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**WHAT DO STUDENTS LEARN ABOUT FUNCTIONS ?
A CROSS CULTURAL STUDY IN ENGLAND AND
MALAYSIA**

A thesis submitted in fulfilment of the requirements

for the degree of Doctor of Philosophy

Md.Nor Bakar

**BSc. Ed. (Honours) Malaya
M. Ed. (Keele)**

at

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ABSTRACT

This research study investigates the concept of function developed by a sample of secondary and university students in England and Malaysia studying mathematics as one of their subjects. It shows that whilst students may be able to do the 'mechanical' parts of this concept, their grasp of the 'theoretical' nature of the function concept may be tenuous and inconsistent.

The hypothesis is that students develop 'prototypes' for the function concept in much the same way as they develop prototypes for concepts in everyday life. The definition of the function concept, though given in the curriculum, proves to be inoperative, with their understanding of the concept reliant on properties of familiar prototype examples: those having regular shaped graphs, such as x^2 or $\sin x$, those often encountered (possibly erroneously), such as a circle, those in which y is defined as an explicit formula in x , and so on.

The results of the study in England revealed that even when the function concept was taught through the formal definition, the experiences which followed led to various prototypical conceptions. Investigations also show significant misconceptions. For example, threequarters of a sample of students starting a university mathematics course considered that a constant function was *not* a function in either its graphical or algebraic forms, and three quarters thought that a circle *is* a function.

The extension of the study in Malaysia was made with the hypothesis that there is a significant difference between the concept as perceived to be taught and as actually learned by the students. Although the intended curriculum emphasises conceptual understanding, in the perceived curriculum (curriculum as understood by the teachers), only 45% of the teachers follow this approach. The tested curriculum as reflected in the public examination questions, only emphasises the procedural skills and the results of the learned curriculum show that learning of functions is more consistent with the theory of prototypical learning. Students in Malaysia develop their own idiosyncratic mental prototypes for the function concept in much the same way as those students in the UK.

CONTENTS

CHAPTER 1

BACKGROUND TO THE STUDY

	Page	
1.1	Introduction and background of the problems	1
1.2	The focus of the study	1
1.3	A review of the related literature	
1.3.1	Overview of previous psychological studies of the function concept	3
1.3.2	Conferences on the function concept	7
1.3.3	Related papers concerning the function concept	8
1.3.4	Overview of some of the official statements concerning the importance of the function concept in mathematics curriculum	15
1.4	Theoretical framework of the study	17
1.5	Objectives of the study	19
1.6	Summary and organisation of the thesis	19

CHAPTER 2

THE FUNCTION CONCEPT

2.1	Overview of the historical development of the function concept	20
2.2	Function definitions	21
2.3	Function concept in the school textbooks	24
2.4	Pedagogical implications	28

CHAPTER 3
THE PROBLEMS OF TEACHING AND LEARNING
MATHEMATICAL CONCEPTS

3.1	Changing views of teaching and learning	29
3.2	Learning theories and conceptual development	30
3.3	The process of conceptual development in mathematics	31
3.4	Toward a theory of prototypical learning of mathematical concepts	34

CHAPTER 4
THE PRELIMINARY STUDY IN ENGLAND

4.1	Analysis of results of preliminary investigation	41
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CHAPTER 5
THE MAIN STUDY IN ENGLAND

5.1	Observation of teaching approach in two English secondary schools	55
5.2	Analysis of results	67

CHAPTER 6
MATHEMATICS EDUCATION IN MALAYSIA

6.1	Modern Mathematics Programmes in Malaysian Secondary Schools	86
6.2	Some of the problems and issues pertaining to mathematics education in Malaysia	87
6.2.1	Problems of Curriculum Adaptation	88
6.2.2	Recent developments of mathematics curriculum	91

CHAPTER 7
THE STUDY IN MALAYSIA .

7.1	Plan for main study in Malaysia	93
7.1.1	Analysis of the intended curriculum	94
7.1.2	Conducting the study in secondary schools	95
7.1.3	Conducting the study at the University of Technology Malaysia, Skudai Campus	96
7.2	The intended curriculum	96
7.2.1	General background of mathematics syllabus and textbooks in Malaysian secondary schools	96
7.2.2	Analysis of curriculum materials with special reference to the function concept	98
7.2.3	Some comments on the intended curriculum related to the function concept	102
7.3	The perceived curriculum	103
7.3.1	The function concept in the school curriculum	104
7.3.2	Teaching approaches	105
7.3.3	Algorithmic skills versus conceptual understanding	107
7.3.4	The effects of public examination	109
7.3.5	Difficulties in teaching the function concept	111
7.4	The tested curriculum	111
7.4.1	An analysis of examination questions	112
7.4.2	Influence of examination on the teaching and learning	114
7.5	The learned curriculum	116
7.5.1	Prototypical conceptions of the function concept	117
7.5.2	What is and what is not a function	122

