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Reasoning about Temporal Properties of Rational Play

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Abstract

This article is about defining a suitable logic for expressing classical game theoretical notions. We define an extension of alternating-time temporal logic (**ATL**) that enables us to express various rationality assumptions of intelligent agents. Our proposal, the logic **ATLP** (***ATL** with plausibility*) allows us to specify sets of rational strategy profiles in the object language, and reason about agents' play if only these strategy profiles were allowed. For example, we may assume the agents to play only Nash equilibria, Pareto-optimal profiles or undominated strategies, and ask about the resulting behaviour (and outcomes) under such an assumption. The logic also gives rise to generalized versions of classical solution concepts through characterizing patterns of payoffs by suitably parameterized formulae of **ATLP**. We investigate the complexity of model checking **ATLP** for several classes of formulae: It ranges from $\mathbf{\Delta_3^P}$ to $\mathbf{PSPACE}$ in the general case and from $\mathbf{\Delta_3^P}$ to $\mathbf{\Delta_4^P}$ for the most interesting subclasses, and roughly corresponds to solving extensive games with imperfect information.

Keywords: game theory, modal and temporal logic, reasoning about agents, rationality.

1 Introduction

Alternating-time temporal logic (**ATL**) [2, 3] is a temporal logic that incorporates some basic game theoretical notions. In **ATL** we can express that a group of agents is able to *bring about* ψ, i.e., they are able to ensure a situation where ψ holds whatever the other agents might do. However, such a statement is weaker than it seems. Often, we know that agents behave according to some rationality assumptions, they are not completely dumb. Therefore we do not have to check *all possible plays* – only those that are *plausible* in some reasonable sense. This has striking similarities to nonmonotonic reasoning, where one considers *default rules* that describe the most plausible behaviour and allow to draw conclusions when knowledge is incomplete.

In general, plausibility can be seen as a broader notion than rationality: One may obtain plausibility specifications e.g. from learning or folk knowledge. In this article, however, we mostly focus on plausibility as rationality in a game-theoretical sense.

Our idea has been inspired by the way in which games are analyzed in game theory. Firstly, game theory identifies a number of *solution concepts* (e.g., Nash equilibrium, undominated strategies, Pareto optimality) that can be used to define rational behaviour of players. Secondly, we usually *assume that players play rationally* in the sense of one of the above concepts, and we *ask about the outcome of the game under this assumption*.

Solution concepts do not only help to determine the right decision for an agent. Perhaps more importantly, they *constrain* the possible (predicted) responses of the opponents to a proper subset of all the possibilities. For many games the number of all possible outcomes is infinite, although only some of them, often finitely many, *make sense*. We need a notion of rationality (like subgame-perfect Nash equilibrium) to discard the *less sensible* ones, and to determine what should happen had the game been played by ideal players.

1.1 Idea and Main Results

While **ATL** is already a logic that incorporates some game theoretical concepts, we claim that extending **ATL** by other useful constructs not only helps us to better understand the classical solution concepts in game theory, but it also paves the way for defining new solution concepts (which we call *general*). We extend **ATL** by the notion of *plausibility*, and call the resulting logic **ATLP**. We claim that this logic is suitable to model and to reason about the rational behaviour of agents.

In this article we discuss the following:

1. We recall from [5, 30] that models of **ATL**, called *concurrent game structures (**CGS**)*, embed *extensive form games with perfect information* in a natural way. This can be done, e.g., by adding auxiliary propositions to the **CGS**, that describe the payoffs of agents. With this perspective, concurrent game structures can be seen as a strict generalisation of extensive games.

2. We discuss informally how these more general games can be "solved", given an appropriate solution concept that defines which plays can be plausibly expected.

3. We extend **ATL** to a new logic **ATLP** that allows to reason about what agents can achieve under an arbitrary plausibility assumption. Analy-

sis of this kind typically starts with assuming that agents are rational in the sense that they only play strategies consistent with a selected solution concept (e.g., they can only play Nash equilibria, or undominated strategies etc.). Then, we can ask which outcomes can be obtained by whom under this assumption.

4. We extend the results from [45, 30], and show that the classical solution concepts (*Nash equilibrium*, *subgame perfect Nash equilibrium*, *Pareto optimality*, and others) can be also characterized in the object language of **ATLP**. That is, we propose expressions of **ATLP** that, given an extensive game, denote exactly the set of Nash equilibria (subgame perfect NE's, Pareto optimal profiles, etc.) in that game. In consequence, **ATLP** can serve both as a language for reasoning about rational play, and for specifying what rational play is. We point out that these characterizations extend traditional solution concepts to the more general class of multi-stage multi-player games defined by concurrent game structures.

5. We also propose an alternative approach to defining solution concepts for games that involve infinite flow of time. In the new approach, path formulae of **ATL** are used to specify the "winning conditions" of each player. This implicitly leads to a normal form game with binary payoffs, where the traditional solution concepts are well defined. We also demonstrate how these "qualitative" solution concepts (parametrized by **ATL** path formulae) can be characterized in **ATLP**.

6. We constructively show that several logics can be embedded into **ATLP**. That is, we demonstrate how models and formulae of those logics can be (independently) transformed to their **ATLP** counterparts in a way that preserves their truth values.

7. Last but not least, we investigate the model checking problem in **ATLP**. We show that, for different subclasses of the new logic, the complexity of model checking ranges from $\mathbf{\Delta_3^P}$-completeness to **PSPACE**-completeness. We also argue that, when the number of plausible strategy profiles is reasonably small, the model checking can be done in polynomial time.

1.2 Related Work

In our approach, some strategies (or rather *strategy profiles*) can be assumed plausible, and one can reason what can be *plausibly* achieved by agents under such an assumption. There are two possible points of focus in this context. Research within game theory understandably favors work on *characterization* of various types of rationality (and defining most appropriate solution concepts). Applications of game theory, also understandably, tend toward *using* the solution concepts in order to predict the outcome in a given game (in other words, to "solve" the game).

The first issue has been studied in the framework of logic, for example in [4, 6, 41, 42]; more recently, game-theoretical solution concepts have been characterized in dynamic logic [21, 20], dynamic epistemic logic [5, 44], and ATL [45, 30].

The second thread seems to have been neglected in logic-based research: papers by Van Otterloo and his colleagues [50, 51, 49, 48] are the only exceptions we know of. Moreover, every proposal from [50, 51, 49, 48] commits to a particular view of rationality (Nash equilibria, undominated strategies etc.). In this paper, we try to generalize this kind of reasoning in a way that allows to "plug in" any solution concept of choice. We also try to fill in the gap between the two threads by showing how sets of rational strategy profiles can be specified in the object language, and building upon the existing work on modal logic characterizations of solution concepts [21, 20, 5, 44, 45, 30].

1.3 Structure of the Article

We begin by introducing some basic notions from game theory and the alternating-time temporal logic (Section 2). In Section 3, we pave the way for Sections 4 and 5: We relate **ATL** and its semantical models to extensive games. Then we do the same for an extension of **ATL**, called **ATLI**, which has been introduced in [30] to characterize solution concepts in extensive games.

Section 4 introduces our logic **ATLP**: We extend **ATL** with a plausibility operator. This constitutes the base language $\mathcal{L}_{ATLP}^{\text{base}}$. The main syntactic novelty are *plausibility terms* that refer to rational strategies. Then, we extend the base language by allowing to *specify sets of rational strategy profiles in the object language*. To do this, we need to define a language with a much richer structure of terms as in $\mathcal{L}_{ATLP}^{\text{base}}$. We achieve this by describing strategy profiles with **ATLI** formulae, and extending $\mathcal{L}_{ATLP}^{\text{base}}$ so that the concepts presented in Section 3.4 can be reused. Finally, we propose the full language $\mathcal{L}_{ATLP}$ where **ATLP** characterizations of solution concepts are "plugged" into **ATLP** formulae that describe the consequences of adopting this or that notion of rationality. Thus, we create a single language for both characterizing rational behaviour and reasoning about its outcome. We define $\mathcal{L}_{ATLP}$ through a hierarchy of sublanguages $\mathcal{L}_{ATLP}^{k}$, each allowing for more levels of plausibility updates than the previous one.

Section 5 lists our main conceptual results. We show how to embed several logics in **ATLP** and how to express several classical solution concepts (such as Nash equilibria and others) already in $\mathcal{L}_{ATLP}^{1}$. Our third result is the generalization of Nash equilibria, Pareto optimality, undominatedness and subgame perfect Nash equilibria as certain parameterized formulae in the language of **ATLP**.

Section 6 contains the results of our study on the complexity of model checking in variants of **ATLP**. Finally, we conclude with Section 7.

Some results reported in this article have been already presented in a preliminary form in several conference and workshop papers. A rough idea of "**ATL** with plausibility" was proposed in [8, 25]. In [26], we studied a more complex language of terms that would allow to specify sets of rational strategy profiles in the object language; still, the language was not expressive enough for our purposes. Some initial complexity results were also reported in that paper. Finally, [11] put forward the idea that rationality specifications can be written in **ATLP** itself, and nested in **ATLP** formulae. The idea of "qualitative" solution concept was also introduced in [11].

2 Preliminaries

In this section, we introduce some concepts that are important for the rest of this article. After recapitulating some machinery of game theory, together with two running examples, we introduce **ATL**, which is the basis for our new logic **ATLP**.

2.1 Concepts From Game Theory

We start with the definition of a *normal form* game, also called *strategic game*, and use the terminology of [35].

Definition 1 (Normal Form (NF) Game) *A (perfect information)* normal form game Γ*, is a tuple of the form* $\Gamma = \langle \mathcal{P}, \mathcal{A}_1, \dots, \mathcal{A}_k, \mu \rangle$*, where*

- $\mathcal{P}$ *is a finite set of* players *(or* agents*), with* $|\mathcal{P}| = k$*,*
- $\mathcal{A}_i$ *are nonempty sets of* actions *(or* strategies*) for player* i*,*
- $\mu : \mathcal{P} \to (\prod_{i=1}^{k} \mathcal{A}_i \to \mathbb{R})$ *is the* payoff function *(which we also write* $\langle \mu_1, \dots, \mu_k \rangle$*).*

A combinations of actions (resp. strategies, payoffs), one per player, will be called an action profile *(resp.* strategy profile*,* payoff profile*) throughout the paper.*

Such games are usually depicted with a payoff matrix. For example, a game with 2 players having 2 strategies each is represented by the matrix in Figure 1.

Example 1 (Classical NF Games) *Some classical NF games with* 2 *players and* 2 *strategies are shown in Figure 2. In the* Matching Pennies *game, player 1 wins when both pennies show the same side. Otherwise player 2 wins. In the*

1\ 2	a_2^1	a_2^2
a_1^1	$\langle \boldsymbol{\mu}_1(a_1^1, a_2^1), \boldsymbol{\mu}_2(a_1^1, a_2^1)\rangle$	$\langle \boldsymbol{\mu}_1(a_1^1, a_2^1), \boldsymbol{\mu}_2(a_1^1, a_2^2)\rangle$
a_1^2	$\langle \boldsymbol{\mu}_1(a_1^2, a_2^2), \boldsymbol{\mu}_2(a_1^2, a_2^2)\rangle$	$\langle \boldsymbol{\mu}_1(a_1^2, a_2^1), \boldsymbol{\mu}_2(a_1^2, a_2^2)\rangle$

Figure 1: Payoff matrix for 2 players and 2×2 strategies

1\ 2	Head	Tail
Head	(1, -1)	(-1, 1)
Tail	(-1, 1)	(1, -1)

1\ 2	C	D
C	(3, 3)	(0, 5)
D	(5, 0)	**(1, 1)**

1\ 2	Dove	Hawk
Dove	(3, 3)	**(1, 4)**
Hawk	**(4, 1)**	(0, 0)

Figure 2: Payoff matrices for Matching Pennies, Prisoner's Dilemma, and Hawk-Dove. Nash equilibria are set in bold font.

Prisoner's Dilemma, *two prisoners can either cooperate or defect with the police. Finally, the* Hawk-Dove *game is similar, but the payoffs are different. The higher the payoff the better it is for the respective player.*

Definition 2 (Solution Concepts in Games) *There are several well-known* solution concepts *such as:*

Nash Equilibrium (NE): *A strategy profile such that no agent can unilaterally deviate from her strategy and get a better payoff;*

Pareto Optimality (PO): *There is no other strategy profile that leads to a payoff profile which is at least as good for each agent, and strictly better for at least one agent;*

Weakly Undominated Strategies (UNDOM): *These are strategies that are not dominated by any other strategy, i.e., such that there is no strategy at least as good for all the responses of the opponent, and strictly better for at least one response.*

We do not repeat the formal definitions here and refer to the literature [35]. We point out, however, that some solution concepts yield sets of individual strategies (UNDOM), while others produce rather sets of strategy profiles *(NE, PO).*

In the examples from Figure 2, there is no Nash equilibrium for the Matching Pennies game, exactly one Nash equilibrium for the Prisoner's Dilemma (namely, the strategy profile $\langle D, D\rangle$), and two Nash equilibria for the Hawk-Dove game ($\langle Hawk, Dove\rangle$ and $\langle Dove, Hawk\rangle$).

In NF games, agents do their moves *simultaneously*: They do not see the move of the opponent and therefore cannot act accordingly. On the other hand, there are many games where the move of one player should depend on the preceding move of the opponent, or even on the *whole history*. This idea is captured in games of *extensive form*.

Definition 3 (Extensive Form (EF) Game) *A (perfect information)* extensive form game Γ *is a tuple of the form* $\Gamma = \langle \mathcal{P}, \mathcal{A}, H, ow, u\rangle$*, where:*

- $\mathcal{P}$ *is a finite set of players,*
- $\mathcal{A}$ *a finite set of actions (moves),*
- H *is a set of finite action sequences (game histories), such that (1)* $\emptyset \in H$*, (2) if* $h \in H$*, then every initial segment of* h *is also in* H*. We use the notation* $\mathcal{A}(h) = \{m \mid h \circ m \in H\}$ *to denote the* moves available at h*, and* $Term = \{h \mid \mathcal{A}(h) = \emptyset\}$*, the set of* terminal *positions,*
- $ow : H \to \mathcal{P}$ *defines which player "owns" history* h*, i.e., has the next move given* h*,*
- $u : \mathcal{P} \times Term \to U$ *assigns agents' utilities to every terminal position of the game.*

We will usually assume that the set of utilities U *is finite.*

Such games can be easily represented as trees of all possible plays.

Example 2 (Bargaining) *Consider bargaining with discount [35, 37]. Two players,* 1 *and* 2*, bargain about how to split goods worth initially* $w_0 = 1$ *EUR. After each round without agreement, the subjective worth of the goods reduces by* discount rates δ_1 *(for player* a_1*) and* δ_2 *(for player* a_2*). So, after* t *rounds, the goods are worth* $\langle \delta_1^t, \delta_2^t\rangle$*, respectively. Subsequently,* a_1 *(if* t *is even) or* a_2 *(if* t *is odd) makes an offer to split the goods in proportions* $\langle x, 1-x\rangle$*, and the other player accepts or rejects it. If the offer is accepted, then* a_1 *takes* $x\delta_1^t$*, and* a_2 *gets* $(1-x)\delta_2^t$*; otherwise the game continues. The (infinite) extensive form game is shown in Figure 3. Note that the tree has infinite depth as well as an inifite branching factor.*

In order to obtain a finite set of payoffs, it is enough to assume that the goods are split with finite precision represented by a rounding function $r : \mathbb{R} \to \mathbb{R}$*. So, after* t *rounds, the goods are in fact worth* $\langle r(\delta_1^t), r(\delta_2^t)\rangle$*, respectively, and if the offer is accepted, then* a_1 *takes* $r(x\delta_1^t)$*, and* a_2 *gets* $r((1-x)\delta_2^t)$*.*

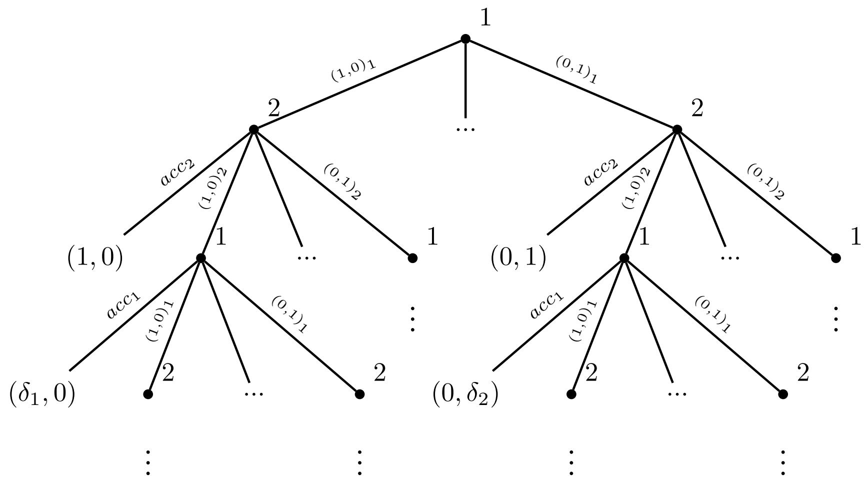


Figure 3: The bargaining game.

A *strategy* for player $i \in \mathcal{P}$ in extensive game Γ is a function that assigns a legal move to each history owned by i. Note that a *strategy profile* (i.e., a combination of strategies, one per player) determines a unique path from the game root ($\emptyset$) to one of the terminal nodes (and hence also a single profile of payoffs). In consequence, one can construct the corresponding normal from game $NF(\Gamma)$ by enumerating strategy profiles and filling the payoff matrix with resulting payoffs.

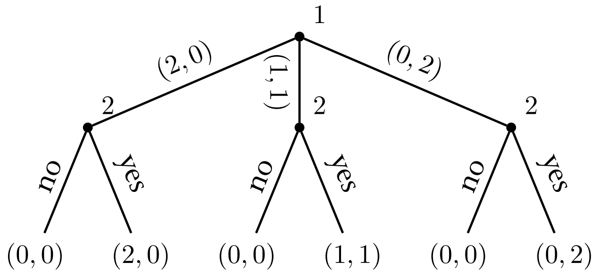
Example 3 (Sharing Game) *Consider the* Sharing Game *in Figure 4A. Its corresponding normal form game is presented in Figure 4B. Firstly, player* 1 *can suggest how to share, say, two 1 EUR coins. E.g.* $(2, 0)$ *means that* 1 *gets two euro and* 2 *gets nothing. Subsequently, player* 2 *can accept the offer or reject it; in the latter case both players get nothing.*

The game includes 3 *strategies for player* 1 *(which can be denoted by the action that they prescribe at the beginning of the game), and* 8 *strategies for player* 2 *(generated by the combination of actions prescribed for the second move), which gives* 24 *strategy profiles in total. However, not all of them seem plausible. Constraining the possible plays to Nash equilibria only, we obtain* 9 *"rational" strategy profiles (cf. Figure 4B), although it is still disputable if all of them really "make sense".*

A *subgame* of an extensive game Γ is defined by a subtree of the game tree of Γ.

Definition 4 (Subgame Perfect Nash Equilibrium (SPN)) *This solution concept is an extension of NE: A strategy is a SPN in* Γ *if it is a NE in* Γ *and, in addition, a NE in all subgames of* Γ.

(A)



(B)

1\ 2	nnn	nny	nyn	nyy	ynn	yny	yyn	yyy
$(2, 0)$	**(0, 0)**	**(0, 0)**	(0, 0)	(0, 0)	**(2, 0)**	**(2, 0)**	**(2, 0)**	**(2, 0)**
$(1, 1)$	(0, 0)	(0, 0)	**(1, 1)**	**(1, 1)**	(0, 0)	(0, 0)	(1, 1)	(1, 1)
$(0, 2)$	(0, 0)	**(0, 2)**	(0, 0)	(0, 2)	(0, 0)	(0, 2)	(0, 0)	(0, 2)

Figure 4: The Sharing game: (A) Extensive form; (B) Normal form. Nash equilibria are set in bold font. A strategy abc ($a, b, c \in \{y, n\}$) of player 2 denotes the strategy in which 2 plays a (resp. b, c) if player 1 has played $(2, 0)$ (resp. $(1, 1)$, $(0, 2)$) where n refers to "no" and y to "yes".

Example 4 (Sharing Game ctd.) *Consider again the from Example 3. While the game has* 9 *Nash equilibria, only two of them are subgame perfect* ($\langle(2, 0), yyy\rangle$ *and* $\langle(1, 1), nyn\rangle$).

Example 5 (Bargaining ctd.) *Consider the bargaining game from Example 7. The game has an immense number of possible outcomes. Still worse, every strategy profile*

$$s^x : \begin{cases} a_1 \textit{ always offers } \langle x, 1-x\rangle\textit{, and agrees to } \langle y, 1-y\rangle \textit{ for } y \geq x \\ a_2 \textit{ always offers } \langle x, 1-x\rangle\textit{, and agrees to } \langle y, 1-y\rangle \textit{ iff } 1-y \geq 1-x \end{cases}$$

is a Nash equilibrium (NE): an agreement is reached in the first round. *Thus, every split* $\langle x, 1-x\rangle$ *can be achieved through a Nash equilibrium; it seems that a stronger solution concept is needed. Indeed, the game has a unique* subgame perfect Nash equilibrium. *Because of the finite precision, there is a minimal round* T *with* $r(\delta_i^{T+1}) = 0$ *for* $i = 1$ *or* $i = 2$. *For simplicity, assume that* $i = 2$ *and agent* a_1 *is the offerer in* T *(i.e.,* T *is even). Then, the only subgame perfect NE is given by the strategy profile* s^κ *with* $\kappa = (1-\delta_2)\frac{1-(\delta_1\delta_2)^{\frac{T}{2}}}{1-\delta_1\delta_2} + (\delta_1\delta_2)^{\frac{T}{2}}$. *The goods are split* $\langle\kappa, 1-\kappa\rangle$*; the agreement is reached in the first round.*[1]

[1]For the standard version of bargaining with discount (with the continuous set of payoffs

2.2 ATL

Alternating-time temporal logic (**ATL**) [2, 3] enables reasoning about temporal properties and strategic abilities of agents. Formally, the language of **ATL** is given as follows.

Definition 5 ($\mathcal{L}_{ATL}$) *Let $\mathbb{A}\mathrm{gt} = \{a_1, \dots, a_k\}$ be a nonempty finite set of all agents, and Π be a set of propositions (with typical element* p*). We use the symbol a to denote a typical agent, and A to denote a typical group of agents from $\mathbb{A}\mathrm{gt}$. The logic $\mathcal{L}_{ATL}(\mathbb{A}\mathrm{gt}, \Pi)$ is defined by the following grammar:*

$$\varphi ::= \mathsf{p} \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\!\langle A \rangle\!\rangle \bigcirc \varphi \mid \langle\!\langle A \rangle\!\rangle \Box \varphi \mid \langle\!\langle A \rangle\!\rangle \varphi \mathcal{U} \varphi.$$

Informally, $\langle\!\langle A \rangle\!\rangle \varphi$ says that agents A have a collective strategy to enforce φ. **ATL** formulae include the usual temporal operators: $\bigcirc$ (*in the next state*), $\Box$ (*always from now on*) and $\mathcal{U}$ (strict *until*). Additionally, $\Diamond$ (*now or sometime in the future*) can be defined as $\Diamond\varphi \equiv \top\mathcal{U}\varphi$. Like in **CTL** [13], every occurrence of a temporal operator is immediately preceded by exactly one cooperation modality (this variant of the language is sometimes called "vanilla" **ATL**). The broader language of **ATL***, where no such restriction is imposed, is not discussed in this article. It should be noted that the **CTL** path quantifiers A, E can be expressed in **ATL** with $\langle\!\langle \emptyset \rangle\!\rangle$, $\langle\!\langle \mathbb{A}\mathrm{gt} \rangle\!\rangle$ respectively. The semantics of **ATL** is defined over *concurrent game structures*.

Definition 6 (CGS) *A* concurrent game structure (**CGS**) *is a tuple:* $M = \langle \mathbb{A}\mathrm{gt}, Q, \Pi, \pi, Act, d, o \rangle$, *consisting of: a set* $\mathbb{A}\mathrm{gt} = \{a_1, \dots, a_k\}$ *of* agents*; a set* Q *of* states*; a set* Π *of* atomic propositions*; a* valuation of propositions $\pi : Q \to \mathcal{P}(\Pi)$*; a set* Act *of* actions*. Function* $d : \mathbb{A}\mathrm{gt} \times Q \to \mathcal{P}(Act)$ *indicates the actions available to agent* $a \in \mathbb{A}\mathrm{gt}$ *in state* $q \in Q$*. We will often write* $d_a(q)$ *instead of* $d(a, q)$*, and use* $d(q)$ *to denote the set* $d_1(q) \times \cdots \times d_k(q)$ *of* action profiles *available in state* q*. Finally,* o *is a* transition function *which maps each state* $q \in Q$ *and action profile* $\overrightarrow{\alpha} = \langle \alpha_1, \dots, \alpha_k \rangle \in d(q)$ *to another state* $q' = o(q, \overrightarrow{\alpha})$.

Remark 1 *In the literature on* ***ATL****, the same symbols for agents (and groups of agents) are used in the semantics and in the object language; we follow this tradition here.*

A *computation* or *path* $\lambda = q_0 q_1 \dots \in Q^\omega$ is an infinite sequence of states such that there is a transition between each q_i, q_{i+1}.We define $\lambda[i] = q_i$ to denote the i-th state of λ. Λ_M denotes all paths in M. The set of all paths starting in q is given by $\Lambda_M(q)$.

Definition 7 (Strategy, outcome) *A (memoryless)* strategy *of agent a is a function $s_a : Q \to Act$ such that $s_a(q) \in d_a(q)$. We denote the set of such functions*

[0, 1]), cf. [35, 37]. Restricting the payoffs to a finite set requires to alter the solution slightly [40, 33], see also Appendix A.

by Σ_a*. A* collective strategy s_A *for team* $A \subseteq \mathbb{A}\mathrm{gt}$ *specifies an individual strategy for each agent* $a \in A$*; the set of* A*'s collective strategies is given by* $\Sigma_A = \prod_{a\in A}\Sigma_a$*. The set of all* strategy profiles *is given by* $\Sigma = \Sigma_{\mathbb{A}\mathrm{gt}}$.

The outcome *of strategy* s_A *in state* q *is defined as the set of all paths that may result from executing* s_A *from state* q *on:* $out(q, s_A) = \{\lambda \in \Lambda_M(q) \mid \forall i \in \mathbb{N}_0\, \exists \overrightarrow{\alpha} = \langle \alpha_1, \ldots, \alpha_k\rangle \in d(\lambda[i])\, \forall a \in A\, (\alpha_a = s_A^a(\lambda[i]) \wedge o(\lambda[i], \overrightarrow{\alpha}) = \lambda[i+1])\}$*, where* s_A^a *denotes agent* a*'s part of the collective strategy* s_A*.*

The semantics of **ATL** can be given by the following clauses:

$M, q \models \mathsf{p}$ iff $\mathsf{p} \in \pi(q)$

$M, q \models \neg\varphi$ iff $M, q \not\models \varphi$

$M, q \models \varphi \wedge \psi$ iff $M, q \models \varphi$ and $M, q \models \psi$

$M, q \models \langle\!\langle A\rangle\!\rangle \bigcirc\varphi$ iff there is $s_A \in \Sigma_A$ such that $M, \lambda[1] \models \varphi$ for all $\lambda \in out(q, s_A)$

$M, q \models \langle\!\langle A\rangle\!\rangle \Box\varphi$ iff there is $s_A \in \Sigma_A$ such that $M, \lambda[i] \models \varphi$ for all $\lambda \in out(q, s_A)$ and $i \in \mathbb{N}_0$

$M, q \models \langle\!\langle A\rangle\!\rangle \varphi\mathcal{U}\psi$ iff there is $s_A \in \Sigma_A$ such that, for all $\lambda \in out(q, s_A)$, there is $i \in \mathbb{N}_0$ with $M, \lambda[i] \models \psi$, and $M, \lambda[j] \models \varphi$ for all $0 \leq j < i$.

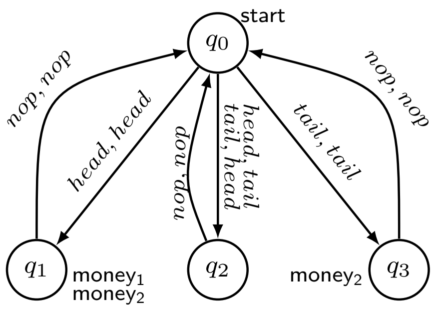
Remark 2 *We somewhat deviate from the original semantics of* ***ATL*** *[2, 3], where strategies assign agents' choices to* sequences *of states (which suggests that agents can recall the whole history of each game). While the choice between the two types of strategies affects the semantics of most* ***ATL*** *extensions, both yield equivalent semantics for* pure ***ATL*** *[38].*

3 Relating Games and ATL-Like Logics

In this section we present some important ideas that form the starting point for later sections. (1) We discuss informally how the notion of strategic ability in **ATL** can be refined so that it takes into account only "sensible" behaviour of agents. (2) We look back on the logic of **GLP** [51] which implements the idea formally, albeit in a very limited way. (3) We summarize a correspondence between extensive games and the models of **ATL**. (4) We recall an extension of **ATL**, called **ATLI** ("**ATL** with Intentions"), which will later serve as an intermediate logical framework and as a motivation for our logic **ATLP**. We also demonstrate how several game-theoretical solution concepts can be expressed in **ATLI**. (5) Finally we present our idea of *qualitative* solution concepts, where **ATL** path formulae are used to define the winning conditions.

