# Facets and Layers of Function for College Students in Beginning Algebra

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# Declaration

I declare that the material in this thesis has not been previously presented for any degree at any university. I further declare that the research presented in this thesis is my unaided work. the only exception is that some parts in Chapter 3 are a modified version of a joint paper by Professor D. Tall and myself which was published in the *Proceedings of the 20th Annual Conference for the Psychology of Mathematics Education.*

# Summary

The first mathematics course for approximately 53 percent of U.S. community college students is a developmental algebra course. Many such students appear to be severely debilitated by their previous encounters with mathematics. Due to numerous misconceptions that dictate against a traditional course, a "reform" beginning algebra course, with function as the unifying concept, was designed. Since there is little research on this population to justify such a approach, the key research question for this thesis becomes: Can adult students who arrive at college having had debilitating prior experiences with algebra acquire at least a process level understanding of function through appropriate instructional treatment? Answering this question provides crucial information for future curricular design in the area of developmental mathematics at the college level.

The theoretical framework considers different aspects that make up the function concept, taking critical account of several current theories of multiple representations and encapsulation of process as object to build a view of function in terms of different facets (representations) and different layers (of development via procedure, process, object, and procept).

Ninety-two students at four community colleges completed written function surveys before and after a "reform" beginning algebra course. Twelve students, representing all four sites, participated in task-based interviews. Comparison of pre- and post-course surveys provided data indicating statistically significant improvement in student abilities to correctly interpret and manipulate function machines, two-variable equations, two-column tables, two-dimensional graphs, written definitions and function notation. The students were divided into three categories (highly capable, capable, and incapable) based on their demonstrated understanding of function. Using the interviews, visual profiles for students in each category were developed. The profiles indicate that the development of the concept image of function in such students is complex and uneven. The cognitive links between facets is sometimes nonexistent, sometimes tenuous, and often unidirectional. The highly capable demonstrated some understanding across all facets while the incapable indicated understanding of the more primitive facets, such as colloquial and numeric, only. Profound differences were noted particularly in the geometric, written, verbal, and notation facets. Overall, the target population appeared able to develop a process layer understanding of function, but this development was far from uniform across facets and across students.

CHAPTER 1

# Thesis Overview

## 1.1 Introduction

A vexing problem facing U.S. college mathematics departments today is the sizable percentage of the student body that must begin their college career in a non-credit mathematics course. Courses that fall under this umbrella include arithmetic, geometry, beginning algebra, and intermediate algebra. A 1995 survey completed by 275 United States community colleges reported that 53 percent of the nearly 1.5 million community college students were enrolled in a developmental mathematics class (Loftsgaarden et al., 1997). Of these, 304,000 were enrolled in beginning algebra, the course that is the focus of this thesis (ibid). Sixty-three percent of mathematics department heads listed remediation as the major problem they face (ibid). Many developmental algebra students have been debilitated by their previous exposure to algebra. The students who participated in the main study for this thesis had, on the average, taken approximately 1.4 years of algebra previously. Fifty-nine percent of these students were younger than 21 years of age while 20 percent were older than 30. On a student attitude survey completed by 285 beginning algebra students, the statement "The mathematics I learn in school is mostly facts and procedures that have to be memorized" registered significantly high agreement. This audience is unique in that the students are mature and experienced and yet commonly exhibit a procedural approach to mathematics. Succeeding with this population may require providing the students with a completely different educational experience. One option makes function the core concept, following the philosophy of Yerushalmy and Schwartz, who state: "... we believe that function is the fundamental object of algebra and that it ought to be present in a variety of representations in algebra teaching and learning from the outset" (1993, p. 41). The American Mathematical Association of Two-Year Colleges (AMATYC), in its Standards document (1995), recommends that "function" be one of the central themes of the Standards for Content (p. 13). A focus on function in developmental algebra might be a viable alternative to the standard skills-based developmental algebra courses.

The research reported in this thesis focuses on the understanding of functions that students acquire as a result of completing a "reform" beginning algebra curriculum. The curriculum, which serves as the independent variable, is defined by a text (DeMarois, McGowen, & Whitkanack, 1996a) that uses context-based problems, function, and technology to develop algebraic understanding. Using "function" as a focal point of their beginning algebra course, the authors hope to provide students with a vehicle to build meaning into their work with algebra. The analysis of the data suggests that function is not beyond the conceptual grasp of students at this level. The understanding of function acquired by these students as a result of a semester with the "reform" curriculum serves as the dependent variable. For beginning algebra students, a "process" level understanding of function is probably more than sufficient for their future mathematical needs. "Process" level means the ability to view, manipulate, and cognitively reflect on the various ways function is presented and described from the point of view that a function consists of a set of inputs and a corresponding set of outputs, each output unique for a given input, in which some process defines how to determine output from input. The evidence suggests that many students can gain at least a "process" level understanding of function.

## 1.2 Background and Statement of the Problem

### 1.2.1 Creating debilitated students

Critical to understanding the nature of the debilitation that has occurred in beginning algebra students is the concept of procept, introduced by Gray and Tall (1994), who write: "An **elementary procept** is the amalgam of three components: a *process* that produces a mathematical *object*, and a *symbol* that represents either the process or the object" (p. 121). For example, the notation $f(x)$ where $f$ is the name of a function contains the ideas of a process to follow to produce output from input, an object–a function, and a symbolism $f(x)$ that can represent either the process or the object. The ambiguity of the symbolism gives the symbolism its power. Those who can move easily between the various meanings of such symbolism have compressed the information inherent in the symbolism. Meaning for symbols often develops by first doing **procedures** such as evaluating a function at a given number. Procedures may then mature into **processes** in which the idea that a function produces output from input is understood by the student without having to apply a specific algorithm to an input to get an output. At some point, the concept may mature in the student's mind to the point where it can be thought about as an **object**. In this case, the student can perform operations on this object "function," such as differentiation. The ability to think flexibly about a concept, such as function, as both a process and an object is referred to as **proceptual thinking**. Contrasting this, thinking that is dependent on the selection and performance of appropriate procedures is called **procedural thinking**. Tall (1996) writes: "Procedures allow individuals to do mathematics, but learning lots of separate procedures and selecting the appropriate one for a given purpose becomes increasingly burdensome. Procepts allow the individual not only to carry out procedures, but to regard symbols as mental objects, so they can not only do mathematics, they can think about the concepts" (p. 12). Students who rely on procedural thinking are doing much more difficult mathematics. Mathematics, for them, consists of disconnected cognitive units of algorithms triggered by a specific problem format. The divergence between those who interpret processes only as procedures and those who see them as flexible procepts is called the **proceptual divide** (Gray & Tall, 1991). In essence, the more able depend on procepts while the less able depend on procedures. As cognitive strain grows, a student, who up to that point may have been successful, encounters difficulty and asks "tell me how to do it," desiring the security of a procedure rather than the flexibility of a procept. From this point on failure is almost inevitable. If someone who has turned to procedural thinking finds security, then additional practice of those procedures does little more than widen the proceptual divide. Developmental algebra instructors regularly encounter symptoms of the proceptual divide. Their students try to memorize so many procedures with so little understanding that algebra is a mishmash of disconnected procedures. For example, students often feel they must perform a procedure when given an algebraic expression. As Tall states, "... the difficulties that average college students have with algebra occur because of previous rule-bound approaches to the subject. When students do not understand what something is, at least they can get temporary success by becoming secure with procedures to do things with it" (1992, p. 3). Developmental algebra students appear to take great security and comfort in procedures. Unfortunately, the procedures are often poorly understood and incorrectly used.

The fact that many students in beginning algebra have had algebra before results in a mental network of concept images rife with misinformation. Such cognitive networks contain **concept images** that conflict with **concept definitions** (Tall & Vinner, 1981). Concept definition refers to the mathematical definition of a concept, while concept image is everything associated in somebody's mind with the concept name. The students' prior exposure to the concepts is detrimental due to the inappropriate existing network. As Ausubel (1968) stated, "The most important single factor influencing learning is what the learner already knows" (p. vi). It is often the case that what the learner knows is replete with misconceptions. Personal observation suggests that students' concept images of variables and equations displays little overlap with the definition of these concepts, for example. What happens when a new idea is presented in this context? Consider what Hiebert & Carpenter (1992) say about existing networks: "If the learner tries hard to fit a new idea, fact, or procedure into a current way of thinking, existing networks constrain the relationships that are created. At the other extreme, a learner may represent new information in a way that does not connect it with existing networks" (p. 70). Two potential problems arise. An existing network that is incorrectly constructed constrains the construction of the web for a new idea. The result is a construction rife with misconceptions. On the other hand, if the new idea does not connect with the existing networks, a lack of connections between old and new occurs leading to disjointed and unusable knowledge.

Due to prior acquaintance with numerous concepts, students exhibit profound learning interferences. Students are often adept at solving linear equations procedurally, and, as a result, resist considering the meaning of such equations or how they arise in various situations. They know how to "solve it," so what's the point in learning more about the concept?

### 1.2.2 Focusing on function

In light of the hindrances placed on students by previous exposure to algebra, one approach to teaching developmental algebra suggests that the course must be radically different from the one students originally took. Using function as the unifying concept is one such approach. However, there are those who worry that introducing function at this level may introduce new obstacles to students. Sierpinska (1992), for example, writes: "Lack of algebraic awareness makes the understanding of function very difficult if not impossible" (p. 45). The curriculum described in this thesis suggests that function be used as a vehicle to develop algebraic awareness.

### 1.2.3 Instructional treatment

The instructional treatment involves a beginning algebra course that focuses primarily on function as process and introduces function in a multi-faceted way including written and verbal definitions, graphs, function machines, tables, equations, and function notation. The text (DeMarois, McGowen, & Whitkanack, 1996a) emphasizes student investigation of problems based on a pedagogical approach that uses a constructivist theoretical perspective of how mathematics is learned (Davis et al., 1990). The materials develop mathematical ideas using a core concept of function. Function is initially defined as "a process that receives input and returns a unique value for output" (DeMarois, McGowen, & Whitkanack, 1996a, p. 92). Each function is based in a problem situation. Functions are investigated numerically, graphically, and with function machines before the symbolic form is created. Tables, equations, graphs, function machines, verbal and written descriptions are all used to analyse functional relationships.

### 1.2.4 Theoretical framework

To develop an appropriate model from which to draw conclusions about student concept images, the researcher combined function research from several different areas. Some researchers (Cuoco, 1994, for example) have focused on the various aspects or representations of a concept while others have analysed the depth of understanding of a concept (See Dubinsky's APOS theory (Cottrill et al., 1996; Breidenbach et al., 1992; Dubinsky & Harel, 1992) or Sfard's (1992) operational-structural theory, discussed in depth later). The researcher chose to follow Schwingendorf et al. (1992) who contrast the vertical development of the concept implying increasing depth and the horizontal development relating different representations and implying breadth.

The elements of the breadth dimension, called **facets**, are conceived as consisting of various representations, including geometric, numeric, and symbolic. The facets of a mathematical entity refer to various ways of thinking about it and communicating to others, including verbal (spoken), written, kinesthetic (enactive), colloquial (informal or idiomatic), notational, numeric, symbolic, and geometric (visual) aspects. These are not intended to be independent or exhaustive, but provide a suitably broad framework to begin an analysis of the function concept.

An area that has received much attention is students' ability to move comfortably between facets implying that students can choose the most appropriate facet to use for a given problem. Cuoco (1994, p. 125) suggests that the connections between "representations" are properties of a "higher-order function". The connections that are established or fail to exist between facets will be a consideration in this research.

The word **layer** is used to refer to each depth dimension. Pre-procedure, procedure, process, concept, and procept are the layers of increasing depth. Pre-procedure assumes that the student is on the ground floor, so to speak, with respect to a concept. A procedure is a coherent sequence of actions—a schema of actions or a specific algorithm while a process is a cognitive entity, not dependent on individual steps, but rather on the result produced. Students at the procedure layer are dependent on the specific steps performed. Students at the process layer can cognitively accept the existence of a process between input and output without needing to know the specific steps. The concept layer aligns closely with the ability to treat the mathematical idea as an object to which a procedure can be applied. After the concept layer, a procept layer is placed, to indicate the flexibility to move easily between process and object layers.

Using this framework, student profiles are created by combining the two dimensions diagrammatically with the layers as concentric circles representing increasing depth, sliced into sectors representing various facets (Figure 1.1). The facets should be viewed as slices that can be moved around–that is why they are disconnected. The nature of the boundaries between various pairs of facets and between successive layers is an important part of this research.

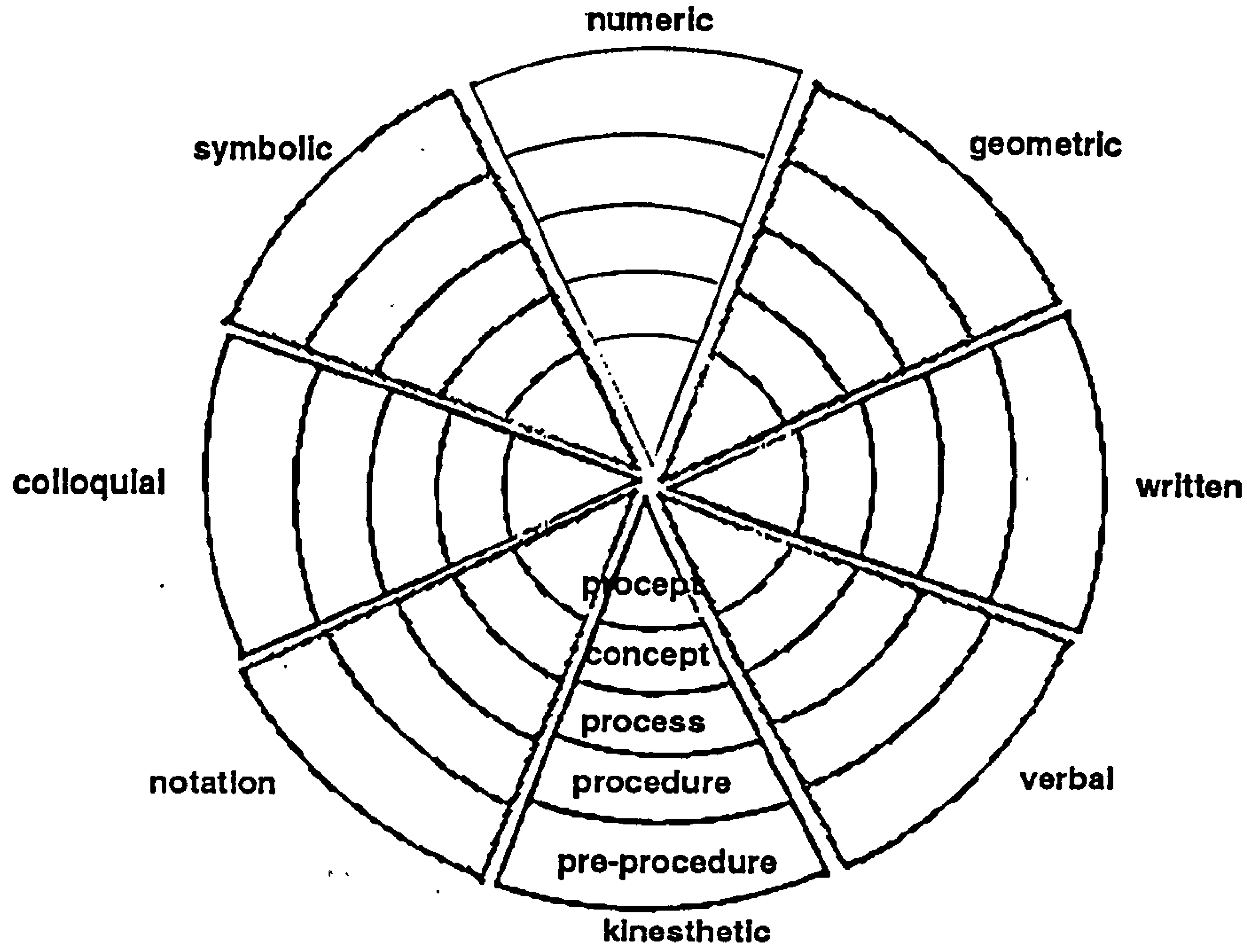



Figure 1.1 Facets and layers

### 1.2.5 Research questions

While using "function" as the unifying concept in a beginning algebra course for college students is supported by several calls for reform (See AMATYC, 1996, for example), there is little research on this particular population to justify this approach. Thus, the key research question becomes: Can adult students who arrive at college having had debilitating prior experiences with algebra develop a process level understanding of function through appropriate instructional treatment?

Before proceeding, a definition of terms is in order. "Appropriate instructional treatment" means a beginning algebra course using text materials, as previously described, that focus primarily on function as process and that introduce function in a multi-faceted way including written definitions, graphs, function machines, tables, and equations. The word "student" refers to an adult community college student enrolled in a beginning algebra course in which the instructional treatment described above is used. As there is no typical such student, a detailed profile of students included in the study is provided.

The main research question is addressed by proposing several sub-questions as follows:

1. Will students demonstrate improved capabilities in interpreting the colloquial facet of function when asked to find output given input and vice versa?
2. Will students demonstrate improved capabilities in interpreting the symbolic facet of function when asked to find output given input and vice versa?
3. Will students demonstrate improved capabilities in interpreting the numeric facet of function when asked to find output given input and vice versa?
4. Will students demonstrate improved capabilities in interpreting the geometric facet of function when asked to find output given input and vice versa?
5. Will students demonstrate improved capabilities in the written facet by writing a definition of function in terms of a dynamic process?
6. Will students demonstrate improved capabilities in the notation facet by interpreting function notation correctly and contextually?
7. Will students exhibit consistency in their concept definition of function across verbal and written facets?
8. Will students demonstrate an ability to adapt alternative concept definitions of function into their own written and verbal definitions?
9. Is the growth of the concept image of function in students uneven? Are the cognitive links between facets sometimes nonexistent, sometimes tenuous, and sometimes unidirectional?

## 1.3 Design and Methodology

Both a pilot study and main study were completed to assess the value of the instructional treatment in light of the stated research questions. For both pilot and main study, students at four United States community colleges completed written pre- and post-course surveys (see appendices) probing their understanding of functions. Several of these students underwent one task-based interview designed to probe, in depth, their understanding of function.

The pilot study was conducted during Spring Semester, 1996. Upon analysis of the data collected, the surveys and interviews for the main study were modified. The main study was conducted during Fall Semester, 1996. Detailed results of both studies and how the pilot study influenced the main study are reported in later chapters.

Data from the pre- and post-course surveys were analysed quantitatively by pairing each student's performance on the two instruments. The interviews were video- and audio-taped, transcribed, and analysed. Using data collected from all three instruments, profiles that visually depict students' understanding of function along both breadth (facets) and depth (layers) dimensions were developed.

Sign and Wilcoxon Tests for Paired Data (Alder & Roessler, 1976), t-tests (related), and Chi-square were used to document changes in student understanding of function from the beginning of the course to the end of the course in the facet (breadth) areas of colloquial (function machine), symbolic (equation), numeric (table), geometric (graph), and notation (function notation). Systemic network charts (Bliss et al., 1983) influenced the measure of the changes along the written definition facet. In addition, task-based interviews were used to develop profiles of student understanding of function along previously described facets, but also the verbal facet. Through these interviews, detailed profiles were developed for each student subject to look at the depth the student can achieve for each facet. The profiles allowed the researcher to assess the connections between facets looking at both a student's consistencies and inconsistencies in their concept image of function with respect to the given facets. Key here is a discussion of what boundaries between facets appear to be rather porous as compared to which boundaries remain impenetrable.

## 1.4 General Conclusions

Comparison of pre- and post-course surveys indicates that a statistically significant number of students were able to demonstrate improved capabilities in:

1. interpreting a function machine both from input to output and vice versa.
2. interpreting a function defined symbolically both from input to output and vice versa.

3. interpreting a function defined by a two-column table both from input to output and vice versa.
4. interpreting a function defined by a graph both from input to output and vice versa.
5. writing a definition of function in terms of a dynamic process.
6. interpreting function notation correctly and contextually.

Based on scores on the post-course survey, the students in the main studied were divided into three groups: highly capable, capable, and incapable. The highly capable students manifested a concept image of function that was minimally at the process layer on all tested facets. The capable students demonstrated concept images that were process layer on most facets and procedure layer on at least one facet. Both these groups demonstrated significant growth in their understanding of function from pre- to post-course survey. The incapable students commonly exhibited pre-procedure knowledge on one or more facets even at the end of the course. The growth of their concept images of function during the course was minimal or nonexistent. One student from each of these three groups who participated in the interview is profiled in the discussion of the results of the main study.

Comparison of pre-course surveys, post-course surveys, and student interviews indicates that highly capable and capable students:

7. exhibit consistency in their concept definition of function across verbal and written facets.
8. demonstrate an ability to adapt alternative concept definitions of function into their own written and verbal definitions.

The interviews indicate that the growth of the concept image of function in students is complex and uneven, even among the highly capable. The cognitive links between facets is sometimes nonexistent, sometimes tenuous, and sometimes unidirectional. This is particularly true of the incapable. For example, the link between colloquial (function machine) and symbolic (equations) is often strong, but the link between the symbolic (equation) and geometric (graph) facets is weak among even the capable students. In fact, links between graphs and other facets, with the possible exceptions of numeric (tables), remain weak or non-existent for all but highly capable students. In addition, incapable students seldom interpret function notation consistently and appropriately.

## 1.5 Thesis Organization

This thesis consists of nine chapters, a bibliography, and appendices.

Chapter 1 contains an overview of the thesis. This includes an introduction, a background and statement of the problem, a brief description of the instructional framework on which the study is based, and the research questions. The procedures used for the study including the target population, general conclusions as a result of the data analysis, and this synopsis of the dissertation conclude the chapter.

Chapter 2 is a general literature review. One focus of the review is on what it means for a student to "understand" a concept and on models for the development and measurement of understanding. Factors contributing to the mathematical debilitation of students by their prior exposure to mathematics are explored. Another major review is on research on the function concept. This includes an historical overview of the development of the concept along with research studies that discuss both the development of the concept in students, but also the difficulties inherent in learning the function concept.

Chapter 3 focuses on the researcher's theoretical perspective and how this perspective is situated among past and current research. A model is presented for measuring the understanding of the function concept and the model is situated among the other major models that have been previously developed. The model will hopefully port to other mathematical concepts and is situated in the idea of the co-development of both breadth and depth when trying to understand a mathematical concept.

Chapter 4 describes the theoretical background and the key components of the instructional treatment on which this study is built. Over the past 10-12 years, many calls for reform in the teaching of algebra have been published. The researcher is co-author of a textbook for beginning algebra college students that radically alters the curriculum and emphasis of the traditional college developmental algebra course. The main theoretical perspective for these materials is that constructivism, use of technology, and multiply-linked representations will lead to better understanding in the target student population. Key to this perspective is the introduction of function as the unifying concept.

The structure and the philosophy of the text is discussed. The nature of the curriculum is described along with a brief overview of other related curriculum projects

Chapter 5 discusses the pilot study. Included will be a description of subjects for the study, the instruments used, a summary of the data, and the observations resulting from the analysis, both quantitative and qualitative, of the data.

Chapter 6 contains the specific statement of the thesis and a description of the main study. The variables for the study are described along with profiles of the participants in the study. The main study consisted of three instruments: a pre-course function survey, a post-course function survey, and task-based interviews with selected students. Each instrument will be described in detail including the aims of each section of each instrument. Discussion follows regarding how the pilot study informed the main study. Finally, the types of triangulation used to support the validity of the findings are discussed.

Chapter 7 presents the quantitative data collected during the main study along with the statistical analysis of the data. The chapter begins by introducing the subjects that participated in the main study. Pre- and post-course survey data, along with the statistical analysis, are presented with a focus on the change in understanding of function that students exhibited from the beginning to the end of the instructional treatment. Additional data collected on the post-course survey are described and analysed to conclude the chapter.

Chapter 8 summarizes the qualitative data collected during the main study along with situating these results within the theoretical framework. Three students concept images of function are profiled in detail along each of the facets. The students represent subjects from the top third, middle third, and bottom third of the participants based on their responses on the post-course survey. A brief description of each student's background is followed by an analysis of the depth of each student's understanding of each facet of function.The strength of the boundaries between facets is explored. The chapter concludes with a visual profile of each student's concept image of function.

Chapter 9 consists of reflections and possible extensions on the main study and thesis. Included is a discussion of future directions. Strengths and weaknesses of the research design are also a topic of this chapter.

The bibliography consists of a list of references used during the creation of this thesis.

Appendices include copies of instruments used during main test.

## 1.6 Conclusion

This chapter summarized the main components of the thesis. The chapter began with some background on the problem investigated in this thesis. The focus is on college students who have been previously unsuccessful at mathematics and who must take beginning algebra. Using an instructional treatment with function as a unifying concept, the thesis asks if this student population can minimally acquire a "process" understanding of function. The instructional treatment is described followed by a discussion of the theoretical framework that focuses on a model for analysing the depth and breadth of understanding of a mathematical concept. The research questions are stated and the design and methodology are summarized. The chapter concludes with a brief synopsis of each chapter of the thesis.

# CHAPTER 2 General Literature Review

## 2.1 Introduction

Students in college developmental algebra courses (beginning and intermediate algebra) have been exposed to the course material previously, often in high school. They arrive at college with, for the most part, a firm dislike of mathematics along with numerous misconceptions and overgeneralizations. It is likely that many of these students think about mathematics proceduraly and that they have been subjected to instruction that is primarily instrumental, in the sense of Skemp (1976).

As an organizational philosophy for this Chapter, the words of Sierpinska (1992) seem appropriate: "It seems to me, however, that any evaluation of a teaching design supposed to promote understanding of functions has to be based on a framework that is external to it. It must be based on a reflection about, first, understanding, and, second, functions" (p. 25). As the thesis for this research addresses the reasonableness and effectiveness of using function as an organizing concept in beginning algebra for college students, this research essentially focuses on the evaluation of a teaching design. This chapter begins with research discussing how the students arrive at college in such a "debilitated" state. The journey begins with a discussion of what it means to understand followed by information on how the brain works and how human beings learn. A discussion of the particular aspects of the students' prior mathematics experience will follow. These aspects contribute to the "debilitated" state many students entering college find themselves in. Included is a look at conceptual, procedural, and proceptual thinking along with instrumental versus relational understanding. The historical development of the algebra curriculum in the United States and its implications on the way algebra has been taught follows. The focus moves to changes that might positively impact these students approach to mathematics. Included in the changes is the use of "function" as the focal concept in their college beginning algebra course. This section begins with a review of three major theoretical perspectives on the concept of function (Sierpinska, Sfard, and Dubinsky). The chapter concludes with a discussion of some of the learning difficulties that have been associated with the function concept.

## 2.2 Definitions

Cognitive Obstacle: A piece of knowledge that in general has been satisfactory for a time for solving certain problems, and so becomes anchored in the student's mind, but subsequently that knowledge proves inadequate and difficult to adapt when the student is faced with new problems.

Cognitive Root: A concept that has the dual role of being familiar to the students and providing the basis for later mathematical development.

Cognitive Unit: Information that has been compressed in such a way as to fit into the focus of attention.

Conceptual Thinking: Thinking that focuses on relationships among objects rather than the procedures performed on the objects.

Concept Definition: The mathematical definition of a concept.

Concept Image: Everything associated in somebody's mind with the concept name. It can be mental pictures, properties, processes, mental representations, contexts of applications, etc.

Constructivism: A theory of learning, introduced by Piaget, in which knowledge is constructed by an individual in her or his mind, as opposed to being intrinsic from birth or existing independently of human interaction.

Distributed Cognition: Knowledge is part of communities and in the interaction between people and their environment.

Encapsulation: Conversion of a dynamic process into a static object.

Epistemological Pluralism: There exist multiple ways of thinking and knowing.

Instrumental Understanding: Rules without reason.

Interiorization: Translating a succession of material actions into a system of interiorized operations.

Procedural Thinking: Thinking that focuses on routine manipulation.

Procept: The amalgam of three components: a process that produces a mathematical object, and a symbol that represents either the process or the object

Proceptual Thinking: The flexible use of symbols as process and concept.

Relational Understanding: Knowing what to do and why.

Selective Construction: Reflection on the result of a procedure preceding the ability to execute the procedure. This is used in the context of computers performing the procedures and students analysing the results.

Situated Cognition: Mental representations are incomplete and that thinking is dependent on the world in which one exists.

## 2.3 Understanding

### 2.3.1 What does it mean to understand?

Schoenfeld states that "mathematics consists of systematic attempts, based on observation, study, and experimentation, to determine the nature or principles of regularities in systems defined axiomatically or theoretically (pure mathematics) or models of systems abstracted from real world objects (applied mathematics). The tools of mathematics are abstraction, symbolic representation, and symbolic manipulation" (1992, p. 344). A critical question is how to define understanding of mathematics, in light of this definition. One might consider understanding as having a sense of coherence about how we operate in the environment. Skemp (1979) suggests that: "To understand a concept, group of concepts, or symbols, is to connect with an appropriate schema" (p. 148). He discusses perceptual learning in which a concept is formed from incoming data via the senses. Once formed the concept can be used to form more complex concepts internally using a process of successive abstraction. As this unfolds, creating symbols for and naming objects serves to bring the concepts together in a schema. Skemp goes on the discuss primary concepts (derived from sensory experiences) and secondary concepts (derived from other concepts) and warns that concept formation may be affected by the frequency of contributing experiences, noise (irrelevant information), and the availability of necessary lower order concepts (ibid, p. 141). To understand is a dynamic, rather then static, process. Skemp suggests that we are constantly updating our understanding of concepts and schemas through a process of realization, assimilation, expansion, differentiation, and re-construction (ibid, p. 125).

Sierpinska has written extensively on the topic of understanding (1992, 1994) providing her view of what it means to understand a mathematical concept:

> It is only when we have seen instances and non-instances of the object defined, when we can say what this object is and what it is not, when we have become aware of its relations with other concepts, when we have noticed that these relations are analogous to relations we are familiar with, when we have grasped the position that the object defined has inside a theory and what are its possible applications, that we can say we understood something about it. (1992, p. 26)

Distinguishing between what an object is and what it is not might be considered a percept which is an amalgam of a perception and a concept. She goes on to describe acts of understanding for the function concept that will be addressed later. However, she classifies the acts of understanding into four categories that clarify her quote above. The category of identification implies the ability to recognize the object in a group of objects. Discrimination implies the ability to recognize the similarities and differences between two distinct objects. Generalization allows the extension of the use of the object. Finally, synthesis implies the existence of appropriate links among objects.

Hiebert and Carpenter (1992) suggest that knowledge is represented internally–the structure of the internal representations is vital. On the other hand, communicating mathematical ideas requires an external representation (p. 66). The student physically interacts with some external representation. So, a person will have both an internal and external representation of some mathematical concept. The nature of external mathematical representations influences the nature of internal mathematical representations and vice versa (Greeno, 1988; Kaput, 1988). How the external representation is integrated into the internal schema substantially affects how the student "understands" the concept. On the other hand, by observing how a student operates on an external representation, the researcher gains insight into how the student's internal schema is constructed.

Once the connection between internal and external representations of a single concept is acknowledged, the next step is to look at several mathematical ideas and how their internal representations might be linked. Hiebert and Carpenter state that: "...when relationships between internal representations of ideas are constructed, they produce networks of knowledge" (1992, pp. 66–67). They go on to suggest that the structure of these networks may be hierarchical or web-like (ibid, p. 67). But how do such networks become formed and what is the role of external representations? Hiebert and Carpenter suggest that one way to stimulate internal connections is by building connections between external representations. In much curriculum reform today (Harvard Consortium calculus project, the American Mathematical Association of Two-Year Colleges Standards, the National Council of Teachers of Mathematics Standards), recommendations related to connections are abundant. Focusing on the function concept, reform curricula commonly emphasize drawing connections between symbolic, graphic, and numeric aspects of function (See the Rule of Three in Hughes-Hallett et al. 1992, for example). Unfortunately, a number of researchers (Sfard,1992, 1995; Cuoco, 1994, for example) have shown that these connections can be notoriously difficult to make. This research will discuss both successes and failures in making connections within the function concept.

With the above background, Hiebert and Carpenter go on to define understanding.

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (1992, p. 67)

Hiebert and Carpenter continue by noting that understanding will grow as the networks increase in size and organization (ibid, p. 69). It is important to note that there is no agreement that understanding can be described in terms of internal representations. Schoenfeld states: "The mainstream idea is that humans are information processors and that in their minds humans construct symbolic representations of the world" (1992, p. 349). But conflicting theories coexist. Pea (1989) argues for **distributed cognition** that knowledge is part of communities and in the interaction between people and their environment. Alternately **situated cognition** theorists argue that the mental representations are incomplete and that thinking is dependent on the world in which one exists. (See Brown et al., 1989, for example). Another theory of cognition that allows for multiple ways of thinking and knowing is **epistemological pluralism** (Turkle & Papert, 1992). Turkle and Papert categorize, and document different styles of thought. They use the words "hard" and "soft" to differentiate two distinct approaches to understanding.

> The ideal typical hard and soft approaches are each characterized by a cluster of attributes. Some involve organization of work (the hards prefer abstract thinking and systematic planning, the soft prefer a negotiational approach and concrete forms of reasoning); other attributes concern the relationship that the subject forms with computational objects. Hard mastery is characterized by a distanced stance, soft mastery by a closeness to objects. (ibid, p. 9)

They go on to suggest that closeness to objects "favours contextual and associational styles of work" (ibid, p. 10) suggesting a more web-like internal structure. The hards, on the other hand, may lean more toward hierarchical connections. In summary, the researcher admits that several approaches to cognition coexist, but that, ultimately, understanding will be viewed in terms of the richness of the internal connections between mathematical ideas. Such connections may be influenced by one's environment or by one's cognitive style.

Regardless of the mode of cognition, the mental representations and connections that a student creates become the basis for future mathematical understanding. The connections may be superficial reflecting no true understanding. Hiebert and Carpenter state that: "Connections that are weak and fragile may be useless in the face of conflicting or nonsupportive situations. Understanding increases as networks grow and as relationships become strengthened with reinforcing experiences and tighter network structuring" (1992, p. 69).

On the other hand, incorrect or inappropriate connections may be strong and resistant to change. Hiebert and Carpenter suggest that "...existing networks influence the relationships that are constructed, thereby helping to shape the new networks that are formed" (ibid, p. 70) implying that prior knowledge could be detrimental. If the learner tries hard to fit a new idea into his/her thought structure, the existing networks affect the relationships that are created. If the learner cannot fit the knowledge into the network at all, the learner may represent new information in a way that does not connect it with existing networks. This results in a lack of connections between old and new. It also explains the lack of connections between various forms of functions, for example.

Building understanding requires a substantive change in the internal cognitive network that may be manifested by a change in the size of the network or in a restructuring of the network. Hiebert and Carpenter suggest that "...the most likely scenarios for building understanding involves increases in either the size or the structure of networks, these processes both building on existing networks" (1992, p. 70). Crucial to both is the need to establish meaningful connections between the old and the new. In many cases, particularly in people who have been unsuccessful with mathematics, the old must be restructured since the new knowledge suggests misconceptions in the old. However, if the new knowledge is supportive of the old, the network strengthens.

(Baddeley, 1976; Bruner, 1960; Hilgard, 1957 as cited by Hiebert & Carpenter, 1992, p. 75) The failure to establish meaningful connections may be a significant part of developmental algebra students' problems with mathematics. It is likely that their knowledge is procedural, rather than conceptual, resulting from years of instrumental instruction using a layer-cake curriculum. This is the topic of a later section of this Chapter.

### 2.3.2 Understanding and the brain

In *Biological Brain and Mathematical Mind* (in preparation), Tall discusses how the brain's operational mode influences how we learn and understand mathematics. The accumulation of a larger and larger collection of schemas will not produce a creatively thinking mind unless they are constructed in a manner that allows them to not only be performed but also to be thought about. Here we see the idea of the existence of mental representations and their connections, but, in addition, the idea of being able to think about the processes. This suggests the idea of a **procept** (Gray & Tall, 1994), the amalgam of process, concept and symbol, which will be discussed extensively later.

Crick (1994) writes that the brain as a multiply-processing organ can only make conscious decisions by suppressing data, thus focusing on a limited quantity of data. This may not involve a specific focus in a specific place in the brain, but a facility for focusing on data. The brain must maximize its efficiency. A primitive way of doing this is to practice a routine sequence of actions that can be performed sequentially, focusing on one step at a time, whenever the right cues trigger it off. Such a strategy is used in procedural rote-learning of mathematical techniques. To be able to do mathematics and to think about it requires, first, that the information be compressed in a form that will fit in the focus of attention. Information that has been compressed in such a way as to fit into the focus of attention will be called a **cognitive unit**. Second, **conceptual links** must be present to relate the current cognitive unit to other appropriate cognitive units stored in long-term memory.

Tall (in preparation) writes that a coherent sequence of actions, such as see-grasp-suck, is called an action schema by Piaget. Such action schema are the basic building blocks of human perception and action. A specific schema of action, or in other words, an algorithm consisting of specific steps is called a **procedure**. The way in which a schema of action (procedure) becomes a mathematical **process** (cognitive process not dependent on individual steps, only on its input and output) and then a concept is crucial the development of many mathematical concepts that are action-based. Some of the ways that function is conceptualized, such as an equation in two variables, are action-based. On the other hand, some mathematical ideas are object-based. Tall (1995a) suggests that their development is quite different in which the visuo-spatial transforms into the verbal-deductive as understanding of the concept grows. Some conceptions of function, such as a two-dimensional graph, may indeed follow this path of development.

Turning to the way the brain represents mathematical ideas leads to the idea of a **concept image**. Tall and Vinner write:

> We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes.... The concept definition [is] a form of words to specify that concept. (1981, p. 152)

Vinner (1992) suggests that the concept image is an entity associated in one's mind with the concept name. To acquire a concept, Vinner suggests that a concept image is formed for the concept's name. He writes: "To understand a concept means to form a concept image for it" (1992, p. 197). This researcher disagrees since the concept image formed may be riddled with misconceptions. A student whose concept image of function consists of the requirement that the graph is a straight line exhibits questionable understanding of function. Vinner notes that the concept image is shaped by examples, resulting in the creation of prototypes (Tall & Bakar, 1992). Vinner goes on to discuss what students might access when working on a task. He suggests that students might access their concept image only, their concept definition only, both their concept image and concept definition (in various orders), or neither. He provides a series of diagrams illustrating how students, receiving input (a cognitive task), might process the task in terms of their mental constructions to produce output (an intellectual behaviour or answer) (1992, p. 199).

The concept image for a particular idea may develop over a lengthy time period. It is likely that such an image will contain many cognitive units and many conceptual connections. This complements the development of the brain nicely. Tall (in preparation) writes:

> What is important is to underline the nature of the biological brain and its profound difference from the order and logic of the mathematical subject which it has developed. Mathematical concepts, on the surface, may seem to be neat and well-organized, but, underneath, in the workings of the brain, all sorts of conflicts and confusions occur. It is for this reason that it is possible to have conflicting portions of concept image. It is only when two conflicting methods are evoked at the same time that the conflict may become apparent.

A key is what happens when a conflict within the concept image surfaces. Among the possibilities are either a reconstruction of the image or a severing of the connections so that the mis-conceptions can co-exist. The latter may be a prescription for failure. Tall suggests: "Success in mathematics involves developing meaningful concept images which connect together in useful ways, Wherever possible it means confronting conflicts squarely and attempting to construct a resolution. Once the child acquiesces and simply accepts the new without reconstructing the old in a way which relates meaningfully to the new context, a step has been made on the slippery slope of failure" (ibid). Such behaviour is commonly observed in students. For example, students often overgeneralize the distributive property so that $(a+b)^n = a^n + b^n$. Developmental algebra teachers will confront such students with the fact that $(a+b)^2$ is equivalent to $(a+b)(a+b)$ and thus equal to $a^2 + 2ab + b^2$. Instead of reconstructing their understanding of the distributive property, students are likely to file the fact that $(a+b)^2 = a^2 + 2ab + b^2$ away as a disconnected cognitive unit soon to be forgotten. When asked to expand $(a+b)^3$, such students respond $a^3 + b^3$.

Developing true mathematical power requires the creation of a cognitive network that complements the working of the brain. One aspect requires sufficient compression of ideas to fit in transient memory. The other requires the necessary connections to ideas in long-term memory so that the focus of attention of the transient memory can be switched efficiently. Tall writes:

> This can be done by two complementary processes: compression to configure data to make it appropriate to fit in the focus of attention; extension to form conceptual linkages with other data to enable it to be brought quickly into the focus of attention as required. In this way, although the short-term memory remains essentially unchanged

> in capacity, the potential working memory—consisting of the conscious focus of attention and all other concepts and processes linked to it—is increased enormously. (ibid)

The use of written symbolism is a powerful means of compression. The symbol "$f(x)$" carries with it the symbol for function, the idea of a process, the concept of function, the name of a function, the variable used for input to the function, and the output of the function. One such symbol captures in a cognitive unit significant mathematical power–the power of a procept.

Once compression into cognitive units has occurred, the focus turns to the establishment of meaningful conceptual connections. Tall discusses the value of procedures and the power of symbols by referring to the visually moderated sequences of Davis.

> If the symbols have an appropriate meaning, the procedure is usually carried out through what Davis (1984, p. 135) calls a "visually moderated sequence." By this he means a sequence of actions where a collection of symbols are written down to be visually scanned and operated on, then the new symbols are operated on; etc. At each stage the symbols are written down and a new decision is taken to operate and move closer to a solution. In this way an individual uses a combination of short-term focus to scan the written material, links with the solution process in long-term memory to decide what to do next, takes the required action by operating in the short-term focus, and then moves to a set of written symbols which are hopefully nearer the solution. (ibid)

He warns, however, against purely procedural methods: "A common problem with procedural thinking is that it only works if the problem is given in recognizable form. Although procedures, which grow out of natural action schemas are an essential part of mathematics, naive practice of "the basics," without a broader conceptual understanding of relationships to bring relevant information into the focus of attention, is a recipe for disaster" (ibid).

It appears that the number of conceptual connections is not a good indicator of a person's understanding. Tall points out that the quality and generality of the links is a key. He cites Krutetskii (1976) when he writes: "Those who grow to succeed compress information, curtail solution processes and evolve links at a higher general level, using the compressed general concepts that they have at their disposal (See Krutetskii, 1976 for example). Those who continue to fail remain working with the detail, attending to irrelevant factors, unable to develop the higher conceptual structures that characterise mathematical thinking (in preparation).

Finally, the quality and sophistication of the conceptual network are improved by reflective thinking. Tall writes: "To develop a sophisticated conceptual structure requires the ability to think about one's knowledge, to reflect on how and why it works and to use this knowledge to move on to solve new problems" (ibid). Thus, just learning procedures and doing *n* problems using these procedures is insufficient for developing the necessary connections. A focus solely on procedures may indeed create a set of disconnected units that can only be accessed in specific instances. No network is created. The links to long-term memory are non-existent. The symbols have little or no meaning. No transfer can take place to other mathematical situations. The student is stuck with a growing list of disconnected facts and, as the number of disconnected units grow, so does the strain on the student. Eventually, a collapse occurs and the student becomes a victim of the proceptual divide.

### 2.3.3 Conceptual and procedural knowledge

Hiebert and Lefevre (1986) define **conceptual knowledge** as "knowledge that is rich in relationships" (p. 3). On the other hand, procedural knowledge is "composed of the formal language, or symbol representation system... [and] the algorithms, or rules, for completing mathematical tasks" (ibid, p. 6). The implication is that conceptual knowledge represents learning with understanding. Procedural knowledge can be stored in memory as isolated facts with few, if any, connections to other relationships. We know, however, that some procedural knowledge is valuable, as discussed in the previous section. Hiebert and Lefevre furthermore assert that procedural knowledge is meaningful only if its is linked to a conceptual base. Hiebert and Lefevre write:

> ...relationships between conceptual and procedural knowledge depend on the connections learners construct between their representations. From an expert's point of view, procedures in mathematics always depend upon principles represented conceptually. If the learner connects the procedure with some of the conceptual knowledge on which it is based, then the procedure becomes part of a larger network, closely related to conceptual knowledge. Procedures connected to networks gain access to all information in the network. (ibid, p. 78)

We see in this statement the importance of connected cognitive units. Too often students learn procedures to solve specific problems. These procedures, due to a layer-cake curriculum and instrumental instruction (discussed in the next section), remain disconnected from other parts of the student's mental network thus creating no real understanding.

Tall (in press) suggests that conceptual knowledge can be thought of as a connected web of knowledge, a network in which the linking of relationships is as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked in some network. In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of the conceptual knowledge only if the holder recognizes its relationship to other pieces of information. (Tall, in preparation)

Conceptual knowledge can be difficult to achieve. Creating conceptual knowledge requires the juggling of multiple cognitive units and the conceptual connections between them. Tall writes: "A learner with inadequate images to cope with the new knowledge may find such an approach too demanding of the limited processing power of his or her focus of attention and seek the comfort and security of formal procedures" (ibid). At this point, the learner has opted for acquiescence to memorizing disconnected facts and is ultimately headed for collapse in their mathematical network.

### 2.3.4 Relational and instrumental understanding

According to Skemp (1976), **relational understanding** is demonstrated by knowing what to do and why while **instrumental understanding** can be thought of as rules without reason. Instrumental understanding often involves a multiplicity of rules rather than fewer principles of more general application. Skemp (1976) writes: "Instrumental understanding necessitates memorising which problems a method works for and which not, and also learning a different method for each new class of problems" (p. 23). The advantages of relational understanding include improved adaptability to new tasks and less dependence on memory. Relational understanding promotes the building of effective connections internally. Skemp makes this point emphatically.

> The kind of learning which leads to instrumental mathematics consists of the learning of an increasing number of fixed plans, by which pupils can find their way from particular starting points (the data) to required finishing points (the answers to the questions). The plan tells them what to do at each choice point, as in the concrete example. And as in the concrete example, *what has to be done next is determined purely by the local situation*.... There is no awareness of the overall relationship between successive stages, and the final goal. And in both cases, the learner is dependent on outside guidance for learning each new 'way to get there'.... In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. (ibid, p. 25)

In discussing the decisions students must make when learning instrumentally, Skemp alludes to the local situation. In the process, we see no significant connections being made in the learner's internal representation. The facts are isolated-discrete bits of information that are only useful if the problem situation is replicated. The knowledge proves useless when the student is confronted with a slight variation in the problem situation. Hiebert and Carpenter lend support when they write: "...learners who possess well-practised automatized rules for manipulating symbols are reluctant to connect the rules with other representations that might give them meaning" (1992, p. 78). The irony for college student in developmental algebra courses is that the best strategy for them to develop understanding is to let go of the old strategies-to unlearn, to break the unwanted connections. Unfortunately, as Heibert and Carpenter suggest, this is a very difficult task! "Students are reluctant to give up familiar strategies, especially if they do not understand them and thus do not recognize their inadequacies" (ibid, p.79). Tall (in preparation) writes: "Forgetting inappropriate connections is a valuable part of learning because it refines the cognitive structure by removing unnecessary clutter." This follows Skemp (1971) who suggested that the making of new connections is an expansion of current knowledge, but the losing of old connections requires a reconstruction of knowledge. Hiebert and Carpenter go on to discuss what must be done instructionally. "Teaching environments should be designed to help students build internal representations of procedures that become part of larger conceptual networks before encouraging the repeated practice of procedures" (1992, p. 79). One such environment and its influence on students' learning of function is a focus of this research.

### 2.3.5 Process-product dilemma

Davis (1975) writes about the **process-product dilemma** that regularly confronts the users of mathematics. He suggests that one difficulty transitioning from arithmetic thinking to algebraic thinking stems from the fact that $3 + 5$ is a question in arithmetic, but in algebra "is both an indication of a process and also a name of the answer" (ibid, p. 18). Davis foreshadows the idea of the proceptual divide by suggesting that the cognitive demands in learning algebra include the ability to use symbols "to indicate both a process and also the result that will be obtained when one carries out the process" (ibid, p. 28). Generally, mathematical symbolism often stands for both a process and a product. The expression $2x + 3$ can be thought of as the process multiply an unknown number by 2 and add 3. It may also be considered a product in that it is an algebraic entity in its own right as determined by context: an expression in one context, a function in another context, to name two. Successful mathematicians do not give a second thought to this "dilemma". They easily are able to see mathematical symbols ambiguously: as both processes and products. Unsuccessful students, however, have great difficulty dealing with this ambiguity. While the researcher observed a basic algebra class, the class discussed the situation in which a store was offering a 10 percent discount and charging a 6 percent tax. The students were asked if it was better to receive the discount first or to be charged the tax first. The discussion proceeded to the point where $p$ was being used for the list price of an item and that a person would pay either $1.06(0.9p)$ or $(1.06p)0.9$. One student said she could not understand the second expression. She said that she didn't know how one could multiply by 0.9 until the value of $p$ was known. She viewed $1.06p$ as a process that required an answer before another process could be executed. She could not see the same expression as a product (or object, as Dubinsky might call it) that could be operated on in its own right. She was unable to get past this process-product dilemma. The explanation of the answer thus made little sense to her.

### 2.3.6 Procedural and proceptual thinking

Previous sections have laid out some basic theory on how the brain assimilates mathematical ideas. The focus in this section is on the problems students encounter when trying to learn mathematics. The researcher notes how instrumental instruction often results in procedural learning in which students acquire a large set of disconnected procedures for dealing with mathematical problems. When this collection becomes unmanageable, the result is the creation and resulting widening of the **proceptual divide** until these students are "debilitated". They enter college with at least one and often more year of high school algebra, but, based on placement results, end up in a beginning algebra course that carries no college credit. The mathematical preparation that led to this problem is the detailed of this section.

Tall (in preparation) discusses how the brain operates to handle mathematical processes suggesting that the brain has evolved in such a way as to create links between successive actions. These then may become routine and, thus, they may either become a cognitive unit or may form links between cognitive units. This may seem to suggest that routine practice of skills is the key to learning mathematics. On the contrary, while some practice of skills seems to be important, another key aspect of learning is the ability to use symbols to compress ideas into cognitive units. The value of compression has been well-documented by Krutetskii (1976). Students who are able to compress mathematical ideas are able to fit richer cognitive units into their focus of attention. For the "less able" student, the lack of compression forces a more difficult mathematics on him or her as suggested by Tall when he writes: "The methods available to so-called slow learners actually force them to learn slower still. The slowness of moving on to more efficient ways places even greater burdens on those who are finding mathematics difficult. The mathematics that they do is not properly compressed, will not fit into their focus of attention, and becomes even more difficult still" (in preparation).

This description seems to profile many of the learners in a college beginning algebra class. It is likely that many of the students that are the subject of this research evidence are primarily procedural thinkers whose strategy in previous mathematics courses has been to study examples and mimic the steps from the examples on other problems. Their cognitive units consist of small bits of actions on objects and, often, these objects have little meaning. There are few, if any, conceptual connections that allow students to compress the actions into cognitive units. Tall (1995a) writes: "However, if the mathematics places too great a cognitive strain, either through failure to compress or failure to make appropriate links, the fall-back position resorts to the more primitive method of routinising sequences of activities —rote-learning of procedural knowledge" (p. 67). These students are victims of the **proceptual divide**, as discussed in Gray and Tall (1994). Gray and Tall argue that more able students do qualitatively different mathematics from the less able students. The more able have knowledge that is capable of generating new knowledge. The less able actually are prone to failure because they are doing more difficult mathematics than the more able (Gray, 1991).

Gray and Tall (1994) go on to define **proceptual thinking** as a combination of *conceptual* and *procedural thinking* (p. 122). Earlier in the paper, they refer to **procedural thinking** as routine manipulation of objects. They write: "Procedural thinking is characterized by a focus on the procedure and the physical or quasi-physical aids that support it" (p. 132). Their definition of conceptual thinking relies on the quality of the cognitive connections linking cognitive units. They quote Hiebert and Lefevre (1986) who wrote that "...by definition [a piece of information] is part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information" (pp. 3–4). The flexible thinking based on this definition of conceptual knowledge, as opposed to thinking based on procedures, is **conceptual thinking**. Finally proceptual thinking depends on the ability to compress stages of the symbol manipulation so that the symbols can be viewed as objects. Gray and Tall later write:

> Proceptual thinking includes the use of procedures. However, it also includes the flexible facility to view symbolism either as a trigger for carrying out a procedure or as the representation of a mental object that may be decomposed, recomposed, and manipulated at a higher level. This ambiguous use of symbolism is at the root of powerful mathematical thinking and makes it possible to overcome the limited capacity of short-term memory. It enables a symbol to be maintained in short-term memory in a compact form for mental manipulation or to trigger a sequence of actions in time to carry out a mathematical process. It includes both concepts to know and processes to do. (1994, pp. 125–126)

Here we see the procept carrying the burden of becoming the cognitive unit. Each aspect of the procept is accessible at appropriate times using conceptual links. Proceptual thinkers are doing a qualitatively different kind of mathematics that is easier for them. The divergence, the "bifurcation of strategy," between those who interpret processes only as procedures... and those who see them as flexible procepts is called the *proceptual divide* (Gray & Tall, 1991). In essence, the more able depend on procepts while the less able depend on procedures. If the cognitive strain on the individual grow too great, it may be that someone, previously successful, founders and asks "tell me how to do it," anxiously seeking the security of a procedure rather than the flexibility of a procept. From this point on, long-term failure is almost inevitable (Gray & Tall, 1991), even though this person may be able to do the procedures for short-term success. If someone who has turned to procedural thinking finds security, then additional practice of those procedures does little more than widen the proceptual divide.

Support for the concept of the proceptual divide is found in Krutetskii's (1976) study of the mathematical abilities of schoolchildren. Krutetskii studied the mathematical abilities of gifted, capable, average, and incapable students. He found that the gifted curtail solutions so they work in a few steps, remember high level strategies, and essential information. Capable and average students only curtail after practice. The incapable remember only incidental information that does not generalize. Rather than being creative and rich with conceptual links, their knowledge is isolated and imitative. According to Krutetskii (1976):

> There are three basic stages of mental activity in solving a mathematical problem: gathering information needed to solve the problem, processing the information so as to obtain a solution, and retaining information about the solution.... Pupils who are especially capable are better able to grasp the essence of a problem at once than are less capable students. The capable students can generalize mathematical material rapidly and easily; they tend to skip over intermediate steps in a logical argument, to switch easily to another solution method, and to strive for an "elegant" solution where possible; and they are easily able to reverse their train of thought if necessary. Finally, capable students tend to remember the relationships in a problem and the principles of a solution, whereas less capable pupils tend to remember only specific details, if anything, about a problem. (p. xiv)

Krutetskii notes the average and incapable pupils at first perceive only disconnected facts when presented with a problem; they focus on the concrete data from the outset. Incapable students tend to be debilitated by the memory of a previous procedure for solution that exerts an inhibiting influence when a process needs to be constructed for a new problem. He writes:

> Incapable pupils are characterized by a poor memory for generalized mathematical material, abstract mathematical relations and symbols, particular problem types, patterns of reasoning and proofs, and generalized methods of problem-solving. Incapable pupils usually recall only specific numerical data or specific facts about a problem....

> In doing examples, the gifted see the shape and focus of the essentials, the capable focus on the details, which they can do, but they lack any sense of more global structures thus leading them to become procedural. (ibid, p. 299)

We see in this statement the distinction between procedural thinking and proceptual thinking. As noted, earlier, the incapable and average are doing much harder mathematics than the gifted or even the capable. The listing of the various differences between the capable and the incapable litters the abyss created by the proceptual divide.

One other reference is worth noting before moving on. The concept of conceptual preparation was introduced by Ali and Tall (1996) when they studied the various methods students used to differentiate and integrate. Instead of the most gifted student producing curtailed solutions with fewer steps, there was little correlation between the level of mathematical ability and the number of steps used in the completion of an algorithm. Instead the better students exhibited more stable knowledge structures and were more likely to use conceptual preparation prior to applying a standard algorithm. For example, when asked to find the derivative of $\frac{1+x^2}{x^2}$, better students were more likely to exhibit a conceptual preparation by performing the differentiation on the equivalent form $\frac{1}{x^2}+1$ causing a simpler algorithmic procedure. Ali and Tall conclude:

> when problems are designed which can be simplified by an initial conceptual preparation, the more successful students are more likely to conceptually prepare than the less successful students. With problems where the preparation involves using a more specific method that is shorter or a generalisable method which happens to be longer, the more successful students are likely to be aware of the alternatives, some using the shorter method, some preferring the more general method and having confidence in their ability to carry out the manipulation. (ibid, p. 2-26)

These results support the difference in doing mathematics between those who think proceduraly versus those who think proceptually. The conceptual connections that allow a student to recognize an equivalent form contribute to the student's overall mathematical understanding.

## 2.4 Factors Contributing to the Creation of "Debilitated" Students

### 2.4.1 The algebra curriculum

The standard U. S. high school algebra curriculum must carry some blame for the misunderstandings and confusions college developmental algebra students have. Kaput (1995) writes:

> School algebra in the U. S. is institutionalized as two or more highly redundant courses, isolated from other subject matter, introduced abruptly to post-pubescent students and often repeated at great cost as remedial mathematics at the post secondary level. Their content has evolved historically into the manipulation of strings of alphanumeric characters guided by various syntactical principles and conventions, occasionally interrupted by "applications" in the form of short problems presented in brief chunks of highly stylized text. All these are carefully organized into small categories of very similar activities that are rehearsed by category before introduction of the next category, when the process is repeated. The net effect is a tragic alienation from mathematics for those who survive this filter and an even more tragic loss of life-opportunity for those who don't. (p. 71)

Davis (1989) provides a similar indictment when he writes: "Our biggest problem is that the wrong kind of learning experiences are being aggregated into the wrong kinds of courses, aimed at producing the wrong kind of mathematical 'knowledge'" (p. 117). Later he states:

> From my perspective, these courses have the following structure: The students is asked to perform some fragmentary piece of a ritual. The student sees no purpose or goal to this activity, other than extrinsic goals (such as pleasing the teacher) or competitive goals (such as doing it better than Joey does). Consequently the student sees no reason why the ritual is performed one way and not another. The theory underlying such courses seems to be: If the students spend enough time practicing dull, meaningless, incomprehensible little rituals, sooner or later something WONDERFUL will happen. I have never shared this optimism. (pp. 117–118)

A look at the historical development of the algebra in the U. S. sheds further light on the problem. Philipp et al. (1993) provide a detailed historical perspective on the forces that have shaped the algebra curriculum. They cite Jones and Coxford (1970) who report that algebra joined the high school curriculum in the first half of the 19th century due to pressure from Harvard, Yale, and Princeton who listed algebra as a required course for entrance. The procedural-oriented college algebra course was ported, virtually unchanged, to the high school supported by the theory of faculty psychology. Philipp et al. write:

> The theory presumed the existence of a few discrete faculties in the mind, including memory, imagination, observation, will, and reasoning. It was believed by mental disciplinarians that the curriculum should include those topics that best developed such faculties of mind. Mathematics was high on their list, because memorizing tables would develop the capacity of memory, constructing proofs would develop reasoning, and solving a lot of tedious exercises would develop the will. (1993, p. 244)

The curriculum remained essentially the same until the 1890s. At that time, Philipp et al. cite Osborne and Crosswhite (1970) who state that a survey of mathematics teachers in 1890 indicated dissatisfaction with the emphasis on manipulation. During the 1890s, national and international organizations began to call for reform, moving away form the emphasis on symbol manipulation in algebra. The theory of faculty psychology was seriously challenge opening up the possibility for a more concept-oriented curriculum. Finally, there were calls to unify the mathematics curriculum around the function concept. In spite of all this, the curriculum changed little. Between 1910 and 1920, the theory of social efficiency replaced the approach to the curriculum promoted by the mental disciplinarians. Philipp et al. write: "Mental discipline dominated curriculum theory of the 19th century offering support for a proceduraly oriented algebra curriculum that made few attempts at developing meaning. The social efficiency model, which dominated between the two world wars, stressed multiple curricula for different segments of the population, depending on career goals" (ibid, p. 247). Neither theory has completely disappeared to date.

The historical development of algebra provides a contrasting viewpoint to the algebra defined by the U. S. high school curriculum. Sfard (1995) presents a detailed survey and interpretation of the development of algebra from both historical and pyschological perspectives. She argues for the value in this since "difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians" (ibid, pp. 15–16). She looks at the development of algebra hierarchically moving from a mathematical idea being initially conceived operationally, and then reified (converting computational operations into object-like entities) into an object arguing that resistance to new objects may "stem from the inability to reify a process" (ibid, p. 16). By an operational viewpoint, Sfard means that a mathematical idea is "conceived as a computational process rather than as a static construct" (1992, p. 60). Treating mathematical ideas as if they are object-like entities represents a structural viewpoint, according to Sfard (1992). Kieran (1992) cites three stages through which algebra developed: rhetorical, syncopated, and symbolic. The first stage, rhetorical, is "characterized by the use of ordinary language descriptions for solving particular types of problems and lacked the use of symbols for special signs to represent unknowns" (p. 391). The second stage was initiated by Diophantus (c. 250 A.D.), who introduced the idea of using letters for unknowns. Vieta (1540–1603) extended Diophantus' idea by using letters to represent givens as well as unknowns. This development ushered in the symbolic stage (See Harper, 1987 for more information on these stages). Kieran concludes: "Vieta's invention of an extremely condensed notation permitted algebra to be more than merely a procedural tool; it allowed the symbolic forms to be used structurally as objects" (ibid, p. 391). Sfard argues that, while the move to symbolic algebra promotes a structural, rather than operational, viewpoint, students with several years of algebra tend to use rhetorical or syncopated algebra rather than symbolic algebra. She continues by arguing that operational conceptions precede structural conceptions, and reification (the transition form operational to structural) is quite difficult. Vieta's invention signalled the move of algebra from operational to structural. Continuing discomfort with the notion of variable eventually resulted in complete dearithmetization of algebra by Peacock (1891–1858), among others, who proposed that a variable be treated as a thing in itself, rather than a generalized number. The result is the "severing of a mathematical idea from its operational origins in order to attain full reification" (Sfard, 1995, p. 29). In a study of students' beliefs about the meaning of symbols and manipulations by Linchevski and Sfard (1991), most students viewed "algebraic expressions as meaningless symbols governed by arbitrary established transformations" (1995, p. 30). While, on the surface, reflecting surprising mathematical maturity, these results suggest that "a student may become quite skilful in manipulating such mathematical objects as number, function, or algebraic expression even without reifiying them" (Sfard, 1995, p. 35). Sfard argues that algebra, as taught in school, is presented structurally when the students really need a treatment that moves them from operational to structural carefully. She paraphrases Picasso in writing: "in mathematics a pupil should be a Platonic realist before turning into a formalist and being able to deal with pure abstraction brought into being by stipulation" (ibid, p. 30). With respect to curriculum and teaching strategy, she states two principles: "New concepts should not be introduced in structural terms. A structural conception should not be required as long as a student can do without it" (1992, p. 69). This historical review concludes with a quote that summarizes the obstacles confronting students when introduced to algebra.

> Thus, the cognitive demands placed on algebra students include, on the one hand, treating symbolic representations, which have little or no semantic content, as mathematical objects and operating upon these objects with processes that usually do not yield numerical solutions, and, on the other hand, modifying their former interpretations of certain symbols and beginning to represent the relationships of word-problem situations with operations that are often inverses of those that they used almost automatically for solving similar problems in arithmetic. It took centuries for the field of algebra to undergo these developments. Yet students beginning their first algebra course are expected to reify (Sfard, 1991) algebraic representations almost immediately. (Kieran, 1992, p. 394)

With this in mind, the next section focuses on the students in a college developmental algebra course who have been "debilitated" by this approach to the algebra curriculum.

### 2.4.2 "Debilitating" prior experiences with algebra

In light of student difficulty in moving from operational to structural conceptions or, likewise in moving from procedural to proceptual thinking, it is not difficult to see why so many students succumb to the proceptual divide. Due to instructional pressures and students' inability to think proceptually, they resort to memorizing so many procedures with so little understanding that algebra is a mishmash of disconnected procedures. For example, when confronted with an algebraic expression such as $2x + 3$, many will set the expression equal to zero and solve for $x$. Wagner, Rachlin, and Jensen (1984) found that many students try to add "= 0" to any expression they were asked to simplify. Kieran (1983) also found that students could not assign meaning to a variable in an expression because the expression was not set equal to something. Students feel they must perform a procedure when given an algebraic expression, rather than seeing the expression as an object in its own right. As Tall states, "... the difficulties that an average college student has with algebra occur because of previous rule-bound approaches to the subject. When students do not understand what something is, at least they can get temporary success by becoming secure with procedures to do things with it" (1992a, p. 3).

Beginning college students prior mathematical experiences often have focused on the procedural aspects of mathematics–routine manipulation of objects. Mathematics is a series of procedures to factor polynomials, solve equations, solve inequalities, simplify radicals, and add rational expressions, to name a few. Little time is spent on foundation concepts such as variable. A quick look at any standard textbook reveals one sentence devoted to the concept of a variable, even though Usiskin (1988) documents at least six different meanings of variable. The overemphasis on mathematics as procedures to follow forces these students to do harder and harder mathematics than their counterparts who are thinking proceptually.

These students have an aversion to contextualized problems, though the type of contextualized problem they have been exposed to are worthy of disdain. Typical contextual problems involve determining how long it will take two planes to be 500 miles apart or how long it will take to repair one of the planes once it lands if you know the average rates of work for repair persons. School mathematics has little relationship with the problem solving students must do in their everyday lives. The students view mathematics as a set of procedures to follow that have little or no relationship to anything beyond the classroom. They are "alien" in the sense of Duffin and Simpson (1995). Few have ever had the need to simplify a radical, solve a quadratic equation, or evaluate an algebraic expression outside of the context of a mathematics class.

Sierpinska (1992) discusses some epistemological obstacles that are present when students encounter the concept of function. She defines an epistemological obstacle as an obstacle that "... is not just ours or maybe a couple of other people's, but is more widespread, or has been widespread for some time or in some culture" (ibid, p. 28). The first obstacle she cites is: "Mathematics is not concerned with practical problems" (ibid, p. 31). It is exactly this obstacle that has predominated the textbook presentations of algebra and that has led students to mindless symbol manipulation. To such students, mathematics has little use beyond the classroom walls.

Helping students develop mathematical literacy and competence based on a foundation of previously-learned mathematics that lacks an understanding of concepts such as order of operations, variable, function, and equation is a major hurdle. These students have a well-defined view of the role of the teacher and the student. The teacher's role is to lecture with examples followed by assignments of 40–50 problems just like the examples. The student's role is to mimic the examples when doing the homework problems. Unfortunately the students develop little or no understanding of the concepts. Furthermore, no connections are constructed among related ideas. As stated by Dubinsky (1991), "... imitation and memorization do not lead to cognitive constructions and the result is that students' desire to learn through growth is suppressed" (p. 120). Students who enrol in these courses have mixed feelings of resentment and antagonism. They expect poor performance and have high test anxiety. Such students see no use for algebra outside of the course and are unable to apply algebraic concepts to problems beyond the classroom.

Another major difficulty is building proceptual understanding on a foundation of procedural learning over a period of twelve years of education. Students have constructed extensive internal networks around such concepts as variable and equation that are loaded with misconceptions. The researcher asked students at the beginning of an intermediate algebra course to describe their understanding of variable. The "best" answer was that a variable is a letter that can be replaced with a number. The majority could provide no explanation of what a variable is. Students were also asked to define "equation". Their examples included $2x+3$ and $x+5>7$. Again, most could provide absolutely no statement or example of what an equation is. These are students that have taken, at the minimum, one year of algebra prior to this course. As instructors begin to deal with such concepts at college they realize all too well that students' "existing networks influence the relationships that are constructed, thereby helping to shape the new networks that are formed" (Hiebert & Carpenter, 1992, p. 70). Papert (1980) alludes to the problems that may arise when new knowledge conflicts with old.

> New knowledge often conflicts with the old, and effective learning requires strategies to deal with such conflict. Sometimes the conflicting pieces of knowledge can be reconciled, sometimes one or the other must be abandoned, and sometimes the two can both be "kept around" if safely maintained in separate compartments. (p. 121)

Unfortunately, it seems that many students who have previously taken algebra keep old misconceptions at the expense of new, correct conceptions or keep both around in their mind. So a college student in beginning algebra often has an extensive network of knowledge about algebra when he or she begins the course. If the student's existing networks were sound, the prior experience positively influences student learning. It appears that many students' existing networks contain aspects of **concept images** that conflict with the **concept definition**. There appears to be some disagreement as to whether the concept definition is a part of, or separate from the concept image. Dubinsky and Harel (1992) define concept image in their glossary as follows: "Everything associated in somebody's mind with the concept name which is not the concept definition. It can be mental pictures, properties, mental representation, contexts of applications and even statements" (p. 17). In this case, the concept image is separate from the concept definition. However, Tall and Vinner (1981) define concept image somewhat differently: "We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (p. 152). Tall and Vinner consider the concept definition to be part of the concept image. It is still entirely possible for the concept definition to conflict with other aspects of the concept image. Tall and Vinner write: "We shall call the portion of the concept image which is activated at a particular time the *evoked concept image*. At different times, seemingly conflicting images may be evoked. Only when conflicting aspects are evoked simultaneously need there be any actual sense of conflict or confusion" (ibid). Thus, if a student 's evoked concept image includes both an aspect of the concept and the concept definition and if these aspects are in conflict, the student may be forced to somehow eliminate the conflict by some modification of his/her concept image.

One might consider the concept image as that which forms the mental representations and connections that are required for understanding. One possible structure views each node of the concept image, if sufficiently simple, as a cognitive unit and treats the conceptual connections as the necessary links between nodes. Within the concept image, we see the possibility of the four categories of understanding alluded to by Sierpinska (1992). The students' prior exposure to the concepts may be detrimental due to the inappropriate existing network–inappropriate cognitive units or conceptual connections in the concept image. The conflicts go unnoticed since they are often not present simultaneously. Two cognitive units may be in distinct conflict, but unless the student has been asked to seriously reflect on this conflict, both remain part of the concept image and co-exist due to the lack of a connection that might expose the conflict. It is often the case that what the learner knows is replete with misconceptions. Students who are the subject of this research hold concept images of variables and equations that display little overlap with the definition of these concepts. What happens when a new idea is presented in this context? Consider what Hiebert and Carpenter say about existing networks: "If the learner tries hard to fit a new idea, fact, or procedure into a current way of thinking, existing networks constrain the relationships that are created. At the other extreme, a learner may represent new information in a way that does not connect it with existing networks" (1992, p. 70). Two potential problems arise. An existing network that is incorrectly constructed constrains the construction of the web for a new idea. The result is an construction rife with misconceptions. On the other hand, if the new idea does not connect with the existing networks, a lack of connections between old and new occurs leading to disjointed and unusable knowledge.

Due to prior acquaintance with numerous concepts, students exhibit profound learning interferences. The researcher has observed students who are adept at solving linear equations, and, as a result, resist considering the meaning of such equations or how they arise in various situations. They know how to "solve it", so what's the point in learning more about the concept?

Contributing to this dilemma are the three kinds of learning described by Duffin and Simpson (1995). These include "natural" which fit into a person's mental structures, "conflicting" which are inconsistent with a person's concept images, and "alien" which cannot be connected into a person's mental structures. The proceptual divide may arise when the mathematical information presented is either "conflicting" or "alien". Worse, an attempt to undo procedural learning in "remedial" courses may be met with resistance because relational instruction may be "conflicting" or "alien" with the previously built algebraic knowledge structures.

Simoneaux and Kirshner (1994) reported research on the "negative consequences of rote (relationship-poor) learning preceding meaningful (relationship-rich) learning. They conducted a study in which one group of U.S. eight graders received rote learning followed by meaningful learning on the perimeter and area of basic geometric shapes. The second group received a brief meaningful instructional treatment. Using a pretest/posttest/retention test design, focusing on the students' comprehension and their ability to transfer their learning to new settings, the researchers report that the first group who received rote instruction initially performed significantly less well than the second group on both the posttest and the retention test. Simoneaux and Kirshner write:

> ... the experience of memorizing material and being able to regurgitate it equated to learning for these students. Memorization inhibits free, open-ended, creative explorations of ideas and materials...Rote learning sets up superficial associations related to solution procedures. These may conflict with subsequent meaningful instruction. In such cases, either prior structures remain, thus making new relationships impossible; or, structures have to be unlearned and new relationships constructed. This unlearning and relearning creates unnecessary obstacles (interferences). Thus when initial mathematics of a concept focuses on memorizing procedures, facts, and definitions, subsequent meaningful learning may be impaired. (1994, pp. 222–223)

One distinction that should be made is a distinction between rote-learning and routinization. While rote-learning relies on memorization and is relationship-rich, routinization (repeating something several times to build up an action schema) may indeed be relationship-rich and contribute to relational understanding, in the spirit of Skemp 1976).

In some sense students like those described above may be working with a poorly constructed cognitive web built on procedural efficiency that must be de-constructed and then re-constructed so that the concepts are understood relationally. Their understanding may be, at best, pseudo-conceptual (Vinner, 1997) allowing them to proceed through prior mathematics courses with little understanding. Students' experiences with mathematics primarily consist of automatized algorithms for the manipulation of symbols. This presents the instructor with a tremendous obstacle since he or she must constantly deal with interferences from the existing cognitive network. One example of such interferences is the overemphasis on equation solving in algebra. Students learn mechanical steps for solving linear and quadratic equations in one variable. When confronted with any mathematical expression or statement, their impulse is to "solve it". The instructor may have a goal of relational teaching in terms of where equations come from, what it means to solve an equation, numerical or graphical solution techniques, and interpretation of solution, but the students desire to execute procedures interferes with attempts to develop deeper understanding. The interferences presented by previous rote learning require a radically different approach to the material. The critical question is how completely can we neutralize these interferences?