CHAPTER 8
SUMMARY, IMPLICATIONS AND SUGGESTIONS
FOR FURTHER RESEARCH

8.1	Introduction	126
8.2	Summary of major findings	
8.2.1	Results pertaining to the analysis of the intended curriculum	126
8.2.2	Results pertaining to the perceived curriculum	127
8.2.3	Results associated with the data on the tested curriculum	127
8.2.4	Results related to the learned curriculum	128
8.3	Discussion	
8.3.1	Questions concerning the suitability of an adaptation of the foreign curricula	129
8.3.2	Questions related to the social and cultural norms	131
8.3.3	Comments concerning the perceived curriculum	131
8.3.4	Questions related to the examinations procedures	132
8.3.5	Questions concerning the learned curriculum	132
8.4	Implications of the study and some recommendations for the improvements of mathematics education in Malaysia.	134
8.5	Limitations of the study	137
8.6	Suggestions for further research	137

APPENDICES

1. Function questionnaire (the study in England)
2. Function questionnaire (the study in Malaysia)
3. Function questionnaire (English version)
4. Questionnaire for teachers (the study in Malaysia)
5. Questionnaire for teachers (English version)
6. Relevant permissions from educational authorities in England and Malaysia.

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DECLARATION

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CHAPTER 1

BACKGROUND OF THE STUDY

1.1 Introduction and background to the study

Much of the recent research on mathematics teaching and learning has focused on students' conceptual development and obstacles in the understanding of basic mathematical concepts and their implications for educational practices. This interest probably reflects a desire amongst mathematics educators and researchers for a more effective and meaningful learning of mathematics at all levels.

Although many concepts can be considered as “basic concepts” in mathematics, it is generally agreed by mathematicians and mathematics educators that the function concept is one of the most important basic concepts which permeates almost every branch of mathematics and occupies a central position in its historical development. During the 1960s and 1970s of the “modern mathematics” era the function concept was regarded as one of the main “integrating themes” or “the central unifying concept” in the modern mathematics curriculum. Its unifying role is often associated with other complex mathematical concepts in algebra, calculus, trigonometry, transformation geometry and other branches of mathematics.

1.2 The focus of the study

In view of the increased recognition of the integrating power of the function concept throughout every level of education, it becomes essential to study the efficacy of its teaching and learning. In recent years, the increasingly documented empirical findings of the problems in teaching and learning of the function concept in many countries especially in the USA (Thomas 1969), Britain (Orton 1970), Australia (Barnes 1988) and Israel (Vinner 1983) clearly indicated significant problems related to students'

understanding, curriculum materials and assessment procedures which needed to be identified and improved.

The set theory in the “new mathematics” of the sixties and seventies introduced the concept of function in the secondary school in terms of domain, range and rule relating each element in the first with a unique element in the second. The notion proves to be one of the most difficult concepts in the mathematics curriculum as widely reported by various researches in the last two decades (for example, Thomas 1969; Orton 1970; Lovell 1971; Vinner 1989; Even 1990). The difficulties experienced by the students were of various kinds.

Somehow the general concept of a function seems to be too general to make much sense (Markovitz *et al.*. 1988). Although we may teach pupils about general concepts such as the domain on which the function is defined and the range of possible values, these terms do not seem to stick in their memories. Instead, they gain their impression of what a function is from its use in the curriculum, implanting deep seated ideas which may be at variance with the formal definition (Vinner *et al.* . 1989).

We hypothesise that the idea of defining a concept is at variance with the child’s everyday experience. Here a concept such as “bird” would be developed through encounters initially with examples and then focusing on salient features. “That is a bird...a bird flies,...it has wings...and feathers...and a beak...and lays eggs”. Then there comes the testing of new creatures against these various criteria. Is a chicken a bird?..It has wings, feathers, a beak and lays eggs, but it doesn’t fly. OK, some birds don’t fly. We will say a chicken is a bird. Is a bat a bird? It flies and it has wings, but it is really a flying mouse, so it is not a bird. Smith (1988) proposes that the individual builds complex interconnected prototypes which help to test whether newly encountered examples can be classified as instances of the general concept.

In everyday life our development of concepts depends on perpetual negotiations of this kind, which are a deep-seated part of the human psyche. It therefore comes as no

surprise that students are likely to apply similar criteria when faced with concepts in the mathematics class.