We illustrate the ideas with two examples from the previous section: *Matching Pennies* and *Bargaining with Discounts*.

(A)



(B)

$1\backslash 2$	s_h	s_t
s_h	$1,1$	$0,0$
s_t	$0,0$	$0,1$

Figure 5: *Asymmetric matching pennies*: **(A)** Concurrent game structure M_1. In q_0 the agents can choose to show $head$ or $tail$. Both agents can only execute action nop (no operation) in states q_1, q_2, q_3. **(B)** The corresponding NF game. We use s_h (resp. s_t) to denote the strategy in which the player always shows head (resp. tail) in q_0 and nop in q_1, q_2, and q_3.

3.1 ATL and Rational Play

Example 6 (Asymmetric matching pennies) *Consider a variant of the matching pennies game, presented in Figure 5A. Formally, the model is given as follows:*

$$M_1 = \langle \{1,2\}, \{q_0, q_1, q_2, q_3\}, \{\mathsf{start}, \mathsf{money}_1, \mathsf{money}_2\}, \pi, \{head, tail, nop\}, d, o\rangle$$

where π is given as in the picture ($\pi(q_0) = \{\mathsf{start}\}$ etc.), $d(a, q_0) = \{head, tail\}$ for $a = 1, 2$, and $d(a, q) = \{nop\}$ for $a = 1, 2$ and $q = q_1, q_2, q_3$. The transition function o can also be read off from the picture. We use nop (no operation) as a "default" action in states q_1, q_2, and q_3 that brings the system back to the initial state. The intuition is that the game is played ad infinitum. Alternatively, one might add loops to states q_1, q_2 and q_3 to model a game that is played only once.

If both players show heads in q_0, both win a prize in the next step; if they both show tails, only player 2 wins. If they show different sides, nobody wins. Note that, e.g., $M_1, q_0 \models \langle\langle 2\rangle\rangle \Box \neg \mathsf{money}_1$, because agent 2 can play $tail$ all the time, preventing 1 from winning the prize. On the other hand, $M_1, q_0 \models \neg\langle\langle 2\rangle\rangle \Diamond \mathsf{money}_2$: Agent 2 has no strategy to guarantee that she will win.

The concurrent game structure in Figure 5A determines the set of available strategy profiles. However, it does not say anything about players' preferences. Suppose now that the players are only interested in getting some money sometime in the future (but it does not matter when and/or how much). The corresponding normal form game under this assumption is depicted in Figure 5B.

Such an analysis of the game is of course correct, yet it appears to be quite coarse. It seems natural to assume that players prefer winning money over losing it. If we additionally assume that the players are rational thinkers, it seems plausible that player 1 should always play head, as it keeps the possibility of getting money open (while playing tail guarantees loss). Under this assumption, player 2 has complete control over the outcome of the game: She can play head too, granting herself and the other agent with the prize, or respond with tail, in which case both players lose. Note that this kind of analysis corresponds to the game-theoretical notion of *weakly dominant strategy*: For agent 1, playing head is dominant in the corresponding normal form game in Figure 5B, while both strategies of player 2 are undominated, so they can be in principle considered for playing.

It is still possible to refine our analysis of the game. Note that 2, knowing that 1 ought to play head and preferring to win money too, should decide to play $head$ herself. This kind of reasoning corresponds to the notion of *iterated undominated strategies*. If we assume that both players do reason this way, then $\langle s_h, s_h \rangle$ is the only rational strategy profile, and the game should end with both agents winning the prize.

3.2 Game Logic with Preferences

Game Logic with Preferences [51] is, to our knowledge, the only logic designed to address the outcome of rational play in games with perfect information. Here, we summarize the idea very briefly.

The central idea of **GLP** is facilitated by the *preference operator* $[a : \varphi]$. Interpretation of $[a : \varphi]\psi$ in model M proceeds as follows: if the truth of φ can be enforced by a, then we remove from the model all the actions of a that do *not* lead to enforcing it, and evaluate ψ in the resulting model. Thus, the evaluation of **GLP** formulae is underpinned by the assumption that *rational agents satisfy their preferences whenever they can*. The requirement applies to all the subtrees of the game tree, and it is called "subgame perfectness" by the authors.

The scope of **GLP**, however, is limited in several respects. Firstly, the models of **GLP** are restricted to finite game trees. Secondly, agents' preferences must be specified with propositional (non-modal) formulae, and they are evaluated only at the terminal states of the game. The temporal part of the language is limited, too. Lastly, and perhaps most importantly, the semantics of **GLP** is based on a very specific notion of rationality (see above). One can easily imagine variants of the semantics, in which other rationality criteria are used (NE, PO, UNDOM) to eliminate "irrational" strategies. Indeed, a preliminary version of **GLP** was based on the notion of Nash equilibrium rather than "subgame perfectness" [50]. In this article, we want to allow as much flexibility as possible with respect to the choice of a suitable solution concept.

3.3 Models of ATL vs. Extensive Games

In this section, we recall the correspondence between extensive form games and the semantical models of **ATL**, proposed in [30] and inspired by [5, 45].

We only consider game trees in which the set of payoffs is finite. Let U denote the set of all possible utility values in a game; U will be finite and fixed for any given game. For each value $v \in U$ and agent $a \in \mathbb{A}\mathrm{gt}$, we introduce a proposition p_a^v into our set Π, and fix $\mathsf{p}_a^v \in \pi(q)$ iff a gets payoff of at least v in q.[2] States in the model represent finite histories in the game. In particular, we us $\emptyset$ to denote the root of the game. The correspondence between an extensive game Γ and a **CGS** M can be captured as follows.

Definition 8 (From Extensive Games to CGS) *A **CGS** $M = \{\mathbb{A}\mathrm{gt}, Q, \Pi, \pi, Act, d, o\}$* corresponds to *an extensive game $\Gamma = \langle \mathcal{P}, \mathcal{A}, H, ow, u\rangle$ if, and only if, the following holds:*

- $\mathbb{A}\mathrm{gt} = \mathcal{P}$,
- $Q = H$,
- Π *and* π *include propositions* p_a^v *to emulate utilities for terminal states in the way described above,*
- $Act = \mathcal{A} \cup \{nop\}$,
- $d_a(q) = \mathcal{A}(q)$ *if* $a = ow(q)$ *and* $d_a(q) = \{nop\}$ *otherwise,*
- $o(q, nop, \ldots, m, \ldots, nop) = q \cdot m$, *and*
- $o(q, nop, nop, \ldots, nop) = q$ *for* $q \in Term$.

*We use $M(\Gamma)$ to refer to the **CGS** which corresponds to Γ.*

Example 7 (Bargaining in a CGS) *We consider the bargaining game from Example 2, but this time as a model of **ATL**. The **CGS** corresponding to the game shown in Figure 6. Nodes represent various states of the negotiation process, and arcs show how agents' moves change the state of the game. A node label refers to the history of the game for better readability. For instance,* $\begin{bmatrix} 0,1 \\ 1,0 \\ acc \end{bmatrix}$ *has the meaning that in the first round* 1 *offered* $\langle 0, 1\rangle$ *which was rejected by* 2*. In the next round* 2*'s offer* $\langle 1, 0\rangle$ *has been accepted by* 1 *and the game has ended.*

Note that, for every extensive game Γ, there is a corresponding **CGS**, but the reverse is not true: Concurrent game structures can include cycles and simultaneous moves of players, which are absent in game trees. Note also that, for those **CGS**'s that correspond to some EF game, we get an implicit correspondence to a normal form game. We will extend this notion of correspondence to all **CGS**'s in Section 3.5.

[2] Note that a state labeled by p_a^v is also labeled by $\mathsf{p}_a^{v'}$ for all $v' \in U$ where $v' < v$.

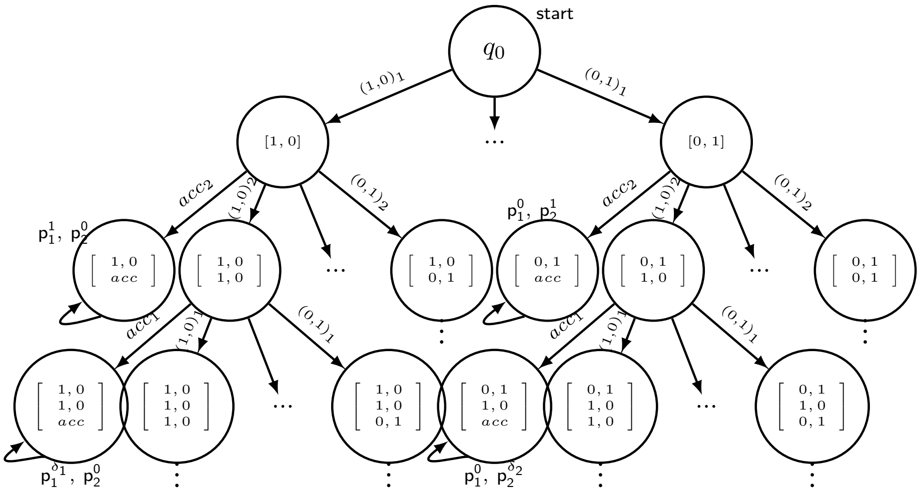


Figure 6: **CGS** M_2 for the bargaining game

3.4 ATLI and Solution Concepts

The correspondence between extensive games and (some) concurrent game structures gives us a way of performing game-theoretical analysis on the latter. In particular, game-theoretical solution concepts become meaningful for these **CGS**'s. This section illustrates how several important notions of rationality from game theory, e.g. Nash equilibria (NE), subgame perfect NE, Pareto optimality etc. can be characterized in a suitable logical language. We use the analysis from [30] where an extension of **ATL**, called **ATLI**, was employed for this purpose. We will later show how these characterizations can be "plugged" into our new logic **ATLP** so that one can reason about the outcome of rational play in a precisely defined sense.

We also point out after [30] that these characterizations give rise to generalized versions of solution concepts which can be applied to *all* **CGS**'s, and not only to those that correspond to some extensive form game.

Alternating-time temporal logic with intentions (**ATLI**) extends **ATL** with formulae $(\mathbf{str}_a\sigma_a)\varphi$ with the intuitive reading: *Suppose that player a intends to play according to strategy σ_a, then φ holds*. Thus, it allows to refer to agents' strategies explicitly via terms σ_a. Let $\mathfrak{Str} = \bigcup_{a\in\mathbb{A}\mathrm{gt}} \mathfrak{Str}_a$ be a finite set of *strategic terms*. $\mathfrak{Str}_a$ are used to denote *individual strategies* of agent $a \in \mathbb{A}\mathrm{gt}$; we assume that all $\mathfrak{Str}_a$ are disjoint.

Definition 9 ($\mathcal{L}_{ATLI}$) *Let $p \in \Pi$, $a \in \mathbb{A}\mathrm{gt}$, $A \subseteq \mathbb{A}\mathrm{gt}$, and $\sigma_a \in \mathfrak{Str}_a$. The language $\mathcal{L}_{ATLI}(\mathbb{A}\mathrm{gt}, \Pi, \mathfrak{Str})$ is defined as:*

$$\theta ::= \mathsf{p} \mid \neg\theta \mid \theta \wedge \theta \mid \langle\!\langle A\rangle\!\rangle \bigcirc \theta \mid \langle\!\langle A\rangle\!\rangle \Box\theta \mid \langle\!\langle A\rangle\!\rangle \theta\mathcal{U}\theta \mid (\mathbf{str}_a\sigma_a)\theta.$$

ATLI Models $M = \langle \mathbb{A}\mathrm{gt}, Q, \Pi, \pi, Act, d, o, \mathcal{I}, \mathfrak{Str}, [\cdot] \rangle$ extend concurrent game structures with *intention relations* $\mathcal{I} \subseteq Q \times \mathbb{A}\mathrm{gt} \times Act$ (where $q\mathcal{I}_a\alpha$ means that a possibly intends to do action α when in q). Moreover, strategic terms are interpreted as strategies according to function $[\cdot] : \mathfrak{Str} \to \bigcup_{a \in \mathbb{A}\mathrm{gt}} \Sigma_a$ such that $[\sigma_a] \in \Sigma_a$ for $\sigma_a \in \mathfrak{Str}_a$ (remember that Σ_a denotes the set of a's strategies). The set of paths consistent with all agents' intentions is defined as

$$\Lambda^{\mathcal{I}} = \{\lambda \in \Lambda_M \mid \forall i\, \exists \alpha \in d(\lambda[i])\, (o(\lambda[i], \alpha) = \lambda[i+1] \wedge \forall a \in \mathbb{A}\mathrm{gt}\ \lambda[i]\mathcal{I}_a\alpha_a)\}$$

We impose on $\mathcal{I}$ the natural requirement that $q\mathcal{I}_a\alpha$ implies that $\alpha \in d_a(q)$ for $a \in \mathbb{A}\mathrm{gt}$; that is, agents only intend to do actions if they are actually able to perform them.

We say that strategy s_A is *consistent with A's intentions* if $q\mathcal{I}_a s_A^a(q)$ for all $q \in Q, a \in A$. The *intention-consistent outcome set* is defined as: $out^{\mathcal{I}}(q, s_A) = out(q, s_A) \cap \Lambda^{\mathcal{I}}$. The semantics of strategic operators in **ATLI** extends and replaces the semantic rules of **ATL** as follows:

$M, q \models \langle\langle A \rangle\rangle \bigcirc \theta$ iff there is a collective strategy s_A consistent with A's intentions, such that for every $\lambda \in out^{\mathcal{I}}(q, s_A)$, we have that $M, \lambda[1] \models \theta$;

$M, q \models \langle\langle A \rangle\rangle \Box\theta$ and $M, q \models \langle\langle A \rangle\rangle \theta\mathcal{U}\theta'$: analogous;

$M, q \models (\mathbf{str}_a\sigma)\theta$ iff $revise(M, a, [\sigma]), q \models \theta$.

The function $revise(M, a, s_a)$ updates model M by setting a's intention relation to

$$\mathcal{I}'_a = \{\langle q, s_a(q)\rangle \mid q \in Q\},$$

so that s_a and $\mathcal{I}_a$ represent the same mapping in the resulting model. Note that a pure **CGS** M can be seen as a **CGS** with the full intention relation

$$\mathcal{I}^0 = \{\langle q, a, \alpha\rangle \mid q \in Q, a \in \mathbb{A}\mathrm{gt}, \alpha \in d_a(q)\}.$$

Additionally, for $A = \{a_{i_1}, \ldots, a_{i_r}\}$ and $\sigma_A = \langle \sigma_{a_{i_1}}, \ldots, \sigma_{a_{i_r}}\rangle$, we define: $(\mathbf{str}_A\sigma_A)\varphi \equiv (\mathbf{str}_{a_{i_1}}\sigma_{a_{i_1}}) \ldots (\mathbf{str}_{a_{i_r}}\sigma_{a_{i_r}})\varphi$. Furthermore, for $B = \{b_1, \ldots, b_l\} \subseteq A$ we use $\sigma_A[B]$ to refer to B's substrategy, i.e. to $\langle \sigma_{b_1}, \ldots, \sigma_{b_l}\rangle$

Example 8 (Asymmetric matching pennies ctd.) *Coming back to our matching pennies example from Figure 5, we have for instance that $M_1, q_0 \models (\mathbf{str}_1\sigma)\langle\langle 2 \rangle\rangle \Diamond \mathsf{money}_2$ if the denotation of σ is set to s_h.*

With temporal logic, it is natural to define outcomes of strategies via properties of resulting paths rather than single states. The notion of *temporal T-Nash equilibrium*, parameterized with a unary operator $T = \bigcirc, \Box, \Diamond, \_\mathcal{U}\psi, \psi\mathcal{U}\_$, was proposed in [30]. Let $\sigma = \langle \sigma_1, \ldots, \sigma_k\rangle$ be a profile of strategic terms, and let T stand for any of the following operators: $\bigcirc, \Box, \Diamond, \_\mathcal{U}\psi, \psi\mathcal{U}\_$ and let a be an agent. Then we consider the following $\mathcal{L}_{ATLI}$ formulae:

$$BR^T_a(\sigma) \equiv (\mathbf{str}_{\mathbb{A}\mathrm{gt}\setminus\{a\}}\sigma[\mathbb{A}\mathrm{gt}\setminus\{a\}]) \bigwedge_{v\in U} \Big((\langle\!\langle a\rangle\!\rangle T\mathsf{p}^{\mathsf{v}}_{\mathsf{a}}) \rightarrow ((\mathbf{str}_a\sigma[a])\langle\!\langle\emptyset\rangle\!\rangle T\mathsf{p}^{\mathsf{v}}_{\mathsf{a}})\Big)$$

$$NE^T(\sigma) \equiv \bigwedge_{a\in\mathbb{A}\mathrm{gt}} BR^T_a(\sigma)$$

$$SPN^T(\sigma) \equiv \langle\!\langle\emptyset\rangle\!\rangle\Box NE^T(\sigma).$$

$BR^T_a(\sigma)$ refers to $\sigma[a]$ being a T-best strategy for a against $\sigma[\mathbb{A}\mathrm{gt}\setminus\{a\}]$; $NE^T(\sigma)$ expresses that strategy profile σ is a T-Nash equilibrium; finally, $SPN^T(\sigma)$ defines σ as subgame perfect T-NE. Thus, we have a family of equilibria: $\bigcirc$-Nash equilibrium, $\Box$-Nash equilibrium etc., each corresponding to a *different temporal pattern* of utilities. For example, we may assume that *agent a gets v* if a utility of at least v is guaranteed for every time moment ($\Box\mathsf{p}^{\mathsf{v}}_{\mathsf{a}}$), is eventually achieved ($\Diamond\mathsf{p}^{\mathsf{v}}_{\mathsf{a}}$), and so on.

The correspondence between solution concepts and their temporal counterparts for extensive games is captured by the following proposition.

Proposition 3 *Let Γ be an extensive game. Then the following holds:*

1. *$M(\Gamma), \emptyset \models NE^{\Diamond}(\sigma)$ iff $[\sigma]_{M(\Gamma)}$ is a NE in Γ [30].*[3]
2. *$M(\Gamma), \emptyset \models SPN^{\Diamond}(\sigma)$ iff $[\sigma]_{M(\Gamma)}$ is a SPN in Γ.*

Proof sketch

1. Since $M(\Gamma)$ corresponds to an EF game, the "payoff" propositions $\mathsf{p}^{\mathsf{v}}_{\mathsf{a}}$ can only become true at the end of each path in $M(\Gamma)$. Thus, $BR^{\Diamond}_a(\sigma)$ in $M(\Gamma), \emptyset$ holds iff, whenever a can achieve the payoff of *at least* v against $\sigma[\mathbb{A}\mathrm{gt}\setminus\{a\}]$ (by any strategy), it can also achieve that by using $\sigma[a]$. That is, a cannot obtain a better payoff by unilaterally changing her strategy.
2. $M(\Gamma), \emptyset \models SPN^{\Diamond}(\sigma)$ iff $M(\Gamma), q \models NE^{\Diamond}(\sigma)$ for every q reachable from the root $\emptyset$ (*). However, Γ is a tree, so every node is reachable from $\emptyset$ in $M(\Gamma)$. So, by the first part, (*) iff σ denotes a Nash equilibrium in every subtree of Γ. ■

We can use the above **ATLI** formulae to express game-theoretical properties of strategies in a straightforward way.

Example 9 (Bargaining ctd.) *We extend the **CGS** in Figure 6 to a **CGS** with intentions; then, we have $M_2, q_0 \models NE^{\Diamond}(\sigma)$, with σ interpreted in M_2 as s^x (for any $x \in [0,1]$). Still, $M_2, q_0 \models SPN^{\Diamond}(\sigma)$ if, and only if, $[\sigma]_{M_2} = s^{\kappa}$.*

[3] The empty history $\emptyset$ denotes the root of the game tree.

We also propose a tentative **ATLI** characterization of *Pareto optimality* (based on the characterization from [45] for normal form games):

$$PO^T(\sigma) \equiv \bigwedge_{v_1} \cdots \bigwedge_{v_k} \Big((\langle\langle \mathbb{A}\mathrm{gt} \rangle\rangle T \bigwedge_i \mathsf{p}_i^{\mathsf{v}_i}) \rightarrow (\mathbf{str}_{\mathbb{A}\mathrm{gt}}\sigma)\big((\langle\langle \emptyset \rangle\rangle T \bigwedge_i \mathsf{p}_i^{\mathsf{v}_i}) \vee (\bigvee_i \bigvee_{\substack{v' \text{ s.t.} \\ v' > v_i}} \langle\langle \emptyset \rangle\rangle T \mathsf{p}_i^{\mathsf{v}'}))\Big).$$

That is, the strategy profile denoted by σ is Pareto optimal iff, for every achievable pattern of payoff profiles, either it can be achieved by σ, or σ obtains a strictly better payoff pattern for at least one player. Note that the above formula has exponential length with respect to the number of payoffs in U. Moreover, it is not obvious that this characterization is the right one, as it refers in fact to the evolution of payoff *profiles* (i.e., combinations of payoffs achieved by agents at the same time), and not temporal patterns of payoff evolution for *each* agent separately. So, for example, $PO^\Diamond(\sigma)$ may hold even if there is a strategy profile σ' that makes each agent achieve eventually a better payoff, as long as not all of them will achieve these better payoffs at the same moment. Still, the following holds.

Proposition 4 *Let* Γ *be an extensive game. Then:*

$$M(\Gamma), \emptyset \models PO^\Diamond(\sigma) \text{ iff } [\sigma]_{M(\Gamma)} \text{ is Pareto optimal in } \Gamma.$$

Proof Let $M(\Gamma), \emptyset \models PO^\Diamond(\sigma)$. Then, for every payoff profile $\langle v_1, \ldots, v_k \rangle$ reachable in Γ, we have that either $[\sigma]$ obtains at least as good a profile,[4] or it obtains an incomparable payoff profile. Thus, $[\sigma]$ is Pareto optimal. The proof for the other direction is analogous. ∎

Example 10 (Asymmetric matching pennies ctd.) *Let* M_1' *be our matching pennies model* M_1 *with additional propositions* $\mathsf{p}_i^1 \equiv \mathsf{money}_i$ *(so, we assign to* money_i *a utility of 1 for* i*). Then, we have* $M_1', q_0 \models PO^\Diamond(\sigma)$ *iff* σ *denotes the strategy profile* $\langle s_h, s_h \rangle$*.*

3.5 General Solution Concepts

In this part we present an abstract formulation of our notion of *general solution concept*. We will elaborate on it later in Section 5.3, using our logic **ATLP**.

We have seen in Section 3.3 that some (but not all!) concurrent game structures can be seen as extensive form games, which in turn defines their correspondence to NF games. These **CGS**'s must be turn-based (i.e., players play by taking turns) and have a tree-like structure; moreover, they must

[4] We recall that $\bigwedge_i \mathsf{p}_i^{\mathsf{v}_i}$ means that each player i gets *at least* v_i.

include special propositions that emulate payoffs and can be used to define agents' preferences. Now, we want to extend the correspondence to arbitrary **CGS**'s. Our idea is to *determine the outcome of a game by the truth of certain path formulae* (e.g., in the case of binary payoffs, we can see the formulae as *winning conditions*). So, we give up the idea of assigning payoffs to leaves in a tree. Instead, we see a concurrent game structure as a game, paths in the structure as plays in the game, and satisfaction of some pre-specified formulae as the mechanism that defines agents' outcome for a given play.

Which formulae can be used in this respect?

Definition 10 (ATL Path Formulae) *By* **ATL** path formulae, *we denote arbitrary* ***ATL*** *formulae that are preceded by a temporal operator* $\bigcirc, \Box, \mathcal{U}$.

Given a ***CGS*** M *and a path* λ *in* M, *satisfaction of path formulae is defined as follows:*

$M, \lambda \models \bigcirc\varphi$ *iff* $M, \lambda[1] \models \varphi$

$M, \lambda \models \Box\varphi$ *iff* $M, \lambda[i] \models \varphi$ *for all* $i \in \mathbb{N}_0$

$M, \lambda \models \varphi\mathcal{U}\psi$ *iff there is* $i \in \mathbb{N}_0$ *with* $M, \lambda[i] \models \psi$, *and* $M, \lambda[j] \models \varphi$ *for all* $0 \leq j < i$.

We propose that player i's preferences can be specified by a finite list of path formulae $\eta_i = \langle \eta_i^1, \dots, \eta_i^{n_i} \rangle$ (where $n_i \in \mathbb{N}$) with the underlying assumption that agent i prefers η_i^1 most, η_i^2 comes second best etc., and the worst outcome occurs when no $\eta_i^1, \dots, \eta_i^{n_i}$ holds for the actual play. Thus, η_i imposes a total order on paths in a **CGS**.

For k players, we need a k-vector of such preference lists $\overrightarrow{\eta} = \langle \eta_1, \dots, \eta_k \rangle$. Then, every concurrent game structure gives rise to the strategic game defined as below.

Definition 11 (From CGS To NF Game) *Let* M *be a* ***CGS***, $q \in Q_M$ *a state, and* $\overrightarrow{\eta} = \langle \eta_1, \dots, \eta_k \rangle$ *a vector of lists of* ***ATL*** *path formulae, where* $k = |\mathbb{A}\mathrm{gt}|$.

Then we define $\mathcal{S}(M, \overrightarrow{\eta}, q)$, *the* NF game associated with M, $\overrightarrow{\eta}$, and q, *as the strategic game* $\langle \mathbb{A}\mathrm{gt}, \mathcal{A}_1, \dots, \mathcal{A}_k, \mu \rangle$, *where the set* $\mathcal{A}_i$ *of* i*'s strategies is given by* Σ_i *for each* $i \in \mathbb{A}\mathrm{gt}$, *and the payoff function is defined as follows:*

$$\mu_i(a_1, \dots, a_k) = \begin{cases} n_i - j + 1 & \text{if } \eta_i^j \text{ is the first formula from } \eta_i \text{ such that } M, \lambda \models \eta_i^j \\ & \text{for all } \lambda \in out(q, \langle a_1, \dots, a_k \rangle), \\ 0 & \text{no } \eta_i^j \text{ is satisfied} \end{cases}$$

where $\eta_i = \langle \eta_i^1, \dots, \eta_i^{n_i} \rangle$, $1 \leq j \leq n_i$ *and we write* μ_i *for* $\mu(i)$.

Below, we present the generalized version of temporal Nash equilibrium and temporal subgame perfect NE.

$$\begin{aligned}
BR_a^{\vec{\eta}}(\sigma) &\equiv (\mathbf{str}_{\mathbb{A}\mathrm{gt}\setminus\{a\}}\sigma[\mathbb{A}\mathrm{gt}\setminus\{a\}])\bigwedge_j\Big((\langle\!\langle a\rangle\!\rangle\eta_a^j)\rightarrow((\mathbf{str}_a\sigma[a])\bigvee_{r\leq j}\langle\!\langle\emptyset\rangle\!\rangle\eta_a^r)\Big)\\
NE^{\vec{\eta}}(\sigma) &\equiv \bigwedge_{a\in\mathbb{A}\mathrm{gt}} BR_a^{\vec{\eta}}(\sigma)\\
SPN^{\vec{\eta}}(\sigma) &\equiv \langle\!\langle\emptyset\rangle\!\rangle\Box NE^{\vec{\eta}}(\sigma).
\end{aligned}$$

The case with a single "winning condition" per agent is particularly interesting. Clearly, it gives rise to a normal form game with binary payoffs (cf., for instance, our informal discussion of the "matching pennies" variant in Example 6). We will stick to such binary games throughout the rest of the paper (especially in Section 5.3 where general solution concepts are studied in more detail), but one can easily imagine how the binary case extends to the case with multiple levels of preference.

4 The Logic ATLP

Agents have limited ability to predict the future. However, some lines of action seem often more sensible or realistic than others. If a rationality criterion is available, we obtain means to focus on a proper subset of possible plays. In game theoretic terms, *we solve the game*, i.e., we determine the most plausible plays, and compute their outcome. In game theory, the outcome consists of the payoffs (or utilities) assigned to players at the end of the game. In temporal logics, the outcome of a play can be seen in terms of temporal patterns that can occur — which allows for much subtler descriptions. In Section 3.4 we explained how rationality can be characterized with formulae of modal logic (**ATLI** in this case). Now we show how the outcome of rational play can be described with a similar (but richer) logic, and that both aspects can be seamlessly combined.

Our logic **ATLP** ("**ATL** with Plausibility") comes in several steps, based on different underlying languages:

$\mathcal{L}_{ATLP}^{\mathbf{base}}$: Sets of plausible/rational strategy profiles can be only referred to via atomic plausibility terms (constants) whose interpretation is "hardwired" in the model. A typical $\mathcal{L}_{ATLP}^{\mathrm{base}}$ statement is $(\mathbf{set\text{-}pl}\ \omega)\mathbf{Pl}\,\varphi$: *Suppose that the set of rational strategy profiles is defined by ω – then, it is plausible to expect that φ holds.* For instance, one can reason about what should happen if only Nash equilibria were played, or about the abilities of players who play only Pareto optimal profiles, had terms for NE and PO been included in the model.