The fact that students bring with them these interferences suggests serious cognitive obstacles. As stated in Tall (1989a) with references to Bachelard (1938/1983) and Brousseau (1986), an **obstacle** is "a piece of knowledge that has in general been satisfactory for a time for solving certain problems, and so becomes anchored in the student's mind, but subsequently that knowledge proves inadequate and difficult to adapt when the student is faced with new problems" (p. 88). Epistemological obstacles as defined by Sierpinska (1992) have been alluded to previously. Here the discussion focuses more on cognitive obstacles. A student may have procedural knowledge of how to solve $3x - 1 = 2$ for $x$, but may be unable to adapt this knowledge to solve $3x - y = 2$ for $x$. If students must solve the latter equation for $x$, they may internalize a new procedure rather than building on their previous ability to solve $3x - 1 = 2$. They have two separate cognitive entities for solving these equations with no conceptual connections. Each cognitive entity may be organized proceduraly as a visually-moderated sequence, in the sense of Davis (1984). These students survived thus far using mimicry and memorization as their primary learning mechanisms. When they enter college, they place in courses much lower than their background would predict due to the inadequacy of their approach to learning (due to the lack of connections in their concept images and to their emphasis on procedural, rather than proceptual, understanding). This is a criticism of both the mode of instruction and the organization of the curriculum. Tall writes that "... the way in which we limit the child to simple cases for a substantial period of time, before passing on to more complex cases, is bound to set up cognitive obstacles....our curricula, designed to present ideas in their logically simplest form, may actually *cause* cognitive obstacles..." (1989a, p. 88). This curriculum design persists through high school and often through college courses as well. A scan of commonly used college texts finds many very short sections that present one mathematical idea with many examples. There is often little or no connection between one section and the next. The principle of optimal complexity may cause problems, as a result, since students will tend to make the simple more complex. Thus, by "dumbing down" the texts, the effect may be to make mathematics even more difficult for the students. Written descriptions are kept to a minimum. Applications appear in their own separate sections at the end of chapters (where they can be avoided because students have trouble with them). This wouldn't necessarily be critical if the text did not define the curriculum. Unfortunately, in many schools, the text is the de facto curriculum. Topics are laid on topics with few connections and little opportunity to look at any idea in depth. It has been said that the U. S. algebra curriculum is a mile wide and an inch deep. The shallow curriculum produces students with little, if any, depth of understanding.

## 2.5 Function as an Organizing Principle

### 2.5.1 Acts of understanding and epistemological obstacles

Function was selected as a focal and organizing topic for the reorganization of the college beginning algebra curriculum because of its central importance in mathematics and because of its power for modelling real-life situations. Dubinsky and Harel (1992) suggest in their Foreword that function may be the single most important concept in mathematics classes from kindergarten up. Eisenberg (1992) writes: "A major goal of the secondary and collegiate curriculum should be to develop in students a sense of functions" (p. 154). In terms of applicability to everyday life, Sierpinska (1992) writes: "the notion of function can be regarded as a result of the human endeavour to come to terms with changes observed and experienced in the surrounding world" (p. 31). Since many students taking beginning algebra in college will take very little additional mathematics, function seems to be the most important mathematical idea for them in terms of applicability to their future lives. From this point of view, a review of the theoretical perspectives for the function concept seems appropriate.

The first theoretical perspective considered is described by Sierpinska (1992) who discusses 19 acts of understanding and 16 acts of overcoming epistemological obstacles related to the function concept. Her epistemological obstacles are inherent difficulties that arose during the evolution of the concept. Sierpinska suggests that the epistemological obstacles cannot be avoided and that they are important in the development of thinking about the concept.

The acts of understanding described by Sierpinska come under four categories–identification, discrimination, generalization, and synthesis–that have been described previously. Using these four broad categories, Sierpinska goes on to list the acts of understanding. The first two focus on recognizing the role of function as a way of describing changes and relationships in everyday life. Students must recognize that quantifiable change is a regular occurrence in everyday life and realize that it is important to have some quantifiable way of discussing the relationship among changes that occur. Her third act of understanding aims at the quantities that are changing. This act suggests that students must clearly identify the variables involved in the relationship. Related to this is the fourth act of understanding: the ability to distinguish between variables and parameters in a given situation. The fifth act requires the learner's ability to discriminate between the independent and dependent variables. Her next two acts of understanding focus on an understanding of the replacement set for variables. She suggests that this falls under the category of synthesis and generalization of the idea of numbers and the ability to discriminate between number and quantity. Act of understanding 8 requires the learner to establish a connection between function and physical laws, seeing that functions are useful in the statement of physical laws. Next, the student must be able to discriminate between the notion of function, in its own right, and the various representations that are used to describe the function. This really goes back to the question: What is a function? Is it the equation that describes the relationship? Is it the graph? Is it the function machine? Or is it the collection of all the ideas united as a single cognitive image? The next two acts focus on the synthesis of the general concept. Act 10 suggests that the student must be able to discriminate between the mathematical definition and the description of the mathematical object, function. Going hand-in-hand with his notion, students must be able to conceive of function as an object, something that will be discussed at great length later. The next four acts focus on further discrimination: the ability to discriminate between functions and relations; the ability to discriminate between functions and sequences (considering sequences as a subset of functions for which the domain is the set of natural numbers); the ability to discriminate between coordinates of points on a curve and the line segments fulfilling some function for the curve; and, discrimination between the various representations of function and the function itself. Act of understanding 16 concludes by pointing out the need for synthesis of the different ways of describing functions, representing functions, and talking about functions. The relationships among all these that students develop is a focal point of this dissertation.

Along with the acts of understanding, Sierpinska describes the attendant epistemological obstacles. The first obstacle is the historical viewpoint that mathematics is not concerned with practical problems. This obstacle stands in the way of making functions meaningful to students and making students want to learn about functions. Many of the recent calls for reform have attacked this obstacle by arguing for mathematics being studied within a contextual framework. The next obstacle addresses a common technique for exploring functions: computing ordered pairs for the relationship. This obstacle suggests that computing ordered pairs for the relationship is somehow "below", in the sense that it is unworthy of consideration by the serious mathematician, the object of study in mathematics. Focusing on how things change rather than on what is changing is the next obstacle. This is parallel to the act of understanding that focuses on the identification of the varying quantities. The next obstacle suggests the problem in discriminating between known and unknown quantities versus discriminating between variables and parameters. Students with some algebra have much experience in discriminating between knowns and unknowns, but have seldom had to deal with the idea of parameter in any detailed way. For example, solving equations such as $x+2 = 5$ is a common occurrence in beginning algebra where the letter $x$ stands for an unknown that soon will be known. However, understanding the significance and different meanings of all the letters in $y = ax+b$ is an entirely different problem. In a similar vein, distinguishing between dependent and independent variables is the next obstacle. What represents the input and what represents the output is a key issue in the definition of a specific function. This is an area that frustrates students since they often mistakenly believe there is one and only one way to specify the input and output. The next obstacle aims at the conception of number. Sierpinska refers to this obstacle as the "heterogeneous conception of number" (ibid, p. 39) in which, for example, proportions are considered something different from equations and a ratio is something different from a quotient. The next obstacle concerns the attitude that everything is number, blocking the discrimination between variables representing physical quantities and variables representing numbers. Viewing functions and physical laws as different entities in separate compartments constitutes the next obstacle blocking the connections between mathematics and real-life applications. Viewing proportions as a relationship different from other kinds of relationships is described as the next obstacle. The next two obstacles target the overemphasis on symbolic manipulations and symbolic forms in representing, defining, and manipulating functions. The next obstacles revolve around the role of formal definition with respect to function suggesting that one does not build a fruitful concept image by focusing only on the formal definition of function. The concept is much richer and must develop over time. The next obstacle strikes at the notion that functions are sequences, thus giving a too-narrow conception of function. Coordinates and graphs are the focus of the next two obstacles. Not seeing coordinates as numbers and allowing graphs to contain points where the function is not defined summarize these two obstacles. The last obstacle concerns the conception of changes in variable. This obstacle suggests that changes in the variable are viewed as changes in time.

Based on this list of acts of understanding and epistemological obstacles, Sierpinska goes on to make some suggestions. She states that students must be interested in studying the relationship between things that change. Functions should initially be introduced as a construct for describing such real-life or physical changes. She states that numerical methods, such as table-building, are valuable activities in the growth of understanding. Next, it is important, instructionally, to focus not only on the change, but what is changing. Sierpinska suggests that some facility with algebraic manipulations is a prerequisite to understanding functions. This research must deal directly with this issue as, in this "reform" curriculum, functions are used to give meaning to algebraic manipulations and to the equivalence of expressions that result from such manipulations. The importance of a variety of representations and the relationships between them is an important part of the teaching of functions.Finally, she suggests that informal definitions are appropriate at the beginning level, something that the curriculum that is the subject of this research supports.

## 2.5.2 Operational versus structural thinking

Sfard (1992) provides a theoretical framework for function that includes a specific path of understanding from operational to structural. While her theory may apply to several mathematical objects, her focus is on the function object. She suggests that the problem students have with certain mathematical concepts stems from the difficulty with reification. Sfard writes: "Awareness of the long and painful processes preceding the birth of mathematical objects may be the key to understanding some of the difficulties experienced by so many learners" (p. 59). She begins establishing this awareness by contrasting structural and operational viewpoints. Sfard says that we are operating from a **structural viewpoint** if we treat mathematical ideas as "object-like entities" (p. 60). An **operational viewpoint** is active if the mathematical idea "is conceived as a computational process rather than a static construct" (p. 60). Sfard suggests that many mathematical ideas are thought about operationally prior to their conception as structural. In fact, in her paper analysing the historical development of algebra (1995), she argues for viewing mathematics "as a hierarchy in which what is conceived operationally (i.e., as a computational process) on one level is reified into an abstract object and conceived structurally on a higher level" (p. 16). Sfard goes on to describe the evolution of algebra "as a constant (but not necessarily conscious) attempt at turning computational procedures into mathematical objects, accompanied by a strenuous struggle for reification" (1995, p. 17).

Central to Sfard's theory are the steps involved in moving from operational to structural conceptions. She begins with familiar objects and discusses performing processes on these objects. For example, a linear polynomial in one variable, such as $2x + 3$, might be a familiar object to a student who has studied polynomials previously. Among the processes we might perform on this polynomial are evaluation for a given value of $x$ and solving when we wish to determine $x$ given a value for the polynomial. In this sense, we think of the polynomial as establishing a process that returns output (a value of the polynomial) for a given input (a value for $x$). Now we have created a function idea in a procedural sense. Sfard refers to this step as **interiorization**. From this conception, one can generalize a function as returning a unique output for a given input without being dependent on the specific procedure. This step is called **condensa-**

**tion** and represents a process phase in the understanding of function. Sfard writes: "**Condensation** means a rather technical change of approach, which expresses itself in an ability to deal with a given process in terms of input/output without necessarily considering its component steps" (1992, p. 64). Finally, the idea of function can be thought of structurally so that a function can be something that one can perform processes on, such as composition or differentiation. Sfard refers to this very important, and difficult step as **reification** writing:

> in the mind of the learner, it [reification] converts the already condensed process into an object-like entity. In other words, while condensation is a gradual quantitative change, reification should be understood as a sudden qualitative jump in the way of looking at things–an ontological shift comparable to a transition from one scientific paradigm to another. The fact that a process has been interiorized and condensed into a compact, self-sustained entity, does not mean, by itself, that a person has acquired the ability to think about it in a structural way. Without reification, her or his approach will remain purely operational. (1992, p. 65)

It may be argued whether the jumps from one stage to another are necessarily gradual or sudden. The path that a student takes in deepening his/her understanding may determine whether there are smooth transitions or sudden jumps between the stages.

Sfard suggests that her theory of concept acquisition has similarities with that of Garcia and Piaget (1989). Their cycle of intraoperational, interoperational, and trans-operational in some sense mirrors the evolution from operational to structural that Sfard describes. Sfard suggests that the operational approach has value when answers to mathematical questions are the goal, but the structural approach in which long sequences of steps are converted into cognitive units with appropriate conceptual connections, in the long run, reduces the **cognitive strain** (Harel & Kaput, 1991) and thus reduces the chances of the proceptual divide occurring. This argument suggests that the structural notion is similar to the proceptual conception of a mathematical idea, but she does not discuss the importance of being able to easily move between operational and structural conceptions as required.

Sfard goes on to discuss some possible sources for the difficulty with reification. First, she references Garcia and Piaget in noting as we attempt to acquire a new mathematical idea, "certain initial properties of objects can no longer be accepted, or else they lead to contradictions in interpretative schema" (from Piaget & Garcia, 1989, p. 204).

Such concessions, especially when the student has interiorized them into his or her concept image, can be difficult to release. In fact, Sfard points out that what must be given up is often exactly what imparted meaning to the concept. In essence, to reify the function concept, students may be forced to let go of the procedural conception. This conception may be the very thing that formed the concept image of function for some students. In fact, Tall (1996) argues that we don't give up the procedural conception, but, rather, develop a conceptual or even a proceptual understanding where one can move flexibly among the various conceptions.

Another possible source of difficulty in reification involves a Catch-22 situation. Sfard writes:

> There is an apparent discrepancy between two conditions which seem necessary for a new abstract object to be born. On the one hand, it appears that reification must precede any mention of higher-level manipulations–of the manipulations to be executed on the concept in question.... On the other hand, before a real need arises for regarding the lower-level processes as fully-blown objects, the student may lack the motivation for putting up with the existence of a new "intangible" thing....The necessary drive will not be created unless an inability to think structurally turns into an obvious hindrance for further progress.... To sum up, higher-level interiorization is a precondition for a lower-level reification, and vice versa... It follows, therefore, that at the crucial junctions in the development of mathematical knowledge a learner may become embroiled in a potentially dangerous vicious circle. (1992, p. 68)

For example, Sfard might view function as a lower-level object to the process of differentiation. As long as function has not been reified, the higher-level process (differentiation) lacks input. On the other hand, the student may have no motivation to view a concept, such as function, as an object, especially if it requires the severing of conceptual links that have previously been a part of the concept image.

Sfard goes on to address instructional issues in light of her theory that the understanding of a new mathematical idea must progress hierarchically from operational to structural. First, she suggests that new mathematical ideas should not be introduced structurally. In fact, bowing to the fact that reification is difficult to achieve, she suggests that the structural conception should not be required until absolutely necessary.

Sfard discusses some difficulties that students have with the function concept in light of her theory. She first argues that an inability to place an object in an appropriate class is a clear sign that a structural conception is not present. We see this kind of behaviour when students rely on prototypes (See Tall & Bakar, 1992) in an attempt to classify objects as functions or not functions. Sfard refers to students' tendency to associate functions with algebraic expressions as **pseudostructural** which indicates a "semantically debased conception" (1992, p. 75). Graphs are another way of thinking about functions, but there is almost no connections between a graph and the underlying algebraic formula. Sfard writes: "Such a flatly conceived notion, lacking operational underpinnings, remains detached from the previously developed system of concepts and does not preserve its identity in transitions from one representation to another and from context to context" (ibid, p. 75). There seems to be a hint of a problem in her theory at this point–the fact that it is difficult to think of graphs operationally. In fact, Tall (1995) suggests that cognitive development of objects, such as graphs, may follow a hierarchy more dependent on visuo-spatial prototypes than on the interiorization, condensation, reification steps proposed by Sfard.

Sfard concludes her paper with a discussion of some ways to stimulate structural thinking. She suggests that incorporating computer programming will help to increase students' understanding of the processes that underlie the mathematical concepts. Tall and Thomas (1991) talk about the conceptual benefits of using computers to promote a more dynamic view of algebra when they write:

> The results of our work suggests differential effects between the computer-based approach to algebra, with its emphasis on letters as generalized numbers and the traditional skill-based type module with its emphasis on acquiring manipulative skills. It seems that the computer work promoted a deeper conceptual understanding, whilst the other work, as expected, initially facilitated better surface skills. However, when the computer module was combined with the skill-based one then it led to a superior overall performance without detrimental effect on skills. (p. 140)

Heid's work in calculus (Heid, 1988a) supports the fact that students using the computer for conceptual work perform better on higher level conceptual problems than students from a traditional calculus course. Additionally, Heid showed that a brief treatment of skills at the end of the course was sufficient in that students from the test group and the traditional group were not significantly different in their ability to do routine manipulations.

The hierarchical nature of learning that Sfard describes suggests that a student must be good at executing certain procedures in order to develop an understanding of the object. With the increasing presence of symbol manipulators, it could be argued that this step may not be necessary. For example, Tall (1991b) introduces the idea of the principle of **selective construction** of knowledge in which a software is used to carry out algorithms (procedures) so that the student can concentrate on the object and the properties of the object. This allows the student at some point to focus selectively on the process and at another point on the objects. It is important to note that both activities are still required. Tall (1991) writes: "*Both* activities remain essential, for the process is needed to be able to *do* the mathematics and the higher level relationships are essential to fit it together in a meaningful way" (p. 258). Sfard supports the use of multiple representations to stimulate reification. She writes: "Tables, symbols, graphs–all these static and integrative ways of picturing functions, may have a reification-stimulating effect....exposing students to many kinds of representations may be helpful in uprooting quasi-structural conceptions" (p. 79). Tall (1992b) suggests that the crucial issue might be in the identification of a cognitive root for the concept. He defines **cognitive root** as a concept that has the "dual role of being familiar to the students and providing the basis for later mathematical development" (p. 497). He goes on that suggest that function as process may act as a cognitive root for the formal concept of function, though warns of obstacles in this approach. Finally, Sfard suggests: "Open discussion on such ontological subjects like the nature of mathematical entities and the difference between processes and objects will put the student face-to-face with her or his implicit beliefs... In the case of function, the presumptions that there must be an algorithm behind every mapping is probably the one which should be attacked with particular force" (1992, p. 79). In conclusion, Sfard believes that reification can be stimulated by external forces, but that for some students, it may be an unreachable goal, and for others, it may be a long time in coming. She states: "For an abstract object to be born, a long period of incubation may sometimes be necessary" (ibid, p. 83). Other researchers have focused on how to increase the depth of understanding of mathematical concepts. We look next at the APOS theory as proposed by Dubinsky and his research group.

### 2.5.3 APOS theory

Like Sfard's hierarchical model for the acquisition of mathematical concepts, a research group headed by Dubinsky has developed and tested a theory, the APOS theory, for the growth of understanding of mathematical concepts along the depth dimension. This theory is explicated in several places in the literature including Dubinsky (1992) and Breidenbach et al. (1992). More recently, the theory has been updated, as described in Cottrill et al. (1996). This section summarizes the theory along with its theoretical underpinnings.

Dubinsky (1992) describes how this theoretical perspective arose out of an interpretation of Piaget's theories, particularly the theory of reflective abstraction. He discusses the three major kinds of abstraction identified by Piaget. **Empirical abstraction** arises by drawing knowledge from the properties of objects (Beth & Piaget, 1966, pp. 188–189). **Pseudo-empirical abstraction** produces properties "that the actions of the subject have introduced into objects" (Quoted from Dubinsky, 1992, p. 97 referencing Piaget, 1985, pp. 18–19). Finally, **reflective abstraction** is drawn from "the general coordinations of actions and, as such, its source is the subject and it is completely internal" (ibid, p. 97 referencing Piaget, 1972, pp. 37–38). Dubinsky goes on to differentiate between the three types of abstraction when he points out that empirical and pseudo-empirical abstraction requires that actions be performed on objects to construct knowledge. Reflective abstraction, on the other hand, "... interiorizes and coordinates these actions to form new actions and, ultimately new objects" (ibid, p. 98). Dubinsky goes on to claim that, based on Piaget, "new mathematical constructions proceed by reflective abstraction" (ibid, p. 98 referencing Beth & Piaget, 1966, p. 205). He concludes by suggesting that reflective abstraction is "a description of the mechanism of the development of intellectual thought" (ibid, p. 99).

Based on the above theoretical perspective from Piaget and several years of research, the APOS theory has evolved to a current form as described in Cottrill et al. (1996). The definition of mathematical knowledge put forth by this research team states:

> Mathematical knowledge is an individual's tendency to respond, in a social context, to a perceived problem situation by constructing, re-constructing, and organizing, in her or his mind, mathematical processes and objects with which to deal with the situation. (p. 171)

This research team identifies **actions, processes,** and **objects** as three general types of mathematical knowledge. Furthermore, these three are organized into structures called **schemas.** Cottrill et al. define **action** as "any mental or physical transformation of objects to obtain other objects" (ibid, p. 171). For example, the ability to plug numbers into an algebraic expression and calculate is an action. Reflection on an action leads to the interiorization of the action as a process. This is similar to the interiorization discussed by Sfard (1992).

A process is a purely cognitive act that does not rely on the performance of a specific sequence of steps. Cottrill et al. write: "A **process** is a transformation of an object (or objects) that has the important characteristic that the individual is in control of the transformation, in the sense that he or she is able to describe, or reflect on, all the steps in the transformation without necessarily performing them" (ibid, p. 171). Some indication that a person has moved from action to process is the demonstrated ability to reverse a process or to combine a process with other processes. With respect to the function concept, a "process-level" understanding might be indicated by a student recognizing that two functions are equal when the same input produces the same output even if two different algorithms are used to produce the output. For example, $f(x) = 5x+10$ and $g(x) = 5(x+2)$ produce different algorithms on $x$, but are equal since the two algorithms are equivalent. A student who says these are different functions might have an action conception while a student who says they are the same might have a process conception of function.

Next, we move to the object phase. A student at the object phase has reified the process as object, in the sense of Sfard. Cottrill et al. define an object as something that "is constructed through encapsulation of a process. This encapsulation is achieved when the individual becomes aware of the totality of the process, realizes that transformations can act on it, and is able to construct such transformations" (ibid). Thus, a function has become an object to a student when he or she realizes that a function itself can be used as an input or an output for some higher-level process. Cottrill et al. go on to point out that objects can be de-encapsulated back to processes and that the ability to move back and forth between process and object is an important part of understanding a mathematical idea. Here we see as hint of procept, as discussed by Gray and Tall

(1994). The student who can view a symbol flexibly as both a process and an object has the flexibility in thinking to choose the appropriate conception in a given problem situation. This is an example of proceptual thinking.

The final piece in this puzzle of conceptual development is the schema. Skemp (1979) writes of "varifocal theory" in which a schema is a concept if one views the schema as an entity and a concept is a schema if one looks at the details of the concept. Skemp writes: "Concepts and schemas are not distinct kinds of mental entities. sometimes one classification is better, sometimes the other, according to the purpose. A schema can be thought of as a concept with interiority" (ibid, p. 141) Skemp notes that interiority refers to the quality or dimensions of a concept (ibid, p. 116). As a counterpoint, Cottrill et al. write: "A **schema** is a collection of actions, processes, objects, and other schemas that are linked in some way and brought to bear on a problem situation" (ibid, p. 172). They go on to suggest that schemas via reflection can be transformed into objects. Finally, objects can be transformed by action into new processes, objects, and schemas. To some degree, schema evokes an image of a cognitive network that has been compressed into a cognitive unit. If not just one cognitive unit, a schema is at least a collection of closely related cognitive units with the appropriate conceptual connections.

Summarizing, the theoretical framework of Dubinsky and his colleagues sees the process of understanding beginning with actions on already-understood objects. These actions become interiorized into processes that are cognitive entities which can be coordinated with other processes and which can be reversed. These processes are encapsulated into new objects on which higher-level actions can be performed. The development of these cognitive structures occurs through the process of reflective abstraction. Finally, a schema is a collection of these actions, processes, and objects. A crucial question relates to the necessity of this sequence of stages in developing an understanding of a concept along the different ways a concept can be thought of. If a particular aspect of a concept lies in a visuo-spatial dimension rather than in an object "to be acted upon" dimension, are these stages of development still appropriate? Such questions are discussed in relation to the theoretical framework in the next chapter.

## 2.6 Difficulties with the Function Concept

Now that some of the major theoretical frameworks for developing an understanding of function have been discussed, the attention turns to research evidence of difficulties in such development. Sierpinska's list of epistemological obstacles has already been discussed. In this section, attention is focused on some of the cognitive obstacles that occur when a student tries to develop an understanding of function.

Given a set of examples, what rules do students use to determine if the example is that of a function? Not surprisingly, Vinner and Dreyfus (1989) suggest that the mathematical objects that students identify as examples of a concept are not necessarily the same as the objects determined by the concept definition. In identifying student beliefs about functions, Vinner (1983) reports that 10th and 11th grade students studying in academically selective high schools in Jerusalem who were taught the function process using the Dirichlet approach manifested the following beliefs: the relationship defined by the function should be systematic; the function must be a term or an equation; the function is identified by only *one* of its representations; the function should be given by one rule; different rules are okay for different domains as long as the domains are "regular," I.e. no singleton domains; a rule that is not algebraic may be a function if it is endorsed by the mathematical community; graphs should be regular and systematic; and, a one-one correspondence is expected. Vinner (1992) goes on to discuss "compartmentalization" in the setting of function. Compartmentalization occurs when two items of knowledge are incompatible and yet reside simultaneously in one's concept image. For example, a student interviewed as part of this research was able to accept a random two-column table as a function, but unwilling to accept a table in which the input was generated by one rule and the output was generated by another rule as a function. Another example occurs when students deal with different representations of function. Students in this research commonly accepted the equation of a circle as a function while identifying its graph as not being a function (most likely blindly applying the "vertical line test"). Compartmentalization suggests the lack of appropriate conceptual links within a concept image. The disconnected cognitive units often act as coping mechanisms in the face of conflict. Instead of dealing with the conflict squarely, the student removes the link that creates the conflict allowing the two opposing ideas to co-exist in the mind. Ultimately the use of examples and isolated, disconnected rules, such as the vertical line test, result in a muddled concept image lacking meaningfulness. Vinner writes: "Students recall words, symbols, or even pictures that are related to the topic which they are asked, but all theses do not make a meaningful idea. They are part of meaningless communication that students try very often in classes or in exams just to make the impression that they know something, an impression they have to make in order to survive" (1992, p. 211). Separating the meaningful from the purely procedural in a students mind is one of the more difficult tasks researchers must deal with in constructing models of student understanding.

Since a concept, such as function, consists of several different representations, researchers have probed the difficulties that arise across these representations. Eisenberg (1992) argues that it is crucial to connect geometric and symbolic representations. In doing so, he suggests that there are two sources of difficulty: epistemological obstacles, in the same vein as Sierpinska (1992), in which symbol manipulation and proofs are viewed as "real" maths while visualization is not; and, cognitive difficulties between visual and analytic processing. In essence, graphs are visuo-spatial based (figural) while symbols tend to be more process-based. In support, Vinner and Dreyfus (1989) suggest that while most students can graph simple functions, they often treat the graph of a function as something external to the function itself and not really part of its essence.

Eisenberg goes on to discuss the importance of the visual, acknowledging the difficulties inherent in thinking visually first. He writes: "The emphasis of getting students to think of concepts first in a visual framework and then in an analytic one, is a switch in direction that is most difficult to obtain" (1991, p. 161). Part of this difficulty may be linked to instrumental instruction and procedural learning, which place their emphasis on manipulating the analytic. In fact, Eisenberg goes on to impugn mathematics more generally: "...functions and their associated notions are not conceived visually, and that this non-visual approach hinders one's development of having a sense for functions...it is the conclusion of this author that this unwillingness to stress the visual aspects of mathematics in general, and of functions in particular, is a serious impediment to students' learning" (ibid, p. 152).

Eisenberg (1992) addresses some of the difficulties inherent in using a visual emphasis. He lists difficulties at three different levels. A visual emphasis is: more difficult (cognitive); is harder to teach (sociological); and, not mathematical (epistemological). Much blame can be placed at the feet of academic knowledge which Eisenberg charges is procedural and disconnected. To defend this charge, he cites the Chevallard's (1985) theory of mathematical didactics. Central to this theory is didactical transposition defined as "the change knowledge undergoes as it is turned from scientific, academic knowledge to instructional knowledge as it is taught in school" (p. 169) Table 2.1 displays the comparison between features of academic and instructional knowledge.

**TABLE 2.1: Didactical transposition**

| Academic knowledge | Instructional knowledge |
|---|---|
| intricate | sequential |
| numerous links | links eliminated |
| nonsequential | "sound bites" |
| contextualized | non-contextualized |
| stresses concepts | stresses procedures |

Eisenberg writes: "New knowledge is formulated, for the purpose of teaching, in a way that stresses computational procedures, and this necessarily allows for sequential presentations" (ibid). This directly impacts the predominance of the analytic over the visual. Analytic processes precisely rely on the sequential nature of the knowledge while the visual tends to be nonsequential, intricate, and with numerous links.

Artigue (1992) lends support to Eisenberg's argument discussing what this researcher calls the didactic contract. This contract includes the assumed role of the teacher and the role of the student in a classroom. Artigue writes: "Traditional teaching is based on the fiction of the possibility of learning through continuity" (1992, p. 111). The impact on students in differential equations is that "most of them [students] are convinced there exists a recipe allowing the exact integration of each kind of differential equation and that the aim of research in this field is to complete the cookbook" (ibid, p. 112) Replace "differential equations" with another mathematical field and the quote remains valid. Artigue argues for "inter-setting" relations that are ultimately settings requiring the use of several different representations of a concept. Three different registers of interaction are mentioned, two of which are pertinent to this research. The first is interpretation in which information is given simultaneously in two different representations. Problem solving requires interaction between the representations. The second register of interaction is prediction in which information is given in one representation, but the solution to the problem requires a different representation. In discussing representations, Artigue pays particular attention to the visual as compared to the analytic suggesting that the visual development is inferior in the minds of mathematicians and thus in the minds of students as well. Artigue writes: "Beliefs and habits about the status and role of a graphic setting act as didactic obstacles and they have to be explicitly questioned in order to obtain the necessary epistemological changes both in teachers and in students" (ibid, p. 132).

Eisenberg (1992) goes on to mention the different processing requirements of the visual versus the analytic. He cites Larkin and Simon (1987) who wrote: "...the diagrammatic [visual] representation preserves explicitly the information about the topological and geometric relations among the components of a problem, while the sentential [analytic] representation does not" (p. 66). Tall (1995a) specifically makes this point in his outline on the development of a cognitive structure. Interaction with the external world produces perceptions that may be primarily objects (visual) or actions (analytic). In the quest toward advanced mathematical thinking the objects develop as visuo-spatial prototypes become more verbal-deductive. On the other hand, actions become symbolized as processes which in turn become encapsulated as objects. the interaction between the two is through conceptual links, if they exist. If such links are absent, two parallel developments may occur, but lack the connections necessary to the development of true mathematical power. Vinner (1992) states: "People remember visual aspects better than analytic aspects" (p. 212) This may be due to the fact that the visual may more easily become a cognitive unit. To remember the analytic, appropriate conceptual connection must be present.

While the preceding discussion has focused on just difficulties between analytic and visual representations, many other student difficulties have been documented. Eisenberg (1991) notes that the notational complexities of the function concept often present obstacles to understanding. Herscovics (1989) agrees stating that the notation "is very efficient because it condenses a great deal of information, but the economy of notation can also create a cognitive obstacle" (p. 76). Eisenberg points out that students do not understand that functions transform every point in the domain to a new position. Finally, he mentions the process-product dilemma of functions acting as operators as well as objects.

Martinez-Cruz (1995) reports on research aimed at "developing a conceptual knowledge of functions in technology-enhanced classes" (p. 279). He used 8 students in a pre-calculus class using a Demana and Waits text in a case study. The question investigated was "What are the concept images and the concept definition of function that students have?" A practice test was given at beginning of study followed by interviews, observations, and other material that was submitted. The author identifies 3 models: a graph model, an equation model, and a unique correspondence model. It is interesting to note the lack of tables-this study was prior to the easy accessibility of tables on graphics calculators. The author appears to focus on the various representations of functions, but have provides little insight into the depth of understanding. For a student with a graph model, there was the often-reported result that the student only identified graphs as functions if they were within his experience. Martinez-Cruz writes: "He [the student] recognized a function when he has seen or graphed a similar or identical graph. Otherwise, he would reject a function based on his experience" (ibid). Here again the emphasis on prototypes is demonstrated. In the equation model, a key issue relates to whether students require functions to be expressible as equations and whether all equations are functions. The unique correspondence model refers to the formal definition of function that emphasized one unique output for each input. The paper concludes by looking at links between the models. One reported link was the interpretation of the vertical line test. The other link is given by translating between representations. Concluding, Martinez-Cruz states that "for some students one single model was more anchored in their mind than others, and they acted accordingly" (ibid). This research suggests a common structural element in the development of a concept image of such a complex concept as function: the tendency to allow one particular aspect or model to dominate all others. This is not such a problem if the model is sufficiently robust, but, if not, it may lead to highly compartmentalized understandings.

Janvier (1978) focuses attention on speed-distance graphs. Given the speed-distance graph of a car on a racetrack, students were asked to select the shape of the racetrack. Students tended to view the graph as a picture of the motion as opposed to a dynamic indication of the relationship between distance and speed. In this way, the contextual situation served as a distracter to an appropriate interpretation of the graph.

Monk (1992) focuses his research on difficulties dealing with the visual representation only. He distinguishes between students who see graphs pointwise as opposed to see graphs across-time. A pointwise view focuses on seeing a specific value for one component given a specific value for another component. An across-time view, rather than focusing on specific points, looks at graphs in terms of the patterns of change that will occur in one component given a pattern of change in the other component. This distinction relates nicely with Eisenberg's difficulty in seeing a function operate across its domain. Another difficulty reported by Monk is that of iconic translation that is the over-literal interpretation of a graph, similar to Janvier (1978). An example is the student who, in attempting to draw a time versus speed graph, draws a graph that displays the movement of the object in time. Dugdale (1993) reinforces the iconic translation when she notes that a frequently observed student error involved ignoring the labels on the axes and looking "for a graph that resembles a "picture" of the event described" (p. 109). She refers to this as the "graphs-as-picture notion" (ibid, p. 110). Kerslake (1977) also noted that students confuse graphs with pictures.

Markovits, Eylon, and Bruckheimer (1988) wrote a chapter specifically focusing on the difficulties students have with function. They begin by listing four components in understanding function. Each component is divided into two parts: a passive part and an active part. The first component relates to the ability to differentiate between functions and nonfunctions. This component seems closely related to the identification and discrimination acts of understanding mentioned in Sierpinska (1992) The second components focuses on the ability to find output given input and vice versa along with the ability to recognize the domain and range of a function. The ability to recognize equal functions and to be able to move between representations is the third component. Finally, the ability to identify and give examples of functions satisfying specific constraints is the last component. Based on these four components, the authors identified the following difficulties based on a study of 14 to 16 year old students. Students had difficulty with the terminology such as domain and range. They often ignored domain and range. Students struggled with locating points on the graphical representation indicating an inability to easily move between analytic and graphical representations. The misconception that every function is linear was documented, as it had been previously by Markovits, Eylon, and Bruckheimer (1983). Certain types of functions, such as the constant function, functions represented by discontinuous graphs, and piecewise-defined functions, give students difficulties. Finally, the authors mention that students often exhibit difficulty in technical manipulations.

Various researchers have identified other difficulties. Herscovics (1989) notes the following: the use and role of literal symbols cause difficulties in moving between table and graphic representations; students believe that functions should be given by only one rule. If two rules are given, there are two functions; and, graphs should be "reasonable" and "regular" with no angles. Karplus (1979) notes, in a study of secondary school students, that students were likely to process data mechanically even though the data arose from a physical context. There was little evidence that the students connected the data to the physical setting. Eisenberg and Dreyfus (1989), in working with secondary students on graphical transformations, mention concern that students viewed the transformations in terms of two static states rather than as dynamic processes. Goldenberg (1988) points out that varying the constant term in a linear function provides the illusion of either a vertical or an horizontal shift, that graphs on graphics calculators are highly dependent on the specific viewing window used, and that students may misinterpret the domain of a function based on the viewing window used. Graham and Ferrini-Mundy (1989) report that students tend to say a graphical representation is not a function unless a formula is associated with the graph. They also report that piecewise-defined functions are a source of difficulty. Moschkovich, Schoenfeld, and Arcavi (1993) point out that students have difficulty knowing what to focus on when they see a graph. A related issue is when to use one representation and when to let go of that representation in favour of another. Finally, Tall and Bakar (1992) discuss the fact that students commonly ignore the definition of function and, in its place, develop a set of prototypes for identifying functions. The paper concludes with a statement of a difficult obstacle:

> The learner cannot construct the abstract concept of function. without experiencing examples of the function concept in action, and they cannot study examples of the function concept in action without developing prototype examples having built-in limitations that do not apply to the abstract concept. (Tall & Bakar, 1992, p. 50)

## 2.7 Conclusion

This chapter surveyed literature on understanding, on factors that contribute to students who have been "debilitated" by their previous mathematical experiences, and on the understanding of the function concept. The chapter began with a set of definitions of key terms. Next, what it means to understand mathematics was explored. In this section, the researcher discussed the relationship of understanding to the biological development of the brain. The distinction between conceptual and procedural knowledge was discussed and related to relational versus instrumental understanding. Next a brief discussion of the process-product dilemma inherent in the symbolism of mathematics led directly to a discussion of the distinction between procedural versus proceptual thinking.

The subjects of this research are college students who have been unsuccessful at mathematics before. This is verified by their enrolment in a beginning algebra course offered at the college. The researcher argues that these students have been "debilitated" by their prior experiences with mathematics. The nature of this "debilitation" is situated in the context of the issues related to understanding described in the preceding section. To understand the debilitation, the researcher places blame on both curriculum and instruction. The section provides an historical perspective on the development of the algebra curriculum in the United States and is followed by a focused discussion on the prior experiences of the target students with this curriculum that contributed to their debilitation. Emphasis is placed on the nature of the students' concept images of basic mathematical ideas.

Since the target students have been unsuccessful with algebra in the past, the researcher postulates that they require an instructional treatment that is markedly different from their prior experiences. The concept of function is the focal point of this revised curriculum. The chapter reviews the major theories related to the acquisition of the function concept. Included are discussions of Sierpinska's epistemological obstacles, Sfard's operational versus structural thinking, and Dubinsky's APOS theory. The chapter concludes with an overview of the difficulties students have in acquiring the function concept, reviewing the findings of several researchers.

This chapter lays the groundwork for the theoretical framework that will structure this research. The next chapter explicates this framework and situates the framework among other conceptual models that have been developed to help researchers assess student understanding of mathematical concepts, and, specifically, of the function concept.

CHAPTER 3

# Facets and Layers of a Concept

## 3.1 Introduction

The previous chapter reviewed the research on understanding and learning along with research in general and with respect to the function concept. One key area of investigation, especially with increased availability of graphics calculators and computers, has been multiple representations. Thompson takes exception with some of the emphasis on multiple representations stating: "I believe that the idea of multiple representations, as currently construed, has not been carefully thought out, and the primary construct needing explication is the very idea of a representation... The core concept of "function" is not represented by any of what are commonly called the multiple representations of function, but instead our making connections among representational activities produces a subjective sense of invariance" (1994, p. 39). To learn a mathematical concept, one must acquire both a breadth and a depth of understanding of the concept. As suggested by Schwingendorf et al. (1992), one might measure both the breadth of the students' concept image (see Tall & Vinner, 1981; Vinner, 1983; Tall, 1989b) and the depth of the student's understanding in order to identify the student's knowledge of the concept. This research uses the word **facet** to describe the breadth dimension, avoiding the preconceptions about representations suggested by Thompson, and the word **layer** to identify the depth dimension. This chapter introduces a framework for a mathematical concept based on facets and layers. Student understanding of the function concept is analysed using this framework as a lens. Another focus of the discussion relates to the nature of the boundaries between facets and between layers. The chapter concludes with a brief description of other conceptual models for understanding that have been proposed and how these relate to the framework proposed in this chapter.

## 3.2 Facets and Layers of a Concept

### 3.2.1 Facets of a mathematical concept

The breadth dimension of function is exemplified by what are commonly listed as the multiple representations, such as symbolic, numeric, and geometric. To avoid the difficulty Thompson refers to with respect to the idea of representation, DeMarois and Tall

(1996) introduced the vocabulary of **facets**. Others have used the word "aspects" to express a similar idea. For example, Selden and Selden (1992), in their review of research on the function concept, write: "we will discuss various aspects (definitions, representations, conceptions) of function. For an expert, each aspect will suggest all the others, and any one aspect of function is convertible to another at will. However, novices often lack the ability to go back and forth between different aspects, and there is some evidence to suggest that familiarity with one aspect of function can interfere with developing an understanding of others" (p. 2). Webster's New World Dictionary defines a **facet** as "any of a number of sides or aspects" (Guralnik, 1980, p. 500). The facets of a mathematical entity refer to the various aspects of the entity. Depending on the mathematical entity, the facets might include a written description, a verbal description, a kinesthetic (enactive) demonstration, notational conventions, numeric aspects, symbolic aspects, geometric aspects, and colloquial (informal or idiomatic) aspects. Each contributes to the horizontal dimension that defines the entity. All mathematical concepts may not exhibit all of the facets, but these serve as a starting point to measure the breadth of understanding of the concept.

The facets of a concept may be visualized by thinking of each as a slice (sector) of a circular object (Figure 3.1). The idea of a slice is important since the pieces may be disconnected and moved around so that any facet may appear "next to" any other facet. That is why the figure displays the slices as disconnected. Also important to note is the "sloppy" nature of the boundaries. While the boundaries between some facets may be virtually non-existent, the boundaries between other facets may be impenetrable, as alluded to by Selden and Selden (1992). One focus of this research will be on the nature of the boundaries between the facets.

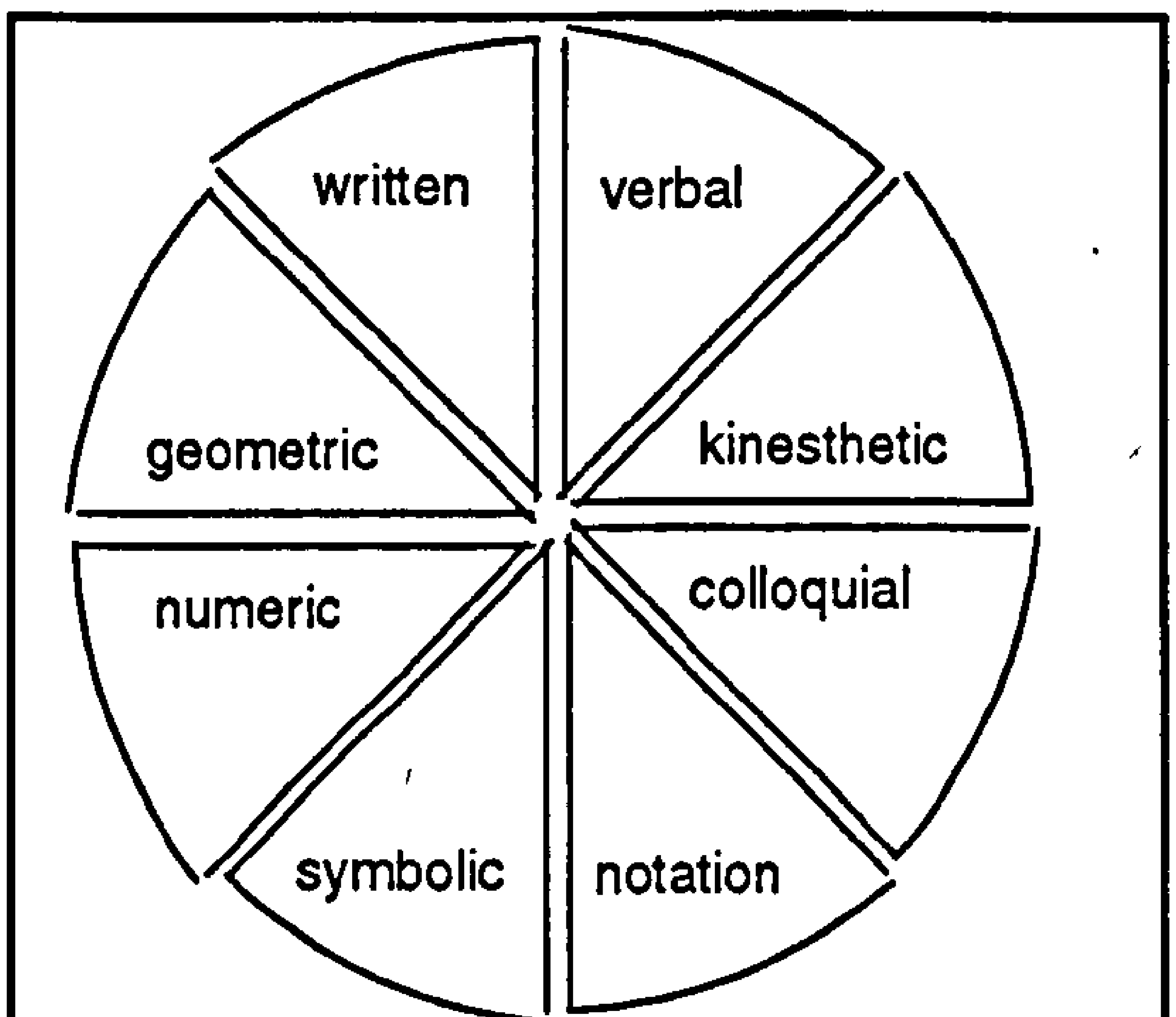


Figure 3.1: Facets of a mathematical entity

### 3.2.2 Layers of a mathematical concept

DeMarois and Tall (1996) use the term **layers** to refer to the various strata of the depth dimension in the development via cognitive process to mental object. Webster's Dictionary describes a "layer" as "a single thickness, coat, or stratum" (Guralnik, 1980, p. 800) In this framework, **pre-procedure, procedure, process, concept,** and **procept** are considered layers of increasing depth. Pre-procedure assumes that the student is on the ground floor, so to speak, with respect to a concept. Such a student exhibits no distinguishable concept image of the entity, at least from a mathematical point of view. For example, most students have encountered the word "function" as used in standard English, but many fewer will have heard this word used in a mathematical context. A procedure is a coherent sequence of actions-a schema of actions. A procedure is a "specific algorithm," as mentioned earlier and can be likened to Davis' visually-moderated sequence (Davis, 1984, p. 35). On the other hand, a process is a cognitive entity, not dependent on individual steps, but rather on the result produced from the original input. While a procedure may be a cognitive entity also, the procedure layer is exemplified by the need to carry out a specific sequence of steps when considering a mathematical entity, similar to Dubinsky's action phase. The process layer manifests itself when the student can think about an entity as a generic activity not reliant on a specific algorithm. The student, given a two-column table, who must know the specific sequence of steps used to convert values in the left column to values in the right column is most likely at the procedure layer while the student, given a two-column table, who can accept the table as a function without knowing the algorithm used may be, at least, at a process layer. As another example, the expressions $2x + 6$ and $2(x + 3)$ represent two different procedures. The results of applying each procedure to a given input are the same. Students who view these as different functions might be classified at the procedure layer while those who classify these as the same function might be placed at the process layer. In fact, Cuoco (1994) suggests that students with this ability have "the necessary cognitive development to think about function as object" (p. 130). According to Cuoco, "Students who view functions as actions think of a function as a sequence of isolated calculations or manipulations" (ibid, p. 122). Students at the procedure layer are dependent on the algorithm performed to obtain output from input. Cuoco suggests that "students who view functions as processes think of functions as dynamic (single-valued) transformations that can be composed with other transformations" (ibid, p. 122) and goes on to suggest that when students can view functions as "atomic structures that can be inputs and outputs to higher-order processes" (ibid, p. 123), such students have an object conception of function. The concept layer aligns closely with the object conception as described in Dubinsky's APOS framework (Cottrill et al., 1996, for example) and with the structural mode of thinking of Sfard (1992). In this framework, the set of concepts is treated as a superset of the set of objects. Concepts can be either nouns or adjectives. For example, both "cup" and "red" are concepts, but "cup" is a noun while "red" is an adjective. Nouns can be acted upon while adjectives cannot. A cup can be filled, lifted, dropped, or washed. These are examples of actions on the noun "cup". No similar actions are apparent for acting on the adjective "red". An object will be considered a concept that can be acted upon. In this sense, a mathematical entity like function is an object that, in turn, is a concept. The procept layer is designed to indicate the flexibility in moving easily between process and object layers as required. The procept layer is closely connected with the ability to exhibit proceptual thinking about the mathematical entity. Students reach the most depth (the procept layer) when they can demonstrate flexibility in viewing a mathematical entity as either a process or a concept, as required by the problem situation.

The depth dimension has been discussed extensively in the literature (see Cottrill et al., 1996; Breidenbach et al., 1992; Dubinsky & Harel, 1992; Goldenberg at al., 1992; Gray & Tall, 1994; Schwingendorf et al., 1992; Sfard, 1992; Thompson, 1994, for example). Layers are closely aligned with the proceptual structure of the entity. The depth development, as described by layers, is related to Sfard's components of concept development. Sfard differentiates between the various depth dimensions of concept acquisition when she writes:

> A constant three-step pattern can be identified in the successive transitions from operational to structural conceptions: first there must be a process performed on the already familiar objects, then the idea of turning this process into a more compact, self-contained whole should emerge, and finally and ability to view this new entity as a permanent object in its own right must be acquired. These three components of concept development will be called **interiorization, condensation,** and **reification,** respectively.
> Condensation means a rather technical change of approach, which expresses itself in an ability to deal with a given process in terms of input/output without necessarily considering its component steps.

> **Reification** is the next step: in the mind of the learner, it converts the already condensed process into an object-like entity. In other words, while condensation is a gradual quantitative change, reification should be understood as a sudden qualitative jump in the way of looking at things–an ontological shift comparable to a transition from one scientific paradigm to another. The fact that a process has been interiorized and condensed into a compact, self-sustained entity, does not mean, by itself, that a person has acquired the ability to think about it in a structural way. Without reification, her or his approach will remain purely operational. (1992, pp. 64–65)

Interiorization occurs as a student moves from the pre-procedure layer to the procedure layer. Condensation implies that a student has moved from the procedure to the process layer. Finally, reification implies the shift from process to concept, similar to Dubinsky's encapsulation.The cumulative measure of the vertical development (Schwingendorf et al., 1992) results in layers of understanding.

Figure 3.2 displays a visualization of the layers of a mathematical concept. Again it is

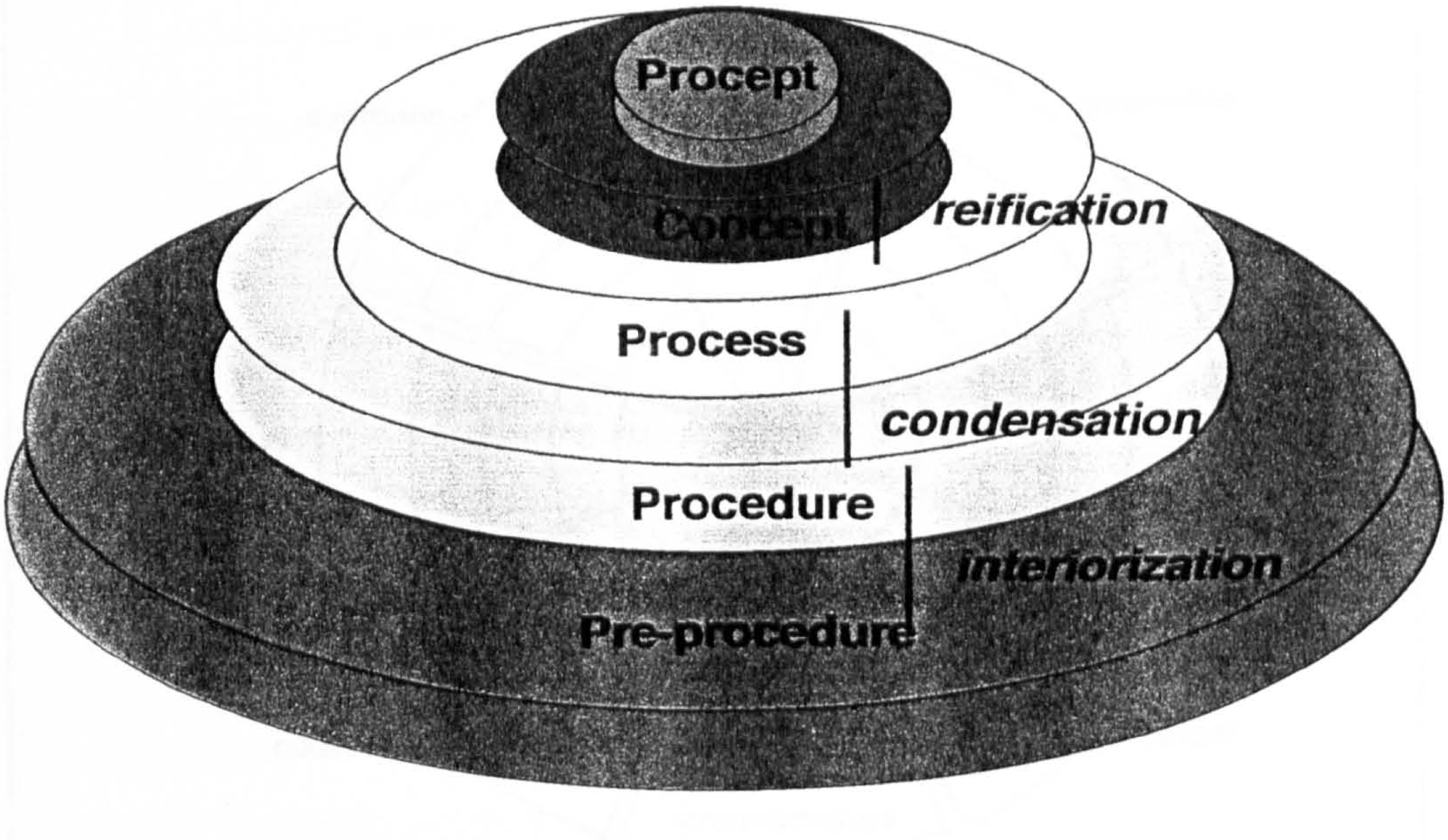



Figure 3.2: Layers of a mathematical entity

important to note that the boundaries between the layers may vary from porous to impenetrable. Sfard (1992) alludes to the fact that condensation (moving from procedure to process) may be gradual and quantitative while reification (moving from process to concept) may be sudden and qualitative. The procedure-process boundary may be rather porous while the process-concept boundary, as suggested by the research by Sfard (1992), Breidenbach et al. (1992), and Dubinsky and Harel (1992), may be impenetrable, at least for some students. Finally, what causes a student to develop the flexibility required to cross the concept-procept boundary? Is traversing this boundary reserved for only our most capable, as discussed by Krutetskii (1976)? Can learning activities be designed to promote the traversing of this boundary? These are questions future research needs to address.

### 3.2.3 Facets and layers combined

Figures 3.1 and 3.2 can be used as visual organizers to analyse a student's concept image of a given mathematical entity. The two dimensions may be combined by viewing the layers from the top, rather than from the side, and slicing the layers into pieces, as represented by the facets.

The visualization in Figure 3.3 provides a convenient way of graphically displaying a student's concept image of a mathematical entity.

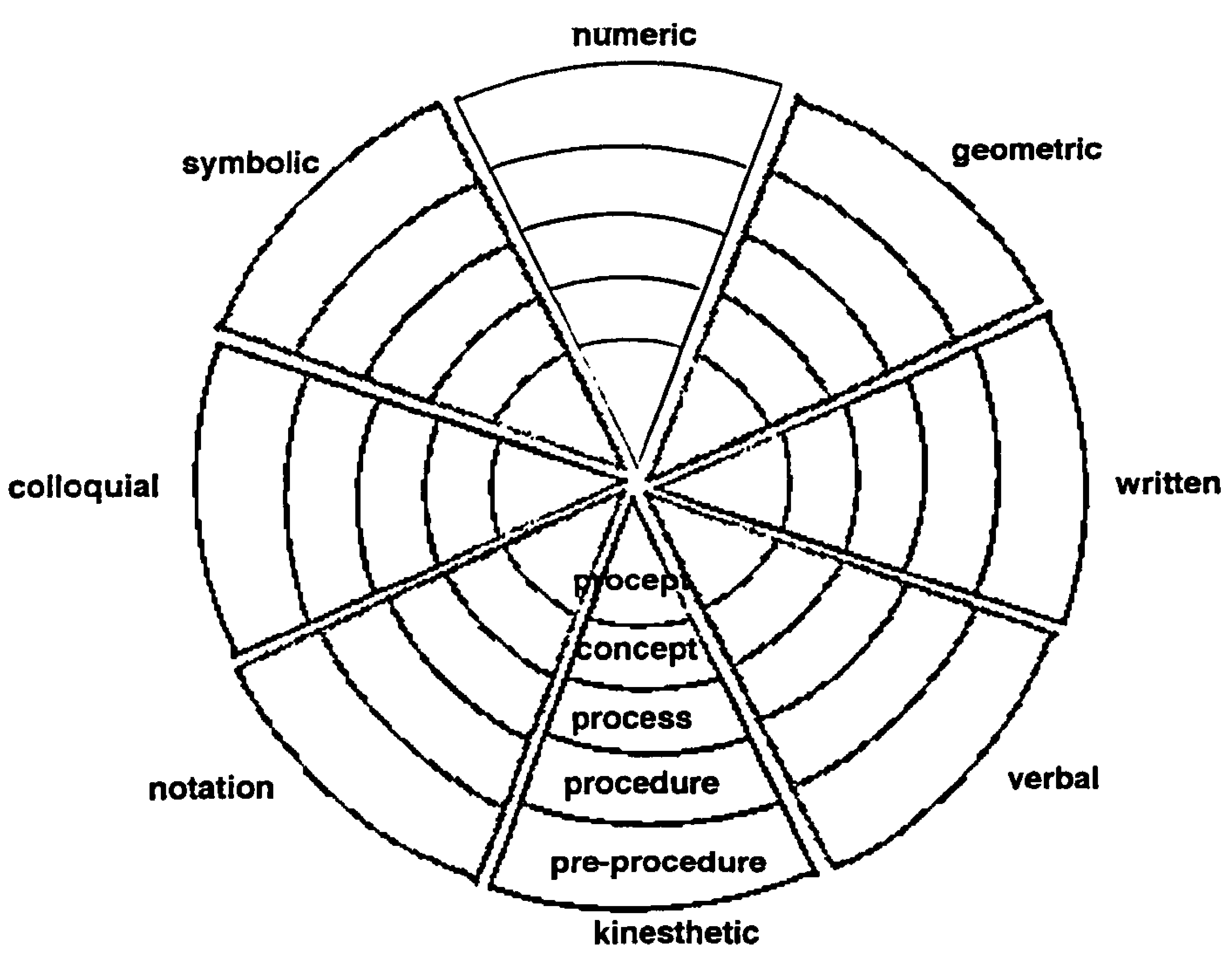



Figure 3.3 Facets and layers

Shading may be used to indicate a student's observed depth of understanding for a given facet. The result is a snapshot of the student's concept image at a given instant in time. Comparing snapshots at various points in a student's development provides insights into the growth (or decay) in the student's understanding of the mathematical entity over time. Note that this visualization may oversimplify the complexity of the cognitive structure. Each facet may not be amenable to the various layers. Some facets, such as numeric, might be essentially more primitive than other facets, such as symbolic. As mentioned previously, the boundaries are porous. Facets flow into each other, such as the symbolic and notation facets. Layers flow into each other such as the procedure and process layers. It may be at these porous boundaries where the most interesting information occurs. Finally, a third dimension could be added to layers and facets called **levels**. Each level might be a collection of layers and facets. Achieving the procept layer for one facet may be a precursor to moving into the procedure layer at the next level. For example, students often encounter evaluation of expressions prior to the function concept. Students may have already encapsulated the algebraic process of evaluation as the concept of expression. By applying this knowledge to function, they may move from the level of expression to the level of function.

### 3.2.4 Co-development of both facets and layers

The co-development of understanding the various facets of a mathematical entity and of the increasing depth of understanding of the entity along each facet is an important issue. Figure 3.3 implies that there may be a pre-procedure, procedure, process, concept, procept path of increasing depth along each facet. This may not be the case. There are at least two distinct paths of development in mathematical growth: one based on objects and being visuo-spatial and a second based on actions on objects that lead through processes finally to procepts. Tall (1995a) indicates that perceptions of and interactions with the external world produce both objects and actions. If action-based, then growth may follow a path described by Sfard's operational-structural sequence or Dubinsky's APOS (Cottrill et al., 1996, for example) or the Procedure-Process-Concept-Procept layers discussed above. However, if object-based, then the layers might be significantly different with increasing depth occurring as a result of, as Tall describes, "visuo-spatial prototypes becoming successively more verbal-deductive" (1995a, p. 64). Mathematical concepts, like function, consist of a number of facets. Some of these facets, such as symbolic, are initially perceived as action-based while others, such as a graph, may be initially perceived as object-based. The distinction between object-based and action-based suggests the possibility of two very different paths in developing increasing depth for these facets. This may suggest why the connections between certain facets prove to be so difficult. For example, Schwartz

(1992) suggests that the symbolic representation reveals the process nature of function while the geometric representation reveals the object nature of function. Contrasting this point of view, Goldenberg et al. (1992) argue that the geometric facet of function should be dynamic in order to construct an appropriate concept image and a conceptual link between the symbolic and geometric facets. In developing snapshots of a student's concept image of a mathematical entity along the lines of Figure 3.3, the issue of connections between facets with different paths of increasing depth for a given facet will be considered.

## 3.3 Facets and Layers of the Function Concept

### 3.3.1 The facets of the function concept

The previous discussion has introduced the facet-layer framework for any mathematical entity. What if that entity is "function"? Three of the facets–numeric using tables, geometric using graphs, and symbolic using equations–have been discussed extensively in the literature. (Cuoco, 1994; Schwingendorf et al., 1994; Sierpinska, 1988; Thompson, 1994, for example) Written and verbal descriptions of function represent two other facets. Function notation defines the notational facet. Function machines are used to explore the colloquial facet. Finally, the kinesthetic aspect might be revealed by asking students to act out their understanding about function. Table 3.1 displays a list of specific facets as applied to the entity "function". However, note that several of

TABLE 3.1: Function facets

| Generic Facets | Function Facets |
|---|---|
| numeric | two-column tables or sets of ordered pairs |
| geometric | rectangular coordinate graphs |
| symbolic | equations in two variables |
| written | definition expressed in writing |
| verbal | definition expressed orally |
| kinesthetic | physical demonstration of what a function is |
| colloquial | function machines |
| notation | function notation |

these facets have sub-facets. For example, there are several ways to represent a function symbolically. Both $f(x) = x + 1$ and $f: x \longrightarrow x + 1$ are symbolic ways of defining the same function. Visually, a two-dimensional coordinate graph provides a visualization for functions of one variable from the real numbers to the real numbers. Other visualizations, such as drawing correspondences from domain to range, can also be used for the geometric facet. Another visualization, dubbed Dynagraphs (Goldenberg, Lewis, & O'Keefe, 1992), allows the user to dynamically manipulate the domain variable and observe the affect on the range variable. Each variable is displayed in its own space as opposed to being viewed in the same plane like the rectangular coordinate system. Verbal and written descriptions can vary greatly, with each definition adequately describing the entity. Crucial here is the ability to assimilate various "correct" definitions into one's concept image and the ability to detect problems with "incorrect" definitions. The function machine, as an example of the colloquial facet, provides an informal entry into the mathematical entity.

Another issue is the information about a function supplied by each facet. Given an equation in two variables which is solvable for one of the variables, one can identify the input variable, the output variable, and the process. A function machine displays similar information. The equation and function machine do not, however, provide significant information about the behaviour, such as extrema, of the function unless one knows about the behaviour of certain classes of function. Equations and function machines have a dynamic feel in that they can be considered manipulable. On the other hand, a two-column table provides information about the input and output, but hides the process. Furthermore, the table is only partially expressive of a function if it is developed from an equation or a graph. The table may be fully expressive of the function if it is created from data. A table may indicate some of the behaviour of the function (intervals on which the function is increasing, for example), but care must be taken in how one interprets the contents of the table. A table on paper is static, but, on a graphing calculator, becomes dynamic. A graph provides information about the input, the output, and the behaviour of the function, but does not, like the table, make the process explicit. A graph is often viewed as static (an object) which causes students problems when they must somehow manipulate a graph. The written, verbal, and kinesthetic facets depend on where the emphasis is placed. Some students may place the emphasis on the input-output pairs, for example, while others may focus on the process.

### 3.3.2 Analysing the layers of the function concept

While the layers–pre-procedure, procedure, process, concept, procept–represent a merging of models for measuring the depth of understanding, this research attempts to analyse each facet along this depth dimension. A key question relates to the evidence needed to support a claim that a students is at a given layer for a given facet. The researcher looks at the work of several others to help define some key points in this development. First, the ability to reverse a path may be considered a distinguishing feature between being at a procedure layer versus a process layer. Dubinsky and Harel (1992), following Breidenbach et al. (1992), use reversibility as a test to differentiate between their action and process stages. Eisenberg (1992) writes: "Levels of understanding exist which can be measured by the ability to reverse the path of development" (p. 174). Reversibility can be applied particularly to the symbolic, numeric, geometric, colloquial, and kinesthetic facets. The ability to reverse, within the framework of one of these facets, is used as evidence that the student may be at a process layer as opposed to a procedure layer for the given facet.

Dubinsky and Harel (1992), based on the work of Breidenbach et al. (1992), discuss the fact that a process conception of function is composed of several aspects. They identified, as a result of interviews with students, four factors that affected students' conception of function. Among these were restrictions students placed on the concept. The manipulation restriction was defined as "the ability to perform explicit manipulations, or you do not have a function" (1992, p. 86). Several facets, especially the symbolic and numeric, are subject to this restriction. Others, such as the written, verbal, and colloquial, may be affected. The quantity restriction, which requires that the inputs be numbers, again cuts across several facets, the most obvious being the numeric and the symbolic. The continuity restriction seems to be unique to the geometric facet. The second factor identified by Dubinsky and Harel is the severity of the restriction. For example, some students with the manipulation restriction must know the specific sequence of steps before admitting to a function while others are comfortable if they are aware of a procedure to follow even if they cannot carry the procedure out. The ability to construct a process when none is explicit is the third factor. The numeric, geometric, and, maybe, the colloquial facets would be most susceptible to this factor. Finally, Harel and Dubinsky cite the "uniqueness to the right" condition as the fourth factor. This condition refers to the fact that, for each input, a function can have one and only one output. They also mention the confusion between this condition and one-to-oneness. This factor crosses all the facets, though it may be less apparent in the kinesthetic facet. Indeed, Dubinsky and Harel report protocols in which the "uniqueness to the right" condition is applied correctly in one setting (facet), incorrectly applied in another setting (facet), and ignored completely in a third setting (facet). This research contains data supporting this phenomena.

Dubinsky and Harel (1992) go on to differentiate some specific behaviours that they suggest provide evidence of a student being process-oriented versus action-oriented. With respect to the symbolic facet, Dubinsky and Harel, referencing equations in two variables, suggest that a student is at an action stage if a student must explicitly solve for a variable before stating whether the equation represents a function or not. On the other hand, students who can state that the equation, such as $3x - 5y = 7$, is a function or not without doing the physical solving are more likely to be at the process stage. With respect to the numeric facet, Dubinsky and Harel analyse how students respond to sets of ordered pairs. Again, the question is to determine if the set of ordered pairs is a function. Students who are likely to be at the action stage look for rules that relate the first component to the second component while students who are most likely process-oriented do not need to know a specific rule in order to answer the question. Finally, with respect to the geometric facet, Dubinsky and Harel note that it may be hard to distinguish an action phase since graphs are given as static objects or, in other words, in a figural form. They suggest that students are likely to be at the action stage if students do not accept discrete points as functions and if students are essentially ruled by prototypes to determine if a graph is a function or not. Students are likely to be process-oriented if they associate the graph with a process that moves from input to output and if they demonstrate the ability to reverse the process by being able to move from output to input.

Dubinsky and Harel use this framework to analyse interviews with students to determine if the student can be classified at the action level or at the process level. They use of combination of all this information to place the student somewhere on the continuum between action and process. The research findings reported herein suggest that an alternate snapshot might view the students at different layers for different facets. For example, a student may be at a procedure layer for the geometric facets, at the process layer for the symbolic facet, and at a concept or even procept layer for the written facet. The diagram in Figure 3.3 will be used to analyse the depth (by layers) of the concept image for each given facet, where possible. The ideas expressed by Dubinsky and Harel looking at the action-process continuum are used in this research to help distinguish between the procedure and process layers.

Schwingendorf et al. (1992) take particular aim at analysing the depth (action, process, object) of understanding along written, symbolic, and geometric facets using students in calculus at an U.S. university as their subjects. As part of the study, the authors spend some time discussing the boundary between the symbolic and geometric facets.

Monk (1992) takes particular aim at the layers of the geometric facet when he contrasts "pointwise questions" with "across-time questions" as mentioned in Chapter 2. Students who are only able to deal with "pointwise questions" may well be at the procedure layer for the geometric facet, while students who are able to correctly answer "across-time questions" may be operating at a process layer for the geometric facet. A third type of question that focuses on the graph as an object might provide insight as to whether a student has reached a concept layer for the geometric facet.

An interesting question asks if there are notions equivalent to pointwise versus across-time conceptions for other facets. For example, considering the symbolic facet, questions asking students to find specific outputs for given inputs might be considered "pointwise" while questions that focus on the behaviour of the output over a range of inputs might be considered "across-time". A similar type of analysis could be applied to the numeric (tables) and colloquial (function machine) facets. Such questions could prove helpful in separating the student who is at the procedure layer from the student who is at the process layer for the given facet.

The preceding contains ideas for classifying students at either the procedure or the process layer for the numeric, symbolic, and geometric facets. The discussion also applies to the colloquial facet. A student could be assigned to the concept layer if he/ she demonstrates an ability to treat each of the facets as manipulable objects (I.e., Use each as an input to a higher level procedure). Evidence that the student has reached the procept layer lies in the student's demonstrated flexibility in moving between thinking of the facet as a process and as a concept. Cuoco (1994) discusses the procept layer when he writes: "... if students are to use functions as objects, they must have at first dealt with these functions as processes. And, if students are to use functions as *both* processes and objects, they must be able to *de-encapsulate* functions-as-objects into the underlying processes" (p. 123).

Little has been said about the layers as applied to the verbal and written facets. In asking students to write about or to talk about functions, a key to determining the depth of their understanding is the flexibility with which they write about or talk about function. If a student's written/verbal description relies on a specific procedure to produce output from input, this student is likely to be at the procedure layer. If the student is able to discuss function as a general process, he/she most likely is at a process layer. This includes seeing a function as a relationship between input and output. A student who can think of function as an object, who can describe it as a "thing" that can be manipulated, might be at the concept layer. The ability of a student to move flexibly between the process conception and the object conception, as demonstrated in written and verbal descriptions, might be evidence that this student is proceptual with respect to verbal and written facets. Finally, an indicator of the strength of a student's verbal conception of function is the student's ability to assimilate alternate function definitions into his/her own definition.

Another facet that has received little attention is the notational facet. How does the student interpret function notation? If a student can only interpret function notation as multiplication, he or she may not even be at the procedure layer. If a student must know a specific algorithm in order to imbue function notation with meaning, he or she is most likely at the procedure layer. A student who sees function notation, such as $f(x)$, and is willing to accept it as a symbol for a process may be at the process layer.

Evidence that a student is at the concept layer lies in the student's ability to manipulate function notation as an object. Is the student comfortable with symbolism such as $f + g$ or $f \circ g$ when no specific functions are given? Assigning a student to a procept layer implies that he or she can flexibly move back and forth between viewing function notation as a process and viewing the notation as representing an object.

Finally, how do layers apply to the kinesthetic facet? Critical to answering this question is how the student demonstrates physically what a function means to him or her. For example, a student who demonstrates a specific algorithm might be at the procedure layer while a student who demonstrates a more generic process might be at a process layer. Students at the concept layer might portray a function as a "thing" rather than as some action to perform. Students who can demonstrate function as both a "thing" and as some action may be proceptual kinesthetically.

### 3.3.3 Boundaries between facets

One area that has received some attention is students' ability to move comfortably between facets. Each facet may be either a conceptual unit or consist of a connected web of conceptual units. In this latter case, each facet may constitute its own concept image. Crossing the boundaries suggests the creation of appropriate conceptual links between the facets. The existence of these links implies that students can choose the most appropriate facet to use for a given problem.

The relationship between the numeric facet, the symbolic facet, and the geometric facet has been the subject of much research. Cuoco suggests that the connections between "representations" are properties of a "higher-order function" (1994, p. 125). In essence, there is a mapping from one facet to another facet. In discussing traditional transformations on functions, Cuoco writes: "Activities such as these can be quite useful in helping students understand the connections between the function [symbolic facet] and the graph [geometric facet], but these connections are properties of the higher-order function G" (ibid, p. 125). He has previously defined G as a function that acts on a function *f* to produce a graph. Of course, seeing the boundary between symbolic and geometric as a function may require the ability to think of function as an object that can be acted upon. Cuoco also argues for the use of the numeric facet, possibly as an intermediary between symbolic and geometric. Instead of thinking of G as a function directly from the symbolic facet to the geometric facet, think of G as a composite function that first converts the symbolic to the numeric and then converts the numeric to the geometric. He suggests that such an approach might reinforce the idea that a graph is "just one method for visualizing a process" (ibid, p. 136).

Demana and Waits (1990) were among the first textbook authors to incorporate, as a major feature, the study of functions using numeric, symbolic, and geometric facets simultaneously. In the Preface, the authors note that their focus is on the connections among the symbolic, the geometric, and the problem situation. Demana, Schoen, and Waits (1993) subsequently discuss the importance of tables as an entry point into functions when they write: "... the frequent use of tables helps to establish function as a major theme of mathematics, a theme that is missing from the typical early curriculum" (p. 22). Kaput (1989) notes the importance of tables foreshadowing the eventual dynamic link, using computer graphing software, between the symbolic, geometric, and numeric facets. Kaput suggests that some representations are display-based while others are action-based. He postulates that the numeric and geometric facets are primarily display-based while the symbolic tends to be action-based. He goes on to detail some software under development to dynamically link these facets. He mentions one other facet–the colloquial facet– when he discusses function machines as good candidates for computer simulations. Finally, the idea of approaching each function using these three facets was carried through to the Calculus by Hughes-Hallett and Gleason (1992) with the "Rule of Three".

Schwartz and Yerushalmy (1992), Confrey (1993), and Kaput (1989, 1993, 1995) particularly address the issue of software in linking, among others, the symbolic, numeric, and geometric facets. Schwartz and Yerushalmy (1992) discuss a curriculum based on software capable of the dynamic linking suggested by Kaput (1989). They focus primarily on the dynamic linking of the symbolic and geometric facet. They argue that the symbolic facet is more viable for understanding function as process and that the geometric facet is more viable for understanding function as an entity. Operations on functions are divided into two camps: those that operate on the symbolic facet and those that operate on the geometric facet. They mention that many graphing packages only allow direct manipulation of the symbolic form, not the graphic form. Instead a better tool is one in which either facet can be directly manipulated and the effects on the other facet can be immediately seen. Schwartz and Yerushalmy write: "A symmetric full-tool would permit users to both manipulate the symbolic representation symbolically and see the graphical consequences of their actions, and to manipulate the graphical representation graphically and see the symbolic consequences of their actions" (1992, p. 264). Software such as The Function Analyzer (1990), The Function Supposer (1990), and The Algebraic Proposer (1987) embody the approach to algebra described by Schwartz and Yerushalmy.