We also hypothesize that the students develop “prototype examples” of the function concept in their mind, such as: a function is like $y=x^2$, or a polynomial, or $1/x$, or a sine function. When asked if a graph is a function, in the absence of an operative definition of a function, the mind attempts to respond by resonating with these mental prototypes. If there is a resonance, the individual experiences the sensation and responds positively. If there is no resonance, the individual experiences confusion, searching in the mind for a meaning to the question, attempting to formulate the reason for failure to obtain a mental match. We shall see that positive resonances may be erroneous because they evoke properties of prototypes which are not part of the formal definition.

For instance, that a function should be describe by a formula, or that the familiar graph of a circle is a function. Negative resonances may be equally incorrect; for instance that a strange-looking graph cannot be a function because it does not match any of the prototypes, or that a function cannot be constant, because a function depends on a variable and it is considered essential that this variable actually appears in the expression.

1.3 A review of the related literature

1.3.1 Overview of the previous psychological studies of the function concept

Among the earliest contributions to the study of the development of the function concept in young children was by Piaget and his associates (1977). Piaget suggest the presence of an intuitive understanding of function among his subjects ranging from ages 4 to 14 which could lead to the understanding of the notion of proportion.

However he was mainly concerned with a very simple algebraic function of the form $f(x)=ax$.

Two early studies on the students' understanding of the function concept during the 1960s and 1970s appear to have come from Thomas (1969) in the USA and Orton (1970) in England.

In the last two decades, many studies related to students' understanding, misconceptions and difficulties, and the problems of teaching and learning of the function concept have been carried out in different countries, for different subjects and at various educational levels from secondary schools through college and university. The studies has been reported in various forms such as articles (e.g. Orton 1971, Vinner 1983, Malik 1980, Markovitz, *et al.* 1988), theses (e.g. Thomas 1969, Orton 1970, Even 1988, Mamona 1987) and conference reports (e.g. Dreyfus and Vinner 1982, Sfard 1987). A comprehensive review of issues related to the function concept has been given recently by Leinhardt *et al.* (1990).

In a Ph.D thesis on stages in the attainment of the function concept, Thomas (1969) administered written tests to 201 grade 7 and 8 students (aged 11 to 14) and followed up with 20 individual test-interviews. All subjects were above average ability (mean IQ:125) and had received specific classroom instruction on functions.

Starting in grade 7, the concept of function had been introduced as a mapping of a set A to a set B, the word "image" referring to the object in B assigned to an element in A. The instructional material used arrow diagrams, rules, ordered pairs, and graphs in developing the concept. Thomas claims that the evidence gathered from the results of the tests does seem to indicate that in mastering the function concept there are certain reasonably clear and identifiable stages of understanding that appear. These stages can be summarized as follows:

Stage 1: ability to handle processes associated with functions when they are arithmetical or when assignments are given in arrow diagrams or in a table;

- Stage 2: ability to find images, pre-images, domain, and range in all representations; improved ordered pair point graph representation;
- Stage 3: ability to discriminate between functions and relations (in any representation), ability to treat functions as conceptual entities;
- Stage 4: high level of performance, integration of all subconcepts.

Of the 201 subjects who took the tests, 55 (27%) were rated as having attained an understanding of the function concept at the two highest stages, while 164 (82%) could be regarded to have attained a minimal level. In his conclusion, Thomas said: "It was... a shock to this investigator to find that, in a group of students who had supposedly been carefully introduced to the concept of function, many could not distinguish functions from non-functions in simple and concrete situations. At the same time these students could carry out many of the processes associated with the function concept."

In England, a similar study was carried out by Orton (1970) to investigate the development of the function concept in secondary school-children. Orton interviewed 72 subjects (age 12 to 17) by selecting 16 students from each grade. These students had been exposed to sets, ordered pairs, and graphs in the second year of their secondary school. They had also been introduced to relations in terms of arrow diagrams and mappings. A function was defined as "a relation in which each member of the domain has only one image". Similar to the stages of attainment in the thesis of Thomas, Orton described the four stages in the development of the function concept as follows:

- Stage 1: concrete, intuitive; can handle processes when arithmetic, or in arrow diagram or table; concept of function as specific type of relation not mastered; limited extension of notions in ordered pair graphs;
- Stage 2: basic criterion for relation to be a function still not mastered; good grasp of relational aspects of function concept in that able to find images, pre - images, sets of images, and domain;
- Stage 3: can identify whether a relation is a function or not in several types of representation; mastery of basic concept of function; care not always taken to check uniqueness of images or correct domain for inverse;
- Stage 4: mastery of concept of function to greater degree of generality than in stage 3; all representation and their inverses classified as functions or not with precise analysis of the uniqueness criterion.

According to the results of the study, only 52% of the subjects achieve the level of understanding described as stage 4. Orton also concluded that some of the misunderstanding in children concerning the function concept were not anticipated (p.151).