$\mathcal{L}^0_{ATLP}$: A mild extension of $\mathcal{L}^{\text{base}}_{ATLP}$. We allow some combinations of the constants of $\mathcal{L}^{\text{base}}_{ATLP}$ to form more complex terms.

$\mathcal{L}_{ATLP^{ATLI}}$: An intermediate language, where rational strategy profiles are characterized by **ATLI** formulae.

$\mathcal{L}^k_{ATLP}$: Here we have nestings of plausibility updates up to level k. It turns out that $\mathcal{L}_{ATLP^{ATLI}}$ is already embedded in $\mathcal{L}^1_{ATLP}$.

$\mathcal{L}_{ATLP}$: Unbounded nestings of formulae are allowed.

The language $\mathcal{L}^{\text{base}}_{ATLP}$ is presented in Sections 4.1 and 4.2. Then, in Section 4.3, we consider an intermediate step, namely plausibility terms written in **ATLI**. They serve as a motivation to extend $\mathcal{L}^{\text{base}}_{ATLP}$ to $\mathcal{L}^1_{ATLP}$, and, more generally, to a hierarchy $\mathcal{L}_{ATLP} = \lim_{k\to\infty} \mathcal{L}^k_{ATLP}$ which we investigate in Section 4.4.

4.1 The Language $\mathcal{L}^{\text{base}}_{ATLP}$

We extend the language of **ATL** with operators $\mathbf{Pl}_A$, $(\textbf{set-pl}\ \omega)$, and $(\textbf{refn-pl}\ \omega)$. The first assumes plausible behaviour of agents in A; the latter are used to fix the actual meaning of plausibility by *plausibility terms* ω. As yet, the terms are simply constants with no internal structure. Their meaning will be given later by a denotation function linking plausibility terms to sets of strategy profiles.

Definition 12 ($\mathcal{L}^{\text{base}}_{ATLP}$) *The base language $\mathcal{L}^{base}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \Omega)$ is defined over nonempty sets: Π of propositions, $\mathbb{A}\text{gt}$ of agents, and Ω of plausibility terms. We use p, a, ω to refer to typical elements of $\Pi, \mathbb{A}\text{gt}, \Omega$ respectively, and A to refer to a group of agents. $\mathcal{L}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \Omega)$ consists of all formulae defined by the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle\langle A\rangle\rangle \bigcirc \varphi \mid \langle\langle A\rangle\rangle \Box\varphi \mid \langle\langle A\rangle\rangle \varphi\,\mathcal{U}\varphi \mid$$
$$\mathbf{Pl}_A\, \varphi \mid (\textbf{set-pl}\ \omega)\varphi \mid (\textbf{refn-pl}\ \omega)\varphi,$$

Additionally, we define $\Diamond\varphi$ as $\top\mathcal{U}\varphi$, $\mathbf{Pl}$ as $\mathbf{Pl}_{\mathbb{A}\text{gt}}$, and $\mathbf{Ph}$ as $\mathbf{Pl}_\emptyset$. We will often use $\mathcal{L}^{base}_{ATLP}$ to refer to the language if the sets are clear from the context.

$\mathbf{Pl}_A$ assumes that agents in A play rationally; this means that the agents can only use strategy profiles that are *plausible* in the given model. In particular, $\mathbf{Pl}$ ($\equiv \mathbf{Pl}_{\mathbb{A}\text{gt}}$) imposes rational behaviour on all agents in the system. Similarly, $\mathbf{Ph}$ disregards plausibility assumptions, and refers to all *physically* available scenarios. The model update operator $(\textbf{set-pl}\ \omega)$ allows to define (or redefine) the set of plausible strategy profiles (referred to by Υ in the model) to the ones described by plausibility term ω (in this sense, it implements *revision* of plausibility). Operator $(\textbf{refn-pl}\ \sigma)$ enables *refining* the set of plausible strategy profiles, i.e. selecting a subset of the previously plausible profiles.

With **ATLP**, we can for example say that $\mathbf{Pl}\,\langle\langle\emptyset\rangle\rangle\Box(\mathsf{closed}\wedge\mathbf{Ph}\,\langle\langle guard\rangle\rangle\bigcirc\neg\mathsf{closed})$: *It is plausible to expect that the emergency door will always remain closed, but the guard retains the physical ability to open it*; or $(\textbf{set-pl}\ \omega_{NE})\mathbf{Pl}\,\langle\langle 2\rangle\rangle\Diamond\mathsf{money}_2$: *Suppose that only playing Nash equilibria is rational; then, agent a can plausibly reach a state where she gets some money*.

We note that, in contrast to [16, 43, 9], the concept of plausibility presented in this article is *objective*, i.e. it does not vary from agent to agent. This is very much in the spirit of game theory, where rationality criteria are used in an analogous way. Moreover, it is *global*, because plausibility sets do not depend on the state of the system. Note, however, that the denotation of plausibility terms depends on the actual state.

4.2 Semantics of $\mathcal{L}^{\mathbf{base}}_{\mathbf{ATLP}}$

To define the semantics of **ATLP**, we extend **CGS**'s to *concurrent game structures with plausibility*. Apart from an actual set of plausible strategies Υ, a *concurrent game structure with plausibility* (**CGSP**) must specify the denotation of plausibility terms $\omega \in \Omega$. It is defined via a *plausibility mapping*

$$[\![\cdot]\!] : Q \to (\Omega \to \mathcal{P}(\Sigma))$$

Instead of $[\![q]\!](\omega)$ we will often write $[\![\omega]\!]^q$ to turn the focus to the plausibility terms. Each term is mapped to a *set* of strategy profiles. Note also, that the denotation of a term depends on the state. In a way, the current state of the system defines the "initial position in the game", and this heavily influences the set of rational strategy profiles for most rationality criteria. For example, a strategy profile can be a Nash equilibrium (NE) in q_0, and yet it may not be a NE in some of its successors.

We will propose a more concrete (and more practical) implementation of plausibility terms in Section 4.4.

Definition 13 (CGSP) *A* concurrent game structure with plausibility (**CGSP**) *is given by a tuple*

$$M = \langle \mathbb{A}\mathrm{gt}, Q, \Pi, \pi, Act, d, o, \Upsilon, \Omega, [\![\cdot]\!]\rangle$$

*where $\langle \mathbb{A}\mathrm{gt}, Q, \Pi, \pi, Act, d, o\rangle$ is a **CGS**, $\Upsilon \subseteq \Sigma$ is a set of plausible strategy profiles (called* plausibility set*); Ω is a set of of plausibility terms, and $[\![\cdot]\!]$ is a plausibility mapping over Q and Ω.*

*By $CGSP(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$ we denote the set of all **CGSP**'s over $\mathbb{A}\mathrm{gt}$, Π and Ω. Furthermore, for a given **CGSP** M we use X_M to refer to element X of M, e.g., Q_M to refer to the set Q of states of M.*

Definition 14 (Compatible model) *Given a formula $\varphi \in \mathcal{L}_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$ a* ***CGSP*** *M is called* compatible *with φ if, and only if, $M \in CGSP(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$. That is, the model interprets all symbols occurring in φ. A model M is called compatible with a set $\mathcal{L}$ of* ***ATLP*** *formulae if, and only if, M is compatible with each formula in $\mathcal{L}$.*

We will assume by default that, given a formula or a set of formulae, the model we consider is compatible with it.

The formula $\mathbf{Pl}\,\langle\!\langle A \rangle\!\rangle\gamma$ implies that A can only play plausible strategies. Thus, A's part of the strategy profiles in Υ is of particular interest which motivates the following definition.

Definition 15 (Substrategy) *Let $A, B \subseteq \mathbb{A}\mathrm{gt}$ be groups of agents such that $A \subseteq B$ and let $s_B \in \Sigma_B$ be a collective strategy for agents B. We use $s_B|_A$ to denote A's* substrategy *t_A contained in s_B , i.e., strategy $t_A \in \Sigma_A$ such that $t_A^a = s_B^a$ for every $a \in A$.*[5] *For a singleton coalition $\{a\}$, we also write $s_B|_a$ instead of $s_B|_{\{a\}}$.*

For a given set $P_B \subseteq \Sigma_B$ of collective strategies of agents B, $P_B|_A$ denotes the set of A's substrategies in P_B, i.e.:

$$P_B|_A = \{s_A \in \Sigma_A \mid \exists s'_B \in P_B \quad (s'_B|_A = s_A)\}.$$

Often, we impose restrictions only on a subset $B \subseteq \mathbb{A}\mathrm{gt}$ of agents, without assuming rational play of all agents. This can be desirable due to several reasons. It might, for example, be the case that only information about the proponents' play is available; hence, assuming plausible behavior of the opponents is neither sensible nor justified. Or, even simpler, a group of (simple minded) agents might be known to not behave rationally.

Consider formula $\mathbf{Pl}_B\,\langle\!\langle A \rangle\!\rangle\gamma$: The team A looks for a strategy that brings about γ, but the members of the team who are also in B can only choose plausible strategies. The same applies to A's opponents that are contained in B. Strategies which comply with B's part of some plausible strategy profile are called B-plausible.

Definition 16 (B-plausibility of strategies) *Let $A, B \subseteq \mathbb{A}\mathrm{gt}$ and $s_A \in \Sigma_A$. We say that s_A is* B-plausible in *M if, and only if, B's substrategy in s_A is part of some plausible strategy profile in M, i.e., if $s_A|_{A\cap B} \in \Upsilon_M|_{A\cap B}$.*

By $\Upsilon_M(B)$ we denote the set of all B-plausible strategy profiles in M. That is, $\Upsilon_M(B) = \{s \in \Sigma \mid s|_B \in \Upsilon_M|_B\}$. Note that s_A is B-plausible iff $s_A \in \Upsilon_M(B)|_A$.

We observe that s_A is trivially B-plausible whenever A and B are disjoint.

As mentioned above, if some opponents belong to the set of agents who are assumed to play plausibly then they must also comply with the actual plausibility specifications when choosing their actions; this is taken into account by the following notion of plausible outcome.

[5] We recall that s_B^a (resp. t_A^a) denotes a's part of s_B (resp. t_A).

Definition 17 (B-plausible outcome) *The B-*plausible outcome*, $out_M(q, s_A, B)$, with respect to strategy s_A and state q is defined as the set of paths which can occur when only B-plausible strategy profiles can be played and agents in A follow s_A:*

$$out_M(q, s_A, B) \quad = \quad \{\lambda \in \Lambda_M(q) \mid \text{ there exists a } B\text{-plausible } t \in \Sigma \text{ such that } t|_A = s_A \text{ and } out_M(q, t) = \{\lambda\}\}.$$

Note that the outcome $out_M(q, s_A, B)$ is empty whenever the $(A \cap B)$'s part of s_A is not part of any plausible strategy profile in Υ_M. For example, assume that all agents in B play only parts of Nash equilibria. Then for a given s_A there are two possibilities for the B-consistent outcome. Either it is empty because $(A \cap B)$'s part of s_A does not belong to any Nash equilibrium, or it consists of all paths which can occur when (1) A stick to s_A, (2) B (including $A \cap B$) play according to some Nash equilibrium, and (3) the other agents behave arbitrarily.

The truth of **ATLP** formulae is given with respect to a model, a state, and a set B of agents. The intuitive reading of $M, q \models_B \varphi$ is: "φ is true in model M and state q if it is assumed that players in B play rationally", i.e., by using only plausible combinations of strategies. No constraints are imposed on the behaviour of agents outside B, but the plausibility operator $\mathbf{Pl}_A$ can be used to change the set of agents (viz A) whose play is restricted. The update/refinement modalities (**set-pl** ω)/(**refn-pl** ω) are used to change the plausibility set Υ_M in the model.

Definition 18 (Semantics of $\mathcal{L}^{\mathbf{base}}_{\mathbf{ATLP}}$) *Let $M \in CGSP(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$ and $A, B \subseteq \mathbb{A}\mathrm{gt}$. The semantics of **ATLP** formulae is given as follows:*

$M, q \models_B p$ *iff* $p \in \pi(q)$ *and* $p \in \Pi$

$M, q \models_B \neg\varphi$ *iff* $M, q \not\models_B \varphi$

$M, q \models_B \varphi \wedge \psi$ *iff* $M, q \models_B \varphi$ *and* $M, q \models_B \psi$

$M, q \models_B \langle\langle A\rangle\rangle \bigcirc \varphi$ *iff there is a B-plausible s_A s.t.* $M, \lambda[1] \models_B \varphi$ *for all* $\lambda \in out_M(q, s_A, B)$

$M, q \models_B \langle\langle A\rangle\rangle \Box \varphi$ *iff there is a B-plausible s_A s.t.* $M, \lambda[i] \models_B \varphi$ *for all* $\lambda \in out_M(q, s_A, B)$ *and all* $i \in \mathbb{N}_0$

$M, q \models_B \langle\langle A\rangle\rangle \varphi \mathcal{U} \psi$ *iff there is a B-plausible s_A such that, for all $\lambda \in out_M(q, s_A, B)$, there is $i \in \mathbb{N}_0$ with $M, \lambda[i] \models_B \psi$, and $M, \lambda[j] \models_B \varphi$ for all $0 \leq j < i$*

$M, q \models_B \mathbf{Pl}_A \varphi$ *iff* $M, q \models_A \varphi$

$M, q \models_B$ (**set-pl** $\omega)\varphi$ *iff $M', q \models_B \varphi$ where the new model M' is equal to M but the new set $\Upsilon_{M'}$ of plausible strategy profiles of M' is set to $[\![\omega]\!]^q_M$.*

$M, q \models_B (\mathbf{refn\text{-}pl}\ \omega)\varphi$ *iff* $M', q \models_B \varphi$ *where* M' *is equal to* M *but* $\Upsilon_{M'}$ *set to* $\Upsilon_M \cap [\![\omega]\!]^q_M$.

The "absolute" satisfaction relation $\models$ *is given by* $\models_\emptyset$.

Definition 19 (Validity) *Let* $\varphi \in \mathcal{L}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \Omega)$ *and* $\mathfrak{M} \subseteq CGSP(\mathbb{A}\text{gt}, \Pi, \Omega)$. *Formula* φ *is* valid *with respect to* $\mathfrak{M}$ *if, and only if,* $M, q \models \varphi$ *for every* $M \in \mathfrak{M}$ *and state* $q \in Q_M$.

Note that an ordinary concurrent game structure (without plausibility) can be interpreted as a **CGSP** with all strategy profiles assumed plausible, i.e., with $\Upsilon = \Sigma$, and empty set of plausibility terms Ω.

Let us clarify the semantics behind $\mathbf{Pl}_B\langle\langle A\rangle\rangle\gamma$ once more. The proponents (A) look for a strategy that enforces γ; some of them ($A \cap B$) are assumed to play a part of a plausible strategy profile while the others ($A \setminus B$) can choose an arbitrary collective strategy. Analogously, some opponents ($B \setminus A$) are supposed to play plausibly (that complies to set Υ_M together with the strategy already chosen by $A \cap B$), while the rest ($\mathbb{A}\text{gt} \setminus (A \cup B)$) have unrestricted choice. In particular, when $B = A$, only the choices of the proponents are restricted; for $B = \mathbb{A}\text{gt} \setminus A$ plausibility restrictions apply to the opponents only.

Remark 5 *We observe that our framework is semantically similar to the approach of* social laws *[39, 34, 46]. However, we refer to* strategy profiles *as rational or not, while social laws define constraints on agents'* individual actions. *Also, our motivation is different: In our framework, agents are expected to behave in a specified way because it is rational in some sense; social laws prescribe behaviour sanctioned by social norms and legal regulations.*

Example 11 (Asymmetric matching pennies ctd.) *Suppose that it is plausible to expect that both agents are rational in the sense that they only play undominated strategies.*[6] *Then,* $\Upsilon = \{(s_h, s_h), (s_h, s_t)\}$. *Under this assumption, agent* 2 *is free to grant itself with the prize or to refuse it:* $\mathbf{Pl}\,(\langle\langle 2\rangle\rangle\Diamond\,\mathsf{money}_2 \wedge \langle\langle 2\rangle\rangle\Box\neg\mathsf{money}_2)$. *Still, it cannot choose to win without making the other player win too:* $\mathbf{Pl}\,\neg\langle\langle 2\rangle\rangle\Diamond(\mathsf{money}_2 \wedge \neg\mathsf{money}_1)$. *Likewise, if rationality is defined via iterated undominated strategies, then we have* $\Upsilon = \{(s_h, s_h)\}$, *and therefore the outcome of the game is completely determined:* $\mathbf{Pl}\,\langle\langle\emptyset\rangle\rangle\Box(\neg\mathsf{start} \rightarrow \mathsf{money}_1 \wedge \mathsf{money}_2)$.

Note that, in order to include both *notions of rationality in the model, we can encode them as denotations of two different plausibility terms – say,* ω_{undom} *and* ω_{iter}, *with* $[\![\omega_{undom}]\!]^{q_0} = \{(s_h, s_h), (s_h, s_t)\}$, *and* $[\![\omega_{iter}]\!]^{q_0} = \{(s_h, s_h)\}$. *Let* M_1' *be model* M_1 *with plausibility terms and their denotation defined as above. Then,*

[6] We recall from Section 2.1 that a strategy $s_a \in \Sigma_a$ is called *undominated* if, and only if, there is no strategy $s'_a \in \Sigma_a$ such that the achieved utility of s'_a is at least as good as for s_a for all counterstrategies $s_{-a} \in \Sigma_{\mathbb{A}\text{gt}\setminus\{a\}}$ and strictly better for at least one counterstrategy $s_{-a} \in \Sigma_{\mathbb{A}\text{gt}\setminus\{a\}}$.

we have that $M_1', q_0 \models (\textbf{\textit{set-pl}}\ \omega_{undom})\mathbf{Pl}\,(\langle\langle 2\rangle\rangle\Diamond \mathsf{money}_2 \wedge \langle\langle 2\rangle\rangle\Box\neg\mathsf{money}_2) \wedge (\textbf{\textit{set-pl}}\ \omega_{iter})\mathbf{Pl}\,\langle\langle\emptyset\rangle\rangle\Box(\neg\mathsf{start} \rightarrow \mathsf{money}_1 \wedge \mathsf{money}_2)$.

Out of many solution concepts, Nash equilibrium is the most widely accepted, especially for non-cooperative games. We briefly extend our working example with game analysis based on Nash equilibrium. Note that, in this case, it is not possible to define rationality with independent constraints on agents' individual strategies (like in normative systems). These are full strategy profiles being rational or not, since rationality of a strategy depends on the actual response of the other players.

Example 12 (Asymmetric matching pennies ctd.) *Suppose that rationality is defined through Nash equilibria. Then,* $\Upsilon = \{(s_h, s_h), (s_t, s_t)\}$. *Under this assumption, agent* 2 *is sure to get the prize:* $\mathbf{Pl}\,\langle\langle\emptyset\rangle\rangle\Box(\neg\mathsf{start} \rightarrow \mathsf{money}_2)$.

Moreover, by choosing the right strategy, 2 *can control the outcome of the other agent:* $\mathbf{Pl}\,(\langle\langle 2\rangle\rangle\Box(\neg\mathsf{start} \rightarrow \mathsf{money}_1) \wedge \langle\langle 2\rangle\rangle\Box\neg\mathsf{money}_1)$. *Note that agent* 1 *can control her own outcome too, if we assume that the players are obliged to play rationally:* $\mathbf{Pl}\,(\langle\langle 1\rangle\rangle\Box(\neg\mathsf{start} \rightarrow \mathsf{money}_1) \wedge \langle\langle 1\rangle\rangle\Box\neg\mathsf{money}_1)$. *This may seem strange, but a Nash equilibrium assumes implicitly that the agents coordinate their actions somehow. Then, assuming a particular choice of one agent in advance* constrains *the other agent responses considerably, which puts the first agent at advantage.*

Example 13 (Bargaining ctd.) *Let* ω_{NE} *denote the set of Nash equilibria (every payoff can be reached by a Nash equilibrium), and* ω_{SPN} *the set of subgame perfect Nash equilibria in the game. Then, the following holds for every* $x \in [0,1]$*:*

$$M_2', q_0 \models$$
$$(\textbf{\textit{set-pl}}\ \omega_{NE})\langle\langle 1,2\rangle\rangle\Diamond(\mathsf{p}_1^{\mathsf{x}} \wedge \mathsf{p}_2^{1-\mathsf{x}}) \wedge (\textbf{\textit{set-pl}}\ \omega_{SPN})\langle\langle\emptyset\rangle\rangle\Diamond(\mathsf{p}_1^{\frac{1-\delta_2}{1-\delta_1\delta_2}} \wedge \mathsf{p}_2^{\frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2}}).$$

where M_2' *is given by* M_2 *extended by plausibility terms and their denotation as introduced above.*

Finally, we observe that the "plausibility refinement" operator ($\textbf{refn-pl}\ \cdot$) can be used to combine several solution concepts, e.g., $(\textbf{set-pl}\ \omega_{NE})(\textbf{refn-pl}\ \omega_{PO})$ restricts plausible play to *Pareto optimal Nash equilibria*. We can also use ($\textbf{refn-pl}\ \cdot$) to *compare* different notions of rationality. For example, $(\textbf{set-pl}\ \omega_{NE})(\textbf{refn-pl}\ \omega_{PO})\langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle \bigcirc \top$ can be used to check if Pareto optimal NE's exist in the model at all.

The base language $\mathcal{L}_{ATLP}^{\text{base}}$ allows to restrict the analysis to a subset of available strategy profiles. One drawback of $\mathcal{L}_{ATLP}^{\text{base}}$ is that we cannot specify sets of plausible/rational strategy profiles *in the object language*, simply because our terms do not have any internal structure — they are just constants. Ideally, one would like to have a flexible language of terms that allows to specify *any sensible rationality assumption*, and then impose it on the system.

Our first step is to employ formulae of **ATLI** and make use of the results in Section 3.4. The second step is to define a proper extension of $\mathcal{L}_{ATLP}^{\text{base}}$ where these concepts can be expressed, thus enabling both specification of plausibility and reasoning about plausible behaviour to be conducted in **ATLP**. The idea is to use **ATLP** formulae θ to specify sets of plausible strategy profiles, with the intended meaning that Υ collects exactly the profiles for which θ holds. Then, we can embed such an **ATLP**-based plausibility specification in another formula of **ATLP**.

4.3 Plausibility Terms based on ATLI

Definition 20 ($\mathcal{L}_{\mathbf{ATLP}^{\mathbf{ATLI}}}$) *Let* $\Omega^* = \{(\sigma.\theta) \mid \theta \in \mathcal{L}_{ATLI}(\mathbb{A}\text{gt}, \Pi, \{\sigma[1], \dots, \sigma[k]\})\}$. *That is,* Ω^* *collects terms of the form* $(\sigma.\theta)$*, where* θ *is an* ***ATLI*** *formula including only references to individual agents' parts of the strategy profile* σ.[7] *The language of* ***ATLP***$^{\mathbf{ATLI}}$ *is defined as* $\mathcal{L}_{ATLP}^{base}(\mathbb{A}\text{gt}, \Pi, \Omega^*)$.

The idea behind terms of this form is simple. We have an **ATLI** formula θ, parameterized with a variable σ that ranges over the set of strategy profiles Σ. Now, we want $(\sigma.\theta)$ to denote exactly the set of profiles from Σ, for which formula θ holds. However – as σ denotes a strategy profile, and **ATLI** allows only to refer to strategies of individual agents – we need a way of addressing substrategies of σ in θ. This can be done by using **ATLI** terms $\sigma[i]$, which are interpreted as i's substrategy in σ.

For example, we may assume that a rational agent does not grant the other agents with too much control over its life: $(\sigma \,.\, \bigwedge_{a\in\mathbb{A}\text{gt}}((\mathbf{str}_a\sigma[a])\neg\langle\langle\mathbb{A}\text{gt}\setminus\{a\}\rangle\rangle\Diamond\mathsf{dead_a}))$. Note that games defined by **CGS**'s are, in general, not determined, so the above specification does not guarantee that each rational agent can efficiently protect her life. It only requires that she should behave cautiously so that her opponents do not have complete power to kill her.

Definition 21 (Denotation of ATLI-based plausibility terms) *Let* M *be a* ***CGS*** *of the form* $M = \langle\mathbb{A}\text{gt}, Q, \Pi, \pi, Act, d, o\rangle$ *and* Ω^* *be as in Definition 20. For each* $s \in \Sigma$ *we define* M^s *to be the following* ***CGS*** *with intentions:*

$$M^s = \langle\mathbb{A}\text{gt}, Q, \Pi, \pi, Act, d, o, \mathcal{I}^0, \mathfrak{Str}, [\cdot]\rangle$$

with $\mathfrak{Str}_a = \{\sigma[a]\}$*, and* $[\sigma[a]] = s[a]$*. We recall from Section 3.4 that* $\mathcal{I}^0$ *represents the full intention relation.*

The plausibility mapping for terms from Ω^* *is defined as:*

$$[\![\sigma.\theta]\!]^q = \{s \in \Sigma \mid M^s, q \models \theta\}.$$

It is now possible to plug in arbitrary **ATLI** specifications of rationality, and reason about their consequences.

[7] σ is the only variable in θ and refers to a strategy profile.

Example 14 (Asymmetric matching pennies ctd.) *It seems that explicit quantification over the opponents' responses (not available in **ATLI**) is essential to express undominatedness of strategies (cf. [45] and Section 5.3). Still, we can at least assume that a rational player should avoid playing strategies that* guarantee *failure if a potentially successful strategy is available. Under this assumption, player* 1 *should never play tail, and in consequence player* 2 *controls the outcome of the game:*

$$M_1'', q_0 \models (\textit{\textbf{set-pl}}\ \sigma. \bigwedge_{a\in\mathbb{A}\mathrm{gt}}(\langle\!\langle \mathbb{A}\mathrm{gt}\rangle\!\rangle\Diamond \mathsf{money_a} \rightarrow (\mathbf{str}_a\sigma[a])\langle\!\langle \mathbb{A}\mathrm{gt}\rangle\!\rangle\Diamond \mathsf{money_a}))$$
$$\mathbf{Pl}\left(\langle\!\langle 2\rangle\!\rangle\Diamond(\mathsf{money}_1 \wedge \mathsf{money}_2) \wedge \langle\!\langle 2\rangle\!\rangle\Box\neg(\mathsf{money}_1 \wedge \mathsf{money}_2)\right).$$

where M_1'' *is the **CGS*** M_1 *extended with propositions* $\mathsf{p}_\mathsf{i}^1 \equiv \mathsf{money_i}$*, **ATLI**-based plausibility terms, and their denotation according to Definition 21.*

Moreover, if only Pareto optimal strategy profiles can be played, then both players are bound to keep winning money:

$$M_1'', q_0 \models (\textit{\textbf{set-pl}}\ \sigma.PO^\Diamond(\sigma))\ \mathbf{Pl}\ \langle\!\langle\emptyset\rangle\!\rangle\Box(\neg\mathsf{start} \rightarrow \mathsf{money}_1 \wedge \mathsf{money}_2).$$

Finally, restricting plausible strategy profiles to Nash equilibria guarantees that player 2 *should plausibly get money, but the outcome of player* 1 *is not determined:*

$$M_1'', q_0 \models (\textit{\textbf{set-pl}}\ \sigma.NE^\Diamond(\sigma))\ \mathbf{Pl}\left(\langle\!\langle\emptyset\rangle\!\rangle\Box(\neg\mathsf{start} \rightarrow \mathsf{money}_2)\right.$$
$$\left.\wedge\neg\langle\!\langle\emptyset\rangle\!\rangle\Diamond\mathsf{money}_1 \wedge \neg\langle\!\langle\emptyset\rangle\!\rangle\Box\neg\mathsf{money}_1\right).$$

Example 15 (Bargaining ctd.) *For the bargaining agents and* $\kappa = (1-\delta_2)\frac{1-(\delta_1\delta_2)^{\frac{T}{2}}}{1-\delta_1\delta_2} + (\delta_1\delta_2)^{\frac{T}{2}}$*, we have accordingly:*

1. $M_2', q_0 \models (\textit{\textbf{set-pl}}\ \sigma.NE^\Diamond(\sigma))\mathbf{Pl}\ \langle\!\langle\emptyset\rangle\!\rangle \bigcirc(\mathsf{p}_1^\mathsf{x} \wedge \mathsf{p}_2^{1-\mathsf{x}})$ *for every* x*;*

2. $M_2', q_0 \models (\textit{\textbf{set-pl}}\ \sigma.SPN^\Diamond(\sigma))\mathbf{Pl}\ \langle\!\langle\emptyset\rangle\!\rangle \bigcirc(\mathsf{p}_1^\kappa \wedge \mathsf{p}_2^{1-\kappa})$*;*

3. $M_2', q_0 \models (\textit{\textbf{set-pl}}\ \sigma.SPN^\Diamond(\sigma))\mathbf{Pl}\ \langle\!\langle\emptyset\rangle\!\rangle\Box(\neg\mathsf{p}_1^{\mathsf{x}_1} \wedge \neg\mathsf{p}_2^{\mathsf{x}_2})$ *for every* $x_1 \neq \kappa$ *and* $x_2 \neq 1-\kappa$

where M_2' *is the **CGSP** obtained from **CGS*** M_2 *by adding **ATLI**-based plausibility terms and their denotation.*

Thus, we can encode a game as a **CGS** M, specify rationality assumptions with an **ATLI** formula θ, and ask if a desired property φ of the system holds under these assumptions by model checking (**set-pl** $\sigma.\theta)\varphi$ in M. Note that the denotation of plausibility terms in Ω^* is fixed. We report our results on the complexity of solving such games in Section 6.