Confrey (1993) describes Function Probe (Confrey, 1991) as a software tool for teaching functions from a contextual viewpoint emphasizing the numeric (tables), geometric (graphs), and symbolic (equations) representations. She notes her particular interest in tables as a vehicle for developing insight about functions when she writes: "Our interest in tables increased as we witnessed students frequently using the table as the primary means of entry to the problem" (1993, p. 57). She references the power of the covariation approach in building tables allowing students to build a column, fill it, build a second column, and fill it. Confrey makes an important point that, in developing a technological tool, no representation should dominate the others. Indeed there are both losses and gains in each representation and the integrity of each must be protected.

Monk (1992) suggests that it is important to notice how students sketch graphs. This directly relates to their kinesthetic sense of the graph and hence serves as a way to look at the boundary between the geometric and kinesthetic facets. Students who focus, during the sketch, on plotting discrete points might be more "pointwise", while students who focus more on the general behaviour of the graph may be more "across-time". So, in one sense, Monk is describing a quality of the boundary between the geometric and kinesthetic facets.

The research discussed in this document will include the nature of boundaries between facets. In particular, the impenetrability of various boundaries will be explored. In some cases, the boundary may be porous in one direction and impenetrable in another.

The boundaries between the symbolic, numeric, and geometric will be explored in light of the availability of graphics calculators (TI–82 or TI–83, for example) that allow for the exploration of each of these facets. Unfortunately, these tools suffer from the restriction mentioned by Schwartz and Yerushalmy; that is, the symbolic form is the only one that is directly manipulatable. Other boundaries that will be particularly interesting are the boundaries between verbal and written facets and the boundaries between the notation facet and the other facets.

### 3.3.4 Boundaries between layers

The boundaries between successive layers again may be very porous for some students while incredibly difficult to pierce for others. The first boundary, between pre-procedure and procedure, may be the one most easily crossed for interiorization may only require that the student be introduced to the subject at hand. Many researchers, including Sfard (1992), Dubinsky and Harel (1992), Schwingendorf et al. (1992), and Breidenbach et al. (1992), have devoted significant time to considering the transition from procedure to process. It can be quite difficult to determine where the student is with respect to this procedure-process boundary as it is not uncommon for students to exhibit understandings that fall in both camps. Sfard maintains that this particular boundary is crossed in a gradual fashion as the student's knowledge of the concept increases. The process-concept boundary is crossed through the process referred to as reification by Sfard and encapsulation by Dubinsky. Sfard argues that this boundary is difficult to cross and that the crossing, when it occurs, is rather sudden. The final boundary, concept-procept, is one with little research behind it. How do we teach to promote proceptual thinking? How do we influence students in such a way as to encourage the flexibility required in proceptual thinking? Tall (1995a) suggests that problem solving approaches (for example, Mason et al., 1982) as part of a curriculum may assist the development of this flexibility as opposed to the "standard" lecture-oriented course packed with curriculum. This area awaits future research.

To help in analysing the boundaries between layers, it may be helpful to view the dimension of increasing depth as a continuum (Figure 3.4). Instead of students being at a particular layer, it is more likely that they will be somewhere along the continuum from pre-procedure to procept. Some behaviours, for example, may suggest the student is at the process layer while others may suggest the student is at the concept layer. Such a student can be placed on the continuum between these two layers.

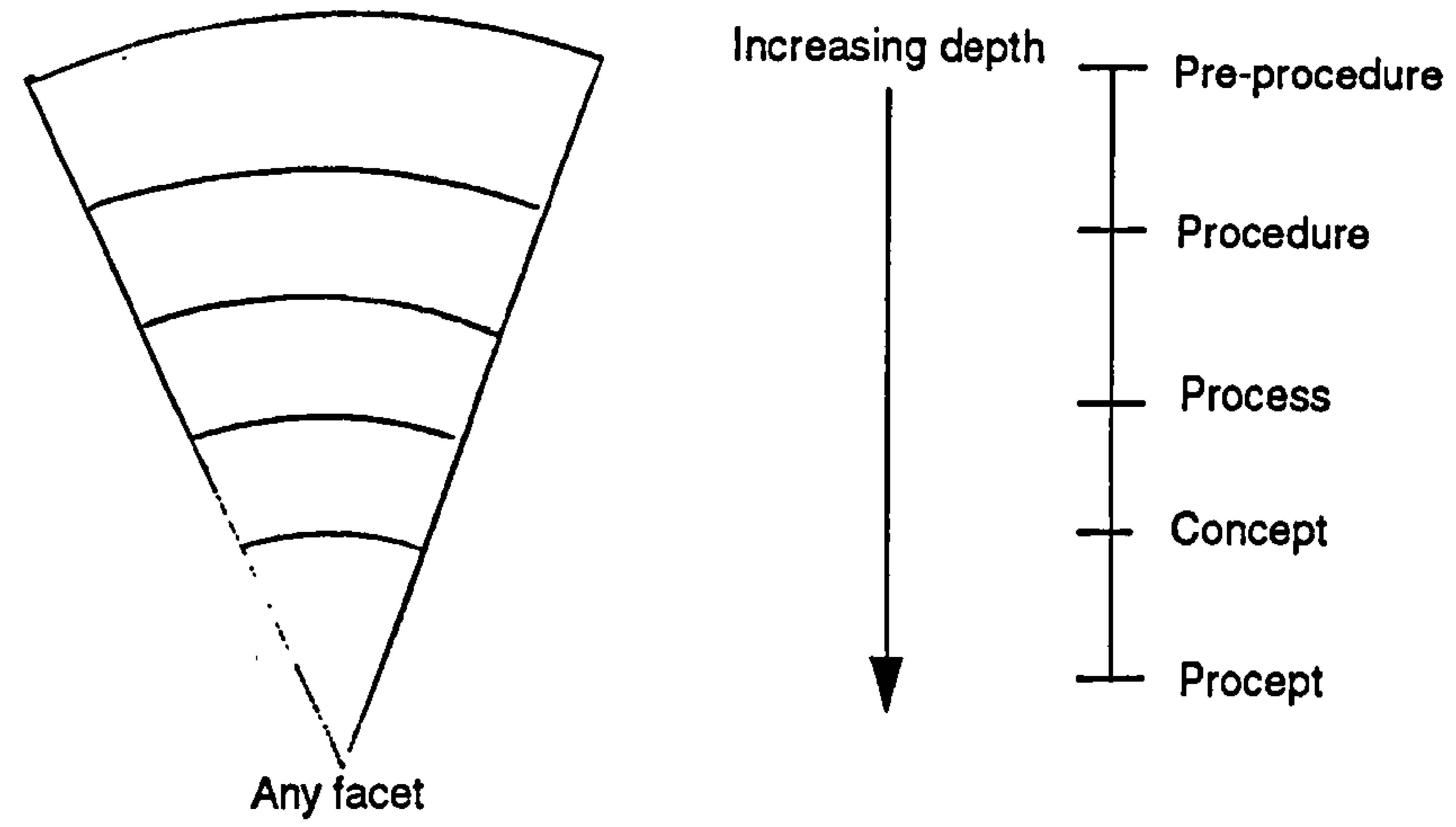



Figure 3.4: Layers viewed along a continuum

## 3.4 Other Models for Concept Development

Other researchers have attempted to model student understanding of mathematical concepts. This section briefly reviews several other models to provide contrast with the theoretical model described in this Chapter. Pirie and Kieren (1994) have created a model for the mathematical understanding of a concept in which students move from a "primitive knowing" stage to, ultimately, an "inventising" stage. In a sense, this model serves as another way to view deepening understanding of a mathematical idea. The model is used to track student understanding over time, using a technique the authors call "mapping" which results in a path for each student. Pirie and Kieren write: "Every student will have a singular path for any topic, and yet all paths will involve 'folding back to move out' in their actualization" (1994, p. 82). "Folding back" means that a student returns to an earlier (inner) stage to extend his/her understanding when confronted with a problem that is not immediately solvable.

The Pirie and Kieren model exhibits the notion of levels, as defined in this research. Using primitive knowing as equivalent to the pre-procedure layer discussed herein, they suggest that the understanding of a concept could serve as the primitive knowing stage of a new concept. Another issue mentioned by the researchers is the existence of "don't need" boundaries. Such boundaries are used "... to convey the idea that beyond the boundary one does not need the specific inner understanding that gives rise to the outer knowing" (ibid, p. 69). The idea of not needing the specific inner understanding is problematic. While the subject may not "need" this understanding, he or she may need to be aware of and be able to access this layer of understanding in order to develop the flexibility required of a proceptual thinker. Perhaps the lack of need for the inner understanding is somehow equivalent to the curtailment that occurs in capable students discussed by Krutetskii (1976).

At each level of their model, Pirie and Kieren suggest the existence of two complementary aspects: acting and expressing. Acting involves such features as doing, seeing, and predicting while expressing involves reviewing, saying, and recording. The authors write: "Acting can encompass mental as well as physical activities and expressing is to do with making overt to others or to yourself the nature of those activities" (ibid, p. 175). In some sense, these two aspects serve as facets or sub-facets for each of their levels (layers). Pirie and Kieren's model focuses on mapping the dynamic development of understanding. The perspective in this research aims more at taking snapshots of student concept images at moments in time.

As mentioned in Chapter 2, Martinez-Cruz (1995) identifies three different models of student thinking about functions as a result of a technology-enhanced precalculus class: the graph model; the equation model; and, the unique correspondence model. Martinez-Cruz constructs a diagram for each student suggesting a network of the images of function. Students may have images that belong to all models, but that Martinez-Cruz suggests that one model takes precedence in each student.

Moschkovich and Schoenfeld (1993) include a framework when they discuss the aspects of understanding in looking at the representations of linear relations. Their model is a rectangular array in which the representation (tabular, algebraic, graphical) creates the columns and the perspective (process, object) defines the rows. The column headings are essentially 3 facets while the row headings are two major layers. Kieran (1993) lists a model that essentially deals with the boundaries between various facets. She also uses a two-dimensional array with both rows and columns labelled: Situations: Pictures or verbal descriptions; Tables of data; Graphs; and, Algebraic expressions. However, to read the array, the rows represent the "From" facet and the columns represent the "To" facet indicating skills necessary to move from one facet to another. She labels the rows "interpretational skills" and the columns "modelling skills". The focus appears to be on crossing the boundaries between facets.

Monk (1992) suggests two general levels of organization as students develop in their understanding of function. One level is specific to discrete points, computing inputs and outputs. This level appears closest to the procedural layer. The second level relates to overall patterns of behaviour of functions. This level relates to the process layer at least, and possibly to the concept and procept layers.

## 3.5 Conclusion

This chapter has situated the theoretical framework for this research within existing research. The chapter begins by considering the development of understanding of a mathematical entity as being composed of both a horizontal dimension, divided into facets, and a vertical dimension, divided into layers. An exhaustive discussion of possible facets and layers for a mathematical entity follows. The two dimensions are combined into a single graphic image that allows for the visual depiction of a student's concept image along these two dimensions. The mathematical entity, function, is analysed using this framework as a lens. The different facets and layers of "function" are discussed in detail along with possible evidence for concluding that students are at a given layer for a given facet. Attention to the nature of the boundaries between both facets and layers is given. Finally, several other models for the development of the understanding of a mathematical entity are mentioned to help provide a contrast to the framework used in this research.

# CHAPTER 4 Instructional Treatment

## 4.1 Introduction

This research is designed to assess the reasonableness of using function as a unifying concept in algebra, as called for in several reform documents (NCTM, 1989; AMATYC, 1995 for example). To do so, the focus has been placed on college students who must take a first course in algebra, subsequently referred to as beginning algebra. As discussed previously, these students have had "debilitating" prior experiences with mathematics in general and with algebra in particular. In order to test whether function is an "acquirable" concept for these students, they were exposed to a one-semester beginning algebra course using a non-traditional text with function as its organizing concept. This chapter details the key components of the instructional materials and describes the treatment of function. Other instructional models, to provide context, are briefly discussed.

## 4.2 Curriculum Overview

The researcher was a co-principal investigator for a U. S. National Science Foundation grant with a focus on writing algebra reform materials (DeMarois, McGowen, & Whitkanack, 1996a; DeMarois, McGowen, & Whitkanack, 1996b) for developmental algebra courses offered at colleges. The authors subscribe to the theoretical perspective that the main concern in mathematics should be "with the students' construction of schemas for understanding concepts. Instruction should be dedicated to inducing students to make these constructions and helping them along in the process" (Dubinsky, 1991, p. 119). Each unit begins with an investigation of a problem situation. Following the gathering of data, students work collaboratively on tasks based on the investigation activities. A discussion in the text summarizes essential mathematical ideas. The instructor orchestrates inter-group and class discussions of the investigations. Explorations are assigned to reinforce the knowledge students are expected to have constructed during the first two steps of the cycle.

Tasks are designed to help students reflect on mental constructions they have made often as a result of working with a computer or calculator. As suggested by Dubinsky and Tall, "Computers can be used in education to help students conceptualize, and construct for themselves, mathematics that has already been formulated by others" (1991, p. 231). Symbolic manipulators are used to allow students to investigate concepts while allowing the computer to execute procedures. While such procedures are often familiar to these students, a student's work is error-prone and limited to simple cases. Procedural errors and intensive computations are seldom problems on the computer. In this way, the computer is used in a way similar to that described by Gray and Tall:

> By using the computer to carry out procedures, the learner can be focused on the *products* and thus the higher level activities can be encouraged earlier and separately from the processes. This reduces the conceptual strain and offers the possibility of the less able breaking out of the proceptual divide wherein they cannot master the procedure because it is too complex and they therefore cannot encapsulate the procedure as a mental object because the procedure causes too much cognitive strain. Thus the computer can be used, by a process of **selective construction**, to encourage the formation of flexible procepts in a wider range of ability. (1991c, p. 12)

Reflection on the result of the procedure can precede facility with the procedure itself. Having students talk with members of their team as they work on a task is another way of getting them to reflect on the problems and the solutions, whether discovered by themselves or presented by someone else.

A significant difference between this approach and that of traditional materials is that the reform materials avoid assigning exercises until after there is a reason to believe that students have constructed the relevant knowledge. Hiebert and Carpenter write: "The emphasis should be placed initially on supporting students efforts to build relationships rather than encouraging them to become proficient executors of procedures" (1992, p. 74). When one is learning something new there is a tendency for early interpretations to be inappropriate as students overgeneralize. Working with too many similar examples could cast these misunderstandings in stone. According to Dubinsky,

> ... working with examples may not be very much help in the *construction* of concepts. Indeed, we agree with Tall (1986) and it is a major aspect of our theory that understanding mathematical ideas come from sources other than looking at many examples and "abstracting their common features", which is what happens if there is only empirical abstractions...it is not clear that more than a very few examples are neces-

> sary to construct a concept...one might well reflect on the contrast between the repetitive examples that seem to be required by conventional wisdom and the single, representative example which so often seems to be in the mind of the mathematician who understand a particular concept. Tall (1986) has referred to this as the *generic example*. ... We would like to go further in our critical view of repetitive examples and suggest that the practice can even be harmful. Yes, the effect of practice will be to reinforce structures that are present. But we would raise the questions, what structures are these? Are they part of the student's concept image which conflict with the concept definition? (1991, pp. 121–122)

Previously-learned misunderstandings need to be identified by the students and reconstructed appropriately. Investigations are designed to enable the instructor to facilitate students discovery of previously-learned misconceptions that contribute to a lack of success. Most students come to such a course lacking an appropriate understanding of arithmetic concepts and basic algebra concepts essential for success in these and subsequent mathematics courses.

In considering the curriculum, the authors agreed that relationships and thus functions would form the core from which other mathematical ideas would follow. Such an approach is supported by Sierpinska: "The most fundamental conception of function is that of a relationship between variable magnitudes. If this is not developed, representations such as equations and graphs lose their meaning and become isolated from one another" (1988, p. 572) Emphasis is placed on the process nature of function. Function as process may be a **cognitive root** (Tall, 1992b) for the development of the more formal definition. But here caution is suggested. As Tall states: "The idea of function as a process may prove to be a more suitable cognitive root for the formal concept, but along the line of cognitive development there are many obstacles to overcome, including the encapsulation of the process as a single concept and the relating of this concept to its many and varied alternative representations" (1992b, p. 501). As each new function arises, investigations encourage the use of multiple facets and call for wise choices in terms of what facet might be best for analysing a specific problem. Tables, equations, graphs, function machines, verbal descriptions are all used to analyse relationships. Graphing calculators provide support for the tables, equations, and graphs. Computer software allows for the manipulation of symbolic forms.

## 4.3 Function Strand

The previous section described the overall goals and some general features of the curriculum project. In this section, the focus turns to the key concept of this research: function. The development of the function concept throughout the curriculum will be detailed.

Chapter 1 of the text (DeMarois, McGowen, & Whitkanack, 1996a) emphasizes problem-solving strategies and the concept of variable. A variation of the M U puzzle (Hofstadter, 1980) is used to build some understanding of a mathematical structure and the rules of the M U puzzle are written using mathematical notation and variables. Variables are used to generalize pattern and rules. Domain for a rule is introduced as a vehicle for determining what can be used as input to the rule. The chapter concludes by comparing arithmetic expressions and the more general algebraic expressions. Two-column tables are introduced to explore the values of given expressions for various values of the input(s) to the expression.

Chapter 2 begins by introducing the whole numbers and exploring algebraic expressions in the context of the set of whole numbers. After investigations involving problems whose domain is the set of whole numbers, order of operations for the four basic operations plus exponentiation is investigated numerically. Students write generalizations for commutative, associative, and distributive properties. Next algebraic expressions (primarily first and second degree polynomials) are investigated both in terms of what they mean and how to evaluate them. Students are encouraged to use calculators to perform evaluations of given expressions. At this point, function machines for the four basic operations are introduced and students complete the chapter by using function machines to parse and evaluate first and second degree polynomials. By the end of Chapter 2, students have been introduced to two-column tables and function machines for expressions even thought the formal concept of function has not been developed yet. Thus they have been introduced to two facets of function before encountering the concept itself.

Chapter 3 is titled Functional Relationships and is devoted to a detailed investigation of function. As with most concepts in the text, function is introduced using a problem to solve. In this case, a number puzzle is used. Students are given an array (Table 4.1) and asked to determine a rule that outputs the column number a whole number will occur in when the input is the whole number greater than 1.

**TABLE 4.1:** Looking for patterns

| | | | | |
|---|---|---|---|---|
| | 2 | 3 | 4 | 5 |
| 9 | 8 | 7 | 6 | |
| | 10 | 11 | 12 | 13 |
| 17 | 16 | 15 | 14 | |
| | | | | |
| | | | | |
| | | | | |

While there are several techniques to solve this puzzle, students, through a series of questions, are led to use division modulo 8 and the resulting patterns to answer the question. As a result, students are introduced to the process of finding finite differences between successive column elements. Function is defined as follows after students complete the initial investigation of the puzzle: "A function is a process that receives input and returns a unique value of output" (DeMarois, McGowen, and Whitkanack, 1996a, p. 92). In this case, the input is the whole number greater than 1 and the output is the column number. The function machine (Figure 4.1) is introduced simultaneously. But the investigation does not end there.

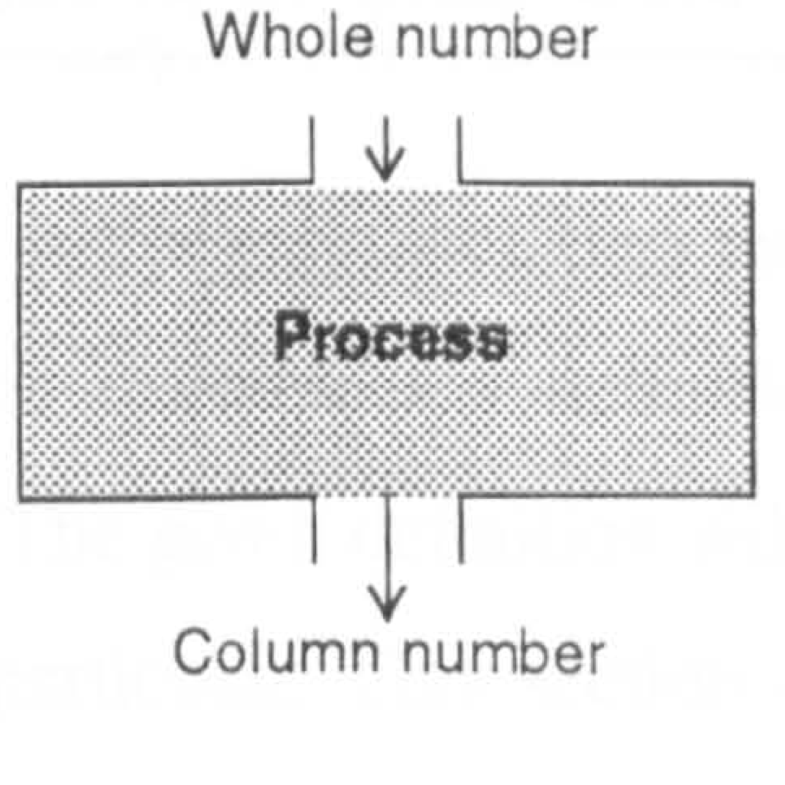



Figure 4.1: Blank function machine

The process in Figure 4.1 must be determined. This is broken into two functions. The first determines the remainder modulo 8 for the input. The second uses the remainder to assign a column number. The result is a composite function (Figure 4.2).

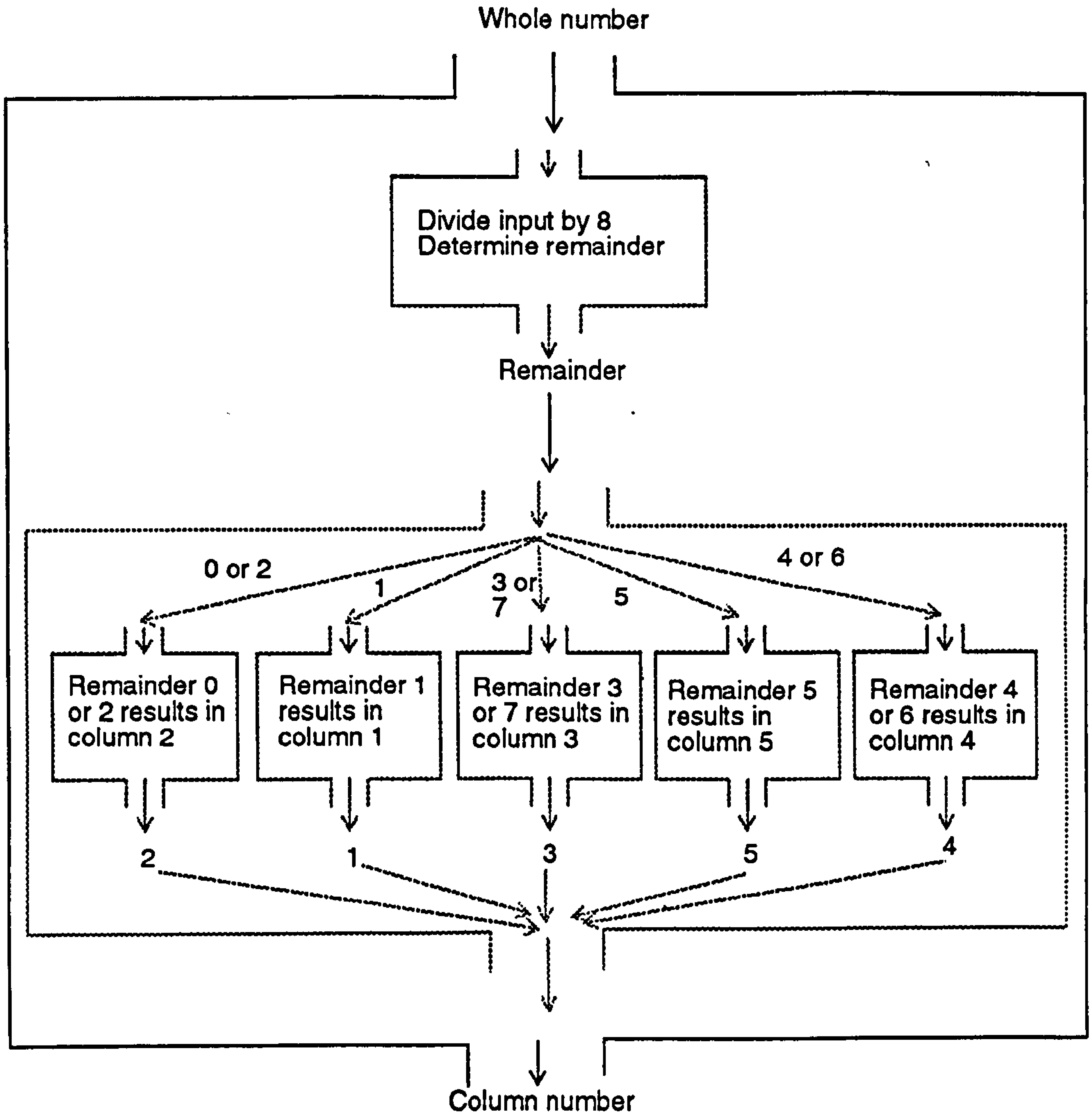



Figure 4.2: Composite function machine

As students are developing this colloquial facet, they are creating corresponding input-output tables. Thus this section introduces a definition of function along with the numeric and colloquial facets. The given definition will contribute to the written, verbal, and kinesthetic facets in particular. This section concludes with a problem that results in a relation that is not a function. In the next section, students build two-column tables to fit particular contextual situations and attempt to write equations in two variables that describe the relationship generally. They are asked to construct function machines and to describe the specific function process in words. The domain of a function is defined for the first time to be "the set of all legal inputs to the function" (ibid, p. 104). The range of a function is defined as the "set of all possible outputs of the function" (ibid, p. 105). Students practice evaluating specific functions and, in a limited number of cases, are asked to find the input given the output for a linear function. This is the students' first introduction to reversing a function's process. The focus in the next section is on function notation–the notation facet. A second definition of function is given at this point as follows: "We will consider a function as a relationship between two quantities that change. In a function, the value of the independent variable (input) uniquely determines a value of the dependent variable (output)" (ibid, p. 113). Students read that function notation has the format *function name* (*independent variable*) = *output or process*. Students are given a contextual problem and must create the symbolic and colloquial forms along with using function notation. The next section introduces the geometric facet. Students plot ordered pairs in Quadrant I of the rectangular coordinate plane. They return to a problem they previously investigated (numeric, symbolic, colloquial, notation facets) and create the graph of the function. The chapter concludes with several other problems that result in functional relationships. Students develop procedures for finding the sum of the first $n$ natural numbers by investigating the triangular numbers. They investigate power functions (basic cubic, basic quartic, etc.) and the factorial function. In each case, students build tables, create function machines, write symbolic forms, and construct graphs. Function notation is used throughout.

Chapter 4 introduces the integers. Students are introduced to the oppositing function in Section 4.1 and to the absolute value function in Section 4.2. The absolute value function is defined in terms of piecewise functions and students investigate one context in which a piecewise-defined function serves as a model prior to dealing with the absolute value function. With each new function, emphasis is placed on symbolic, numeric, geometric, colloquial, and verbal descriptions. During the previous investigations, students have met one other major function: the identify function. Section 4.3 extends graphing to four quadrants as students revisit functions by working with the geometric facet. Section 4.4 introduces graphing using a graphing calculator. Students deal with issues such as viewing window, appropriate scaling of axes, and discrete versus continuous graphs. They are introduced to parametric functions and their graphs. The chapter concludes with an in-depth investigation of the similarities and differences among the identity, oppositing, and absolute value functions.

Chapter 5 focuses on the rational number system. Students are introduced to the verbal, numeric, symbolic, geometric, and colloquial facets of the reciprocal function. The other major function idea explored in Chapter 5 is rate of change. Rates of change are initially treated using a graph that displays changing rates. An example of a linear relationship (constant rate of change) is introduced as a special case at the end of the section. Later in the chapter, power and reciprocal functions are further analysed. Particular emphasis is placed on the geometric facet with examples of direct variation, inverse variation, accelerated variation, cyclic variation, and stepped variation provided visually for students.

Chapter 6 introduces students to the real number system. One new major class of function, the square root function, is investigated. Section 6.3 serves as a review of all major classes of function that the students have encountered. The constant, identity, linear, quadratic, absolute value, oppositing, reciprocal, and square root functions are all revisited. Students are ask to create numeric, symbolic, geometric, and colloquial facets for each of these classes of basic function. Domain and range for each function are discussed.

Chapter 7 focuses exclusively on linear and quadratic functions. Students learn that linear and quadratic equations in which one side is constant can be thought of as symbolic forms in which the output of a function is given and the task is to find the input. Techniques for solving such equations involve the use of tables, the use of graphs, and, in the linear case, manipulation of symbolic forms. Systems of linear equations are investigated in Section 7.3. Linear equations in which the variable appears on both sides of the "=" are viewed as the result of trying to find the input to a system of two equations so that the outputs of the equations in the system are the same. Emphasis is placed on solving numerically, geometrically, and symbolically.

## 4.4 Pedagogy and Instructional Practice

The materials cause the role of the teacher and the role of the student to be redefined. The teacher acts as a guide, facilitator, and resource. The teacher provides activities that promote the construction of the conceptual web in the students. The teacher assists in the creation of connections between the ideas. The student is active rather than passive. The student is a team member who contributes his or her expertise to the mathematical growth of the team. The student is empowered to be an independent learner who has the capabilities and the access to resources to construct his or her own knowledge. The classroom is a community of learners with the teacher as guide.

Prior to teaching this course, a majority of the instructors participated in a week-long workshop facilitated by the authors. During the workshop, instructors became aware of some the mathematics education research that was used to shape the materials. Secondly, the instructors became familiar with the course content by role-playing as students and working through significant portions of the book. Time during the workshop was devoted to moving the teacher away from lecturing and more toward facilitating. Techniques for forming and using groups effectively were discussed. Much time was spent discussing assessment, including portfolios, oral exams, group exams, etc. The effective integration of appropriate technology (graphing calculators, symbol manipulators) was addressed. Participants discussed their beliefs about critical skills and the role of skill development versus concept development in such a course. The instructors who did not participate in the workshop were in departments in which at least one other faculty member did attend the workshop. All instructors were provided with an Instructor Resource Manual (DeMarois, McGowen, & Whitkanack, 1996c) that details the suggested learning environment, the goals and philosophy of each section, data-collection instruments, assessment techniques, and forms of assessment. Included is a technology reference guide for the TI-82, the calculator all instructors were going to use with their students. Thus all instructors involved in the project were acquainted with the background and goals of the curriculum's authors.

## 4.5 Intended, Implemented, and Attained Curriculum

The preceding analysis of the textbook describes the **intended curriculum** (National Research Council, 1996). This is the curriculum that students are intended to do, as viewed by the authors of the algebra reform project. Another significant factor in the intended curriculum is the definition of the course as specified by the local department, the school, the state, the upper-division institution, or some other group. The influence of these groups was minimized during the implementation of these reform materials, as most sections were taught as "pilot" sections with few restrictions. However, in most cases, students still were held accountable for course outcomes defined by the aforementioned agencies. This meant some variation in what was "covered" in the textbook. However, in interviews with all instructors who contributed student data, the material, especially that on functions, was faithfully included. This leads to the **implemented curriculum** (ibid) which is defined by instructional practice and classroom activity. Of the 12 instructors who contributed student data for this research, 10 went through the in-depth one week workshop described previously. The two other instructors are in departments with faculty who did attend the workshop and had previously implemented the curriculum. Much of the workshop was devoted to discussions of pedagogy. As a result, the classroom practices of the instructors were influenced by the curriculum's authors. As such, there was a degree of homogeneity in terms of how classes were run, including the role of the instructor and the role of the student. No other attempt beyond the workshop was made to control the implemented curriculum.

Of major importance to this research is the **attained curriculum** (ibid). The post-course surveys and the follow-up interviews were designed to measure this for the one topic that was the focus of the materials: function. The authors created an intended curriculum that used function as a focus. The key point of this research is to determine if the attained curriculum includes a "process" conception of function that empowers "debilitated" students who have been previously unsuccessful with mathematics. The intended curriculum was described earlier. The pedagogical elements of the project contribute to the implemented curriculum. The analysis of the data defines a key portions of the attained curriculum.

## 4.6 Selected Reform Curriculum Projects With Function as a Focus

Since the release of the first hand-held graphing calculator in the mid-1980s, curriculum projects have been under development in hopes of effectively utilizing this new tool. Symbol manipulators, such as MuMath, Derive, Maple, Mathematica, etc., have also driven the creation of new curriculum materials. Several curriculum projects that bear resemblance to the one described in this chapter are briefly discussed in this section. The purpose is to situate the curriculum for this research among some major curriculum projects of the last ten years.

The technological advances of the last ten years have changed what it means to "do mathematics". First, graphs in two and three dimensions require little effort to create. Second, complex symbol manipulation can be performed in exact form almost instantaneously using a symbol manipulator. One current hand-held device, the TI–92, has the capability to do both and is easily portable. Both of these advances allow for the investigation of function at an earlier stage in the curriculum than was previously possible. This, in turn, may lead to significant curricular changes. Philipp, Martin, and Richgels (1993) write: "Textbooks might reflect a new organizational structure in which problems are sequenced by the sophistication of their functional representation rather than by their specific type of algebraic representation" (p. 267).They go on to discuss skill development in this revised curricular environment: "Because there will be a greatly reduced need to develop extensive manipulative skills prior to dealing with important mathematical concepts, necessary skills can be organized in conjunction with the study of interesting ideas" (ibid, p. 268).

*College Algebra: A Graphing Approach* by Demana and Waits (1990) was the first text to influence the curriculum project described in this chapter. Assuming access to a graphing utility, Demana and Waits restructured the standard College Algebra curriculum to make more effective use of technology in the study of mathematics. Many traditional skill manipulations, such as solving complex equations symbolically, were replaced with graphical algorithms for approximating the results. Function took on a more central role with relationships, presented in writing, between changing quantities playing a key role in problem development. Where a typical traditional text might focus only on symbolic forms, this text treated symbolic and geometric facets equally. Some tables were introduced, but the numeric facet was a "second class citizen" in the text, as compared to the symbolic and geometric facets. A distinct difference between the Demana-Waits project and the one discussed in this chapter is the clientele. Many students using the Demana-Waits text had prior successful experiences with algebra and would be continuing on to calculus in the future. The students in this research were unsuccessful in their previous attempts at algebra and few will even proceed to a college algebra course, let alone calculus.

The Triple Representation Model (TRM) curriculum by Schwartz (1988) has a goal of promoting the transfer between symbolic, numeric, and geometric facets. This curriculum is designed for first-year algebra students with the hope of avoiding the development of common misconceptions surrounding the function concept. Eisenberg and Dreyfus (1991) state that the TRM curriculum "has been specifically designed to focus attention on within-representation and between-representation relationships. The student learns by operating in the algebraic, graphical, and tabular representations and by measuring and comparing the effect of an operation in various representations" (p. 35). Five key aspects of the curriculum are identified by Schwartz, Dreyfus, and Bruckheimer (1990): intuitive understanding of the concept of function including the collection of experimental data and finding of sequential rules by guessing; graphical representation of a function including transfer between verbal, graphical, and tabular representations, interpretation of graphs, construction of graphs corresponding to a collection of data, and limitations of the graphical representation; algebraic representation of a function including emphasis on discrete functions, transfer between tabular and algebraic representations, and inductive guessing of algebraic rules; transfers between all three representations; and problem solving encouraging transfer between the three representations (p. 251). The curriculum was tested in a ninth-grade class in which there was one computer for every pair of students. Schwartz et al. (1990) reported success in the development of curved-line reasoning as compared to the results of previous studies by Markovitz et al. (1986) and Karplus (1979). However, Schwartz et al. (1990) reported intense teacher guidance in order to assure that the implemented curriculum matched the intended curriculum. One key part of the implementation of the TRM curriculum that is significantly different from the curriculum for this research is the target student. Schwartz aims at students who have few misconceptions about algebra since his curriculum is aimed at the first year high school course. He suggested that these students have "a much cleaner educational story line" (Schwartz & Yerushalmy, 1992, p. 261) something that is just the opposite for college students in beginning algebra.

Haimes (1996) discusses the mathematics curriculum in Western Australia (Western Australian Ministry of Education, 1990) which, since 1987, has focused on teaching introductory algebra with an emphasis on patterns, sequences, and functions. Functions act as one of the five major strands for grades 8–10. Haimes describes the function strand as follows: "The "Function" strand begins with number sequences in Stage 1. Linear, quadratic, exponential, reciprocal, and periodic relationships are introduced progressively over the next five stages, thereby emphasizing the importance of functions as a unifying theme of the algebra component of the curriculum" (1996, p. 585). Later he writes: "Underlying the curriculum document is a constructivist theory of learning, for it envisages the learner, through investigation and exploration, incorporating new learning into personal knowledge structures" (ibid, p. 586). Haimes' research focuses on the role of the teacher in the implemented curriculum as compared to the intended curriculum. Haimes conducted a qualitative case study of one ninth-grade teacher in Perth. He reports that the teacher's beliefs and pedagogical practices were quite different from that of the intended curriculum. In particular, her classroom was teacher-centred while the curriculum called for a student-centred environment. Where the curriculum viewed maths as a way of thinking, the teacher implemented the curriculum as discrete content sections. Haimes draws a key conclusion that effects many reform efforts: "The impact of the curriculum on the actions of the teacher was found to be minimal" (ibid, p. 601). The result is an implemented curriculum that is quite different from the intended curriculum. Similar issues are a concern as the appropriateness of the reform curriculum described in this chapter is researched.

Another project aimed at the secondary algebra curriculum is the "Algebra with Computers" project of Fey and Heid. Using a computer-intensive environment, this curriculum is notable for "making the concept of function a central organizing theme for theory, problem solving, and technique in algebra, and developing students' understanding of algebra concepts, and their ability to solve word problems requiring algebra, before they master symbol manipulation techniques" (Heid, 1988b, p. 1). End-of-course interviews show that the students exposed to this curriculum were more adept at working within and between representations and at problem solving. The results suggest that a course focused on concepts first, rather than skills, does not adversely impact traditional manipulation skill as compared to a traditional curriculum (Heid, 1985, 1988a) This project eventually became the Computer-Intensive Algebra (CIA) project (Fey et al., 1991) and is currently distributed as *Concepts in Algebra: A Technological Approach* (Fey et al., 1995). Again this is a curriculum heavy on technology that focuses on function and is designed as a student's first exposure to algebra. Connections between the numeric, symbolic, and geometric facets play an important role. A key difference between this curriculum and the curriculum used for this research is the focus of this research on "debilitated" students.

Finally, Kaput (1992) describes a teaching experiment with 13 "weak" college students using functions and Function Probe (Confrey, 1991). He writes:

> Their instruction included deliberate teaching of function categories definable by coefficient-parameters, which resulted in almost universal learnability of non-rhetorical approaches to fairly general classes of polynomials. In this approach students are taught to look at numerous growth patterns in order to characterize the degree of the polynomial, and then work from this characterization to determine the polynomial's coefficients. Interestingly, in this approach parameters appear as rhetorical devices to enable efficient discussion and especially comparison of families of functions. ... The style of instruction was mainly guided inquiry, with class time split between teacher-centered class and computer laboratory. (p. 316)

Kaput reports that "almost all of these students were able to do the guess-my-rule problems involving elementary polynomials in the contexts of both tables and graphs" (ibid). Guess-my-rule is software "that puts the student in the position of inferring a rule from either a numerical or graphical representation of the input-output relations and formalizing that rule in algebraic form" (ibid, p. 291). Subsequently Kaput taught a more complete curriculum for an entire semester using the same pedagogy to 10 more "weak" college freshmen. He reports that the eight students completing the course actually appeared to treat polynomial and exponential functions as objects.

Kaput warns that care must be exercised in drawing conclusions about student understanding, however:

> In particular, we must first distinguish between our own mathematically mature understanding that may be projected onto the students' behavior and that of the student's actual understanding–always a ticklish problem, especially when the observer is a teacher who fervently wants to believe that the student has high quality understanding. And more importantly, behavior that may be readily interpretable as evidence of cognitive encapsulation in one context may change radically if the activity structure is changed so that the student is put at the edge of her/his competence. This can be done by changing the representation system in which the student is asked to work .... (ibid, p. 317)

This is precisely the reason for analysing student understanding by looking at the various facets and layers of a concept. While a student may demonstrate great depth while working with one facet, this same student may become hopelessly confused when looking at another facet. Hence, the complexity in discussing understanding.

## 4.7 Conclusion

This chapter describes the curriculum and instructional treatment that form the basis for this research. The chapter begins with an overview of the curriculum designed as a technologically-rich "reform" curriculum for college students enrolled in a beginning algebra course. The curriculum's philosophy and structure, with its foundation in contructivism, its goal of actively engaging the students, and the redefinition of instructor-student roles, are described. Since function is the unifying concept and forms the focus of this research, the function strand is traced throughout the text. The discussion turns to the role of the instructor within the curriculum which leads to a brief discussion of the intended curriculum, the implemented curriculum, and the attained curriculum. While the authors of the text create the intended curriculum, the teacher defines the implemented curriculum. The students' demonstrated understanding forms the data for the attained curriculum. The chapter concludes with a brief survey of several other "reform" curriculum projects that have similarities to the described curriculum.

# CHAPTER 5 *Pilot Study: Methodology and Results*

## 5.1 Introduction

The purpose of this chapter is to describe the procedures and methodology the researcher used to complete a pilot study using the theoretical framework described in a prior chapter. After discussing how the study was set up, an analysis of the results will follow.

## 5.2 The Participants

Students attending one of four community colleges and enrolled in beginning algebra during Spring Semester, 1996 were participants in the pilot study. The community colleges involved were William Rainey Harper College in Palatine, Illinois, College of Lake County in Grayslake, Illinois, Northwestern Michigan College in Traverse City, Michigan, and Lakeland Community College in Kirkwood, Ohio. The subjects of the study were enrolled in a "reform" beginning algebra course, as described in Chapter 4. The instructors had previously attended a week-long National Science Foundation workshop organized and led by the authors of the materials. This workshop focused on the implementation of the "reform" curriculum.

The students self-selected to enrol in the "reform" course as opposed to the more traditional "skills-based" course that was offered on all classes. Prior to registration, the course listing indicated that the section the student was enrolling in was a "reform" section using a different text from traditional sections and requiring the use of a graphic calculator. While this information was made available to students prior to enrolment, many ignored or did not notice it and selected the "reform" course based on the time it was offered or knowledge of the instructor. In all cases, students were offered the option of switching to a traditional section during the first week of the term. Several students took advantage of this offer and are not a part of the study.

### 5.2.1 Student profile

Since the students span such a wide spectrum, some characteristics of the target population are provided to create a profile of the study participants. These profiles result from data collected during a study conducted as part of a National Science Foundation grant to support the implementation of the afore-mentioned curriculum. The following analysis of the data was provided by Professor Carole Burnett, Department of Mathematical Sciences at William Rainey Harper College, who holds a Ph.D. in Statistics.

In a survey of 285 beginning algebra students in Fall, 1995,

- 59% were female
- 72% rated themselves as fair to disastrous in algebra
- 64% were enrolled for 12 or more semester hours
- 36% took mathematics the previous term
- 16% had not had maths in at least 5 years
- 60% were between 17 and 20 years old while 18% were 30 or older
- 12% had never used a calculator.

### 5.2.2 Pre-course survey: student attitude findings

On a 60-question pre-course attitude survey with responses ranging from 1 meaning strongly agree to 5 meaning strongly disagree, there was significantly high agreement on 14 statements and significant disagreement on 1 statement. The questions resulting in significantly high agreement follow.

1. The mathematics I learn at school is mostly facts and procedures that have to be memorized.
2. Good mathematics teachers show you lots of ways to look at the same question.
3. Taking good notes during class is the best way to learn mathematics.
4. When given a maths problem I don't know how to do, I often can't even get started.
5. I use a calculator when taking tests.

6. I think it is important to share my ideas and attitudes about the class with my instructor.
7. Problems on a mathematics test should be like the problems we've done for homework.
8. A good mathematics teacher explains things first, before giving out assignments.
9. I am willing to try a different approach when the first one fails.
10. A calculator is helpful when trying to find errors in my work.
11. Good mathematics teachers show you the exact way to answer questions you will be tested on.
12. When I get an answer on the calculator, I check to see if it seems reasonable.
13. Being able to explain how I got an answer to others is an essential part of learning mathematics.
14. Investigating a problem on my own or with another member of the class before the teacher explains it helps me identify what I already know and what question(s) I need to ask.

The question resulting in significantly high disagreement follows.

15. I learn maths by just learning the rules—I don't need to know why.

The results of this last statement are difficult to interpret since the statement has two parts: "just learning the rules" and "don't need to know why".

### 5.2.3 Analysis of post-course findings

At the end of the semester, students completed a post-course attitude survey. The same 60 questions were asked and a statistical analysis was run to try to detect any significant shifts in learning algebra through the use of group learning and technology. The following significant ($p < 0.05$) shifts were noted.

There was a significant shift towards more agreement on the following statements:

- Using a graphing calculator allows me to think about what is happening mathematically, instead of worrying about getting the problem wrong.
- I sometimes use a graphic calculator to check my algebraic work.
- Using a graphing calculator makes it easier to do mathematics.
- I feel confident in my ability to solve mathematical problems.
- I use a calculator when taking a test.
- It is important to use multiple representations (table, graph, algebraic form) in order to fully understand a problem and have confidence in my answer.
- Weekly journals help me focus on what I have learned and still need to know.
- Journals help me reflect on what mathematics I'm supposed to know.

There was a significant shift towards more disagreement on the following statements.

- The mathematics I learn at school is mostly facts and procedures that have to be memorized.
- When a teacher asks a question in mathematics class, I hope I am not called upon to give an answer.
- I am afraid of trying a mathematical problem I have not seen before.
- Time spent using a graphing calculator could be better spent practising the mathematics skills I need for the next course.
- I get confused about mathematics in a problem when it is analysed more than one way (algebraically, graphically, and/or numerically, using a table.)
- Learning to do mathematics means just learning the procedures.
- I get confused when trying to read $x$ and $y$ values from a graph.

### 5.2.4 Course evaluation form

At the end of the "reform" course, students, in addition to re-taking the 60-question attitudinal survey, also completed a 19-question course evaluation form. Many statements were paired, one asking about a student attitude before the course and the other asking about a change in that attitude after the course. Significant mean shifts were noted on the following pairs of statements.

a. How would you rate your ability to interpret mathematical notation and symbols at the BEGINNING OF THE SEMESTER?

| very poor | somewhat poor | fair | somewhat good | very good |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

b. To what degree do you think this course has improved your ability to interpret mathematical notation and symbols?

| not at all | a little | somewhat | a good bit | very much |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

The mean on a. was 2.6 while the mean on b. was 3.5.

a. How would you rate your ability to interpret and analyze data at the BEGINNING OF THE SEMESTER?

| very poor | somewhat poor | fair | somewhat good | very good |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

b. To what degree do you think this course has improved your ability to interpret and analyze data?

| not at all | a little | somewhat | a good bit | very much |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

The mean on a. was 2.7 while the mean on b. was 3.5.

a. How would you rate your willingness to attempt to solve a problem you have never seen before at the BEGINNING OF THE SEMESTER?

| very poor | somewhat poor | fair | somewhat good | very good |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

b. To what degree do you think this course has improved your willingness to attempt to solve a problem you have never seen before?

| not at all | a little | somewhat | a good bit | very much |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

The mean on a. was 2.8 while the mean on b. was 3.7.

a. How would you rate your ability to solve a problem you have never seen before at the BEGINNING OF THE SEMESTER?

| very poor | somewhat poor | fair | somewhat good | very good |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

b. To what degree do you think this course has improved your ability to solve a problem you have never seen before?

| not at all | a little | somewhat | a good bit | very much |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

The mean on a. was 2.8 while the mean on b. was 3.6.

The above statistics are provided to give a profile of the type of student enrolled in the "reform" beginning algebra course. While there is no "typical" student, these statistics should help in forming a mental picture of the student who was the target of this research.

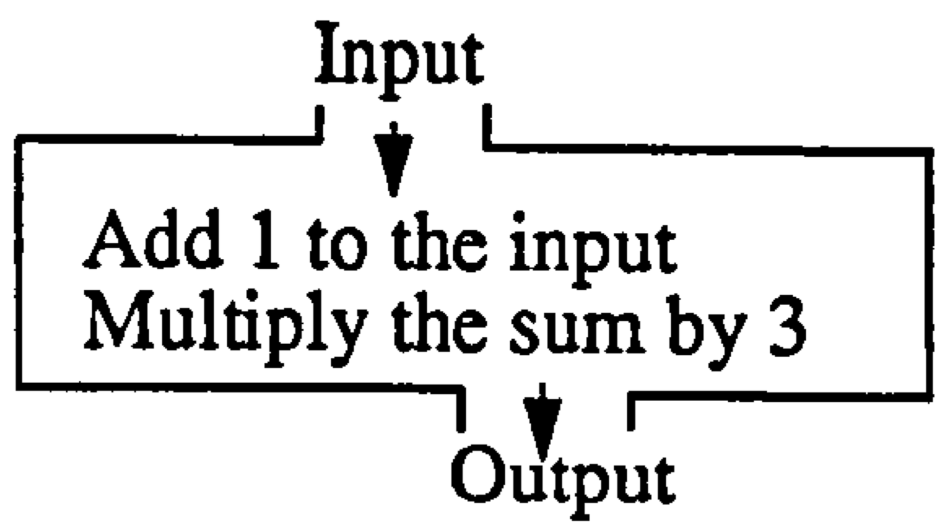
## 5.3 The Study

Students at four community colleges completed written function surveys at the beginning and the end of a "reform" beginning algebra course, as described in Chapter 4, during the spring semester of 1996. Subsequently, several students at each site participated in task-based interviews. One hundred forty-nine beginning algebra students completed a pre-course function survey. The post-course survey was completed by 82 students. The number of surveys in which students completed both pre- and post-course surveys was 70. Both the written surveys and interviews were used to profile students' concept image of function. A primary purpose of the study was to test the research design in preparation for the main study.

## 5.4 Methodology

The study has both a quantitative and a qualitative component. Students were asked the same 6 questions on both the pre-course and post-course survey. The following 4 questions were analysed quantitatively:

1. Consider the diagram.



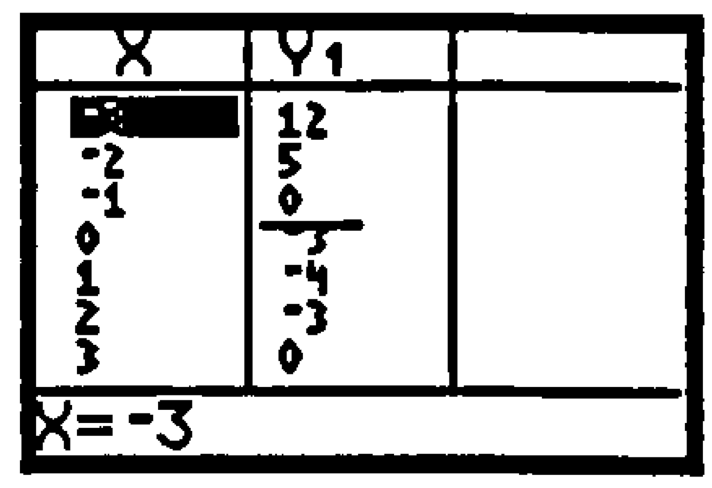

   a. What is the output if the input is 7?

   b. What is the input if the output is 18?

2. Use the equation $y = 3x - 7$ to answer each question. The "$x$" represents input and the "$y$" represents output.
   a. What is the output if the input is 5?

   b. What is the input if the output is 0?

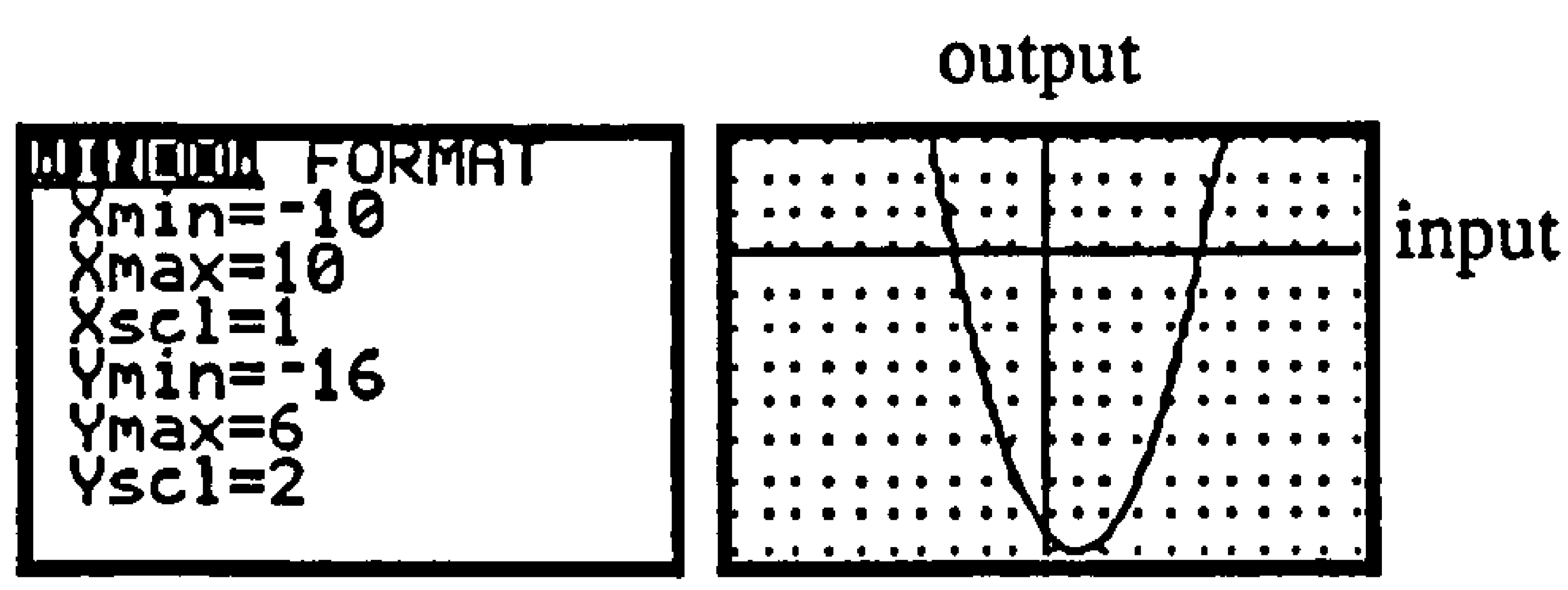
3. Use the following input-output table to answer each question. The "X" represents input and the "$Y_1$" represents output. If the answer does not appear in the table, write "not possible."




   a. What is the output if the input is –2?

   b. What are the input(s) if the output is –3?

4. Use the following graph to answer each question. If the answer does not appear on the graph, write "not possible."




   a. What is the output if the input is 3?

   b. What are the input(s) if the output is 0?

These four questions were designed to measure change in students ability to apply a process and reverse a process for the colloquial (function machine), symbolic (equation in two variables), numeric (table), and geometric (graph) facets. Question 1 admittedly involves the numerical facet also. Though difficult to read, Question 4 used calculator screen display since this would be the primary mode for studying graphs in the course. Students were graded on each part and their scores from pre- to post-course on each question were compared. A Sign Test for Paired Data and a Wilcoxon Test for Paired Data were used to test for significance in shifts in student scores from pre- to post-course.

Changes in the written facet were noted from pre- to post-course by asking students to define "function". These responses were analysed using systemic network-charts based on the "systemic networks" described by Bliss et al. (1983). Answer evolution charts were used similar to those in Garcia-Mila et al. (1996) to chart changes in students' written definition during the course.

Additional data from the post-course surveys and the interviews were analysed qualitatively. The information from the written surveys and the interviews is used to develop a profile of each student's concept image of function. Survey and interview questions were designed to measures students' understanding along all facets discussed in the theoretical framework. In addition, questions were used to measure the strength of the boundaries between pairs of facets. A profile, with justification, for one student is included.

## 5.5 Quantitative Analysis

In a series of 4 questions, students were given functions in the form of a function machine, an equation in two variables, a two-column table, and a graph. In each case they were asked to find a specific output, given the input and vice versa. The number of correct responses along with percentages appear in Table 5.1.

**TABLE 5.1:** Questions 1–4: correct responses (n = 70)

| Question | Pre-course Number correct (% correct) | Post-course Number correct (% correct) |
|---|---|---|
| Colloquial facet: function machine input given | 62 (89%) | 64 (91%) |
| Colloquial facet: function machine output given | 48 (69%) | 49 (70%) |
| Colloquial facet: function machine both parts correct | 46 (66%) | 45 (64%) |
| Symbolic facet: equation input given | 54 (77%) | 62 (89%) |
| Symbolic facet: equation output given | 14 (20%) | 32 (46%) |
| Symbolic facet: equation both parts correct | 13 (19%) | 31 (44%) |

**TABLE 5.1:** Questions 1–4: correct responses (n = 70)

| Question | Pre-course Number correct (% correct) | Post-course Number correct (% correct) |
|---|---|---|
| Numeric facet: table input given | 42 (60%) | 66 (94%) |
| Numeric facet: table output given | 22 (30%) | 39 (56%) |
| Numeric facet: table both parts correct | 21 (29%) | 37 (53%) |
| Geometric facet: graph input given | 0 (0%) | 14 (20%) |
| Geometric facet: graph output given | 2 (3%) | 30 (43%) |
| Geometric facet: graph both parts correct | 0 (0%) | 11 (16%) |

A graphical comparison of the results appears in Figure 5.1.

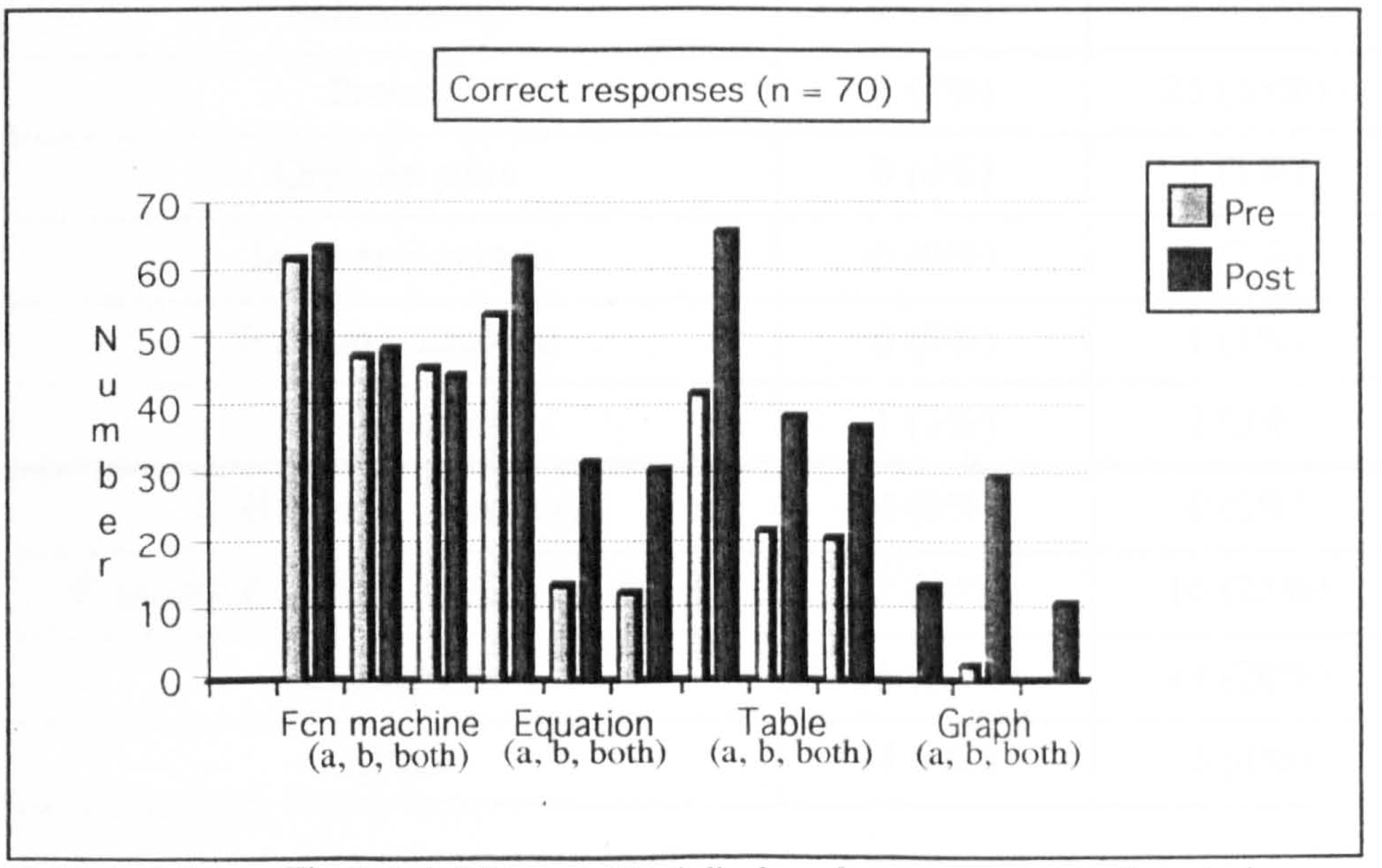



Figure 5.1: Pilot: graphical display of responses to questions 1–4

Both a Sign Test for Paired Data and a Wilcoxon Test for Paired Data were performed comparing the pre- and post-course performances of each student. Each question was scored using 1 point for a correct answer when the input was given and 2 points for a correct answer when the output was given. Thus, each question had a possible total of

3 points. The changes from pre- to post-course on the symbolic, numeric, and geometric facets were all significant at a 0.001 level. The data show little change using the function machine, but this may be somewhat due to the fact that a high percentage of students were able to correctly use the function machine on the pre-course survey. Not surprisingly, the geometric facet presented the most difficulty. In fact, the post-course result for Question 4 remains amazingly low possibly indicating how complicated it is to interpret a graph on a graphing calculator

To measure the written facet, students were asked to define "function" on both pre- and post-course surveys. The definitions provided were categorized based on the main focus of what the student wrote. Six of the 10 categories were judged to be acceptable definitions. These 6 are listed first in Table 5.2.

**TABLE 5.2:** Function definitions (n = 70)

| Category | Pre-course Number (%) | Post-course Number (%) |
|---|---|---|
| Relationship | 1 (1%) | 5 (7%) |
| Process | 3 (4%) | 23 (33%) |
| Ordered pairs | 0 (0%) | 1 (1%) |
| Input and output | 0 (0%) | 5 (7%) |
| Function machine | 0 (0%) | 1 (1%) |
| Dependency | 1 (1%) | 2 (3%) |
| Colloquial– "Purpose" | 4 (6%) | 0 (0%) |
| Colloquial– "Way to do something" | 31 (44%) | 16 (23%) |
| No response | 26 (37%) | 14 (20%) |
| Other | 4 (6%) | 3 (4%) |

The data is displayed visually in Figure 5.2. The categories "Ordered pairs", "Input and output", "Function machine", and "Dependency" were grouped together under "Other" category of correct definitions. The two "Colloquial" categories were grouped together under "Colloquial" category of incorrect.

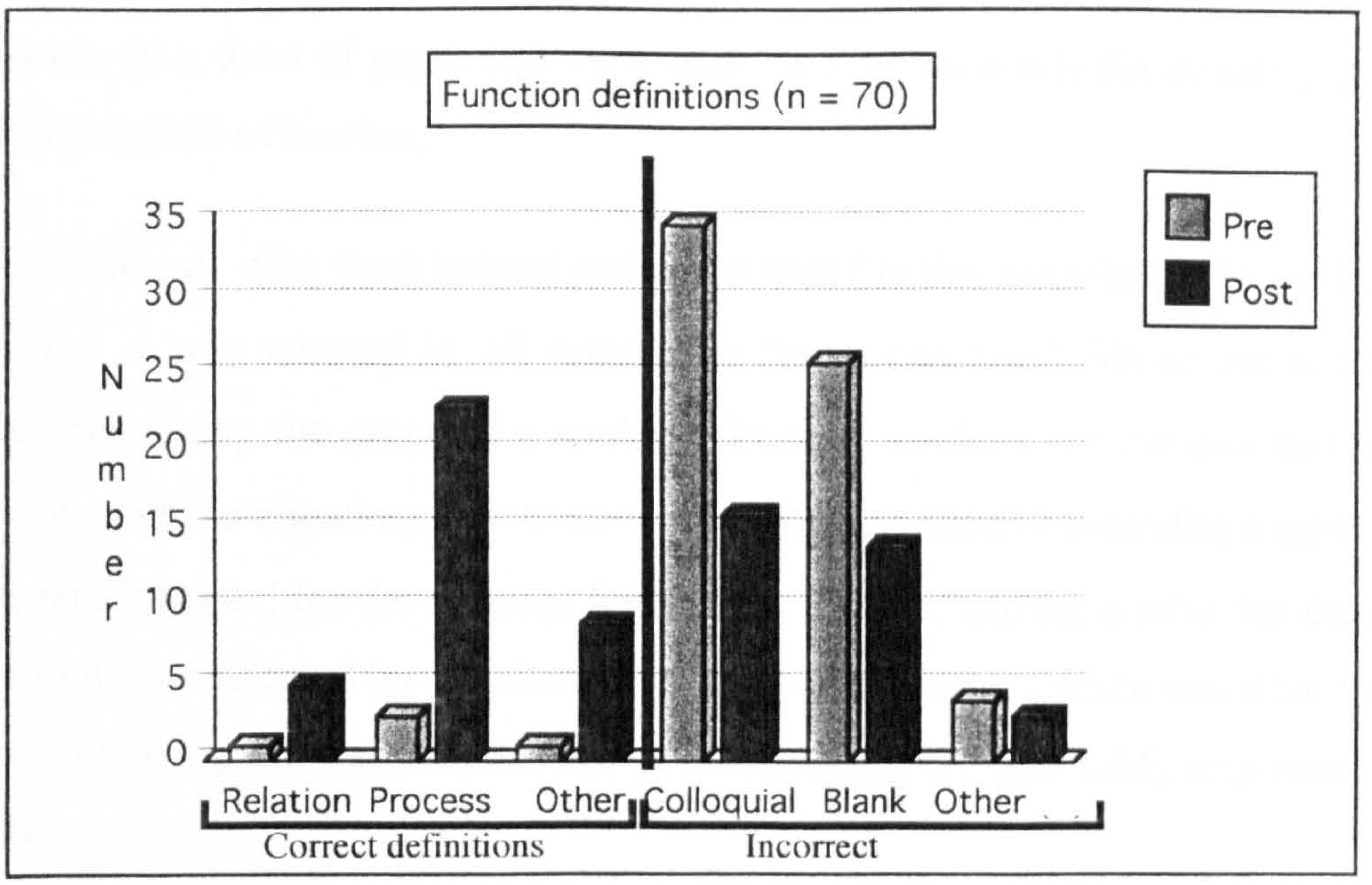



**Figure 5.2:** Pilot: graphical display of function definitions

On the pre-course survey, 93 percent were unable to provide a satisfactory definition of function. Fifty percent gave "colloquial" responses–that is, responses based on non-mathematical uses of the word. On the post-course survey, 52 percent wrote satisfactory definitions while only 23 percent still wrote answers described as "colloquial".

Summarizing, the pre- and post-course surveys supplied information on development of student concept image of function in the colloquial, symbolic, numeric, geometric, and written facets. Many, but not all, students demonstrated a statistically significant growth from pre- to post-course in all facets except the colloquial. However, a majority of students demonstrated a high degree of comfort with this facet on the pre-course survey.

## 5.6 Qualitative Analysis

### 5.6.1 Function cards

Students were given 24 cards, each containing either an equation, a table, a graph, or a function machine (See appendix D for cards). Equations in both one and two variables were included. In several cases, the same function appeared on four different cards using the four different facets listed above. Students were asked to place the cards in two piles: one for "functions" and one for "not functions". They recorded their answers on a sheet of paper and were asked to state their rule for deciding if a given card contained a function.

Seven students who were interviewed participated in this exercises. The card containing $x = 4$ was selected by all students as "not a function". Six of the seven chose cards containing the graph of a circle, a function machine for the constant function $y = 4$, and the equation $y = 4$ as "not functions". Cards containing a table for the constant function, for the constant function as a graph, and for a table for the median function were selected by only 2 students as "not functions". When asked for their reasons in making their selections, two of the students, SC and LAF, responded as follows:

SC: Function machines "are functions". Also, on the graphs if there was an input and output. Tables—I'm not real sure of because I can't tell. I put them on the function pile but I'm not really sure.

LAF: Well, these 2 ($x = 4$ and $y = 4$) aren't functions since there is no process—just variable equal number. This one ($2x + 1 = 7$) because you have multiple inputs to get the same output. I don't see how you could do the same process on five different numbers and got the same output (the median function).

Some observations based on the function cards exercise include:

- Students did not make distinction between equations in one variable and equations in two variables.
- Inconsistent responses occurred on a constant function. Only 1 student felt that the function machine and the equation were functions. But 5 students said the table and the graph were functions.
- Inconsistency occurred in applying the "uniqueness of output" qualifier to functions. This is what Dubinsky and Harel (1992) refer to as the "uniqueness from the right" condition. Six students said the graph of a circle was not a function. All stu-

dents said that the equation and the table for the circle were functions. If a student can discern a process between $x$ and $y$ in a two-variable equation, the student will often say this is a function regardless of whether the output is unique.

- Lack of transfer across representations as indicated by the inconsistent responses to both the cards with constant functions (4 different cards) and the cards with the circle (3 different cards).

### 5.6.2 Creating student profiles

By collecting all the data (pre-course survey, post-course survey, and interview) for each student interviewed, the researcher hoped to build profiles of student concept images of function using the theoretical framework described previously. The following is a profile of one students developed as part of this study.

### 5.6.3 Student background

DB is a female student in her mid-twenties. She had taken two years of high school algebra previously, barely scraping by. She was among the top students in her beginning algebra course.

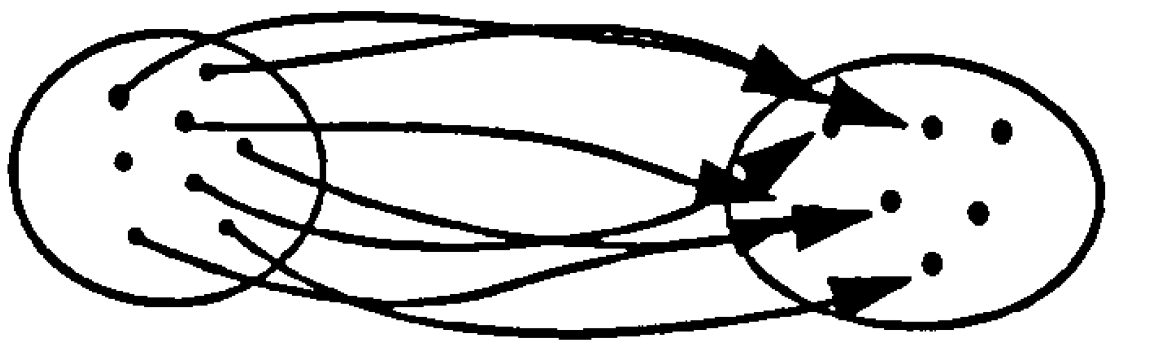
### 5.6.4 Layers of the written facet

DB was asked to write the definition of a function on both the pre- and post-course surveys. On the pre-course survey, she wrote: "A function is similar to an activity. A function in maths would be to add or subtract some numbers" indicating she was at the pre-procedural layer at the beginning of the course. On the post-course survey, DB wrote: "A function is a relationship between two changing quantities." While she did not include the provision of an unique output for given input, she may still be at the process layer.

### 5.6.5 Layers of the verbal facet

DB was asked during the interview to state her verbal definition of function and to compare this definition to other definitions. Her verbal definition matched the one she wrote on the post-course survey. The following transcript probes her comfort level with alternate definitions.

Intvw: Consider the definition: A **function** is a **correspondence** that assigns to each element of one set one and only one element of a second set. A diagram appears at right. Discuss the relationship between **your** definition of function and this definition.



DB: Well, it's still putting one thing in and coming out with something different after going through some change or process.

DB states that "something different" comes out. How would she respond to the identity function? Does she really mean the output must be different from the input? The interview continues.

Intvw: Okay so where is the input?

DB: This is the input (pointing to left circle).

Intvw: And the output?

DB: This is the output (pointing to right circle).

Intvw: Where would the process lie?

DB: Right here (pointing to arrows).

Intvw: Okay. Is that pretty much equivalent to your definition of function?

DB: Yes, I would say so.

Intvw: Consider the following definition: A **function** is a set of ordered pairs $(a, b)$ in which for each value of $a$ in the domain of the function, there is one and only one value of $b$ in the range of the function. Discuss the relationship between your definition of function and this definition.

DB: I would say that they are the same definition. Given one input, only one possible output.

Notice that DB mentions uniqueness of output for a given input for the first time.

Intvw: What is acting as the input?

DB: Well the domain—$a$.

Intvw: And the output?

