+s\Delta_j t))\circ X(y(t_j+s\Delta_j t)) -$
$- DX(y_j)\circ X(y_j)](\Delta_j W, \Delta_j W)ds.$

Using 5.1,

$$(5.4)\quad y_k - x_k = A_k + \sum_{j=1}^{k-1} (X(y_j) - X(x_j))\Delta_j W$$
$$+ \sum_{j=1}^{k-1} (DX(y_j)\circ X(y_j) - DX(x_j)\circ X(x_j))(\Delta_j W, \Delta_j W)$$
$$+ S_k + T_k$$

where

$$S_k = \sum_{j=1}^{k-1} X(x_j)\Delta_j W - \int_0^{t_k} X(x(t))dW(t)$$

and $T_k = \sum_{j=1}^{k-1} DX(x_j)\circ X(x_j)(\Delta_j W, \Delta_j W) - \int_0^{t_k} tr_L(DX(x(t))\circ X(x(t)))dt.$

We have seen, in the remarks preceding the statement of the theorem, that both S_k and T_k tend , in $\mathcal{L}^2$-norm, to zero as mesh π tends to zero. It is not hard to see that this convergence is uniform in k. We call terms satisfying this convergence condition $\mathcal{E}(\pi)$ (cf. $\mathcal{E}(\beta)$ in the proof of Theorem 4.1.1).

For $0 \le s \le 1$,

$$E[\|y(t_j+s\Delta_j t) - y(t_j)\|_E^4] =$$
$$E\left\|\int_0^s X(y(t_j+r\Delta_j t))\Delta_j W dr\right\|_E^4$$

$$\leq E\left(\int_0^s K\|\Delta_j W\|_{E_0} dr\right)^4$$

$$\leq K^4 E\|\Delta_j W\|_{E_0}^4$$

(5.5) $$= K^4(\Delta_j t)^2 m_4.$$

Now,

$$\|A_k\| = \Big\|\sum_{j=1}^{k-1}\int_0^1 (1-s)[DX(y(t_j+s\Delta_j t))\circ X(y(t_j+s\Delta_j t)) - DX(y_j)\circ X(y_j)](\Delta_j W, \Delta_j W)\,ds\Big\|$$

$$\leq \sum_{j=1}^{k-1}\int_0^1 \|(DX(y(t_j+s\Delta_j t))\circ X(y(t_j+s\Delta_j t)) - DX(y_j)\circ X(y_j))(\Delta_j W, \Delta_j W)\|\,ds$$

$$\leq \sum_{j=1}^{k-1}\int_0^1 (E[(JK+NF)^2\|y(t_j+s\Delta_j t)-y_j\|_E^2\|\Delta_j W\|_{E_0}^4])^{\frac{1}{2}}ds$$

$$\leq \sum_{j=1}^{k-1}\int_0^1 (JK+NF)(E[\|y(t_j+s\Delta_j t)-y_j\|_E^4]E[\|\Delta_j W\|_{E_0}^8])^{\frac{1}{4}}\,ds$$

$$\leq (JK+NF)\sum_{j=1}^{k-1}\int_0^1 (K^4(\Delta_j t)^2 m_4\ (\Delta_j t)^4 m_8)^{\frac{1}{4}}ds$$

$$= (JK+NF)K\sum_{j=1}^{k-1}(m_4 m_8)^{\frac{1}{4}}(\Delta_j t)^{3/2}$$

$$\longrightarrow 0 \text{ as mesh } \pi \longrightarrow 0,$$

where $m_8 = \int_{E_0}\|y\|_{E_0}^8 d\nu(y) < \infty$ by Fernique's theorem.

Hence we have shown that A_k is $\mathcal{E}(\pi)$.

Next we consider,

$$U_k = \sum_{j=1}^{k-1}(X(y_j)-X(x_j))\Delta_j W.$$

Set $N(s) = \sup_{0\leq r\leq s}\|y(r)-x(r)\|$, $0\leq s\leq R$.

Then $\|U_k\|^2 \leq A_E\sum_{j=1}^{k-1}E\|X(y_j)-X(x_j)\|_\nu^2\Delta_j t$ by (vii) of 1.3.1

$$\leq A_E F^2\sum_{j=1}^{k-1}\|y_j-x_j\|^2\Delta_j t$$

$$\leq A_E\ F^2 \sum_{j=1}^{k-1} N(t_j)^2 \Delta_j t$$

$$\leq A_E\ F^2 \int_0^{t_k} N(s)^2 ds$$

since $N(s)$ is non-decreasing.