4.4 Language $\mathcal{L}^k_{ATLP}$ and $\mathcal{L}^\infty_{ATLP}$

As we have already explained, our main idea is to use **ATLP** for both specification of rationality assumptions and describtion of the outcome of rational play. Thus, we need a possibility to embed an **ATLP** formula φ (that defines the rationality condition) in a "higher-level" formula of **ATLP**, as a part of plausibility term (**set-pl** $\sigma.\varphi$). The reading of (**set-pl** $\sigma.\varphi)\psi$ is, again: "Let the plausibility set consist of profiles σ that satisfy φ; then, ψ holds". Apart from the possibility of nesting formulae via plausibility updates, we also propose to add quantifier-like structures to the language of terms. Consider, for example, the term $\sigma_1.(\exists\sigma_2)\varphi$. We would like to *collect* all strategies s_1 such that there is a strategy s_2 for which φ holds (we use σ_i to refer to s_i). Thus, $\sigma_1.(\exists\sigma_2)\varphi$ is supposed to act in a similar way as the first order logic-based set specification $\{x \mid \exists y : \varphi(x,y)\}$. It is easy to see that e.g. the set of all undominated strategies can now be specified in a straightforward way.

As before, the new version of **ATLP** is given over a set $\mathbb{A}\mathrm{gt} = \{a_1,\dots,a_k\}$ of agents, a set Π of propositions, and a set Ω of *primitive plausibility terms* (cf. Section 3.4). In addition to these sets, we also include a set $\mathcal{V}ar$ of *strategic variables*. Variables in $\mathcal{V}ar$ range over strategy profiles; we need them to characterize specific rationality criteria, in a way similar to first order logic specifications.

The definition of $\mathcal{L}_{ATLP}$ is given recursively. In each step the structure of plausibility terms becomes more sophisticated. At first, we only consider terms out of Ω; their interpretation is given in the model. On the next level, we also allow plausibility terms to be quantified **ATLP** formulae which contain strategic variables *and* elements from Ω. Plausibility terms of subsequent levels can again be based on terms from the previous levels, and so forth. In consequence, the *core 0-level language* of our new **ATLP** is almost the same as the base language $\mathcal{L}^{\mathrm{base}}_{ATLP}$ defined in Section 4.1: It extends it with simple combinations of terms.

In general, all the levels of the language can be seen as containing ordinary formulae of the original **ATLP**, the only thing that changes as we move to higher levels is the complexity of plausibility terms. We begin with defining simple combinations of plausibility terms, and then present the hierarchy of languages $\mathcal{L}^k_{ATLP}$, with the underlying idea that $\mathcal{L}^k_{ATLP}$ allows for at most k ($k \in \mathbb{N}_0$) nested plausibility updates. The full language $\mathcal{L}_{ATLP}$ allows for any arbitrary finite number of nestings.

Definition 22 (Strategic combination) *Let $\mathbb{A}\mathrm{gt}$ denote a set of agents and X be a non-empty set of symbols. We say that y is a* strategic combination *of x if it is generated by the following grammar:*

$$y ::= x \mid \langle y,\dots,y\rangle \mid y[A]$$

where $x \in X$, $\langle y,\dots,y\rangle$ is a vector of length $|\mathbb{A}\mathrm{gt}|$, and $A \subseteq \mathbb{A}\mathrm{gt}$. The set of strategic combinations *over X is defined by $\mathcal{T}(X)$. It is easy to see that operator $\mathcal{T}$ is idempotent ($\mathcal{T}(X) = \mathcal{T}(\mathcal{T}(X))$).*

The intuition is that elements of $x \in X$ are symbols in the object language that refer to sets of strategy profiles, and the elements of $\mathcal{T}(X)$ allow to combine these sets to new sets.[8] Let x refer to a set of strategy profiles $\chi \subseteq \Sigma$. Then, $x[A]$ refers to all the profiles in Σ in which A's substrategy agrees with some profile from χ. Similarly, if $x_1, \ldots, x_k$ denote sets of strategy profiles $\chi_1, \ldots, \chi_k$, then $\langle x_1, \ldots, x_k\rangle$ refers to all the profiles that agree on a_i's strategy with at least one profile from χ_i for each $i = 1, \ldots, k$.

Definition 23 ($\mathcal{L}^k_{ATLP}$) *Let $\mathbb{A}\text{gt}$ be a set of agents, Π a set of propositions, Ω be a set of primitive plausibility terms, and $\mathcal{V}ar$ a set of* strategic variables *(with typical element σ). The logics $\mathcal{L}^k_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{V}ar, \Omega)$, $k = 0, 1, 2, \ldots$, are recursively defined as follows:*

- $\mathcal{L}^0_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{V}ar, \Omega) = \mathcal{L}^{base}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \Omega_0)$, *where* $\Omega_0 = \mathcal{T}(\Omega)$*;*
- $\mathcal{L}^k_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{V}ar, \Omega) = \mathcal{L}^{base}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \Omega_k)$, *where:*

$$\Omega_k = \mathcal{T}(\Omega_{k-1} \cup \Omega^k),$$
$$\Omega^k = \{\sigma_1.(Q_2\sigma_2)\ldots(Q_n\sigma_n)\varphi \mid n \in \mathbb{N}, \forall i\, (1 \le i \le n \Rightarrow \sigma_i \in \mathcal{V}ar,\ Q_i \in \{\forall, \exists\}, \varphi \in \mathcal{L}^{base}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{T}(\Omega_{k-1} \cup \{\sigma_1, \ldots, \sigma_n\})))\ \}.$$

Thus, plausibility terms on level k (i.e., Ω_k) augment terms from the previous level (Ω_{k-1}) with new terms Ω^k that combine *quantification over strategic variables $\sigma_1, \ldots \sigma_n$ with formulae possibly containing these strategic variables*. Such terms are used to *collect* (or describe) specific strategy profiles (referred to by variable σ_1 which plays a distinctive role in comparison with the other variables).

Definition 24 ($\mathcal{L}_{ATLP}$) *The set of **ATLP** formulae with arbitrary finite nesting of plausibility terms is defined by*

$$\mathcal{L}_{ATLP} = \mathcal{L}^{\infty}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{V}ar, \Omega) = \lim_{k\to\infty} \mathcal{L}^k_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{V}ar, \Omega).$$

Definition 25 (k-formula, k-term) *Formula $\varphi \in \mathcal{L}^{\infty}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{V}ar, \Omega)$ is called an **ATLP**^{k}-formula (or simply k-*formula*) if, and only if, $\varphi \in \mathcal{L}^k_{ATLP}(\mathbb{A}\text{gt}, \Pi, \mathcal{V}ar, \Omega)$. Analogously, a plausibility term occurring in a k-formula is called a k-*term.

Remark 6 *We use the acronym **ATLP** to refer to both the full language $\mathcal{L}^{\infty}_{ATLP}$ and the basic sublanguage $\mathcal{L}^{base}_{ATLP}$.*

[8] This correspondence will be given formally in Definition 26 (Section 4.5).

Example 16 (Illustrating plausibility terms in $\mathcal{L}^k_{\textbf{ATLP}}$) *Below we present some simple formulae illustrating the different levels of our logic.*

$\mathcal{L}^{\textbf{base}}_{\textbf{ATLP}}$**:** $(\textbf{set-pl}\ \omega_{NE})\mathbf{Pl}\,\langle\langle A\rangle\rangle\gamma$; *group A can enforce γ if only Nash equilibria are played (we assume that ω_{NE} denotes exactly the set of Nash equilibria in the model).*

$\mathcal{L}^{0}_{\textbf{ATLP}}$**:** $(\textbf{set-pl}\ \langle\omega_{NE},\ldots,\omega_{NE}\rangle)\mathbf{Pl}\,\langle\langle A\rangle\rangle\gamma$; *plausibility terms can be combined. Note the difference to the previous formula, agents are assumed to play a strategy which is part of* some *NE. The resulting strategy profile does not have to be a Nash equilibrium, though.*

$\mathcal{L}^{1}_{\textbf{ATLP}}$**:** $\varphi \equiv (\textbf{set-pl}\ \sigma.\exists\sigma_1\varphi'(\sigma,\sigma_1))\mathbf{Pl}\,\langle\langle A\rangle\rangle\gamma$ *where $\varphi'(\sigma,\sigma_1)$ is a formula possibly containing operators $(\textbf{set-pl}\ \omega)$ with $\omega \in \mathcal{T}(\Omega\cup\{\sigma,\sigma_1\})$; e.g. $\varphi' \equiv (\textbf{set-pl}\ \langle\sigma,\ldots\sigma,\sigma_1,\omega_{NE}\rangle)\mathbf{Pl}\,\langle\langle A\rangle\rangle\gamma'$. We will have a closer look at the $(\textbf{set-pl}\ \cdot)$ operator in φ. The operator collects all strategies σ such that there exists another strategy profile σ_1 for which $\mathbf{Pl}\,\langle\langle A\rangle\rangle\gamma'$ holds if all but the last 2 agents play according to σ, the second to last agent plays according to σ_1, and the last one according to a fixed strategy out of ω_{NE}.*

$\mathcal{L}^{2}_{\textbf{ATLP}}$**:** *Consider the previous formula φ again, but this time $\varphi'(\sigma,\sigma_1)$ can also contain quantification; e.g. $\varphi' \equiv ((\textbf{set-pl}\ \langle\sigma,\ldots,\sigma,\sigma_1,\omega_{NE}\rangle)\mathbf{Pl}\,\langle\langle B\rangle\rangle\gamma') \rightarrow ((\textbf{set-pl}\ \sigma'.\exists\sigma_1'\varphi''(\sigma',\sigma_1'))\mathbf{Pl}\,\langle\langle A\rangle\rangle\gamma)$ where $\varphi''(\sigma',\sigma_1')$ is a base formula with plausibility terms taken from $\mathcal{T}(\Omega\cup\{\sigma',\sigma_1'\})$.*

In the next section we show how the denotation of complex terms is constructed, and how it is plugged into the semantics of **ATLP** from Section 4.2.

4.5 Semantics of $\mathcal{L}^k_{\textbf{ATLP}}$ and $\mathcal{L}^\infty_{\textbf{ATLP}}$

$\mathcal{L}^k_{ATLP}$ does not change the very structure of **ATLP** formulae, it only extends $\mathcal{L}^{\text{base}}_{ATLP}$ by more ornate plausibility terms. Therefore, it seems natural that the plausibility mapping for theses terms is of particular interest; the denotation reflects the construction of strategic combinations given in Definition 22.

Definition 26 (Extended plausibility mapping $\widehat{[\![\cdot]\!]}$) *Let $M \in CGSP(\mathbb{A}\text{gt},\Pi,\Omega)$. The* extended plausibility mapping $\widehat{[\![\cdot]\!]}_M$ *with respect to $[\![\cdot]\!]_M$ is defined as follows:*

1. *If $\omega\in\Omega$ then $\widehat{[\![\omega]\!]}^q_M = [\![\omega]\!]^q_M$;*
2. *If $\omega = \omega'[A]$ then $\widehat{[\![\omega]\!]}^q_M = \{s\in\Sigma \mid \exists s' \in \widehat{[\![\omega']\!]}^q_M\ s|_A = s'|_A\}$;*

3. *If* $\omega = \langle \omega_1, \ldots \omega_k \rangle$ *then* $\widehat{[\![\omega]\!]}^q_M = \{s \in \Sigma \mid \exists t_1 \in \widehat{[\![\omega_1]\!]}^q_M, \ldots, \exists t_k \in \widehat{[\![\omega_k]\!]}^q_M \forall i = 1, ..., k\ s|_{a_i} = t_i|_{a_i})\}$*;*

4. *If* $\omega = \sigma_1.(Q_2\sigma_2)\ldots(Q_n\sigma_n)\varphi$ *then*

$$\widehat{[\![\omega]\!]}^q_M = \{s_1 \in \Sigma \mid Q_2 s_2 \in \Sigma, \ldots, Q_n s_n \in \Sigma \quad (M^{s_1,\ldots,s_n}, q \models \varphi)\},$$

where $M^{s_1,\ldots,s_n}$ *is equal to* M *except that we fix* $\Upsilon_{M^{s_1,\ldots,s_n}} = \Sigma$*,* $\Omega_{M^{s_1,\ldots,s_n}} = \Omega_M \cup \{\sigma_1, \ldots, \sigma_n\}$*,* $[\![\sigma_i]\!]^q_{M^{s_1,\ldots,s_n}} = \{s_i\}$*, and* $[\![\omega]\!]^q_{M^{s_1,\ldots,s_n}} = [\![\omega]\!]^q_M$ *for all* $\omega \neq \sigma_i$*,* $1 \leq i \leq n$*, and* $q \in Q_M$*. That is, the denotation of* σ_i *in* $M^{s_1,\ldots,s_n}$ *is set to strategy profile* s_i*.*[9]

Consider, for instance, plausibility term $\sigma_1.\forall\sigma_2\varphi$. The extended plausibility mapping $\widehat{[\![\sigma_1.\forall\sigma_2\varphi]\!]}_q$ collects all strategy profiles $s_1 \in \Sigma$ (referred to by σ_1) such that *for all* strategy profiles $s_2 \in \Sigma$ (referred to by σ_2) φ is true in model M^{s_1,s_2} and state $q \in Q$, i.e. $M^{s_1,s_2}, q \models \varphi$ for all $s_2 \in \Sigma$.

Remark 7 *Note that if the language includes a term* $\omega_\top$ *that refers to* all *strategy profiles, then* $x[A]$ *can be expressed as* $\langle \omega_1, \ldots, \omega_k \rangle$*, where* $\omega_a = x_a$ *for* $a \in A$*, and* $\omega_a = \omega_\top$ *otherwise. We also observe that in* $\mathcal{L}^k_{ATLP}$*,* $k > 0$*,* $\omega_\top$ *can be expressed as* $\sigma.\top$*.*

In Definition 18 we defined the semantics of the base language of **ATLP**. Truth of $\mathcal{L}^k_{ATLP}$ formulae is defined in the same way, we only need to replace the previous (simple) plausibility mapping by the extended one in the semantics of plausibility updates.

Definition 27 (Semantics of $\mathcal{L}^k_{ATLP}$ and $\mathcal{L}^\infty_{ATLP}$) *The semantics for* $\mathcal{L}_{ATLP}$ *formulae is given as in Definition 18 with the extended plausibility mapping* $\widehat{[\![\cdot]\!]}_M$ *used instead of* $[\![\cdot]\!]_M$*. I.e., only the semantic clauses for* (***set-pl*** ω)φ *and* (***refn-pl*** ω)φ *change as follows:*

$M, q \models_B$ (**set-pl** ω)φ *iff* $M', q \models_B \varphi$ *where the new model* M' *is equal to* M *but the new set* $\Upsilon_{M'}$ *of plausible strategy profiles is set to* $\widehat{[\![\omega]\!]}^q_M$*;*

$M, q \models_B$ (**refn-pl** ω)φ *iff* $M', q \models_B \varphi$ *where the new model* M' *is equal to* M *but the new set* $\Upsilon_{M'}$ *of plausible strategy profiles is set to* $\Upsilon_M \cap \widehat{[\![\omega]\!]}^q_M$*.*

Remark 8 *By a slight abuse of notation, we will refer to the extended plausibility mapping with the same symbol as to the simple plausibility mapping, i.e., with* $[\![\cdot]\!]$*.*

We will discuss some important examples of $\mathcal{L}_{ATLP}$ formulae and terms (together with their interpretation) in Sections 5.2 and 5.3 where **ATLP** characterizations of solution concepts are presented.

[9] It should be emphasized that model $M^{s_1,\ldots,s_n}$ in which plausibility of profile s_1 is evaluated does *not* presuppose any notion of plausibility, i.e., $\Upsilon_{M^{s_1,\ldots,s_n}} = \Sigma$.

5 Properties of ATLP

This section contains our main conceptual results. We show:

1. That several logics can be embedded into **ATLP** by means of polynomial translation of models and/or formulae (Section 5.1),
2. That several classical solution concepts for extensive games (Nash equilibria, subgame perfect Nash equilibria, Pareto Optimality), can be characterized in **ATLP** already in the language $\mathcal{L}^1_{ATLP}$ (Section 5.2),
3. That these solution concepts can be also re-formulated in a qualitative way, through appropriate formulae of **ATLP** parameterized by **ATL** path formulae (Section 5.3).

5.1 Embedding Existing Logics into ATLP

In this section, we compare **ATLP** with several related logics and show their formal relationships. To this end, we first define notions that allow to compare expressivity of logical systems. *Embedding* takes place on the level of satisfaction relations ($\models$): Logic L_1 embeds L_2 if models and formulae of L_2 can be simulated in L_1 in a truth-preserving way. *Subsumption* refers to the level of valid sentences: L_1 subsumes L_2 if all the validities of L_2 are validities of L_1 as well.

Definition 28 (Embedding) Logic L_1 embeds logic L_2 *iff there is a translation* tr *of* L_2 *formulae into formulae of* L_1*, and a transformation* TR *of* L_2 *models into models of* L_1*, such that* $M, q \models_{L_2} \varphi$ *iff* $TR(M), q \models_{L_1} tr(\varphi)$ *for every pointed model* M, q *and formula* φ *of* L_2.

Note that the translation of formulae and transformation of models are supposed to be independent. This prevents translation schemes that transform triples $M, q \models \varphi$ in L_2 to $M', q \models \top$, and triples $M, q \not\models \varphi$ in L_2 to $M', q \not\models \bot$ (with an arbitrary model M'), that would yield embeddings between any pair of logics.

It is important to point out that all the transformation and translation schemes proposed in this section can be computed in polynomial time and incur only polynomial increase in the size of models and the length of formulae. Thus, we are in fact interested in *polynomial* embeddings of logics in **ATLP**.

Definition 29 (Subsumption) Logic L_1 subsumes logic L_2 *iff the set of validities of* L_1 *subsumes validities of* L_2.

Proposition 9 ***ATLP*** *embeds* ***ATL***.

Proof We use the identity translation of formulae: $tr(\varphi) \equiv \varphi$. As for models, $TR(M) = M'$ that extends M with an arbitrary set of plausible strategy profiles Υ. It is easy to see that the plausibility assumptions Υ will never be used in the evaluation of φ since φ includes no **Pl** operators. Thus, the result of the evaluation will be the same as for $M, q \models \varphi$. ■

The above reasoning implies also that **ATL** validities hold for all **ATLP** models.

Corollary 10 ***ATLP** subsumes **ATL**.*

The relationship of **ATLP** to most other logics can be studied only in the context of embedding, as they use different modal operators (and thus yield incomparable sets of valid formulae). We begin with embedding "**ATL** with Intentions" [30] in **ATLP**. Then, we show that "**CTL** with Plausibility" from [10] can be embedded in **ATLP** for a limited (but very natural) class of models. Finally, we propose an embedding of the two existing versions of Game Logic with Preferences [50, 51] which allow to reason about what can happen under *particular* game-theoretical rationality assumptions.

Proposition 11 ***ATLP** embeds **ATLI**.*

Proof sketch For an **ATLI** model $M = \langle \mathbb{A}\mathrm{gt}, Q, \Pi, \pi, Act, d, o, \mathcal{I}, \mathfrak{Str}, [\cdot] \rangle$, we construct the corresponding concurrent game structure with plausibility $TR(M) = \langle \mathbb{A}\mathrm{gt}, Q, \Pi, \pi, Act, d, o, \Upsilon, \Omega, [\![\cdot]\!] \rangle$ with the set of plausible strategy profiles $\Upsilon = \{s \in \Sigma \mid s \text{ is consistent with } \mathcal{I}\}$, plausibility terms $\Omega = \{\omega_\sigma \mid \sigma \in \mathfrak{Str}\} \cup \{\omega_\top\}$, and their denotation $[\![\omega_\top]\!]^q = \Sigma$ and $[\![\omega_\sigma]\!]^q = \{s \in \Sigma \mid s|_a = [\sigma]\}$ for each $\sigma \in \mathfrak{Str}_a$.

For an **ATLI** formula φ, we construct its **ATLP** translation by transforming strategic assumptions (about agents' intentions) imposed by $(\mathbf{str}_a\sigma)$ to plausibility assumptions (about strategy profiles that can be plausibly played) defined by (**set-pl** ω_σ) and applying them to the appropriate set of agents (i.e., those for whom strategic assumptions have been defined). Formally, the translation is defined as $tr(\varphi) = \mathbf{Pl}\ tr_{\langle \omega_\top, \ldots, \omega_\top \rangle}(\varphi)$, where $tr_{\langle \omega_1, \ldots, \omega_k \rangle}$ is defined as follows:

$$
\begin{array}{rcl}
tr_{\langle \omega_1, \ldots, \omega_k \rangle}(p) & = & p, \\
tr_{\langle \omega_1, \ldots, \omega_k \rangle}(\neg\varphi) & = & \neg tr_{\langle \omega_1, \ldots, \omega_k \rangle}(\varphi), \\
tr_{\langle \omega_1, \ldots, \omega_k \rangle}(\varphi_1 \wedge \varphi_2) & = & tr_{\langle \omega_1, \ldots, \omega_k \rangle}(\varphi_1) \wedge tr_{\langle \omega_1, \ldots, \omega_k \rangle}(\varphi_2), \\
tr_{\langle \omega_1, \ldots, \omega_k \rangle}(\langle\!\langle A \rangle\!\rangle \bigcirc \varphi) & = & \langle\!\langle A \rangle\!\rangle \bigcirc tr_{\langle \omega_1, \ldots, \omega_k \rangle}(\varphi), \\
 & & \text{(for } \langle\!\langle A \rangle\!\rangle \Box\varphi \text{ and } \langle\!\langle A \rangle\!\rangle \varphi_1 \mathcal{U} \varphi_2 \text{ analogously)} \\
tr_{\langle \omega_1, \ldots, \omega_k \rangle}((\mathbf{str}_a \sigma'_a)\varphi) & = & (\textbf{set-pl}\ \overrightarrow{\omega}) tr_{\overrightarrow{\omega}}(\varphi), \\
\text{where } \overrightarrow{\omega} & = & \langle \omega_1, \ldots, \omega_{\sigma'_a}, \ldots, \omega_k \rangle.
\end{array}
$$

Note that, for "vanilla" **ATLI**, $\langle\langle A\rangle\rangle\gamma$ holds iff γ can be enforced *against every response strategy from* $\mathbb{A}\mathrm{gt} \setminus A$. Thus, e.g., $M, q \models_{\mathbf{ATLI}} (\mathbf{str}_a\sigma_a)\langle\langle A\rangle\rangle\Box p$ iff $TR(M), q \models_{\mathbf{ATLP}} \mathbf{Pl}\,(\mathbf{set\text{-}pl}\ \langle\omega_\top, \ldots, \omega_{\sigma_a}, \ldots, \omega_\top\rangle)\langle\langle A\rangle\rangle\Box p$, and analogously for the other cases. ■

CTLP, i.e., "**CTL** with Plausibility" [10], is an extension of the branching-time logic **CTL** with a similar notion of plausibility as the one we use here. The main difference lies in the fact that **CTLP** formulae refer to plausible *paths* rather than strategy profiles.

Proposition 12 ***ATLP** embeds **CTLP** in the class of transition systems.*

Proof sketch To transform models, we first observe that every transition system M can be seen as a concurrent game structure that includes only a single agent a_1. Furthermore, we can transform M to a **CGSP** $TR(M)$ by adding $\Upsilon = \Sigma$ and $\Omega = \emptyset$ (cf. Section 4.1). To translate **CTLP** formulae, we use the scheme below:

$$
\begin{aligned}
tr(\varphi) &= tr_{(\mathbf{set\text{-}pl}\ \sigma.\top)}(\varphi)\\
tr_\omega(p) &= p, \quad tr_\omega(\neg\varphi) = \neg tr_\omega(\varphi), \quad tr_\omega(\varphi_1 \wedge \varphi_2) = tr_\omega(\varphi_1) \wedge tr_\omega(\varphi_2),\\
tr_\omega(\mathsf{E}\gamma) &= \langle\langle \mathbb{A}\mathrm{gt}\rangle\rangle tr_\omega(\gamma),\\
tr_\omega(\bigcirc\varphi) &= \bigcirc tr_\omega(\varphi) \quad \text{(for } \Box\varphi \text{ and } \varphi_1\mathcal{U}\varphi_2 \text{ analogously);}\\
tr_\omega(\mathbf{Pl}\,\varphi) &= (\mathbf{set\text{-}pl}\ \omega)\mathbf{Pl}\,tr_\omega(\varphi),\\
tr_\omega(\mathbf{Ph}\,\varphi) &= \mathbf{Ph}\,tr_\omega(\varphi),\\
tr_\omega((\mathbf{set\text{-}pl}\ \gamma)\varphi) &= tr_{\omega'}(\varphi),\\
\text{where} \quad \omega' &= \sigma.(\mathbf{set\text{-}pl}\ \sigma)\mathbf{Pl}\,\langle\langle\emptyset\rangle\rangle\gamma.
\end{aligned}
$$

Now, $M, q \models_{\mathbf{CTLP}} \varphi$ iff $M, q \models_{\mathbf{ATLP}} tr(\varphi)$.

Note that we cannot use the above construction for arbitrary models of **CTLP**, as not every set of (plausible) paths can be obtained by memoryless strategy profiles. ■

Proposition 13 ***ATLP** cannot be polynomially embedded in neither **ATL**, nor **ATLI**, nor **CTLP**.*

Proof Suppose that any of these logics polynomially embeds **ATLP**. Then, the embedding provides a polynomial reduction of model checking from **ATLP** to that logic. Since model checking of **ATL**, **ATLI**, and **CTLP** can be done in polynomial deterministic time [3, 30, 10], we get that the problem for **ATLP** is in P, too. But model checking **ATLP** is $\mathbf{\Delta_3^P}$-hard already for $\mathcal{L}_{ATLP}^{\text{base}}$ (see Section 6).

There is not much work on logical descriptions of behaviour of agents under rationality assumptions based on game-theoretical solution concepts. In fact, we know only of one such logic for agents with perfect information, which is **GLP** from [51]. There, agents can be assumed qualitative preferences (i.e., a propositional formula φ_0 that they supposedly want to make eventually true). Moreover, they are assumed to play rationally in the sense that if they have some strategies that guarantee $\Diamond\varphi_0$, they can use only those strategies in their play. Interestingly enough, the preference criterion was different in a preliminary version of **GLP** [50], where it was based on the notion of Nash equilibrium. Both versions of **GLP** can be embedded in **ATLP**. One may embed game logics with other preference criteria in an analogous way.

Proposition 14 *GLP can be embedded in ATLP.*

Proof sketch For the translation of models, we transform game trees of **GLP** to concurrent game structures using the construction from Section 3.3, and transform the **CGS** to **CGSP** by taking $\Upsilon = \Sigma$ and $\Omega = \emptyset$. Then, we use the following translation of **GLP** formulae:

$$
\begin{aligned}
tr(\varphi) &= \mathbf{Pl}\ tr_{(\mathbf{set\text{-}pl}\ \sigma.\top)}(\varphi),\\
tr_\omega(p) &= p, \quad tr_\omega(\neg\varphi) = \neg tr_\omega(\varphi), \quad tr_\omega(\varphi \vee \psi) = tr_\omega(\varphi) \vee tr_\omega(\psi),\\
tr_\omega(\Box\varphi_0) &= \langle\langle\emptyset\rangle\rangle\Diamond\varphi_0,\\
tr_\omega([a:\varphi_0]\psi) &= (\mathbf{set\text{-}pl}\ \omega')tr_{\omega'}(\psi),\\
\text{where}\quad \omega' &= \sigma.\mathrm{Pl}\,(\mathbf{set\text{-}pl}\ \omega)\langle\langle\emptyset\rangle\rangle\Box\big(plausible(\sigma) \wedge prefers(a,\sigma,\varphi_0)\big)\\
plausible(\sigma) &\equiv (\mathbf{refn\text{-}pl}\ \sigma)\langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle \bigcirc\top\\
prefers(a,\sigma,\varphi_0) &\equiv \langle\langle a\rangle\rangle\Diamond\varphi_0 \rightarrow (\mathbf{refn\text{-}pl}\ \sigma[a])\langle\langle\emptyset\rangle\rangle\Diamond\varphi_0.
\end{aligned}
$$

That is, with each subsequent preference operator $[a : \varphi_0]$, only those from the (currently) plausible strategy profiles are selected that are preferred by a. The preference is based on the (subgame perfect) enforceability of the outcome φ_0 at the end of the game: if φ_0 can be enforced at all, then a prefers strategies that do enforce it.