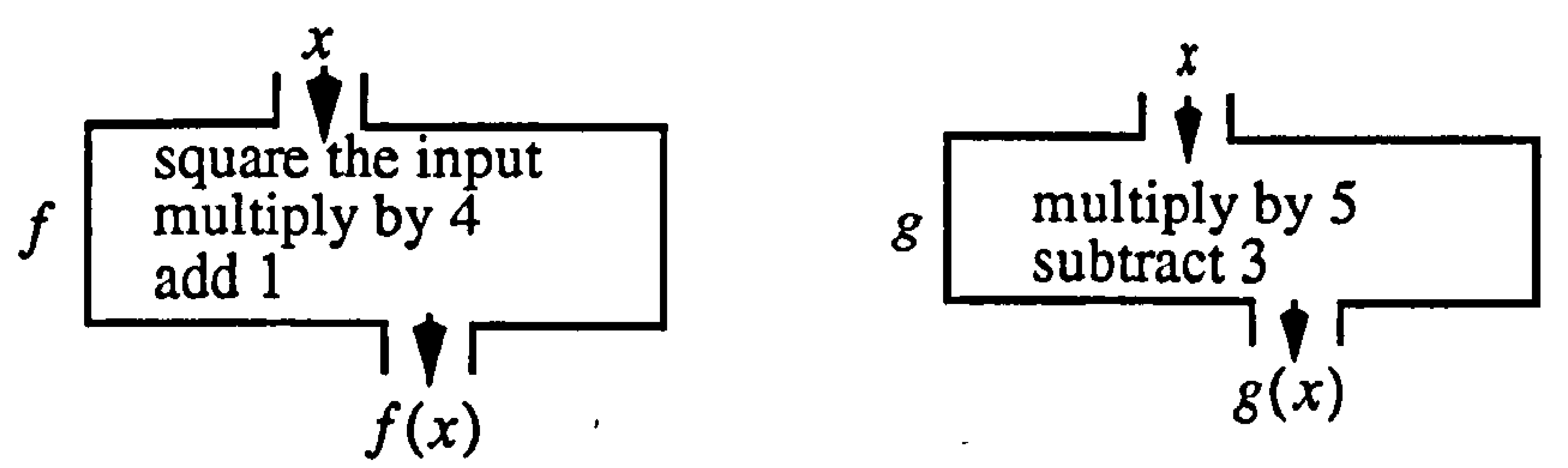
DB: $b$.

When asked about the input, DB is not clear on the distinction between the input and the domain. DB demonstrates good flexibility in her concept image of a function's definition. She appears able to identify the similarities between the definitions and easily merge them with her own definition. She is minimally at the process layer here and may well have sufficient understanding to be classified at the concept or even procept layer with respect to her verbal definition of function. It is important to note that her verbal and written definitions match indicating some consistency in her concept image.

### 5.6.6 Layers of the colloquial facet

The function machine is used to investigate the colloquial facet of function. Given a function machine, students were asked on both the pre- and post-course surveys to find the output if the input is given and vice versa. DB answered both of these correctly on both surveys indicating an understanding of the diagram prior to the course. Since she demonstrated procedural understanding being able to find output and reversibility by being able to find input, she appears to be at a process layer for this facet.

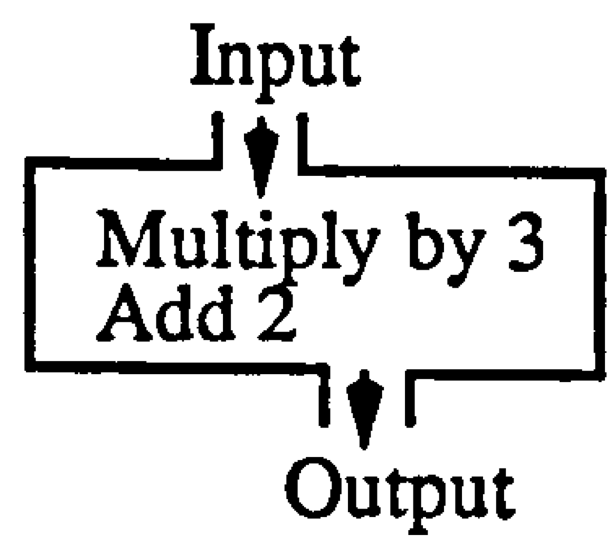
Several questions involving function composition were asked on both the post-course survey and during the interview. Students received no instruction on function composition during the course nor prior to the interview. All function composition questions share aspects of the notation facet, since function notation was used to denote the requested composition. During the interview, DB was asked, given the following function machines for functions $f$ and $g$, to find $f(g(2))$ and to describe what she did.




Without hesitation, she responds: "197. First of all I find g(2) so I input 2 into function machine g multiplied 2 by 5 got 10, subtract 3 to get 7. That becomes the input to *f*. Square to get 49. Multiply that by 4; 196; adding 1 is 197." Her facility in making the transition from 7 as an output of $g$ to 7 as an input to $f$ is impressive.

Two additional questions tested her ability to accept an object as input rather than a number.

Consider the function machine at right. What is the output if the input is $y(x) = x^2 - 5x$ ? What did you do?

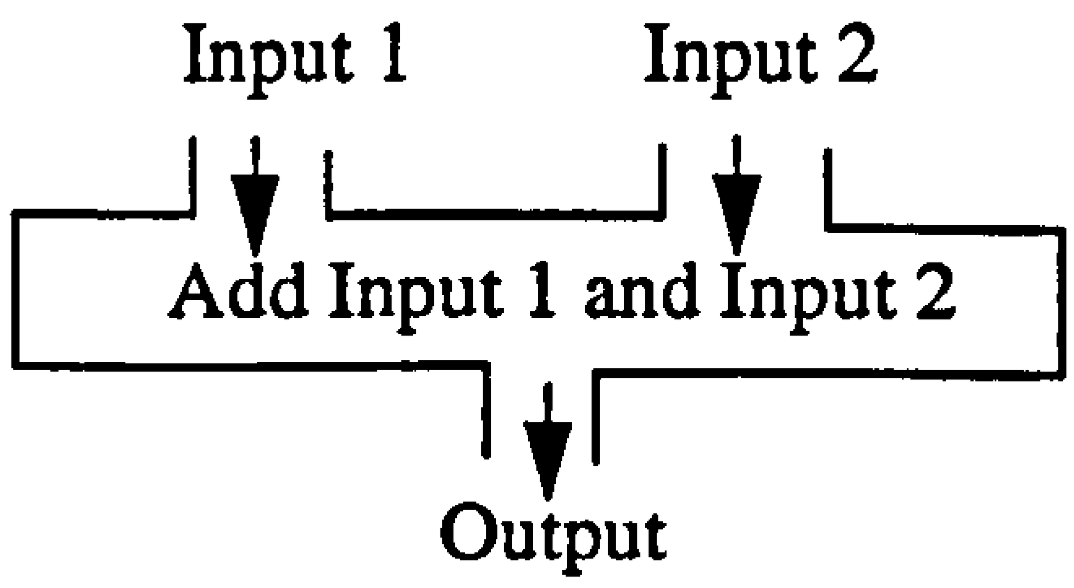



DB: That would be $3x$ squared minus $15x$; take that and add 2 to it.

Intvw: Is that an acceptable output to you?

DB: Yes

Intvw: Consider the function machine at right. What is the output if Input 1 is $f(x)$ and Input 2 is $g(x)$ where $f(x) = 3x - 5$ and $g(x) = x^2 + 1$ ? What did you do?




DB: Hmmm I think this is fruit salad. You have to add the like terms so you would get $3x$ + $x$ squared minus 4

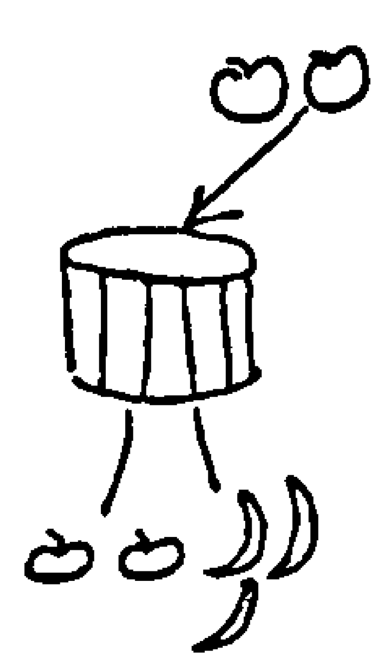
Intvw: Is that an acceptable output?

DB: Yes.

DB demonstrated no more difficulty when the input involved functions than if the input involved numbers. Her flexibility with function notation is impressive. The interviewer could have tested her ability to reverse the process using a function as output, but did not.

### 5.6.7 Layers of the kinesthetic facet

DB was asked to demonstrate what a function is without using words by using a physical motion. She was unable to do this resorting to drawing the picture at right. She states: "I guess this is following along the lines of a function machine. You have 2 apples and you add 3 bananas and you get fruit salad." The emphasis is on process, but she could not demonstrate such physically. She seemed to rely on a "way to think about functions" given by her instructor instead of creating her own dynamic conception. While more data would be needed, this suggests that her kinesthetic facet is not as well developed as some of the other facets.



### 5.6.8 Layers of the symbolic facet

Students were asked on both the pre- and post-course surveys, given the equation $y = 3x - 7$, to find the output if the input is given and to find the input if the output is given. DB answered both correctly on both surveys. On the post-course survey, students were asked to identify each of the following as "a function" or "not a function". The question was repeated during the interview.

a. $y = 3x - 2$ b. $y = 9 - x^2$ c. $y = 5$ d. $x^2 + y^2 = 1$

e. $y = \begin{cases} 1 & \text{if } x < -3 \\ x^2 & \text{if } x \geq -3 \text{ and } x < 4 \\ 2 & \text{if } x \geq 4 \end{cases}$ f. $y = \pm\sqrt{x+2}$

g. If $x$ is rational, then $y = 0$ h. $xy = 7$

i. $y = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is not rational} \end{cases}$ j. $x = 2 + t$ and $y = 3t^2 - 5t + 1$

DB marked only a, b, and e as functions on the written survey. Her interview response follows.

DB: Hmm. I would say a and b are both functions. Each gives you the process for getting the output. C is not a function since it is just giving you the output, but not how to get it. D is a function because it is showing you the process on how to get the output. E hmm well I would say that is a function. I have to concentrate on that one. It is giving you the process to go through given certain values of $x$.

The interviewer gave her a couple of inputs and asked her to find the outputs for the function in e. After doing so, she responded that she was much more confident that this was a function.

Intvw: Okay what about the rest?

DB: F—I want to say it is still a function but with that square root I would have to change my mind because square root could have two different outputs.

Intvw: So would that be a function if that happened?

DB: No, it would be a relation.

Notice she mentions uniqueness of outputs again even though this condition was not part of her written or verbal definitions. DB did not recognize that the same issue occurs in d, the equation of a unit circle. She distinguishes between a relation and a function for the first time.

DB: I say that g is not a function either. No it could be because you only have one output depending on the value of $x$.

Again the interviewer gave DB some examples to work through. After some discussion about what constitutes a rational number, DB felt g was a function.

Intvw: Okay, how about h?

DB: h is not a function because you don't really know how to get the values of $x$ and $y$.

Intvw: So what if I said $x$ was 5?

DB: Then you could figure it out. 5 times $y$ would be 7

Intvw: What if I said $x$ was 8?

DB: Then you could do it to. I think you need to know the value of one of these before you can do it.

Intvw: Do you need that kind of thing to get a function?

DB: No. I guess it could be a function.

Intvw: What about i?

DB: It's a function. J is a function since there is a way to find what the values are.

Her first impulse, as noted in the survey, is to say "not a function". After more reflection, she often revises her response. DB appears to be process-oriented when confronted with a symbolic definition of a function. Her criteria throughout depends on the existence of a rule for finding the output, given the input. Finally, DB demonstrated the ability to compute the value of a function composition symbolically.

Intvw: Consider functions $f$ and $g$ defined as $f(x) = 3x-5$ and $g(x) = x^2+1$. What is $g(f(3))$ ? Describe what you did.

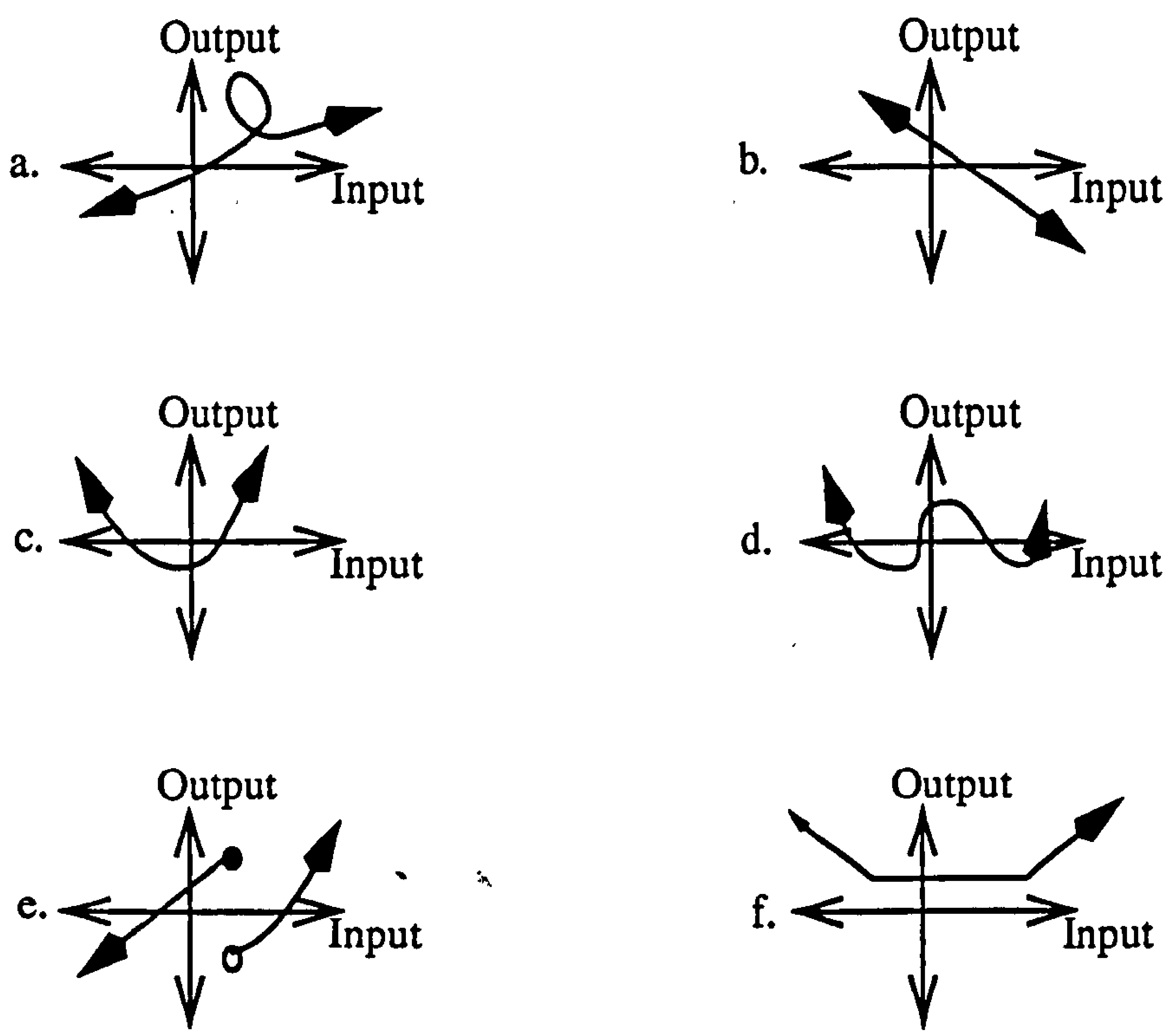
DB: $f(3)$ is 4 and then that makes the input to $g$ be 4. Four squared is 16 plus 1 is 17.

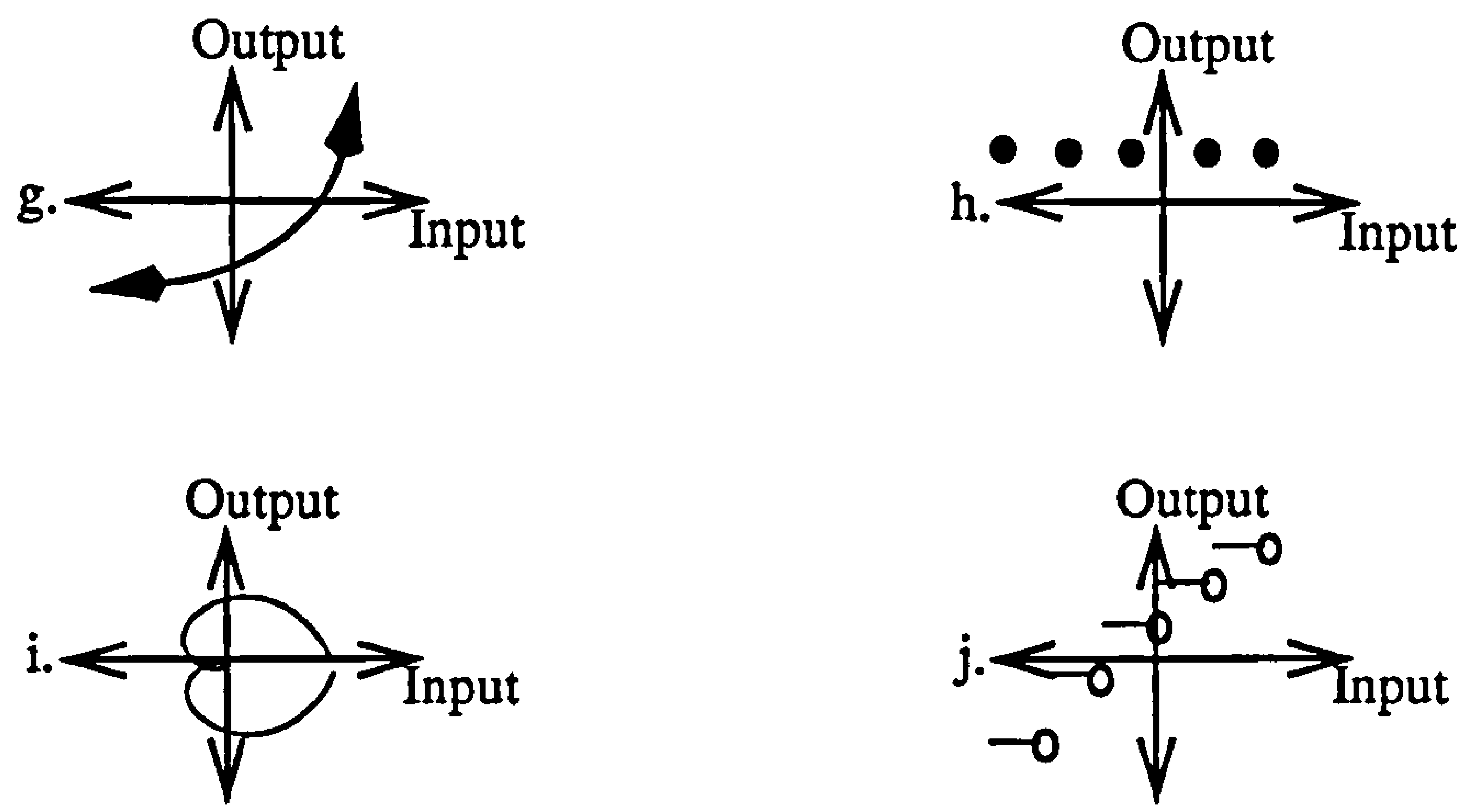
Summarizing, DB is quite flexible when identifying or working with symbolic definitions of functions. She is not bound by prototypes she has seen before. She is at least at the process layer for this facet demonstrating the need for the existence of a process to identify a function.

### 5.6.9 Layers of the geometric facet

Rectangular coordinate graphs are used to investigate the layers of this facet. Given a graph of a quadratic function and an appropriate viewing window, students were asked on both the pre- and post-course surveys to find output given input and vice versa. These questions caused students the most problems. Less than 5% answered these questions correctly on the pre-course survey and less than 40% answered them correctly on the post-course survey. DB was unable to answer either question on the pre-course survey. On the post-course survey, she correctly found input given output, but made an error finding output given input. Based on her answer, her error appears to be that of scale.

On the post-course survey, students were asked to identify each of the following graphs as "a function" or "not a function".






The question was repeated during the interview. DB identified all but a, f and j as functions on her post-course survey. Her response on the interview follows.

DB: Well, I know b is a function since it is linear. If a is a function I don't know it because it is like nothing I have ever seen. I don't think that's a function. C is a function, d is, e is also, f—I'm not too sure about being that there is that long straight line in the middle. I suppose it is possible, but I'm not sure. G is a function, h is not; those look more like a discrete function which I suppose could be derived from a function. I— we haven't really seen anything like this yet but I would say it is a function. J [is] not a function-it doesn't seem to connect anywhere.

DB exhibits little inconsistency with her survey response. She has changed her mind about h and is uneasy about f.

Intvw: Let's go back. H—you said it is not a function but maybe it's derived from a function. What do you mean?

DB: Well, I would say no because anytime we have done discrete points it's strictly an input-output. I would say looking at it just this way, I would say it is not a function.

DB seems to be saying no because the process is not known though input-output pairs are given.

Intvw: What about j?

DB: It looks to me more like discrete points rather than like a continuous curve.

Intvw: Okay, but you don't have a problem with e which also has a break in it?

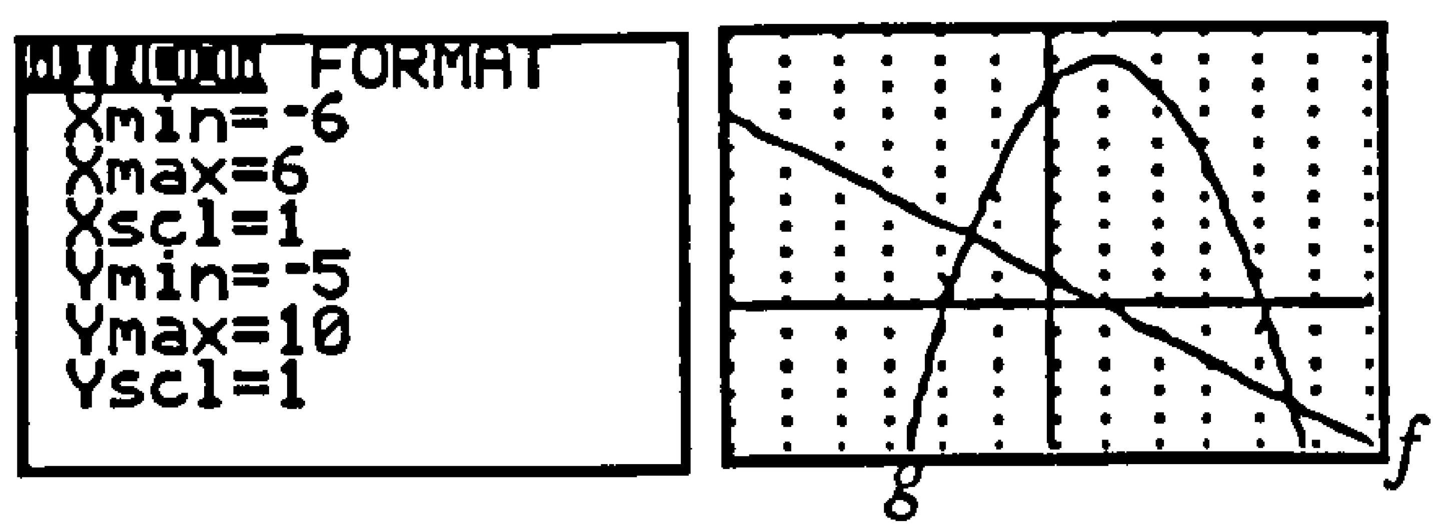
DB: Right. Well I remember seeing that as a piecewise function

Intvw: What about f?

DB: Well, I suppose I would say a function although the continuous straight line throws me off. I suppose a function could have a series of outputs that are the same.

DB's interpretation of graphs displays more of a tendency to identify with prototypes (See Tall & Bakar, 1992) than the other facets we have discussed. There is some indication of a problem with the lack of continuity (See Markovits et al., 1993). She demonstrates that she is thinking about input-output pairs, but needs to rely on the existence of a process. While her image of the symbolic and verbal facets demonstrated a knowledge of the restriction of one output for a given input, that restriction does not appear to be part of her concept image for the geometric facet suggesting different developments of figural representations versus proceptual representations.

One final question on graphs required DB to do a function composition. She was asked to consider the graphs for functions *f* and *g* at right and to approximate the value of $g(f(2))$ .




DB: Ooh. $g(f(2))$ . You have to find what *f*(2) is; –1. The input to *g* is –1. Somewhere up there (pointing to the parabola at an input of –1). About positive 4.

The ease with which she was able to move between inputs and outputs and between curves was impressive. While her criteria for recognizing functions from graphs is very process-oriented, she demonstrates good facility at interpreting specific information from a graph.

### 5.6.10 Layers of the numeric facet

Tables were used to investigate the numeric facet. Students were asked on both surveys to find output given input and vice versa for a function defined by a table. DB was unable to answer either question on the pre-course survey but answered both correctly on the post-course survey. When given two random two-column tables, she identified both as functions even though the second table contained two different outputs for the same input. It is curious that she did not require the existence of a process in order to identify these tables as functions. She was later asked the following question on the post-course survey: "A table has two columns. The left column begins at 0 and increases in increments of 2. The right column begins at 1. Each entry in the right column is computed by multiplying the preceding entry by 3. Part of the table appears below. Is $y$ a function of $x$?"

| $x$ | $y$ |
|---|---|
| 0 | 1 |
| 2 | 3 |
| 4 | 9 |
| 6 | 27 |
| 8 | 81 |

DB marked that this was not a function. The same question was asked on the interview. DB supplies reasoning for her answer.

DB: Hmmm. I would say no because the values of $y$ are not taking the values of $x$ into consideration. It is just taking the previous number and multiplying by 3. I don't see where the $x$ s come in to play in finding $y$.

Intvw: So you are missing a process from $x$ to $y$?

DB: Yes.

Intvw: And that makes you say you don't have a function?

DB: Yes.

Again DB's process orientation comes through. It might be a good idea to go back and ask her about the two random tables she identified as functions. As with the colloquial, symbolic, and geometric facets, DB was asked to use tables for function composition. Again her transcript indicates her ease at doing so.

Intvw: Consider the following tables for functions $f$ and $g$. What is the value of $f(g(2))$ ? Why?

| $x$ | $f(x)$ |
|---|---|
| 1 | 3 |
| 2 | –1 |
| 3 | 1 |
| 4 | 0 |
| 5 | –2 |

| $x$ | $g(x)$ |
|---|---|
| –2 | 3 |
| –1 | 1 |
| 0 | 5 |
| 1 | 2 |
| 2 | 4 |

DB: First I find $g(2)$ which is 4. That becomes the input for $f$. So it is 0.

Intvw: What is the value of $g(f(2))$ ? Why?

DB: $f(2)$ is –1. That becomes the input to $g$. So it is 1.

DB exhibits a facility with tables equal to that which she demonstrated for the other facets. The key point is her "process" view. DB's notion of "process" may actually interfere with the "simpler" process of reading from a left column entry to the corresponding right column entry. An alternative teaching sequence such as "read tables" first might yield different results.

### 5.6.11 Layers of the notation facet

On the post-course written survey, students responded "true" or "false" to the following statements: Suppose that $f$ is the name of a function and $x$ is the input to that function.

a. $f(x)$ represents the output of the function when $x$ is input.

b. $f(x)$ represents the product of $f$ and $x$.

c. $f(x)$ represents the rule you follow to find the output.

DB responded "true" to part a and "false" to parts b and c. A similar question was asked during the interview.

Intvw: What do you think when you see the notation $y(x)$?

DB: Well when I first saw it I thought $y$ times $x$. Now it is easier to figure out that it is $y$ is dependent on the value of $x$.

Intvw: Do you identify this symbolism with a process?

DB: Well if you see just $y(x)$ that really doesn't mean anything. Usually you see an equal sign and an equation following it.

The response seems to be consistent with the fact that DB did not associate the notation with a rule to follow to find the output. The interviewer dug a bit further.

Intvw: If I just write $y(x)$, would you identify a process with it?

DB: I guess I would because I know that it has to go through some process to get the value of $y$.

Notice that DB identifies the output with $y$ only, not with $y(x)$. DB exhibits an image of the notation as having meaning only if the process is defined. Next the interviewer probed her interpretation of the notation as an output of a function.

Intvw: Do you identify that symbolism $y(x)$ with output?

DB: Yes. $y$ is definitely the output.

While DB interprets the notation as function notation, she demonstrates some conceptual difficulties. She finds it difficult to accept $y(x)$ alone without setting it equal to an algebraic expression that describes the process. Thompson (1994) suggests that "the predominant image evoked in a student by the word "function" is of two written expressions separated by an equal sign" (p. 24). DB seems to attach the output to $y$ only, rather than $y(x)$.The interviewer chose to delve further by providing two specific examples.

Intvw: If I say $y(x) = 4$, does this represent a process, an output, or both to you?

DB: That would be an output because you're not giving the equation you are just giving basically $y = 4$ when $x$ is input.

DB is uncomfortable with the given statement. Student difficulty with constant functions is well-documented. (See Markovits et al., 1993 and Tall & Bakar, 1992, for example.) DB thinks 4 is an output. DB believes there is some process performed on $x$ to obtain 4. She sees the notation as a specific ordered pair rather than as a general statement of a function.

The interviewer followed with a question about a more prototypical function.

Intvw: If $y(x) = 3x - 7$, does this represent a process, an output, or both?

DB: That's a process because now you are giving the equation as to how you would establish the value of $y$.

DB is much more comfortable with this more common, prototypical form. DB demonstrates little flexibility in shifting between process and output.

Another question asked on the post-course survey was: "Assume that $f$ is the name of a function. Is there a difference between $3f(2)$ and $2f(3)$?" DB wrote: "Yes, the value of the number in the parentheses is the independent variable which would affect the value of the function." Let's look at how DB responded to the same question in an interview setting.

DB: Different. The way I would read this since *f* is name of function. The 3 and 2 in parentheses are input. How I am reading this is that I would find the value of *f* for the given input and then multiply that output by the number in front of *f*.

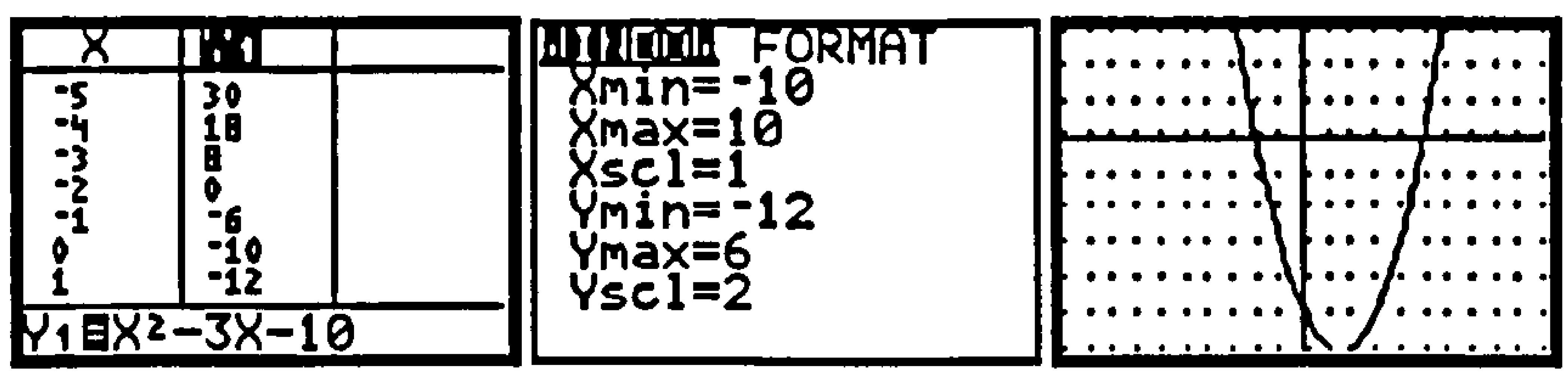
DB's interpretation is consistent with her written answer. DB's understanding of the notation is developing, but she has trouble seeing the notation flexibly. She does demonstrate a good grasp of the relationship between input and output implied in the notation. She does not appear to be at the stage where she interprets the notation as an object.

### 5.6.12 Connections between symbolic, geometric, and numeric facets

One last interview question explores the relationship between the symbolic, geometric, and numeric facets. The transcript is presented in full.

Intvw: An equation, a table, and a graph are displayed below for the same function.

$$y(x) = x^2 - 3x - 10$$




What is the output if the input is –1? Did you use the equation, the table, or the graph to answer the question?

DB: –6 using the table.

Intvw: What is the output if the input is 4? Did you use the equation, the table, or the graph to answer the question?

DB: Now I would use the graph. 1, 2, 3, 4 (counting on the horizontal axis of the graph). Around –8, I guess.

DB is very careful to look at the scale while answering this question.

Intvw: What is the output if the input is 12? Did you use the equation, the table, or the graph to answer the question?

DB: Um, I guess you'd have to just multiply that out since the 12 doesn't appear in either the table or the graph.

Intvw: What do you mean by "multiply it out?"

DB: Go up to the equation and input 12 for the $x$'s and figure it out.

Intvw: What is the output if the input is $h$? Did you use the equation, the table, or the graph to answer the question?

DB: Um, well, again I mean all you would do is change the $x$'s to $h$'s.

Intvw: Would that be an acceptable output to you?

DB: Yes.

Intvw: What are the input(s) if the output is 0? Did you use the equation, the table, or the graph to answer the question?

DB: –2 using the table.

Intvw: Are there any others?

DB: Oh, obviously there would have to be (looking at the graph). Also, 5 on the graph.

Intvw: What are the input(s) if the output is 44? Did you use the equation, the table, or the graph to answer the question?

DB: There would be a possible negative or positive. You'd have to figure that out from the equation. You would replace $y$ with 44 and solve for $x$.

It is impressive, watching the video, how easily DB moves between the various facets. She demonstrates equal facility in reading information off of the appropriate facet. In each case, her choice proves to be the most efficient choice. DB appears to have good mental connections between tables, equations, and graphs of functions. She is equally facile in moving between input and output in either direction.

### 5.6.13 DB's profile

Based on DB's written and verbal responses to the surveys and interview, a visual profile of her concept image of function is developed (Figure 5.3). In essence, these are snapshots of her understanding according to the theoretical perspective. The shading indicates the number of layers the student has demonstrated in her understanding of each facet. The student's knowledge of a specific facet has not been assessed if the outermost layer (pre-procedure) is unshaded. Profiles of DB at the beginning of the beginning algebra course and at the completion of the interview are displayed. In essence,

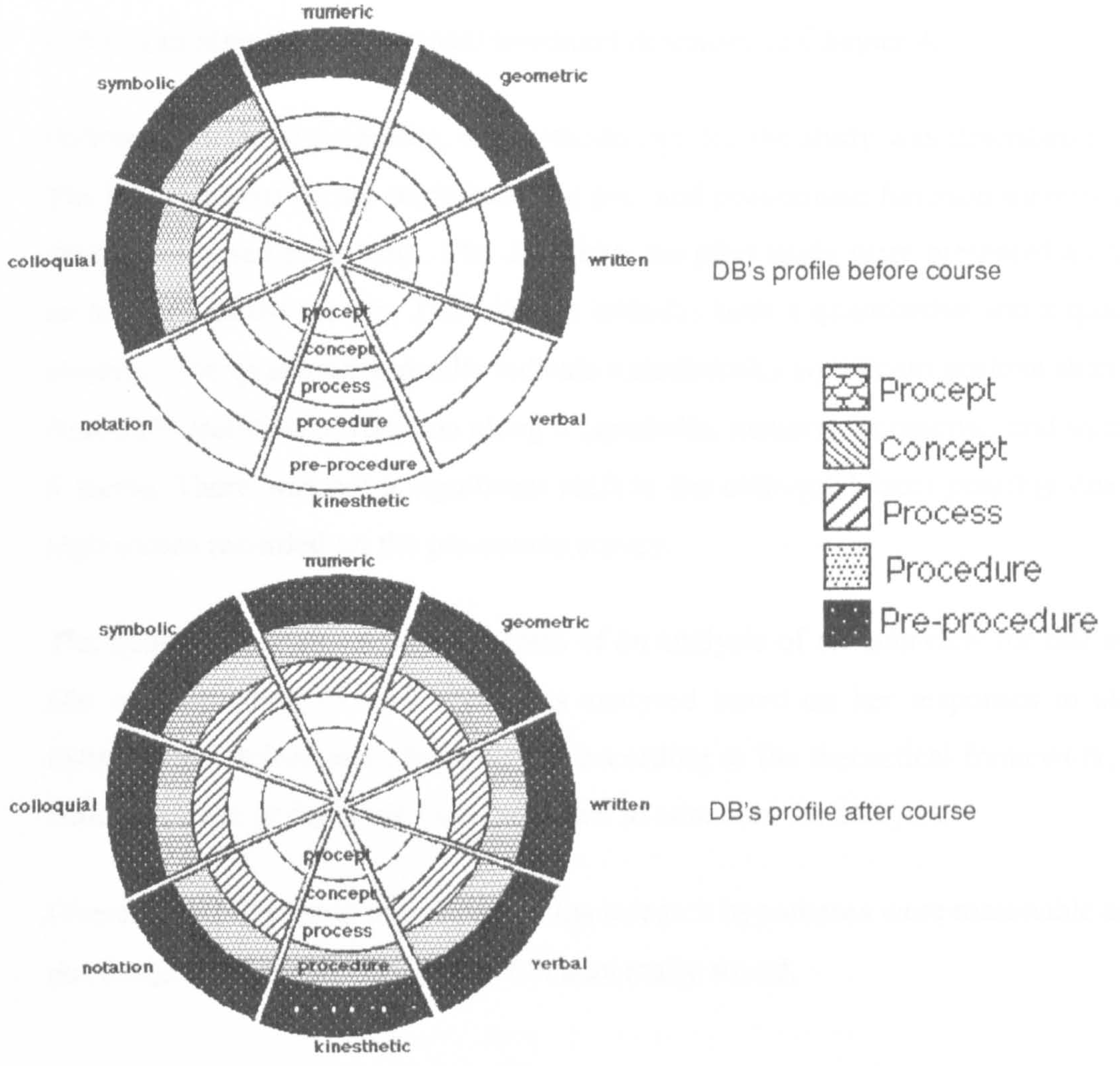



Figure 5.3: DB's concept image profiles

DB appears to be at least at the process layer for all facets, except possibly the kinesthetic, where more data must be collected. She has shown substantial growth as a result of the course. It is possible that she has reached the concept or even procept layer for several facets, but, again, more data is necessary to conclude this.

## 5.7 Conclusion

This chapter summarized the methods and results of the pilot study performed during spring term, 1996. The first section provided information on the students who formed the population from which the sample for the study was drawn. The population was profiled especially with respect to the attitudes they manifested toward mathematics before and after the instructional treatment described in Chapter 4.

Following the student profile, the methodology for the study was described in detail. The key elements of the study included pre- and post-course function surveys and in-depth task-based interviews. The data from the pilot study were presented along with an analysis of the results. The analysis includes both a quantitative and a qualitative element. The quantitative results indicate a statistically significant positive shift in student understanding of function along 4 (symbolic, numeric, geometric, and written) of 5 facets. There was not a significant shift in the colloquial facet possibly due to the high scores recorded on the pre-course survey.

The qualitative analysis took the form of an analysis of an interview for one student. Her understanding of each facet was analysed based on her responses to all three instruments. Before and after profiles, according to the theoretical framework, of her concept image of function were presented to conclude the analysis.

Overall, the pilot study indicated that the research hypotheses were reasonable and that the design of the instruments was fundamentally sound.

# CHAPTER 6 Thesis and Main Study

## 6.1 Introduction

This chapter states the hypotheses of the thesis and describes the procedures and methodology used by the researcher to complete the main study. Included is a discussion of the purpose of the research, the variables, the research design, and the purpose of each instrument used in the main study. The individual components of each instrument are described along with their purpose within the main study. The influences of the pilot study in the design of the main study are described. Various types of triangulation used to establish the validity of the results are briefly discussed to end the chapter.

## 6.2 Thesis

A significant number of American students must enrol in a beginning algebra course in college even though they have had the same course, and often several courses higher, in high school. As described previously, many of these students exhibit behaviour that indicates they have become "debilitated" by their prior mathematical experiences. Rather than forcing them to take the same beginning algebra course again, a course with a re-designed curriculum using the unifying thread of function has been implemented in an effort to assist students past the hurdles that have previously blocked their paths. This leads to the main purpose of this research as addressed in the following question:

**Research Question:** Can adult students who arrive at college having had debilitating prior experiences with algebra develop a process level understanding of the function concept through appropriate instructional treatment?

The definition of each term in the previous statement follows.

- "Appropriate instructional treatment" means a beginning algebra course using text materials, as previously described, that focus primarily on function as process and that introduce function in a multi-faceted way including written definitions, graphs, function machines, tables, and equations. Function notation along with graphing calculators are used throughout the course.

- "student" refers to an adult community college student enrolled in a beginning algebra course in which the instructional treatment described above is used. As there is no typical such student, a profile of students included in the study was provided in Chapter 5 as part of the description of the pilot study.
- "process level understanding" means that students have demonstrated the ability to answer questions about the various facets of "function" that indicate they are, minimally, at a process layer for the majority, if not all, of the facets. The questions used along with their accompanying rubric will be discussed later in this chapter.
- "debilitating prior experiences" were described in Chapter 2. In essence, such students evidence the symptoms of procedural knowledge (Hiebert & Lefevre, 1986), instrumental understanding, at best (Skemp, 1976), and the proceptual divide (Gray & Tall, 1994).

The main research question is addressed by proposing several sub-questions as follows:

1. Will students demonstrate improved capabilities in interpreting the colloquial facet of function, as exemplified by a function machine, when asked to find output given input and vice versa?
2. Will students demonstrate improved capabilities in interpreting the symbolic facet of function, as exemplified by an equation in two variables, when asked to find output given input and vice versa?
3. Will students demonstrate improved capabilities in interpreting the numeric facet of function, as exemplified by a two-column table, when asked to find output given input and vice versa?
4. Will students demonstrate improved capabilities in interpreting the geometric facet of function, as exemplified by a two-dimensional coordinate graph, when asked to find output given input and vice versa?
5. Will students demonstrate improved capabilities in the written facet by writing a definition of function in terms of a dynamic process?

6. Will students demonstrate improved capabilities in the notational facet by interpreting function notation correctly and contextually?
7. Will students exhibit consistency in their concept definition of function across verbal and written facets?
8. Will students demonstrate an ability to adapt alternative concept definitions of function into their own written and verbal definitions?
9. Is the growth of the concept image of function in students uneven? Are the cognitive links between facets sometimes nonexistent, sometimes tenuous, and sometimes unidirectional?

In addition to measuring changes in students' concept images of function as a result of the previously-described instructional treatment, the researcher creates three "template" profiles of student concept images as a result of classifying students highly capable, capable, and incapable (Krutetskii, 1976) with respect to the function concept based on the results of written surveys and interviews.

## 6.3 Variables

### 6.3.1 Prior variables

The prior variables revolve around both students enrolled in a beginning algebra course at a U. S. community college and the instructors who teach this course. Such students have been previously classified as "debilitated" by their prior encounters with mathematics. The instructors often are frustrated by the fact that their students have entered college with so little mathematical understanding. These instructors usually are devoted to teaching, but the traditional approach to the beginning algebra courses has been to focus on instrumental understanding (Skemp, 1976) through the teaching of endless skills and procedures.

Profiles of the students who enrol in the target course were provided in the Chapter 5. The instructors who taught the classes involved in the main study have indicated their dedication to a radically different approach to the course by attending at least one week-long workshop on teaching with the non-traditional materials and by volunteering to teach the course using the non-traditional materials.

### 6.3.2 Independent variables

The independent variables include the "reform" curriculum as described in Chapter 4 and the extensive use of technology, especially a graphing calculator. The curriculum that serves as an independent variable is the intended curriculum, not the implemented curriculum nor the acquired curriculum. Students were required to purchase a graphing calculator and the text integrates the calculator as a tool to explore mathematics extensively.

### 6.3.3 Intervening variables

There are numerous intervening variables that must be acknowledged. The first is the implemented curriculum. The number of sections in the text that students actually studied and the topics where emphasis was placed significantly impact the formation of the students' concept images of function. What the instructors chose to assess is a second intervening variable. Assessment choices place an emphasis on certain aspects of the curriculum and on the use of technology at the expense of other parts of the curriculum. The use of the technology within the class is another factor influencing student understanding. Each instructor incorporates technology in their own unique way thus affecting student understanding of the concepts. A fourth intervening variable is the role of the student and the role of the instructor in the classroom community. The curriculum was built on the philosophy that students should be actively engaged in doing mathematics rather than watching someone else (the teacher) do mathematics. The instructors who taught these courses professed belief in this principle, but when the end of the term began to draw near, some reverted to a lecture-based approach to "cover" the material. Student effort and dedication to the course is a final intervening variable. The students in this study are adults with jobs and families. The level of commitment to this course will vary widely. For those who exert little effort, the outcomes are going to be marginal at best.

Little beyond the training the instructors received prior to the course can be done to control the intervening variables. As the results are presented in the next two chapters, it is appropriate to keep these variations in mind.

### 6.3.4 Dependent variables

The key dependent variable is the students' concept images of function as a result of the independent variables. The understanding of function is a key part of the acquired curriculum. Using the nontraditional text and having ready access to powerful graphing technology, can such students develop an appropriately rich concept image of function? This variable is measured using the three instruments described later in this chapter.

### 6.3.5 Consequent variables

The students' future success in mathematics courses and in the ability to use function to reason quantitatively in daily life serve as the consequent variables in this study. Measuring such variables is beyond the scope of this study.

## 6.4 Research Design

### 6.4.1 Subjects

The subjects were drawn from the same student pool used for the pilot study. The students were enrolled in a section of the "reform" beginning algebra course (as described in Chapter 4) at one of four community colleges during Fall Term, 1996. As in the pilot, all instructors had previously attended a week-long National Science Foundation workshop organized and led by the authors of the materials and focusing on the implementation of this "reform" curriculum. None of the instructors were authors of the materials. The student profile for the main study approximates that described in detail in Chapter 5. Specific characteristics of the population will be further described in Chapter 7 when the data are analysed.

### 6.4.2 Instruments

The main study consists of both quantitative and qualitative components, similar to those described in the pilot study. Pre- and post-course surveys during Fall Term, 1996 were used to collect quantitative data. The pre-course survey was administered on the first day of class and the post-course survey was administered during the last week of the term. Scores on the post-course survey were used to divide students into three categories: highly capable, capable, and incapable. Task-based structured interviews were conducted within two weeks of the end of the course to collect qualitative data. Students from each category were interviewed. Profiles of one student from each category were subsequently created. A discussion of each instrument along with the rationale for the questions appears below.

### 6.4.3 Pre-course survey

There are a total of 7 questions on the pre-course survey (Appendix A). The rationale for each follows.

- Question 1 measures students' ability to manipulate a function machine by asking students to find the input given the output and vice versa. This question provides data about the colloquial facet of function. The computations in the function machine (essentially a linear function) were kept simple to reduce the chances of arithmetic error.
- Question 2 measures students' ability to manipulate a two variable equation by asking students to find the input given the output and vice versa. This question provides data about the symbolic facet of function. The computations were kept simple to reduce the chances of arithmetic error.
- Question 3 measures students' ability to use a two-column table by asking students to find the input given the output and vice versa. This question provides data about the numeric facet of function. A function that is not one-to-one was used to assess students' ability to find multiple answers to the question.
- Question 4 measures students' ability to use a two-dimensional rectangular coordinate graph by asking students to find the input given the output and vice versa. This question provides data about the geometric facet of function. A quadratic function was used to assess students' ability to find multiple answers to the question.

For Questions 1–4, student ability to find the output, given the input, suggests a procedure layer of understanding of the facet. Ability to find the input, given the output, requires students to reverse the procedure and is used to suggest a process layer of understanding of the given facet.

- Question 5 measures students' depth of understanding along the written facet by asking students to write a definition of function.
- Question 6 asks students to write definitions of four common terms: variable, two-column table of numbers, graph, and equation. The definition of variable provides data on student understanding of a key concept preliminary to the function concept. The other three definitions were used to determine if students recognize them as three of the facets of the function concept.
- Question 7 measures students' recognition of and flexibility with function notation.

In summary, the pre-course survey was designed to provide data before the course began about student understanding of function focusing on the colloquial, symbolic, numeric, geometric, verbal, and notation facets. No other more sophisticated function questions were asked since the researcher assumed that most, if not all, students in the study would have no prior exposure to the concept of "function". Based on the results of the pre-course survey discussed in the next chapter, this assumption appears to be appropriate.

### 6.4.4 Post-course survey

Students completed the post-course survey (Appendix B) during the last week of the term of the beginning algebra course. The first seven questions are the same as those on the pre-course survey described in Section 6.4.3. These provide the opportunity to measure change in students' understanding from the beginning to the end of the term. In addition, student scores on these questions provided data to divide the students into the top third (highly capable), middle third (capable), and bottom third (incapable). Students were asked several other questions to further assess the depth of their knowledge of the facets of function. The rationales for the remaining questions follow:

- Questions 8 and 9 further tested student understanding of the notation aspect. Question 8 primarily relies on students' ability to recognize the notation $f(2)$ as an output determined by the input 2 and not as multiplication. Students who do not see a difference between $3f(2)$ and $2f(3)$ are not recognizing function notation as producing an output that may vary based on the value of the input. The ability to see

$f(2)$ and $f(3)$ as outputs suggests a student minimally at the process layer. Question 9 checks for the ability to deal flexibly with function notation as either a process or an object. Those students who view the notation as multiplication may be pre-procedural. Students who see $f(x)$ as an output may be at least at the process layer while those that see it as a rule or process may be at the concept layer. Those who can view $f(x)$ flexibly as both output and process exhibit the traits of proceptual thinking with respect to this facet.

- Question 10 asks students to select tables and sets of ordered pairs that are functions from a collection of such objects. This question is used to measure depth of understanding of the numeric facet. In particular, students at the procedure layer may only select as functions those tables in which a specific procedure to move from input to output is apparent. More flexible students may focus on the existence of an output for a given input and not need to know the specific procedure used to move from input to output. A secondary issue concerns students' recognition of the "uniqueness of the right" condition (Dubinsky & Harel, 1992) and their ability to differentiate this condition from one-to-oneness.
- Question 11 asks students to select equations that are functions from a collection of same objects. This question is used to measure depth of understanding of the symbolic facet. In particular, students at the procedure layer may only select as functions those equations in which a specific procedure to move from input to output is apparent. More flexible students will focus on the existence of a relationship between the input and output and not need to know the specific procedure used to move from input to output. One interesting question revolves around students' treatment of equations that they had never seen before. More information on this issue was collected via the interviews. Another issue is whether students could distinguish between relations and functions.
- Question 12 asks students to select graphs that are functions from a collection of same objects. This question is used to measure depth of understanding of the geometric facet. In particular, students at the procedure layer may only select as functions those graphs in which a specific procedure to move from input to output is apparent. Another indication of lack of depth of understanding is the reliance on

prototypes (Tall & Bakar, 1992). More flexible students may focus on the existence of a relationship between the input and output and not need to know the specific procedure used to move from input to output. One interesting question revolves around students' treatment of graphs that they had never seen before. Again would they rely on prototypes in answering the question? Interviews were used to collect more data on this issue. Similar to Question 10, how do students deal with graphs of relations versus functions and do they require one-to-oneness of a function?

- Question 13 contains information about several facets: colloquial, numeric, verbal definition vis-a-vis concept definition, and symbolic. The main point of the question relates to the student's definition of function. Are functions "equal" only if they use the same procedure to produce output from input or are they "equal" if the sets of ordered pairs that define the function are equal? Students are more likely at the procedure layer if two functions can only be "equal" when the procedures are equal. Students at the process layer may only require that the two functions produce the same output for the same input in order for the functions to be the same–to be equal.

Table 6.1 summarizes the questions on the post-course survey with respect to which facet(s) they provide information about.

**TABLE 6.1:** Post-survey questions and facets tested

| Question | Facet |
|---|---|
| 1 | colloquial |
| 2 | symbolic |
| 3 | numeric |
| 4 | geometric |
| 5 | written |
| 6 | numeric, geometric, symbolic |
| 7 | notation |
| 8 | notation |
| 9 | notation |
| 10 | numeric |
| 11 | symbolic |
| 12 | geometric |
| 13 | colloquial, numeric, symbolic, written |

Thus, the pre- and post-course surveys provide data on six of the eight facets of function. The remaining two, kinesthetic, and verbal, were addressed though interview.

### 6.4.5 Interviews

As stated previously, selected students participated in one hour interviews within two weeks of the end of their beginning algebra course. The students interviewed were selected by their instructors based on their willingness to participate and on the requirement that the researcher interview as wide a range of students as possible, as defined by their score on the post-course survey. As a result, the researcher was able to develop profiles of students in the categories of "highly capable" (top third on post-course survey), "capable" (middle third on post-course survey), and "incapable" (bottom third on post-course survey). Such profiles are detailed in the Chapter 8. A summary of the questions asked appears in Appendix C. Note that some questions were omitted when interviewing particular students. If the student appeared completely unprepared to answer the question, the interviewer chose to omit the question. All interviews were video- and audio-taped. Additionally, the interviewer wrote detailed notes and collected any writings made by the students.

The first 9 interview questions appeared on either both the pre- and post-course surveys or on just the post-course survey. The purpose was to establish some triangulation between students written responses and their verbal responses in an interview setting. A brief discussion of the interview questions and their purpose follows.

- The students were initially given a collection of 24 cards (Appendix D), each containing an equation, a table, a graph, or a function machine. Several functions were represented on four different cards using four different facets. Students were asked to divide the cards into two piles: functions and nonfunctions. After completing the task the interviewer asked students to describe the algorithm they used to determine which pile to put a particular card in. The purpose of this question was to probe student understanding of function along 5 facets: numeric, symbolic, colloquial, geometric, and notation. Would students correctly identify functions for each facet? Would students recognize different facets of the same function? What consistency would there be across facets when students selected their functions?

- Questions 1–5 matched the corresponding questions on the pre- and post-course surveys. The purpose was to develop a profile across time of student understanding of colloquial, symbolic, numeric, and geometric facets. In addition, would students define functions verbally in the same way they defined them in writing? Such a comparison allows the researcher to look at the boundary between the written and verbal facets.
- Question 6 matched question 7 on the pre- and post-course surveys investigating the notation facet.
- Question 7 matched question 9 on the post-course survey investigating the numeric facet.
- Question 8 matched question 10 on the post-course survey investigating the symbolic facet.
- Question 9 matched question 11 on the post-course survey investigating the geometric facet.
- Questions 10 and 11 were taken from Norman (1992) and were designed to evaluate the flexibility of students' verbal definition of function by providing settings that involve nonnumeric pairings and a function in which the output is a point in the plane.
- Question 12 matched question 8 on the post-course survey investigating the notation facet.
- Question 13 asks for a physical demonstration of what a function is and was used to investigate the kinesthetic facet.
- Question 14 again tests the flexibility of the students' verbal definition of function by asking them to respond to alternate forms of the definition. This question was asked during the pilot study, but the definitions were presented as correct definitions. For the main study, the students did not know if the definitions were correct definitions or not. Thus, they were not forced to assimilate the definition into theirs if they did not wish to.

- Question 15 begins a set of questions that focus on the boundaries between facets. Students are required to move flexibly between the symbolic, numeric, and geometric facets to answer the question. The interviewer focused on the ease with which students were able to transfer from one facet to another and which facet was used to answer a particular question.
- Question 16 deals with both the colloquial and verbal facets. This question is similar to question 13 on the post-course survey. One part aims at the correct interpretation of function machines. The second issue focuses on "equality" of functions. Are two functions equal only if they have the same procedure to move from input to output or are two functions equal if they produce the same output, regardless of procedure used, for the same input? The question is designed to separate students into either the procedure layer or the process layer.
- Questions 17–20 deal directly with the boundaries between symbolic, numeric, colloquial, and geometric facets. Students are given a function in terms of one of the facets and asked to create the function's representation in the other 3 facets. Which boundary would students cross first in each case? Which boundaries seemed to be most difficult to cross? Which boundary crossings were uni-directional?
- Questions 21–24 and 27 deal with students ability to perform composition of functions for various facets. Answering the question requires facility with function notation and with the ability to think of a function as an object that can be used as input to another function. Can students take an output of one function and think of it as an input to another function.
- Finally, questions 25 and 26 use function machines and symbolic forms to test if students can see functions as objects. In essence, students are using a function machine to perform a process on a function. Can students use functions as inputs and accept functions as output?

Table 6.2 summarizes the questions on the interview with respect to which facet(s) they provide information about.

TABLE 6.2: Interview questions and facets tested

| Question | Facet |
|---|---|
| 1 | colloquial |
| 2 | symbolic |
| 3 | numeric |
| 4 | geometric |
| 5 | verbal (boundary with written) |
| 6 | notation |
| 7 | numeric |
| 8 | symbolic |
| 9 | geometric |
| 10 | verbal |
| 11 | verbal, geometric |
| 12 | notation |
| 13 | kinesthetic |
| 14 | verbal |
| 15 | symbolic, numeric, geometric (boundary) |
| 16 | colloquial, numeric, symbolic, verbal |
| 17–20 | colloquial, numeric, symbolic, geometric (boundary) |
| 21 | numeric, notation |
| 22 | colloquial, notation |
| 23 | geometric, notation |
| 24 | symbolic, notation |
| 25–26 | colloquial, symbolic (boundary) |
| 27 | numeric, notation |

### 6.4.6 Combining the instruments

The previous sections have detailed the specifics of each of the three instruments used in this research study. Table 6.3 displays the facets and the questions on each instrument that provided data to analyse the students' understanding of the facet.

TABLE 6.3: Questions by facet

| Facet | Pre-course survey | Post-course survey | Interview |
|---|---|---|---|
| numeric | 3, 6 | 3, 6, 10, 13 | 3, 7, 15–21 |
| symbolic | 2, 6 | 2, 6, 11, 13 | 2, 8, 15–20, 24–26 |
| geometric | 4, 6 | 4, 6, 12 | 4, 9, 11, 15, 17–20, 23 |
| colloquial | 1 | 1, 13 | 1, 16–20, 22, 25, 26 |
| written | 5 | 5, 13 | |
| verbal | | | 5, 10, 11, 14, 16 |
| kinesthetic | | | 13 |
| notation | 7 | 7, 8, 9 | 6, 12, 21–24, 27 |

Additionally, Table 6.4 displays the boundaries that we investigated as a result of the interview questions.

TABLE 6.4: Boundaries investigated

| Boundary | Interview question |
|---|---|
| verbal–written | 5 as compared to pre- and post-course 5 |
| symbolic–numeric | 15–20 |
| numeric–geometric | 15, 17–20 |
| geometric–symbolic | 15, 17–20 |
| colloquial–symbolic | 16–20, 25, 26 |
| colloquial–numeric | 15–20 |
| colloquial–geometric | 17–20 |
| numeric–notation | 21, 27 |
| symbolic–notation | 24 |
| geometric–notation | 23 |
| colloquial–notation | 22 |

In each case, an attempt is made to investigate the porousness of the boundary moving in each direction.

## 6.5 Student Profiles

As indicated in the research questions, the researcher is interested in measuring change in students' concept images of function over the period of the target course. Using just pre- and post-course surveys, models of students' concept images can be developed for 6 of the 8 facets. The interview data are used to build more complete profiles for 3 students along all 8 facets. Note that the focus of the research was on the pre-procedure, procedure, process boundaries (where layers, rather than facets, are considered). While some students may develop greater depth of understanding along one or several facets, the population being "debilitated developmental algebra students" suggests that progress toward a process-layer understanding is an appropriate goal. Hopefully, later courses can help students refine their concept image further.

## 6.6 Influences of Pilot Study

All three instruments were modified based on the results of the pilot study. The pre-course survey was revised to include information on the algebra background of students entering the target course. Other changes on the pre-course survey included:

- The variables $x$ and $y$ were more clearly defined to represent input and output, respectively, on questions 1–4. The lack of consistency across function machine, table, equation, and graph representations caused some confusion on the pilot study.
- Wording was changed to allow for the possibility of multiple outputs given an input in questions 1–4. In this way, students were not given advance cues about how many answers might be expected.
- The survey made it clear that the table and graph displays were from a TI–82 graphing calculator, thus identifying the source of the display. These frames were used since the primary device students would use to create and view tables and graphs throughout the instructional treatment was either a TI–82 or a TI–83 calculator.
- The question "What is a function" replaced a similar question on the pilot study that did not directly ask students to describe what a function is.
- A question on function notation (7) was added to gather information before the course about students' understanding of the notation facet.

The post-course survey was similarly modified so that the first seven questions matched those on the pre-course survey. Other changes include:

- A question was added asking students to indicate their age (within a given set of ranges). The purpose was to see if there was any correlation between age and the growth of a student's concept image of function.
- The form of the questions asking students to identify functions given a set of tables, a set of equations, and a set of graphs was modified. In addition, students were asked to write their rule for determining if a given representation was a function or not. There was simply a checklist on the pilot study questionnaire in which students checked off if the given situation was a function or not. Analysis of the pilot study data provided little information on the rule used. Unfortunately, few students on the main study could identify their rule for identifying if a given representation was a function or not. Many students left the questions blank suggesting they had little idea how to discriminate between various forms of tables, equations, and graphs. More specifics will be discussed in the next two chapters.
- The pilot study post-course survey contained questions on function composition using tables and graphs. Student responses on the pilot indicated very little success with the question suggesting that the question was not a useful discriminator. The question was removed from the main study post-course survey, though retained for the interview where it was possible to probe students understanding of the notation.

The interview for the main study was modified from that used in the pilot study in the following ways:

- The common questions on the pre- and post-course surveys were added to the interview to allow for triangulation across all three instruments.
- All other questions that were asked on the post-course survey were asked on the interview again to allow for triangulation between the written and the verbal responses.

- Both the pilot study interview and the main study interview had a question that gave the student two different function machines that were essentially the same function, though they contained different procedures. The function machines were simplified for the main study since the complexity of those on the pilot study seemed to get in the way of the focus of the question.
- Students were presented with two different definitions of function on the pilot interview and asked to compare and contrast those definitions with their own definition. A colleague suggested that the question be modified so that the students would not know if the given definitions were correct or not. The way the question was asked during the pilot suggested that the student had to assimilate the definition into their own. Using the colleague's suggestion, students had the flexibility to refute the definition if they felt it was at odds with their own. This change resulted in more useful answers, in terms of data analysis, than did the original question.

## 6.7 Triangulation

Several types of triangulation were used to further support the validity of the findings. Data triangulation "involves collecting accounts from different participants involved in the chosen setting, from different stages in the activity of the setting and from different sites of the setting" (Bannister et al., 1996, p. 146). Ninety-two students participated in both the pre- and post-course surveys. Data were collected at the beginning and end of the instructional treatment. Additionally, several students were interviewed within two weeks of the end of the instructional treatment. Students from 4 different sites and from 11 different instructors were studied.

Method triangulation "entails the use of different methods to collect information" (Bannister et al., 1996, p. 147). Both written surveys and task-based interviews were used to collect data. Several questions were asked on all three instruments to allow comparison among instruments.

Theoretical triangulation "embraces multi-theories and breaks through the parameters and limitations that inevitably frame an explanation which relies on one theory" (Bannister et al., 1996, p. 148). The theory of facets and layers of the function concepts arises from the melding of Dubinsky's APOS theory (Cottrill et al., 1996 for example) and the procept theory of Gray and Tall (1994). In turn, the theoretical framework proposed in this document further draws on the work of Schwingendorf et al. (1992), Confrey (1993), Sfard (1992, 1995), Sierpinska (1992, 1994), Schwartz & Yerushalmy (1992), Davis (1975), and Cuoco (1995), to name a few.

## 6.8 Conclusion

The purpose of this chapter was to clearly state the researcher's thesis and related research questions and to discuss how the main study was designed to collect data for the thesis. The chapter begins by stating the key research question.

**Research Question:** Can adult students who arrive at college having had debilitating prior experiences with algebra develop a process level understanding of the function concept through appropriate instructional treatment?

Each term used is carefully defined and the sub-questions that the main study was designed to answer are clearly stated. The design of the main study including the subjects, the variables, the instruments used, and what the instruments were supposed to measure are described in detail. In particular, each question on each instrument (pre-course survey, post-course survey, interview) is discussed in terms of its purpose and its relationship to the theoretical framework. Tables are included to display the relationship between questions on various instruments and the relationship between the questions and the facets described in the theoretical framework. The next section discusses how the instruments for the main study were modified based on the results of the pilot study. The chapter concludes with a discussion of the various forms of triangulation that were used to provide validity to the findings.

CHAPTER 7

# Main Study: Quantitative Data Analysis

## 7.1 Introduction

This chapter presents the quantitative data collected during the main study. Results of pre- and post-course surveys will be presented and analysed by focusing on significant quantitative changes that occurred in student understanding of the numeric, symbolic, colloquial, geometric, written, and notation facets of function before and after the instructional treatment described in Chapter 4. The resulting data provide a classification of types of students' understanding that may result when students who have been previously unsuccessful with mathematics are exposed to a course in which function is a unifying theme. The chapter concludes with a look at how the research questions that are the centre of this dissertation have been answered in light of the quantitative data collected.

## 7.2 Subject Specifics

Two hundred eighty-eight students completed the pre-course survey. Four community colleges participated in the pre-course survey as noted in Table 7.1.

TABLE 7.1: College demographics

| College | Location | Number of instructors | Number of students completing pre-course survey | Number of students completing post-course survey |
|---|---|---|---|---|
| William Rainey Harper College | Palatine, IL | 2 | 32 | 19 |
| College of Lake County | Grayslake, IL | 2 | 80 | 42 |
| Lakeland Community College | Kirtland, Ohio | 3 | 72 | 31 |
| Northwestern Michigan College | Traverse City, MI | 4 | 104 | 0 |

Only 92 completed the post-course survey. The post-course surveys were never collected by the instructors at Northwestern Michigan College. While some data will be reported for the original 288 students, all comparisons between pre- and post-course surveys are based on a sample of size 92, the number of students completing both surveys.

Aside from the instructor and the college, students were tracked using the last 5 digits of their social security numbers. Other personal information collected included their age and the average number of years of algebra the students had taken prior to their enrolment in the target course. Table 7.2 displays a summary of students' prior exposure to algebra.

TABLE 7.2: Prior exposure to algebra

| Statistic | All students (n = 288) | Students completing both surveys (n = 92) |
|---|---|---|
| Mean number of years of algebra prior to target course | 1.28 | 1.40 |
| Standard deviation | 0.94 | 1.00 |

Table 7.3 displays the age classifications for students completing both surveys.

TABLE 7.3: Age distribution of students completing both surveys (n = 92)

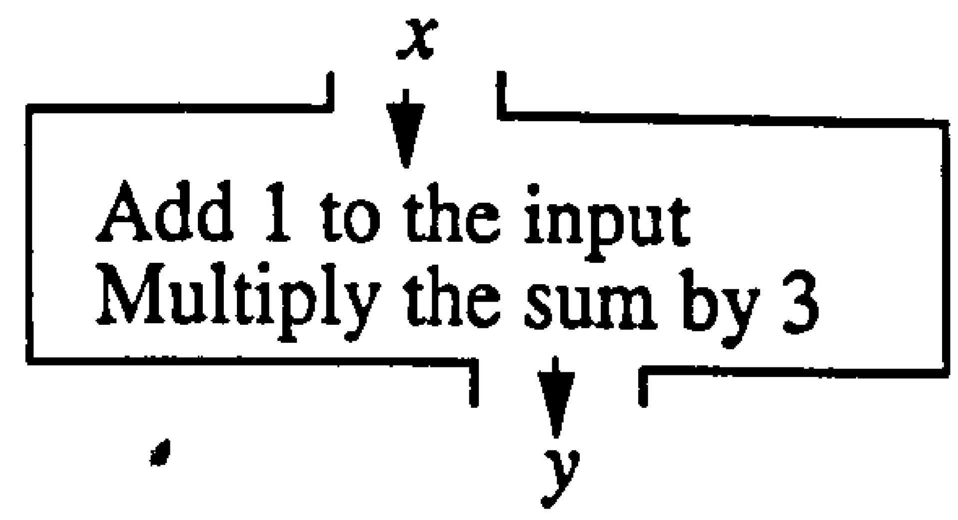
| Category | Number | Percent |
|---|---|---|
| 17–20 | 54 | 59% |
| 21–25 | 13 | 14% |
| 26–30 | 7 | 8% |
| 31– | 18 | 20% |

## 7.3 Pre- and Post-Course Survey Quantitative Data

### 7.3.1 Colloquial, symbolic, numeric, and geometric facets

The following 4 questions were asked on both the pre- and post-course surveys.

1. Consider the diagram.
   a. What are the output(s) if the input is 7?
   b. What are the input(s) if the output is 18?

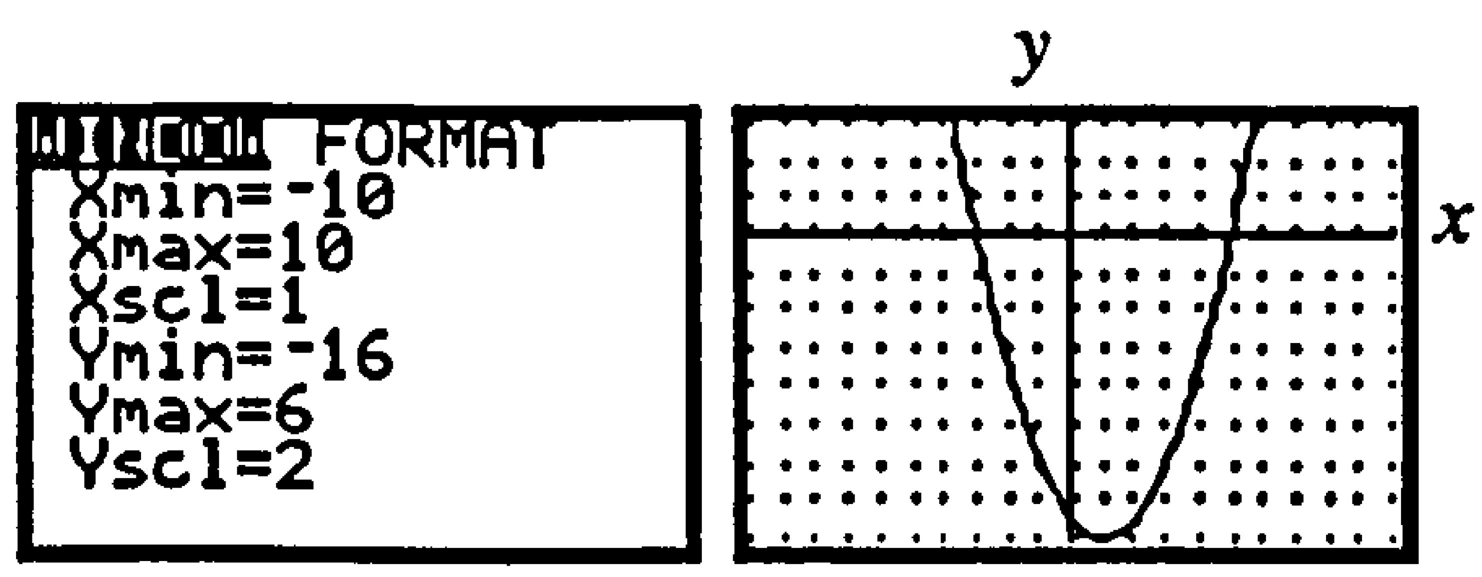



2. Consider the equation $y = 3x - 7$.
   a. What are the output(s) if the input is 5?
   b. What are the input(s) if the output is 0?

3. Consider the following table copied from a TI–82 graphics calculator. The "X" stands for $x$ and the "$Y_1$" stands for $y$.
   a. What are the output(s) if the input is –2?
   b. What are the input(s) if the output is –3?

| X | $Y_1$ | |
|---|---|---|
| [illegible] | 12 | |
| -2 | 5 | |
| -1 | 0 | |
| 0 | -3 | |
| 1 | -4 | |
| 2 | -3 | |
| 3 | 0 | |

4. Consider the following viewing window and graph copied from a TI–82 graphics calculator.
   a. What are the output(s) if the input is 3?
   b. What are the input(s) if the output is 0?




As was the case in the pilot study, these questions were designed to measure change in students' ability to apply a process and reverse a process for the colloquial (function machine), symbolic (equation in two variables), numeric (table), and geometric (graph) facets. Students were graded on each part and their scores from pre- to post-course on each question were compared. Students were given 1 point for a correct answer to part a of each of the questions. Students were given 2 points for a correct answer to part b. The additional weighting for parts b was used to reflect the added difficulty inherent in the "reversal" of the function. Since 3b and 4b have 2 answers, student were given 1 point for each correct answer.

The distribution of responses is reported in Table 7.4.

TABLE 7.4: Responses to pre- and post-course questions 1–4

| Question | Pre-course number correct (% correct) n = 288 | Pre-course with post-course number correct (% correct) n = 92 | Post-course number correct (% correct) n = 92 |
|---|---|---|---|
| 1a Colloquial facet: function machine input given | 205 (71%) | 62 (67%) | 79 (86%) |
| 1b Colloquial facet: function machine output given | 138 (48%) | 44 (48%) | 64 (70%) |
| 1 Colloquial facet: function machine both parts correct | 133 (46%) | 43 (47%) | 61 (66%) |
| 2a Symbolic facet: equation input given | 211 (73%) | 67 (73%) | 84 (91%) |
| 2b Symbolic facet: equation output given | 36 (13%) | 17 (18%) | 38 (41%) |
| 2 Symbolic facet: equation both parts correct | 35 (12%) | 16 (17%) | 37 (40%) |
| 3a Numeric facet: table input given | 165 (57%) | 63 (68%) | 84 (91%) |
| 3b Numeric facet: table output given (one answer) | 124 (43%) | 47 (51%) | 60 (65%) |
| 3b Numeric facet: table output given (both answers) | 11 (4%) | 5 (5%) | 14 (15%) |
| 3 Numeric facet: table both parts correct (one answer to b) | 121 (42%) | 46 (50%) | 60 (65%) |
| 3 Numeric facet: table both parts correct (both answers to b) | 11 (4%) | 5 (5%) | 14 (15%) |
| 4a Geometric facet: graph input given | 6 (2%) | 1 (1%) | 38 (41%) |

TABLE 7.4: Responses to pre- and post-course questions 1–4

| Question | Pre-course number correct (% correct) n = 288 | Pre-course with post-course number correct (% correct) n = 92 | Post-course number correct (% correct) n = 92 |
|---|---|---|---|
| 4b Geometric facet: graph output given (one answer) | 5 (2%) | 3 (3%) | 15 (16%) |
| 4b Geometric facet: graph output given (both answers) | 6 (2%) | 0 (0%) | 20 (22%) |
| 4 Geometric facet: graph both parts correct (one answer to b) | 2 (1%) | 1 (1%) | 7 (8%) |
| 4 Geometric facet: graph both parts correct (both answers to b) | 2 (1%) | 0 (0%) | 19 (21%) |

The data suggest little difference between the total population that took the pre-course survey and the sample that participated in both the pre- and post-course survey.

