

# Some forgotten equilibria of the Bertrand duopoly!?

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## Abstract

This note analyzes the Bertrand duopoly with constant but asymmetric marginal costs on a market with homogenous products. It is shown that there exist some equilibria that are ignored in the literature on IO. In addition, in this setting (perfectly or nearly perfectly) competitive equilibria exist.

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The Bertrand duopoly with homogenous products plays an important role in modern industrial organization. Firstly, because of its analytical simplicity and the characteristics of the equilibria it is a standard part of modern textbook contents. Secondly, it shows that under some particular conditions even duopolies can induce perfectly competitive results. Thirdly, perhaps due to its comfortable handling it is often used as an essential part in multistage games. "Bertrand competition is interesting because it depicts a polar case. It represents what we have in mind when we think of sharp small-number competition." (Tirole, 1988, 212).

One popular and often used variation of the basic Bertrand model is of special interest. This is the Bertrand duopoly with homogenous products and unequal production costs of the duopolists. It has commonly been argued that cost inequalities reduce the degree of competitiveness in equilibrium as the more efficient supplier can realize positive profits in equilibrium. In this note I shall try to show that this is not necessarily the case.<sup>1</sup> In the following

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<sup>1</sup>Somewhat conversely to this paper Baye and Morgan (1997) and Baye and Morgan (1999) show that under certain circumstances there exist equilibria with strictly positive profits in the symmetric Bertrand duopoly.

section I introduce the basic Bertrand model with cost inequality and present the usual equilibria. In section 2 I show that there exist further equilibria with more competitive outcomes. The note is closed with some concluding remarks.

## 1 The usual analysis of Bertrand duopoly with homogenous products and cost inequality

Let us analyze an industry with two suppliers, firm 1 and firm 2. Their production costs are given by  $C_i = c_i q_i$  for  $i = 1, 2$  with  $c_2 > c_1$ . Firms simultaneously compete in prices and offer homogenous products so that consumers always buy from the firm that offers the lowest price. Let market demand be given by  $q = D(p)$  with  $p = \min(p_1, p_2)$ . Firms individual demand functions are given by

$$q_i = \begin{cases} 0, & p_i > p_j \\ \frac{D(p_i)}{2}, & p_i = p_j \\ D(p_i), & p_i < p_j \end{cases}, \quad i = 1, 2 \text{ and } i \neq j.$$

Thus the firms' profit functions are given by  $\pi_i = (p_i - c_i) q_i(p_i, p_j)$  with  $i, j = 1, 2$  and  $i \neq j$ . Assume further that  $\Delta$  is the smallest monetary unit so that costs and prices have to be a multiple of  $\Delta$ , i.e.  $c_i, p_i = \lambda \Delta$  with  $\lambda$  being an integer. This assumption, of course, only serves to circumvent problems with the existence of equilibria that arise if we allow for continuous prices. Let  $\Delta$  be arbitrarily small so that the following two conditions are met:

1.  $c_2 - c_1 \gg \Delta$ ,
2.  $(p - \Delta - c_1) D(p - \Delta) > (p - c_1) \frac{D(p)}{2}$  for all  $p \geq c_1 + 2\Delta$ .

The first condition ensures that there are multiple feasible prices between  $c_1$  and  $c_2$ . Condition 2 implies that underbidding the other firm by  $\Delta$  is profitable as long as profits remain strictly positive. Finally assume that there is no drastic cost advantage, i.e.  $c_2 < p_1^{mon}$  with  $p_1^{mon}$  being the monopoly price of the low cost firm. As usual, Nash equilibria in pure strategies are used as the equilibrium concept. The standard solution of this game is given by

**Proposition 1** *Given the assumptions above, there exist two equilibria.*

1. *Equilibrium 1 is characterized by  $p_1 = c_2$ ,  $p_2 = c_2 + \Delta$ . Consequently, the quantities of firm 1 and firm 2 are given by  $q_1 = D(c_2) > 0 = q_2$  and profits are  $\pi_1 = (c_2 - c_1) D(c_2) > 0 = \pi_2$ .*
2. *Strategies in the second equilibrium are given by  $p_1 = c_2 - \Delta$  and  $p_2 = c_2$ . The corresponding quantities and profits are  $q_1 = D(c_2 - \Delta) > 0 = q_2$  and  $\pi_1 = (c_2 - \Delta - c_1) D(c_2 - \Delta) > 0 = \pi_2$ , respectively.*