Now, we have that $\Gamma \models_{\mathrm{GLP}} \varphi$ iff $TR(\Gamma), \emptyset \models_{\mathrm{ATLP}} tr(\varphi)$.[10] ■

Proposition 15 *Preliminary GLP can be embedded in ATLP.*

Proof Analogous to Proposition 14. The translation only differs in the characterization of agents' preferences. The agents are now assumed to stick to their individual parts of Nash equilibria defined by a zero-sum game where a

[10] Again, $\emptyset$ denotes the position with empty history, i.e., the initial state of the game.

wins iff φ_0 is enforced:[11]

$$
\begin{aligned}
\omega' &= \sigma.\exists\sigma' \mathbf{Pl}\,(\mathbf{set\text{-}pl}\ \omega)\langle\langle\emptyset\rangle\rangle\Box \\
&\quad \big(\mathit{plausible}(\sigma) \wedge NE(\sigma', a, \varphi_0) \wedge \mathit{coincides}(\sigma, \sigma', a)\big) \\
\mathit{plausible}(\sigma) &\equiv (\mathbf{refn\text{-}pl}\ \sigma)\langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle \bigcirc \top \\
\mathit{coincides}(\sigma, \sigma', a) &\equiv (\mathbf{set\text{-}pl}\ \sigma[a])(\mathbf{refn\text{-}pl}\ \sigma'[a])\langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle \bigcirc \top \\
NE(\sigma, a, \varphi_0) &\equiv \bigwedge_{i\in\mathbb{A}\mathrm{gt}} BR_i(\sigma, a, \varphi_0), \\
BR_i(\sigma, a, \varphi_0) &\equiv \begin{cases} (\mathbf{refn\text{-}pl}\ \sigma[\mathbb{A}\mathrm{gt}\setminus\{i\}])\langle\langle i\rangle\rangle\Diamond\varphi_0 & \\ \quad \rightarrow (\mathbf{refn\text{-}pl}\ \sigma)\langle\langle\emptyset\rangle\rangle\Diamond\varphi_0 & \text{for } i = a \\ (\mathbf{refn\text{-}pl}\ \sigma[\mathbb{A}\mathrm{gt}\setminus\{i\}])\langle\langle i\rangle\rangle\Box\neg\varphi_0 & \\ \quad \rightarrow (\mathbf{refn\text{-}pl}\ \sigma)\langle\langle\emptyset\rangle\rangle\Box\neg\varphi_0 & i \neq a \end{cases}
\end{aligned}
$$

■

A couple other logics were defined for various solution concepts with respect to incomplete information games [49, 48]. We do not study them here, since our framework lacks the notions of knowledge and uncertainty – but it seems a promising area of future research.

Remark 16 *We have presented embeddings of several quite different logics into **ATLP**, which suggests substantial gain in expressive power. Most of them (**ATL**, **ATLI**, and **CTLP**) are embedded already in the lowest levels of the **ATLP** hierarchy (i.e., $\mathcal{L}^{base}_{ATLP}$ or $\mathcal{L}^{1}_{ATLP}$ with no quantifiers). **GLP** formulae with at most k preference operators are embedded in $\mathcal{L}^{k}_{ATLP}$, which is inevitable given their semantics that combines model update and irrevocable strategic quantification (cf. the discussion and the complexity results in [1, 7]).*

5.2 Classical Solution Concepts in $\mathcal{L}^1_{ATLP}$

In Section 3.3 we showed how extensive games Γ (with a finite set of utilities) can be expressed by **CGS**'s: each Γ can be transformed in a **CGS** $M(\Gamma)$ such that they correspond (in the sense of Definition 8).

The following terms rewrite the specification of best response profiles, Nash equilibria, and the specification of subgame-perfect Nash equilibria from Section 3.4. Note that the new specifications use only **ATLP** operators.

$$
\begin{aligned}
BR^T_a(\sigma) &\equiv (\mathbf{set\text{-}pl}\ \sigma[\mathbb{A}\mathrm{gt}\setminus\{a\}])\mathbf{Pl} \bigwedge_{v\in U}\Big((\langle\langle a\rangle\rangle T\mathsf{p}^{\mathsf{v}}_{\mathsf{a}}) \rightarrow (\mathbf{set\text{-}pl}\ \sigma)\langle\langle\emptyset\rangle\rangle T\mathsf{p}^{\mathsf{v}}_{\mathsf{a}}\Big) \\
NE^T(\sigma) &\equiv \bigwedge_{a\in\mathbb{A}\mathrm{gt}} BR^T_a(\sigma) \\
SPN^T(\sigma) &\equiv \langle\langle\emptyset\rangle\rangle\Box NE^T(\sigma)
\end{aligned}
$$

[11] Note the similarity of the scheme below to the characterization of qualitative Nash equilibrium in Section 5.3.

Recalling briefly the ideas behind the above specifications, $BR_a^T(\sigma)$ holds iff $\sigma[a]$ is the best response to $\sigma[\mathbb{A}\mathrm{gt} \setminus \{a\}]$. That is, after we fix the $\mathbb{A}\mathrm{gt} \setminus \{a\}$'s collective strategy to $\sigma[\mathbb{A}\mathrm{gt}\setminus\{a\}]$, agent a cannot obtain a better temporal pattern of payoffs than by playing $\sigma[a]$. Then, σ is a *Nash equilibrium* if each individual strategy $s[a]$ is the best response to the opponent's strategies $\sigma[\mathbb{A}\mathrm{gt} \setminus \{a\}]$ (cf. [35]). The formalization of a subgame perfect Nash equilibrium is straightforward: We require profile σ to be a Nash equilibrium in all reachable states (seen as initial positions of particular subgames).

The following propositions are simple adaptations of the results from Section 3.4.

Proposition 17 *Let Γ be an extensive game with a finite set of utilities. Then the following holds:*

1. $s \in [\![\sigma.NE^{\diamond}(\sigma)]\!]^{\emptyset}_{M(\Gamma)}$ *iff s is a Nash equilibrium in Γ.*

2. $s \in [\![\sigma.SPN^{\diamond}(\sigma)]\!]^{\emptyset}_{M(\Gamma)}$ *iff s is a subgame perfect Nash equilibrium in Γ.*

In Section 3.4 we defined a quantitative version of *Pareto optimality* formulated in **ATLI**. However, as we pointed out, the **ATLI** formula had exponential length and some counterintuitive implications. Quantification allows to propose a more compact and intuitive specification:

$$PO^T(\sigma) \equiv \forall\sigma' \,\mathbf{Pl}\Big(\bigwedge_{a\in\mathbb{A}\mathrm{gt}} \bigwedge_{v\in U} \big((\mathbf{set\text{-}pl}\ \sigma')\langle\!\langle\emptyset\rangle\!\rangle T\mathsf{p}_\mathsf{a}^\mathsf{v} \to (\mathbf{set\text{-}pl}\ \sigma)\langle\!\langle\emptyset\rangle\!\rangle T\mathsf{p}_\mathsf{a}^\mathsf{v}\big) \vee \bigvee_{a\in\mathbb{A}\mathrm{gt}} \bigvee_{v\in U} \big((\mathbf{set\text{-}pl}\ \sigma)\langle\!\langle\emptyset\rangle\!\rangle T\mathsf{p}_\mathsf{a}^\mathsf{v} \wedge \neg(\mathbf{set\text{-}pl}\ \sigma')\langle\!\langle\emptyset\rangle\!\rangle T\mathsf{p}_\mathsf{a}^\mathsf{v}\big)\Big).$$

This definition of Pareto optimality is more intuitive than the one given in Section 3.4 because it does not focus on temporal evolution of whole payoff profiles, but rather on the interaction between temporal patterns of individual patterns.

Proposition 18 *Let Γ be an extensive game with a finite set of utilities. Then:*

$$s \in [\![\sigma.PO^{\diamond}(\sigma)]\!]^{\emptyset}_{M(\Gamma)} \textit{ iff } s \textit{ is Pareto optimal in } \Gamma.$$

Let $\langle x^A, y^{\mathbb{A}\mathrm{gt}\setminus A}\rangle$ be a shorthand for the term $\langle z_1, \ldots, z_k\rangle$ with $z_a = x$ for $a \in A$ and $z_a = y$ otherwise. The following specification, formulated as an $\mathcal{L}^1_{ATLP}$ formula, characterizes the set of strategy profiles that include undominated strategies for agent a:

$$UNDOM^T(\sigma) \equiv \quad \forall\sigma_1\forall\sigma_2\exists\sigma_3$$
$$\mathbf{Pl}\Big(\bigwedge_{v\in U}\big((\textbf{set-pl}\ \langle\sigma_1^{\{a\}},\sigma_2^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle T\mathsf{p}_a^v \rightarrow (\textbf{set-pl}\ \langle\sigma^{\{a\}},\sigma_2^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle T\mathsf{p}_a^v\big)$$
$$\vee \bigvee_{v\in U}\big((\textbf{set-pl}\ \langle\sigma^{\{a\}},\sigma_3^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle T\mathsf{p}_a^v \wedge \neg(\textbf{set-pl}\ \langle\sigma_1^{\{a\}},\sigma_3^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle T\mathsf{p}_a^v\big)\Big).$$

Proposition 19 *Let Γ be an extensive game with a finite set of utilities. Then*

$$s \in [\![\sigma.UNDOM^\Diamond(\sigma)]\!]^\emptyset_{M(\Gamma)} \text{ iff } s|_a \text{ is undominated in } \Gamma.$$

5.3 General Solution Concepts in $\mathcal{L}^1_{ATLP}$

In this section, we return to the idea of *general solution concepts* from Section 3.5 and show how qualitative versions of NE, SPN, PO and UNDOM can be captured in **ATLP**. Like for temporalized solution concepts, it turns out that their qualitative counterparts can be already specified in $\mathcal{L}^1_{ATLP}(\mathbb{A}\mathrm{gt},\Pi,\emptyset)$. That is, we need only one level of nested plausibility updates (and no "hard-wired" plausibility terms) to effectively capture classical notions of rationality and extend them to more general games that we study in this paper.

We only consider one "winning condition" per agent to represent agents' preferences, but this view can be naturally extended to full preference lists, as in Section 3.5. In what follows, let $\overrightarrow{\eta} = \langle\eta_1,\dots,\eta_k\rangle$ be a vector of $\mathcal{L}_{ATL}$ path formulae.

Definition 30 (Transforming a CGSP into a NF Game) *Let $M \in$ **CGSP**$(\mathbb{A}\mathrm{gt},\Pi,\Omega)$ and $q \in Q_M$. The* associated NF game $\mathcal{S}(M,\overrightarrow{\eta},q)$ *with respect to $\overrightarrow{\eta}$ is given as in Definition 11 with M interpreted as a pure **CGS** by removing Υ and $[\![\cdot]\!]$ from it.*

Our aim is to define analogues of classical solution concepts (Nash equilibria and such) that are based on explicit "winning conditions" η_i instead of numerical payoffs. We can build on our results from the previous section; we only need to replace temporal patterns of payoffs with the formulae η_i:

$$BR_a^{\overrightarrow{\eta}}(\sigma) \equiv (\textbf{set-pl}\ \sigma[\mathbb{A}\mathrm{gt}\setminus\{a\}])\mathbf{Pl}\left(\langle\langle a\rangle\rangle\eta_a \rightarrow (\textbf{set-pl}\ \sigma)\langle\langle\emptyset\rangle\rangle\eta_a\right)$$
$$NE^{\overrightarrow{\eta}}(\sigma) \equiv \bigwedge_{a\in\mathbb{A}\mathrm{gt}} BR_a^{\overrightarrow{\eta}}(\sigma)$$
$$SPN^{\overrightarrow{\eta}}(\sigma) \equiv \langle\langle\emptyset\rangle\rangle\Box NE^{\overrightarrow{\eta}}(\sigma)$$
$$PO^{\overrightarrow{\eta}}(\sigma) \equiv \forall\sigma'\ \mathbf{Pl}\Big(\bigwedge_{a\in\mathbb{A}\mathrm{gt}}((\textbf{set-pl}\ \sigma')\langle\langle\emptyset\rangle\rangle\eta_a \rightarrow (\textbf{set-pl}\ \sigma)\langle\langle\emptyset\rangle\rangle\eta_a) \vee$$
$$\bigvee_{a\in\mathbb{A}\mathrm{gt}}((\textbf{set-pl}\ \sigma)\langle\langle\emptyset\rangle\rangle\eta_a \wedge \neg(\textbf{set-pl}\ \sigma')\langle\langle\emptyset\rangle\rangle\eta_a\Big).$$

$$UNDOM^{\vec{\eta}}(\sigma) \equiv \quad \forall\sigma_1\forall\sigma_2\exists\sigma_3\, \mathbf{Pl}$$
$$\Big(\big((\textbf{set-pl}\ \langle\sigma_1^{\{a\}},\sigma_2^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle\eta_a \rightarrow (\textbf{set-pl}\ \langle\sigma^{\{a\}},\sigma_2^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle\eta_a\big)$$
$$\vee \big((\textbf{set-pl}\ \langle\sigma^{\{a\}},\sigma_3^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle\eta_a \wedge \neg(\textbf{set-pl}\ \langle\sigma_1^{\{a\}},\sigma_3^{\mathbb{A}\mathrm{gt}\setminus\{a\}}\rangle)\langle\langle\emptyset\rangle\rangle\eta_a\big)\Big).$$

The intuitions behind these concepts are the same as in the quantitative case. Note that we did not have to include the big conjunctions/disjunctions over all possible utility values in the case of Pareto optimal and undominated strategies. This is because the corresponding NF game can be seen as a game with only *two* possible outcomes per agent.

The following proposition shows that $NE^{\vec{\eta}}$, $PO^{\vec{\eta}}$, and $UNDOM^{\vec{\eta}}$ indeed extend the classical notions of Nash equilibrium, Pareto optimal strategy profile, and undominated strategy.

Proposition 20

1. *The set of Nash equilibrium strategies in $\mathcal{S}(M,\vec{\eta},q)$ is given by $[\![\sigma.NE^{\vec{\eta}}(\sigma)]\!]_M^q$.*
2. *The set of Pareto optimal strategies in $\mathcal{S}(M,\vec{\eta},q)$ is given by $[\![\sigma.PO^{\vec{\eta}}(\sigma)]\!]_M^q$.*
3. *The set of a's undominated strategies in $\mathcal{S}(M,\vec{\eta},q)$ is given by $\big([\![\sigma.UNDOM^{\vec{\eta}}(\sigma)]\!]_M^q\big)|_a$.*

Subgame perfect Nash equilibria cannot be directly related to normal form games, but we can state the following.

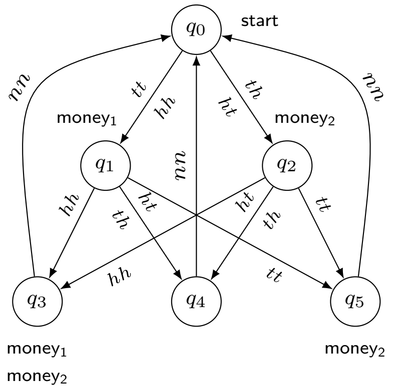
Proposition 21 *Let Q' be the set of states reachable from q in M. Then, $[\![\sigma.SPN^{\eta}(\sigma)]\!]_M^q = \bigcap_{q\in Q'}[\![\sigma.NE^{\eta}(\sigma)]\!]_M^q$.*

Example 17 (Extended matching pennies) *In Figure 7 we consider a slightly more complex version of the* asymmetric matching pennies game *presented in Figure 5. The new game consists of two phases (played* ad infinitum*). Firstly, player 1 wins some money if the sides of the pennies match, otherwise the money goes to player 2. In the second phase, both win a prize if both show heads; if they both show tails, only player 2 wins. If they show different sides, nobody wins.*

We denote particular strategies as $s_{\alpha_1\alpha_2}$, where α_1 is the action played at state q_0, and α_2 is the action played at states q_1, q_2 (it is not necessary to consider strategies that specify different actions in q_1 and q_2, since the outgoing transitions in q_2 are exact copies of those in q_1). Note that every combination of strategies (i.e., every strategy profile) determines a single temporal path. For example, if agent 1 plays s_{ht} and agent 2 plays s_{tt}, then they both ensure the (infinite) temporal path $q_0q_2q_5q_0q_2q_5\ldots$

Let us additionally assume that the winning conditions are: $\eta_1 \equiv \Box(\neg\mathsf{start} \rightarrow \mathsf{money}_1)$ for player 1 and $\eta_2 \equiv \Diamond\mathsf{money}_2$ for player 2. That is, agent 1 is only happy

(A)



(B)

$\eta_1 \backslash \eta_2$	s_{hh}	s_{ht}	s_{th}	s_{tt}
s_{hh}	$\mathbf{1,1}$	$0,0$	$0,1$	$0,1$
s_{ht}	$0,0$	$0,1$	$0,1$	$0,1$
s_{th}	$0,1$	$0,1$	$\mathbf{1,1}$	$0,0$
s_{tt}	$0,1$	$0,1$	$0,0$	$0,1$

Figure 7: "Extended matching pennies": (A) **CGS** M_3; again, action profile xy refers to action x played by player 1 and action y played by 2. (B) Strategies and their outcomes for $\eta_1 \equiv \Box(\neg\mathsf{start} \rightarrow \mathsf{money}_1)$, $\eta_2 \equiv \Diamond\mathsf{money}_2$. Pareto optimal profiles are indicated with bold font, Nash equilibria with grey background.

if she gets money all the time (whenever possible). Agent 2 is more minimalistic: it is sufficient for him to win money once, sometime in the future. So, for instance, the play that results from strategy profile $\langle s_{ht}, s_{tt}\rangle$ satisfies the second player, but not the first one. This way, it is easy to construct a table of binary payoffs that indicates which strategy profiles are "winning" for whom, like the table in Figure 7B. Now, we can for instance observe that profile $\langle s_{ht}, s_{tt}\rangle$ is a Nash equilibrium (player 1 cannot make herself happy by unilaterally changing her strategy), but it is not Pareto optimal ($\langle s_{hh}, s_{hh}\rangle$ and $\langle s_{th}, s_{th}\rangle$ yield strictly better payoff profiles). As before, the ***CGS*** *M_3 in Figure 7A can be seen as a* ***CGSP*** *by adding $\Upsilon = \Sigma$ and $\Omega = \emptyset$. Now, we have that:*

- $[\![\sigma.NE^{\eta_1,\eta_2}(\sigma)]\!]^{q_0}_{M_3} = \{\langle s_{hh}, s_{hh}\rangle, \langle s_{hh}, s_{tt}\rangle, \langle s_{ht}, s_{ht}\rangle, \langle s_{ht}, s_{tt}\rangle, \langle s_{th}, s_{ht}\rangle, \langle s_{th}, s_{th}\rangle, \langle s_{tt}, s_{ht}\rangle, \langle s_{tt}, s_{tt}\rangle\}$*, and*

- $[\![\sigma.PO^{\eta_1,\eta_2}(\sigma)]\!]^{q_0}_{M_3} = \{\langle s_{hh}, s_{hh}\rangle, \langle s_{th}, s_{th}\rangle\}$.

Suppose that agent 1 wants money always, and 2 wants money eventually, and only Pareto optimal Nash equilibria are played. Then, agent 1 is bound to get money at the beginning of each round of the game. Formally:

$$M_3, q_0 \models (\textbf{\textit{set-pl}}\ \sigma.NE^{\eta_1,\eta_2}(\sigma))(\textbf{\textit{refn-pl}}\ \sigma.PO^{\eta_1,\eta_2}(\sigma))\mathbf{Pl}\,(\mathsf{start} \rightarrow \langle\!\langle\emptyset\rangle\!\rangle \bigcirc \mathsf{money}_1).$$

In **ATLP**, we can also describe relationships between different solution concepts in a **CGS**. For example, in the "extended matching pennies" game, all Pareto optimal profiles happen to be in Nash equilibrium, which is equivalent to the following formula:

$$(\textbf{set-pl}\ \sigma.PO^{\eta_1,\eta_2}(\sigma))(\textbf{refn-pl}\ \sigma.\neg NE^{\eta_1,\eta_2}(\sigma))\mathbf{Pl}\,\neg\langle\langle\mathbb{A}\mathrm{gt}\rangle\rangle\bigcirc\top,$$

and the formula does indeed hold in M_3, q_0.

6 Model Checking ATLP

In this section we discuss the model checking complexity of **ATLP**. The *model checking problem* refers to the question whether a given formula holds in a given model and state. The size of the input is usually measured in the number of transitions in the model (m) and the length of the formula (l). Note that the problem of checking **ATLP** with respect to the size of the *whole* **CGSP** (including the plausibility set Υ), is trivially linear in the size of the model: The model size is *exponential with respect to the number of states and transitions*. Hence, model checking **CGSP**'s does not make sense if the set of plausible strategies is stored explicitly. The set should be stored implicitly; for instance, by means of some decision procedure. We will assume throughout this section that the plausibility set Υ does not discriminate any strategy profiles (i.e., all strategy profiles are initially plausible), and actual plausibility assumptions must be specified in the object language through (simple or complex) plausibility terms.

The same remark applies to the denotations of primitive ("hard-wired") plausibility terms. In this respect, we will consider two subclasses of **CGSP**'s in which the representation of plausibility assumptions of plausibility assumptions does not overwhelm the complexity of the rest of the input – namely, pure concurrent game structures and so called "well-behaved" **CGSP**'s. In pure **CGS**'s, plausibility terms and their denotations are simply absent. In well-behaved **CGS**'s, we put a limit on the complexity of the *plausibility check*, i.e., the computational resources needed to determine whether a given strategy is plausible according to a given plausibility term and plausibility mapping.

Definition 31 (CGS as CGSP) *As before, we will take each* ***CGS*** *to be an implicit representation of* ***CGSP*** *where all strategy profiles are initially plausible ($\Upsilon = \Sigma$) and there are no "hardwired" plausibility terms ($\Omega = \emptyset$).*

Definition 32 (Well-Behaved CGSP) *A* ***CGSP*** M *is called* well-behaved *if, and only if,*

1. $\Upsilon_M = \Sigma$*: all the strategy profiles are plausible in* M*;*

2. *There is an* $\mathbf{NP}$*-algorithm (with respect to* l *and* m*) which determines whether* $s \in [\![\omega]\!]_M^q$ *for every state* $q \in Q_M$*, strategy profile* $s \in \Sigma$*, and plausibility term* $\omega \in \Omega$*.*

Remark 22 *We note that, if a list (or several alternative lists) of plausible strategy profiles is given* explicitly in the model *(via the plausibility set* Υ *and/or the denotations of abstract plausibility terms* ω *from Section 4), then the problem of guessing an appropriate strategy from such a list is in* $\mathbf{NP}$ *(memoryless strategies have polynomial size with respect to* m*). Consequently, we assume that, if such a list is given explicitly, that it is stored* outside *the model.*

We begin our study with the complexity of model checking the basic language $\mathcal{L}_{ATLP}^{\text{base}}$ in Section 6.1. Then, we investigate the complexity for the intermediate language $\mathcal{L}_{\boldsymbol{ATLP^{ATLI}}}$ (Section 6.2). It turns out that the problem is in both cases $\mathbf{\Delta_3^P}$-complete in general, which seems in line with existing results on the complexity of solving games. In particular, it is known that if both players in a 2-player imperfect information game have imperfect recall, and chance moves are allowed, then the problem of finding a max-min pure strategy is $\mathbf{\Sigma_2^P}$-complete [31].[12] That is, there are established results within game theory which show that reasoning about the outcome of a game where the strategies of both parties are restricted cannot be easier than $\mathbf{\Sigma_2^P}$ (resp. $\mathbf{\Delta_3^P}$ when nesting of game specifications is allowed). In the light of this, our complexity results are not as pessimistic as they seem, especially as **ATLP** allows specification of much more diverse restrictions than those imposed by imperfect information in 2-player turn-based games.[13]

Moreover, we show in Sections 6.1 and 6.2 that model checking $\mathcal{L}_{ATLP}^{\text{base}}$ and $\mathcal{L}_{\boldsymbol{ATLP^{ATLI}}}$ is $\mathbf{\Delta_2^P}$-complete if only the proponents' strategies are restricted. This, again, corresponds to some well-known $\mathbf{NP}$-hardness results for solving extensive games with imperfect information and recall [12, 17, 31].

Finally, in Section 6.3 we study the model checking complexity of $\mathcal{L}_{ATLP}^{k}$ and $\mathcal{L}_{ATLP}^{\infty}$. We summarize the results in Section 6.4.

6.1 Model Checking $\mathcal{L}_{\boldsymbol{ATLP}}^{\textbf{base}}$

In this section we show that model checking $\mathcal{L}_{ATLP}^{\text{base}}$ is $\mathbf{\Delta_3^P}$-complete in general, and $\mathbf{\Delta_2^P}$-complete when only the proponents' strategies are restricted. Moreover, model checking $\mathcal{L}_{ATLP}^{\text{base}}$ over *rectangular models* and models with *bounded plausibility sets* can be done in polynomial time.

[12]Note that strategic operators can be nested in an **ATLP** formula, thus specifying a sequence of games, with the outcome of each game depending on the previous ones—and solving such games requires adaptive calls to a $\mathbf{\Sigma_2^P}$ oracle.

[13] In particular, imperfect information strategies (sometimes called *uniform* strategies) can be characterized in **ATLP** for a relevant subclass of models, cf. Section 6.1.2.

function $mcheckATLP(M, q, \varphi)$**;**
Model checking **ATLP**: the main function.

- Return $mcheck(M, q, \varphi, \emptyset, \emptyset)$;

function $mcheck(M, q, \varphi, \overrightarrow{\omega}, B)$**;**
Returns "true" iff φ plausibly holds in M, q. The current plausibility assumptions are specified by a sequence $\overrightarrow{\omega} = [\langle\omega_1, q_1\rangle, \ldots, \langle\omega_n, q_n\rangle]$ of plausibility terms with interpretation points. The set of agents which are assumed to play rational are denoted by B.

cases $\varphi \equiv p$**,** $\varphi \equiv \neg\psi$**,** $\varphi \equiv \psi_1 \wedge \psi_2$ **:** proceed as usual;
case $\varphi \equiv$ (**set-pl** ω')ψ **:** return($mcheck(M, q, \psi, [\langle\omega', q\rangle], B)$);
case $\varphi \equiv$ (**refn-pl** ω')ψ **:** return($mcheck(M, q, \psi, \overrightarrow{\omega} \oplus \langle\omega', q\rangle, B)$);
case $\varphi \equiv$ **Pl** $_A\psi$ **:** return($mcheck(M, q, \psi, \overrightarrow{\omega}, A)$);
case $\varphi \equiv \langle\langle A\rangle\rangle \bigcirc\psi$**, where** ψ **includes some** $\langle\langle B\rangle\rangle$ **:** Label all $q' \in Q$, in which $mcheck(M, q, \psi, \overrightarrow{\omega}, B)$ returns "true", with a new proposition yes. Return $mcheck(M, q, \langle\langle A\rangle\rangle \bigcirc \mathsf{yes}, \overrightarrow{\omega}, B)$;
case $\varphi \equiv \langle\langle A\rangle\rangle \bigcirc\psi$**, where** ψ **includes no** $\langle\langle C\rangle\rangle$ **:** Remove all operators **Pl**, **Ph**, (**set-pl** $\cdot$) from ψ (they are irrelevant, as no cooperation modality comes further), yielding ψ'. Return $solve(M, q, \langle\langle A\rangle\rangle \bigcirc\psi', \overrightarrow{\omega}, B)$;
cases $\langle\langle A\rangle\rangle \Box\psi$ **and** $\langle\langle A\rangle\rangle \psi_1 \mathcal{U} \psi_2$ **:** analogously ;
end case

function $solve(M, q, \varphi, \overrightarrow{\omega}, B)$**;**
Returns "true" iff φ holds in M, q under plausibility assumptions specified by $\overrightarrow{\omega}$ and applied to B. We assume that $\varphi \equiv \langle\langle A\rangle\rangle \Box\psi$, where ψ is a propositional formula, i.e., it includes no $\langle\langle B\rangle\rangle$, **Pl**, **Ph**, (**set-pl** $\cdot$).

- Label all $q' \in Q$, in which ψ holds, with a new proposition yes;
- Guess a strategy profile s;
- **if** $plausiblestrat(s, M, \overrightarrow{\omega}, B)$ **then** return(not $beatable(s[A], M, q, \langle\langle A\rangle\rangle \Box \mathsf{yes}, \overrightarrow{\omega}, B)$);
  else return(false);

Figure 8: Model checking **ATLP**

6.1.1 Model Checking $\mathcal{L}_{ATLP}^{\mathbf{base}}$: Upper Bounds

Well-behaved CGSP. A detailed algorithm for model checking $\mathcal{L}_{ATLP}^{\text{base}}$ formulae in well- behaved concurrent game structures with plausibility is presented in Figure 8. Apart from model M, state q, and formula φ to be checked, the input includes a plausibility specification vector $\overrightarrow{\omega}$ and a set B of agents which are assumed to play rationally. The plausibility vector $\overrightarrow{\omega} = [\langle\omega_1, q_1\rangle, \ldots, \langle\omega_n, q_n\rangle]$ is a sequence of plausibility terms together with states at which the terms are evaluated; this is because we need to keep track of applications of the refinement operators (**refn-pl** $\cdot$). The intuition is that the vector represents the

function $beatable(s_A, M, q, \langle\langle A\rangle\rangle\gamma, \overrightarrow{\omega}, B)$**;**
Returns "true" iff the opponents can beat s_A so that it does not enforce γ in M, q under plausibility assumptions specified by $\overrightarrow{\omega}$ and imposed on B. The path formula γ is of the form $\bigcirc\psi$, $\Box\psi$, $\psi\mathcal{U}\psi'$ with propositional ψ, ψ'.