A graphical comparison of the results for students who completed both surveys appears in Figure 7.1.

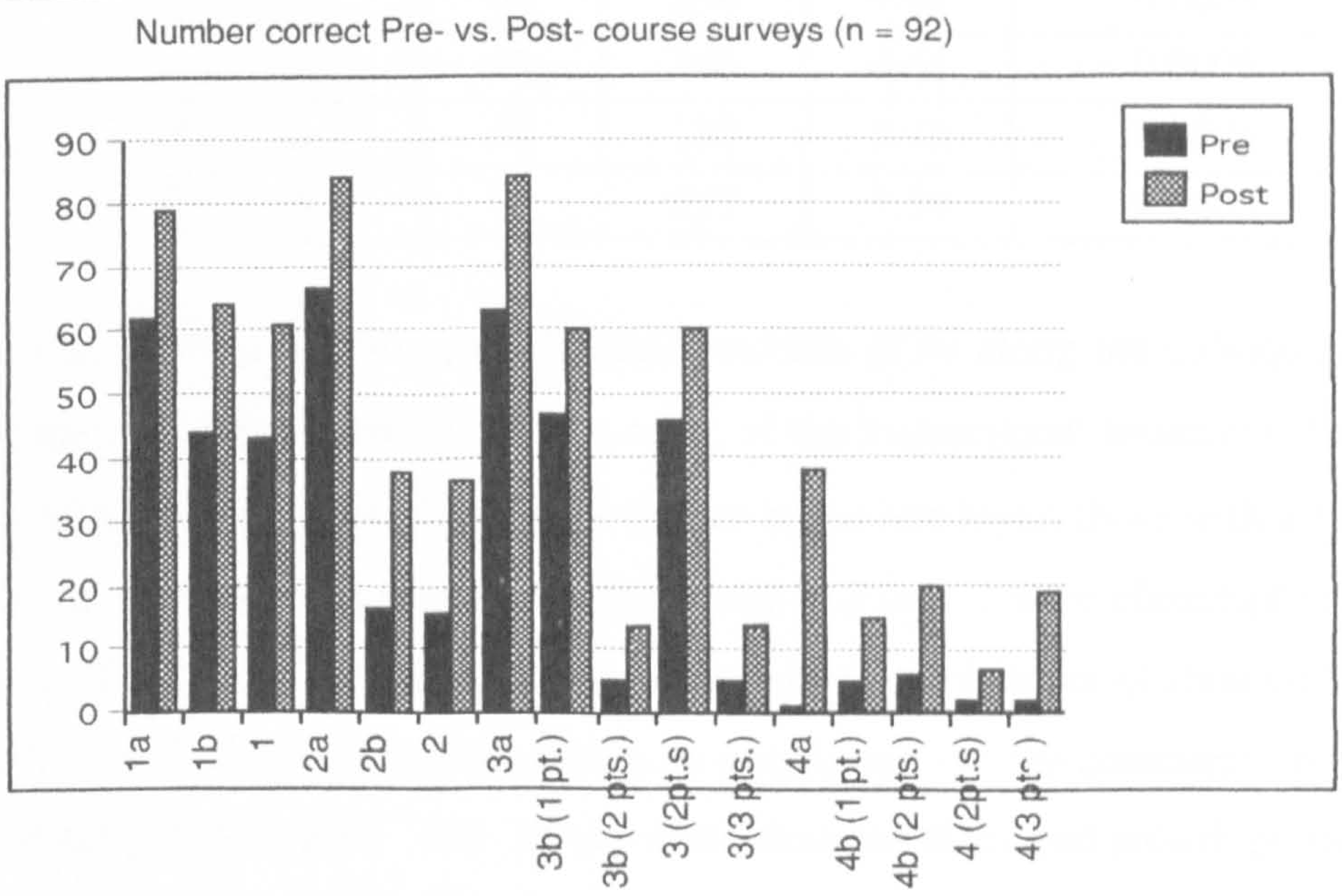



Figure 7.1 Main: graphical display of responses to questions 1–4

The number of students capable of finding output, given input, for the colloquial (Question 1), the symbolic (Question 2), and the numeric (Question 3) facets was quite high at the beginning of the term indicating either prior familiarity or a representation that was "natural" to the students. On the other hand, student performance on the geometric facet (Question 4) was poor both at the beginning and the end of the term, even though significant improvement occurred over the course of the term.

The non-parametric tests, Sign Test for Paired Data and a Wilcoxon Test for Paired Data, were used to test for significance in shifts in student scores from pre- to post-course on the pilot study. The stronger parametric test, t-test for related cases, was used to test for significant shifts in the data for the main study. Each question had a possible total of 3 points (1 point for part a and 2 points for part b). In each case, the null hypothesis was: student scores from the pre-course survey to the post-course survey will not improve significantly. This results in a one-tailed test on questions 1–4. Table 7.5 displays the results of the statistical analysis where $d$ is the signed difference in score on the question between pre-course survey and post-course survey and n = 92.

TABLE 7.5: t-scores measuring changes from pre- to post-course

| Question | $\sum d$ | $\sum d^2$ | t | level of significance |
|---|---|---|---|---|
| 1 | 60 | 254 | 4.07 | 0.0005 |
| 2 | 56 | 180 | 4.61 | 0.0005 |
| 3 | 52 | 130 | 5.16 | 0.0005 |
| 4 | 89 | 223 | 7.56 | 0.0005 |

Just as important, how did individual students grow along the colloquial, symbolic, numeric, and geometric facets as a result of the instructional treatment? Students with a 0 on a question were classified at the pre-procedure layer; those with a 1 were classified at the procedure layer; and, those with a 2 or a 3 were classified at the process layer. Table 7.6 is an evolution chart that displays the number of students in each category from pre- to post-course survey on questions 1–4. The columns in bold font indicate growth during the term. Thirty-seven students displayed growth on the colloquial facet. In addition, 37 started and ended at the process layer. So 74 students (80%) either stayed at the process layer or demonstrated growth along the colloquial facet.

TABLE 7.6: Evolution of colloquial, symbolic, numeric, and geometric facets (n = 92)

| Pre to Post | Colloquial | Symbolic | Numeric | Geometric |
|---|---|---|---|---|
| pre-procedure to pre-procedure | 5 | 5 | 7 | 46 |
| **pre-procedure to procedure** | **8** | **10** | **4** | **18** |
| **pre-procedure to process** | **16** | **9** | **17** | **25** |
| procedure to pre-procedure | 3 | 2 | 1 | 0 |
| procedure to procedure | 2 | 32 | 1 | 1 |
| **procedure to process** | **13** | **16** | **11** | **1** |
| process to pre-procedure | 1 | 0 | 0 | 0 |
| process to procedure | 7 | 5 | 5 | 0 |
| process to process | 37 | 13 | 46 | 1 |

Thirty-five students demonstrated growth along the symbolic facet, 32 along the numeric facet, and 44 along the geometric facet. Fifty-two percent stayed at the process layer or demonstrated growth along the symbolic facet, 85 percent were at the process layer or demonstrated growth along the numeric facet, and 49 percent were at the process layer or demonstrated growth along the geometric facet. Thus more than 80 percent of students improved or stayed at the process layer for the colloquial and numeric facets. Approximately 50 percent of students improved or stayed at the process layer for the symbolic and geometric facets. These results are summarized in Figure 7.2.

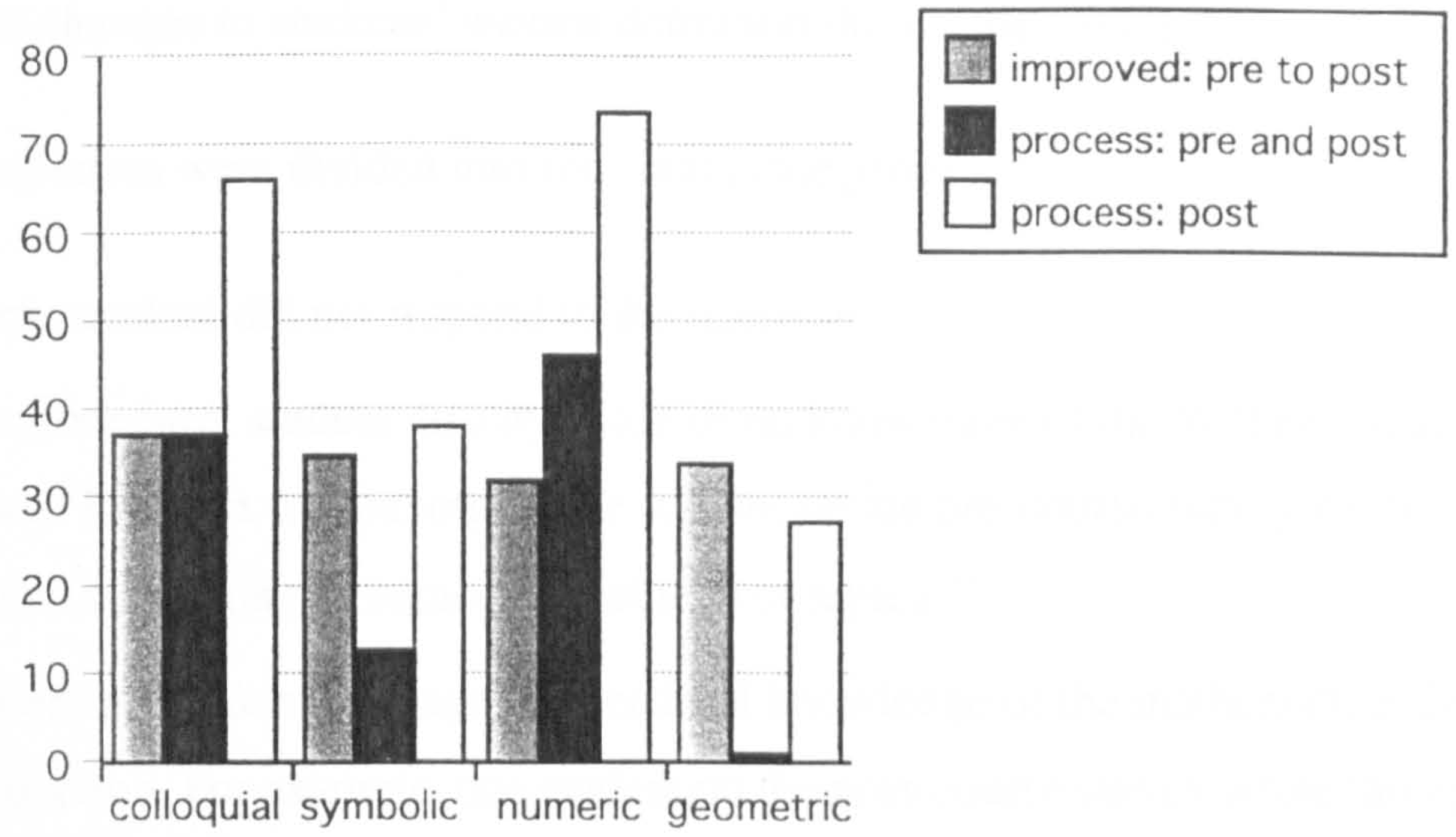



Figure 7.2: Evolution along 4 facets

These results allow the subsequent conclusions related to the first 4 sub-questions of the research hypothesis.

Students will demonstrate significantly-improved capabilities in interpreting the

1. colloquial facet of function, as exemplified by a function machine, when asked to find output given input and vice versa.
2. symbolic facet of function, as exemplified by an equation in two variables, when asked to find output given input and vice versa.
3. numeric facet of function, as exemplified by a two-column table, when asked to find output given input and vice versa.
4. geometric facet of function, as exemplified by a two-dimensional coordinate graph, when asked to find output given input and vice versa.

### 7.3.2 Written Facet

As with the pilot study, changes in the written facet were noted from pre- to post-course by asking students to respond to the question "What is a function?" These responses were analysed based on the "systemic networks" described by Bliss et al.

(1983). Answer evolution charts were used similar to those in Garcia-Mila et al. (1996) to chart changes in students' written definition during the course.

The responses were divided into four main categories:

- blank: student did not respond to the question.
- pre-procedure: student displays little or no knowledge of the mathematical definition of function. For example, one student on the pre-course survey wrote; "an ability that something or someone is able to complete."
- procedure: student displays a procedural knowledge of the mathematical definition of function. For example, one student on the post-course survey wrote "an operation or a rule." Essentially, students who placed the emphasis on specific operations in their definition were classified at this layer.
- process: student displays a process-oriented knowledge of the mathematical definition of function. For example, one student on the post-course survey wrote: "a process that receives input and produces output." The process category could be further subdivided based on where the student placed the emphasis in the definition. There were 3 common subcategories: process; relationship; and, input-output. Examples of each from the post-course survey follow.

Process layer–process emphasis: "the process that receives input and produces a unique output."

Process layer–relationship emphasis: "a relationship between two quantities that change." Included in this category are those who emphasized the idea of a dependency between two variables.

Process layer–input-output emphasis: "list of inputs and outputs."

Table 7.7 displays the number of students responding to each category. While only 2

TABLE 7.7: Written function definitions (n = 92)

| Category | Pre-course Number (%) | Post-course Number (%) |
|---|---|---|
| Blank | 53 (58%) | 21 (23%) |
| Pre-procedure | 28 (30%) | 14 (15%) |
| Procedure | 9 (10%) | 9 (10%) |
| Process layer–process emphasis | 1 (1%) | 20 (22%) |
| Process layer–relationship emphasis | 1 (1%) | 15 (16%) |
| Process layer–input-output emphasis | 0 (0%) | 13 (14%) |

percent of students responded at the process layer on the pre-course survey, 52 percent responded at the process layer on the post-course survey. Eighty-eight percent indicated no knowledge of the written definition of function of the pre-course survey. Sixty-two percent were at least at the procedure layer on the post-course survey.

More importantly, how did individual students grow along the written facet as a result of the instructional treatment? Table 7.8 is an evolution chart of individual students from pre- to post-course survey on this question.

TABLE 7.8: Evolution of the written facet

| From (pre-course) | To (post-course) | Number |
|---|---|---|
| blank | blank | 14 |
| blank | pre-procedure | 9 |
| blank | procedure | 7 |
| blank | process | 23 |
| pre-procedure | blank | 6 |
| pre-procedure | pre-procedure | 4 |
| pre-procedure | procedure | 1 |
| pre-procedure | process | 17 |
| procedure | blank | 0 |
| procedure | pre-procedure | 1 |
| procedure | procedure | 1 |
| procedure | process | 7 |
| process | blank | 1 |
| process | pre-procedure | 0 |
| process | procedure | 0 |
| process | process | 1 |

Noteworthy is the number of students (40 or 43%) who evolved from either blank or pre-procedure to process. Note also that very few students (8) regressed and, of these, 7 were a regression to blank which may indicate that they did not take the time to answer the question as opposed to having no knowledge of how to answer the question.

These results allow the subsequent conclusion related to the sub-question 5 of the research hypothesis.

5. Students will demonstrate significantly-improved capabilities in the written facet by writing a definition of function in terms of a dynamic process.

### 7.3.3 Notation facet

Changes in the notation facet from pre- to post-course survey were measured using the following question on both surveys:

Briefly state what $f(x)$, $y(x) = 4$, and $a(b+c)$ mean to you.

Table 7.9 displays the categorized responses for $f(x)$ at the beginning and at the end of the course.

TABLE 7.9: Meaning of $f(x)$ (n = 92)

| Category | Pre-course Number (%) | Post-course Number (%) |
|---|---|---|
| Blank | 15 (16%) | 13 (14%) |
| Multiplication | 69 (75%) | 14 (15%) |
| Function $f$ of $x$ | 4 (4%) | 60 (65%) |
| Output | 0 (0%) | 2 (2%) |
| Other (incorrect) | 4 (4%) | 3 (3%) |

Figure 7.3 displays the data in Table 7.9 graphically. While 75 percent of the students interpreted the notation as multiplication on the pre-course survey, 62 percent interpreted the notation as function notation by the end of the course. These data were analysed using Chi-Square. There was a significant shift toward a "function" interpretation of $f(x)$ from pre- to post-course survey ($\chi^2 = 87.7$, d. f. = 3, $p < 0.001$).

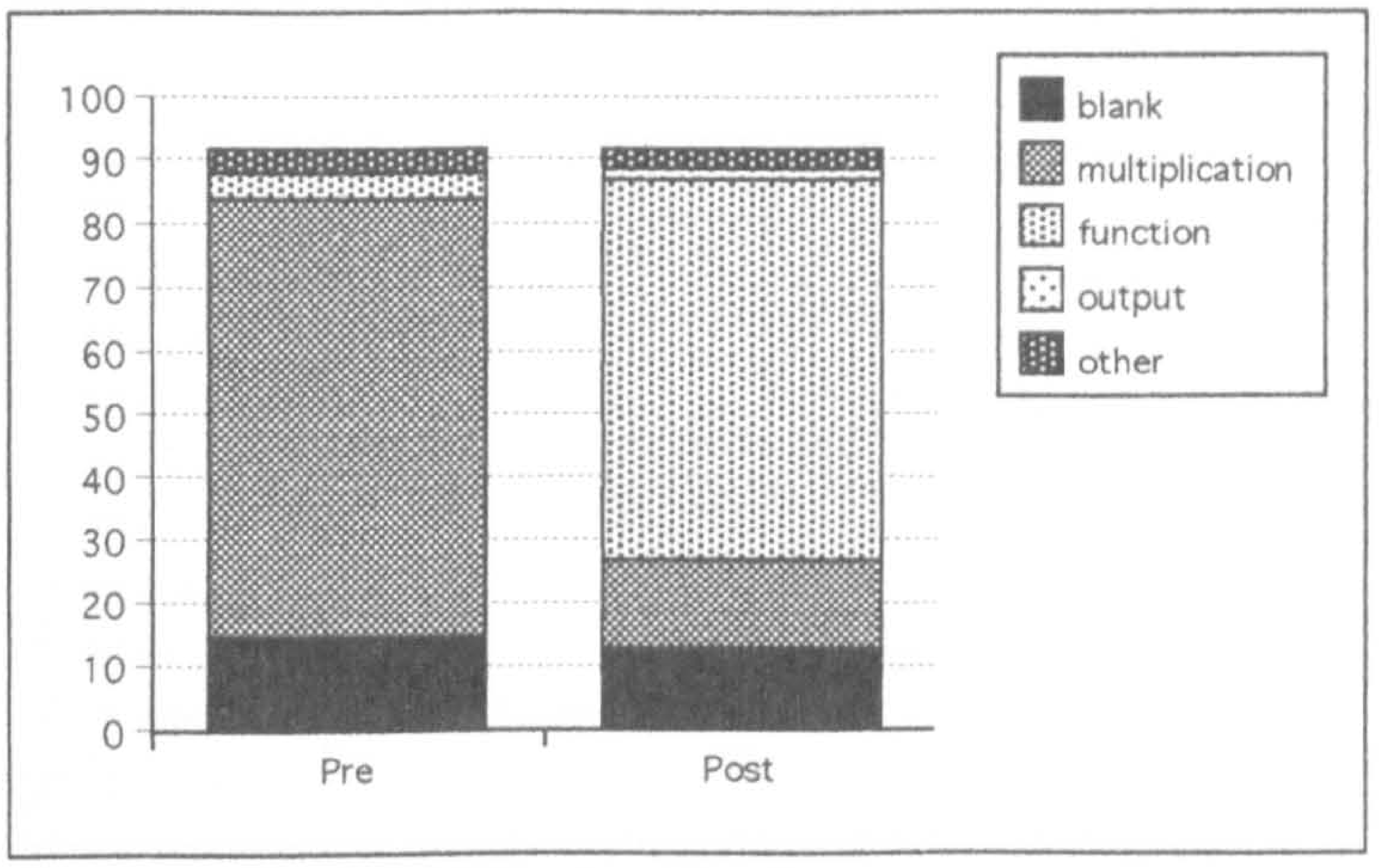



Figure 7.3 Main: meaning of $f(x)$

Table 7.10 displays the categorized responses for $y(x) = 4$ at the beginning and at the end of the course.

TABLE 7.10: Meaning of $y(x) = 4$ (n = 92)

| Category | Pre-course Number (%) | Post-course Number (%) |
|---|---|---|
| Blank | 13 (14%) | 21 (23%) |
| Multiplication | 70 (76%) | 15 (16%) |
| $x = 4$ | 2 (2%) | 9 (10%) |
| Function notation | 1 (1%) | 33 (36%) |
| $y = 4$ | 0 (0%) | 8 (9%) |
| Other | 6 (7%) | 6 (7%) |

Figure 7.4 displays the data in Table 7.10 graphically. Seventy-six percent of the students interpreted the notation as multiplication on the pre-course survey, including several who insisted that both $x$ and $y$ must be 2 in order for the product to be 4. Approximately 45 percent interpreted the notation correctly as either $y$ of $x$ equal to 4 or the constant function $y = 4$. These data were analysed using Chi-Square. There was a significant shift toward a "function" interpretation of $y(x) = 4$ from pre- to post-course survey ($\chi^2 = 79.5$, d. f. = 4, $p < 0.001$).

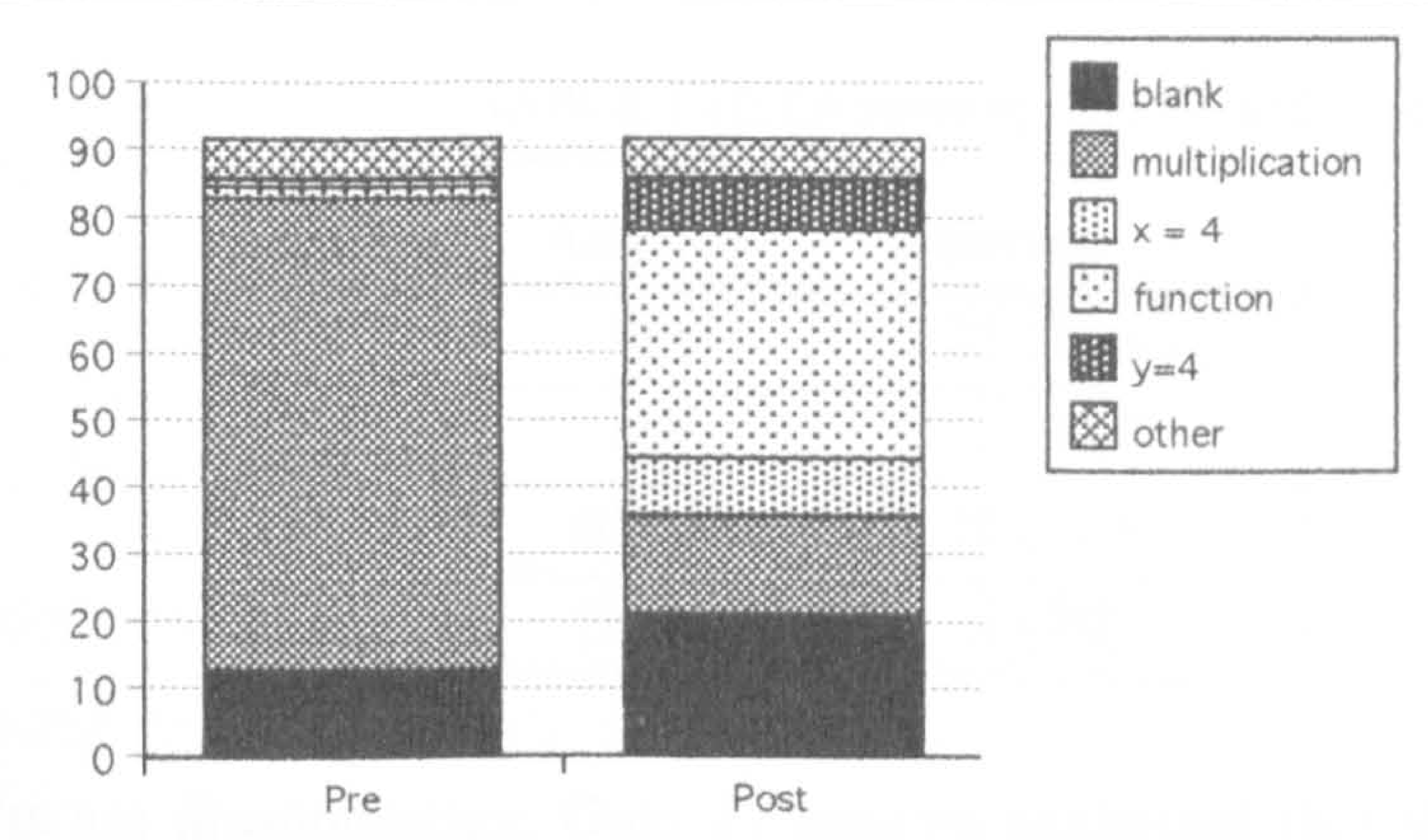



Figure 7.4 Main: meaning of $y(x) = 4$

Finally, when asked about $a(b + c)$, no student interpreted the symbolism as function notation on either the pre- or post-course survey. The apparent familiarity of the symbolism eliminated any possible cognitive link to function notation.

These results allow the subsequent conclusion related to the sub-question 6 of the research hypothesis.

6. Students will demonstrate significantly-improved capabilities in interpreting the function notation correctly and contextually.

## 7.4 Post-Course Survey Data

There were 6 questions asked on the post-course survey and during the interview that were not included on the pre-course survey. These questions were omitted from the pre-course survey since the researcher assumed that students were likely to have little or no idea how to answer the questions since the questions require some background on the function concept. The data on these questions is included to provide a glimpse at student concept image of function at the end of the semester.

### 7.4.1 Notation facet

To further investigate student understanding of function notation, the following question was included: Assume that $f$ is the name of a function. Is there a difference between $3f(2)$ and $2f(3)$? If yes, what is the difference?

Table 7.11 displays the responses to the first question.

TABLE 7.11: Interpreting $3f(2)$ and $2f(3)$ (n = 92)

| Answer | Number (%) |
|---|---|
| Yes | 25 (27%) |
| No | 31 (34%) |
| Blank | 28 (30%) |
| Other | 8 (9%) |

The results are disappointing. Only 27 percent suggested that there was a difference between $3f(2)$ and $2f(3)$. Of these, only 15 (16%) indicated that they recognized that there were different inputs to the function *f* and thus the two expressions would most likely be different based on what the function *f* does. A sample response was: "Yes, because different inputs can change the output of the function entirely." Six (7%) students stated that all they saw going on was multiplication indicating serious problems with function notation. A sample response from this group was: "No difference because of the commutative property of multiplication."

Continuing the focus on function notation, students were asked the following three-part question:

Suppose that *f* is the name of a function and *x* is the input to that function. Consider the notation $f(x)$. Check **true** or **false**.

a. $f(x)$ represents the output of the function when *x* is input.

b. $f(x)$ represents the product of *f* and *x*.

c. $f(x)$ represents the rule you follow to find the output.

Table 7.12 displays the responses.

TABLE 7.12: Interpreting $f(x)$ (n = 92)

| Question | True (%) | False (%) |
|---|---|---|
| a | 69 (75%) | 23 (25%) |
| b | 25 (27%) | 67 (73%) |
| c | 57 (62%) | 35 (38%) |

These results are a bit more promising. Only 27 percent interpreted the notation as multiplication. However, there were 35 (38%) students who answered True, False, True to parts a, b, and c respectively. This group showed proceptual flexibility in interpreting the notation as both an object (output) and as a process (rule).

### 7.4.2 Numeric facet

On the post-course survey and the interview, students were asked to identify each of the following as "a function" or "not a function."

**a.**

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | –6 |
| 2 | 9 |
| –5 | 3 |
| 8 | –6 |

**b.**

| Input | Output |
|---|---|
| 3 | 5 |
| 4 | 6 |
| 3 | 2 |
| 8 | –1 |
| 2 | 0 |

**c.**

| Input | Output |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 8 |
| 5 | 10 |

**d.**

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | 4 |
| 2 | 4 |
| –5 | 4 |
| 8 | 4 |

**e.** $\{(-1, 5), (7, 2), (-3, -8), (4, -1)\}$

**f.** A table has two columns. The left column begins at 0 and increases in increments of 2. The right column begins at 1. Each entry in the right column is computed by multiplying the preceding entry by 3. Part of the table appears below.

| Input | Output |
|---|---|
| 0 | 1 |
| 2 | 3 |
| 4 | 9 |
| 6 | 27 |
| 8 | 81 |

Table 7.13 indicates the number of students who selected each answer as a function.

TABLE 7.13: Tables as functions (n = 92)

| Question | Number (%) who said it was a function |
|---|---|
| a | 24 (26%) |
| b | 18 (20%) |
| c | 59 (64%) |
| d | 27 (29%) |
| e | 22 (24%) |
| f | 41 (46%) |

Twenty-eight (30%) did not answer the question, 6 (7%) chose all, and 5 (5%) selected all but part b, which was the only true non-function. The percentages choosing function in each case are surprisingly low. C may have been chosen by 64% since it is a table for a linear function in which a pattern is easily discernible. The next most popular choice (f) also specifies a process, but not between input and output. The lack of any apparent pattern in the others probably contributes to the low percentages. It appears that many students were procedure oriented when answering this question. The existence of a clear procedure was a necessary condition for this group. Only 20 percent chose part b, but it is unclear how many recognized the multiple outputs for input 3 as opposed to the lack of a pattern.

Only 33 students stated the rule they used to determine which tables were functions. Table 7.14 categorizes their responses.

TABLE 7.14: Tables as functions-reasons

| Category | Number |
|---|---|
| Input/output | 11 |
| Finite differences | 10 |
| Looked for pattern | 4 |
| Makes sense | 2 |
| Listed specific function types | 2 |
| Guess | 1 |

A student who selected only c wrote: "Only one that shows a correlation between input and output." This student needed to know there was a procedure to get from input to output and would, most likely, be classified at the procedure layer. A second student was also pattern dependent choosing only c and f writing: "They follow a pattern." Another student displayed confusion between unique output and unique input choosing a, c, e, and f and stating: "Look to see if a number occurs more than one time in the $x$ or $y$ column." A student who said "Checked finite differences" selected only c and d. A student who chose all but the constant function d wrote: "They have different inputs that match different outputs." Another showed little discrimination choosing all and writing: "All gave a set of inputs and outputs."

### 7.4.3 Symbolic facet

Students were given a list of equations in two variables and asked to identify those that were functions. The number of students that selected each as a function appears in Table 7.15.

TABLE 7.15: Equations as functions (n = 92)

| Question | Number (%) who said it was a function |
|---|---|
| a. $y = 3x - 2$ | 61 (66%) |
| b. $y = 9 - x^2$ | 54 (59%) |
| c. $y = 5$ | 22 (24%) |
| d. $x^2 + y^2 = 1$ | 27 (29%) |
| e. $y = \begin{cases} 1 & \text{if } x < -3 \\ x^2 & \text{if } x \geq -3 \text{ and } x < 4 \\ 2 & \text{if } x \geq 4 \end{cases}$ | 26 (28%) |
| f. $y = \pm\sqrt{x+2}$ | 34 (37%) |
| g. If $x$ is rational, then $y = 0$ | 19 (21%) |
| h. $xy = 7$ | 27 (29%) |
| i. $y = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is not rational} \end{cases}$ | 34 (37%) |
| j. $x = 2 + t$ and $y = 3t^2 - 5t + 1$ | 34 (37%) |
| k. $x = 4$ | 21 (23%) |

Twenty-eight (30%) did not answer the question, 4 (4%) chose all, and no student responded "all but d, f, and k." Not surprisingly, the most popular choices for functions were the linear and quadratic functions, which are forms the students would have seen in their course. The constant function causes the usual problem with only 24% selecting it. No other response stands out.

Only 31 students stated the rule they used to determine which equations were functions. Table 7.16 categorizes their responses.

TABLE 7.16: Equations as functions-reasons

| Category | Number |
|---|---|
| Input/output | 13 |
| Looked for process | 4 |
| Definition of function | 4 |
| Guess | 2 |
| Must have $x$ and $y$ | 2 |
| Makes sense | 1 |
| Listed specific function types | 1 |

A look at some student responses is enlightening.

"One input for one output" chose a, b, e, i, and k.

"If they have both an input and output" chose a, b, d, f, and j.

"Each input determines an output" chose a, b, d, e, f, g, i, j, and k.

It is interesting how all 3 of these omitted c, the constant function, and h in which the product of $x$ and $y$ is 7. It is surprising that the first student did not eliminate $x = 4$.

"Would input produce unique output?" chose a, b, d, f, and h. This student avoided the piecewise-defined functions, the single variable equations, and the parametric equations, but the plus or minus on the square root function was accepted as a function.

"An equation must have an $x$ and $y$" chose a, b, d, e, f, i, and j. This student omitted equations that did not contain both $x$ and $y$ including g.

"Math process that has input to create one unique output" eliminated only $y = 5$ and $x = 5$ apparently not seeing any process.

Finally, "all gave a set of inputs and outputs" chose all as functions.

### 7.4.4 Geometric facet

Students were given a list of two-dimensional, rectangular coordinate graphs and asked to identify those that were function. The number of students that selected each as a function appears in Table 7.17.

TABLE 7.17: Graphs as functions (n = 92)

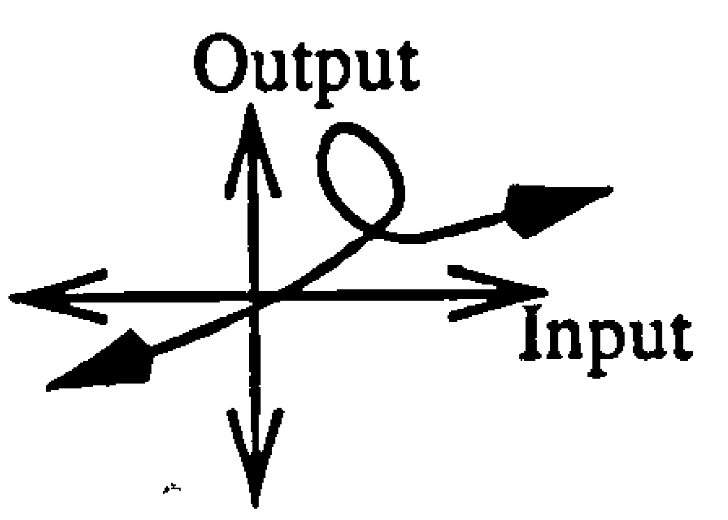
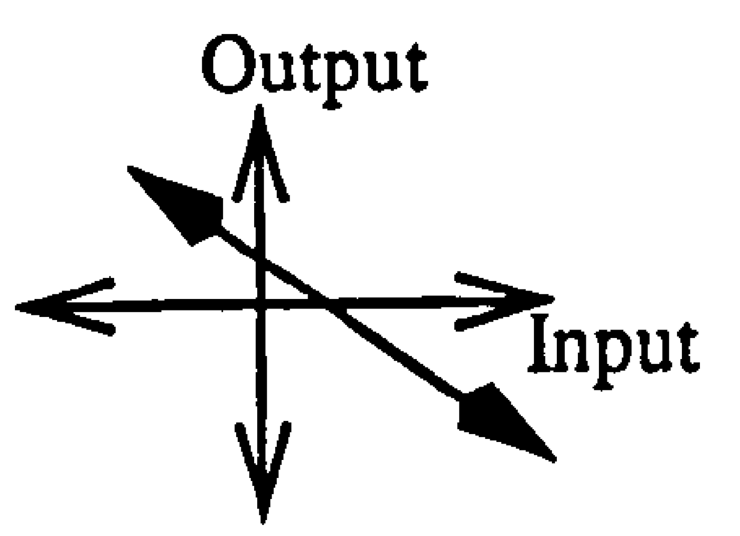
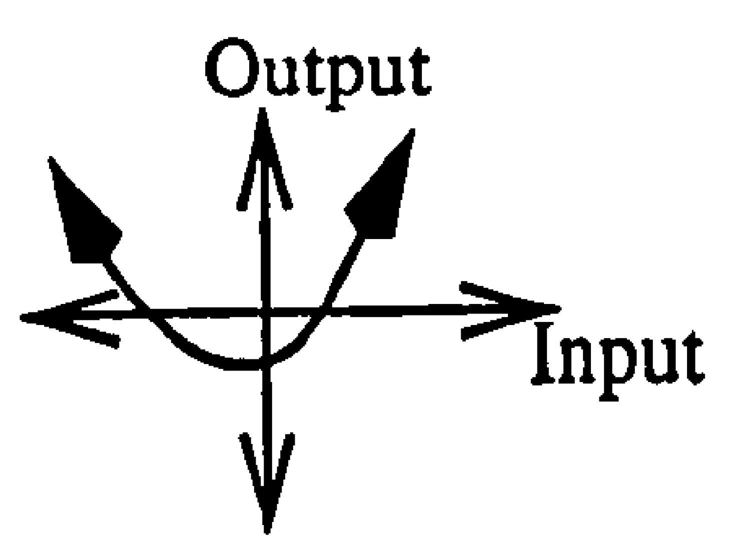
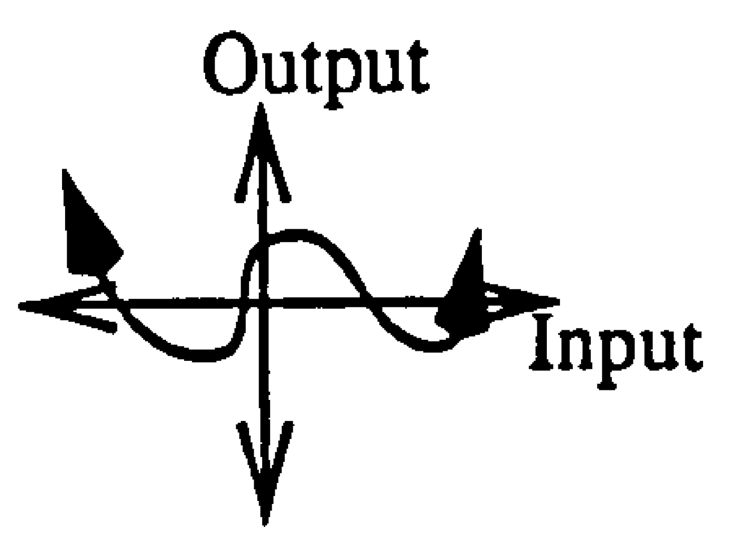
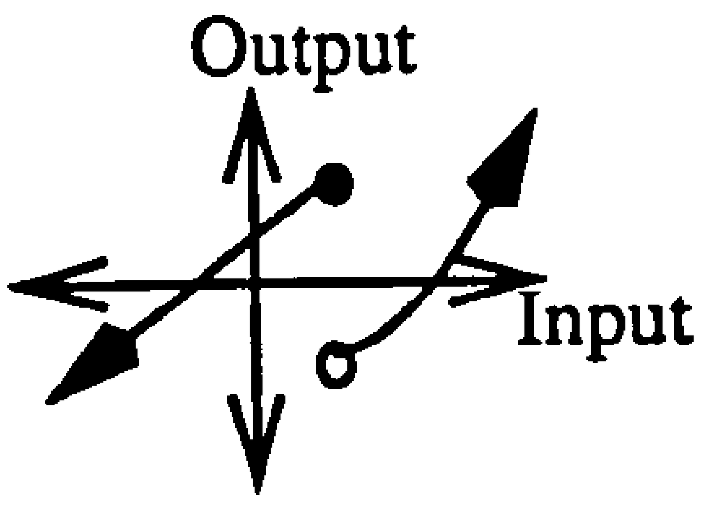
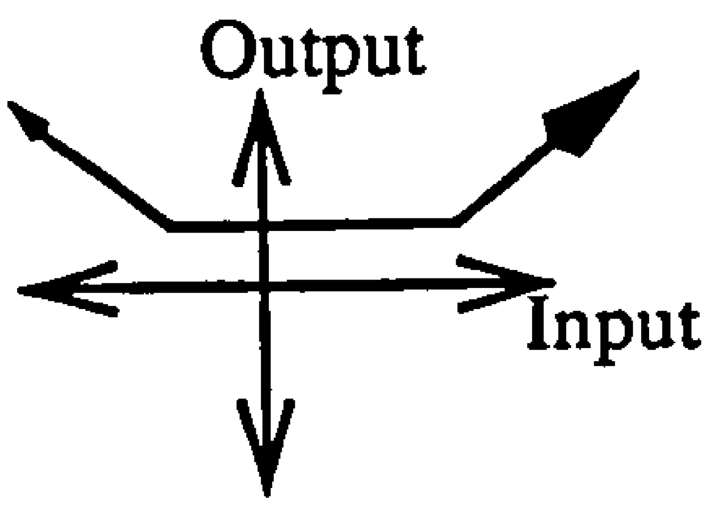
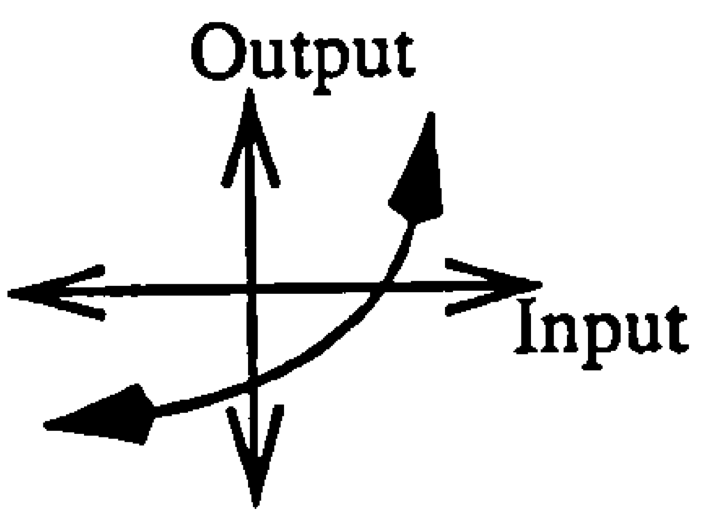
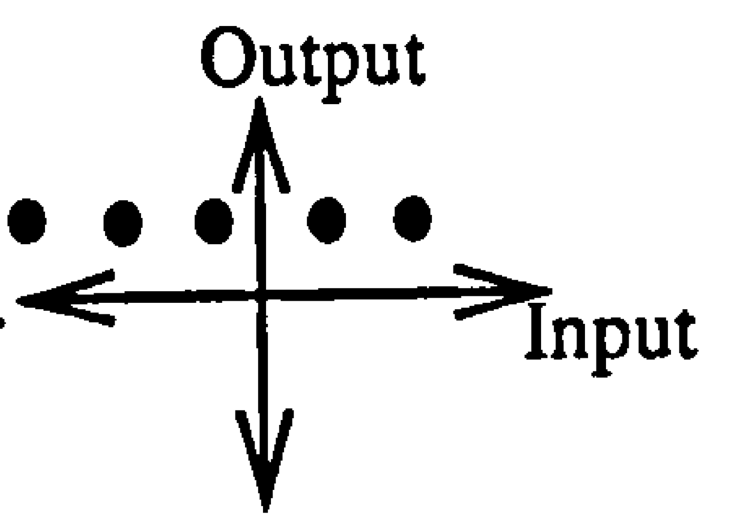
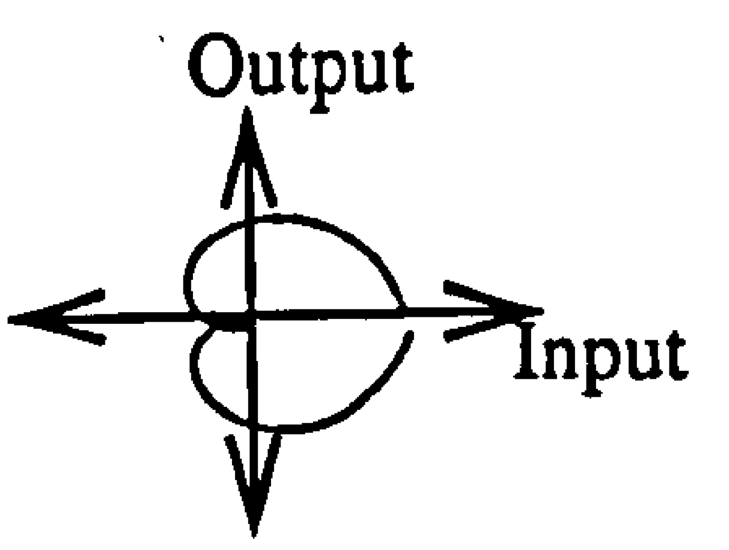
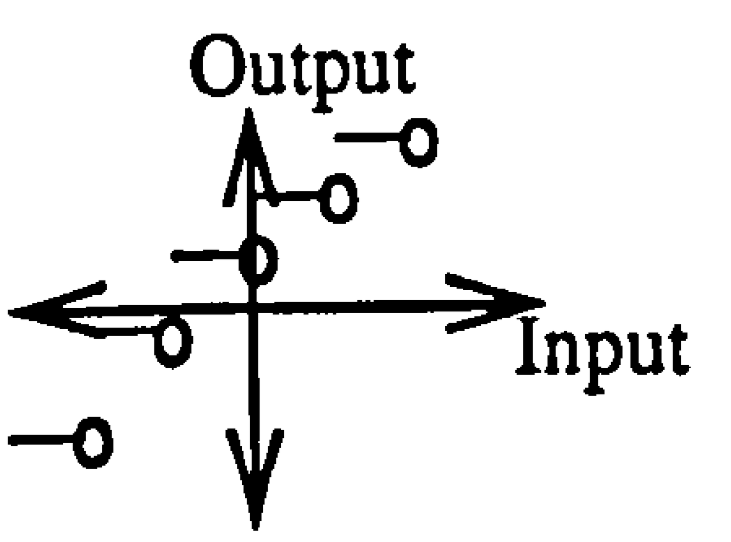
| Question | Number (%) who said it was a function |
|---|---|
| a.   | 9 (10%) |
| b.   | 62 (67%) |
| c.   | 54 (59%) |
| d.   | 33 (36%) |

TABLE 7.17: Graphs as functions (n = 92)

| Question | Number (%) who said it was a function |
|---|---|
| e.   | 19 (21%) |
| f.   | 19 (21%) |
| g.   | 43 (47%) |
| h.   | 44 (48%) |
| i.   | 13 (14%) |
| j.   | 42 (46%) |

Twenty-six (28%) did not answer the question, 5 (5%) chose all, and 3 students responded "all but a and i." Again it is not surprising that the graphs chosen by more than half the students as functions appeared to be graphs of a linear and a quadratic function. Graphs g, h, and j were selected by almost half the students. There is a section on reading graphs in the text and graphs like g and j are included. The popularity of graph h may stem from the fact that the students constructed scatter plots during the course. The constant nature of the outputs may have limited the number of students making this choice. Likewise graph f was chosen by very few as a function. It is impossible to tell if this was due to the horizontal segment or the piecewise nature of the graph. Similarly, the fact that graphs a and i were selected by few students may be due to the fact these are not functions because of the multiple outputs for certain inputs or to the fact that students had not encountered graphs like these in the course. The lack of continuity may have also contributed to lowering the percentages on graphs e, h, and j.

Only 28 students stated the rule they used to determine which graphs were functions. Table 7.18 categorizes their responses.

TABLE 7.18: Graphs as functions-reasons

| Category | Number |
|---|---|
| Input/output | 6 |
| Some version (correct and incorrect) of the vertical line test | 6 |
| Familiarity (use of prototypes) | 5 |
| Guess | 2 |
| Used graphing calculator | 2 |
| Pattern apparent | 2 |
| "Others have weird or unconnected lines." | 1 |

One point that is interesting is the fact that a number of students tried to use some version, mostly incorrect, of the vertical line test. Some examples follow.

"When a line hits the *x* or *y* axis more than one time" chose only c and d. Why this student did not include e is unclear unless the discontinuity caused concern.

"Hits $x$-axis exactly once" chose a, b, g, i, and j. The inclusion of i and j is unclear.

"If they crossed over both the $x$ and $y$-axis" chose b, c, g, i, and j. Again it is unclear why a, d, and e have been eliminated.

"Use vertical line test" chose only c. Why?

Several students' responses suggested that their choices were based on familiarity (prototypes). For example, a student who chose only b, c, d, and j wrote: "Have I seen them before?" Finally, as in the case of the previous two facets, some students selected all as functions with a sample statement like "All had processes."

The data on this question reinforce the conjecture that the geometric facet, as represented by the graphs, causes students significant trouble. Students attempting to use a blind rule, like the vertical line test, exhibit little consistency between the statement of the test (even if it is inaccurate) and the graphs they choose. Note that the vertical line test is never mentioned in the materials. However, it appears that since it is a nice, easy rule to teach, instructors give this tool to students who quickly internalize it incorrectly and constantly call on and misuse it. The vertical line test is a good example of a piece of procedural knowledge that carries different, incorrect meanings in student concept images. More than the numeric and symbolic facets, students seem to depend more on prototypes in their selections choosing by example rather than by any connection to some internal definition that has a geometric manifestation. There appears to be little connection in the concept image between the concept definition and the geometric facet. Some even state an appropriate concept definition, but their selection of graphs bears little connection to the definition.

### 7.4.5 Connection between colloquial, numeric, symbolic, and written facets

Students were given the two function machines that appear in Figure 7.5 and asked to find the output if the input is 7 and to write the symbolic form of the function.

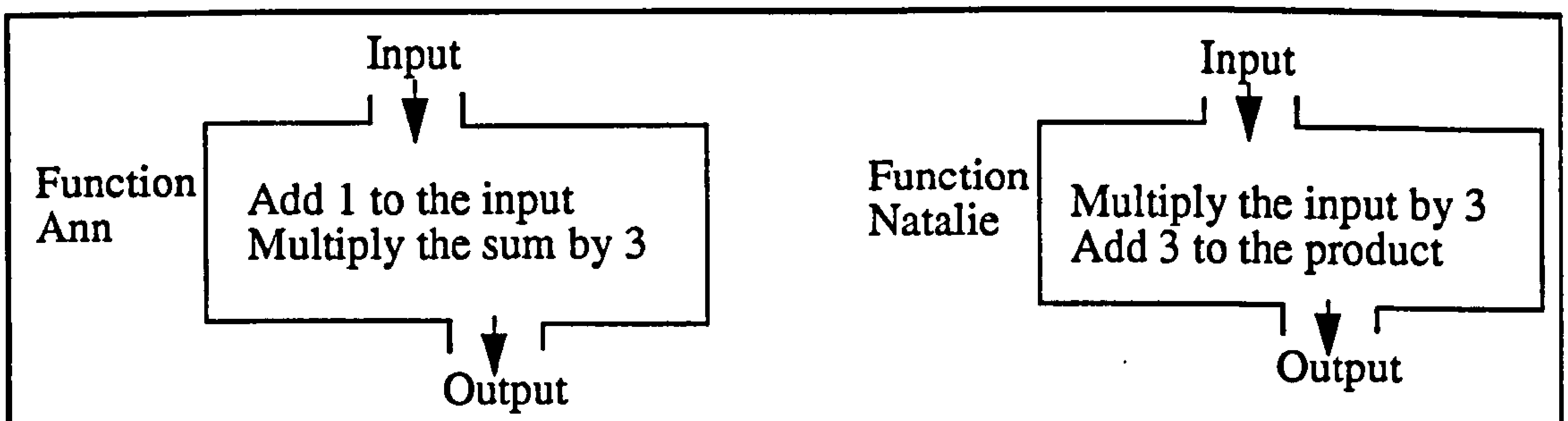



**Figure 7.5** Equivalent function machines: quantitative analysis

Additionally, students were asked if Ann and Natalie were the same function and to give a reason for their answer. Table 7.19 displays the result of inputting 7 to the two functions. The same 11 students left both parts blank. Of the 77 who correctly wrote 24 for Ann, 76 wrote 24 for Natalie and the other wrote 27. Two students who correctly wrote 24 for Natalie wrote either 27 or 10 for Ann. Ultimately, 76 (83%) students correctly answered both parts.

TABLE 7.19: Evaluating Ann and Natalie when input is 7 (n = 92)

| Output | Ann Number | Natalie Number |
|---|---|---|
| 24 | 77 (84%) | 78 (85%) |
| 21 | 2 (2%) | 0 (0%) |
| 27 | 1 (1%) | 1 (1%) |
| 10 | 1 (1%) | 0 (0%) |
| 25 | 0 (0%) | 1 (1%) |
| 1 | 0 (0%) | 1 (1%) |
| Blank | 11 (12%) | 11 (12%) |

Students were asked to write the symbolic representations for both Ann and Natalie. These results are summarized in Tables 7.20 and 7.21.

TABLE 7.20: Symbolic forms for Ann (n = 92)

| Expression | Number (percent) |
|---|---|
| $3(x+1)$ | 20 (22%) |
| $(x+1)3$ | 18 (20%) |
| $x+1(3)$ | 15 (16%) |
| Blank | 11 (12%) |

TABLE 7.20: Symbolic forms for Ann (n = 92)

| Expression | Number (percent) |
|---|---|
| $3x$ | 7 (8%) |
| $3x+1$ | 5 (5%) |
| $6x$ | 2 (2%) |
| $x+(1(3))$ | 2 (2%) |
| $x(3)$ | 2 (2%) |
| Other | 8 (9%) |

Notice that only 38 (42%) students were able to write the symbolic form correctly with almost half of them exhibiting a "process-oriented order" (Crowley et al., 1994). In this framework, this is more an example of procedural orientation. Remember that 77 students were able to calculate the correct output for a given input even though only 38 can write the correct symbolic expression. Now let's look at the corresponding table for Natalie. 71 (77%) students were able to write the correct symbolic form for Natalie with 25 exhibiting a procedural orientation. The better response here, most likely, is due to the fact that the expression can be written directly from the procedural steps with no need to insert grouping symbols, unlike that of Ann.

TABLE 7.21: Symbolic forms for Natalie (n = 92)

| Expression | Number (percent) |
|---|---|
| $3x+3$ | 46 (50%) |
| $x(3)+3$ | 25 (27%) |
| Blank | 11 (12%) |
| $6x$ | 2 (2%) |
| $3(x+3)$ | 2 (2%) |
| Other | 6 (7%) |

Finally, the students were asked if Ann and Natalie were the same function. Thirty-five (38%) students said "yes" even though 6 of these wrote non-equivalent algebraic procedures for Ann and Natalie. Some sample reasons follow:

"You have the same outputs for the same inputs."

"It's the same function just different things in the function machines."

"Since $3(x+1) = 3x+3$."

"They will be the same function, but with different steps."

One student drew a table to compare inputs and outputs and said he/she checked them graphically. Some students who said "yes" wrote questionable reasons.

"They are both inputting and outputting." .

"The domain and range are the same."

"They are both linear functions."

There were 33 (36%) students who said Ann and Natalie are not the same function. Sample responses indicate a procedural orientation.

"Different order."

"They have different processes."

"Different order of operations."

"Even though they have the same answers, the functions are different."

"The function machines are not the same."

"You do different things to the numbers."

Of the 68 students that answered the question, they are almost evenly split between "yes" and "no". The no's seem to be procedure dependent in the definition of function. Some of the yes's indicated a process orientation, but others give answers that bring their concept image structure into question.

## 7.5 Conclusion

This chapter's purpose was to present the quantitative data collected from the pre- and post-course surveys and to discuss the results in terms of the thesis and of the theoretical framework. Students exhibited statistically significant shifts from pre- to post-course surveys in their ability to find output given input and vice versa for the colloquial, symbolic, numeric, and geometric facets. Many students were adept at the colloquial facet at the beginning of the course possibly suggesting this facet as a good entry point to functions. A small percentage were adept at the geometric facet by the end of the course. The shifts were also statistically significant in their ability to define a function in writing (written facet) and to recognize and correctly interpret function notation. The ability to correctly interpret function notation in context was inconsistent, however.

The post-course survey provided data regarding the students' concept images of function for the numeric, symbolic, and geometric facets by allowing students to select functions from a list of tables, equations, and graphs. For the numeric facet, students seemed to separate into four major categories:

- those who correctly invoked an appropriate concept definition of function;
- those who were process-oriented in that they did not need to see a pattern or know a rule for generating output from input but, rather, just needed to see input-output pairs;
- those who required that a pattern or procedure be apparent; and,
- those who exhibited no discrimination.

Of these, the first two are most likely at the process layer while the third may be at the procedure layer.

For the symbolic facet, a key distinction occurred between those students who accept the equations as stating generic processes between input and output versus those who need to know the specific procedure used to generate output from input. Traditional problems with piecewise-defined functions and constant functions were apparent.

The geometric facet proved to be the most difficult for students. Unlike the numeric and symbolic facets, students placed more emphasis on familiarity or prototypes. Processes are less apparent on graphs, which appear static, than on tables or with equations. Some students exhibited incorrect procedural knowledge by attempting to apply some incorrect version of the vertical line test. Traditional problems with discontinuous functions and constant functions occurred.

The last question on the post-course survey was designed to determine if students would view functions that were algebraically equivalent as the same if the functions were different procedurally. In addition, the opportunity to watch students cross boundaries between colloquial, numeric, symbolic, and written facets was available. While most students were able to deal with the function machine numerically, supporting the results from earlier questions on the pre- and post-course surveys, less than half were able to successfully cross the boundary from colloquial to symbolic. About half of the students responding accepted two functions that used different algorithms to produce equal outputs for equal inputs as the same function, though some stated questionable reasons for their answer.

An analysis of the qualitative data occurs in the next chapter. The results from the two surveys will be combined with interview data to build profiles of student understanding of function. The results of this chapter allow the formation of partial profiles; the interviews allow for the construction of much more complete profiles for three specific students.

CHAPTER 8

# Main Study: Qualitative Data Analysis

## 8.1 Introduction

This chapter presents the qualitative data collected during the main study. Results of pre-course surveys, post-course surveys, and interviews will be analysed to develop profiles of students' concept images of "function". The resulting profiles provide a classification of types of students' understanding that may result when students who have been previously unsuccessful with mathematics are exposed to a beginning algebra course in which function is a unifying theme. Three separate profiles are developed illustrating a student who is classified as highly capable, a student who is classified as capable, and a student who is classified as incapable. In-depth analysis of each students concept image of function is provided.

## 8.2 Subject Specifics

At least three students from each of the participating colleges were interviewed within two weeks of the end of the instructional treatment. After analysing the interviews along with the pre- and post-course surveys of the interviewed students, the researcher chose to develop profiles for three students that serve as representative samples of the students interviewed. To control as many variables as possible, the three profiles are for students from the same college, College of Lake County, who had the same instructor for the beginning algebra course in Fall, 1996. This guarantees that the students' profiled had equivalent instructional experiences. Furthermore the three students include a student from each category: highly capable, capable, and incapable. Some brief background for each student follows.

### 8.2.1 Specific student attributes

AF (A for grade in course; F for female) is a female between 21 and 25 years of age. She had taken 1.5 years of algebra prior to coming to college. The beginning algebra course was her first mathematics course at college. AF is pursuing a Liberal Arts degree, but is undecided about where she wishes to concentrate. At the beginning of the course, AF equated the word "math" with frustration. At the end of the course, she equated word "math" with fear. AF earned an A in the beginning algebra course and was classified in the highly capable group as a result of the post-course survey. AF appears relaxed and comfortable during the interview. She is often smiling and appears to enjoy the challenge of answering questions about functions.

BF (B for grade in course; F for female) is a female between 26 and 30 years of age. She had taken 1 year of algebra prior to coming to college. The beginning algebra course was her first mathematics course at college. BF is pursuing a degree in Business. At the beginning of the course, BF equated the word "math" with ugh. At the end of the course, she equated the word "math" with money. BF earned a B in the beginning algebra course and was classified in the capable group as a result of the post-course survey. BF appears a bit nervous and shy during the interview. Her responses are often difficult to hear. She seems to be less confident than she should be.

CM (C for grade in course; M for male) is a male who is older than 30 years. He had taken 1 year of algebra prior to coming to college. The beginning algebra course was his second mathematics course at college, the first being a basic skills course. CM is pursuing a degree in Biology. At the beginning of the course, CM did not equate the word "math" with any word. At the end of the course, he equated word "math" with misery. CM barely earned a C in the beginning algebra course and was classified in the incapable group as a result of the post-course survey. CM is immediately defensive during the interview. He wears a pained expression throughout and regularly protests that he just doesn't understand math.

### 8.2.2 Creating student profiles

By collecting all the data (pre-course survey, post-course survey, and interview) for each student interviewed, the researcher hoped to build profiles of student concept images of function using the theoretical framework described previously. A discussion and analysis of each facet follows.

## 8.3 Colloquial Facet

The function machine is used to investigate the colloquial facet of function. Given a function machine, students were asked on the pre-course survey, the post-course survey, and the interview to find the output if the input is given (part a: 1 = full credit) and vice versa (part b: 2 = full credit). The results for each student appear in Table 8.1.

TABLE 8.1: Function machine data

| Student/ item | Pre- | Post- | Interview |
|---|---|---|---|
| AF a | 1 | 1 | 1 |
| AF b | 2 | 2 | 2 |
| BF a | 0 | 1 | 1 |
| BF b | 0 | 2 | 2 |
| CM a | 1 | 1 | 1 |
| CM b | 0 | 0 | 2 |

AF answered both correctly on all 3 instruments indicating an understanding of the diagram prior to the course. BF showed the most growth being unable to answer either part at the beginning of the course and then answering both parts correctly at the end of the course. This ability was retained during the interview. CM was initially able to find the output, given the input, but unable to reverse the process until the interview. By the time of the interview, all 3 students indicated proficiency with this question.

The following two questions probe student understanding of both the colloquial and symbolic facets. Students were given the function machines displayed in Figure 8.1 on the post-course survey. Students were asked to write expressions for each function machine and asked whether the two function machines represented the same function.

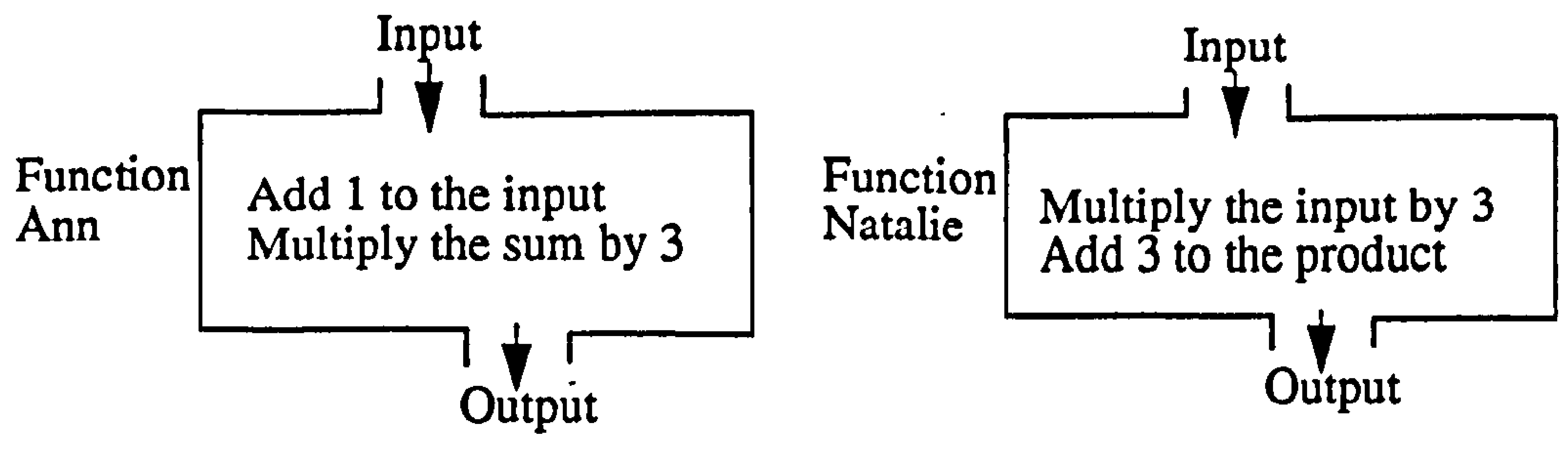



Figure 8.1 Equivalent function machines: qualitative analysis

Table 8.2 displays the responses on the post-course survey.

TABLE 8.2: Equivalent function machines (survey)

| Student | Ann | Natalie | Are functions equal? |
|---|---|---|---|
| AF | $3(x+1)$ | $3x+3$ | Yes, distribute the 3 in Ann and the function is the same as Natalie |
| BF | $(x+1)3$ | $(x3)+3$ | Yes, because they are both linear functions |
| CM | $x+1\cdot 3$ | $3x+3$ | No, even though they have the same answers, the functions are different. |

AF is the only student to see the algebraic equivalence. BF is uncomfortable writing the algebraic symbolism preferring a process-oriented order (Crowley, Thomas, & Tall, 1994) suggesting that she is more procedural. Her reason for equality displays little understanding that the two machines produce the same input-output pairs. CM appears to be procedural especially when writing function Ann, even up to ignoring order of operations. He recognizes that the input-output pairs are the same, but insists the functions are different probably because they involve different procedures.

They were given the function machines displayed in Figure 8.2 during the interview.

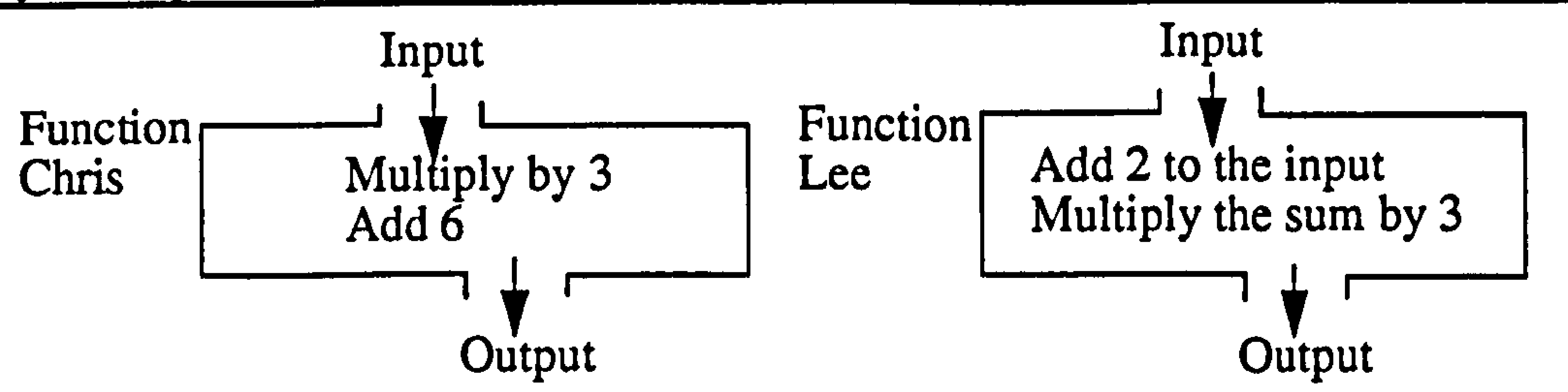



Figure 8.2 Equivalent function machines: interview

Again, students were asked to write equations for each function machine and asked whether the two function machines represented the same function. Table 8.3 displays the responses on the interview.

TABLE 8.3: Equivalent function machines (interview)

| Student | Chris | Lee | Are functions equal? |
|---|---|---|---|
| AF | $3x+6$ | $3(x+2)$ | Yes, if I distribute the 3 in Lee, I get the same function as Chris |
| BF | $x3+6$ | $(x+2)3$ | Yeah, but different processes |
| CM | $3x-6$ | $x+2(3x)$ | No, you come up with the same answer, but they are different processes |

The interviewer asked AF if it bothered her that the processes were different even when she said it was the same function. She said "No." On the other hand, BF was unsure when asked the same question. She still appears procedural in her interpretation of the machines and is unsure whether to say that the functions are equal because of the different procedures. CM reinforces his belief that equal functions require equal procedures.

Intvw: Are those the same functions?

CM: No.

Intvw: How come?

CM: You'll come up with the same answer but they are different processes. In one your input gets multiplied by 3 then you add 6. The other one, you add 2 then multiply by 3. They are different. The end result might be the same but they are two different things.

As in the pilot study, students were given function cards (Appendix D) and asked to identify those cards that contained functions. Five of the cards contained function machines. All 3 students placed the function machine cards in the function pile.

In conclusion, by the time of the interview, all three students are able to find output given input and vice versa for a function machine. Only one, AF, is comfortable crossing the boundary from function machine to symbolic form.

## 8.4 Symbolic Facet

Equations in two variables are used to investigate the symbolic facet of function. Students were asked on both surveys and the interview, given the equation $y = 3x - 7$, to find the output if the input is given (part a: 1 = full credit) and vice versa (part b: 2 = full credit). The results for each student appear in Table 8.4. AF and BF were able to perform the procedure at the beginning of the course. All 3 students were able to reverse the procedure at the end of the course, but CM did not retain this ability in the interview.

TABLE 8.4: Equation data

| Student/ item | Pre- | Post- | Interview |
|---|---|---|---|
| AF a | 1 | 1 | 1 |
| AF b | 0 | 2 | 2 |
| BF a | 1 | 1 | 1 |
| BF b | 0 | 2 | 2 |
| CM a | 0 | 1 | 1 |
| CM b | 0 | 2 | 0 |

On the post-course survey and the interview, students were asked to identify each of the following as "a function" or "not a function".

a. $y = 3x-2$ b. $y = 9-x^2$ c. $y = 5$ d. $x^2+y^2 = 1$

e. $y = \begin{cases} 1 & \text{if } x<-3 \\ x^2 & \text{if } x\geq -3 \text{ and } x<4 \\ 2 & \text{if } x\geq 4 \end{cases}$ f. $y = \pm\sqrt{x+2}$

g. If $x$ is rational, then $y = 0$ h. $xy = 7$

i. $y = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is not rational} \end{cases}$ j. $x = 2+t$ and $y = 3t^2-5t+1$

k. $x = 4$

Table 8.5 displays student choices on the survey and during the interview. Notice that the constant function was not included in the answers to the survey and only BF said it was a function on the interview.

TABLE 8.5: Selecting functions from equations

| Student | Survey | Interview |
|---|---|---|
| AF | all but c | a, b, d, e, f, h, j |
| BF | a, b, d, f, h | a, b, c, d, g, h, j |
| CM | a, b, d, f, g, j | a, b, d, e, f, g, i, j |

The interviewer did some probing with AF, especially about the piecewise functions, the constant function, and h since AF said she was unsure about it.

AF: Looking at h at first I thought no because it looked like $x$ times $y = 7$ but I guess you could add parentheses and say $x(y) = 7$.

It seems like she is trying to create function notation, but the interviewer did not pursue this.

Intvw: But what if it was $x$ times $y = 7$?

AF: Yeah that is a function.

Intvw: Why'd you change your mind?

AF: That $x$ times $y$ is a process and 7 is output and you could say $x$ and $y$ are inputs.

She seems to be dependent either on reading the statement from left to right and interpreting the "=" as "makes" or on knowing the procedure to answer this question. AF doesn't appear to recognize implicit procedures as still defining a relationship between $x$ and $y$.

Intvw: You said c is not a function–that seems to conflict with something you did on the function cards.

AF: Actually it is. You just don't know the process.

Notice that she was more dependent on knowing procedure here than on the tables.

Intvw: You said e was a function but i was not. How come?

AF: I think what threw me off was the rational and not rational.

Intvw: It's a decision-making process. If I say a rational number you make $y = 0$ and if I say an irrational number then you say $y = 1$.

AF: Yeah that would be a function because no matter what the process is you get one output.

We see the first instance that she is aware that there is only one output for a given input.

Intvw: Does that change your answer to g?

AF: Yeah basically it's stating the same thing.

The interviewer went on at this point even though it is unclear that AF really understood g and i. Ultimately, it seems that AF needed to understand the procedure in order to identify an equation as a function.

BF indicated discomfort with piecewise functions also.

Intvw: Why was e eliminated?

BF: Hmm let me think. Umm well there wouldn't be one answer. It could have 3 different outputs.

The interviewer then went through several numeric cases to help her understand what the equation was saying.

Intvw: Does it fit your idea of a function?

BF: Yes, I guess it does.

Intvw: Would that change your impression of any of the others?

BF: (Thinks) I'm not sure about f.

Intvw: That means + or –.

BF: I'm not sure about this.

Intvw: Suppose the input is 2. Then we'd have...

BF: square root of 4.

Intvw: Which is?

BF: 2.

Intvw: But then we'd have a + or – there.

BF: Two outputs so it wouldn't be a function.

Intvw: Good. Any others?

BF: I would be a function too.

BF does not appear to be procedure-dependent on this question, unlike her method of writing an equation from a function machine. She appears to primarily depend on the requirement that an input can have only one output.

CM's response displays some of his emotions about mathematics along with his apparent procedural viewpoint.

Intvw: Say a few words about the ones you didn't list.

CM: Well nothing is really going on for c. It's just $y = 5$. For h it's just $xy = 7$, and for k it's just $x = 4$. I don't see any process going on. It's just variable equal a number.

Intvw: Yet you find g acceptable?

CM: Right. There is some kind of process you do. If $x$ is rational $y$ is equal to 0 and if $x$ is irrational then $y$ is not zero. Something is going on there.

Intvw: I do want to compare because this is very interesting to me. Umm when you did the cards you rejected card 1 (quadratic equation in two variables). I wonder how that is different from a and b?

CM: It's not. It's just when it comes to me and math I don't know what's my problem... I fight and struggle. Something is going on there on the card so it is probably a function. When it comes to math I'm not real sure of myself and I just don't comprehend.

CM displays his frustrations in the last statement. His responses are inconsistent across different instruments and he can't easily resolve the conflict. He needs to see a procedure in order to say something is a function and displays no awareness of the uniqueness to the right condition (Dubinsky & Harel, 1992).

Eight of the function cards contained equations. Included were a linear function, a quadratic function, a constant function, an exponential function, 3 equations in just the variable $x$, and the equation of a circle centred at the origin. AF and BF said all were functions except $x = 4$. In addition to $x = 4$, CM eliminated the constant and exponential functions. When asked about his choices, he responded as follows.

Intvw: Are you looking for a process of some kind?

CM: Yeah. I'm sure there is a process going on but just to look at this (card 21: the exponential function) I can't see the process. Y =. I was undecided by these—you're multiplying and squaring, but it is just y = this so I don't think it is a process

Intvw: Now I see card 1 (quadratic function) in the function pile and that seems similar to 21 that you just rejected.

CM: Right. I was kind of undecided. I should put card 1 in with the non-functions. I'm not really sure it just says y is equal to blah blah. Same with $x = 4$. Nothing whatsoever is going on; $y = 4$ same thing.

No student differentiated between 1 and 2 variable equations nor used the "uniqueness to the right" condition. CM was constantly looking for a procedure.