**Proof.** Omitted.<sup>2</sup> ■

If we model prices as a continuous variable in the above model, then, strictly speaking, there exists no equilibrium in pure strategies (e.g. Wolfstetter (1999, 73)). However, if we change the rationing rule for the case of equal prices of firm 1 and firm 2 then the existence problem vanishes. For example, assume that the individual demand functions are changed as follows:

$$q_1 = \begin{cases} 0, & p_1 > p_2 \\ D(p_1), & p_1 = p_2 \\ D(p_1), & p_1 < p_2 \end{cases} \quad \text{and} \quad q_2 = \begin{cases} D(p_2), & p_1 > p_2 \\ 0, & p_1 = p_2 \\ 0, & p_1 < p_2 \end{cases} .$$

Here it is arbitrarily assumed that in case of  $p_1 = p_2$  the high cost firm does not produce and the low cost firm gets all demand. With this new rationing rule and continuous prices the following proposition holds:

**Proposition 2** *In the continuous version of the Bertrand duopoly with the modified rationing rule the strategy combination  $p_1 = p_2 = c_2$  constitutes an equilibrium. The corresponding quantities and profits are given by  $q_1 = D(c_2) > 0 = q_2$  and  $\pi_1 = (c_2 - c_1) D(c_2) > 0 = \pi_2$ .*

**Proof.** Omitted. ■

Furthermore, Wolfstetter (1999, 73) emphasizes that one can regard the equilibrium in continuous strategies as the limit equilibrium of the discrete strategies model with an ever decreasing value of  $\Delta \rightarrow 0$ .

This is where the story typically ends.<sup>3</sup> At least the author is not aware of any contribution that goes beyond the analysis that has been presented so far. However, there are numerous other equilibria of the Bertrand duopoly that have not been mentioned so far.

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<sup>2</sup>E.g., see the standard texts of industrial organization such as Wolfstetter (1999, 72-74), or Shy (1995, 109-110), in conjunction with the Errata for the first edition, available under <http://econ.haifa.ac.il/~ozshy>.

<sup>3</sup>E.g. see Shapiro (1989, 344), where he describes "the" Bertrand equilibrium.

## 2 Some other equilibria

In many discussions of the Bertrand duopoly scenarios with prices below marginal costs are not analyzed in detail. Often simple plausibility arguments are presented with the aim of ruling out such a behavior. Two textbook examples may serve as evidence: (1) "At any such price, firm 2 would choose to produce zero ..." (Varian, 1992, 292); (2) "In equilibrium, each firm must make nonnegative profit. Hence,  $p_i^b \geq c_i, i = 1, 2$ ." (Shy, 1995, 110). Both arguments seem to be intuitively clear and convincing. However, a closer look makes clear that they do not suffice. The first of these statements is certainly correct but still misses the point! In Bertrand duopolies players do not choose quantities but prices! Therefore, a quantity of zero does not imply prices at or above marginal costs. The second statement is incorrect. It is true that both players have nonnegative profits in equilibrium. However, the "hence" is not valid. For example, if  $p_1 = c_1$  then  $c_2 > p_2 = c_1 + \Delta$  induces nonnegative profits, too.

The basic insight is that *any*  $p_2 > p_1$  makes sure that firm 2 does not produce and that its profits are equal to zero. This is independent from the absolute magnitude of  $p_2$ . Consequently, it may be possible that in equilibrium  $p_2 < c_2$  as is shown in the following proposition.

**Proposition 3** *In the discrete version of the Bertrand duopoly there are  $(c_2 - c_1)/\Delta$  equilibria in pure strategies. In particular, any strategy combination  $(p_1^N, p_2^N)$  that satisfies  $c_2 \geq p_1^N \geq c_1 + \Delta$  and  $p_2^N = p_1^N + \Delta$  constitutes a Nash equilibrium.*

**Proof.** If  $c_2 \geq p_1^N$  then firm 2 cannot gain by underbidding firm 1 because it will make losses. Furthermore, all  $p_2 > p_1^N$  induce zero profits. Consequently,  $p_2 = p_1^N + \Delta$  is a best reply of firm 2 to firm 1 playing  $p_1^N \leq c_2$ .

If firm 2 plays  $p_2^N = p_1^N + \Delta \geq c_1 + 2\Delta$  then firm 1 gets all demand and realizes a strictly positive profit. As prices are already below their monopoly level further price decreases cannot lead to higher profits. A price increase by  $\Delta$  leads to an equal division of demand among the duopolists. However, we assumed that  $\Delta$  is sufficiently small so that  $(p - \Delta - c_1) D(p - \Delta) > (p - c_1) \frac{D(p)}{2}$  for all  $p \geq c_1 + 2\Delta$ . Consequently, firm 1 cannot increase profits by increasing its price and sharing demand with its competitor. Even further price increases by firm 1 would induce a demand of zero and zero profits, too. Hence,  $p_1 = p_2^N - \Delta$  is a best reply of firm 1 to firm 2 playing  $c_2 + \Delta \geq p_2^N \geq c_1 + 2\Delta$ .