- Guess a strategy profile t;
- **if** $plausiblestrat(t, M, \overrightarrow{\omega}, B)$ and $t|_A = s_A$ **then**
  - M' := "trim" M, removing all transitions that cannot occur when $t|_B$ is executed;
  - return($mcheck_{CTL}(M', q, \neg \mathsf{A}\gamma)$);

  else return(false);

function $plausiblestrat(s, M, \overrightarrow{\omega}, B)$**;**
Checks whether B's part of strategy profile s is part of some profile in $\bigcap_{\langle\omega,q\rangle\in\overrightarrow{\omega}} [\![\omega]\!]_M^q$.

- return true if $s|_B \in \bigcap_{\langle\omega,q\rangle\in\overrightarrow{\omega}} [\![\omega]\!]_M^q|_B$; and false otherwise.

Figure 9: Model checking **ATLP**

incremental plausibility updates. Moreover, by $[\langle\omega_1, q_1\rangle, \ldots, \langle\omega_n, q_n\rangle] \oplus \langle\omega, q\rangle$ we denote the vector $[\langle\omega_1, q_1\rangle, \ldots, \langle\omega_n, q_n\rangle, \langle\omega, q\rangle]$.

Since **CTL** model checking is linear in the number of transitions in the model and the length of the formula [14] and as long as $plausiblestrat(s, M, q, \omega, B)$ can be computed in polynomial time, we get that $mcheckATLP$ runs in time $\mathbf{\Delta_3^P}$, i.e., the algorithm can be implemented as a deterministic Turing Machine making adaptive calls to an oracle of range $\mathbf{\Sigma_2^P} = \mathbf{NP^{NP}}$. In fact, it suffices to require that $plausiblestrat(s, M, q, \omega, B)$ can be computed in *nondeterministic* polynomial time, as the witness for $plausiblestrat$ can be guessed together with the strategy profile s in function $solve$, and with the strategy profile t in function $beatable$, respectively. The intersection of plausibility terms can also be neglected as the vector of plausibility terms can contain at most l terms (length of the formula). Schematically, we can describe the main part of the algorithm by $\exists s\neg(\exists t)$: s is guessed first, then t is guessed (and its answer is negated, so we have $\exists s\forall t$). This schematic view will be useful in Section 6.3 to give an intuition about the complexity of nested formulae together with quantification over strategic terms.

Proposition 23 *Let M be a well-behaved **CGSP**, q a state in M, and φ a formula of $\mathcal{L}_{ATLP}^{base}(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$. Then $M, q \models \varphi$ iff $mcheckATLP(M, q, \varphi)$. The algorithm runs in time $\mathbf{\Delta_3^P}$ with respect to the number of transitions in the model and the length of the formula.*

Proof in Appendix D.1.

Note that the requirement that the set of plausible strategies is given by Σ is not a real restriction. Specific plausibility specification can always be set using operator (**set-pl** $\cdot$), by adding a new plausibility term that denotes the desired set of strategy profiles. The only restriction is that inclusion in the set must be verifiable in nondeterministic polynomial time.

Finally, we observe that the complexity can be improved if only the strategies of the proponents are restricted.

Proposition 24 *Let γ be an $\mathcal{L}^{base}_{ATLP}$ path formula without cooperation modalities. Then the model checking problem for formulae of the form $\mathbf{Pl}_A\langle\langle A\rangle\rangle\gamma$ is in $\mathbf{\Delta_2^P}$ (instead of $\mathbf{\Delta_3^P}$).*

Proof Sketch We consider the case $\varphi \equiv \langle\langle A\rangle\rangle \bigcirc\psi$, where ψ includes no $\langle\langle C\rangle\rangle$. In $solve$ a plausible strategy s_A for A is guessed (**NP**-call). Then, in function $beatable$ the model is directly trimmed according to s_A (without guessing another profile t) and the **CTL** model checking algorithm is executed. In this case, function $beatable$ can be executed in polynomial time. ■

Corollary 25 *Let $\varphi \in \mathcal{L}^{base}_{ATLP}$. If for each cooperation modality $\langle\langle A\rangle\rangle$ occurring in φ it is specified that only agents A' where $A' \subseteq A$ play plausibly then model checking is in $\mathbf{\Delta_2^P}$.*

Pure CGS. This is a somewhat degenerate case because in $\mathcal{L}^{\text{base}}_{ATLP}$ only primitive plausibility terms can be used. With no such terms, (**set-pl** $\cdot$) and (**refn-pl** $\cdot$) operators cannot be used, so all strategy profiles will be considered plausible in the evaluation of every subformula. In consequence, model $\mathcal{L}^{\text{base}}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \emptyset)$ can be done in the same way as for **ATL**. Since model checking **ATL** lies in **P** [3] we get the following result.

Proposition 26 *Let M be a **CGS**, q a state in M, and $\varphi \in \mathcal{L}^{base}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \emptyset)$. Model checking φ in M, q is in $\mathbf{P}$ with respect to the number of transitions in the model and the length of the formula.*

Proof Remove all $\mathbf{Pl}_A$ operators from φ and check whether $M'q, \models_{\mathbf{ATL}} \varphi$ where M' is the **CGS** obtained from M by leaving out Υ, Ω, and $[\![\cdot]\!]$. ■

Special Classes of Models. We will now consider the special case in which each plausibility term refers to at most polynomially many strategies.

Definition 33 (Bounded Models $\mathfrak{M}^c$) *Given a fixed constant $c \in \mathbb{N}$ we consider the class $\mathfrak{M}^c \subseteq CGSP(\mathbb{A}\text{gt}, \Pi, \Omega)$ of models such that for all $M \in \mathfrak{M}^c$, $\omega \in \Omega_M$, and $q \in Q_M$ it holds that $|[\![\omega]\!]^q_M| \leq l^c \cdot m^c$ where l (resp. m) denotes the length of the input formula (resp. number of transitions of M).*

Proposition 27 *Let $c \in \mathbb{N}$ be a constant. Model checking $\mathcal{L}^{base}_{ATLP}$ formulae with respect to the class of well-behaved bounded models $\mathfrak{M}^c$ can be done in polynomial time with respect to the number of transitions in the model and the length of the formula.*

Proof in Appendix D.1.

Even with arbitrarily many strategies the complexity can be improved if the set of plausible profiles has a specific structure, namely if the set can be (and is) represented in a *rectangular* way. Intuitively, such a set of profiles can be represented by behavioral constraints [46]. That is, we restrict the actions that can be performed independently for each state and agent, and then consider all strategy profiles generated from the constrained repertoire of actions.

Definition 34 (Rectangularity, $\mathfrak{M}^{\mathbf{rect}}$) *Let $S_a \subseteq \Sigma_a$ be a set of strategies of agent a. We say that S_a is* rectangular *if it is represented by a function $d'_a : Q_M \to \mathcal{P}(Act)$ such that for all states $q \in Q_M$ it holds that $d'_a(q) \subseteq d_a(q)$; then, S_a is taken to be the set $\{s_a \in \Sigma_a \mid \forall q \in Q_M \ (s_a(q) \in d'_a(q))\}$.*

A set of collective strategies (resp. strategy profiles) $S_A \subseteq \Sigma_A$ is rectangular if it represented as a collection of rectangular sets of individual strategies. Then, S_A is to the Cartesian product of the individual sets, i.e., $S_A = \prod_{a \in A} S_a$.

*A set of plausibility terms Ω is rectangular in a model M if all terms in $\omega \in \Omega$ have rectangular denotations $[\![\omega]\!]^q_M$. Finally, we say that a **CGSP** M is rectangular if the set Υ_M is rectangular and terms Ω are rectangular in M. We denote the class of such models by $\mathfrak{M}^{rect}$.*

Note, for example, that each Σ_A is rectangular.

Proposition 28 *Model checking $\mathcal{L}^{base}_{ATLP}$ formulae in the class $\mathfrak{M}^{rect}$ can be done in $\mathbf{P}$ with respect to the number of transitions in the model and the length of the formula.*

Proof The algorithm is very simple; we present the procedure for $\varphi \equiv \langle\!\langle A \rangle\!\rangle \Box \psi$ being in the scope of (**set-pl** ω) and **Pl** $_B$. Other cases are analogous.

Firstly, we model-check (**set-pl** ω)**Pl** $_B\psi$ recursively and label the states where the answer was "true" with a new proposition yes. Then, we take $[\![\omega]\!]^q_M$ (recall that it is represented in a rectangular way, i.e., by function $d' : \mathbb{A}\mathrm{gt} \times Q \to \mathcal{P}(Act)$), and replace function d in M by d'' such that $d''(a,q) = d'(a,q)$ for $a \in B$ and $d''(a,q) = d(a,q)$ for $a \notin B$. Finally, we use any **ATL** model checker to model-check $\langle\!\langle A \rangle\!\rangle \Box \mathsf{yes}$ in the resulting model, and return the answer. ■

We observe that strategic combinations of rectangular plausibility terms are also rectangular. In consequence, the results extends to $\mathcal{L}^0_{ATLP}$ in a straightforward way, which will prove useful in Section 6.3.[14]

Lemma 29 *If $S \subseteq \Sigma_a$ (resp. $S \subseteq \Sigma_A$) contains only a single strategy (resp. strategy profile) then it is rectangular.*

Lemma 30 *Let Ω be a rectangular set of plausibility terms, then $\tau(\Omega)$ is rectangular as well.*

Corollary 31 *Model checking $\mathcal{L}^0_{ATLP}$ formulae in the class $\mathfrak{M}^{rect}$ can be done in* $\mathbf{P}$ *with respect to the number of transitions in the model and the length of the formula.*

6.1.2 Model Checking $\mathcal{L}^{\mathbf{base}}_{\boldsymbol{ATLP}}$: Hardness and Completeness

Well-behaved CGSP. We prove $\mathbf{\Delta^P_3}$-hardness through a reduction of $\mathbf{SNSAT}_2$, a typical $\mathbf{\Delta^P_3}$-complete variant of the Boolean satisfiability problem. The reduction is done in two steps.

1. Firstly, we define a modification of $\mathbf{ATL}_{ir}$ [38], in which *all* agents are required to play only uniform strategies. We call it "uniform $\mathbf{ATL}_{ir}$" ($\mathbf{ATL}^u_{ir}$ in short), and show that model checking $\mathbf{ATL}^u_{ir}$ is $\mathbf{\Delta^P_3}$-complete by means of a polynomial reduction of $\mathbf{SNSAT}_2$ to $\mathbf{ATL}^u_{ir}$ model checking.
2. Then, we point out that each formula and model of $\mathbf{ATL}^u_{ir}$ can be equivalently translated (in polynomial time) to a **CGSP** and a formula $\mathcal{L}^{\text{base}}_{ATLP}$, thus yielding a polynomial reduction of $\mathbf{SNSAT}_2$ to model checking $\mathcal{L}^{\text{base}}_{ATLP}$.

Parts of our construction reuse techniques presented in [19, 27, 23, 28].

In "uniform $\mathbf{ATL}_{ir}$" ($\mathbf{ATL}^u_{ir}$), where we assume that all the players have limited information about the current state, and each agent can only use *uniform* strategies (i.e., ones that assign same choices in indistinguishable states). The syntax of $\mathbf{ATL}^u_{ir}$ is the same as that of **ATL**, only cooperation modalities are annotated with additional tags ir and u to indicate the imperfect **i**nformation and **r**ecall, and **u**niformity of all agents' strategies. The semantics of $\mathbf{ATL}^u_{ir}$ is defined over *concurrent epistemic game structures* (**CEGS**), i.e. **CGS** extended with epistemic relations that represent indistinguishability of states for agents. Details of the semantics and more thorough presentation can be found in Appendix B. The following proposition summarizes the complexity results from Appendix B.2.

Proposition 32 *Model checking $\mathbf{ATL}^u_{ir}$ is $\mathbf{\Delta^P_3}$-complete with respect to the number of transitions in the model and the length of the formula.*

[14] Recall, that $\mathcal{L}^0_{ATLP}$ consists of all base formulae in which plausibility terms form $\tau(\Omega)$ can be used (instead of plain terms from Ω only).

Remark 33 *We have thus proven that checking strategic abilities when* all *players are required to play uniformly is $\mathbf{\Delta_3^P}$-complete (that is, harder than ability compared with the worst line of events captured by $\mathbf{ATL}_{ir}$ formulae, which is "only" $\mathbf{\Delta_2^P}$-complete). We believe it is an interesting result with respect to verification of various kinds of agents' abilities under incomplete information. We note that the result from [31] for extensive games with incomplete information can be seen as a specific case of our result, at least in the class of games with binary payoffs.*

Now we show how $\mathbf{ATL}_{ir}^u$ model checking can be reduced to model checking of $\mathcal{L}_{ATLP}^{\text{base}}$. We are given a **CEGS** M, a state q in M, and an $\mathbf{ATL}_{ir}^u$ formula φ. Let Σ^u be the set of all uniform strategy profiles in M. We take **CGSP** M' as M (sans epistemic relations) extended with plausibility mapping $[\![\cdot]\!]$ such that $[\![\omega]\!]^q = \Sigma^u$. Then:

$$M, q \models_{\mathbf{ATL}_{ir}^u} \langle\langle A\rangle\rangle_{ir}^u \varphi \quad \text{iff} \quad M', q \models_{\mathbf{ATLP}} (\mathbf{set\text{-}pl}\ \omega)\mathrm{Pl}\,\langle\langle A\rangle\rangle \varphi,$$

which completes the reduction.

Remark 34 *We note in passing that, technically, the size of the resulting model M' is not entirely polynomial. M' includes the plausibility set Υ, which is exponential in the number of states in M (since it is equal to the the set of all uniform strategy profiles in M). This is of course the case when we want to store Υ explicitly. However, checking if a strategy profile is uniform can be done in time linear wrt the number of states in M, so an* implicit *representation of Υ (e.g., the checking procedure itself) requires only linear space.*

As a result of this and Proposition 23, we obtain the following theorem.

Theorem 35 *Model checking $\mathcal{L}_{ATLP}^{base}$ for well-behaved **CGSP**'s is $\mathbf{\Delta_3^P}$-complete with respect to the number of transitions in the model and the length of the formula.*

For the special case when only the proponents have to follow plausible strategies, a reduction from model checking $\mathbf{ATL}_{ir}$ (instead of $\mathbf{ATL}_{ir}^u$) is sufficient. Since model checking $\mathbf{ATL}_{ir}$ is $\mathbf{\Delta_2^P}$-complete [38, 28], we get the following.

Theorem 36 *Let $\mathcal{L}$ the subset of $\mathcal{L}_{ATLP}^{base}$ in which every cooperation modality $\langle\langle A\rangle\rangle$ occurs in the scope of Pl_B with $B \subseteq A$. Then, model checking $\mathcal{L}$ in the class of well-behaved **CGSP**'s is $\mathbf{\Delta_2^P}$-complete.*

Proof sketch The inclusion in $\mathbf{\Delta_2^P}$ has been already shown in Section 6.1.1. We prove the lower bound by a reduction of model checking Schobbens' $\mathbf{ATL}_{ir}$ [38] to model checking of our sublanguage $\mathcal{L}$. Let M be a **CEGS**, q a state in M, and $\varphi \equiv \langle\langle A\rangle\rangle_{ir}\gamma$ a formula of $\mathbf{ATL}_{ir}$. Moreover, let Σ_A^u be the set of all strategy profiles in M that are uniform for A. We take **CGSP** M' as M (sans epistemic relations) extended with plausibility mapping $[\![\cdot]\!]$ such that $[\![\omega]\!]^q = \Sigma_A^u$. Then:

$$M, q \models_{\mathbf{ATL}_{ir}} \langle\langle A\rangle\rangle_{ir}\gamma \quad \text{iff} \quad M', q \models_{\mathbf{ATLP}} (\mathbf{set\text{-}pl}\ \omega)\mathrm{Pl}\,\langle\langle A\rangle\rangle\gamma,$$

which completes the reduction. ∎

Pure CGS and Special Classes of Models. In order to show lower bounds for model checking $\mathcal{L}_{ATLP}^{\mathrm{base}}$ for pure concurrent game structures, well-behaved bounded models, and rectangular models, we observe that **ATL** is a subset of $\mathcal{L}_{ATLP}^{\mathrm{base}}$ even if the latter does not use plausibility terms – and model checking **ATL** is **P**-complete [3]. Thus, we conclude with the following.

Theorem 37 *Let $c \in \mathbb{N}$ be a constant. Model checking $\mathcal{L}_{ATLP}^{base}$ with respect to well-behaved bounded models $\mathfrak{M}^c$, rectangular models $\mathfrak{M}^{rect}$, and pure **CGS**'s is $\mathbf{P}$-complete.*

6.2 Model Checking $\mathcal{L}_{ATLP^{ATLI}}$

Here, we show that model checking **ATLP** with plausibility terms based on **ATLI** is also $\mathbf{\Delta_3^P}$-complete. Note that the only primitive terms occurring in formulae of **ATLP$^{\mathbf{ATLI}}$** are used to simulate strategic terms of **ATLI** (which denote individual strategies of particular agents. Thus, the results in this section refer to model checking with rectangular **CGSP**'s.

6.2.1 Model Checking $\mathcal{L}_{ATLP^{ATLI}}$: Upper Bound

The algorithm in Figure 8 uses abstract plausibility terms but it can also be used for **ATLI**-based plausibility terms presented in Section 4.3. In [30] it was shown that the model checking problem for **ATLI** is polynomial with respect to the number of transitions and length of the formula. Thus, we get another immediate corollary of Proposition 23.

Proposition 38 *Model checking **ATLP** with **ATLI**-based plausibility terms in rectangular well-behaved **CGSP**'s is in $\mathbf{\Delta_3^P}$ with respect to the number of transitions in the model and the length of the formula.*

In Section 4.4 we have used $\mathcal{L}_{ATLP}^1$ formulae to characterize game theoretic solution concepts. For this purpose it was not necessary to have hard-wired plausibility terms in the language. Indeed, the absence of such terms positively influences the model checking complexity of higher levels of **ATLP**.

6.2.2 Model Checking $\mathcal{L}_{ATLP^{ATLI}}$: Hardness and Completeness

Like in Section 6.1.2, we show the lower bound by a reduction from model checking $\mathbf{ATL}^{u}_{ir}$. That is, we demonstrate how uniformity of strategy profiles can be characterized by formulae of **ATLI** for a relevant class of concurrent game structures. The actual reduction is quite technical and can be found in Appendix C. The following result is an immediate corollary of Proposition 50, presented in Appendix C.

Theorem 39 *Model checking $\mathcal{L}^{base}_{ATLP}$ with **ATLI**-based plausibility terms is $\mathbf{\Delta^P_3}$-complete with respect to the number of transitions in the model and the length of the formula.*

Moreover, if plausibility restrictions apply only to proponents, then the complexity improves (the proof is analogous to Theorem 36).

Theorem 40 *Let $\mathcal{L}$ the subset of $\mathcal{L}_{ATLP^{ATLI}}$ in which every cooperation modality $\langle\langle A\rangle\rangle$ occurs in the scope of $\mathbf{Pl}_B$ with $B \subseteq A$. Then, model checking $\mathcal{L}$ in the class of well-behaved rectangular **CGSP**'s is $\mathbf{\Delta^P_2}$-complete.*

Proof sketch We prove the lower bound (again) by a reduction of model checking $\mathbf{ATL}_{ir}$ to model checking $\mathcal{L}$. The reduction is very similar to the one shown in Appendix C except that only the "verifier" decides upon the values of the propositions (cf. [27]). ■

6.3 Model Checking $\mathcal{L}^{k}_{ATLP}$

In this section we present our results regarding the model checking complexity of the full logic $\mathcal{L}_{ATLP}$. The complexity depends on both the nesting level of **ATLP** formulae and on the structure and alternations of strategic quantifiers. Before we state our results we introduce some additional definitions needed to classify such complex formulae.

6.3.1 Classifying $\mathcal{L}_{ATLP}$ Formulae: Some Definitions

The complexity of model checking formulae in $\mathcal{L}_{ATLP}$ does not only depend on the actual nesting depth of plausibility terms but also on the structure of strategic quantifiers used inside (**set-pl** ·) and (**refn-pl** ·) operators. The latter structure is quite complex and cannot solely be described by the number of quantifiers. Often, a specific position of quantifiers can be used to combine two "guessing" phases, improving complexity.

Firstly, not the number of quantifiers is important but rather the number of alternations. We introduce function $\text{ALT} : \{\exists,\forall\}^+ \rightarrow \{\exists,\forall\}^+$ which modifies a word over $\{\exists,\forall\}$ such that each quantifier following a quantifier of the same type is removed; for example, $\textsc{Alt}(\exists\forall\forall\exists\forall) = \exists\forall\exists\forall$. Moreover, existential quantifiers at the beginning and end of a quantifier series can, under some conditions, be ignored without changing the model checking complexity. For example, let us assume that the first quantifier is existential. Then it follows a guess of the proponents (resp. opponents) strategy and both guesses can be combined. Analogously, an existential quantifier at the end usually follows another existential guess. To take these issues into account, we define function $\textsc{RAlt} : \{\exists, \forall\}^+ \to \mathbb{Z}$ that counts the number of the *relevant alternations of quantifiers* in a sequence:

$$\textsc{RAlt}(\overrightarrow{Q}) = \begin{cases} n & \text{if } \textsc{Alt}(\overrightarrow{Q}) = Q_1 \dots Q_n \text{ and } Q_1 \neq \exists \neq Q_n; \\ n-1 & \text{if } \textsc{Alt}(\overrightarrow{Q}) = Q_1 \dots Q_n \text{ and } Q_1 = \exists \text{ xor } Q_n = \exists; \\ n-2 & \text{if } \textsc{Alt}(\overrightarrow{Q}) = Q_1 \dots Q_n \text{ and } Q_1 = \exists = Q_n \text{ and } n > 2; \\ -1 & \text{else.} \end{cases}$$

Function $\textsc{RAlt}$ characterizes the "hardness" of the outermost level in a given term. The next two functions take into account the recursive structure of terms, due to possibly nested (**set-pl** $\cdot$) or (**refn-pl** $\cdot$) operators. Firstly, $\mathcal{UO}(\varphi)$ returns the set of all the *update operations* (**set-pl** ω) and (**refn-pl** ω) within formula φ. Secondly, *ql* takes a set of update operations and returns the *quantifier level* in these operations as follows:

$$ql(S) = \begin{cases} \max_{s \in S} ql(\{s\}) & \text{if } |S| > 1 \\ ql(\mathcal{UO}(\varphi')) & \text{if } S = \{(\mathbf{Op}\ \sigma.\varphi')\} \\ \textsc{RAlt}(Q_1 \dots Q_n) + ql(\mathcal{UO}(\varphi')) & \text{if } S = \{(\mathbf{Op}\ \sigma.Q_1\sigma_1 \dots Q_n\sigma_n\varphi')\} \text{ and} \\ & (\varphi' \notin \mathcal{L}^0_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \mathcal{V}ar, \mathcal{V}ar) \text{ or } Q_n = \forall) \\ \textsc{RAlt}(Q_1 \dots Q_n) + ql(\mathcal{UO}(\varphi')) + 1 & \text{if } S = \{(\mathbf{Op}\ \sigma.Q_1\sigma_1 \dots Q_n\sigma_n\varphi')\} \text{ and} \\ & \varphi' \in \mathcal{L}^0_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \mathcal{V}ar, \mathcal{V}ar) \text{ and } Q_n = \exists \\ 0 & \text{if } S = \emptyset \text{ or } (S = \{(\mathbf{Op}\ \omega)\} \text{ and } \omega \in \Omega) \end{cases}$$

where (**Op** $\cdot$) is either (**set-pl** $\cdot$) or (**refn-pl** $\cdot$).

The intuition behind *ql* is that it determines the maximal sum of relevant alternations in each sequence of nested update operators (**set-pl** $\cdot$), (**refn-pl** $\cdot$). Intuitively, the nested operators represent a tree. Given an $\mathcal{L}^k_{ATLP}$ formula we add arcs from the root of the tree to nodes representing update operators operators in the kth level. Then, from such a new node representing (**set-pl** ω) or (**refn-pl** ω), we add arcs to nodes representing update operators inside ω (i.e., on the $k-1$th level) and so on. Leaves of the tree consist of nodes representing operators whose terms contain no further update operators. Now, each node represented by e.g. (**set-pl** $\sigma.Q_1\sigma_1 \dots Q_n\sigma_n\varphi'$) is labeled by $\textsc{RAlt}(Q_1 \dots Q_n)$. Function *ql* returns the maximal sum of such numbers along all paths from the root to some leaf.

Given an operator ($\textbf{set-pl}\ \sigma.Q_1\sigma_1 \ldots Q_n\sigma_n\varphi'$) on the second to last level without hard-wired plausibility term (i.e., for $\varphi' \in \mathcal{L}^0_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \mathcal{V}ar, \mathcal{V}ar)$) and which ends with an existential quantifier $\exists$, the very operator Q_n cannot be ignored in the calculation of the characteristic number, as it is usually done. The reason for that is that model chekcing φ' can be done in $\mathbf{P}$ (cf. Corollary 31) and this does not allow to combine the existential quantifier of the last strategic term with another one. This is reflected in the third case of the definition of *ql*.

Definition 35 (Level i Formula) *We say that φ is a* level i formula *iff $ql(\mathcal{UO}(\varphi)) = i$.*

Example 18 *Formula:*

$$\varphi \equiv (\textbf{\textit{set-pl}}\ \sigma.\forall\sigma_1\exists\sigma_2\exists\sigma_3(\textbf{\textit{set-pl}}\ \sigma.\forall\sigma'_1\exists\sigma'_2\exists\sigma'_3\forall\sigma'_4\varphi''))\mathbf{Pl}\,\langle\!\langle A\rangle\!\rangle \bigcirc \mathsf{p}$$

is 4-level since $\mathcal{UO}(\varphi) = \{(\textbf{set-pl}\ \sigma.\forall\sigma_1\exists\sigma_2\exists\sigma_3(\textbf{set-pl}\ \sigma.\forall\sigma'_1\exists\sigma'_2\exists\sigma'_3\forall\sigma'_4)\varphi'')\}$ and $ql(\mathcal{UO}(\varphi)) = \mathrm{RALT}(\forall\exists\exists) + ql(\mathcal{UO}(S)) = 1 + 3 = 4$ where $S = \{(\textbf{set-pl}\ \sigma.\forall\sigma'_1\exists\sigma'_2\exists\sigma'_3\forall\sigma'_4)\varphi''\}$ and $ql(\mathcal{UO}(S)) = \mathrm{RALT}(\forall\exists\exists\forall) + ql(\mathcal{UO}(\varphi'')) = 3 + 0$ where $\mathcal{UO}(\varphi'') = \emptyset$.

Moreover, $(\textbf{set-pl}\ \sigma.\forall\sigma_1\exists\sigma_2\exists\sigma_3(\textbf{set-pl}\ \sigma.\forall\sigma'_1\exists\sigma'_2\forall\sigma'_3\exists\sigma'_4)\varphi'')\mathbf{Pl}\,\langle\!\langle A\rangle\!\rangle \bigcirc \mathsf{p}$ is a 4-level formula as well.

6.3.2 Model Checking $\mathcal{L}^k_{ATLP}$: Upper Bounds

Plausibility terms are quite important for the base language $\mathcal{L}^{\mathrm{base}}_{ATLP}$; it does not make much sense to consider the logic without them. In fact, when $\mathcal{L}^{\mathrm{base}}_{ATLP}$ formulae are considered in the context of pure **CGS**'s, the whole logic degenerates to pure **ATL**. This observation does not apply to higher levels of **ATLP** any more. Indeed, all characterizations of game theoretic solutions concepts that we have presented are expressed as $\mathcal{L}^1_{ATLP}$ formulae *without* hard-wired terms. Moreover – as we will see – not using hard-wired terms yields an improved model checking complexity.

Below we state the main results of this section. The intuition is the following. For each level i formula we have i quantifier alternations; in addition to that, in each level there can be two more implicit quantifiers due to the cooperation modalities (*there is* a plausible strategy of the proponents such that *for all* plausible strategies of the opponents ...). It must also be ensured that the quantifiers of two nested levels are separated from each other, otherwise they can be combined; the term $\max\{0, k-i-1\}$ accounts for that.

Theorem 41 (Model Checking $\mathcal{L}^k_{ATLP}$ in Pure CGS) *For $k \geq 1$, $i \geq 0$ let φ be a level-i formula of $\mathcal{L}^k_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \emptyset)$. Moreover, let M be a **CGS**, and q a state in M. Then, model checking $M, q \models \varphi$ can be done in time $\mathbf{\Delta^P_{i+2k+1-\max\{0,k-i-1\}}}$.*

Proof in Appendix D.2.1.

Note, that the restriction to pure **CGS** is essential because defining a given set of strategies Υ might require checking whether a strategy is plausible in the final nesting stage. And that case the advantage of not having hard-wired plausibility terms would vanish and the complexity would increase. So, if plausibility terms are available the last level of an **ATLP** formula cannot be verified in polynomial time anymore (according to Corollary 31). The complexity can increase as shown in the following result.

Theorem 42 (Model Checking $\mathcal{L}^k_{ATLP}$ in Well-Behaved CGSP) *Let φ be a level-i formula of $\mathcal{L}^k_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$, M a well-behaved **CGSP**, and q a state in M. Model checking $M, q \models \varphi$ can be done in $\mathbf{\Delta^P_{i+2(k+1)+1-\max\{0,k-i\}}}$.*

Proof in Appendix D.2.1.