Now we consider,

$$V_k = \sum_{j=1}^{k-1} (DX(y_j)\circ X(y_j) - DX(x_j)\circ X(x_j))(\Delta_j W, \Delta_j W)$$

$$\|V_k\| \leq \sum_{j=1}^{k-1} (JK+NF)(E[\|y_j-x_j\|_E^2 \|\Delta_j W\|_{E_0}^4])^{\frac{1}{2}}$$

$$= \sum_{j=1}^{k-1} (JK+NF)(E[E_{t_j}[\|y_j-x_j\|_E^2 \|\Delta_j W\|_{E_0}^4\ ]])^{\frac{1}{2}}$$

$$= \sum_{j=1}^{k-1} (JK+NF)(E[\|y_j-x_j\|_E^2\ E_{t_j}\|\Delta_j W\|_{E_0}^4\ ])^{\frac{1}{2}}$$ since y_j and x_j are $\mathcal{F}_{t_j}$-measurable

$$=(JK+NF)\sum_{j=1}^{k-1} \|y_j-x_j\|\,\Delta_j t\,(m_4)^{\frac{1}{2}}$$

$$\leq (JK+NF)(m_4)^{\frac{1}{2}} \int_0^{t_k} N(s)\,ds.$$

Returning to 5.4 we now have,

$$\|y_k-x_k\| \leq \|\mathcal{E}(\pi)\| + [FA_E^{\frac{1}{2}} + R(JK+NF)(m_4)^{\frac{1}{2}}]\sqrt{\int_0^{t_k} N(s)^2 ds}.$$

Now, if $t_k \leq t < t_{k+1}$ then $x(t)-x_k$ is $\mathcal{E}(\pi)$ by the uniform continuity of $x : [0,R] \longrightarrow \mathcal{L}^2(\Omega, \mathcal{F}; E)$ and $y(t)-y_k$ is $\mathcal{E}(\pi)$ by 5.5.

Thus,

$$\|y(t) - x(t)\| \leq \|\mathcal{E}(\pi)\| + [FA_E^{\frac{1}{2}} + R(JK+NF)(m_4)^{\frac{1}{2}}]\sqrt{\int_0^{t_k} N(s)^2 ds}$$

and so,

$$\|y(t)-x(t)\| \leq \|\mathcal{E}(\pi)\| + [FA_E^{\frac{1}{2}} + R(JK+NF)(m_4)^{\frac{1}{2}}]\sqrt{\int_0^{t} N(s)^2 ds}$$

giving,

$$N(t)^2 \leq 2\|\mathcal{E}(\pi)\|^2 + 2[FA_E^{\frac{1}{2}}+R(JK+NF)(m_4)^{\frac{1}{2}}]^2\int_0^t N(s)^2ds.$$

Then an application of Gronwall's lemma shows,

N(t) tends to 0,, uniformly in [0,R] as mesh π tends to zero, as required.

#

(5.7) Theorem

Under the conditions of Theorem 5.2,

$$\mu\{\omega: \sup_{0\leq t\leq R}\|x_\pi(t)-x(t)\|_E > \varepsilon\} \longrightarrow 0 \text{ as mesh } \pi \text{ tends to zero.}$$

Proof

We retain the notation of the proof of the previous theorem. We have,

$$y_k-x_k = A_k + S_k + T_k + U_k + V_k.$$

Property (x) of 1.3.1 gives that

$$\mu\{\omega : \sup_k\|U_k\|_E>\varepsilon\} \leq \frac{4A_E}{\varepsilon^2}\sum_{j=1}^{k-1} E\|X(x_j)-X(y_j)\|_\nu^2 \Delta_j t$$

$$\leq \frac{4A_E}{\varepsilon^2}F^2\sum_{j=1}^{m-1} E\|x_j-y_j\|_E^2\Delta_j t(\text{FL})$$

tends to zero as mesh π tends to zero by Theorem 5.2

$$\sup_k\|V_k\|_E \leq \sum_{j=1}^{m-1} (JK+NF)\|y_j-x_j\|_E\|\Delta_j W\|_{E_0}^2 .$$