Summarizing, within the given range,  $p_1^N$  and  $p_2^N$  are mutually best replies and thus constitute a Nash equilibrium. ■

We can derive a corresponding result for the model with continuous prices, too. Again, assume that the modified rationing rule is valid, i.e. if both duopolists offer the same price duopolist 1 (with the lower costs) gets all demand. In this case we get the following equilibria:

**Proposition 4** *In the continuous version of the Bertrand duopoly there are multiple equilibria. In particular, any strategy combination  $(p_1^N, p_2^N)$  that satisfies  $c_2 \geq p_2^N = p_1^N \geq c_1$  constitutes a Nash equilibrium in pure strategies.*

**Proof.** If  $c_2 \geq p_1 = p_1^N \geq c_1$  then all  $p_2 \geq p_1^N$  are best replies of duopolist 2 to firm 1 playing  $p_1^N$ . Hence,  $p_2 = p_2^N = p_1^N$  is one of the best replies of duopolist 2. Furthermore, if  $p_2 = p_2^N \geq c_1$  then it is always optimal for firm 1 to offer a price  $p_1 = p_1^N = p_2^N$ , i.e.  $p_1^N$  is a best reply to  $p_2^N$ . As  $p_1^N$  and  $p_2^N$  are mutually best replies they constitute a Nash equilibrium. ■

**Remark 1** *Note that the equilibria that have been shown in the preceding four propositions often have firm 2 equilibrium strategies that are weakly dominated.*

- All "new" equilibria with  $p_2 < c_2$  are characterized by firm 2 equilibrium strategies that are weakly dominated by other strategies, e.g. by strategies with  $p_2 > c_2$ .
- However, the same is true for all but one of the "traditional" equilibria. In particular, equilibrium 2 of Proposition 1 (discrete strategies) and the only equilibrium in Proposition 2 (continuous strategies) are constituted by weakly dominated firm 2 strategies.

We have shown that in the simple Bertrand duopoly with asymmetric costs there exist many more equilibria than commonly stated. Equilibrium prices include all feasible prices between  $c_1$  and  $c_2$  (in the continuous case). As a consequence, the Bertrand paradox need not vanish when costs differ between duopolists as is commonly argued in the literature.<sup>4</sup>

### 3 Conclusion

The main result of this note is that in the asymmetric Bertrand duopoly with constant marginal costs there are numerous equilibria in weakly dominated strategies that have been ignored in the literature on Industrial Organization. As these equilibria are derived quite easily it is rather implausible that they

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<sup>4</sup>E.g. see Tirole (1988, 211).

have not been found by somebody else before. Unfortunately, the author was unable to find a source in which these equilibria are mentioned<sup>5</sup> and, quite obviously, most other authors are not aware of their existence either. Therefore, it seems justified to present this finding (once again?).

It remains to be discussed how important the "new" equilibria may be. First, the use of weakly dominated strategies in a Nash equilibrium always looks somewhat implausible. Still, the strategy combination remains a Nash equilibrium and as the literature on refinements has not been overwhelmingly successful there does not exist a widely accepted procedure to sort out such equilibria. Furthermore, in many games weakly dominated strategies need not be regarded as implausible.<sup>6</sup> This is particularly true when "Bertrand behavior" is used as a punishment strategy in multistage games. Finally, and maybe most important, it is inadequate to discriminate against the "new" equilibria by the criterion that they comprise of weakly dominated strategies because this criterion is also valid for widely accepted equilibria of the Bertrand game!

Obviously, the adequacy/plausibility of weakly dominated equilibrium strategies depends on the details of the game in question. Without the availability of a reliable general rule about how to select among equilibria we also have to take into account the embeddedness of the game theoretic model into the real economic situation that is going to be explained. Consequently, we must not ignore some equilibria *ex ante*. This, in turn, means that the "new" equilibria presented in this note deserve sufficient attention.

## References

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<sup>5</sup>If any reader of this note knows a source in which these equilibria are mentioned the author would be very happy about being informed.

<sup>6</sup>See, e.g., "the Modeller's Game" in Rasmusen (2001, 27-8).

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