6.3.3 Model Checking $\mathcal{L}^k_{ATLP}$: Hardness and Completeness

As it turns out, model checking $\mathcal{L}_{ATLP}$, and even each $\mathcal{L}^k_{ATLP}$ for $k \geq 1$ is in general **PSPACE**-complete. To show the lower bounds for $\mathcal{L}^k_{ATLP}$ (with arbitrary $k \geq 1$) we show that $\mathcal{L}^1_{ATLP}$ is **PSPACE**-hard, implying that all logics $\mathcal{L}^k_{ATLP}$ (for $k \geq 1$) are **PSPACE**-hard too. That the general model checking problem for $\mathcal{L}_{ATLP}$ formulae is in **PSPACE** follows directly from the algorithm shown in Figure 8.

The hardness proof, similar to the one for $\mathcal{L}_{ATLP^{ATLI}}$ is rather technical and can be found in Appendix D.2.2. As a corollary of Proposition 52, we get the following.

Theorem 43 ($\mathcal{L}^k_{ATLP}$ is PSPACE-complete) *The model checking problems for $\mathcal{L}_{ATLP}$ and for $\mathcal{L}^k_{ATLP}$ (for each $k \geq 1$) are **PSPACE**-complete.*

Proof Easiness is immediate since the model checking algorithm presented in Figure 8 can be executed in polynomial space with respect to the input (cf. Theorem 41 and Proposition 26). Hardness is shown by the polynomial space reduction from **QSAT** (Proposition 52). ■

Finally, we turn to classes in which the number of alternations is restricted by a fixed upper bound, and we conjecture that the model checking problem for i-level formulae of $\mathcal{L}^k_{ATLP}$ is in fact complete in its complexity classes determined in Theorems 41 and 42.

Conjecture 44 *Let φ be a level-i formula of $\mathcal{L}^k_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \emptyset)$, $k \geq 1$, $i \geq 0$. Moreover, let M be a **CGS**, and q a state in M. Then, model checking $M, q \models \varphi$ is $\mathbf{\Delta^P_{i+2k+1-\max\{0,k-i-1\}}}$-complete.*

Conjecture 45 *Let φ be a level-i formula of $\mathcal{L}^k_{ATLP}(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$, M a well-behaved* ***CGSP****, and q a state in M. Model checking $M, q \models \varphi$ is $\mathbf{\Delta^P_{i+2(k+1)+1-\max\{0,k-i\}}}$-complete.*

6.4 Summary of Complexity Results

Throughout Section 6, we have analyzed the model checking complexity of $\mathcal{L}_{ATLP}$. The base language was shown to lie in $\mathbf{\Delta^P_3}$ with both abstract and **ATLI**-based plausibility terms. We also proved that model checking both logics is complete regarding this class. The complexity of model checking $\mathcal{L}^k_{ATLP}$ formulae was shown to depend on three factors:

1. The *nesting level* k of plausibility terms;
2. the *quantifier level*; and
3. whether abstract plausibility terms were present or not.

The quantifier level is influenced by the number of alternations and with which quantifiers – existential or universal – sequences start and end. In general, an i-level $\mathcal{L}^k_{ATLP}$ formula without plausibility terms was shown to be in

$$\mathbf{\Delta^P_{i+2k+1-\max\{0,k-i-1\}}}$$

where its counterpart with hard-wired terms was marginally more difficult to check:

$$\mathbf{\Delta^P_{i+2(k+1)+1-\max\{0,k-i\}}}.$$

The results for formulae without (resp. with) primitive plausibility terms are summarized in Figure 10 (resp. Figure 11).

Note that all our game theoretic characterizations could already be expressed by $\mathcal{L}^1_{ATLP}$ formulae without hard-wired terms.

7 Conclusions

We proposed a logic in which one can study the outcome of rational play in a logical framework, under various rationality criteria. Although solving game-like scenarios with help of various solution concepts is arguably the main application of game theory, to our knowledge, there has been very little work on this issue. We are *not* discussing the merits of one rationality criterion or the other, nor the pragmatics of using particular criteria to predict the actual behaviour of agents. Our aim was to propose a *conceptual tool* in which the consequences of accepting one or another criterion can be studied.

	0	1	2	...	i	...	unbounded
$\mathcal{L}_{ATLP}^{basic}$	$\mathbf{P}$	-	-	-	-	...	-
$\mathcal{L}_{ATLP}^{0}$	$\mathbf{P}$	-	-	...	-	...	-
$\mathcal{L}_{ATLP}^{1}$	$\mathbf{\Delta_3^P}$	$\mathbf{\Delta_4^P}$	$\mathbf{\Delta_5^P}$	...	$\mathbf{\Delta_{i+3}^P}$	...	**PSPACE**
$\mathcal{L}_{ATLP}^{2}$	$\mathbf{\Delta_4^P}$	$\mathbf{\Delta_6^P}$	$\mathbf{\Delta_7^P}$	...	$\mathbf{\Delta_{5+i-\max\{0,1-i\}}^P}$	...	**PSPACE**
$\vdots$						...	$\vdots$
$\mathcal{L}_{ATLP}^{k}$ $i > k+1$	$\mathbf{\Delta_{k+2}^P}$	$\mathbf{\Delta_{k+4}^P}$	$\mathbf{\Delta_{k+6}^P}$	...	$\mathbf{\Delta_{i+2k+1-\max\{0,k-i-1\}}^P}$	...	**PSPACE**

Figure 10: Summary of the model checking results for pure concurrent game structures (i.e., without hard-wired plausibility terms). All $\mathbf{P}$, $\mathbf{\Delta_3^P}$, and **PSPACE** results are completeness results.

	0	1	2	...	i	...	unbounded
$\mathcal{L}_{ATLP}^{basic}$	$\mathbf{\Delta_3^P}$	-	-	...	-	...	-
$\mathcal{L}_{ATLP}^{0}$	$\mathbf{\Delta_3^P}$	-	-	...	-	...	-
$\mathcal{L}_{ATLP}^{1}$	$\mathbf{\Delta_4^P}$	$\mathbf{\Delta_6^P}$	$\mathbf{\Delta_7^P}$	...	$\mathbf{\Delta_{i+5-\max\{0,1-i\}}^P}$	...	**PSPACE**
$\mathcal{L}_{ATLP}^{2}$	$\mathbf{\Delta_5^P}$	$\mathbf{\Delta_7^P}$	$\mathbf{\Delta_9^P}$	...	$\mathbf{\Delta_{7+i-\max\{0,2-i\}}^P}$	...	**PSPACE**
$\vdots$							$\vdots$
$\mathcal{L}_{ATLP}^{k}$ $i > k$	$\mathbf{\Delta_{k+3}^P}$	$\mathbf{\Delta_{k+5}^P}$	$\mathbf{\Delta_{k+7}^P}$	...	$\mathbf{\Delta_{i+2(k+1)+1-\max\{0,k-i\}}^P}$	...	**PSPACE**

Figure 11: Summary of the model checking results in well-behaved **CGSP**'s. All $\mathbf{\Delta_3^P}$ and **PSPACE** results are completeness results.

We believe that the logic we propose provides much flexibility and modeling power. The results presented in Sections 5 and 6 also suggest that the expressive power of the language is quite high. Our main technical results are as follows:

ATLP: The very definition of the logic **ATLP** in Section 4 and the study of its expressive power in Section 5.1.

Classical Solution Concepts: There are several *classical* solution concepts for extensive games: Nash equilibrium, subgame perfect Nash equilibrium, undominated strategies, and Pareto optimality. We show, by *relating models of our logic (**CGSP**'s) to extensive form games*, that these solution concepts can be formulated as formulae in **ATLP** (in fact, already in $\mathcal{L}^1_{ATLP}$). This is shown in Section 5.2

General Solution Concepts: While the classical solution concepts for games are formulated using *payoffs* (which was the reason to extend models by additional propositions), we propose to formulate *generalized solution concepts* as formulae in our logic **ATLP**. More precisely, we propose to use $\mathcal{L}_{ATL}$-path formulae η_i as *winning conditions* for agent i. Thus, instead of computing payoffs in an extensive form game, we consider **CGSP** models plus a vector of $\mathcal{L}_{ATL}$-path formulae η_i (representing the payoff for agent i). We demonstrate $\mathcal{L}^1_{ATLP}$ formulae that correctly express in **ATLP** our generalized solution concepts. This is elaborated in Section 3.5.

Model Checking in ATLP: An extensive study of the model checking complexity in several classes of models and variants of the language is presented in Section 6. On the way, we also define another interesting variant of **ATL** (where both proponents and opponents are required to use only uniform strategies) and we establish its model checking complexity.

Our ultimate goal is to come up with a logic that would allow us to study strategies, time, knowledge, and plausible/rational behaviour under both perfect and imperfect information. However, putting so many dimensions in one framework at once is usually not a good idea – even more so in this case because the interaction between abilities and knowledge is non-trivial (cf. [29, 24, 22]). In [10], we have investigated *time, knowledge and plausibility*. In this article, we studied *strategies, time and rationality*. We hope to integrate both views into a single powerful framework in the future.

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Appendix

A Bargaining with Discount

In Example 7 we presented bargaining with discount. After each round the worth of the goods is reduced by δ_i. In round t the goods have a value of $r(\delta_i^t)$. Because we use a rounding function r, there is a minimal round T such that $r(\delta_i^{T+1}) = 0$ for $i = 1$ or $i = 2$. We can treat this case as finite horizon bargaining game [40, 33].

Now, consider the case that a_i's opponent, denoted by a_{-i}, is the offerer in T. It can offer 0 and a_i should accept, because in the next round the goods are worthless for a_i.

On the other hand, if a_i is offerer in T we have to distinguish two cases. If $r(\delta_{-i}^{T+1}) = 0$ then following the same reasoning as before a_i can offer 0 to a_{-i}. In the other case, namely $r(\delta_{-i}^{T+1}) \neq 0$, we consider the subsequent round $T + 1$ in which a_{-i} takes the role as offerer and can successfully offer 0 to i.

Now, it is possible to solve the game starting from the end. Solutions for $\delta_1 = \delta_2$ can be found in the literature [33]. Here, we recall the idea for different discount rates.

At first, let a_1 be the last offerer and $r(\delta_2^{T+1}) = 0$. This implies, that T is even (the initial round is 0). In T, a_1 offers $\langle 1, 0\rangle$ and a_2 accepts. Knowing this, in $T-1$ agent a_2 can offer $\langle \delta_1, 1-\delta_1\rangle$, since in the next round the value of the good for a_1 would become reduced by δ_1. Following the same reasoning, in $T-2$ a_1 could successfully offer $\langle 1-\delta_2(1-\delta_1), \delta_2(1-\delta_1)\rangle$. Finally, in round $t = 0$ a_1 can offer $\langle \zeta, 1-\zeta\rangle$ where

$$\zeta := (1-\delta_2)\sum_{i=0}^{\frac{T}{2}-1}(\delta_1\delta_2)^i + (\delta_1\delta_2)^{\frac{T}{2}} = (1-\delta_2)\frac{1-(\delta_1\delta_2)^{\frac{T}{2}}}{1-\delta_1\delta_2} + (\delta_1\delta_2)^{\frac{T}{2}}$$

Secondly, consider the case in which a_2 is the last offerer in T and $r(\delta_1^{T+1}) = 0$. This time T is odd but the reasoning stays the same. In round 0 a_1 can offer $\langle \zeta', 1-\zeta'\rangle$ where

$$\zeta' := (1-\delta_2)\frac{1-(\delta_1\delta_2)^{\frac{T+1}{2}}}{1-\delta_1\delta_2}$$

B Uniform $\mathbf{ATL}_{ir}$

In this section, we introduce and investigate the logic of "uniform $\mathbf{ATL}_{ir}$" ($\mathbf{ATL}_{ir}^{u}$). We use the logic only for technical reasons, namely it provides the intermediate step in the completeness proof for the complexity of model checking **ATLP**. Still, we believe that the logic can be interesting in itself.

Moreover, the technique we use for proving the completeness is interesting too (and gives insight into the complexity as well as the relationship between the problem we study and known complexity from game theory).

The idea is based on Schobbens's $\mathbf{ATL}_{ir}$ [38], i.e., **ATL** for agents with imperfect information and imperfect recall. There, it was assumed that the coalition A in formula $\langle\langle A\rangle\rangle_{ir}\varphi$ can only use strategies that assign same choices in indistinguishable states (so called *uniform* strategies). Then, the outcome of every strategy of A was evaluated in every possible behaviour of the remaining agents $\mathbb{A}\mathrm{gt} \setminus A$ (with no additional assumption with respect to that behaviour). In $\mathbf{ATL}^u_{ir}$, we assume that the opponents ($\mathbb{A}\mathrm{gt} \setminus A$) are also required to respond *with a uniform memoryless strategy*. The syntax of $\mathbf{ATL}^u_{ir}$ is the same as that of **ATL**, only cooperation modalities are annotated with additional tags ir and u to indicate the imperfect **i**nformation and **r**ecall, and **u**niformity of all agents' strategies.

B.1 Semantics

The semantics of $\mathbf{ATL}^u_{ir}$ can be defined as follows. Firstly, we define models as *concurrent epistemic game structures* (**CEGS**), i.e. **CGS** with epistemic relations $\sim_a \subseteq Q \times Q$, one per agent. (The intended meaning of $q \sim_a q'$ is that agent a cannot distinguish between between states q and q'.) Secondly, we require that agents have the same options in indistinguishable states, i.e., that $q \sim_a q'$ implies $d_a(q) = d_a(q')$. A (memoryless) strategy s_A is *uniform* if $q \sim_a q'$ implies $s^a_A(q) = s^a_A(q')$ for all $q, q' \in Q, a \in A$. To simplify the notation, we define $[q]_a = \{q' \mid q \sim_a q'\}$ to be the class of states indistinguishable from q for a; $[q]_A = \bigcup_{a \in A}[q]_a$ collects all the states that are indistinguishable from q for some member of the group A; finally, $out(Q, s_A) = \bigcup_{q \in Q} out(q, s_A)$ collects all the execution paths of strategy s_A from states in set Q.

Now, the semantics is given by the clauses below:

$M, q \models p$ iff $p \in \pi(q)$

$M, q \models \neg\varphi$ iff $M, q \not\models \varphi$

$M, q \models \varphi \wedge \psi$ iff $M, q \models \varphi$ and $M, q \models \psi$

$M, q \models \langle\langle A\rangle\rangle^u_{ir} \bigcirc \varphi$ iff there is a uniform strategy s_A such that, for every uniform counterstrategy $t_{\mathbb{A}\mathrm{gt}\setminus A}$, and $\lambda \in out([q]_A, \langle s_A, t_{\mathbb{A}\mathrm{gt}\setminus A}\rangle)$,[15] we have $M, \lambda[1] \models \varphi$;

$M, q \models \langle\langle A\rangle\rangle^u_{ir} \Box \varphi$ iff there is a uniform strategy s_A such that, for every uniform counterstrategy $t_{\mathbb{A}\mathrm{gt}\setminus A}$, and $\lambda \in out([q]_A, \langle s_A, t_{\mathbb{A}\mathrm{gt}\setminus A}\rangle)$, we have $M, \lambda[i] \models \varphi$ for all $i = 0, 1, \ldots$;

[15] Note that the definition of concurrent game structures, that we use after [3], implies that **CGS** are deterministic, so there is in fact exactly one such path λ.

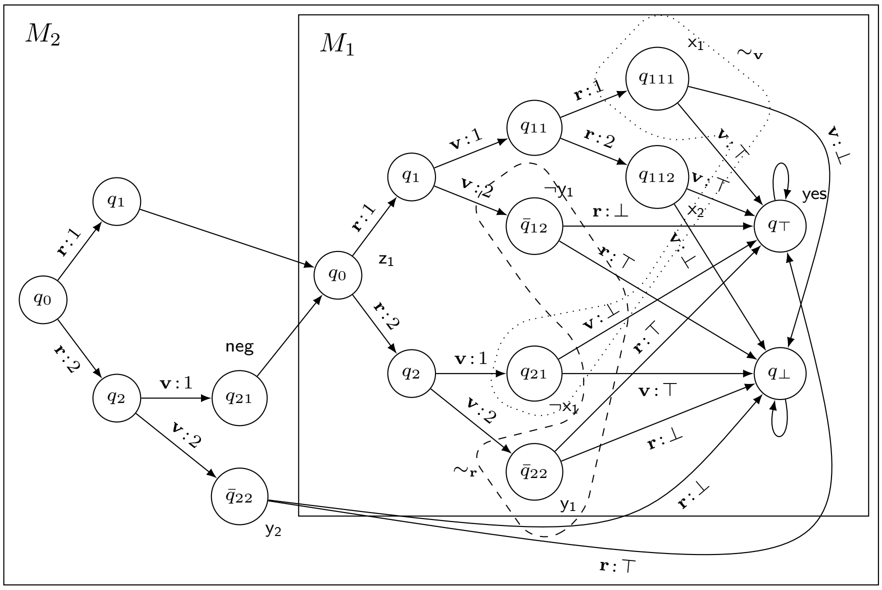

Figure 12: CEGS M_2 for $\varphi_1 \equiv ((x_1 \wedge x_2) \vee \neg y_1) \wedge (\neg x_1 \vee y_1)$, $\varphi_2 \equiv z_1 \wedge (\neg z_1 \vee y_2)$.

$M, q \models \langle\langle A \rangle\rangle_{ir} \varphi \mathcal{U} \psi$ iff there is a uniform strategy s_A such that, for every uniform counterstrategy $t_{\mathbb{A}\mathrm{gt}\setminus A}$, and $\lambda \in out([q]_A, \langle s_A, t_{\mathbb{A}\mathrm{gt}\setminus A}\rangle)$, there is $i \in \mathbb{N}_0$ with $M, \lambda[i] \models \psi$, and $M, \lambda[j] \models \varphi$ for all $0 \le j < i$.

B.2 Model Checking Complexity

We show the lower bound by reduction of **SNSAT**$_2$, a typical $\mathbf{\Delta_3^P}$-complete problem. We recall the definition of **SNSAT**$_i$ after [32].

Definition 36 (SNSAT$_i$)
Input: *p sets of propositional variables $X_r^j = \{x_{1,r}^j, ..., x_{k,r}^j\}$ for each $j = 1, \ldots, i$; p propositional variables z_r, and p Boolean formulae φ_r in positive normal form (i.e., negation is allowed only on the level of literals). Each φ_r involves only variables in $\bigcup_{j=1}^{i} X_r^j \cup \{z_1, ..., z_{r-1}\}$, with the following requirement: $z_r \equiv \exists X_r^1 \forall X_r^2 \exists X_r^3 \ldots Q X_r^i . \varphi_r(z_1, ..., z_{r-1}, X_r^1, \ldots, X_r^i)$ where $Q = \forall$ (resp. $Q = \exists$) if i is even (resp. odd).*
Output: *The value of z_p.*

In this section we focus on **SNSAT**$_2$ where we set $X_r^1 = X_r = \{x_{1,r}, ..., x_{k,r}\}$ and $X_r^2 = Y_r = \{y_{1,r}, ..., y_{k,r}\}$.

Our reduction of **SNSAT**$_2$ is an extension of the reduction of **SNSAT** presented in [27, 28]. That is, we construct the **CEGS** M_r corresponding to z_r

with two players: *verifier* $\mathbf{v}$ and *refuter* $\mathbf{r}$. The **CEGS** is turn-based, that is, every state is "governed" by a single player who determines the next transition. Each subformula $\chi_{i_1\dots i_l}$ of φ_r has a corresponding state $q_{i_1\dots i_l}$ in M_r. If the outermost logical connective of φ_r is $\wedge$, the refuter decides at q_0 which subformula χ_i of φ_r is to be satisfied, by proceeding to the "subformula" state q_i corresponding to χ_i. If the outermost connective is $\vee$, the verifier decides which subformula χ_i of φ_r will be attempted at q_0. This procedure is repeated until all subformulae are single literals. The states corresponding to literals are called "proposition" states.

The difference from the construction from [27, 28] is that formulae are in positive normal form (rather than CNF) and that we have two kinds of "proposition" states now: $q_{i_1\dots i_l}$ refers to a literal consisting of some $x \in X_r$ and is governed by $\mathbf{v}$; $\bar{q}_{i_1\dots i_l}$ refers to some $y \in Y_r$ and will be governed by $\mathbf{r}$. Now, the values of the underlying propositional variables x, y are declared at the "proposition" states, and the outcome is computed. That is, if $\mathbf{v}$ executes $\top$ for a positive literal, i.e. $\chi_{i_1\dots i_l} = x$, (or $\bot$ for $\chi_{i_1\dots i_l} = \neg x$) at $q_{i_1\dots i_l}$, then the system proceeds to the "winning" state $q_\top$; otherwise, the system goes to the "sink" state $q_\bot$. For states $\bar{q}_{i_1\dots i_l}$ the procedure is analogous. Models corresponding to subsequent z_r are nested like in Figure 12.[16] "Proposition" states referring to the same variable x are indistinguishable for $\mathbf{v}$ (so that he has to declare the same value of x in all of them), and the states referring to the same y are indistinguishable for $\mathbf{r}$. A sole $\mathbf{ATL}^u_{ir}$ proposition yes holds only in the "winning" state $q_\top$. As in [27, 28], we have the following result which concludes the reduction.

Proposition 46 *The above construction shows a polynomial reduction of* $\textbf{\textit{SNSAT}}_2$ *to model checking* $\textbf{\textit{ATL}}^u_{ir}$ *in the following sense. Let*

$$\begin{aligned}\Phi_1 &\equiv \langle\langle\mathbf{v}\rangle\rangle^u_{ir}(\neg\mathsf{neg})\mathcal{U}\mathsf{yes}, \quad \textit{and}\\ \Phi_r &\equiv \langle\langle\mathbf{v}\rangle\rangle^u_{ir}(\neg\mathsf{neg})\mathcal{U}(\mathsf{yes} \vee (\mathsf{neg} \wedge \langle\langle\varnothing\rangle\rangle^u_{ir} \bigcirc \neg\Phi_{r-1})) \quad \textit{for } r = 2,\dots,p.\end{aligned}$$

Then, we have z_p *iff* $M_p, q_0^p \models_{\mathbf{ATL}^u_{ir}} \Phi_p$*.*

As for the upper bound, we note that there is a straightforward $\mathbf{\Delta_3^P}$ algorithm that model-checks formulae of $\mathbf{ATL}^u_{ir}$: when checking $\langle\langle A\rangle\rangle^u_{ir} T\varphi$ in M, q, it first recursively checks φ (bottom-up), and labels the states where φ held with a special proposition yes. Then, the algorithm guesses a uniform strategy s_A and calls an oracle that guesses a uniform counterstrategy $t_{\mathbb{A}\mathrm{gt}\setminus A}$. Finally, it trims M according to $\langle s_A, t_{\mathbb{A}\mathrm{gt}\setminus A}\rangle$, and calls a **CTL** model checker to check formula $\mathsf{A}T\mathsf{yes}$ in state q of the resulting model. This gives us the following result.

[16] All states in the model for z_r are additionally indexed by r.

Theorem 47 *Model checking $\textbf{\textit{ATL}}^u_{ir}$ is $\mathbf{\Delta_3^P}$-complete with respect to the number of transitions in the model and the length of the formula. It is $\mathbf{\Delta_3^P}$-complete even for turn-based **CEGS** with at most two agents.*

C From $\mathbf{ATL}^u_{ir}$ to ATLP with ATLI-Based Plausibility Terms

The reduction of $\mathbf{ATL}^u_{ir}$ model checking to model checking of $\mathbf{ATLP^{ATLI}}$ in "pure" **CGS** is rather sophisticated. We do not present a reduction for full model checking of $\mathbf{ATL}^u_{ir}$; it is enough to show the reduction for the kind of models that we get in Appendix B.2 (i.e., turn-based models with two agents, two "final" states $q_\top, q_\bot$, no cycles except for the loops at the final states, and uncertainty appearing only in states one step before the end of the game, cf. Figure 12).

Firstly, we reconstruct the concurrent epistemic game structure M_p from Section B.2 so that the last action profile is always "remembered" in the final states. Then, we show how uniformity of strategies can be characterized with a formula of **ATLI** extended with epistemic operators. Thirdly, we show how the model and the formula can be transformed to get rid of epistemic links and operators (yielding a "pure" **CGS** and a formula of "pure" **ATLI**). Finally, we show how the resulting characterization of uniformity can be "plugged" into an **ATLP** formula to require that only uniform strategy profiles are taken into account.

Adding More Final States to the Model. To recall, the input of $\mathbf{ATL}^u_{ir}$ model checking consists in our case of a concurrent epistemic game structure M_p (like the one in Figure 12) and an $\mathbf{ATL}^u_{ir}$ formula Φ_p (cf. Proposition 46). We begin the reduction by reconstructing M_p to M'_p in which the last action profile is "remembered" in the final states. The idea is based on the construction from [19, Proposition 16] where it is applied to all states of the system, cf. Figure 13.

In our case, we first create copies of states $q_\top, q_\bot$, one per incoming transition. That is, the construction yields states of the form $\langle q, \alpha_1, \ldots, \alpha_k\rangle$, where $q \in \{q_\top, q_\bot\}$ is a final state of the original model M_p, and $\langle \alpha_1, \ldots, \alpha_k\rangle$ is the action profile executed just before the system proceeded to q. Each copy has the same valuation of propositions as the original state q, i.e., $\pi'(\langle q, \alpha_1, \ldots, \alpha_k\rangle) = \pi(q)$. Then, for each action $\alpha \in Act$ and agent $i \in \mathbb{A}\mathrm{gt}$, we add a new proposition $\mathsf{i} : \alpha$. Moreover, we fix the valuation of $\mathsf{i} : \alpha$ in M'_p so that it holds exactly in the final states that can be achieved by an action profile in which i executes α (i.e., states $\langle q, \alpha_1, ..., \alpha_i, ..., \alpha_k\rangle$). Note that the number of both states and transitions in M'_p is linear in the transitions of M_p. The transformation produces model M'_p which is equivalent to M_p in the following sense. Let φ

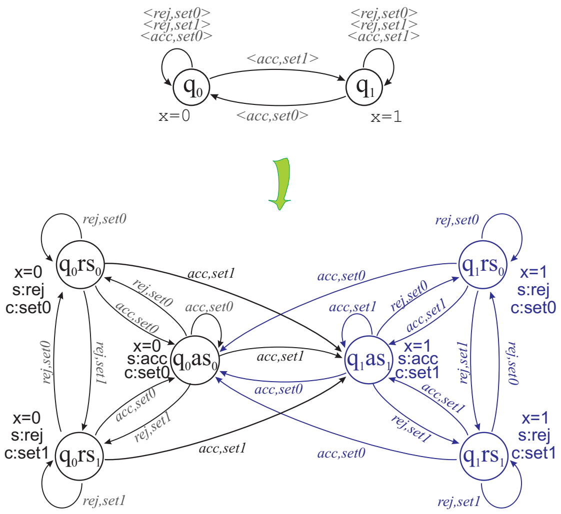

Figure 13: Memorizing the last action profile in a simple 2-agent system

be a formula of $\mathbf{ATL}^u_{ir}$ that does not involve special propositions $\mathsf{i} : \alpha$. Then, for all $q \in Q$: $M_p, q \models_{\mathbf{ATL}^u_{ir}} \varphi$ iff $M'_p, q \models_{\mathbf{ATL}^u_{ir}} \varphi$.

In M'_p, agents can "recall" their actions executed at states that involved some uncertainty (i.e., states in which the image of some indistinguishability relation $\sim_i$ was not a singleton). Now we can use **ATLI** (with additional help of knowledge operators, see below) to characterize uniformity of strategies.

Characterizing Uniformity in ATLI+K. We will now show that uniformity of a strategy can be characterized in **ATLI** *extended with epistemic operators* K_a (that we call **ATLI+K**. $K_a\varphi$ reads as "agent a knows that φ". The semantics of **ATLI+K** extends that of **ATLI** by adding the standard semantic clause from epistemic logic:

$$M, q \models K_a\varphi \text{ iff } M, q' \models \varphi \text{ for every } q' \text{ such that } q \sim_a q'.$$

We note that **ATLI+K** can be also seen as **ATEL** [47] extended with intentions.

Let us now consider the following formula of **ATLI+Knowledge**:

$$\mathit{uniform}(\sigma) \equiv (\mathbf{str}\,\sigma)\langle\langle\emptyset\rangle\rangle\Box \bigwedge_{i\in\mathbb{A}\mathrm{gt}} \bigvee_{\alpha\in d(i,q)} K_i\langle\langle\emptyset\rangle\rangle\bigcirc \mathsf{i} : \alpha.$$

The reading of $\mathit{uniform}(\sigma)$ is: suppose that profile σ is played $(\mathbf{str}\,\sigma)$; then, for all reachable states $(\langle\langle\emptyset\rangle\rangle\Box)$, every agent has a single action $(\bigwedge_{i\in\mathbb{A}\mathrm{gt}} \bigvee_{\alpha\in d(i,q)})$

that is determined for execution ($\langle\langle\emptyset\rangle\rangle \bigcirc \mathsf{i} : \alpha$) in every state indistinguishable from the current state (K_i). Thus, formula $uniform(\sigma)$ characterizes the ***uniformity*** of strategy profile σ. Formally, for every concurrent epistemic game structure M, we have that $M, q \models_{\mathbf{ATLI+K}} uniform(\sigma)$ iff $[\sigma[a]]$ is uniform for each agent $a \in \mathbb{A}\mathrm{gt}$ (for all states reachable from q). Of course, only reachable states matter when we look for strategies that should enforce a temporal goal.