Thus,

$$\mu\{\omega: \sup_k\|V_k\|_E>\varepsilon\} \leq \frac{1}{\varepsilon^2}\left\|\sum_{j=1}^{m-1}(JK+NF)\|y_j-x_j\|_E\|\Delta_j W\|_{E_0}^2\right\|^2$$

$$\leq \frac{(JK+NF)^2}{\varepsilon^2}\left(\sum_{j=1}^{m-1}(E[\|y_j-x_j\|_E^2\|\Delta_j W\|_{E_0}^4])^{\frac{1}{2}}\right)^2$$

$$\leq \frac{(JK+NF)^2}{\varepsilon^2}\left(\sum_{j=1}^{m-1}(E[\|y_j-x_j\|_E^2(\Delta_j t)^2 m_4])^{\frac{1}{2}}\right)^2$$

$$\leq \frac{(JK+NF)^2}{\varepsilon^2}\left(\sum_{j=1}^{m-1}\|y_j-x_j\|(m_4)^{\frac{1}{2}}\Delta_j t\right)^2$$

$$\leq \frac{(JK+NF)^2}{\varepsilon^2} Rm_4 \int_0^R N(t)^2 dt$$

tends to zero as mesh π tends to zero by Theorem 5.2.

$$\sup_k\|A_k\|_E \leq \sum_{j=1}^{m-1}\int_0^1\|(DX(y(t_j+s\Delta_j t))\circ X(y(t_j+s\Delta_j t)) -$$

$$- DX(y_j)\circ X(y_j))(\Delta_j W,\Delta_j W)\|_E\, ds$$

$$\leq \sum_{j=1}^{m-1}\int_0^1 (JK+NF)\|y(t_j+s\Delta_j t) - y_j\|_E\|\Delta_j W\|_{E_0}^2 ds$$

and hence

$$\mu\{\omega : \sup_k\|A_k\|_E > \varepsilon\} \leq \frac{1}{\varepsilon^2}\Big\|\sum_{j=1}^{m-1}\int_0^1 (JK+NF)\|y(t_j+s\Delta_j t)-y_j\|_E \cdot \|\Delta_j W\|_{E_0}^2 ds\Big\|^2$$

$$\leq (m_4 m_8)^{\frac{1}{2}}(JK+NF)^2 K^2 R^2 . \text{mesh}\pi$$

(by the proof of Theorem 5.2)

$$\longrightarrow 0 \text{ as mesh } \pi \text{ tends to } 0.$$

$$S_k = \sum_{j=1}^{k-1}\int_{t_j}^{t_{j+1}} (X(x(t_j))-X(x(t)))dW(t)$$

$$= \int_0^{t_k} (X(x_\pi(t)) - X(x(t)))dW(t) \text{ where } x_\pi(t) = x(t_j) \text{ for } t_j \leq t < t_{j+1}$$

Thus (x) of 1.3.1 gives,

$$\mu\{\omega : \sup_k\| S_k\|_E > \varepsilon\} \leq \frac{4A_E}{\varepsilon^2}\int_0^R\|X(x_\pi(t))-X(x(t))\|_y^2 dt$$

$$\leq \frac{4A_E}{\varepsilon^2} F^2\int_0^R\|x(t)-x_\pi(t)\|^2 dt \;(\text{t-L})$$

which tends to zero by the uniform continuity of x.

We are left with T_k. We know that T_k tends to zero as mesh π tends to zero , in $\mathcal{L}^2$. The results of Elworthy [ ], particularly Lemma 5A of Chapter IV give that $\sup_k \| T_k \|_E$ tends to zero in measure as mesh π tends to zero.

For $\gamma > 0$ we set $\sigma_\gamma(W)(\omega) = \sup\{\|W(t,\omega)-W(s,\omega)\|_{E_0} : |t-s| < \gamma\}$ a.e. Then $\sigma_\gamma(W)$ tends to zero in measure as γ tends to zero, and it is easy to see that the same is true of $\sigma_\gamma(x)$.

Thus given $\gamma_1 > 0, \gamma_2 > 0$ there exists $\Omega_0 \in \mathcal{F}$ and $\varepsilon > 0$ such that $\mu(\Omega_0) > 1 - \gamma_1$ and for mesh $\pi < \varepsilon$, $\omega \in \Omega_0$, $k=1,\ldots,m$.

$$\|\Delta_k W\|_{E_0} \leq \gamma_2/K$$

$$\|x(t) - x_k\|_E < \gamma_2 , \; t_k < t < t_{k+1}.$$

Thus, by the mean value theorem, for $\omega \in \Omega_0$ and mesh $\pi < \varepsilon$, $\|y(t) - y_k\| < \gamma_2, t_k \leq t < t_{k+1}$

which completes the proof. $\#$

<u>Remark</u> Using this result we obtain a theorem whose statement and proof are almost the same as those of Theorem 4.3.4 (cf. Theorem 4.4.14), regarding the approximation of piecewise linear paths on manifolds to Brownian motions on manifolds.

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