## 8.5 Numeric Facet

Two-column tables and sets of ordered pairs are used to investigate the numeric facet of function. Students were asked on both surveys and the interview, given a table, to find the output if the input is given (part a: 1 = full credit) and vice versa (part b: 1 = 1 correct answer; 2 = both correct answers). The results for each student appear in Table 8.6. AF and CM were able to perform the procedure at the beginning of the course. AF and BF were able to reverse the procedure at the end of the course, though only AF gave both answers. All 3 were able to reverse the procedure during the interview, but CM and BF needed to be prompted to give the second answer.

TABLE 8.6: Table data

| Student/ item | Pre- | Post- | Interview (* indicates prompting) |
|---|---|---|---|
| AF a | 1 | 1 | 1 |
| AF b | 1 | 2 | 2 |
| BF a | 0 | 1 | 1 |
| BF b | 0 | 1 | 2* |
| CM a | 1 | 1 | 1 |
| CM b | 0 | 0 | 2* |

AF had an interesting response when asked if two inputs for one output was acceptable according to her understanding of function.

Intvw: Is that okay? Can a function have the same output for 2 inputs?

AF: Umm let me think. A function means no well gosh (laughs) the difference between a function and relation I would say I feel right now no just because there is one unique output but if 0 has a unique output of –3 and if 2 has a unique output of –3 but it doesn't say they can't have the same output so I would say its okay but I am not totally comfortable with it.

AF has effectively argued the distinction between the "uniqueness to the right condition" and one-to-oneness.

On the post-course survey and the interview, students were asked to identify each of the following as "a function" or "not a function".

a.

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | –6 |
| 2 | 9 |
| –5 | 3 |
| 8 | –6 |

b.

| Input | Output |
|---|---|
| 3 | 5 |
| 4 | 6 |
| 3 | 2 |
| 8 | –1 |
| 2 | 0 |

c.

| Input | Output |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 8 |
| 5 | 10 |

d.

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | 4 |
| 2 | 4 |
| –5 | 4 |
| 8 | 4 |

e. $\{(-1, 5), (7, 2), (-3, -8), (4, -1)\}$

f. A table has two columns. The left column begins at 0 and increases in increments of 2. The right column begins at 1. Each entry in the right column is computed by multiplying the preceding entry by 3. Part of the table appears below.

| Input | Output |
|---|---|
| 0 | 1 |
| 2 | 3 |
| 4 | 9 |
| 6 | 27 |
| 8 | 81 |

Table 8.7 displays student choices on the survey and during the interview.

TABLE 8.7: Selecting functions from tables

| Student | Survey | Interview |
|---|---|---|
| AF | all but b | all but b |
| BF | c, d, f | all but b |
| CM | all | c |

AF was consistent on both the survey and the interview, eliminating the only non-function.

AF: At first glance seeing input-output I thought all were but then I noticed in b there were 2 outputs for b and based on the function definition I wasn't comfortable with that.

Intvw: Is that basically what you were looking for in all of them?

AF: Yeah just to make sure there was an input and an output and that there was a unique output.

Intvw: So you feel a table represents a function even though you don't know the process to go from input to output

AF: Yes. Right.

AF appears to have a very stable idea of the use of tables to represent functions. BF, on the other hand, seems to need to recognize some kind of process on the survey. The ones she chose all had some kind of discernible process. However, on the interview, she has become more flexible. CM exhibits no discrimination on the survey, but completely reverses his direction on the interview.

CM: C is a function. I'm not too sure about the others.

Intvw: Okay why did you pick c?

CM: Well the inputs are just doubling to get the output–the others I'm not too sure what they are doing. I mean 3 goes in and 4 comes out; 7 goes in and 4 comes out (looking at d)...

Intvw: So, in c, you can really see what is happening?

CM: Yes.

CM appears to be looking for a pattern in the outputs–in other words, he is again trying to identify a procedure.

Six of the function cards contained tables. Included were a linear function, a quadratic function, a constant function, an exponential function, a median function, and a non-function. AF and BF said all were functions ignoring the fact that one table contained an input with two different outputs. CM indicated that none of them were functions.

CM: All the ones with a list (table), I don't see anything be done. There's nothing at the top of the list that tells me what has been done.

Intvw: Are you looking for a process of some kind?

CM: Yeah. I'm sure there is a process going on but just to look at these I can't see the process.

It is interesting that CM has indicated that he is most comfortable with tables, and yet does not equate tables with functions.

## 8.6 Geometric Facet

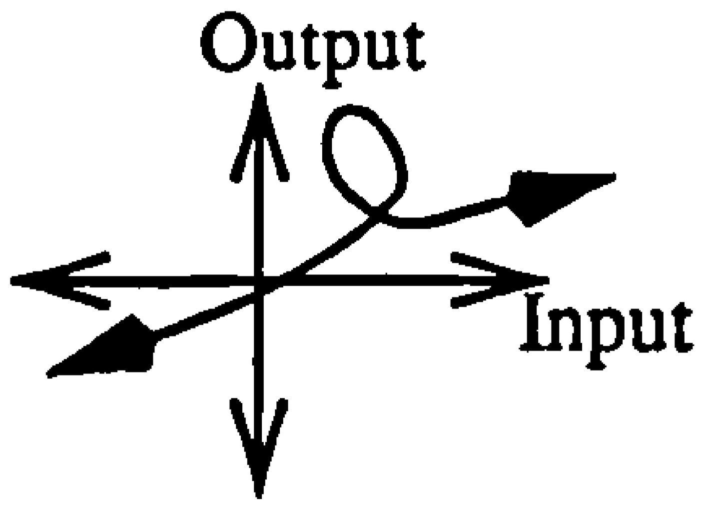
Rectangular coordinate graphs are used to investigate the geometric facet of function. Students were asked on both surveys and the interview, given a graph, to find the output if the input is given (part a: 1 = full credit) and vice versa (part b: 1 = 1 correct answer; 2 = both correct answers). The results for each student appear in Table 8.8. All 3 students were pre-procedural at the beginning of the course, but both AF and BF were able to exhibit process-layer knowledge by the end of the course and repeat this ability on the interview. CM displayed procedural ability at the end of the course, but returns to pre-procedural layer knowledge by the time the interview commenced.

TABLE 8.8: Graphical data

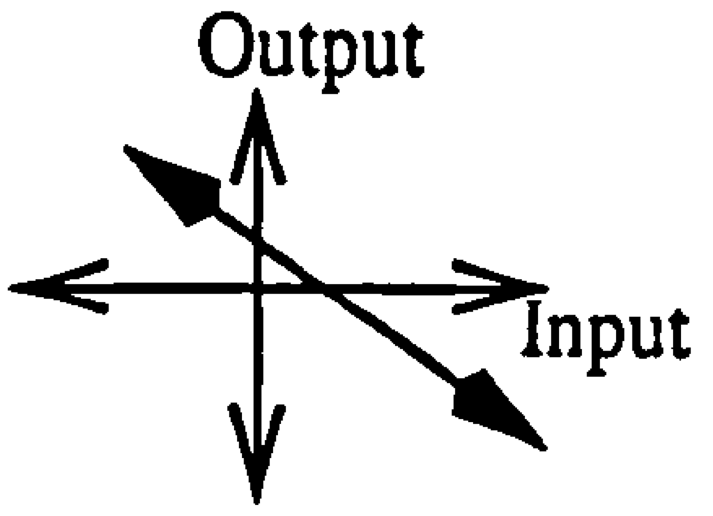
| Student/ item | Pre- | Post- | Interview |
|---|---|---|---|
| AF a | 0 | 1 | 1 |
| AF b | 0 | 2 | 2 |
| BF a | 0 | 1 | 1 |
| BF b | 0 | 2 | 2 |
| CM a | 0 | 1 | 0 |
| CM b | 0 | 0 | 0 |

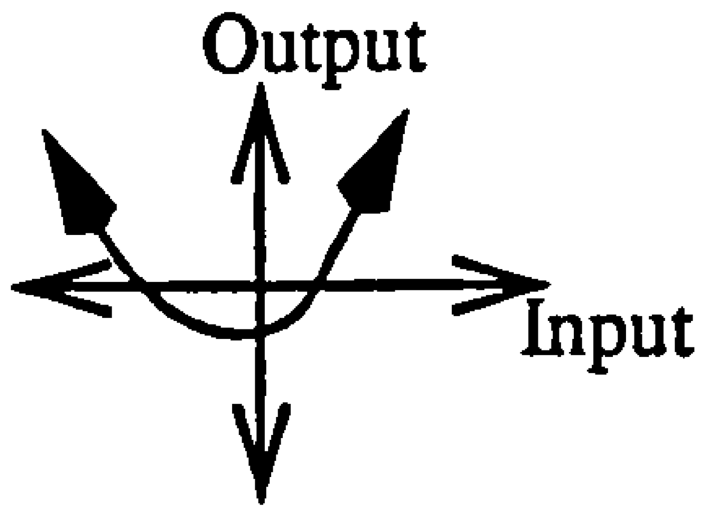
On the post-course survey and the interview, students were asked to identify each of the following as "a function" or "not a function".

a.




b.



c.
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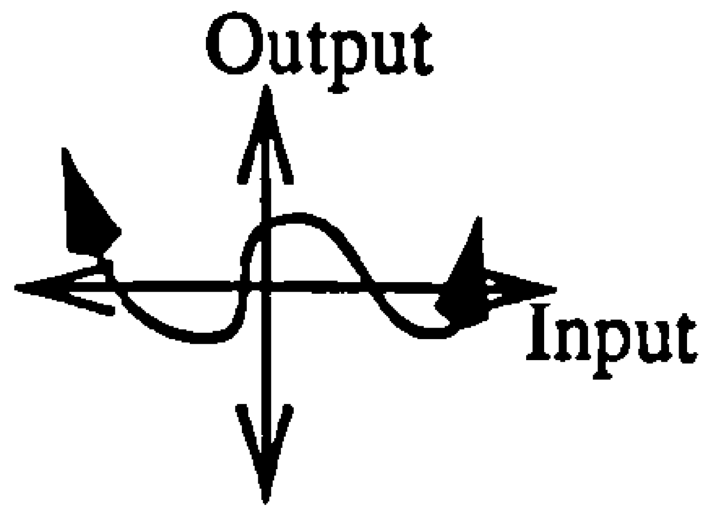


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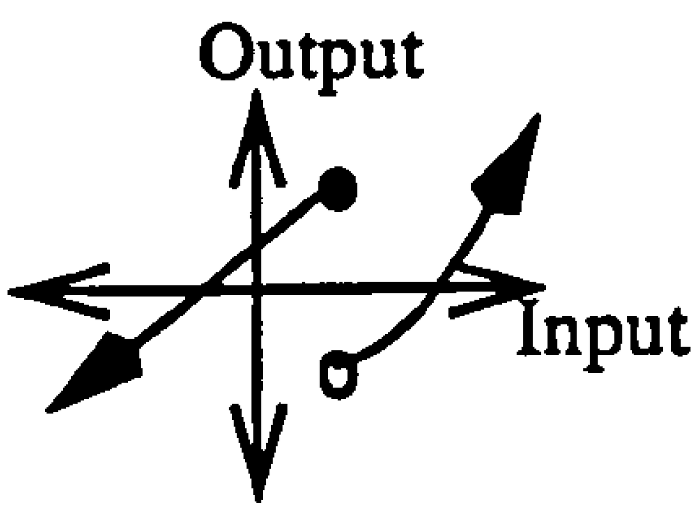


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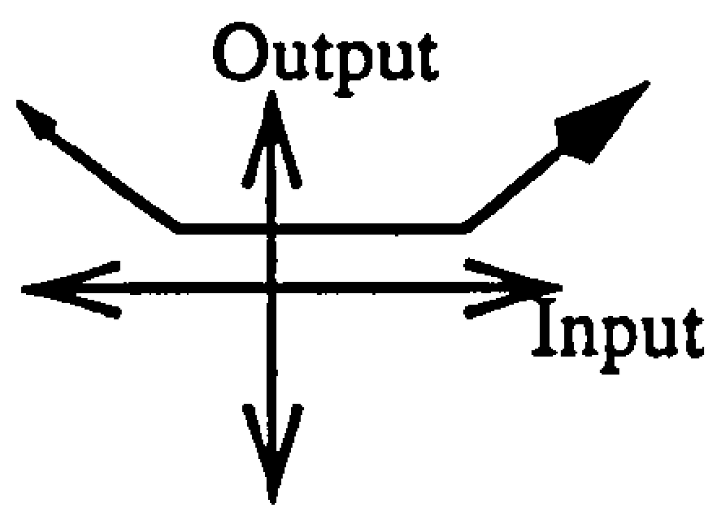


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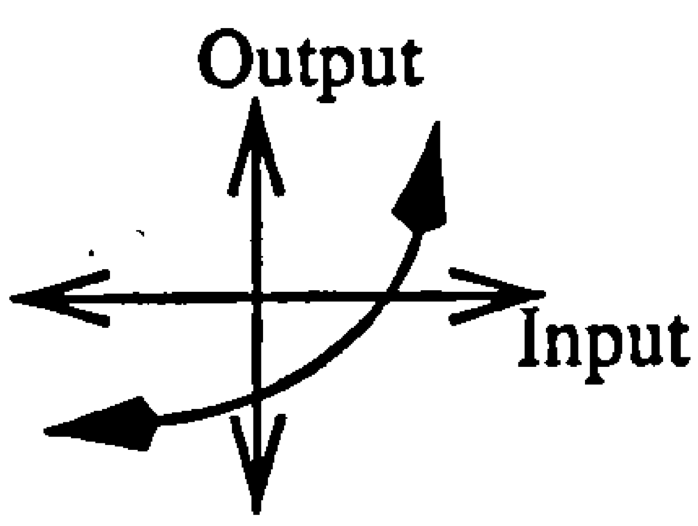


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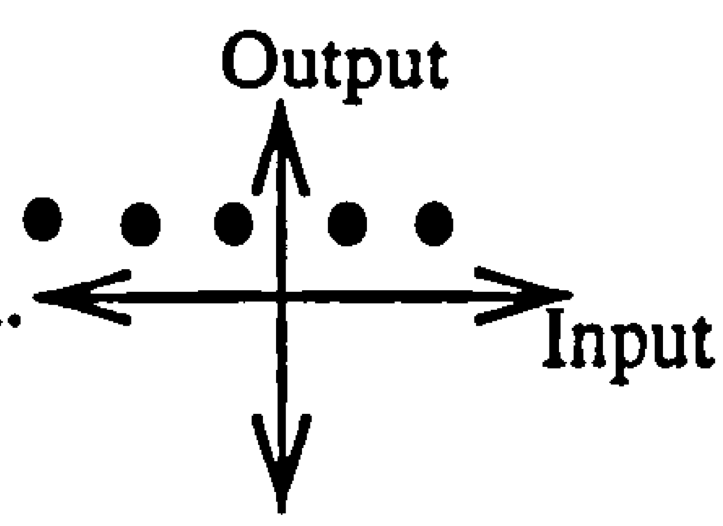
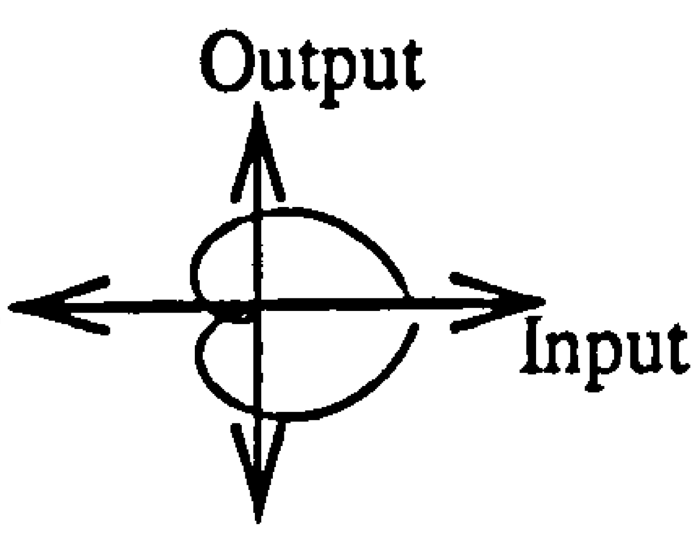



i.


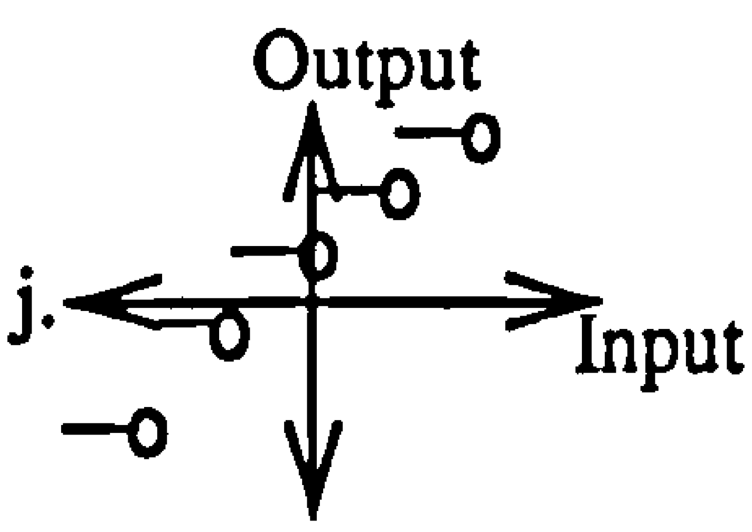

j.



Table 8.9 displays student choices on the survey and during the interview.

TABLE 8.9: Selecting functions from graphs

| Student | Survey | Interview |
|---|---|---|
| AF | all but a and j | all but f |
| BF | b, c, d, g, h, i | b, c, d, g, h, j |
| CM | b, c, d, e, f, g, j | all |

All 3 students were inconsistent in their responses from post-course survey to interview. Looking at the transcripts provides some insight.

AF: I wasn't sure about f since I hadn't seen anything like this before. I mean basically looking at it as input and output I would say it is a function but these lines going up sort of like a flat parabola. I wanted to say that it was but I just wasn't sure.

Intvw: So just basically the fact that you hadn't seen it before affected your answer?

AF: Right.

Intvw: If the pressure was on what would you say?

AF: I'd say that it was.

Intvw: Let's go down to i. (The interviewer shows her an input.) What's the output?

AF: The positive and the negative one.

Intvw: So that input has two outputs. Is that okay?

AF: Hm hm wait! (laughs) No, by the definition, it wouldn't be okay. Yeah I would say that's not a function.

Intvw: Yes you are right ... any others like that?

AF: Probably this loop (pointing to a). Wow.

With some coaxing, AF discovers the vertical line test. BF demonstrated use of the vertical line test in eliminating a and i as functions.

Intvw: Why didn't you select a and i as function?

BF: For this one (a), if I choose an input here there would be two outputs. This one (e) I don't know. You're supposed to move left to right, but the arrows are pointing the opposite way. Hmm I don't see where the output would be if I'm here (she points to the input axis where no output exists). And this one (i), I get two outputs at an input like this (she points interior to cardioid).

Both AF and BF had some trouble recognizing e and f as functions indicating that the arrowheads somehow bothered them. This convention of using arrows both ways on a graph appears to have created an obstacle for these two students. On the other hand, CM used no discrimination assuming that anything can be graphed using a graphing calculator.

CM: I think they all are.

Intvw: And what rule are you using to say they all are?

CM: Umm just from the graphing calculator. If you want to graph something you put it in the y=. It seems like even if you put y = 4 you'd get a straight line so they are all functions.

AF and BF appear to be on equal footing for this facet. While AF had not used the "uniqueness to the right" condition on graphs before, she quickly discovered how she could use it. BF seemed very confident with this question. It is curious why she said i was a function on the post-course survey. It's possible she meant to write j rather than i. CM even accepts the constant function when viewed graphically.

Five of the function cards contained graphs. Included were a linear function, a quadratic function, a constant function, an exponential function, and a circle. AF said all were functions ignoring the fact that the circle contained inputs with two different outputs. It should be noted that she did the function cards prior to discovering the vertical line test. BF indicated all but the circle were functions confirming her knowledge of the vertical line test prior to the interview. CM indicated that the linear, quadratic, and exponential graphs were functions. He continued to exhibit problems with the constant function. Why he omitted the circle is unclear. He may have been using prototypes (Tall & Bakar, 1992). Finally, there is no indication that the lack of continuity (See Markovits et al., 1993) was a problem.

## 8.7 Written and Verbal Facets

The next pair of facets deal with written and verbal definitions of functions. They will be discussed together to allow for comparison of responses. Students were asked on both the pre- and post-course survey to write their definition of function. Subsequently, on the interview, they were asked to complete the sentence: "A function is ...." The results for each student appear in Table 8.10.

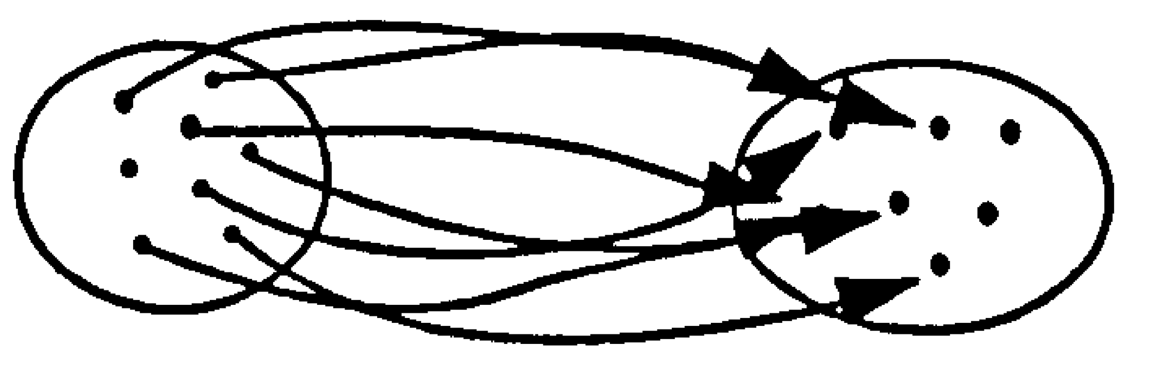
TABLE 8.10: Function definition

| Student | Pre- | Post- | Interview |
|---|---|---|---|
| AF | a symbol to represent something else | a math process that has an input to create one output | a process which has an input and returns one output |
| BF | - | a process that receives input and returns a unique output | one input, then there is a process to achieve one unique output |
| CM | - | a math operation involving input and output | a process involving 2 inputs to get one output |

All three appear to be at the pre-procedure layer at the beginning of the course. Both AF and BF write a process-oriented definition indicating a relationship between input and output on the post-course survey. These written definitions seems to closely match the verbal definitions on the interview. CM, on the other hand, places his emphasis on operation in his written answer and on a binary operation in the interview. Such a response suggests that CM has not advanced beyond the procedure layer for these two facets.

Another measure of these facets is the ability to assimilate alternate definitions into the student's definition. The students were given two other function definitions and told the definitions came from a friend. This was to assure that they would not know ahead of time if the definitions are acceptable or not. They were asked to decide how the given definition "fit" with their definition. The first definition is as follows.

Sue's definition: A **function** is a **correspondence** that assigns to each element of one set one and only one element of a second set. A diagram appears at right.



AF: I would say that it is acceptable because basically what she is saying is that every dot in this first one is assigned to a dot in the oval so to me it's saying that if this is the input (points to left circle) then this is the unique output (points to right oval).

BF: Hmm mm. She is saying that [for] each element of one set you only get one from the second set.

Both students were asked to identify the input, process, and output in the picture. They both pointed to the left circle for input, the arrows for process, and the right circle for output. AF and BF appear to have had little problem assimilating the definition with her own.

CM: I don't agree with it. I don't understand it. I don't think it's correct. It doesn't fit in. You've got one thing and you're assigning it to another and...I just don't know.

Intvw: It doesn't fit your definition of function?

CM: No.

The second definition is as follows. Gail's definition: A **function** is a set of ordered pairs $(a, b)$ in which for each value of *a* in the domain of the function, there is one and only one value of *b* in the range of the function.

AF: I would say yes. It sounds like she is saying the same thing. For every *a* there is a *b*. You don't have more than one *b* for each *a*

BF: She saying that *a* is the domain.There is one and only one value of *b*. Then if you only have one value there wouldn't just be one value. I don't know how to explain it.

Intvw: So we have your definition of function and these two. Do they seem to be on the same track.

BF: They do but as long as these (pointing to range in Sue's definition) are different values.

Intvw: So you have some restrictions. You'd like to ask some questions first?

BF: Yeah?

Both AF and BF were asked to identify the input and the output. Both pointed to *a* and *b* respectively. BF called *a* the domain rather than an element of the domain. She also demonstrated some confusion between unique output and one-to-oneness. She seems to be saying that the outputs must be different. Her concern that all the dots in the output set be different and the *b* not having different values suggest some confusion about what unique output really means. CM, on the other hand, responds to Gail's definition as follows.

CM: Not really. Ummm, it's just like the one above. I don't really understand either one of them

Intvw: Okay. In order for a definition of function to make sense to you what does it have to have in it? Is there some essential part of it?

CM: Ummm to me there just needs to be some kind of process not just assigning a number or space to. You have to take it and do something to it even if the function process is to ignore it. Even if you are ignoring it you are doing something with it.

AF demonstrates good flexibility in her concept image of a function's definition. She appears able to identify the similarities between the definitions and easily merge them with her own definition. She is minimally at the process layer and may well have sufficient understanding to be classified at the concept or even procept layer with respect to her verbal definition of function. BF has a bit more trouble assimilating the definitions seemingly struggling with the difference between the uniqueness to the right condition and one-to-oneness. CM has great difficulty with the alternate definitions. Even his own definition is incomplete. He emphasizes that he needs to have a procedure to follow. Finally, both AF and BF demonstrated verbal and written definitions that match indicating a smooth crossing of the boundary between the verbal and written facets. CM's written and verbal definitions may indicate a more solid boundary between the two facets or may simply reflect a changing viewpoint depending on when he responds to the question.

## 8.8 Notation Facet

The next facet investigated tests student understanding of function notation. Students were asked on both the pre- and post-course surveys and on the interview to identify the meaning of $f(x)$, of $y(x) = 4$, and of $a(b+c)$. The results for each student appear in Table 8.11.

TABLE 8.11: Function notation

| Student | Symbol | Pre- | Post- | Interview |
|---|---|---|---|---|
| AF | $f(x)$ | $f$ multiplied by $x$ | $f$ of $x$ | $f$ of $x$ |
| BF | $f(x)$ | $f$ times $x$ | $f$ of $x$ | $f$ of $x$ |
| CM | $f(x)$ | $f$ times $x$ | $f$ depends on $x$ | function notation |
| AF | $y(x) = 4$ | $y$ multiplied by $x = 4$ | output of function $y(x)$ is 4 | $y$ of $x$ is a process and the output is 4 |
| BF | $y(x) = 4$ | $y$ times $x = 4$ | $y$ of $x$ is 4; 4 is output | $y$ of $x$ is 4; 4 is output |
| CM | $y(x) = 4$ | $y$ times $x = 4$ | $y$ times $x = 4$ | $x = 4$ |

All three appear to be at the pre-procedure layer at the beginning of the course. By the end of the course and during the interview, both AF and BF interpret the notation appropriately. CM seems to have progressed little, if at all.

All 3 students consistently interpreted $a(b+c)$ as multiplication across all 3 instruments. The interviewer chose to probe this with AF.

Intvw: Could this $(a(b+c))$ ever mean function notation instead of multiplication?

AF: No because a function is something that has one input. If I were to look at this as function notation there would be two inputs $b$ and $c$.

Intvw: But can't $b+c$ be thought of as one thing?

AF: Yeah if you know what they were you'd know their sum. So yeah.

Intvw: In fact I could define a function $f(x) = 2x+3$ and ask you to compute $f(b+c)$. Do you know what you would do?

AF: I would just input $b+c$ for $x$. So $2(b+c)+3$. Wow.

AF demonstrated true wonderment at this idea. Note, however, that she initially sees $b+c$ as two inputs indicating a temporary inability to accept the expression as a single entity. When she calculates $b+c$, is she only performing a rote process or does she really understand why $a(b+c)$ could be function notation in an appropriate context? Vinner (1997) might suggest that her demonstrated understanding is pseudo-conceptual, rather than conceptual.

The post-course survey provided two more pieces of information on function notation. Students responded "true" or "false" to the following statements: "Suppose that $f$ is the name of a function and $x$ is the input to that function.

a. $f(x)$ represents the output of the function when $x$ is input.
b. $f(x)$ represents the product of $f$ and $x$.
c. $f(x)$ represents the rule you follow to find the output."

All three students responded true, false, true respectively to this question indicating flexibility in viewing the notation as both an output and as a name for a process.

Another question asked on the post-course survey was: "Assume that $f$ is the name of a function. Is there a difference between $3f(2)$ and $2f(3)$?" AF did not respond, but both BF and CM indicated that there was no difference citing the commutative property thus exhibiting an inconsistency with their answer to the previous question. AF and CM were asked the same question during the interview. Unfortunately, the question was overlooked during the interview of BF. CM said he had no idea. AF's response follows.

AF: They are different–*f* of 2 is the function you would take the output and multiply it by 3. Down here (pointing to 2*f*(3)) 3 is the input to the function you'd take the output of the function and multiply it by 2 so they generally would not be the same.

Only AF demonstrates an ability to understand this more complex notation. The ability to recognize $f(2)$ and $f(3)$ as outputs is critical to answering this question. This ability suggests that AF may even be at the concept layer with respect to the notation facet. Overall, AF demonstrates the most understanding of function notation, though her application of it is not consistent. BF appears to recognize and interpret function notation appropriately in basic cases, but reverts to a multiplication interpretation in a more complex case. CM recognizes $f(x)$ as function notation but cannot apply this to other situations. Overall, the understanding of function notation seems quite fragile with inconsistent interpretations when new situations arise.

## 8.9 Kinesthetic Facet

Each student was asked to demonstrate what a function is without using words by using a physical motion. Unfortunately, none were willing to demonstrate a physical motion. The investigation of the kinesthetic facet remains for a future study.

## 8.10 Boundaries Between Facets

### 8.10.1 Boundaries between the notation, numeric, symbolic, and geometric facets

One specific interview question was designed to investigate the boundaries that exist between tables, equations, and graphs. Since function notation is used in the symbolic form, the notation facet also enters this discussion. An equation, a table, and a graph are displayed in Figure 8.3 for the same function. Students were asked a series of questions involving finding outputs given inputs and vice versa. Careful attention was noted as to which facet the students used.

Students were first asked to find the output if the input is –1. All 3 answered correctly with BF and CM immediately indicating they used the table. Strangely, AF used the graph.

$y(x) = x^2 - 3x - 10$

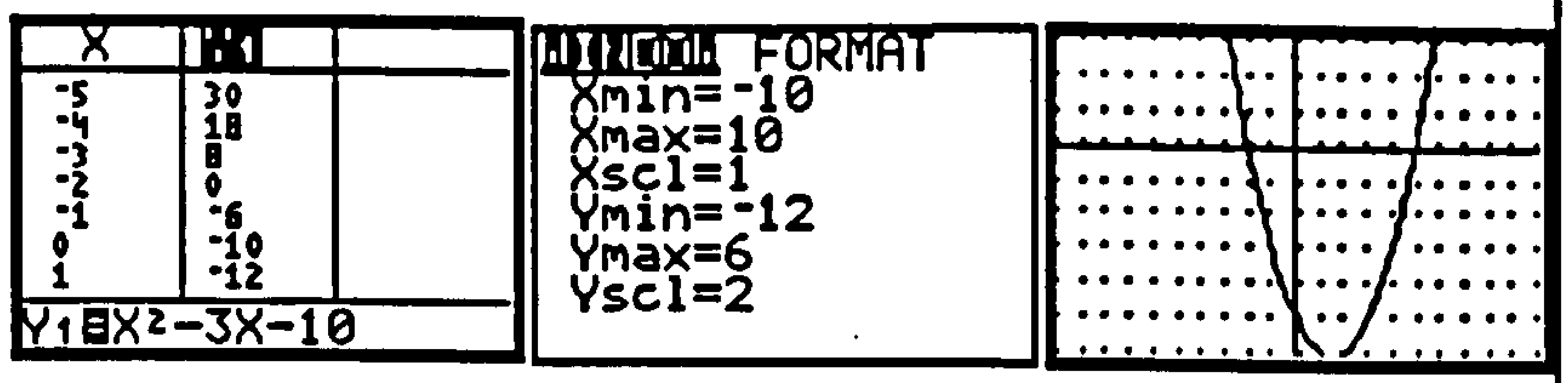



Figure 8.3 Equation, table, and graph: interview

AF: –6.

Intvw: What did you use?

AF: The graph.

Intvw: Is there something else you could use?

AF: The equation.

Intvw: And what would you do?

AF: I would input –1 for $x$.

Intvw: And is there anything else you could use?

AF: The table like on your calculator.

Intvw: Well look at table.

AF: Oh (laughs) –6.

It seems that AF had overlooked the table looking for something harder to do (Principle of Optimal Complexity: when something appears too simple, a person will naturally add unnecessary complexity to it). She may be so confident with this question that the representation is inconsequential. Next they were given an input of –5. All 3 responded with 30 immediately and indicated they used the table. Notice CM's comment regarding his preference.

CM: 30. Same as last one. I used the table. It's a lot easier than messing around with the graph or using trace or zoom in. I like the tables.

Now they are asked to find the output if the input is 4. Note that this input does not appear in the table. Both AF and BF responded correctly using the graph, though BF initially made a sign mistake. CM's response is most interesting.

CM: It's not showing on my table. If I had the calculator I'd scroll up or down then I'd hafta ...–6.

Intvw: Tell me what you are doing.

CM: Well the input is 4 so first I'd square it and then 3 times 4 is 12 subtract it and subtract 10 to get –6.

Intvw: Okay so you used the equation to plug in 4 for $x$?

CM: Right.

Noteworthy is the fact that he avoided the graph, but was able to evaluate the function at 4 mentally quite easily. The students were then given an input of 12. All indicated that they would plug 12 in for $x$. Next the researcher explored responses when the input is a variable.

Intvw: Suppose that the input is $h$?

AF: It would be $y$ of $h$. $h$ squared minus $3h$ minus 10.

Intvw: Okay. Is that an acceptable output to you?

AF: Hm hmm.

BF: Input is $h$. I don't know. You can't evaluate it.

Intvw: But can you still give me an output?

BF: I'm not sure.

Intvw: When I said the input was 12, you put 12 in for $x$. What if the input is $h$?

BF: Okay. Then I put in $h$ for $x$.

Intvw: And what would the output be?

BF: $h$ squared minus $3h$ minus 10.

Intvw: Okay. Is that an acceptable output to you?

BF: It is but it's just a variable. It's the same as $x$.

Intvw: So you see it as just replacing $x$?

BF: Yeah.

CM: Input is $h$? Um then you just plug in $h$ ... $h$ squared minus 3 times $h$ minus 10.

Intvw: In your mind is that an acceptable output?

CM: Yeah.

It is interesting to note that BF had the most problem with this. She seemed to have difficulty accepting a non-numeric input and a non-numeric output. Probing questions only seemed to increase the confusion. Meanwhile AF dealt with the question effortlessly while CM demonstrated that he was comfortable with function evaluation ("plug-and-chug"), but how much does he really understand? Next the focus shifts to finding inputs given the output.

Intvw: What are the input(s) if the output is 0?

AF: (Looks in table.) –2. (Switches to graph) And 5. First I use the table and then I looked at graph, saw the parabola, and saw there was another answer.

BF: –2. Table.

Intvw: Are there any others?

BF: Hmm. Move to the graph. –10 (pointing to the y intercept)

Intvw: Remember I said the output is 0.

BF: Oh the output is 0. –2 and 5.

CM: Input is a –2. Table.

Intvw: Are there any others?

CM: Not that I can see from this table.

Intvw: Okay. Any possibilities from the other forms?

CM: Probably but I just don't know.

While AF handled the question beautifully, BF needed prodding to look beyond the first answer. CM was unable to move to the graph and thus was unable to find the second input. AF demonstrates great flexibility in shifting between the table and the graph. BF is able to shift from table to graph, but exhibits confusion between $x$- and $y$-intercepts.

Intvw: Suppose the output is 44?

AF: I would have to put 44 in place of $y$.

Intvw: What would you write?

AF: 44 of $x = x$ squared minus $3x$ minus 10

Intvw: How would you do that?

AF: I would add 10 divide by 3 and square root it.

Intvw: If you had you calculator could you do this?

AF: Yeah. I would scroll up on the table to find the output of 44 or you could possibly zoom on the graph.

Intvw: Can you tell by what kind of function you have how many answers there would be?

AF: I would say 1.

Intvw: Just 1? When you had an output of 0 there were 2 answers.

AF: (looking at graph) So there would be 2 but they would be a lot further apart.

AF knows that 44 must be substituted for $y$, but then says "44 of $x$" indicating a misreading of function notation. She attempts to find $x$ by reversing, unsuccessfully, the operations implicit in the quadratic function. Students have not solved quadratic equations in the course so AF would not know an appropriate symbolic solution method. The interviewer, rather than dealing with the algebraic solution technique, turned her attention to the calculator which allowed her to state more appropriate algorithms for finding the answers. Finally, she did not think carefully about the graph initially when asked about the number of solutions, though immediately saw the flaw in her thinking when referred to a previous problem.

Next BF attempts to address the problem.

BF: If the output is 44 (holds her head; goes to equation) it would be $44x$. I'd go up to 44 on the graph.

Intvw: So your focus would be to use the graph.

BF: Yeah.

Intvw: Could you use the table or the equation also?

BF: Oh sure. If I had to use the table, I punch in y = 44 and... I'm not sure. I just scale it til I got 44.

Intvw: Is there any way to use the equation?

BF: This equation?

Intvw: Hmm mm.

BF: I'd reverse the process of the function machine beginning with 44 (she tries to do it but quickly runs into trouble).

While AF went to the equation first, BF looks at the equation, but, rather than continuing, suggests that she would use the graph to find the answers. She does say "$44x$" which might suggest some confusion with the function notation, similar to that exhibited by AF. When asked about the table she appears confused about how to get the table to do what she wants. She seems to recognize that she can't "punch in y = 44" and is unclear about what she would do when she says: "I just scale it til I got 44." Finally, when asked to return to the equation, she attempts to reverse the operations just like AF, but at least recognizes that this isn't as easy at it seems. It is not clear that AF ever recognized this problem.

Finally, CM attempts to answer the question.

CM: Ummm. Well I couldn't use this table but if I had the calculator I'd scroll up or down and look for it.

Though questioned by the interviewer, CM could not identify any way to use either the equation or the graph to answer the question. He remained firmly tied to his favourite facet: numeric.

AF seems to be the most comfortable of the 3 students moving between tables, equations, and graphs flexibly. Her ability when given the output of 0 to use the table and graph simultaneously without prompting supports this suggestion. BF is not far behind though she wasn't as flexible in using several facets to answer a question. She wasn't comfortable with nonnumeric inputs and outputs and exhibited some confusion about how to use a table to find an input. CM was the least flexible. He was unable to use the graph at all and was only able to use the equation procedurally. Finally, both AF and BF indicated some misunderstanding of function notation when given a value for the output.

### 8.10.2 Boundaries between the colloquial, numeric, symbolic, and geometric facets

All three students were asked a series of 4 questions that probed their ability to create 3 other representations from a given representation of a linear function. Specifically, they were asked to:

- given an equation, create a table, a graph, and a function machine;
- given a table, create an equation, a graph, and a function machine;
- given a function machine, create a table, a graph, and an equation; and,
- given a graph, create a table, an equation, and a function machine.

They were encouraged to create the other forms in any order they wished. Tables 8.12-8.14 display the results along with the order the student used (the "X" indicates successful creation and the number indicates the order each was created).

TABLE 8.12: Creating representations: AF

| AF From/To | Equation | Table | Function machine | Graph |
|---|---|---|---|---|
| Equation | | X (1) | X (2) | X (3) |
| Table | X (2) | | X (3) | X (1) |
| Function machine | X (1) | X (2) | | X (3) |
| Graph | X (2) | X (1) | X (3) | |

After doing the first problem in which the equation is given, AF commented:

AF: I am much more comfortable with the function machine and the table as opposed to creating a graph on my own. I'm not as comfortable doing a graph on my own.

When given the table, AF first created the graph, but then went back to the table to create the equation. She actually used the graph to determine the type of equation but then used the table to determine the slope by using finite differences.

AF: I'm trying to find the finite difference. I know from the graph it looks like it will be a line so I think it will be linear which I know is $y(x) = ax + b$. So for that I need the slope and the 0 input which I already have which is –3. It looks like the slope is 2 so I get $y(x) = 2x - 3$.

Figure 8.4 displays a diagram illustrating the connections between facets that AF demonstrated answering this series of questions.

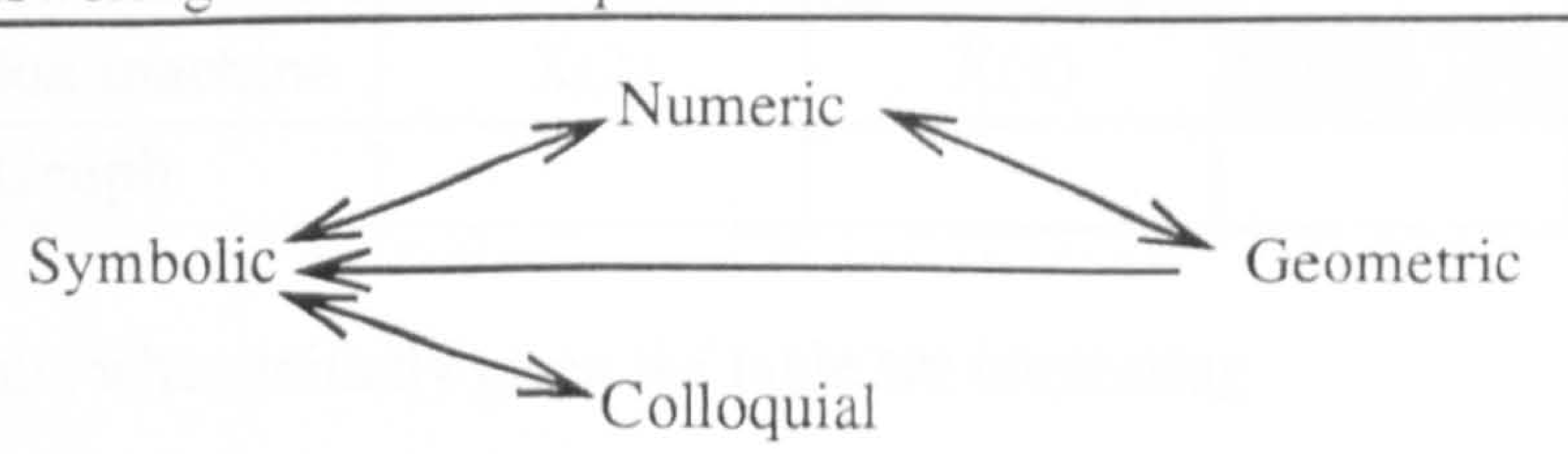



Figure 8.4 Boundaries between numeric, symbolic, colloquial, and geometric facets crossed by AF

While BF was able to start from the equation or function machine and generate the other facets, she was only able to move between table and graph when starting from a table or a graph. She kept trying to generate either an equation or a function machine using only one point. Consequently, she was unable to determine the slope. Figure 8.5 displays a diagram illustrating the connections between facets that BF demonstrated answering this series of questions.

TABLE 8.13: Creating representations: BF

| BF From/To | Equation | Table | Function machine | Graph |
|---|---|---|---|---|
| **Equation** | | X(2) | X (1) | X(3) |
| **Table** | | | | X(1) |
| **Function machine** | X(1) | X(2) | | X(3) |
| **Graph** | | X(1) | | |

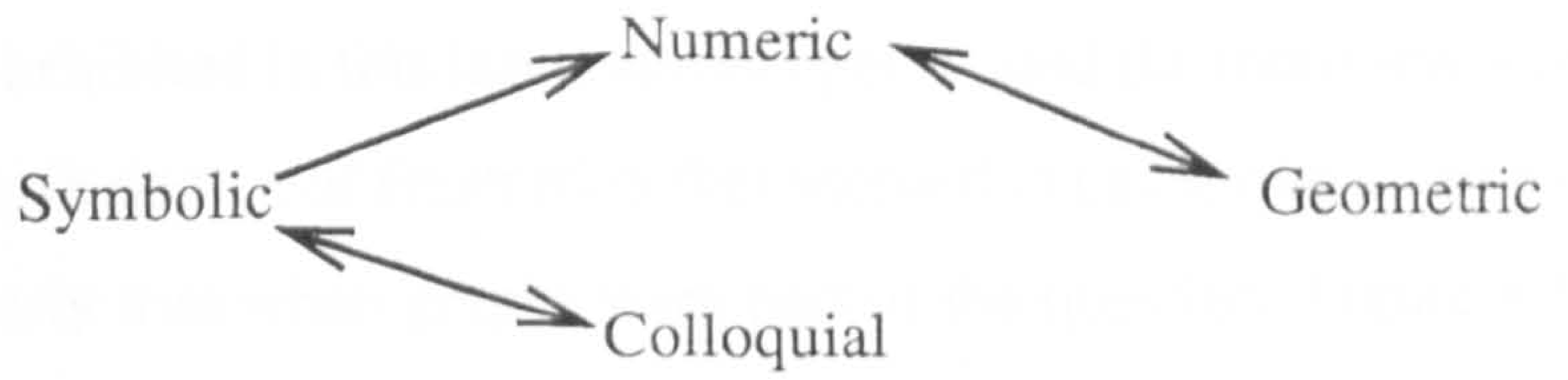



Figure 8.5 Boundaries between numeric, symbolic, colloquial, and geometric facets crossed by BF

Similar to BF, CM was able to start from the equation or function machine and generate the other facets. He was also able to move from table to graph, but unable to reverse this direction.

TABLE 8.14: Creating representations: CM

| CM From/To | Equation | Table | Function machine | Graph |
|---|---|---|---|---|
| **Equation** | | X(1) | X(3) | X(2) |
| **Table** | | | | X(1) |
| **Function machine** | X(2) | X(1) | | X(3) |
| **Graph** | | | | |

His comments when initially given the table are interesting.

CM: I'm not real sure on equation or function machine.

Intvw: If you had to choose between the two, which would you prefer?

CM: It doesn't matter. I don't like either. I really don't like anything that has to do with math.

The pained look on his face and the nervous body language speak volumes.

Intvw: You like tables.

CM: Yeah. Tables are a little bit easier for me. I trust those more than having to figure out stuff.

Notice he suggests that "stuff doesn't have to be figured out" if he sticks to tables. Given the graph he draws the structure for the table but gives up.

CM: No. I can't do it.

Intvw: You started to do a table.

CM: Yeah ummm. If I were to sit down and think about it for a while I probably could. That's the way a lot of math is to me. I just keep trying different ways until I hit upon one that works. To save my life I probably could, but I'm not real sure.

The attitude exhibited in this last comment permeated the interview with CM. He demonstrated a high degree of frustration that seemed to cause him to give up quickly. This was particularly true when graphs were part of the question. Figure 8.6 displays a diagram illustrating the connections between facets that CM demonstrated answering this series of questions.

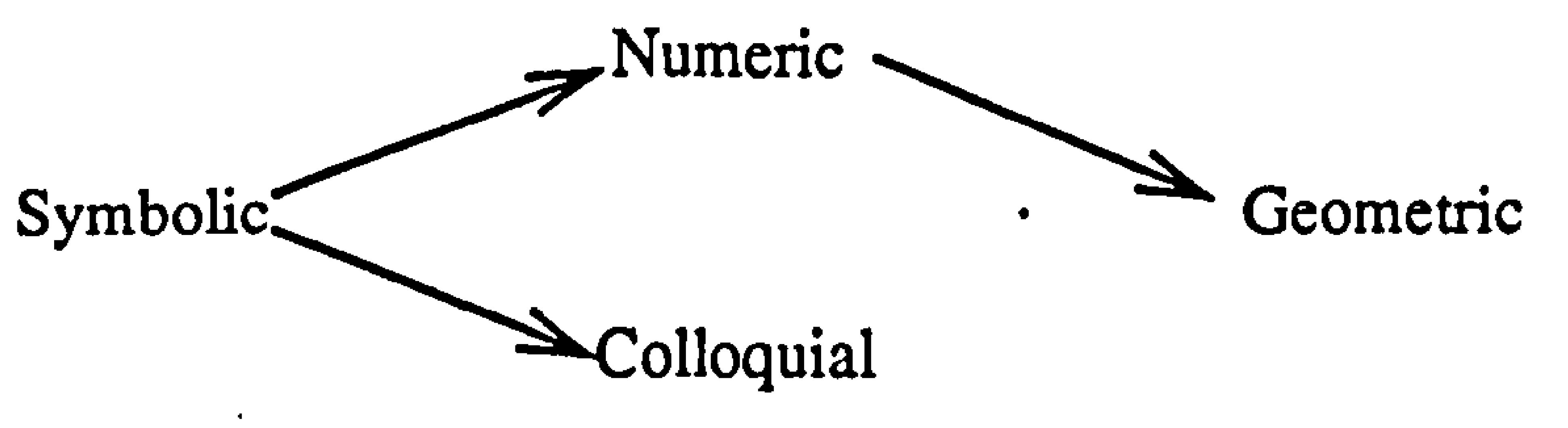



Figure 8.6 Boundaries between numeric, symbolic, colloquial, and geometric facets crossed by CM

Overall, AF's performance on this series of questions was flawless. She demonstrated the most ability in crossing boundaries between facets. BF demonstrated good connections between symbolic and colloquial and between numeric and geometric, but was unable to establish solid connections between these pairs. CM established good connections between symbolic and colloquial, but any connection to graphs was tenuous at best.

### 8.10.3 Boundaries between the written and verbal facets

This boundary was explored in Section 8.7. Suffice it to say that AF and BF demonstrated consistency between their written and verbal definitions of function. CM supplied quite different definitions in writing and verbally. This may indicate an ever changing concept image since the questions were asked about two weeks apart.

## 8.11 Profiles

In this section, visual profiles of the concept images of function for each of the 3 students are created. These are snapshots of their understanding according to the theoretical perspective. The shading indicates the number of layers the student has demonstrated in his/her understanding of each facet. The student's knowledge of a specific facet has not been assessed if the outermost layer (pre-procedure) is unshaded. Profiles for each student at the beginning of the beginning algebra course, at the end of the beginning algebra course, and at the completion of the interview are displayed. The profiles include a network of directed segments indicating the porousness of boundaries between facets. This network begins with the connections between the symbolic, numeric, geometric, and colloquial facets displayed previously. Connections to the notation, written, and verbal facets are added. Arrowheads indicate the direction that the boundary was perceived to be crossed. If a boundary was not crossed in either direction, the segment connecting the facets is omitted. These networks indicate the connections students were able to make by the time of the interview.

The pre-course survey provided information on the colloquial, symbolic, numeric, geometric, written, and notation facets. Thus each student is minimally at the pre-procedure layer for these facets at the beginning of the instructional treatment. Sufficient data were collected to analyse whether the student was at the pre-procedure, procedure, or process layer for each facet. The post-course survey provided additional information on the colloquial, symbolic, numeric, geometric, written, and notation facets. In addition, one question on the post-course survey provided some initial information on the boundaries between the notation, colloquial, numeric, and symbolic facets. Finally, the interview provided additional data on the verbal facet and the nature of the boundaries between numeric, symbolic, geometric, and notation facets and between verbal and written facets. The additional information provided by the post-course survey and the interview permits the refinement of the profiles.

## 8.11.1 AF's profile

Figures 8.7 and 8.8 display the facet-layer and facet boundary profiles for AF.

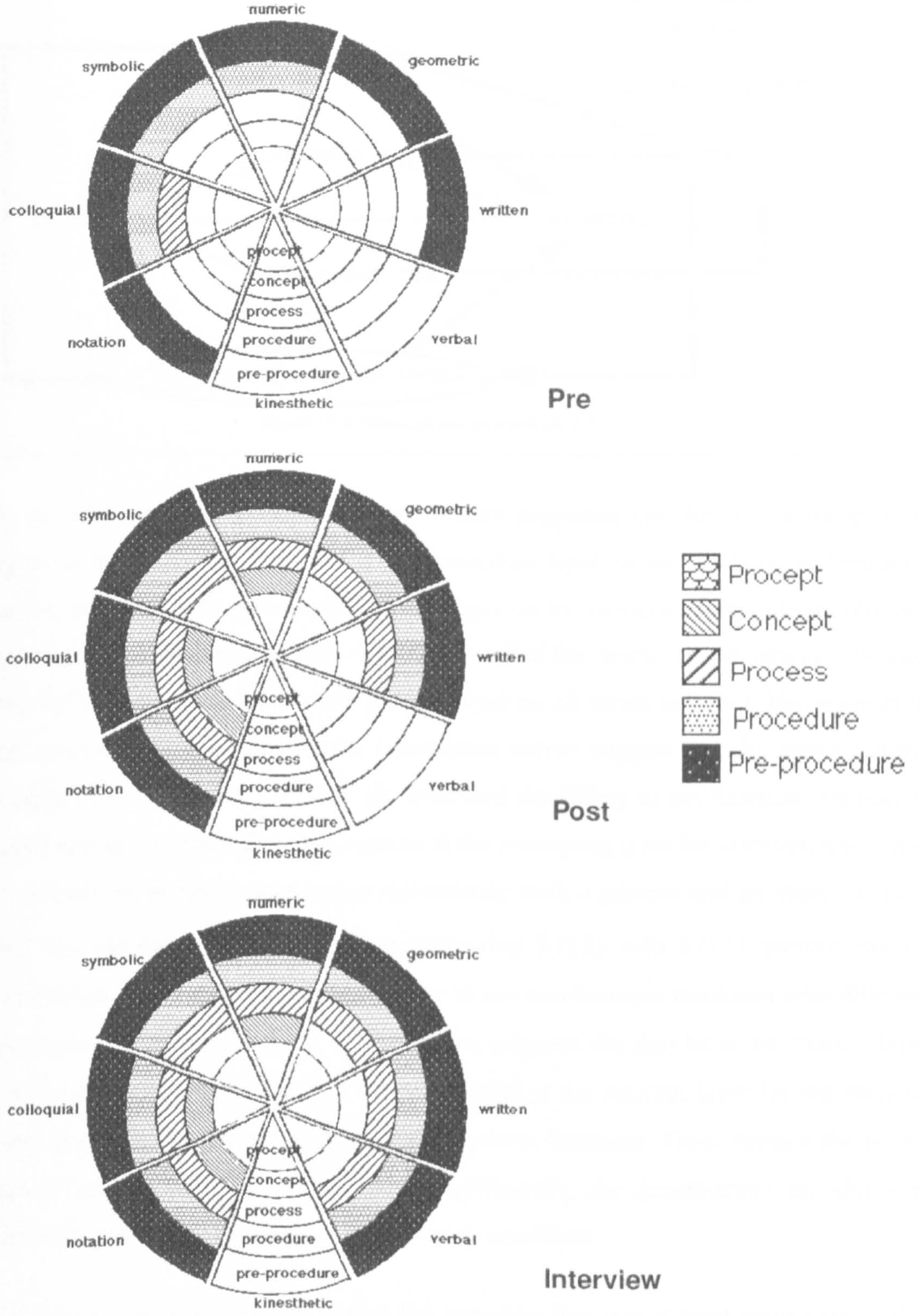



Figure 8.7 Concept image profiles of AF

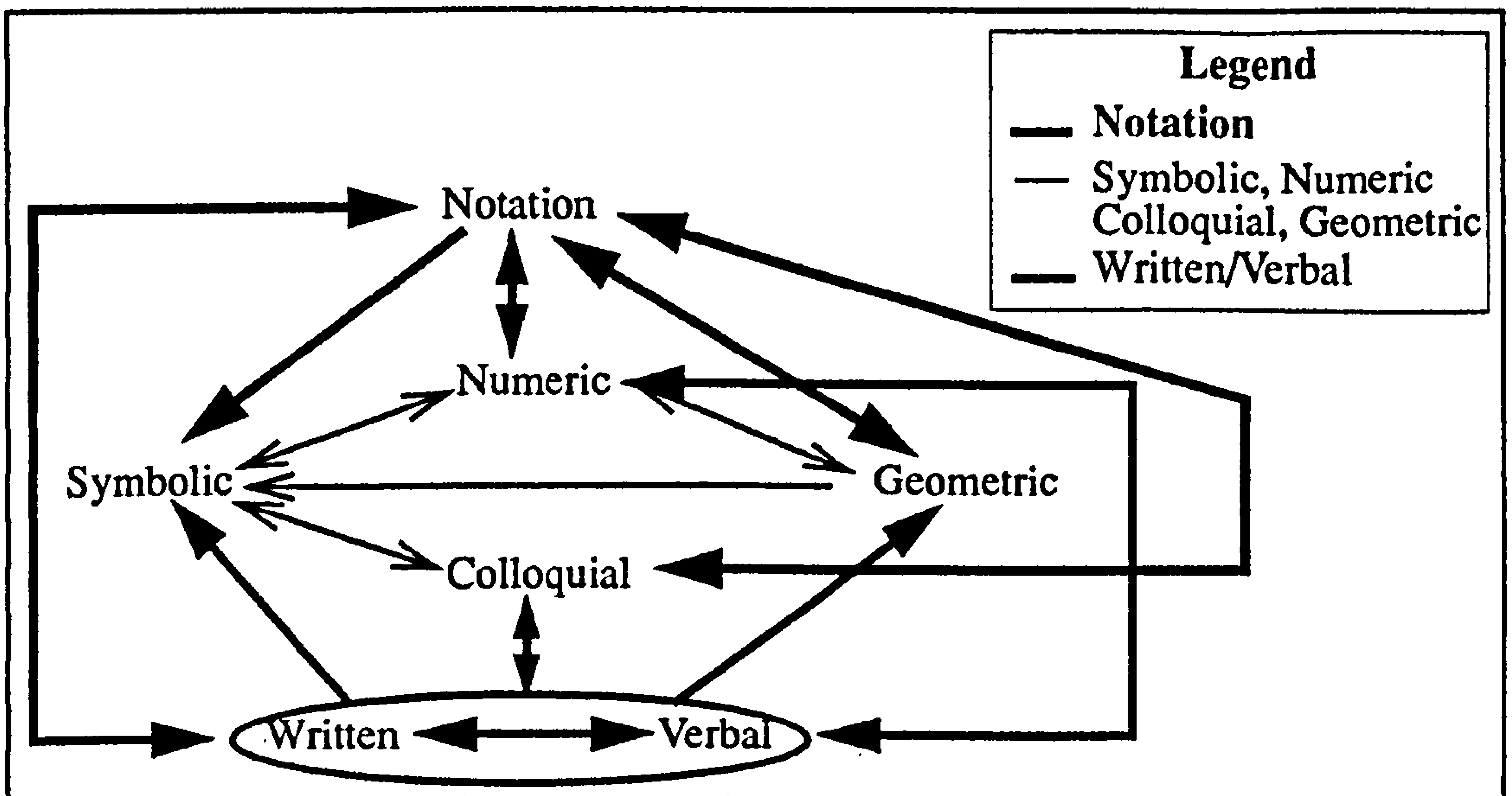



Figure 8.8 Boundaries crossed by AF

At the beginning of the course, survey results suggested that AF was at the process layer on the colloquial facet and at the procedure layer on the symbolic and numeric facets. She may have been at the process layer on the numeric facet, but the information was insufficient to conclude this. By the end of the course, survey results indicated that AF had moved at least to the process layer on all facets assessed. Her answers to the two notation questions on the post-course survey suggest that she may be at the concept layer for this facet since she indicated the ability to see function notation as representing an object. She may even be at the proceptual layer for notation, being able to flexibly think about notation as representing both a process and an object, but the fact that she did not do the question comparing $3f(2)$ with $2f(3)$ prevents such a conclusion from being drawn. Her ability to see two function machines with different procedures as representing the same function suggests she may be at the concept layer for the colloquial facet. Finally, she was placed at the concept layer for the numeric facet based on her choices of tables that represent functions. These suggest she is neither procedure nor process dependent. Additionally, she demonstrates the ability to correctly apply the "uniqueness on the right" condition.

AF demonstrated knowledge during the interview that was at least equivalent to that displayed on the post-course survey. Her knowledge of the verbal facet matched that of the written facet since her verbal and written descriptions of function were identical. In addition, she was able to assimilate alternate definitions easily into her own concept image.

She was placed at the process layer on the symbolic facet. AF did exhibit difficulty during the interview identifying implicit equations as functions of one variable in terms of a second variable. She did not use the "uniqueness on the right" condition in her selection of functions from a set of equations. She initially eliminated the constant function as a function, but later changed her mind. She displays proceptual abilities working with both tables and function machines. She is easily able to think of them as functions (static objects) and as processes (dynamic objects). AF's understanding of graphs was developing even as we conducted the interview. She did not need to know a specific procedure recognizing each graph as representing a set of input-output pairs. She was not prototype-driven and did not initially realize how to apply the "uniqueness to the right" condition. However, after some instruction, she demonstrated an ability to check the existence of this condition. The researcher did not place her at the concept layer for the geometric facet because she demonstrated a process orientation in looking at graphs as opposed to seeing a graph as a function object. Her knowledge of the notation facet appeared strong and consistent except for the time she substituted 44 for $y$ and said "44 of $x$." She did quickly back off from this statement and thought of 44 as replacing $y(x)$. AF was the only student interviewed able to appropriately describe the distinction between $3f(2)$ and $2f(3)$.

AF displayed the most advanced abilities in crossing facet boundaries. She was able to begin with an equation, a table, a function machine, or a graph and develop the other three facets, something that the other two students could not do. She was able to move between a graph, a table, and an equation in the same problem choosing the representation that was most appropriate, in most cases, to answer the question. She demonstrated consistency between the verbal and written facets which is why they are grouped as one in Figure 8.8. AF appeared unable, at least initially, during the interview to apply her written and verbal definition of function to the symbolic or geometric facets. She demonstrated one problem interpreting function notation when given a symbolic form of a function as indicated by the "44 of $x$" mentioned above. Beyond these two problems, she appeared to cross boundaries among the facets with little difficulty.

### 8.11.2 BF's profile

Figures 8.9 and 8.10 display the facet-layer and facet boundary profiles for BF.

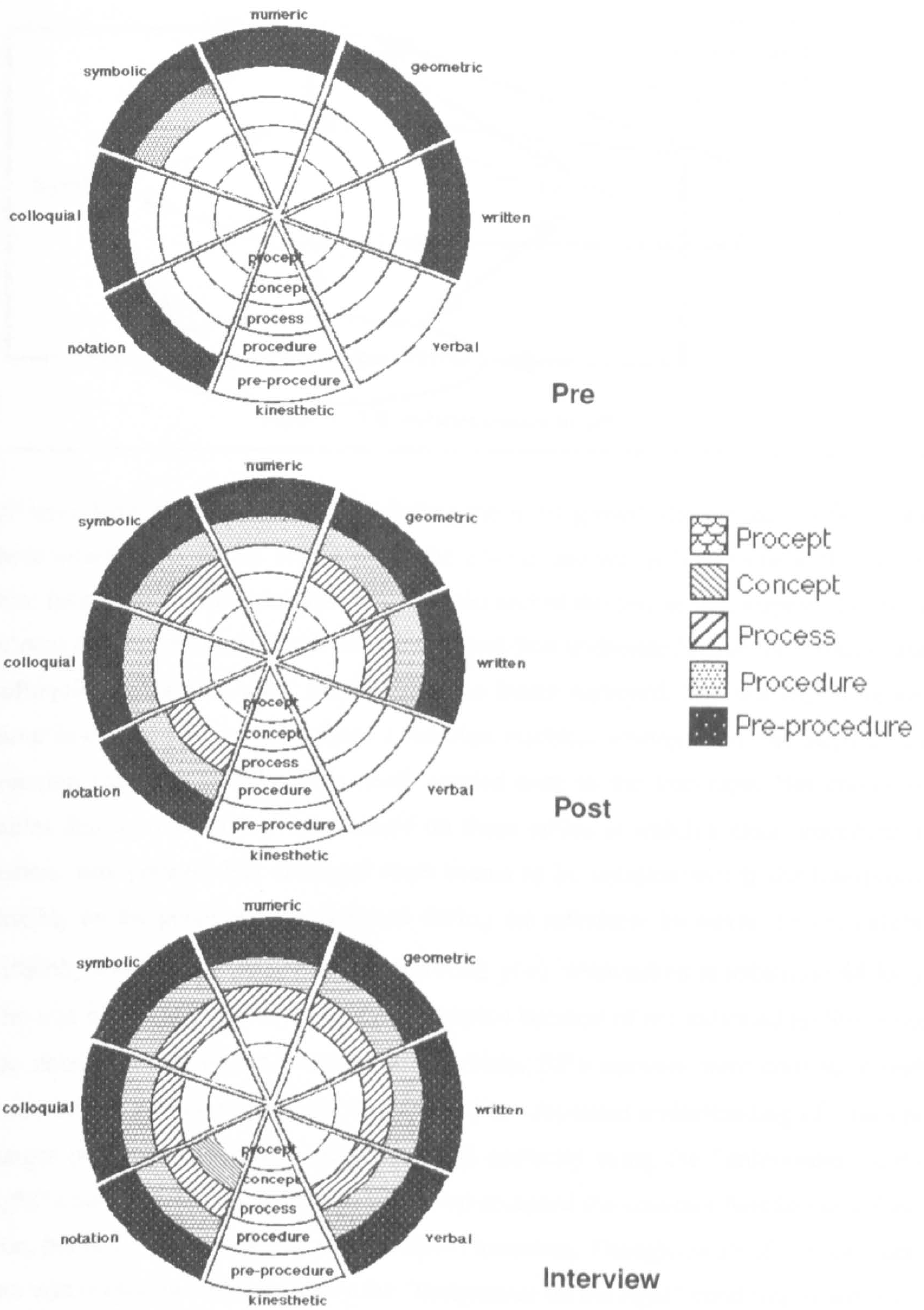



Figure 8.9 Concept image profiles of BF

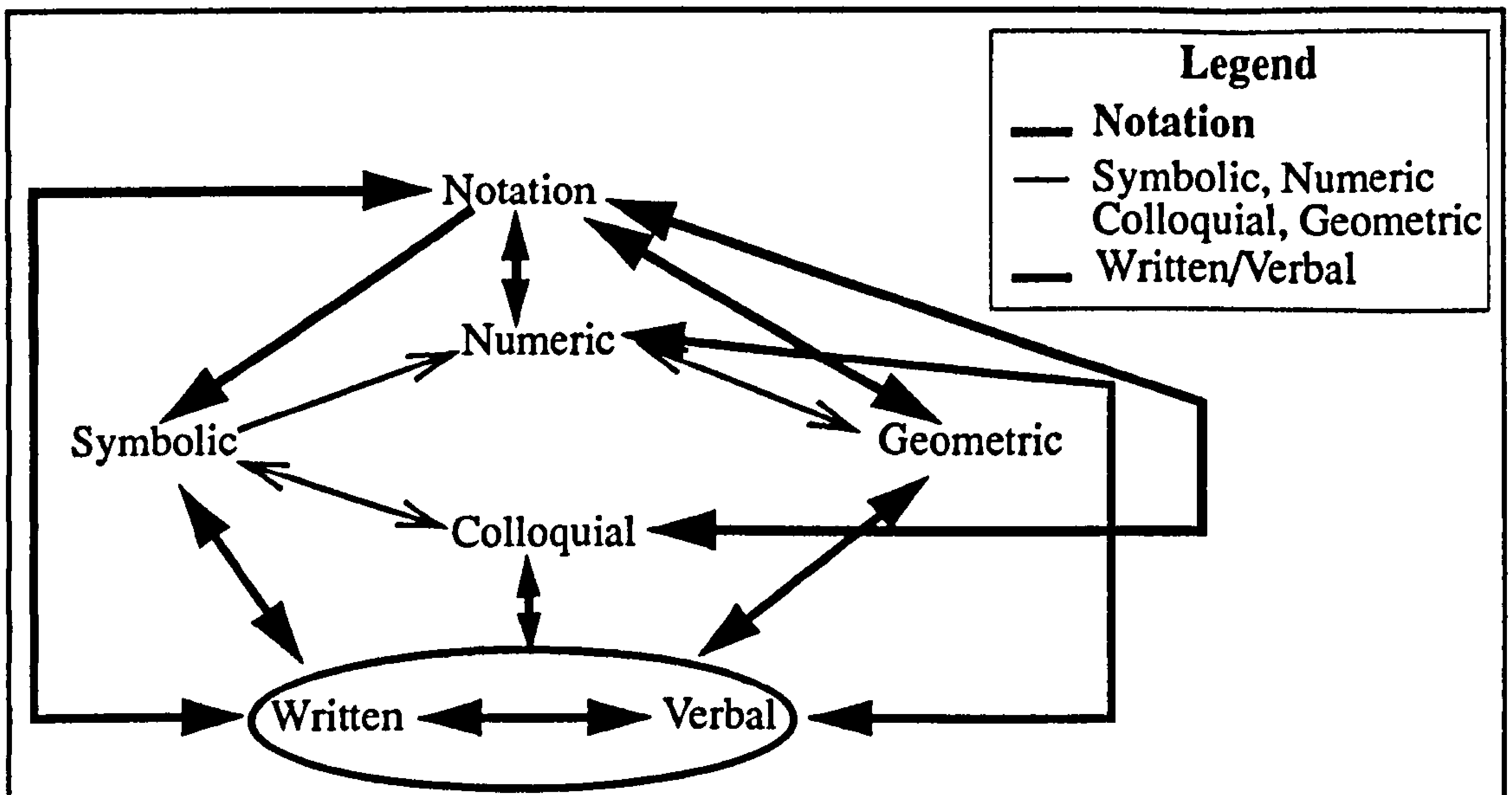



Figure 8.10 Boundaries crossed by BF

BF may have been the student exhibiting the most growth during the course of the three interviewed. At the beginning of the course, she was judged to be at the procedure layer only on the symbolic facet. By the end of the course, she appeared to be at or near the process layer on all facets surveyed. Her understanding of the numeric and colloquial facets were most suspect of those facets surveyed. She was highly procedural in creating an equation from a function machine writing down the steps of the function machine literally. This result carried over to the interview. Her choice of tables that represent functions focused on those tables in which a clear procedure or pattern was present. Her strongest facet seems to be notation which she interpreted flexibly on the post-course survey and during the interview. However she did exhibit difficulty interpreting $3f(2)$ and interpreting $y(x)$ when asked to substitute 44 for $y$. She was placed at the concept layer for notation because of her indicated ability to see the notation as an object. During the interview, BF's answers were consistent with those on the post-course survey. She did display improved understanding of tables no longer putting an emphasis on pattern and correctly using the "uniqueness on the right" condition. On the symbolic facet, she accepted the constant function as a function, but had trouble with piecewise-defined functions. Though aware of the condition, she was unable to correctly apply the "uniqueness on the right" condition when given equations. BF was the only student of the 3 that was able to correctly apply the vertical line test to graphs both on the post-course survey and during the interview. While consistent in her verbal and written definitions, BF was not as comfortable as AF in adopting alternate definitions. She exhibited confusion between the "uniqueness on the right" condition and one-to-oneness when looking at other definitions.

BF had more difficulty crossing boundaries than did AF. She did not easily move from a function machine to an equation and was quite procedural in using equations. This caused difficulty when given a variable input. She was unsure what to do and was not sure the output made much sense. Unlike AF she was unable to move from either a table or a graph to an equation or a function machine. The boundaries are porous between numeric and geometric facets since she easily moved between the two. The boundaries are also porous among the notation, colloquial, and symbolic facets since BF was able to move between these facets, but in a way that suggested dependency on procedure rather than the more generic process. BF exhibited a similar difficulty to AF when she said "$44x$" when she had to substitute 44 for $y$ in $y(x)$. She quickly corrected her error showing she understood what had to be done next, but recognized she did not legitimately know a process for answering the question using the equation. Similar to AF, BF exhibited consistency between her verbal and written definition of function. She appeared more able than AF to apply her definition to symbolic and geometric facets.

### 8.11.3 CM's profile

CM was the least successful of the students profiled. At the beginning of the course, he demonstrate procedure layer knowledge in both numeric and colloquial facets placing him slightly ahead of BF. However, by the end of the course, he demonstrated process layer knowledge only in the symbolic facet since he was able to reverse a symbolic procedure. The post-course results suggest he might be at the process layer on the numeric facet since he indicated partial ability to reverse the table and was not procedure-dependent when selecting tables as functions. The interview results, however, discount this. CM appears to be at the procedure layer on all facets except geometric where he is at the pre-procedure layer. In addition, his interview answers on the symbolic, geometric, numeric, and verbal facets were highly inconsistent with those on the post-course survey.

Figures 8.11 and 8.12 display the facet-layer and facet boundary profiles for CM.