Note that the epistemic operator K_a refers to incomplete information, but σ is now an arbitrary (i.e., not necessarily uniform) strategy profile. We observe that the length of the formula is linear in the number of agents and actions in the model.

Translating Knowledge to Ability. To get rid of the epistemic operators from formula $uniform(\sigma)$ and epistemic relations from model M'_p, we use the construction from [23] (which refines that from [19, Section 4.4]). The construction yields a concurrent game structure $tr(M'_p)$ and an **ATLI** formula $tr(uniform(\sigma))$. The idea can be sketched as follows. The set of agents becomes extended with ***epistemic agents*** e_i (one per $a_i \in \mathbb{A}\mathrm{gt}$), yielding $\mathbb{A}\mathrm{gt}'' = \mathbb{A}\mathrm{gt} \cup \mathbb{A}\mathrm{gt}^e$. Similarly, the set of states is augmented with ***epistemic states*** q^e for every $q \in Q'$ and $e \in \mathbb{A}\mathrm{gt}^e$; the states "governed" by the epistemic agent e_a are labeled with a special proposition $\mathsf{e_a}$. The "real" states q from the original model are called "action" states, and are labeled with another special proposition act. Epistemic agent e_a can enforce transitions to states that are indistinguishable for agent a (see Figure 14 for an example).[17] Then, "a knows φ" can be rephrased as "e_a can only effect transitions to epistemic states where φ holds". With some additional tricks to ensure the right interplay between actions of epistemic agents, we get the following translation of formulae:

$$
\begin{aligned}
tr(p) &= p, \qquad \text{for } p \in \Pi \\
tr(\neg\varphi) &= \neg tr(\varphi) \\
tr(\varphi \vee \psi) &= tr(\varphi) \vee tr(\psi) \\
tr(\langle\langle A\rangle\rangle \bigcirc \varphi) &= \langle\langle A \cup \mathbb{A}\mathrm{gt}^e\rangle\rangle \bigcirc (\mathsf{act} \wedge tr(\varphi)) \\
tr(\langle\langle A\rangle\rangle \Box \varphi) &= \langle\langle A \cup \mathbb{A}\mathrm{gt}^e\rangle\rangle \Box (\mathsf{act} \wedge tr(\varphi)) \\
tr(\langle\langle A\rangle\rangle \varphi \mathcal{U} \psi) &= \langle\langle A \cup \mathbb{A}\mathrm{gt}^e\rangle\rangle (\mathsf{act} \wedge tr(\varphi)) \mathcal{U} (\mathsf{act} \wedge tr(\psi)) \\
tr(K_i\varphi) &= \neg\langle\langle e_1, ..., e_i\rangle\rangle \bigcirc \big(\mathsf{e_i} \wedge \langle\langle e_1, ..., e_k\rangle\rangle \bigcirc (\mathsf{act} \wedge \neg tr(\varphi))\big).
\end{aligned}
$$

Note that the length of $tr(\varphi)$ is linear in the length of φ and the number of agents k. Two important facts follow from [23, Theorem 8]:

Lemma 48 *For every **CEGS** M and a formula of $\mathbf{ATL}^u_{ir}$ that does not include the*

[17] The interested reader is referred to [23] for the technical details of the construction.

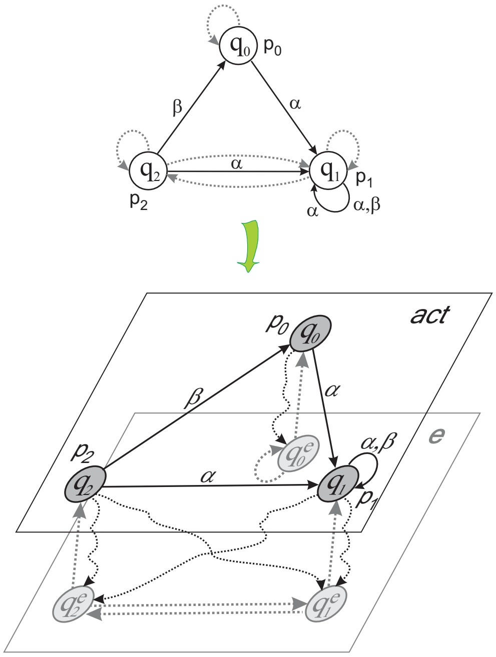

Figure 14: Getting rid of knowledge and epistemic links

special propositions act, $e_1, \ldots, e_k$*, we have* $M, q \models_{\boldsymbol{ATL}_{ir}^{u}} \varphi$ *iff* $tr(M), q \models_{\boldsymbol{ATL}_{ir}^{u}} tr(\varphi)$.

Lemma 49 *For every* ***CEGS*** M*, we have* $M, q \models_{\boldsymbol{ATLI+K}} uniform(\sigma)$ *iff* $tr(M), q \models_{\boldsymbol{ATLI+K}} tr(uniform(\sigma))$.

Putting the Pieces Together: the Reduction. We observe that $\mathbf{ATL}_{ir}^{u}$ can be seen as **ATL** where only uniform strategy profiles are allowed. An **ATLI** formula that characterizes uniformity has been defined in the previous paragraphs. It can be now plugged into our "**ATL** with Plausibility" to restrict agents' behaviour in the way the semantics of $\mathbf{ATL}_{ir}^{u}$ does. This way, we obtain a reduction of $\mathbf{SNSAT}_2$ to model checking of $\mathbf{ATLP}^{\mathbf{ATLI}}$.

Proposition 50
z_p *iff* $tr(M_p'), q_0^p \models_{\boldsymbol{ATLP}^{\boldsymbol{ATLI}}} (\textbf{set-pl}\ \sigma.tr(uniform(\sigma)))\mathbf{Pl}\, tr(\Phi_p)$.

Proof. We have z_p iff $M_p', q_0^p \models_{\mathbf{ATL}_{ir}^{u}} \Phi_p$ iff $tr(M_p'), q_0^p \models_{\mathbf{ATL}_{ir}^{u}} tr(\Phi_p)$ iff $tr(M_p'), q_0^p \models_{\mathbf{ATLP}^{\mathbf{ATLI}}} (\textbf{set-pl}\ \sigma.tr(uniform(\sigma)))\mathbf{Pl}\, tr(\Phi_p)$. ■

D Some Model Checking Complexity Proofs

D.1 Results in Section 6.1

Proposition 23: Let M be a well-behaved **CGSP**, q a state in M, and φ a formula of $\mathcal{L}^{\text{base}}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \Omega)$. Then $M, q \models \varphi$ iff $mcheckATLP(M, q, \varphi)$. The algorithm runs in time $\mathbf{\Delta_3^P}$ with respect to the number of transitions in the model and the length of the formula.

Proof Function $mcheck$ is called recursively, at most l times. All cases apart from $\varphi \equiv \langle\langle A \rangle\rangle \bigcirc \psi$ where ψ includes no $\langle\langle C \rangle\rangle$ (analogously for the other temporal operators) can be performed in polynomial time. Now, there is a nondeterministic Turing machine A_B which implements function $beatable$: Firstly, it guesses a strategy t possibly together with another witness necessary for $plausiblestrat$ (by assumption the latter is in **NP**) and verifies if t is plausible, the verification can be done in polynomial time (by the same assumption). Finally, if t is plausible A_B has to perform **CTL** model checking which lies in **P**.

It remains to show that there is a nondeterministic oracle Turing machine A_S with oracle A_B implementing $solve$. (Formally, the machine requires two oracles, one answering the question whether s is plausible, and the other is given by A_B. However, the former is computationally less expensive then the latter and can be ignored since we are interested in the oracle with the highest complexity.) A_S works as follows: Firstly, it guesses a profile s (again possibly together with a witness for $plausiblestrat$); secondly, it verifies whether s is plausible and then calls oracle A_B and inverts its answer. Altogether, there are polynomial many calls to machine $A_S^{A_B} \in \mathbf{NP^{NP}}$. This renders the algorithm to be in $\mathbf{\Delta_3^P}$. ■

Proposition 27: Let $c \in \mathbb{N}$ be a constant. Model checking $\mathcal{L}^{\text{base}}_{ATLP}$ formulae with respect to the class of well-behaved bounded models $\mathfrak{M}^c$ can be done in polynomial time with respect to the number of transitions in the model and the length of the formula.

Proof sketch We modify the original **ATL** model checking procedure as follows. Consider the formula $\varphi \equiv \langle\langle A \rangle\rangle \gamma$ where γ is a pure **ATL** path formula. Let B be the set of agents assumed to play plausibly and let $\Upsilon \neq \Sigma$ be the current set of plausible strategies described by some term and state. For each $s_B \in \Upsilon|_B$ we remove from M all transitions which cannot occur according to s_B, yielding model M^{s_B}, and check whether $M^{s_B}, q \models_{\mathbf{ATL}} \langle\langle A \rangle\rangle \gamma$. We proceed like this for all $s \in \Upsilon|_B$ (there are only polynomially many). This procedure is incorporated into our **ATLP** model checking algorithm and applied bottom up. ■

D.2 Results in Section 6.3

D.2.1 Upper Bounds

First, we recall a basic complexity result that will be used in the rest of this section. Then, we present proofs of upper bounds for model checking $\mathcal{L}^k_{ATLP}$ for pure **CGS**'s and well-behaved **CGSP**'s.

Remark 51 *A relation* $R \subseteq \times_{i=1}^{k+1}\Sigma^*$ *(*$k \geq 1$*) is called* polynomial decidable *whenever there is a deterministic Turing machine (DTM) which decides* $\{(x, y_1 \dots, y_k) : (x, y_1 \dots, y_k) \in R\}$ *in polynomial time; furthermore,* R *is called* polynomial balanced *if there is a* $k \in \mathbb{N}$ *such that for all* $(x, y_1 \dots, y_k) \in R$*:* $|y_i| \leq |x|^k$ *for all* $i = 1, \dots k$*.*

For a language L *and* $k \geq 1$ *the following holds:* $L \in \Sigma^P_k$ *if, and only if, there is a polynomial decidable and balanced* $(k+1)$*-ary relation* R *such that* $L = \{x \mid \exists y_1 \forall y_2 \exists y_3 \dots Q y_k\ ((x, y_1 \dots, y_k) \in R)\}$ *where* $Q = \forall$ *(resp.* $Q = \exists$*) if* k *is odd (resp.* k *even) [36, Corollary 2 of Theorem 17.8].*

Theorem 41: Let φ be a level-i formula of $\mathcal{L}^k_{ATLP}(\mathbb{A}\text{gt}, \Pi, \emptyset)$, $k \geq 1$, $i \geq 0$. Moreover, let M be a **CGS**, and q a state in M. Then, model checking $M, q \models \varphi$ can be done in time $\mathbf{\Delta^P_{i+2k+1-\max\{0,k-i-1\}}}$.

Proof By induction over k. In the following we restrict ourselves to (**set-pl** $\cdot$) without loss of generality.

Case $k = 1$. Let φ be a level-i $\mathcal{L}^1_{ATLP}$ formula, (**set-pl** ω) an operator occurring in φ such that $l(\{(\textbf{set-pl}\ \omega)\}) = i$ and $\omega = \sigma.Q_1\sigma_1 Q_2\sigma_2 \dots Q_n\sigma_n\varphi'$ where

$$\varphi' \in \mathcal{L}^{\text{base}}_{ATLP}(\mathbb{A}\text{gt}, \Pi, \{\sigma, \sigma_1, \dots, \sigma_n\}).$$

Note that $M^{s,s_1,\dots,s_n}, q \models \varphi'$ can be checked in polynomial time since all constructible plausibility terms are rectangular and the representation is directly given (see Corollary 31). Moreover, let q' denote the state in which ω is evaluated. W.l.o.g. we can assume that φ has the following structure:

$$\varphi \equiv (\textbf{set-pl}\ \omega)\mathbf{Pl}\ \langle\!\langle A\rangle\!\rangle \Box \mathsf{yes}$$

Now, φ is true in M and q if and only if there is a plausible strategy s_A for A and *no* plausible strategy t with $t|_A = s$ such that $M', q \models_{\mathbf{CTL}} \neg\mathsf{A}\Box\mathsf{yes}$ where M' is the trimmed model of M wrt t. In the following we neglect the complexity needed to verify whether s_A is plausible since the method $\mathit{beatable}$ also verifies this property and its complexity is as least as high (cf. proof of Proposition 23). Thus, φ is true if, and only if

$$\exists s_A \neg\big(\exists t\ (t \in \widehat{[\![\omega]\!]}^{q'} \text{ and } R_{\models}(M, q, s_A, t, \Box\mathsf{yes}))\big)$$
$$\text{iff } \exists s_A \neg\left(\exists t Q_1 s_1 Q_2 s_2 \dots Q_n s_n\ (M^{t,s_1,\dots,s_n}, q' \models \varphi' \text{ and } R_{\models}(M, q, s_A, t, \Box\mathsf{yes}))\right)$$
$$\text{iff } \exists s_A \forall t \bar{Q}_1 s_1 \bar{Q}_2 s_2 \dots \bar{Q}_n s_n\ (M^{t,s_1,\dots,s_n}, q' \not\models \varphi' \text{ or } \neg R_{\models}(M, q, s_A, t, \Box\mathsf{yes}))$$

where $R_{\models}(M, q, s_A, t, \Box\mathsf{yes}) = \mathit{true}$ iff $t|_A = s_A$ and $M', q \models_{\mathbf{CTL}} \neg\mathsf{A}\Box\mathsf{yes}$ where M' is the "trimmed" model of M wrt t, and $\bar{Q}$ is the dual operator to Q.

Now, the latter conditions can be verified in polynomial time. We consider the number of quantifier alternations. Subsequent strategies which are quantified by quantifiers of the same type can be guessed together. The same holds if the sequence starts with existential quantifiers. These strategies can be guessed together with strategy t. A quantifier level of $l(\{(\textbf{set-pl}\ \omega)\}) = i$ denotes that it is sufficient to alternatingly guess i witnesses. We obtain the following structure:

$$\exists s_A \forall x_t \exists x_1 \forall x_2 \dots Q x_i$$

where $Q = \exists$ (resp. $Q = \forall$) if i is even (resp. odd). Where x_i denotes a witness for a strategy or several strategies if guessing can be combined.

Thus, according to Remark 51 checking whether φ is satisfied can be determined in time Σ_{i+2} and the complete model checking algorithm for level-i $\mathcal{L}^1_{ATLP}$ formula can be performed in time $\mathbf{\Delta^P_{i+3}}$ (there can be polynomial many such constructs).

Induction step: $k \mapsto k+1$ **(**$k > 1$**).** Let φ be a level-i $\mathcal{L}^{k+1}_{ATLP}$ formula and let ω be a term in φ of the form $\omega = \sigma_1.Q_1\sigma_1 Q_2\sigma_2 \dots Q_n\sigma_n\varphi'$ such that $l((\textbf{set-pl}\ \omega)) = i$. Furthermore, let $\text{RALT}(Q_1 \dots Q_n) = j$; then, $l_{\varphi'} := ql(\mathcal{UO}(\varphi')) = i - j$ and φ' is an $\mathcal{L}^k_{ATLP}$ formula. Thus, by induction hypothesis we have that φ' can be model checked in time

$$\mathbf{\Delta^P_{r+1}} \quad \text{where } r := l_{\varphi'} + 2k - \max\{0, k - l_{\varphi'} - 1\}.$$

Again, w.l.o.g. we can assume that φ has the following structure:

$$\varphi \equiv (\textbf{set-pl}\ \omega)\mathbf{Pl}\,\langle\langle A\rangle\rangle\Box\mathsf{yes}.$$

We proceed as in case $k = 1$. Firstly, a profile s is guessed, then a profile t and it is checked whether t is plausible and coincides with s wrt A and whether the trimmed model (wrt t) satisfies $\neg\mathsf{A}\Box\mathsf{yes}$. We obtain the following structure:

$$\exists s_A \neg \left(\exists t\ (t \in \widehat{[\![\omega]\!]}^{q'} \text{ and } R_{\models}(M, q, s_A, t, \Box\mathsf{yes})) \right)$$

$$\text{iff } \exists s_A \neg \left(\exists t Q_1 s_1 Q_2 s_2 \dots Q_n s_n\ (\underbrace{M^{t,s_1,\dots,s_n}, q' \models \varphi'}_{\in \mathbf{\Delta^P_{r+1}}} \text{ and } \underbrace{R_{\models}(M, q, s_A, t, \Box\mathsf{yes})}_{\in \mathbf{P}}) \right)$$

Since $M^{t,s_1,\dots,s_n}, q' \models \varphi'$ is invoked by a nondeterministic polynomial Turing machine we can assume that its model checking problem can be solved in $\mathbf{\Sigma_r^P}$ instead of $\mathbf{\Delta_{r+1}^P}$; the polynomial effort of the deterministic machine can also be done by the invoking nondeterministic machine. Hence to verify $M^{t,s_1,\dots,s_n}, q' \models \varphi'$ witnesses according to

$$\exists x_1 \forall x_2 \exists x_3 \dots Q_r x_r$$

have to be guessed; then, the question whether φ is satisfied with respect to the witnesses $x_1, \dots, x_r$ can be solved in polynomial time.

Because $\text{RALT}(Q_1 Q_2 \dots Q_n) = j$ it suffices to guess witnesses according to the following structure:

$$\forall x'_1 \exists x'_2 \forall x'_3 \dots \forall x'_j.$$

If $Q_1 Q_2 \dots Q_n$ would start (resp. end) with existential quantifiers the corresponding witnesses could be guessed together with the one for profile t (resp. witness x_1). Putting things together the following witnesses have to be guessed:

$$(\star) \qquad \exists s_A \forall x_t \; \exists x'_1 \forall x'_2 \exists x'_3 \dots \exists x'_j \; \forall x_1 \exists x_2 \forall x_3 \dots \bar{Q}_r x_r$$

It remains to show that the number of alternations in $(\star)$ does never exceed $i + 2(k+1) - \max\{0, (k+1) - i - 1\}$

We distinguish two cases $(k+1) - i - 1 \leq 0$ and $(k+1) - i - 1 > 0$.

Case: $k + 1 - i - 1 \leq 0$**.** That is, $k \leq i$. We are going to determine the maximal possible number of alternations in $(\star)$.

Firstly, assume that $j \geq 1$. That is the number of alternation is given by $2 + j + r = i + 2(k+1) - \max\{0, k - i + j - 1\}$. This expression is maximal whenever $k - i + j - 1 \leq 0$. Because of $k \leq i$ this is always the case for $j = 1$. In this case the formula has at most

$$i + 2(k+1) - \max\{k + 1 - i - 1\} \quad \text{alternations.}$$

For $j = 0$ there is at least one alternation less, since the witness x_t can be guessed together with x_1.

Case: $k + 1 - i - 1 > 0$**.** That is, $k > i$. Firstly, we consider the case $j \geq 1$. There are at most $i + 2(k+1) - \max\{0, k - i + j - 1\}$ alternations, where the number becomes maximal for $j = 1$; i.e. we have at most

$$i + 2(k+1) - \max\{k + 1 - i - 1\} \quad \text{alternations.}$$

Now, we consider the case $j = 0$. In this case there are at most $i + 2(k+1) - 1 - \max\{0, k - i - 1\}$ alternations. Because of $k > i$, we have that $k - i - 1 \geq 0$ and, hence $i + 2(k+1) - 1 - \max\{0, k - i - 1\}$ is equivalent to $i + 2(k+1) - \max\{0, (k+1) - i - 1\}$.

Thus, $i + 2(k+1) - \max\{k+1-i-1\}$ alternations denotes the maximal possible number of alternations which proofs our claim the model checking algorithm for level-i $\mathcal{L}_{ATLP}^{k+1}$ can be performed in time $\mathbf{P}^{\mathbf{\Sigma}^{\mathbf{P}}_{\mathbf{i+2(k+1)-\max\{k+1-i-1\}}}} = \mathbf{\Delta}^{\mathbf{P}}_{\mathbf{i+2(k+1)+1-\max\{k+1-i-1\}}}$.

■

Theorem 42: Let φ be a level-i formula of $\mathcal{L}_{ATLP}^{k}(\mathbb{A}\mathrm{gt}, \Pi, \Omega)$, M a well-behaved **CGSP**, and q a state in M. Model checking $M, q \models \varphi$ is in $\mathbf{\Delta}^{\mathbf{P}}_{\mathbf{i+2(k+1)+1-\max\{0,k-i\}}}$.

Proof The proof is similar to the one of Theorem 41. In comparison to the claim of Theorem 41, $2k$ has changed to $2(k+1)$ and $\max\{0, k-i-1\}$ to $\max\{0, k-i\}$. The reason for this is that the final nesting (i.e. formulae in $\mathcal{L}_{ATLP}^{\mathrm{base}}$) might contain hard-wired terms and it can not be verified in polynomial time anymore. This causes the change from k to $k+1$ (it requires to guess s_A and verify it against all responses t). However, now the complexity might be increased too much since the final strategy s_A of A could be guessed together with the next to last strategy t' of the opponents $(\exists s'_A \neg(\exists t' \exists s_A \neg(\exists t)))$ if there is no further alternation between t' and s_A, caused by a plausibility term. Such an “interfering” alternation is only possible if the given formula is at least an level-k formula; this is reflected by $\max\{0, k-i\}$. ■

D.2.2 PSPACE-completeness of $\mathcal{L}_{ATLP}^{k}$ Model Checking

We use *quantified satisfiability* (**QSAT**) to show **PSPACE**-completeness of model checking $\mathcal{L}_{ATLP}^{k}$ and $\mathcal{L}_{ATLP}$.

Definition 37 (QSAT [36])
Input: *A boolean formula φ in conjunctive normal with i variables $x_1, \dots, x_i$.*
Output: *True if $\exists x_1 \forall x_2 \dots Q_i x_i\ \varphi$ is satisfiable, false otherwise (where $Q = \forall$ if i is even, and $Q = \exists$ if i is odd).*

Given an instance φ of **QSAT** we construct an $\mathcal{L}_{ATLP}^{1}$ formula θ_φ and a **CGSP** M_φ (both are constructible in polynomial space regarding the length of φ) such that φ is satisfiable if, and only if, $M_\varphi, q_0 \models \theta_\varphi$. In the following we sketch the constructions which are based on the reduction of $\mathbf{SNSAT}_2$ to model checking $\mathbf{ATL}_{ir}^{u}$ proposed in Appendix B.2, and the translation of $\mathbf{ATL}_{ir}^{u}$ to $\mathcal{L}_{\boldsymbol{ATLP}^{\boldsymbol{ATLI}}}$ proposed in Appendix C.

Let $\varphi \equiv \exists x_1 \forall x_2 \dots Q_n x_n\ \psi$ be an instance of **QSAT**. Firstly, we sketch the construction of the **CEGS** M'_φ which will then be transformed into a **CGSP** M_φ. In comparison to the construction in Appendix B.2, we consider n agents one for each quantifier (in fact, we consider $\max\{2, n\}$ agents; however, for the rest of this section we assume that $n \geq 2$). The agent belonging to quantifier i is named a_i. Except for the proposition states the procedure is completely analogous to the construction given in Appendix B.2 where agent a_2

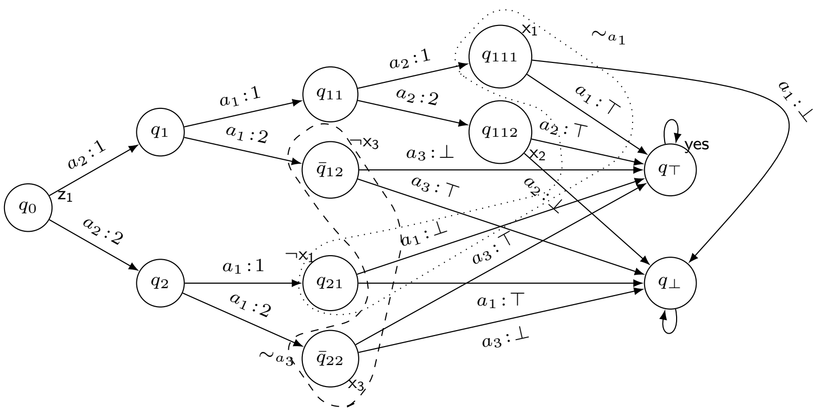

Figure 15: Construction of the intermediate model M'_φ for $\varphi \equiv \exists x_1 \forall x_2 \exists x_3((x_1 \land x_2) \lor \neg x_3) \land (\neg x_1 \lor x_3)$.

is considered as *refuter* and a_1 as *verifier*. (Alternatively, two additional agents could be added.) The procedure at the proposition states changes as follows: In such a state, say q, referring to a literal l, say $l = x_i$, agent a_i can decide on the value of x_i. Note again that the agent is required to make the same choice in indistinguishable states. In Figure 15 the construction is shown for the formula $\varphi \equiv \exists x_1 \forall x_2 \exists x_3((x_1 \land x_2) \lor \neg x_3) \land (\neg x_1 \lor x_3)$. Finally, the model M_φ is obtained from M'_φ by following the same steps as described in Appendix B.2.

Secondly, we construct formula θ_φ from φ as follows:

$$\theta_\varphi \equiv (\textbf{set-pl}\ \sigma_1.\forall\sigma_2\exists\sigma_3 \ldots Q_n\sigma_n\chi)\mathbf{Pl}\,\langle\langle \mathbb{A}\mathrm{gt} \rangle\rangle \bigcirc \top$$

where

$$\chi \equiv \left(\bigwedge_{i=1,\ldots,n} \mathit{uniform}^i_{\mathbf{ATLP}}(\sigma_i) \right) \land (\textbf{set-pl}\ \langle \sigma_1[1], \ldots, \sigma_n[n] \rangle)\mathbf{Pl}\,\langle\langle \emptyset \rangle\rangle \Diamond \mathsf{yes}.$$

Next, we will give the intuition behind θ_φ. Firstly, it is easy to see that $\mathbf{Pl}\,\langle\langle \mathbb{A}\mathrm{gt} \rangle\rangle \bigcirc \top$ is true whenever the set of plausible strategy profiles is *not empty*. Hence, the actual set of strategies described by the preceding ($\textbf{set-pl}\ \cdot$) operator is not particularly important, rather if *some* strategy is plausible or not.

Secondly, note that ($\textbf{set-pl}\ \langle \sigma_1[1], \ldots, \sigma_n[n] \rangle$) in χ describes a single strategy profile and that all individual strategies can be considered independently (the set is rectangular, cf. Definition 34 and Lemma 29). Furthermore, an individual strategy is mainly used to assign $\top$ or $\bot$ to propositional variables in the proposition states. (Except for agents a_1 and a_2 which also take on the refuter and verifier role; they can also perform actions in non-proposition states.) Hence, a given strategy profile can be seen as a valuation of the propositional variables.

Thirdly, we analyze χ with respect to a given profile $\sigma := \langle\sigma_1[1], \ldots, \sigma_n[n]\rangle$ taking into account the previous points. By formula $\mathit{uniform}^i_{\mathbf{ATLP}}(\sigma_i)$ it is ensured that agent i assigns the same valuation to propositions in indistinguishable states. Now, χ is true if the "winning state" $q_\top$ is reached by following the strategy described by σ (it describes a unique path in the model). In other words, χ is true if, and only if, the valuation described by σ satisfies φ.

Finally, due to the previous observations, if $[\![\sigma_1.\forall\sigma_2\widehat{\exists\sigma_3\ldots} Q_n\sigma_n\chi]\!]$ is nonempty it can be interpreted as follows: There is a valuation of x_1 such that for all valuations of x_2 there is a valuation of x_3, and so forth such that φ is satisfied.

The following proposition states that the construction is correct.

Proposition 52 *Let φ be a **QSAT** instance. Then it holds that φ is satisfiable if, and only if, $M_\varphi, q_0 \models \theta_\varphi$ where M_φ and θ_φ are effectively constructible from φ in polynomial space with respect to the length of the formula φ.*

Proof sketch Let φ be a **QSAT** instance. We use the construction above to obtain M'_φ and θ_φ where $\mathit{uniform}^i_{\mathbf{ATLP}}(\sigma)$ is obtained as follows: Firstly, we take the **ATLI** +K formula $\mathit{uniform}(\sigma|_i)$ (where $\sigma|_i$ refers to agent i's startegy in σ) as described in Appendix C; then, we use the polynomial translation to change knowledge to ability, yielding a pure **ATLI** formula. Finally, we use the polynomial translation from **ATLI** to **ATLP** given in Section 5.1 (Proof of Proposition 11) to obtain a pure **ATLP** formula $\mathit{uniform}^i_{\mathbf{ATLP}}(\sigma)$. Hence, the latter formula is true if agent i's strategy contained in the complete profile σ is a uniform strategy. This shows that θ_φ can be constructed in polynomial space.

Model M_φ is obtained from M'_φ by the same scheme. Firstly, the construction from [23] referred to in Appendix C is applied. Secondly, the resulting **CGS** with intentions is transformed to a **CGSP** using the construction from Section 5.1 (Proposition 11) again. The constructed model M_φ is also polynomial with respect to φ.

We get that φ is satisfiable if, and only if, $M_\varphi, q_0 \models \theta_\varphi$. ∎