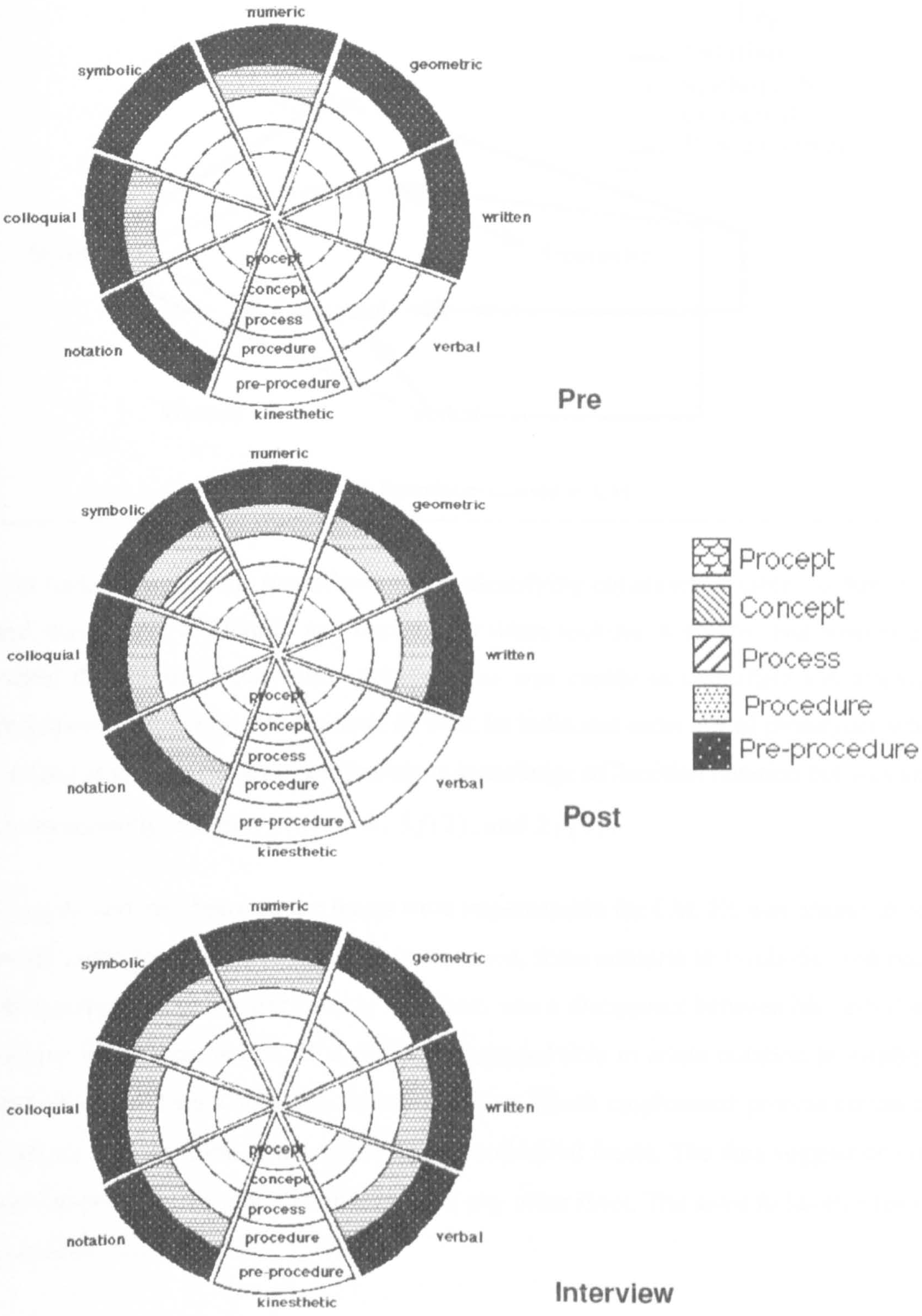



Figure 8.11 Concept image profiles of CM

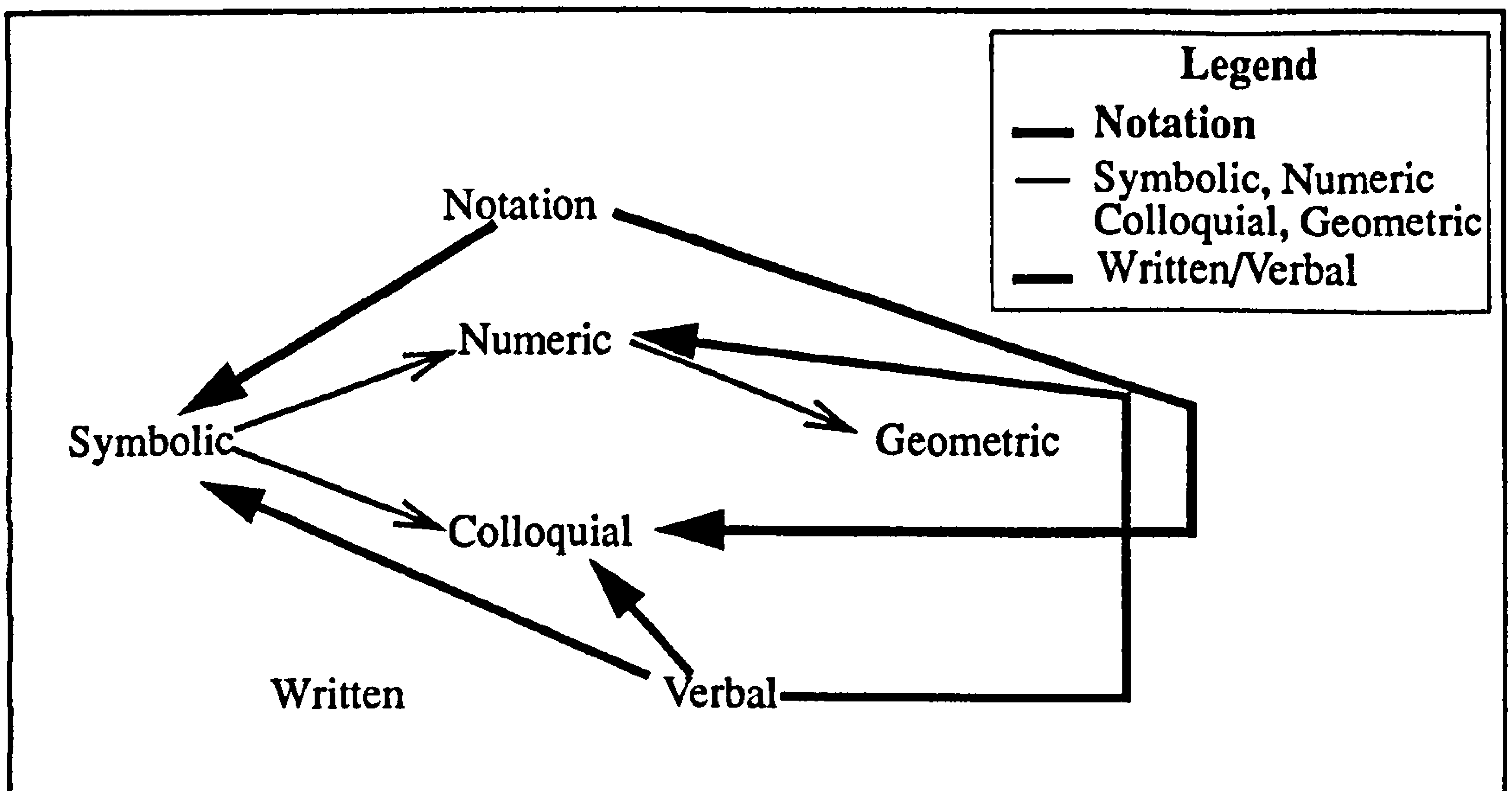



Figure 8.12 Boundaries crossed by CM

CM looked for specific procedures when identifying equations or tables as functions and was unable to identify any usable rule when looking at graphs. His written and verbal definitions of functions varied and he was unable to assimilate any alternate definitions of function into his own. At best, he indicated some use of prototypes when looking at graphs. He demonstrated some knowledge of function notation but was unable to correctly interpret $y(x) = 4$, $3f(2)$, and $2f(3)$.

Many boundaries between the facets were impenetrable for CM. He was unable to correctly move from function machine to equation, from numeric to symbolic, and could not interpret the geometric facet at all. There was a disconnect between his verbal and written definitions, unlike AF and BF. He seemed able to relate notation to symbolic and colloquial facets only. His verbal definition which emphasized process connected with his interpretation of the symbolic and colloquial facets. The data suggest no connection between his written definition and any other facet. The same holds true for the geometric facet.

## 8.12 Conclusion

This chapter has profiled the growth in three students' understanding of function as a result of the instructional treatment described in Chapter 4. Using the results of pre-course surveys, post-course surveys, and interviews, profiles as described in the theoretical framework are developed for each student at three different points: at the beginning of the course; at the end of the course; and, at the end of the interview. In-depth analysis and comparison of the each student's responses to each question provides the opportunity to compare and contrast developing concept images of function. Both AF and BF developed at least a process-layer understanding of function along all facets tested. In some facets, their understanding may be at the conceptual or even at the proceptual layer. Each demonstrates some weaknesses and AF demonstrates a somewhat more developed concept image based on her ability to more easily cross boundaries. Ultimately, she seems to be the more flexible of the two students across facets. Unfortunately, CM seems to be at the procedure layer across facets, except the geometric, and appears to have even slid backwards between the end of the course and the interview. His understanding of graphs seems particularly weak.

AF demonstrates the most connections between pairs of facets, though BF is not far behind. BF demonstrates a better connection among verbal, written, and symbolic facets and between verbal, written, and geometric facets. However, AF appears stronger at making the crucial connections among tables, equations, and graphs. CM makes few connections between facets. In particular, the geometric and written facets appear isolated. No connections are bi-directional. His understanding of the various facets of function seems to be very compartmentalized and fragile.

The creation of profiles demonstrates the complexity of students' concept images of function. The written surveys provide only a partial glimpse at student understanding while the interview allow for a more complete pictures. In several places, students' answers conflict across facets or across instruments indicating the fragility of the concept image over time. Places where the cognitive connections are tenuous or non-existent become apparent. One place to look for helpful information lies in areas of common difficulties. This will be the focus of the last chapter as the information gained is summarized and suggestions for future research and for modifications in current curriculum are made.

CHAPTER 9

# Reflections on Future Directions

## 9.1 Introduction

This chapter will discuss several implications of the findings presented in the previous two chapters. Some conclusions will be suggested about the relationship between the "reform" curriculum and the achievement of students as measured by their demonstrated understanding of the function concept. In particular, the relationship among the intended curriculum (an independent variable), the implemented curriculum (an intervening variable) as best as can be surmised, and the attained curriculum (a dependent variable at least as far as the function concept) must be considered.

The implications of this research are discussed. In particular, the reasonableness of using function as an organizing principle in a college beginning algebra course is reviewed. How must the intended curriculum be modified to better develop student understanding of function? What were the weaknesses of the research design that could be altered to inform future studies? Included is a reflection on the theoretical framework. The usability and adaptability of the facets-layers construct for mathematical entities will be discussed. First, the researcher discusses how well the framework served as a perspective for analysing student understanding of function. Secondly, the researcher attempts to apply the framework to other mathematical entities.

The chapter concludes with some recommendations for future directions. These will include both curricular recommendations and suggestions for future research in this area.

## 9.2 Intended Curriculum and Student Understanding

The purpose of this research is to determine if a refocused beginning algebra curriculum, with function as an organizing concept, makes mathematics more accessible to students who have been previously unsuccessful in learning mathematics. This thesis is a first step. If the curriculum is entered on function, can students be expected to acquire, minimally, a process layer understanding of the concept? This understanding should be sufficient to help them appreciate how knowing mathematics and function are crucial to a quantitatively literate citizenry. In addition, for those that continue in mathematics, the understanding of function developed in this course should result in a concept image that can be effectively expanded in later courses leading to an ever deepening understanding of function and its pivotal role in mathematics.

### 9.2.1 Intended curriculum

One independent variable for this thesis is the "reform" curriculum as defined by the text (DeMarois, McGowen, & Whitkanack, 1996a) used in the course. The other independent variable is the use of technology, in particular, the extensive use of graphing calculators which students were required to purchase. The text is organized so that students answer questions about mathematical situations first. The questions are structured to assist them in the discovery of key mathematical ideas. These questions, called Investigations, are followed by a Discussion which explains the key mathematical ideas "discovered" in the previous set of Investigations. The Discussions serve two purposes: for students who were able to answer the Investigations, the Discussions help them check the accuracy of their discoveries; for students who were unable to answer the Investigations, the Discussions help them answer the questions. After exploring the meaning of variable and algebraic expressions, including the use of operation machines to parse algebraic expressions, students are introduced to function as a process that receives input and produces one and only one output for a given input. The students encounter functions through function machines, equations in two variables, two-column tables, and graphs. Most functions arise from problem situations given to students. This approach continues throughout of the text.

Students are required to have a graphing calculator from the beginning of the course. The calculator is initially used to parse expressions and assist students in developing some understanding of variable. Students learn to enter functions symbolically, to explore functions numerically using a dynamic table feature, and to graph functions both from two-column tables and from equations. The calculator is a ready tool throughout the course.

Ultimately, in such a hypothetical environment, how would students' concept image of function develop?

### 9.2.2 Prior variables

The prior variables include profiles of the students in a beginning algebra course and of the instructors who teach such a course. The researcher assumed that students entering such a course would have had some prior exposure to algebra, but would have little or no prior exposure to function. Under this assumption, the prior exposure to algebra may serve as a hindrance to student development of reasonable concept images in this course. The lack of prior exposure to function provides a "clean slate" from which to build an appropriate concept image of one of the key concepts in mathematics. Both of these assumptions were verified by the data. The students entered the course with some prior exposure to algebra ($\bar{x}$ = 1.4 years, $\delta$ = 1.0). The responses on the pre-course survey suggest little or no prior exposure to function. For example, only 11 percent of 92 students were able to write a definition of function that indicated any prior familiarity with the mathematical concept prior to the course. Significant is the students' belief about mathematics and education in general. For example a student wrote: "I had been living so long in the valley of defeat that the last thought I had was to reattempt college, especially a dreaded maths course."

The instructors constitute a second prior variable. They were selected based on their prior exposure to workshops designed to help them understand the philosophies and approaches of the material. All instructors were willing participants in the workshops and in teaching from the reform materials. All expressed dissatisfaction with the traditional curriculum and with the traditional "lecture-approach" for these students. All instructors received training in appropriate uses of the technology.

### 9.2.3 Implemented curriculum

Due to the careful selection of instructors who were highly motivated by the material, the implemented curriculum did not vary greatly from the intended curriculum in terms of topic coverage. There was, however, some variation in the delivery of the curriculum. While all students were exposed to the first 5 chapters of the material where function is introduced and carefully presented, there was variation in what students were exposed to in the last two chapters. As these two chapters include an in-depth treatment of linear and quadratic functions, there were some variations across instructors in their emphasis on this material. Secondly, as time became short, some instructors resorted to more of a lecture approach to "cover" the necessary material. Thus, it was not uncommon for the discovery process to become disabled by the end of the semester.

The use of the technology varied from instructor to instructor. While all had received basic training in the use of technology and had received instructor manuals suggesting where and how to use the technology, some were ultimately more comfortable with its use than others.

An intervening variable that was not controlled were assessment and evaluation practices. Each instructor set up his/her own assessment policies. As a result, expectations of students varied by instructor though no instructor limited the use of technology in any way on any assessment instrument.

Another intervening variable is the additional ideas teachers chose to introduce to students that are not presented in the text. For example, one instructor in the main study choose to introduce and place some emphasis on the vertical line test as a way to determine if a graph is a function. This technique does not appear in the materials. Since this technique is often presented as a procedure with no conceptual understanding, it becomes another piece of procedural knowledge that students use and misuse. Some evidence of this appears in the data for the main study.

Ultimately, the key intervening variable is the role of the student and the role of the teacher in such a classroom. The curriculum was designed to empower students to explore mathematics alone and within groups and to develop a conceptual understanding of important concepts by developing an ownership in the material. The hope is that such students would be more flexible in their understanding of what mathematics is and how to use it. Most questions used on the data-collection instruments were not questions students would have encountered during their course. The questions were designed to cause them to draw upon their concept image of function rather than upon some memorized and poorly-understood procedure. The success in creating empowered students varied greatly. Some instructors find it very difficult to relinquish their role as the dispenser of knowledge. The degree to which this occurred certainly influences the ultimately success of this curricular approach.

### 9.2.4 Attained curriculum

A dependent variable in this thesis is the student concept image of function that results from exposure to the curriculum. Understanding of function is a key component of the attained curriculum.

**The data indicate significant growth in student concept image of function along the colloquial, symbolic, numeric, geometric, verbal, written, and notation facets. These results suggest that the function concept is accessible to this level of student.**

In particular, based on results of the pre-course survey, students enter such a course able to:

- manipulate function machines effectively; and,
- use equations to evaluate functions.

The function machine appears to be a sufficiently primitive structure so as to serve as an excellent entry point into the function concept. The ability to evaluate seems to stem back to prior exposure to mathematics. The researcher has suggested that much of the students' prior exposure to algebra has been procedural and something they seemed to learn rather well along the way is "plugging" in a number and computing the result.

While the results point to the accessibility of function for these students, there are several areas of concern.

- While the function machine appears to be a good entry point to function, the growth in the understanding of this facet is slight as students proceed through the course. On the pilot study, the demonstrated understanding of this facet did not change significantly from beginning to end of the course.

- Demonstrated understanding of the colloquial facet does not necessarily ease the transition to other facets. Some students exhibited real difficulty in crossing the boundary from the colloquial to the symbolic facet, to the numeric facet, and to the notation facet. There was little connection demonstrated between function machines and graphs. While both are essentially visual, function machines appear to be quite primitive representations for which increasing depth does not easily follow.
- Student difficulty with the geometric facet remained at the end of the instructional treatment. Virtually, no student entered the course able to answer input-output questions about graphs. While a significantly greater number of students improved by the end of the semester, the percentage of students able to appropriately interpret this facet was low. At the end of the semester, only 41 percent of 92 students were able to find output given input and only 19 percent were able to reverse the process. During the interviews, the highly capable student showed good depth in understanding of this facet, but the capable and incapable students both struggled.
- One problem seemingly introduced by the technology became apparent. A graphing calculator requires a student to look at both the graphing window and the window that determines the settings for the rectangular portion of the plane that defines the viewing window. It appears to be quite difficult for students to keep a focus on both at once. These appear to be two different representations with tenuous connections. On survey questions, the viewing window and graph were side by side making the transition between the two a bit easier. When a student actually uses a graphing calculator, the student cannot see these views simultaneously possibly causing a disconnect between the two pieces of information.
- The use of prototypes (Tall & Bakar, 1992, for example) to recognize functions was apparent though students did not generally exhibit a problem with lack of continuity (Breidenbach et al., 1992, for example). This may be due to the number of discrete graphs present in the materials. Such a result suggests how experience and context affects the concept image.

- Function notation was interpreted inconsistently. While students demonstrate an ability to interpret the notation correctly in some settings, this ability sometimes does not translate to a new, similar setting. Even the most capable students on the surveys and interviews demonstrated some inconsistency with this facet.
- Students exhibited the use of prototypes when asked to select equations as functions. However, the more capable students tend to look for the presence of some process or relationship between variables.
- The well-documented problem with the constant function (Markovits, Eylon, & Bruckheimer, 1988, for example) arose though, again, the more capable student was able to work her way through it though her interpretation of the constant function in various representations was inconsistent.
- The requirement that a function have one and only one output for a given input—the "uniqueness to the right" condition (Breidenbach et al., 1992, for example)—was interpreted inconsistently depending on the facet. While easily recognized with tables, confusion arose related to one-to-oneness. In the interview, the highly capable student "learned" how to recognize this condition on a graph exhibiting conceptual knowledge while the capable student was aware of how to recognize this condition on graphs, but seemed to apply the methods procedurally. The incapable student exhibited no ability to differentiate using this condition on a graph. The symbolic facet caused the most problem of the three with this condition. No student consistently applied this condition to equations.
- It appears that it is significantly more difficult than expected to neutralize the interferences of prior procedural, instrumental learning. Two points are at issue here. First, the prior learning about topics, such as equations, seemed to cause problems when students had to view equations in two variables more flexibly as defining relationships between two variables. Such a result suggests that students may be viewing the "=" as the action "makes" which results from interpreting the notation as a process rather than an object. This problem seemed particularly acute when trying to develop flexible understanding of the symbolic facet. Secondly, by introducing such a radically different curriculum to students, they may believe that it has no relevance ("alien" in the sense of Duffin and Simpson, 1995). Even if the material

appears relevant, this student population is often driven by external goals that revolve around completing their mathematics requirement, not understanding mathematics.

- Students are often good at "plug and chug" mathematics and use this ability to hide the weaknesses in their understanding. CM, for example, resorted to using the more abstract symbolic facet when the more primitive table failed him. He indicated little understanding of the symbolism, but demonstrated several times that he could evaluate a function. This appears to be an example of "pseudo-conceptual" understanding (Vinner, 1997).
- Finally, the ability to establish connections between facets varied greatly. Some major differences between highly capable, capable, and incapable student arose in this context. Both highly capable and capable students exhibited consistency between verbal and written facets. The incapable student did not. The highly capable student was most able to assimilate alternate definitions into her own. The capable student did this with difficulty while the incapable student was unable to adapt alternate definitions. The highly capable student most easily moved between symbolic, numeric, colloquial, and geometric facets exhibiting the most connections of the three students. The capable student made fewer connections and the incapable student fewer yet.

### 9.2.5 Consequent variables

The consequent variables revolve around the students' next step. Most will have to take at least one more mathematics course. All will have to enter a society that is overloaded with quantitative information. Another phase of this research would study the continued growth of the function concept in subsequent courses. Another key element is the role that function plays in the concept image that the student has of mathematics.

- Will the early introduction of function ease or prevent student development of more sophisticated views of function?
- Will student understanding of function contribute to or inhibit student understanding of other key mathematical ideas?

Another area of interest is the students' entry into the appropriate mathematical community based on their role in society. Even if they take no more mathematics, have they been sufficiently empowered to think quantitatively and participate in the crucial quantitative decisions of their life? As one student wrote on an end-of semester evaluation: "A proof of understanding, or learning, is that these ideas are integrated into everyday life, even outside of class."

## 9.3 Implications of the Research

### 9.3.1 Reasonableness of function as a core concept

This research seems to suggest that a majority of students in the "reform" college beginning algebra course can achieve a process layer understanding of function. The concept is not "beyond" them. However, numerous caveats are necessary. The development of their concept images is truly uneven. A much more careful look at how graphs are dealt with is an example since the growth in this facet was minimal for so many students. The research was not designed to test for concept or procept layer understanding of various facets, though some conjectures regarding this depth can be made by studying the data. The belief of the researcher is that this depth of understanding is not necessary for students exiting a beginning algebra course. Should they continue with their mathematical training, the hope is that such depth will develop. It appears that students do not need to be proficient in certain procedural aspects of algebra, as suggested by Sierpinska (1992), to develop a process layer understanding of function. As indicated in previous research (Vinner & Dreyfus, 1989. for example), connections between facets prove to be troublesome. In revising the curriculum, the authors must further analyse difficulty in moving among the facets, particularly among the symbolic, numeric, colloquial, notation, and geometric facets. Finally, will the concept image of function formed in this course be an adequate building block for deeper understanding of function in later courses or will it serve to create misconceptions that will hinder students' understanding of function at a deeper level?

### 9.3.2 Modifications of the intended curriculum

The analysis of the data suggests some changes in the curriculum that may allow students to develop a more complete concept image of function.

- The intended curriculum introduces functions as a process that receives input and returns one and only one output. Some time should be devoted to see functions more generally as objects—possibly, as sets of ordered pairs. From student responses, it seems this would not be beyond the grasp of some students. Too much emphasis on process may strengthen the procedural layer and make it difficult for students to develop a more flexible concept of function.
- The curriculum could spend more time discussing ways of recognizing functions when given a table, an equation, a function machine, or a graph. Students in the study developed their own techniques to do this, but such techniques appeared to be strewn with misconceptions.
- Connections between various facets could be more explicitly drawn. In particular, more effort should be devoted to exploring the pros and cons of using a function machine, a table, an equation, or a graph to describe a function. What is the best use of each facet? How can one move from one facet to the other or create one facet from another?
- As mentioned above, the geometric facet proved most difficult for students. Understanding what they are seeing when given a graph of a function is a key to developing materials that will help them view the representation in the most advantageous way. While extensive research exists on this subject (Romberg et al., 1993, for example), little has been done with this population and the impact of the graphing calculator on this population is unknown. The data suggest that the curricular materials assume too much about student understanding and interpretation of graphs.

### 9.3.3 Facets and layers

The theoretical framework designed around the ideas of facets and layers of a mathematical entity appears to be a valuable way to analyse the function concept. The concept itself proves quite complex and the technique of dividing the concept via two dimensions to allow for micro-analysis seems helpful. Some important observations include:

- Analysing the depth of each facet can prove problematic for some facets. The more primitive facets seem to be the colloquial and the numeric. It is these facets that students seem to enter the course with some innate understanding of how to interpret. However, incapable students appear to find it difficult to move beyond these facets to more complex ones, such as symbolic or geometric.
- By beginning the course with a focus on these primitive facets, the curriculum quickly gives students access to function, but, on the other had, may create misconceptions by oversimplifying the concept in such a way so that student growth along other dimensions is inhibited.
- Some facets tend to be more figural in the sense of appearing to be an object already. These include the function machine, the graph, and sometimes the table. Students can intuitively grasp the idea behind the table and the function machine, but may have a difficult time seeing these as procedures or processes. Graphs on the other hand appear to carry an added complexity. Other facets, such as the symbolic, tend to be more proceptual than figural in that there are definite stages when a student may see an equation as a procedure, other times as representing a generic process, and still other times as an object that can be used as input to higher level procedures.
- Methods for describing the nature of the boundaries need to be further defined and developed.

While the facets-layers profile proves helpful in analysing student understanding of function, is it generalizable to other mathematical entities? Table 9.1 suggests how the mathematical entities "rational expression" and "instantaneous rate of change" may be viewed in terms of facets.

**TABLE 9.1:** Facets of two mathematical entities

| Facet | Rational expression (in one variable) | Instantaneous rate of change |
|---|---|---|
| symbolic | $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials | $\lim_{h \to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$ |
| numeric | fractions or decimals resulting from evaluation | table of $x$ versus $\frac{d}{dx}(f(x))$ |

TABLE 9.1: Facets of two mathematical entities

| Facet | Rational expression (in one variable) | Instantaneous rate of change |
|---|---|---|
| colloquial | fraction, decimal, or division | average rate of change |
| geometric | graphs of rational functions | local straightness |
| notation | meaning of $\frac{f(x)}{g(x)}$ | meaning of $\frac{d}{dx}(f(x))$ |
| verbal | definition/description stated in words | definition/description stated in words |
| written | written definition/description | written definition/description |
| kinesthetic | physical ways of demonstrating division | physical ways of illustrating instantaneous change |

As mentioned in the theoretical framework and also suggested by Sfard (1992, 1995) developing a procedural understanding of one concept may initially require the reification of a more elementary concept. The idea of levels, each consisting of facets and layers, was previously mentioned as a way of thinking about the prior concepts and how they must be understood before moving to the more advanced concept. For example, to begin developing an understanding of rational expressions, one first needs understanding of rational number, division, and polynomials, at least. To begin the development of a concept image of instantaneous rates of change, one first need some understanding of ratios, of rates, and of function. The framework suggests that a facet-layer analysis of concepts that are needed for the next level may point toward potential problems when the student attempts to build a concept image of a mathematical entity one level up.

Thus it appears that the facet-layer construct is applicable to other mathematical entities. It may be the case that some of these entities have slightly different facets from those discussed for function, but by thinking of the entity in terms of the depth of development along each facet and the connections between the facets, the researcher may better be able to interpret the level of student understanding and to identify gaps in that understanding. Seeing new concepts as being built upon old concepts supports the part of the structure called levels. If a student is having trouble building an understanding of a particular concept along a particular facet, it may be due to inadequate understanding of a mathematical entity one level down. These thoughts are subjects for future research.

### 9.3.4 The future

The future depends on how the developmental mathematics curriculum changes to meet the needs of the next century. A key variable is how the curriculum incorporates technology. This research project was designed around the developmental mathematics curriculum as it is currently manifested in the United States. This curriculum usually involves a semester of basic skills (arithmetic), a semester of plane geometry, and two semesters of algebra (beginning and intermediate algebra). While few students take the entire sequence of courses, a majority of community college students take at least one course. Is this an appropriate way to structure a developmental curriculum designed to prepare students to take college credit mathematics courses along with preparing students to be informed, contributing citizens? The answer, if based on success rates alone, is "no". AMATYC, in its Standards document (AMATYC, 1995) outlines the basic structure of a mathematical curriculum, called the Foundation, that is radically different from the existing model. Such a curriculum would be built around the content areas of number sense, symbolism and algebra, geometry and measurement, functions, discrete mathematics, probability and statistics, and deductive proof (ibid, pp. 26–28). Instead of the content being divided into discrete courses, the content areas would be integrated into one or several courses. In terms of pedagogy, teaching with technology would be paramount and cooperative learning strategies would be employed. Increased attention would be given to actively involving students in problem analysis, skills in context, mental arithmetic and estimation, conceptual understanding with the ability to use valid arguments, and appropriate use of technology (ibid, p. 29). Paper-and-pencil drill, contrived problems, isolated topics, and single method, single answer problems would receive reduced attention (ibid, p. 30).

The curriculum used in this research study exhibits some of the characteristics of this new, recommended curriculum. However, as changes are made in the curriculum, careful research must be done to measure the success of such change. This was the case with this study. If the curriculum is to focus on function, is it reasonable to expect students to develop an appropriate level of understanding of the concept? As the changes are made, research is needed to judge the effectiveness of the change and to suggest future avenues.

It is this researcher's opinion that the traditional beginning algebra curriculum in mathematics must be eliminated. As new curricula are gradually implemented, the corresponding research must be done in order to inform curriculum developers of the success or failure of such changes. By success and failure, this researcher means the level of understanding the student demonstrates and the usefulness of that understanding as a future member of society. It is hoped that the framework described in this project can serve as a useful guideline for measuring understanding that occurs as a result of future changes in curriculum and instruction.

## 9.4 Conclusion

This chapter has served as a reflection on this research study and what the future may hold. The chapter began with a discussion of the relationship between the "reform" curriculum (independent variable) and student understanding of function (the dependent variable). The results of this research suggest that, with some reservation, the function concept is accessible to beginning algebra students via a curriculum that uses the concept as an organizing principle. In the process of discussing these results, the researcher looked back at the intended curriculum as included in the text, the implemented curriculum as devised and actualized by the teacher, and the attained curriculum as demonstrated by students on the three instruments that were part of this study. The section concluded with a look at the consequent variables that relate to the next step students take after this instructional treatment and whether the concept image of function developed during the beginning algebra course will prove helpful as they continue their education.

The chapter next turned to a look at the implications of this research. Included was a discussion of the effects of the function concept on students' mathematical knowledge and suggestions for where the intended curriculum can be modified due to weaknesses exposed by this research. The researcher looked at the theoretical framework entered on facets and layers and discussed the effectiveness of this framework in the analysis of understanding of function. In addition, how the framework might be used to analyse understanding of other concepts, such as rational expression and instantaneous rate of change, was demonstrated. The researcher looked at mathematical understanding on a broader level by discussing a superstructure, called levels, to the facet-layer structure. Finally, the researcher reflected on future directions in curriculum for the developmental mathematics student, including recommendations for major reorganization suggested by the American Mathematical Association of Two-Year Colleges (AMATYC, 1995).

Williams (1993) states that "learning mathematics is profitably viewed in terms of enculturation into a society of users of mathematics. ... successful learning is learning that enables students to participate in a reasonable way both in the practices and in the language games of a mathematical community" (p. 316). He goes on to list several types of mathematical communities: research mathematicians, technical users, mathematics teachers, and the casual, occasional user of mathematics. This researcher suggests that it is this last community that the great majority of students in developmental mathematics courses will join. For these students, a process layer understanding of function is probably more than sufficient for their future mathematical needs. The research discussed in this thesis supports the theory that students can acquire such an understanding, albeit sometimes fragile and uneven.

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# Pre-Course Function Survey

## A note to the student

You are enrolled in an algebra course where one of the key mathematical ideas is the concept of function. I am conducting a research study on this course and I need your help. I need your honest answers to two surveys. This survey is designed to assess your knowledge of the mathematical concept "function" prior to this course. Do not worry if you are unable to answer the following questions since you are not expected to have studied this concept before. I hope to learn how your concept image of "function" grows during the course. A concept image is everything that is associated in your mind with a concept. Each person's concept image is unique, like a fingerprint. You will be asked to complete another survey at the end of the course. Thank you very much for your cooperation in this venture.

Sincerely,

Phil DeMarois

## Identification information

Please print the **last five digits** of your social security number:__________________

Name of

school:__________________________________________________________

Name of instructor:__________________________________________________

How much algebra have you had prior to this course (1/2 year, 1 year, etc.)?_________

What is the first word that comes to your mind when you hear the word "math"?_____

## Directions for survey

Next to each question is a box like the one at the right. Read the question. Check the box below the word that best describes your confidence about answering the question using the following guidelines.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

**Know** I can answer this question correctly.

**Afraid** of making a simple mistake, but I know how to answer the question.

**Unsure** of how to answer the question though I have some ideas about how to answer.

**No idea** how to answer this question.

For example, **Afraid** has been checked in the box at the right.

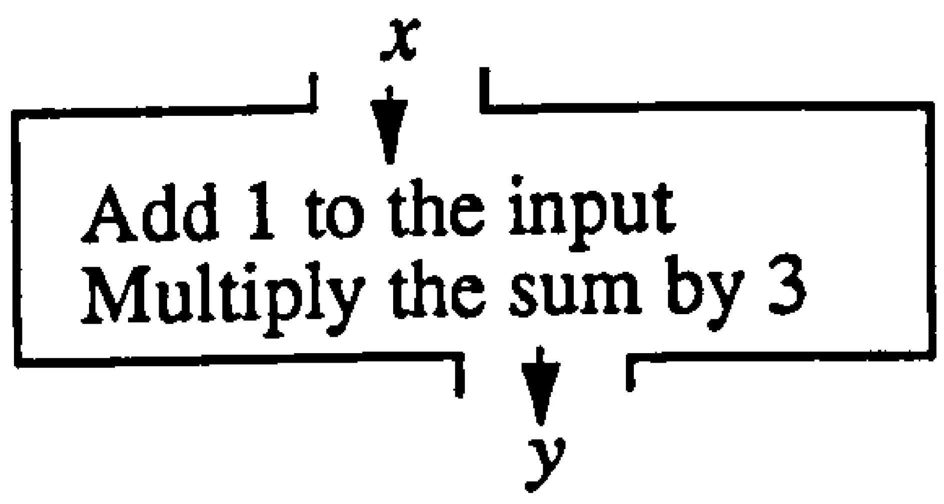
| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | ✓ | | |

**After checking the box, attempt to answer the question, if you can.**

When you see the variable *x*, assume that it stands for **input**.

When you see the variable *y*, assume that it represents **output**.

1. Consider the diagram.




a. What are the output(s) if the input is 7?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. What are the input(s) if the output is 18?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

2. Consider the equation $y = 3x - 7$.

a. What are the output(s) if the input is 5?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. What are the input(s) if the output is 0?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

3. Consider the following table copied from a TI–82 graphics calculator. The "X" stands for *x* and the "$Y_1$" stands for *y*.

| X | $Y_1$ | |
|---|---|---|
| [illegible] | 12 | |
| -2 | 5 | |
| -1 | 0 | |
| 0 | -3 | |
| 1 | -4 | |
| 2 | -3 | |
| 3 | 0 | |

a. What are the output(s) if the input is –2?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

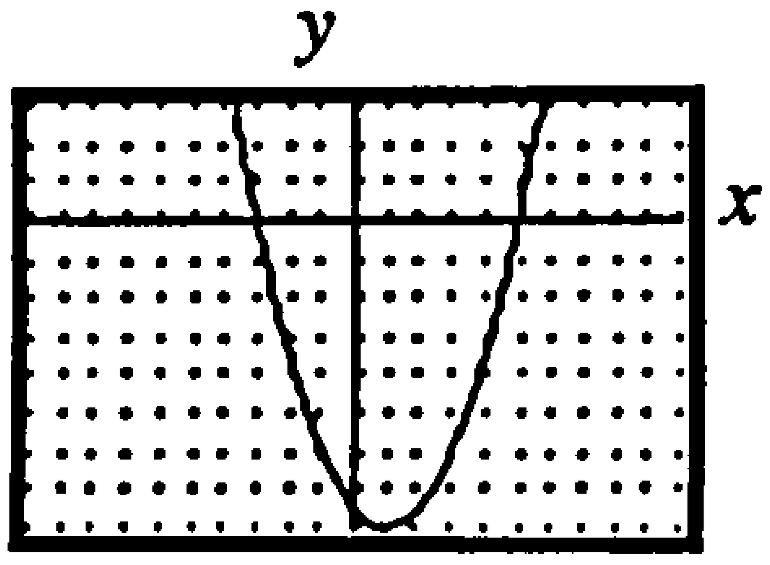
b. What are the input(s) if the output is –3?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

4. Consider the following viewing window and graph copied from a TI–82 graphics calculator.

```
WINDOW FORMAT
Xmin=-10
Xmax=10
Xscl=1
Ymin=-16
Ymax=6
Yscl=2
```




a. What are the output(s) if the input is 3?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. What are the input(s) if the output is 0?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

5. What is a function?

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

6. Briefly state what each of the following words or phrases mean to you.

a. a variable.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. a two-column table of numbers.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

c. a graph.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

d. an equation.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

7. Briefly state what each of the following symbols mean to you.

a. $f(x)$

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. $y(x) = 4$

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

c. $a(b+c)$

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

**Thank you for your help and cooperation. I appreciate it!**

# Post-Course Function Survey

## A note to the student

You have completed an algebra course in which one of the key mathematical ideas was the concept of function. This survey is designed to assess your concept image of function at the end of the course. You may recall that you completed a Pre-Course Function Survey when you began this course. Your answers established a basis for your knowledge of function at the beginning of the course. This survey will be used to measure the growth in your knowledge of function as a result of the course. Thank you very much for your cooperation in this venture.

Sincerely,

Phil DeMarois

## Identification information

Please print the **last five digits** of your social security number:________________

Name of school:________________________________________

Name of instructor:______________________________________

Circle your age group 17–20 21–25 26–30 31–?

What is the first word that comes to your mind when you hear the word "math"?_____

## Directions for survey

Next to each question is a box like the one at the right. Read the question. Check the box below the word that best describes your confidence about answering the question using the following guidelines.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

**Know** I can answer this question correctly.

**Afraid** of making a simple mistake, but I know how to answer the question.

**Unsure** of how to answer the question though I have some ideas about how to answer.

**No idea** how to answer this question. If you circle N, go on to the next problem.

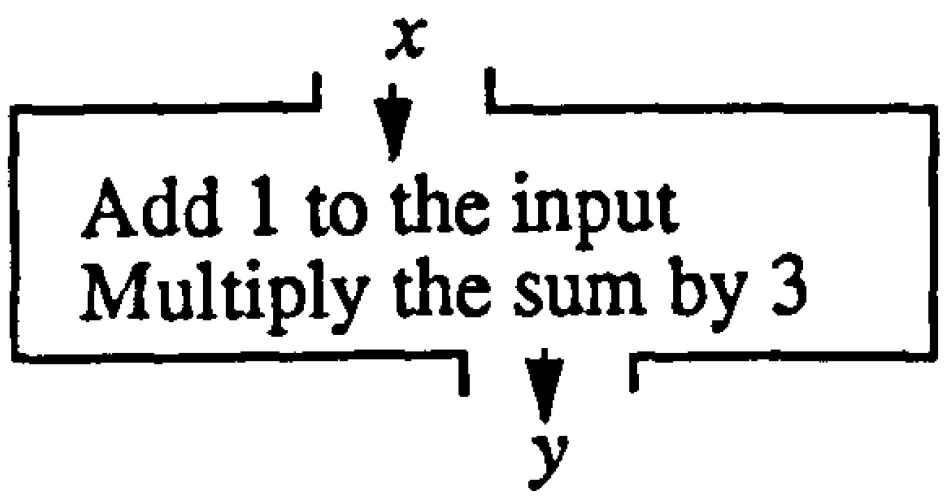
For example, **Afraid** has been checked in the box at the right. Then, answer the question, if you can.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | ✓ | | |

When you see the variable $x$, assume that it stands for **input**.

When you see the variable $y$, assume that it represents **output**.

1. Consider the diagram.




a. What are the output(s) if the input is 7?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. What are the input(s) if the output is 18?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

2. Consider the equation $y = 3x - 7$.

a. What are the output(s) if the input is 5?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. What are the input(s) if the output is 0?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

3. Consider the following table copied from a TI–82 graphics calculator. The "X" stands for $x$ and the "$Y_1$" stands for $y$.

| X | Y1 | |
|---|---|---|
| [illegible] | 12 | |
| -2 | 5 | |
| -1 | 0 | |
| 0 | -3 | |
| 1 | -4 | |
| 2 | -3 | |
| 3 | 0 | |

a. What are the output(s) if the input is –2?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

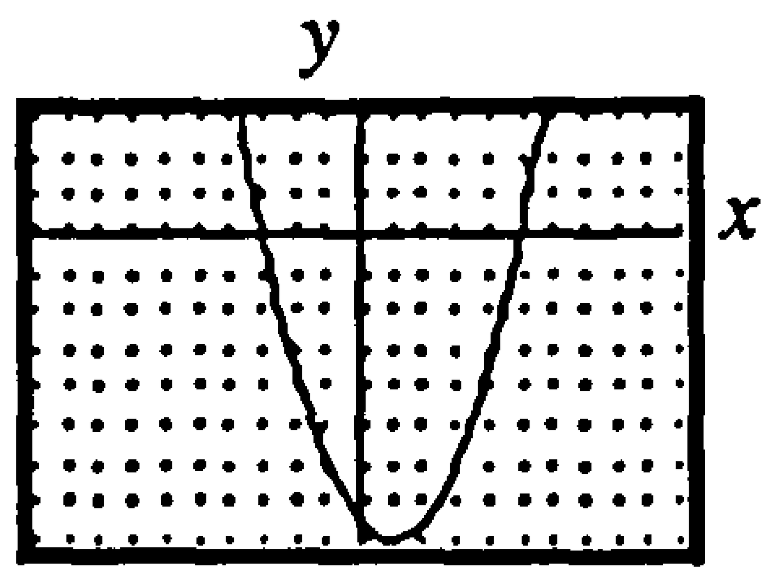
b. What are the input(s) if the output is –3?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

4. Consider the following viewing window and graph copied from a TI–82 graphics calculator.

```
WINDOW FORMAT
Xmin=-10
Xmax=10
Xscl=1
Ymin=-16
Ymax=6
Yscl=2
```




a. What are the output(s) if the input is 3?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. What are the input(s) if the output is 0?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

5. What is a function?

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

6. Briefly state what each of the following words or phrases mean to you.

a. a variable.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. a two-column table of numbers.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

c. a graph.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

d. an equation.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

7. Briefly state what each of the following symbols mean to you.

a. $f(x)$

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. $y(x) = 4$

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

c. $a(b+c)$

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

8. Assume that $f$ is the name of a function. Is there a difference between $3f(2)$ and $2f(3)$ ? If yes, what is the difference?

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

9. Suppose that $f$ is the name of a function and $x$ is the input to that function. Consider the notation $f(x)$ . Check **true** or **false.**

a. $f(x)$ represents the output of the function when $x$ is input.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

True_____ False_____

b. $f(x)$ represents the product of $f$ and $x$.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

True_____ False_____

c. $f(x)$ represents the rule you follow to find the output.

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

True_____ False_____

10. Write the letters of those tables/sets listed below that you believe are functions. What rule did you use to decide if the given table/ set is a function?

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

**Functions:**________________________________

**Rule used:**

a.

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | –6 |
| 2 | 9 |
| –5 | 3 |
| 8 | –6 |

b.

| Input | Output |
|---|---|
| 3 | 5 |
| 4 | 6 |
| 3 | 2 |
| 8 | –1 |
| 2 | 0 |

c.

| Input | Output |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 8 |
| 5 | 10 |

d.

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | 4 |
| 2 | 4 |
| –5 | 4 |
| 8 | 4 |

e. $\{(-1,5),(7,2),(-3,-8),(4,-1)\}$

f. A table has two columns. The left column begins at 0 and increases in increments of 2. The right column begins at 1. Each entry in the right column is computed by multiplying the preceding entry by 3. Part of the table appears below.

| Input | Output |
|---|---|
| 0 | 1 |
| 2 | 3 |
| 4 | 9 |
| 6 | 27 |
| 8 | 81 |

11. Write the letters of those equations listed below that you believe are functions. What rule did you use to decide if the given equation is a function?

| Know | Afraid | Unsure | No Idea |
|---|---|---|---|
| | | | |

**Functions:**______________________________

**Rule used:**

a. $y = 3x - 2$

b. $y = 9 - x^2$

c. $y = 5$

d. $x^2 + y^2 = 1$

e. $y = \begin{cases} 1 & \text{if } x < -3 \\ x^2 & \text{if } x \geq -3 \text{ and } x < 4 \\ 2 & \text{if } x \geq 4 \end{cases}$

f. $y = \pm\sqrt{x+2}$

g. If $x$ is rational, then $y = 0$

h. $xy = 7$

i. $y = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is not rational} \end{cases}$

j. $x = 2 + t$ and $y = 3t^2 - 5t + 1$

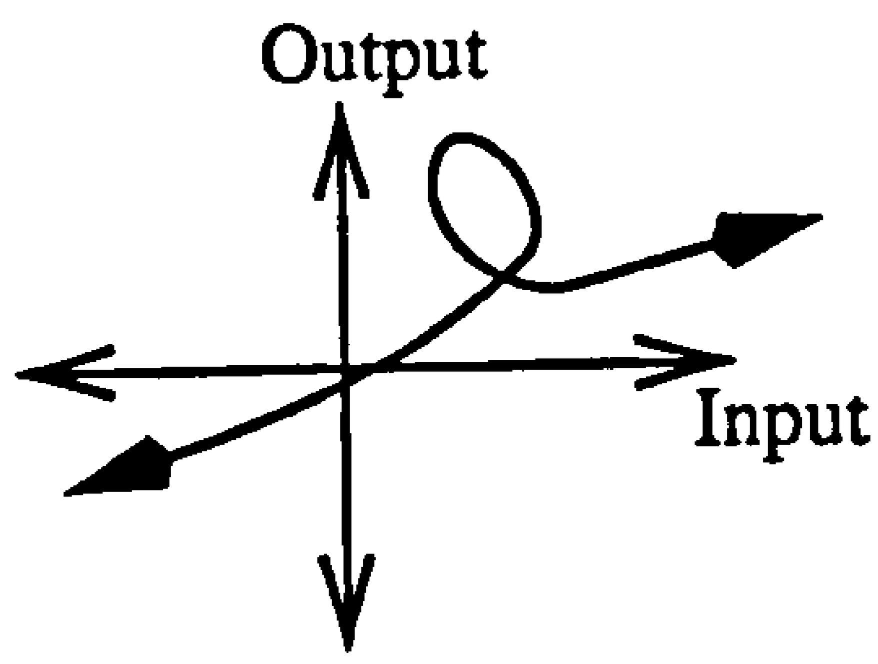
k. $x = 4$

12. Write the letters of those graphs listed below that you believe are functions. What rule did youusetodecideifthegivenequationisafunction?

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

**Functions:**____________________________________

**Rule used:**

a.
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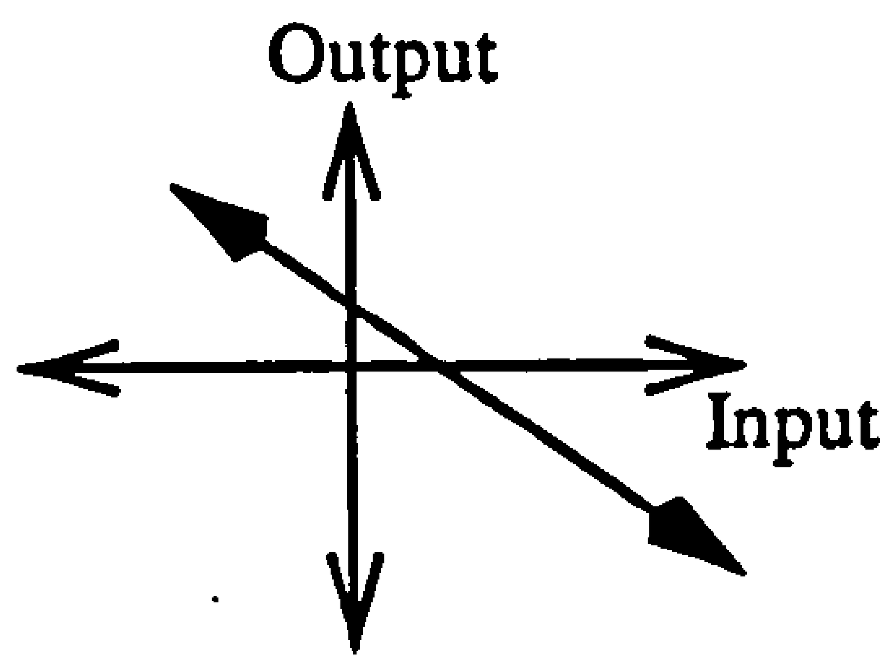


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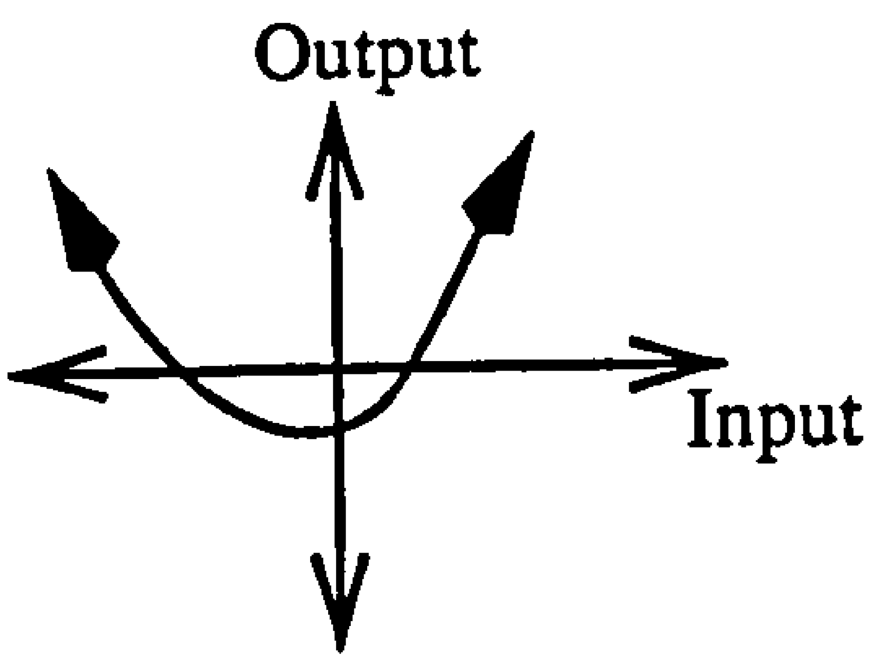


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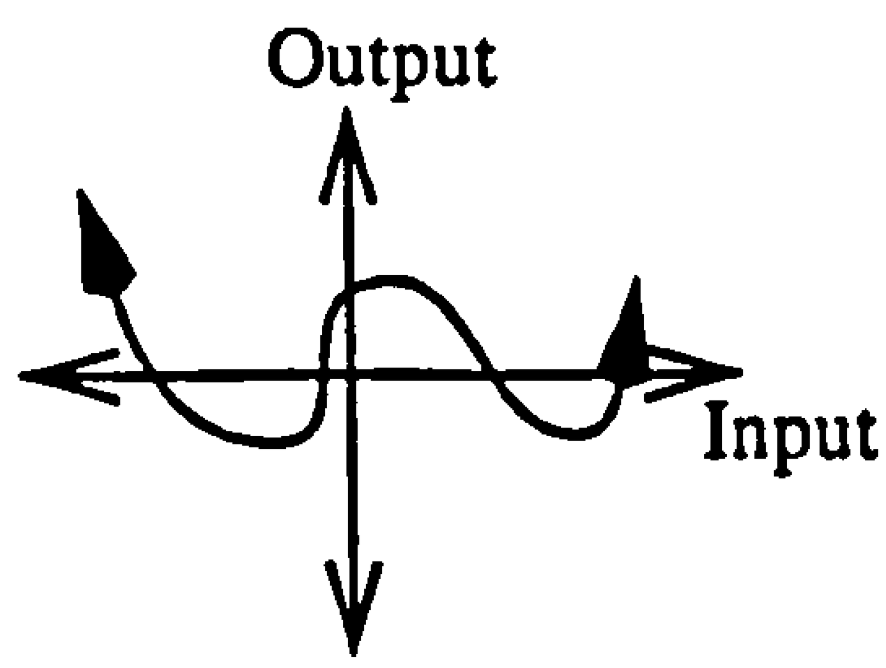


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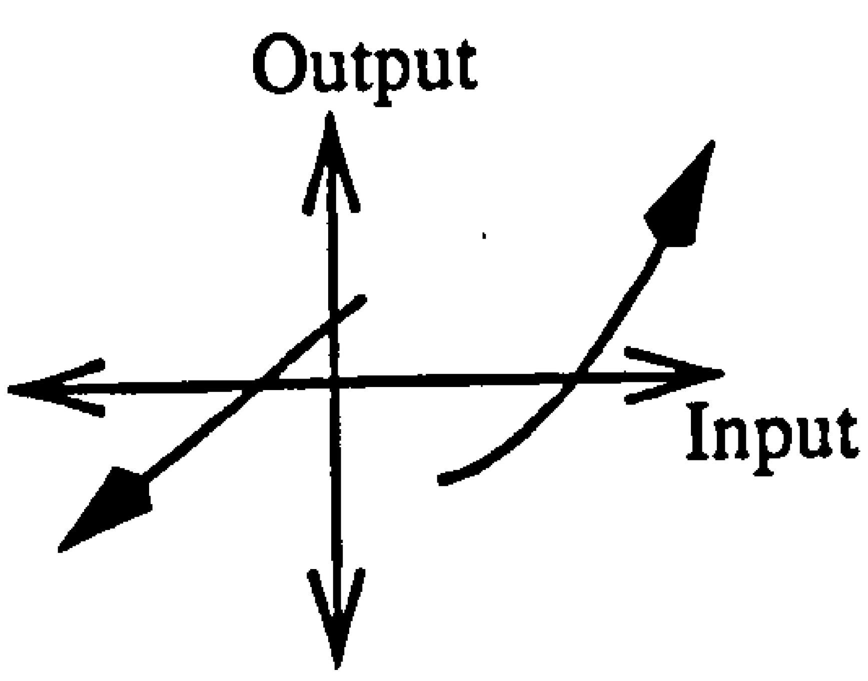


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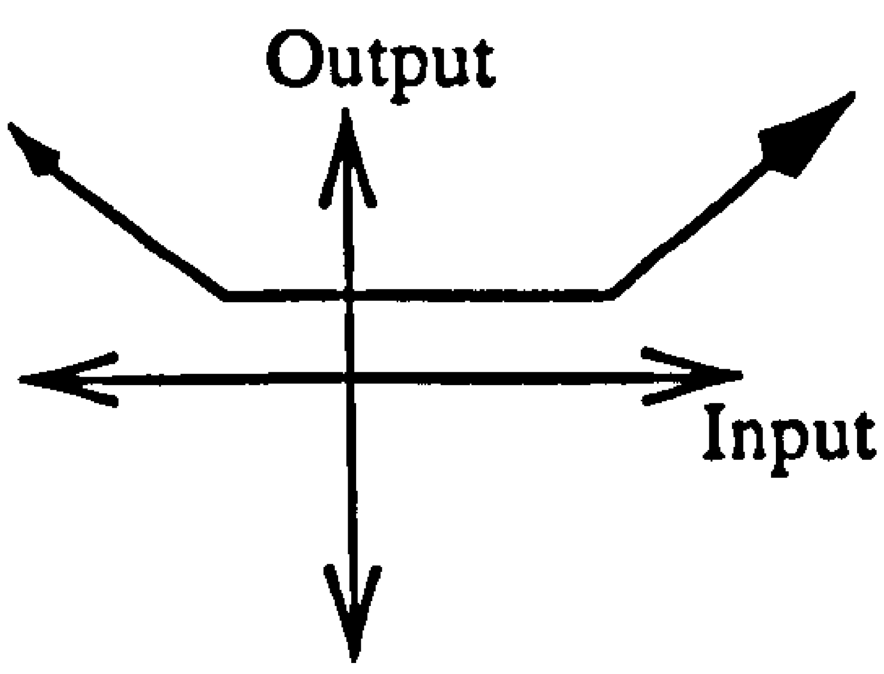


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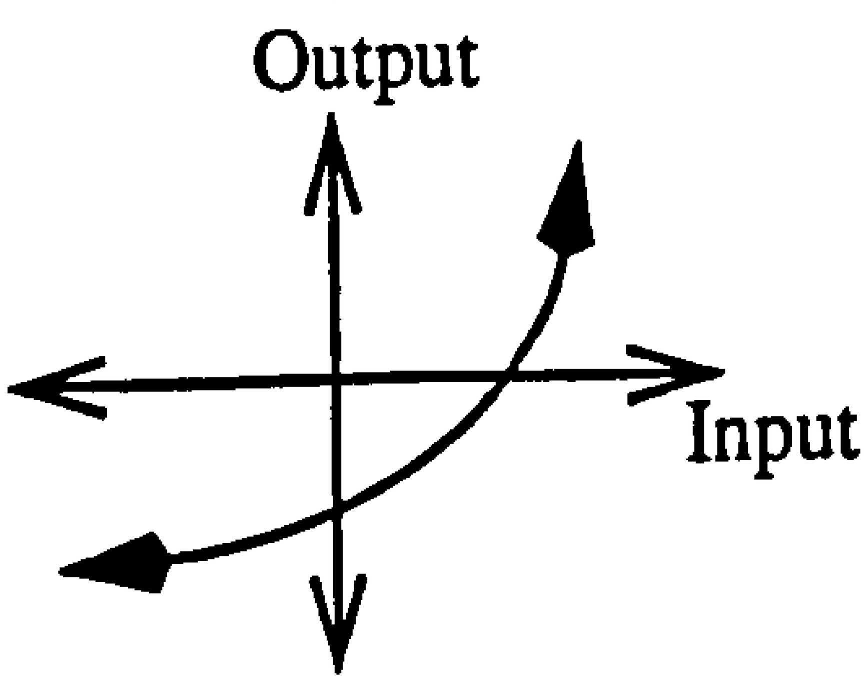


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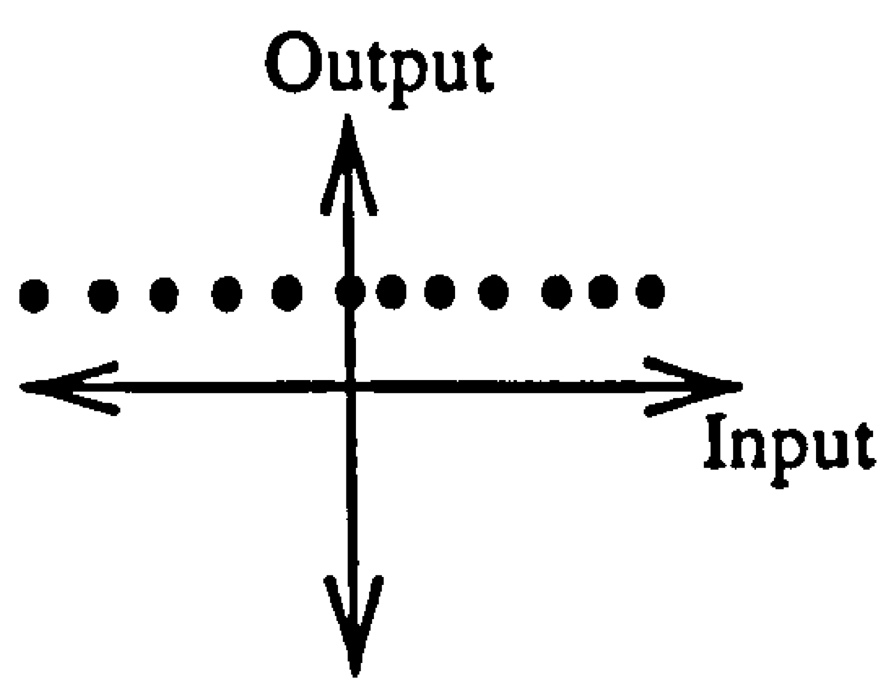


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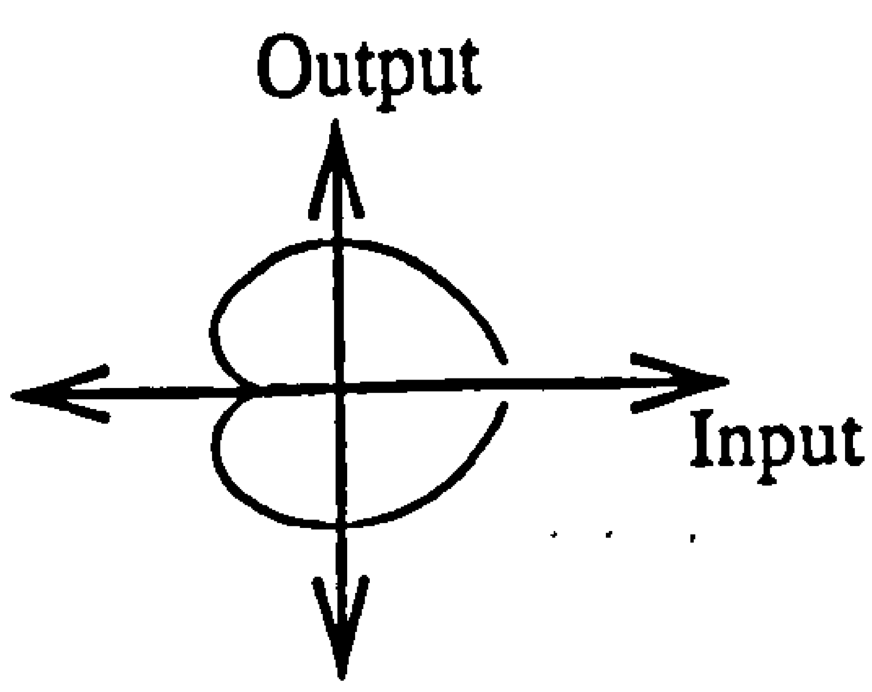
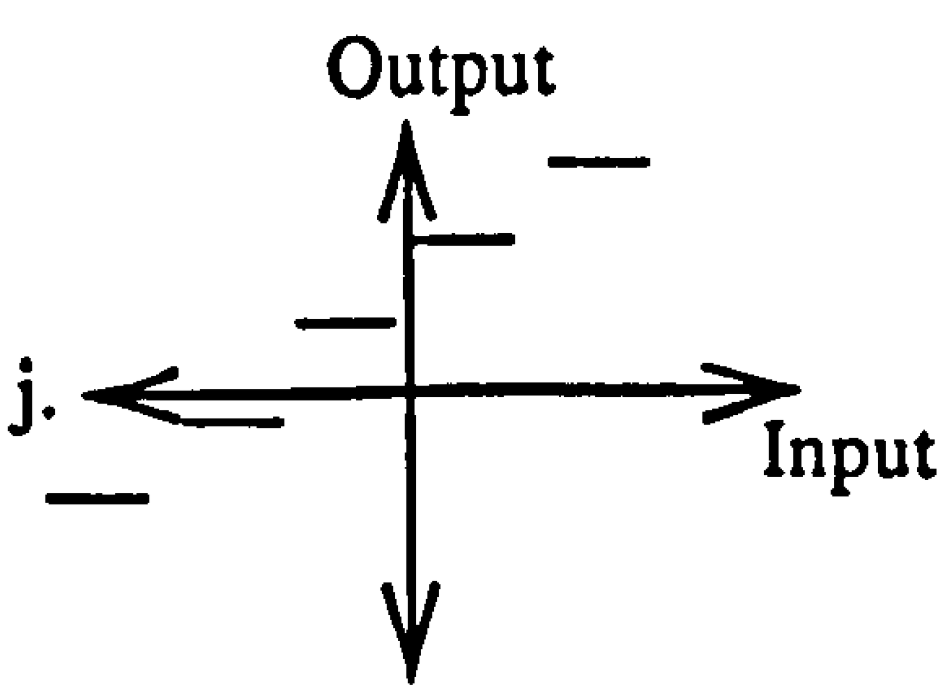
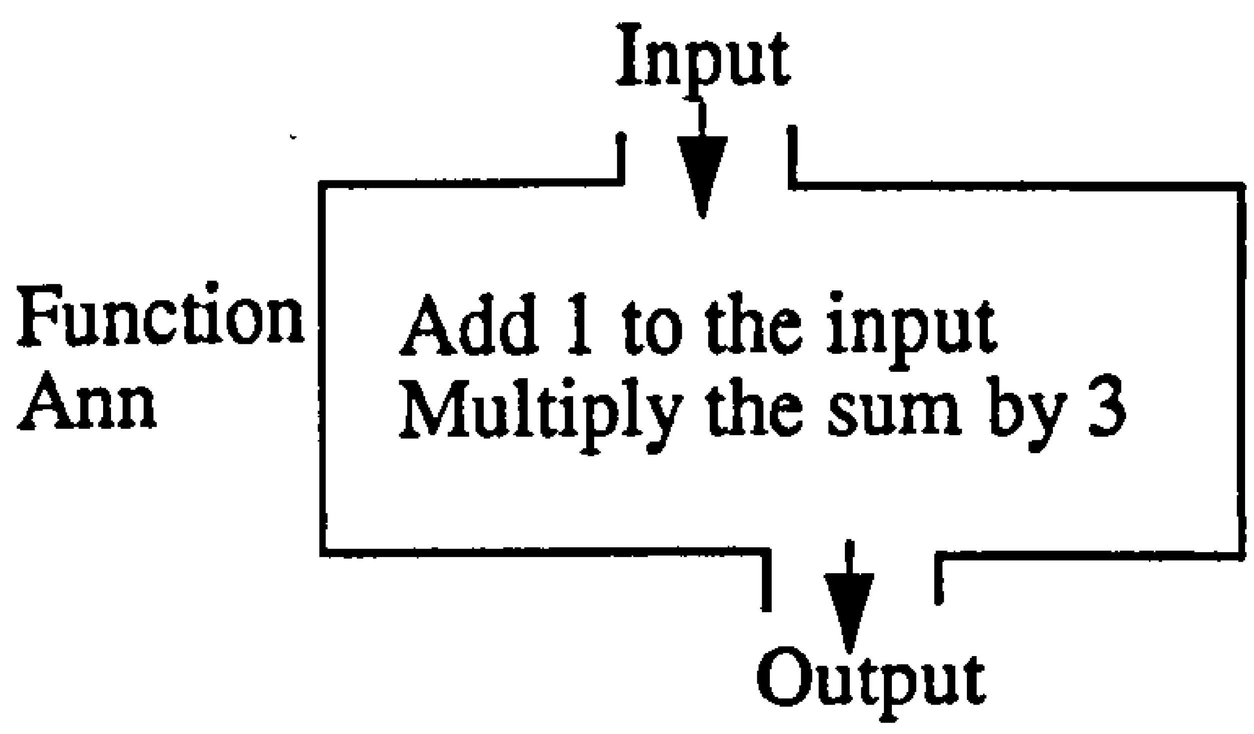
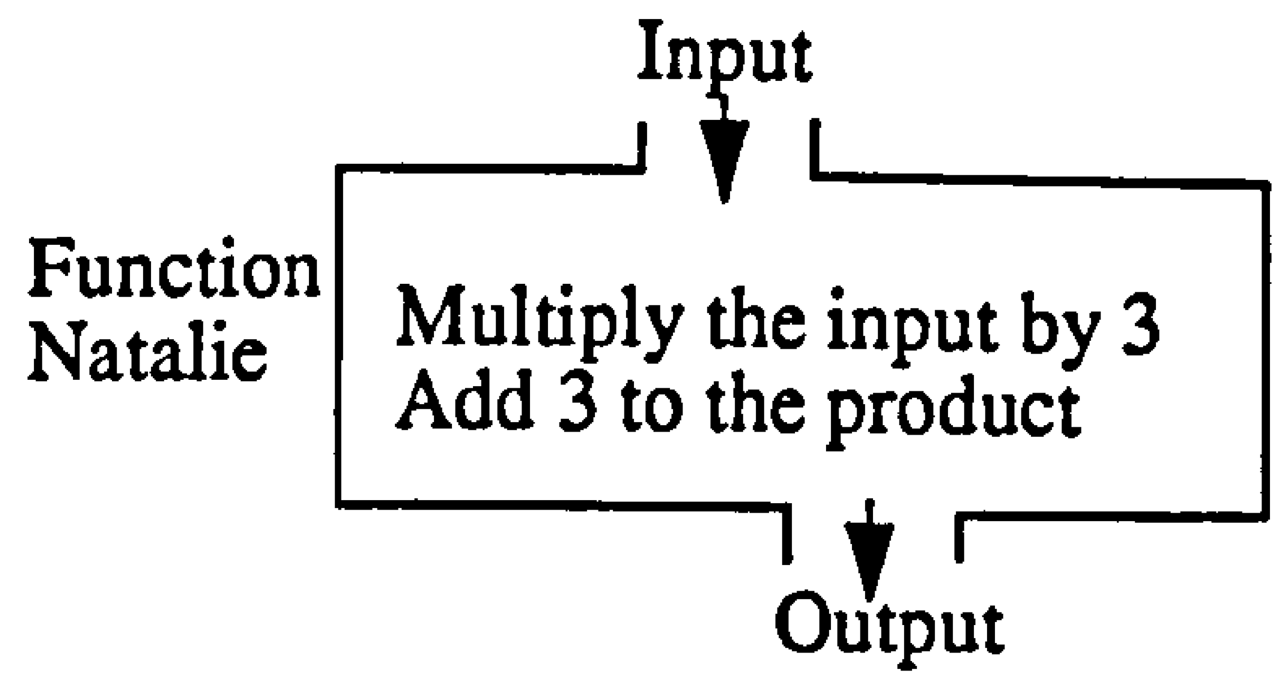



j.


13. Consider the function machines for function Ann and function Natalie.




a. What is the output of function Ann if the input is 7?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

b. What is the output of function Natalie if the input is 7?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

c. What is the output of function Ann if the input is $x$?

Answer:________

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

d. What is the output of function Natalie if the input is $x$?

Answer:_______

| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

e. Do you consider Ann and Natalie the same function? Why or why not?

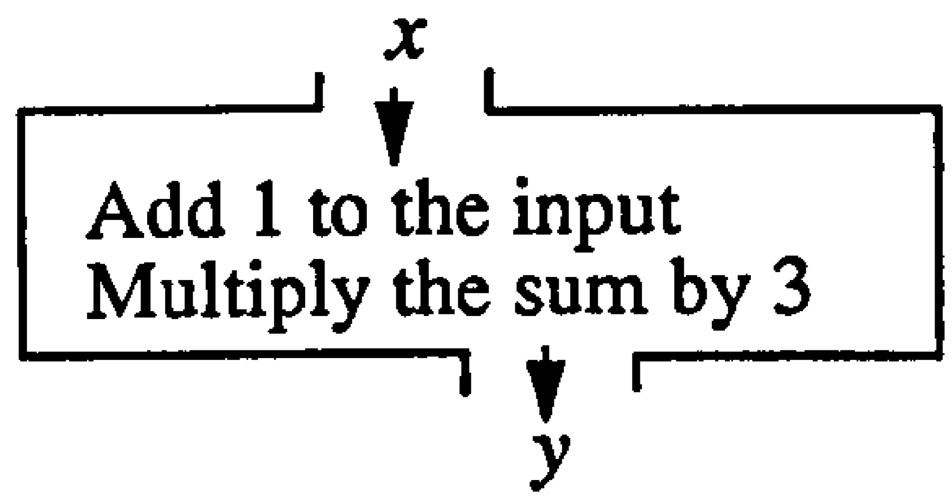
| Know | Afraid | Unsure | No idea |
|---|---|---|---|
| | | | |

**Thank you for your cooperation.**

**I hope the course was a valuable experience for you!**

# Function Questions for Student Interview

1. Consider the diagram.




   a. What are the output(s) if the input is 7?

   b. What are the input(s) if the output is 18?

2. Consider the equation $y = 3x - 7$.

   a. What are the output(s) if the input is 5?

   b. What are the input(s) if the output is 0?

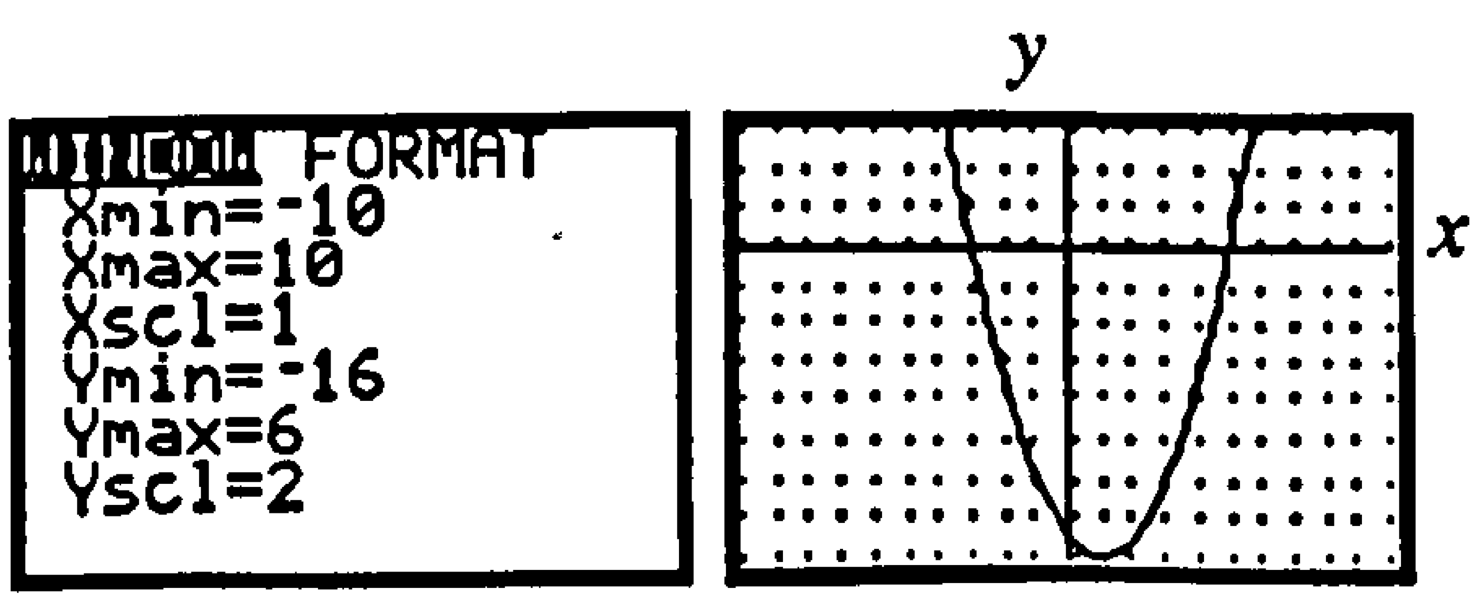
3. Consider the following table copied from a TI–82 graphics calculator. The "X" stands for $x$ and the "$Y_1$" stands for $y$.

| X | $Y_1$ | |
|---|---|---|
| [illegible] | 12 | |
| -2 | 5 | |
| -1 | 0 | |
| 0 | -3 | |
| 1 | -4 | |
| 2 | -3 | |
| 3 | 0 | |

   a. What are the output(s) if the input is –2?

   b. What are the input(s) if the output is –3?

4. Consider the following viewing window and graph copied from a TI–82 graphics calculator.




   a. What are the output(s) if the input is 3?

   b. What are the input(s) if the output is 0?

5. Complete my sentence: A function is ....

6. Briefly state what each of the following symbols mean to you.

   a. $f(x)$. Do you identify a process with the notation? Do you identify an output with the notation?

   b. $y(x) = 4$. do you identify the statement with a "process", do you identify the statement with an "output, or do you identify the statement with both a "process" and an "output?"

   c. $a(b+c)$ Could this ever mean function notation?

7. Which tables/sets listed below that you believe are functions. Why?

a.

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | –6 |
| 2 | 9 |
| –5 | 3 |
| 8 | –6 |

b.

| Input | Output |
|---|---|
| 3 | 5 |
| 4 | 6 |
| 3 | 2 |
| 8 | –1 |
| 2 | 0 |

c.

| Input | Output |
|---|---|
| 1 | 2 |
| 2 | 4 |
| 3 | 6 |
| 4 | 8 |
| 5 | 10 |

d.

| Input | Output |
|---|---|
| 3 | 4 |
| 7 | 4 |
| 2 | 4 |
| –5 | 4 |
| 8 | 4 |

e. $\{(-1,5),(7,2),(-3,-8),(4,-1)\}$

f. A table has two columns. The left column begins at 0 and increases in increments of 2. The right column begins at 1. Each entry in the right column is computed by multiplying the preceding entry by 3. Part of the table appears below.

| Input | Output |
|---|---|
| 0 | 1 |
| 2 | 3 |
| 4 | 9 |
| 6 | 27 |
| 8 | 81 |

8. Which equations listed below that you believe are functions. Why?

a. $y = 3x - 2$

b. $y = 9 - x^2$

c. $y = 5$

d. $x^2 + y^2 = 1$

e. $y = \begin{cases} 1 & \text{if } x < -3 \\ x^2 & \text{if } x \geq -3 \text{ and } x < 4 \\ 2 & \text{if } x \geq 4 \end{cases}$

f. $y = \pm\sqrt{x+2}$

g. If $x$ is rational, then $y = 0$

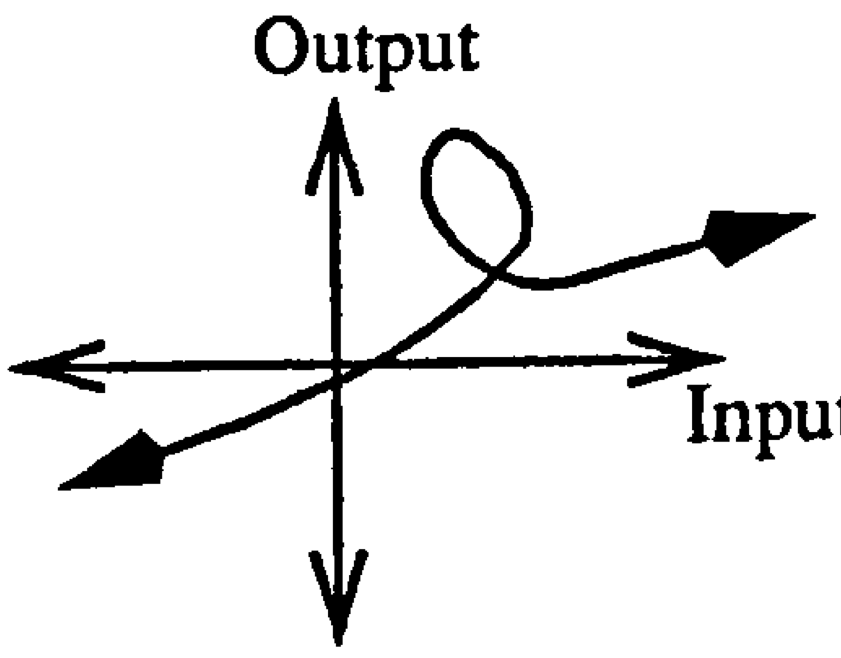
h. $xy = 7$

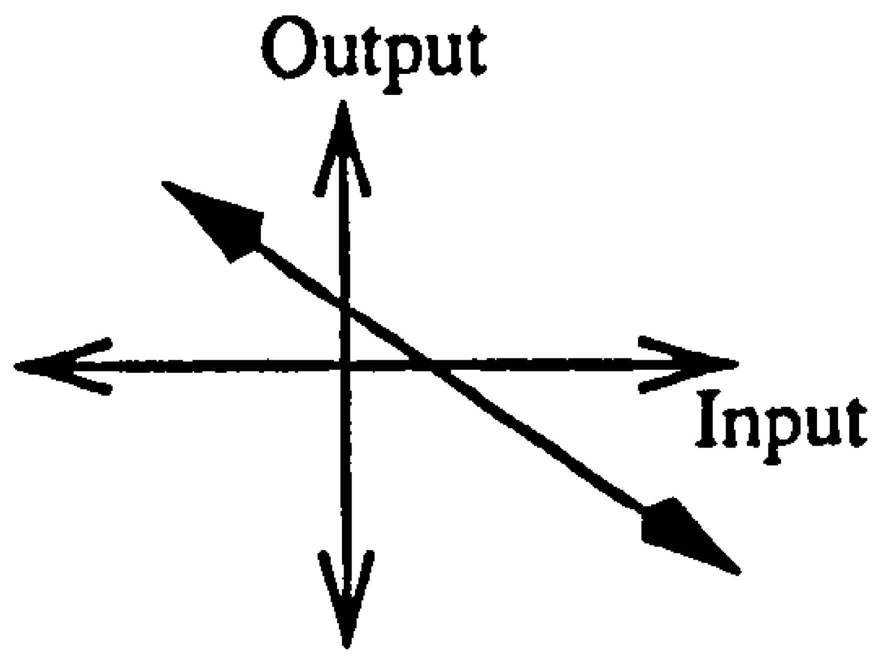
i. $y = \begin{cases} 0 \text{ if } x \text{ is rational} \\ 1 \text{ if } x \text{ is not rational} \end{cases}$

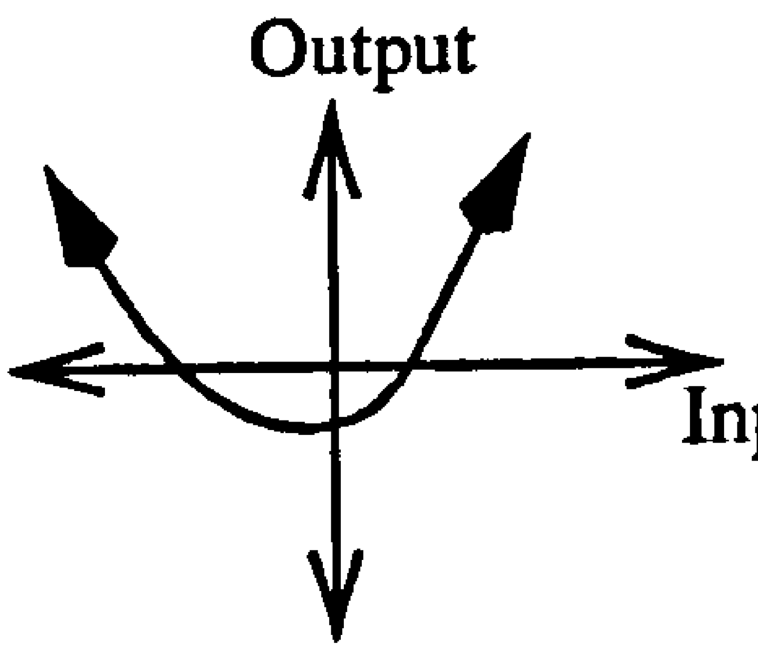
j. $x = 2 + t$ and $y = 3t^2 - 5t + 1$

k. $x = 4$

9. Which graphs listed below that you believe are functions. Why?

a.



b.



c.
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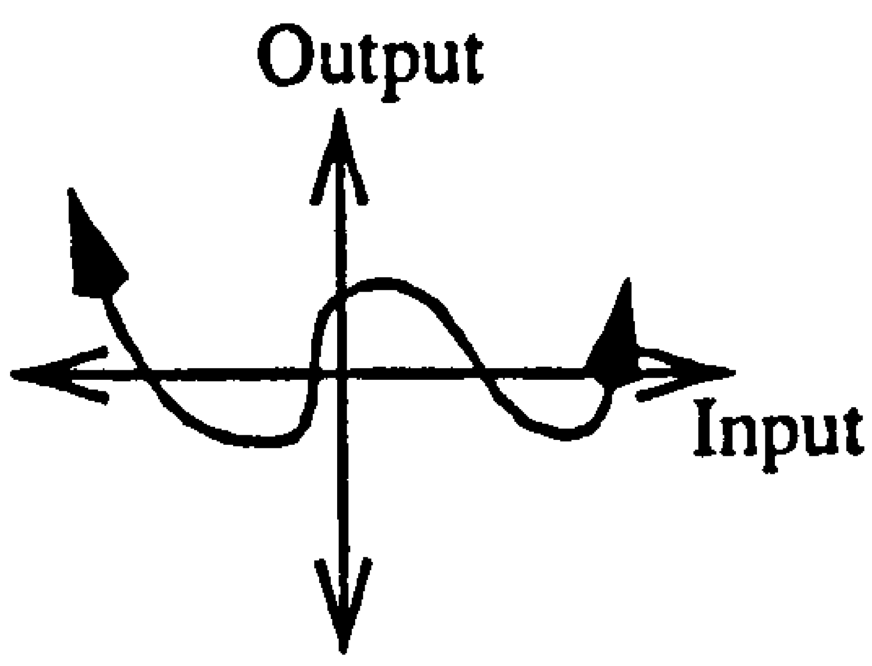
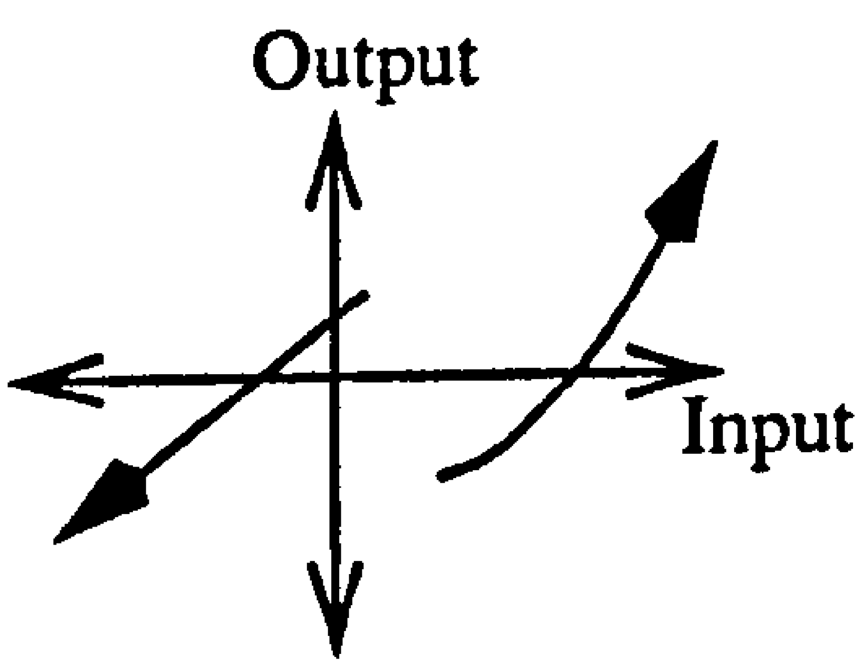
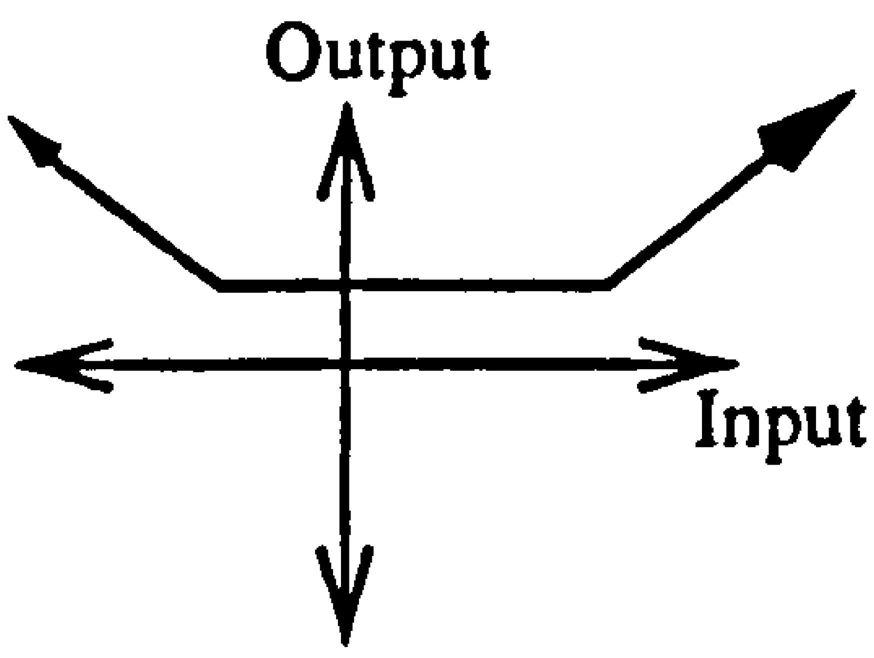
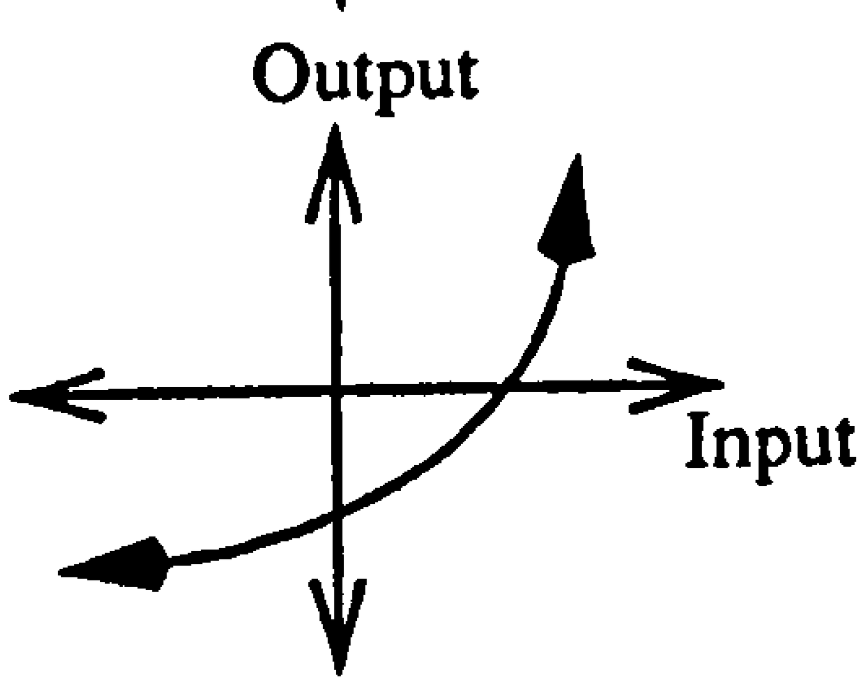
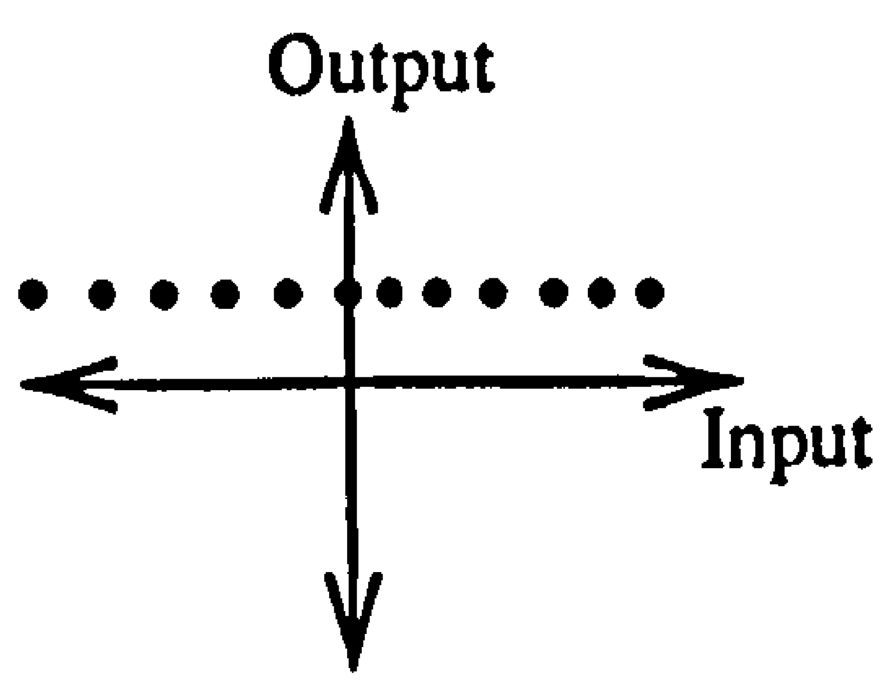
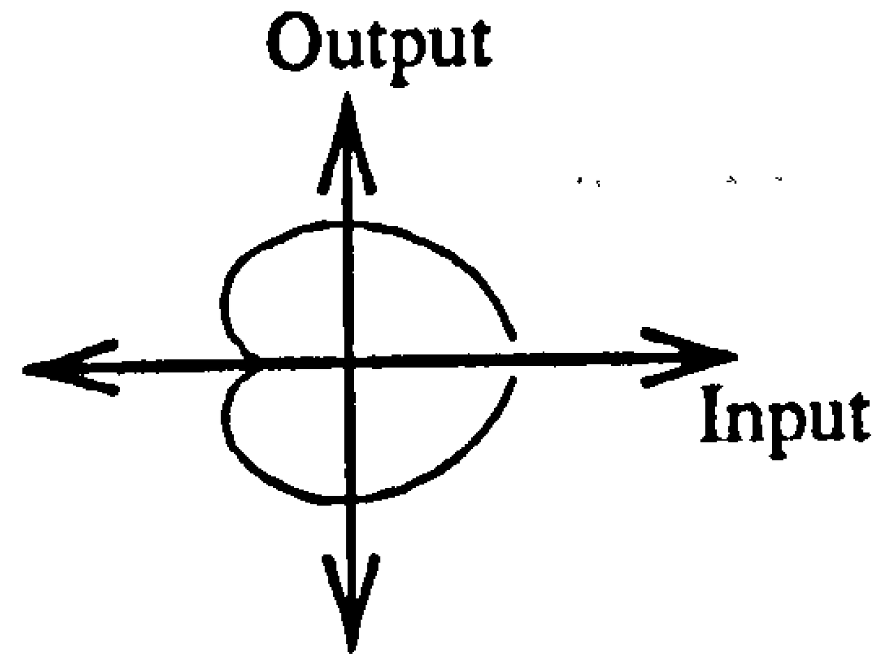


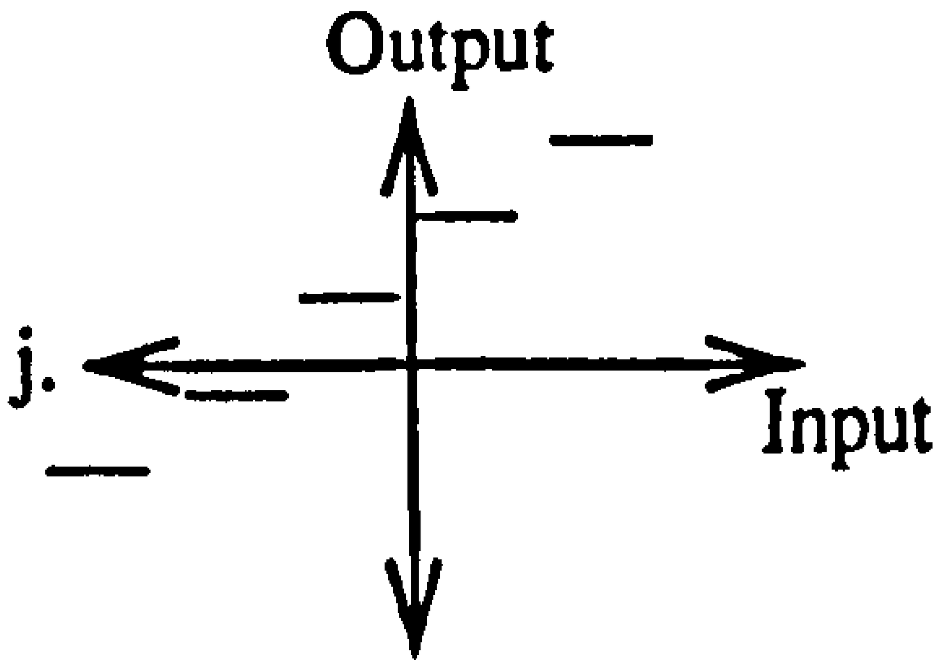

e.



f.



g.



h.



i.



j.



10. Suppose you were to make a list pairing each student's first name with that student's score on a test. Could that pairing represent a function? Why or why not? (Norman, in Harel & Dubinsky, p. 232)

11. A catepillar is crawling around on a piece of coordinate graph paper like so. If we were to determine the creatures location on the paper with repsect to time, would location be a function of time? Why or why not? (ibid).

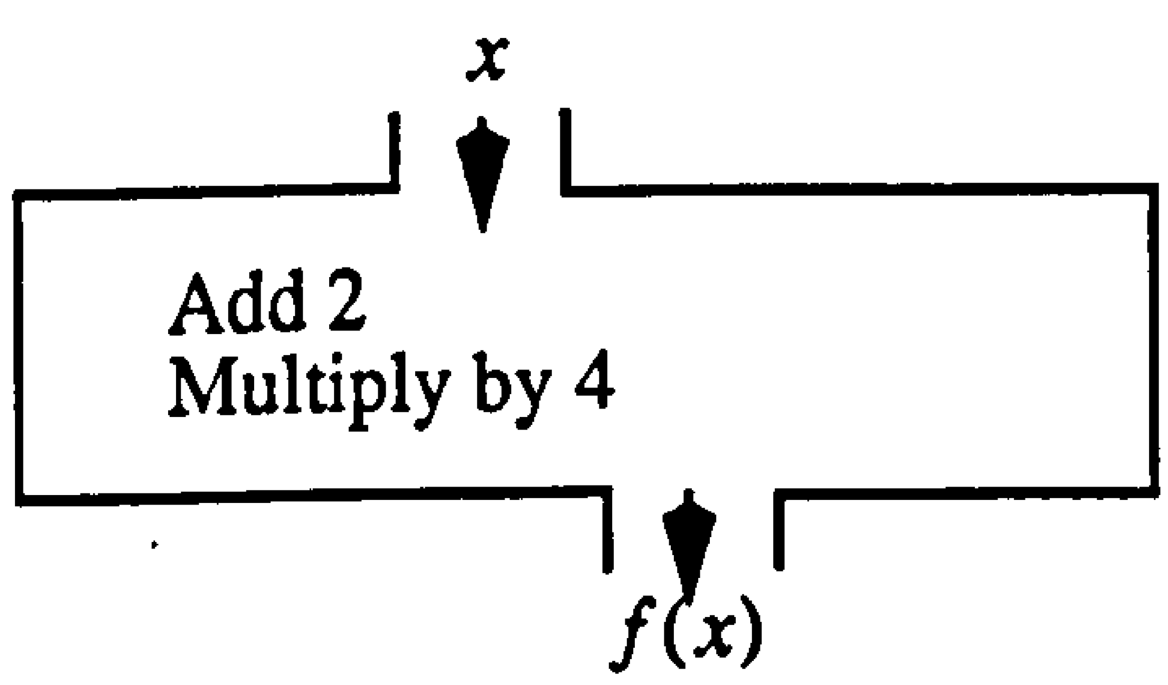
12. Assume that $f$ is the name of a function. Is there a difference between $3f(2)$ and $2f(3)$ ? Explain. Relate to each of the following.

a.

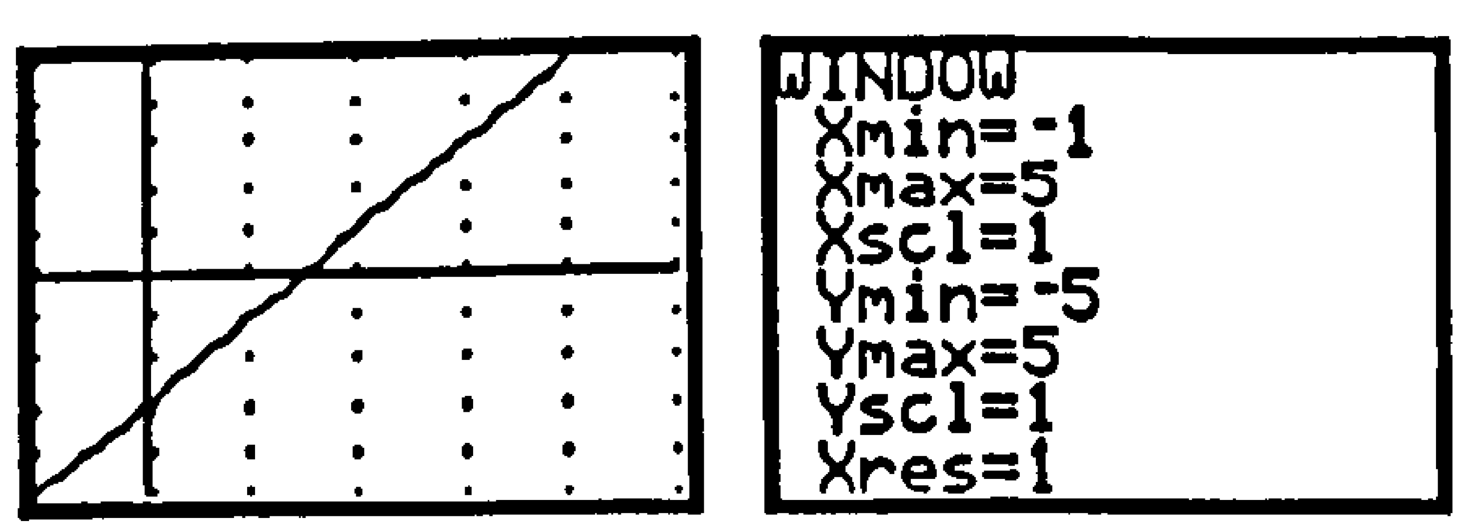
| $x$ | $f(x)$ |
|---|---|
| 0 | 7 |
| 1 | 15 |
| 2 | 5 |
| 3 | 4 |
| 4 | 9 |

b. $f(x) = x+2$

c.



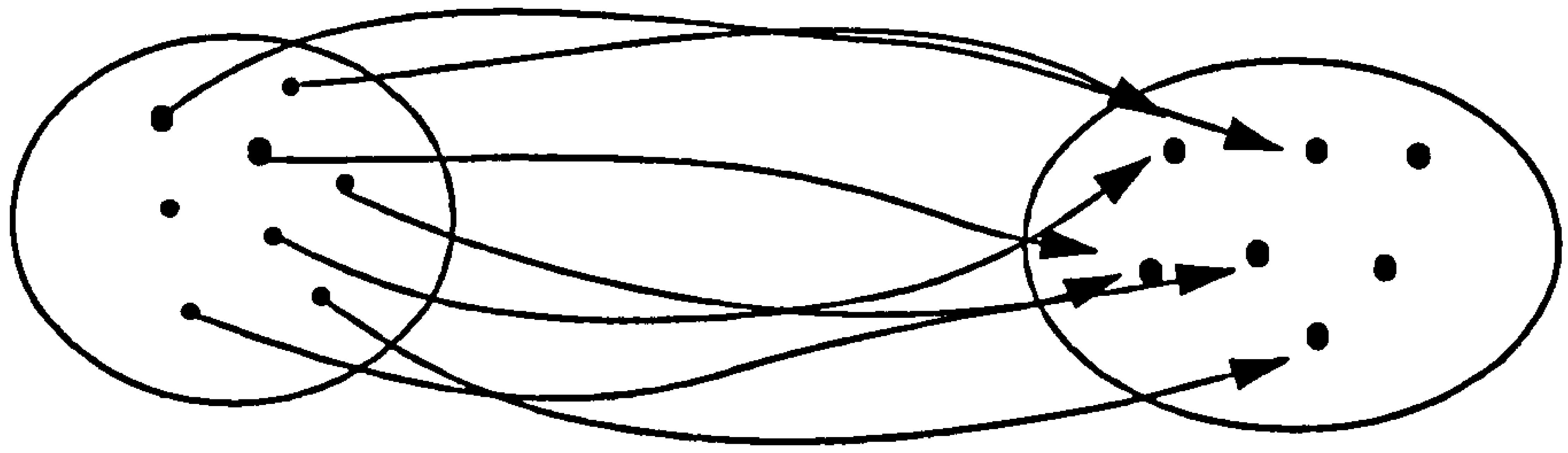

d.




13. Can you demonstrate what a function is through actions rather than words. Act out a physical motion that demonstrates what a function means to you.

14. A friend comes to you with her definition of function. Is her definition acceptable? Why or why not? How does it "fit" with your definition?

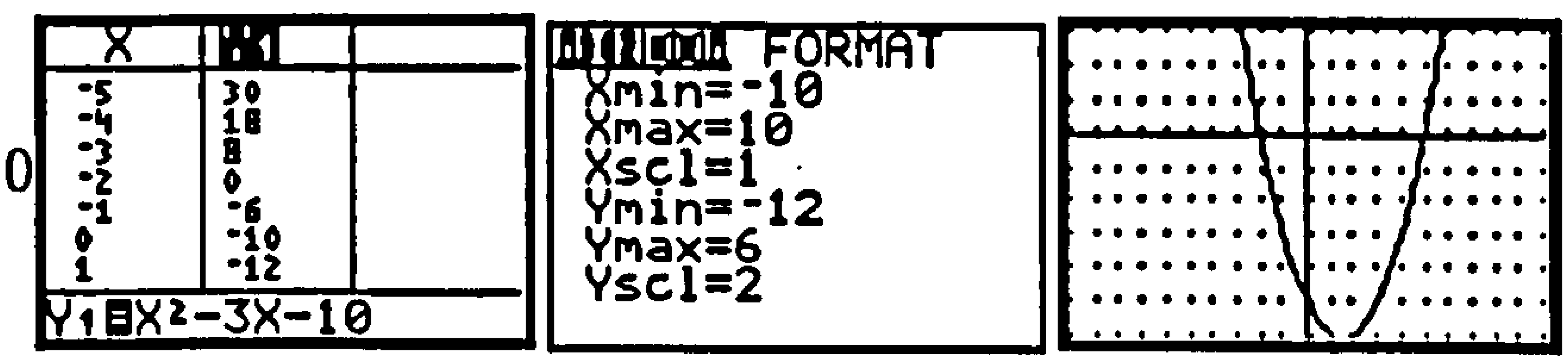
a. Sue's definition: A **function** is a **correspondence** that assigns to each element of one set one and only one element of a second set. A diagram appears below.



b. Gail's definition: A **function** is a set of ordered pairs $(a, b)$ in which for each value of $a$ in the domain of the function, there is one and only one value of $b$ in the range of the function.

15. An equation, a table, and a graph are displayed below for the same function.

$y(x) = x^2 - 3x - 10$




a. What is the output if the input is –1? Did you use the equation, the table, or the graph to answer the question?

b. What is the output if the input is –5? Did you use the equation, the table, or the graph to answer the question?

c. What is the output if the input is 4? Did you use the equation, the table, or the graph to answer the question?

d. What is the output if the input is 12? Did you use the equation, the table, or the graph to answer the question?

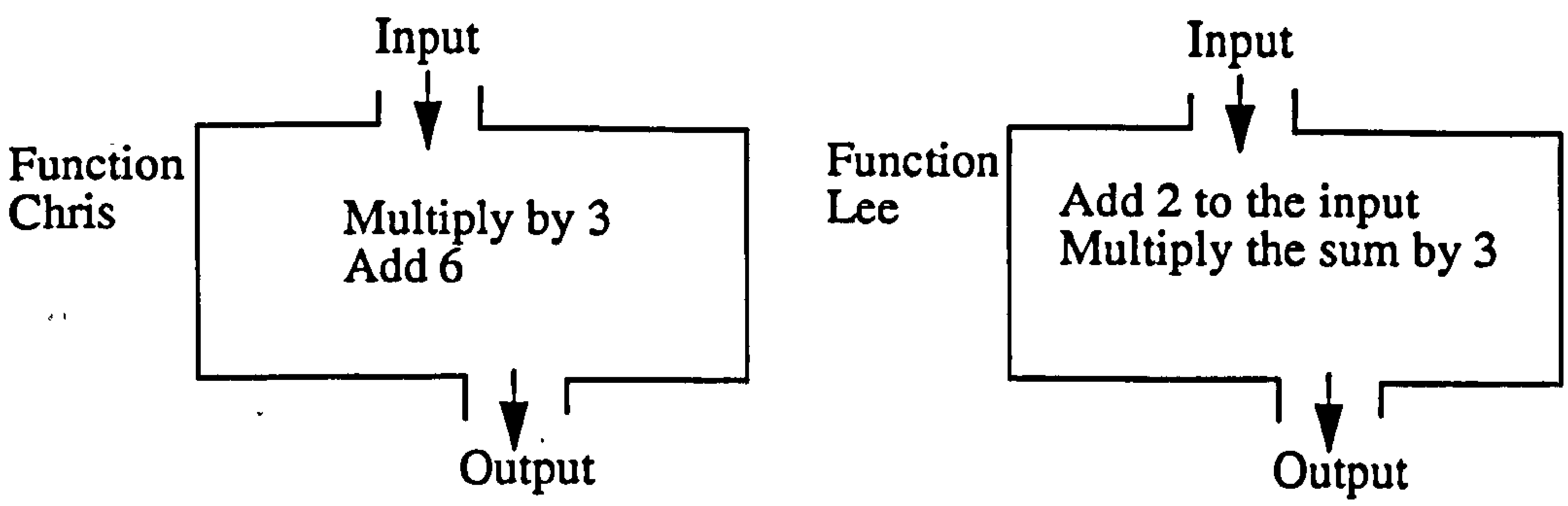
e. What is the output if the input is $h$? Did you use the equation, the table, or the graph to answer the question?

f. What are the input(s) if the output is 0? Did you use the equation, the table, or the graph to answer the question?

g. What are the input(s) if the output is 44? Did you use the equation, the table, or the graph to answer the question?

16. a. In your opinion, what does it mean to say that two functions are the same?

b. The function machines for two functions appear below. Are the two functions the same? Why or why not?




c. Write the algebraic form for each of these functions. Are the two functions the same? Why or why not?

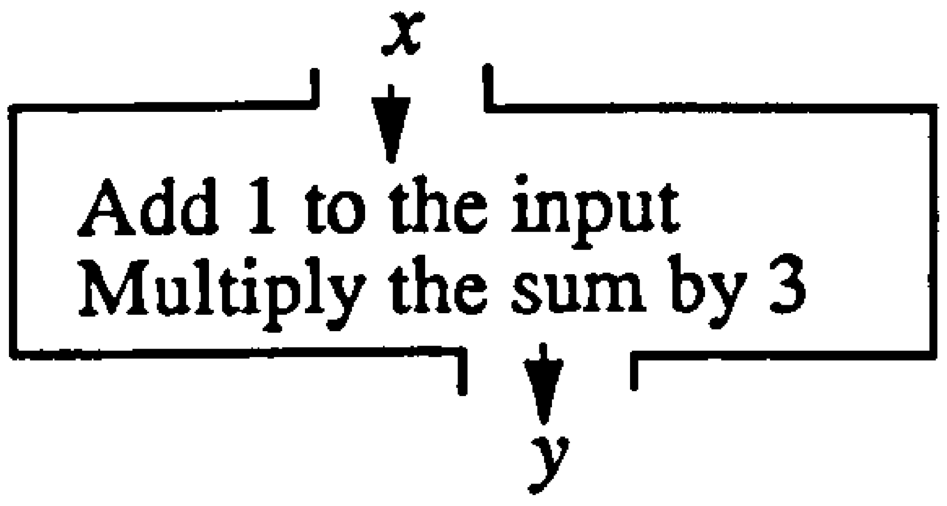
17. Given the function $y(x) = 4x + 5$, create a table, a graph, and a function machine. Describe the function verbally.

18. Given the table,

| $x$ | $y$ |
|---|---|
| 0 | –3 |
| 1 | –1 |
| 2 | 1 |
| 3 | 3 |
| 4 | 5 |
| 5 | 7 |
| 6 | 9 |
| 7 | 11 |

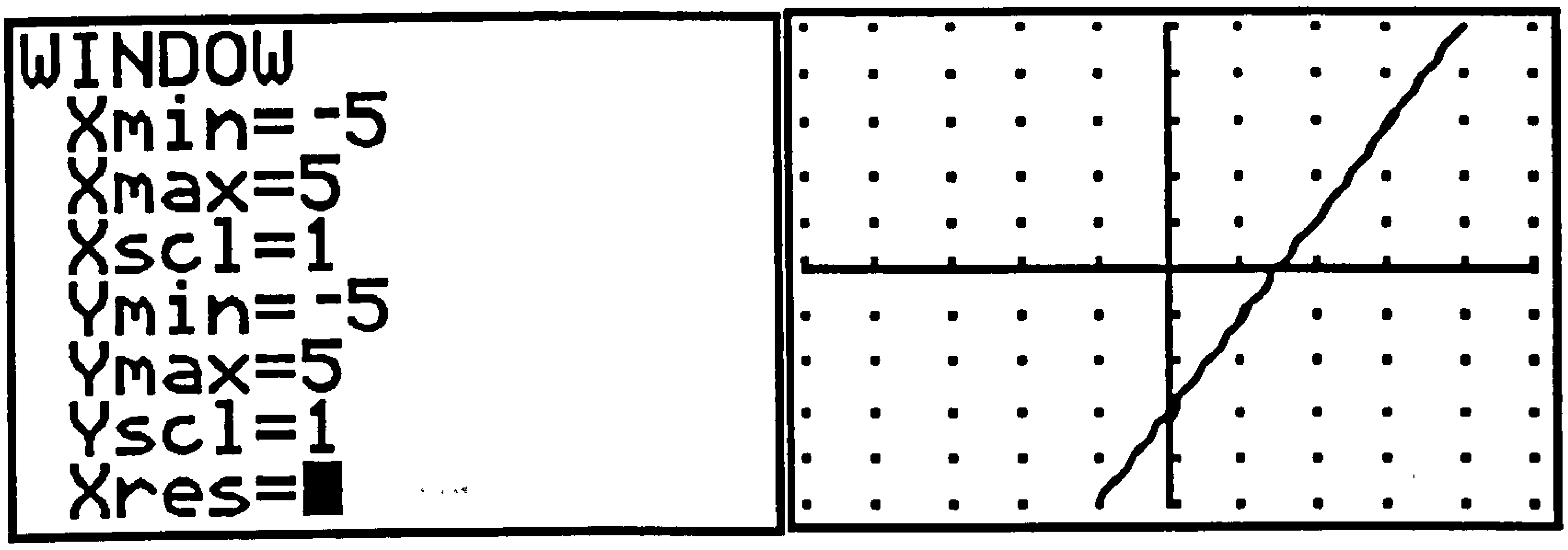
create an equation, a graph, and a function machine. Describe the function verbally.

19. Given the function machine,




create a table, a graph, and an equation. Describe the function verbally.

20. Given the graph,




create a table, an equation, and a function machine. Describe the function verbally.

21. Consider the following tables for functions $f$ and $g$.

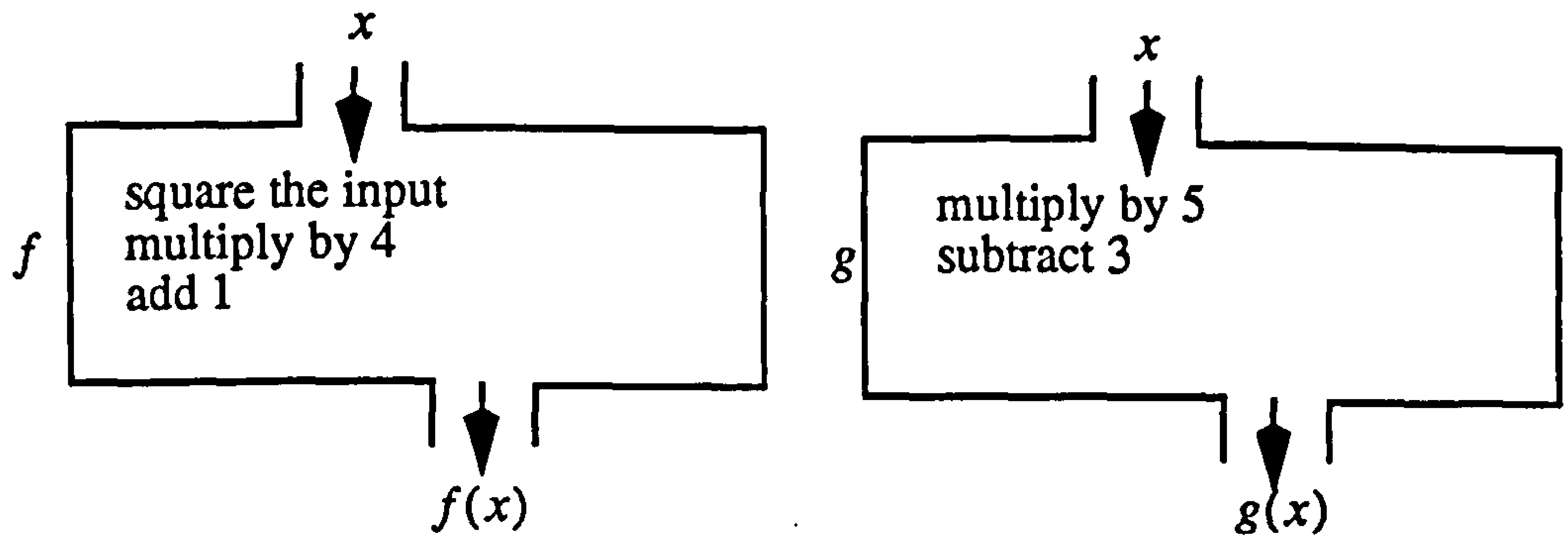
| $x$ | $f(x)$ |
|---|---|
| 1 | 3 |
| 2 | –1 |
| 3 | 1 |
| 4 | 0 |
| 5 | –2 |

| $x$ | $g(x)$ |
|---|---|
| –2 | 3 |
| –1 | 1 |
| 0 | 5 |
| 1 | 2 |
| 2 | 4 |

a. What is the value of $f(g(2))$ ? Why?

b. What is the value of $g(f(2))$ ? Why?

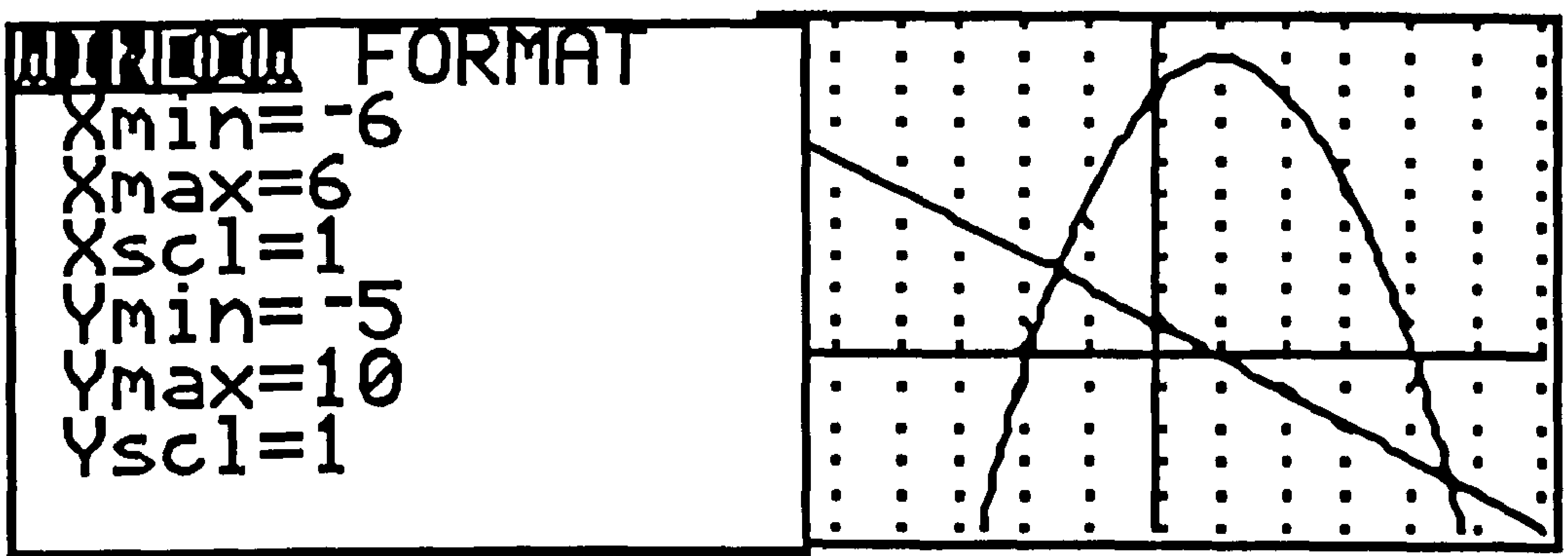
22. Consider the following function machines for functions *f* and *g*.




a. What is the value of $f(g(2))$ ? Why?

b. What is the value of $g(f(2))$ ? Why?

23. Consider the following graphs for functions *f* and *g*. The graph of *f* is the line. The graph of *g* is the parabola.




*g*

a. Approximate the value of $f(-3)$ .

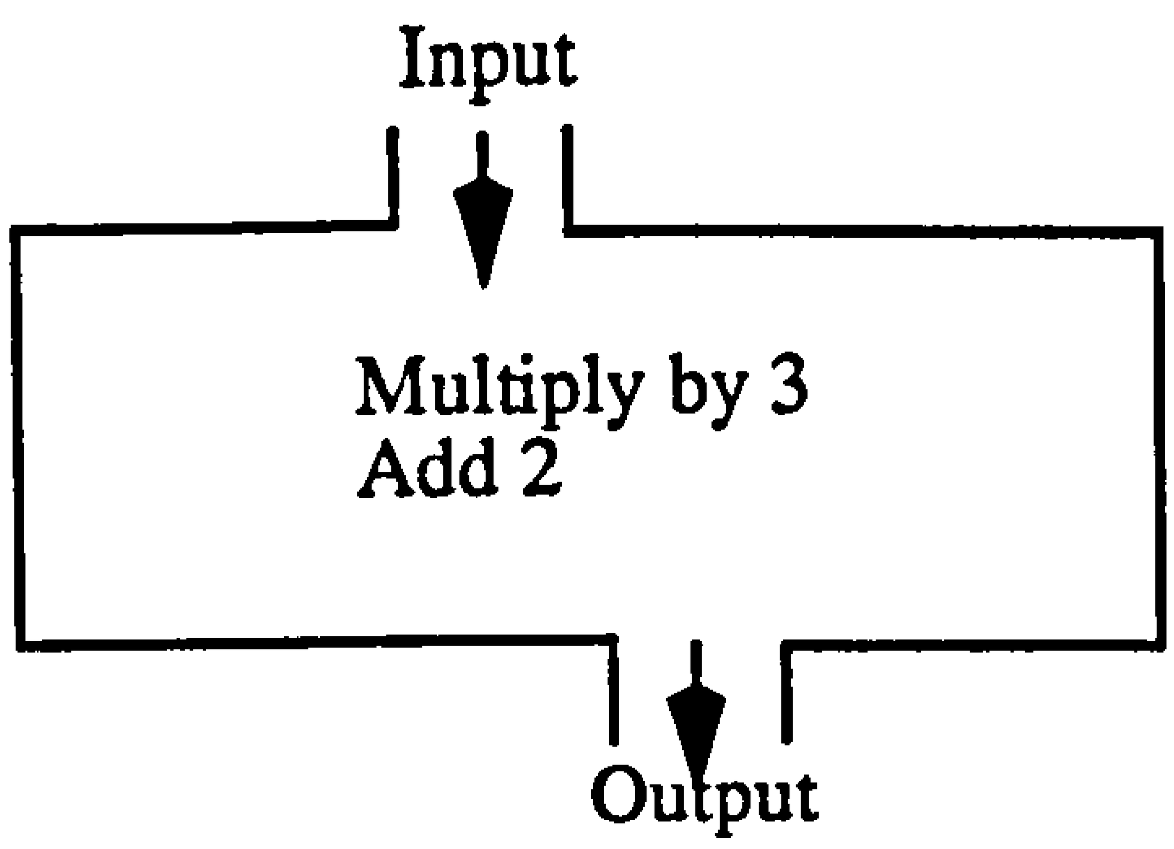
b. Approximate the value of $g(1)$ .

c. Approximate the value of $g(f(2))$ . Describe what you did.

24. Consider functions *f* and *g* defined as $f(x) = 3x-5$ and $g(x) = x^2+1$ .

What is $g(f(3))$ ? Describe what you did.

25. Consider the following function machine.

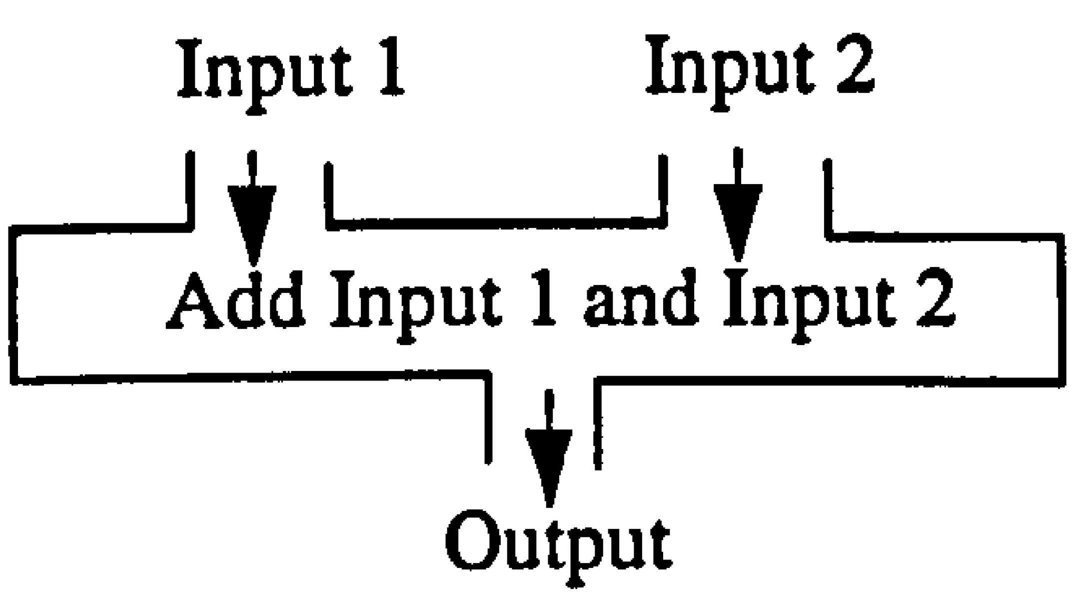



a. What is the output if the input is 5? What did you do?

b. What is the input if the output is 5? What did you do?

c. What is the output if the input is $y(x) = x^2 - 5x$ ? What did you do?

26. Consider the following function machine.




a. What is the output if Input 1 is 7 and Input 2 is 11? What did you do?

b. What is the output if Input 1 is $f(x)$ and Input 2 is $g(x)$ where $f(x) = 3x - 5$ and $g(x) = x^2 + 1$ . What did you do?

27. For each part of this question, *f*, *g*, and *h* represent functions and $h(x) = f(g(x))$ .

a. If only the information in the following table were known, would it be possible to find $h(0)$ ? If so, find it and if not explain why not.

| $x$ | $f(x)$ | $g(x)$ |
|---|---|---|
| –1 | 2 | –3 |
| 0 | –3 | –1 |
| 4 | 1 | 2 |

b. If only the information in the following table were known, would it be possible to find $f(2)$ ? If so, find it and if not explain why not.

| $x$ | $h(x)$ | $g(x)$ |
|---|---|---|
| –1 | 1 | –3 |
| 4 | 7 | 1 |
| 7 | 0 | 2 |

c. If only the information in the following table were known, would it be possible to find $g(4)$ ? If so, find it and if not explain why not.

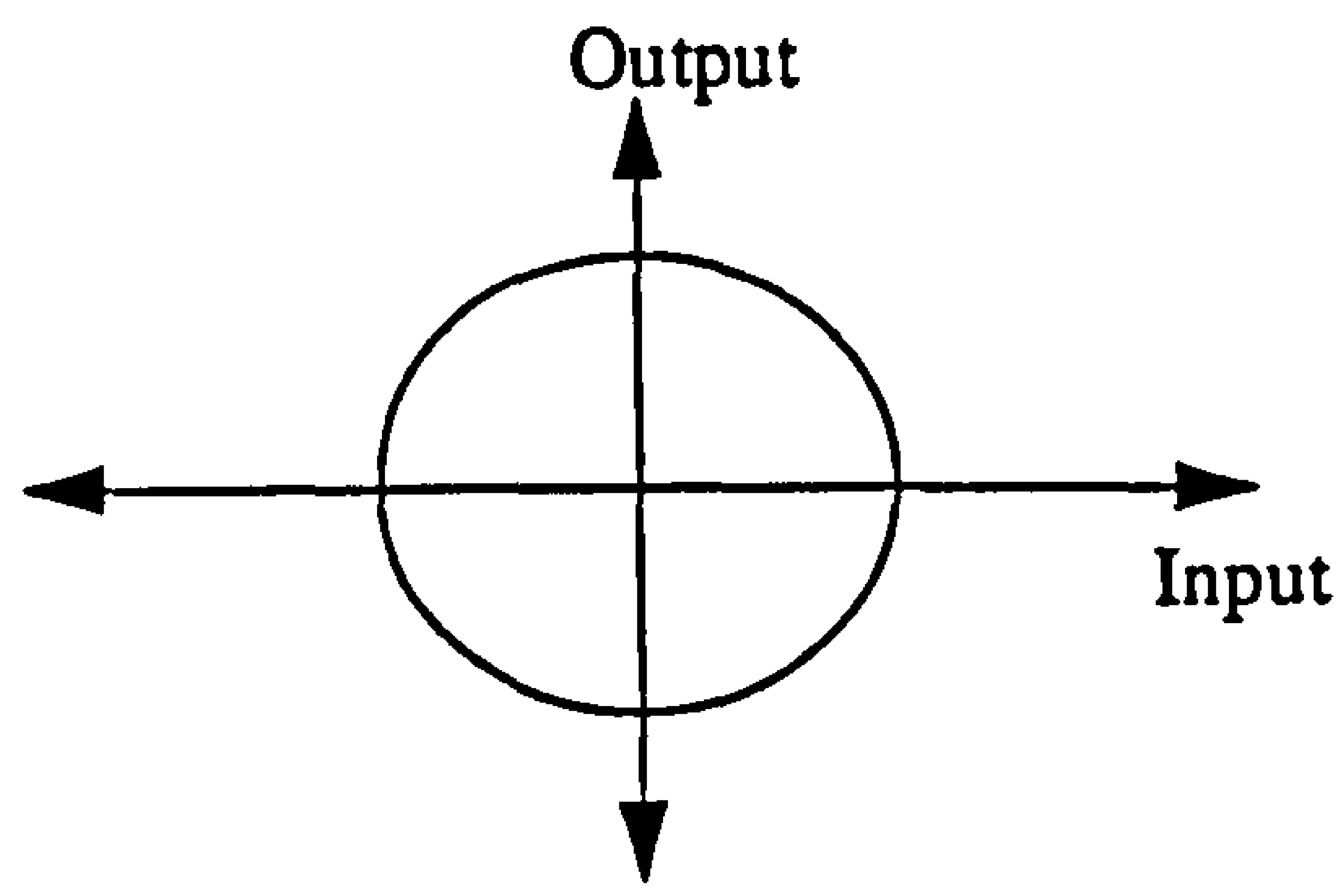
| $x$ | $h(x)$ | $f(x)$ |
|---|---|---|
| –1 | 1 | –2 |
| 2 | 3 | 1 |
| 4 | –2 | 7 |

28. Describe a real-world situation that is modeled by a function.

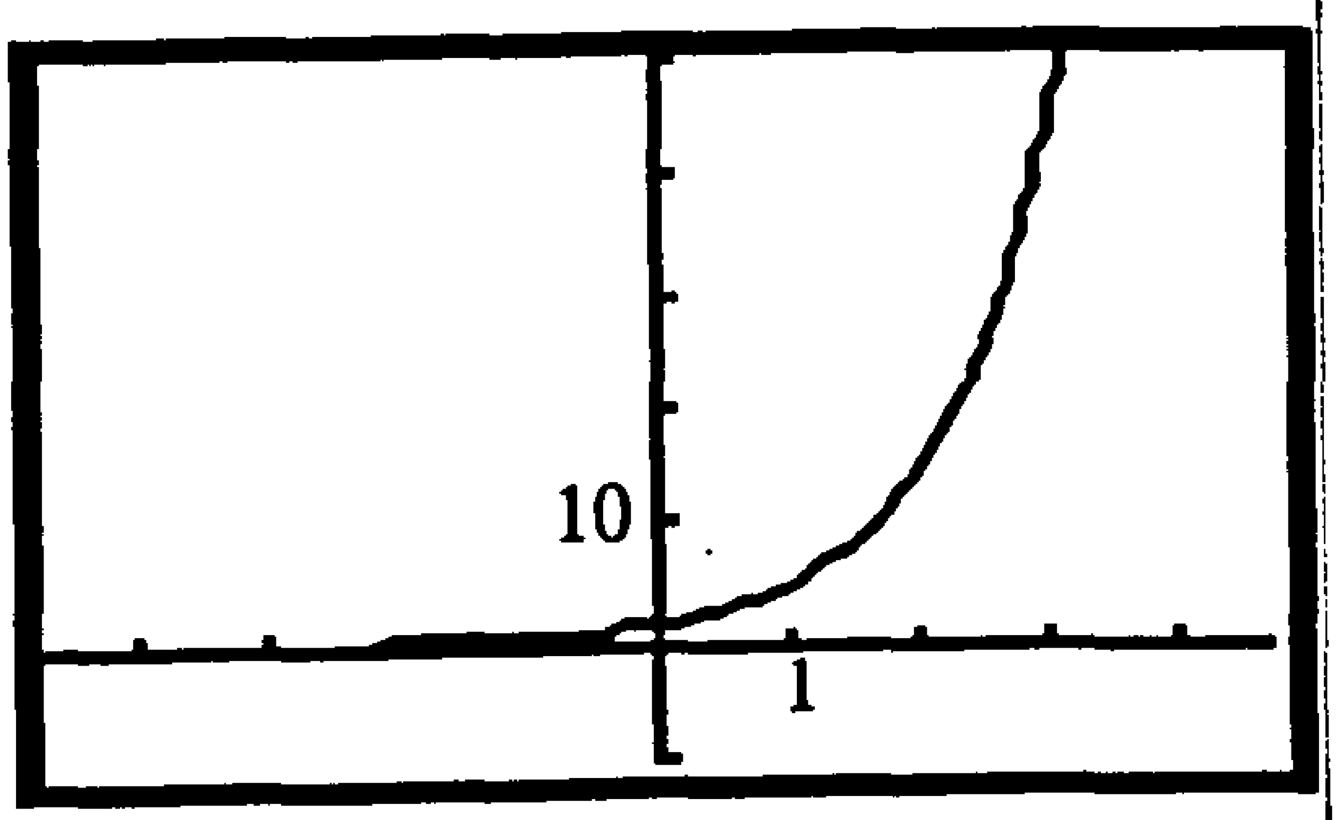
1

$$y = 3x^2 - x$$

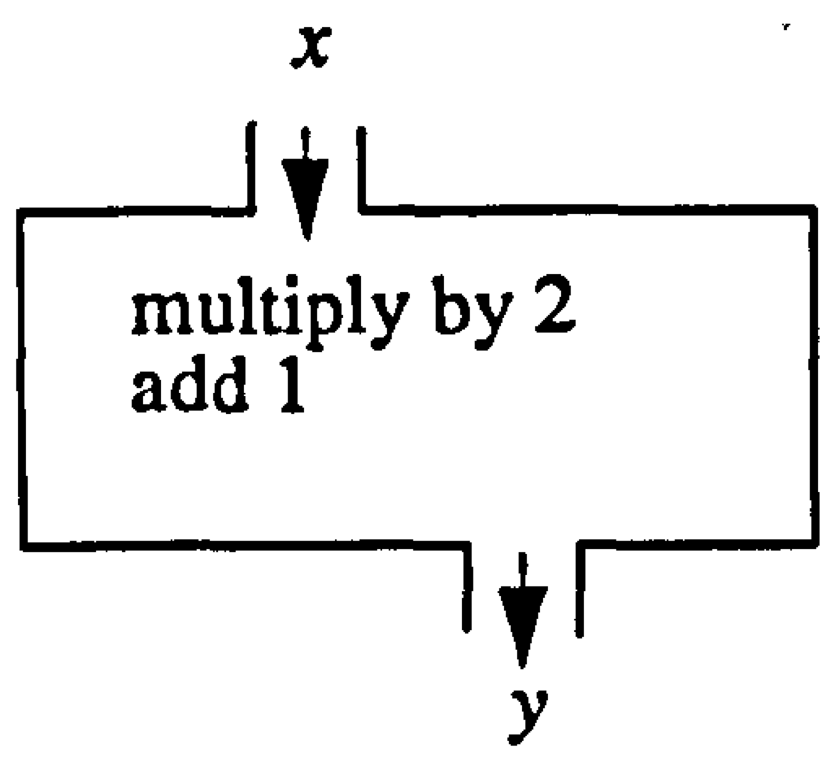
2




3




4




5

| $x$ | |
|---|---|
| –2 | 2/3 |
| –1 | 2 |
| 0 | 6 |
| 1 | 18 |
| 2 | 54 |

6

$$x = 4$$

7

| $x$ | |
|---|---|
| –2 | 4 |
| –1 | 4 |
| 0 | 4 |
| 1 | 4 |
| 2 | 4 |

8

$$y = 2x + 1$$

9

$x$

raise 3 to power equal to input
multiply result by 6

output

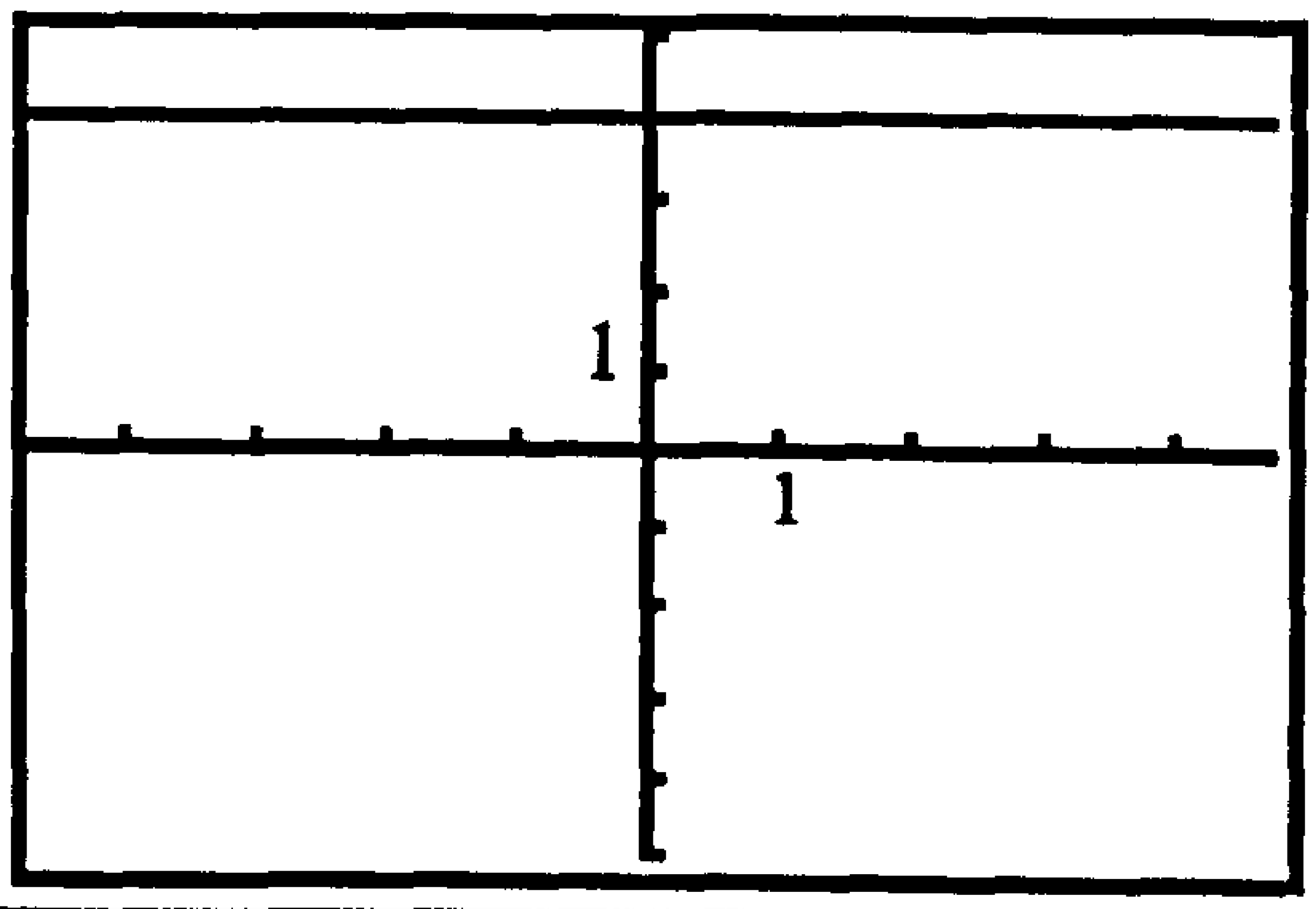
10

| $x$ | |
|---|---|
| –2 | 9 |
| –1 | 5 |
| 0 | 4 |
| 1 | 6 |
| 2 | 11 |

11

$$2x + 1 = 7$$

12



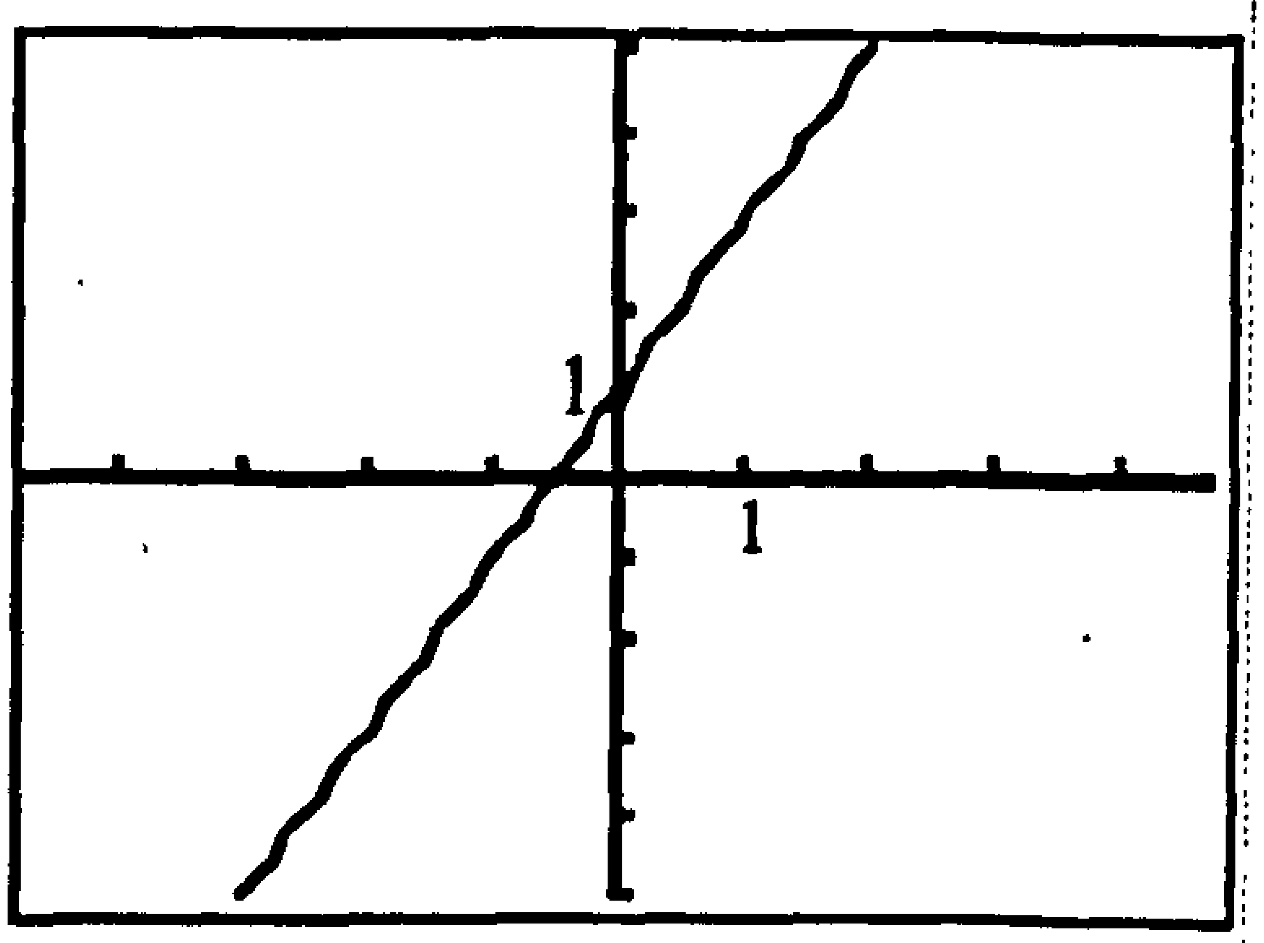

13

$$x^2 + y^2 = 25$$

14

| $x$ | |
|---|---|
| {5, 7, 11, 19, 34} | 11 |
| {–2, 0, 5, 6, 8, 12} | 5.5 |
| {1, 5, 19} | 5 |
| {6, 9, 11, 45} | 10 |
| {8, 17, 4, 76, 21} | 17 |

15




16

$x$

Ignore input

4

17

$y = 4$

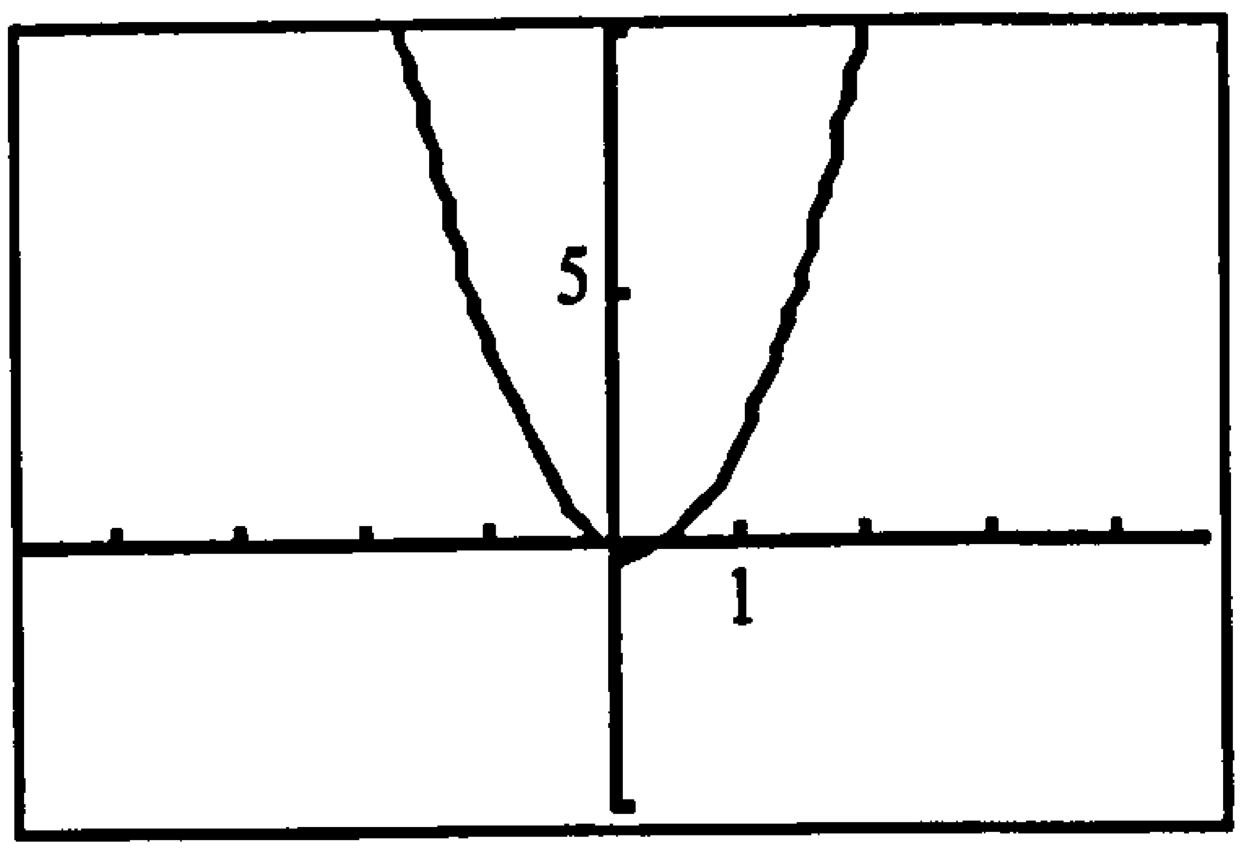
18

$x$

square input
multiply by 3
subtract input from product

$y$

19




20

list of numbers

add numbers in list
divide sum by number of numbers

output

21

$y = 6(3)^x$

22

$3x^2 - x = 2$

23

| $x$ | |
|---|---|
| –2 | –3 |
| –1 | –1 |
| 0 | 1 |
| 1 | 3 |
| 2 | 5 |

24

| $x$ | |
|---|---|
| –3 | 4 |
| 0 | –5 |
| 0 | 5 |
| –3 | –4 |
| 5 | 0 |