The results of this research are interesting and suggestive especially in pin-pointing some of the particular difficulties encountered by the children on various tasks designed to distinguish the types of relations which represent a function.

Another recent study that investigated the students' understanding of mathematical functions, particularly amongst prospective secondary schools mathematics teachers in a few universities in the USA, was by Even (1988). Her sample consists of 162 prospective teachers in the last stage of their preservice preparation at eight mid-western universities. Students were given a free response questionnaire on various aspects of the function concept including some non-standard mathematics problems addressing the seven interrelated aspects of the function concept. The questionnaire also asked respondents to comment on examples of students' work which represented some misunderstanding or error related to functions.

This study provides strong evidence that the prospective teachers had a very limited understanding of important aspects of the function concept (p. 235). Even also reports that prospective teachers tended not to use modern terms such as relation, mapping, correspondence, domain and range, when defining a function, instead they tried to describe a function by using terms which are already familiar to them.

While this study certainly provides more information and extra evidence of the problems related to the understanding of the function concept, there are nonetheless some deficiencies with the study. First, questions can be raised about the suitability of treating all the subjects from eight different universities as one group without acknowledging the differences in separate teacher education programmes. Secondly it

did not provide any indications about the possible causes of the problems in understanding of function concept amongst the students involved.

In another recent work, Mamona (1987) reported that the subtlety of the function concept is often underestimated by pupils at the sixth-form level. This is mostly because of the “well-behaved” and concrete nature of all the functions they are likely to have met so far. Mamona argued that pupils’ perception of the notion of function is in terms of a smooth relation between two varying quantities. This idea apparently constituted Euler’s definition of function which was traditionally adopted in England as “an analytic expression representing the relation between two variables with its graph having no corners”.

1.3.2 Conferences on the function concept

Up till now, there have been two major conferences on functions; the first one was held in the Netherlands in 1982, organized by the Mathematics Department of the National Institute for Curriculum Development and the second in 1990 was organized by the Mathematics Department of the Purdue University in the USA.

The conference in the Netherlands was attended by eleven specialists from the Netherlands and sixteen specialists from abroad. The accent of the conference was to be on teaching functions to mixed ability groups of 12–13 year olds (Berneveld *et al.* 1982). According to the organizer, “for us, the conference constituted the starting point for the development of teaching methods on functions.” (ibid, p. 4).

Most of the papers presented in this conference were mainly on the theoretical perspective with very little empirical evidence from practical work in the classroom. The only paper which provided any empirical evidence of teaching of functions in schools was by Malcolm Swan from the Shell Centre for Mathematical Education, University of Nottingham. Swan proposed an alternative model for curriculum

development, based on the concept of “Diagnostic Teaching” which he claimed to be a more effective method of teaching in schools.

The second conference on the teaching and learning of functions was held in Purdue University in 1990. This conference was attended by a number of specialists who had already been engaged in work on functions and most of them had published papers in this field. In contrast with the first conference, most of the papers submitted to this conference were mainly the reports of empirical research including current work using computers. A significant paper by Schwingendorf *et al.* describes the use of the new computer language ISETL in mathematics teaching. Schwingendorf shows how the ISETL exercises ask students to program the function as a procedure which can then be used as an object in its own right, thus helping students to encapsulate the notion of a function as both a process and an object.

1.3.3 Related papers concerning the function concept

It can be said that most of the papers related to the concept of function were published only in the 1980s and 1990s. However, the paper by Orton (1971), together with his thesis (Orton, 1970) can probably be considered the first to give an impetus for a growing interest surrounding the function concept in later years. From a theoretical perspective Malik (1980), for example, argued that teachers engaged in teaching the function concept face enormous difficulties in communicating this abstract concept in the classroom. But the root of the problem is probably due to the modern approach of teaching about functions which is based on ideas of sets and relations. According to Orton (1970),

“From the point of view of a logical development of ‘modern’ mathematics this is an essential means of approach. From the pupil’s point of view, the concept of a function may now be much more difficult to understand.” (p.45)

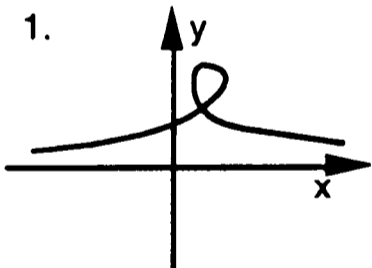
Orton also argued that the problem is further compounded by the number of new definitions which involve some of the following words: domain, codomain, range, relation, function, mapping, image, into, onto, inverse, many-to-one, one-to-one, one-to-many, and many-to-many, which all are useful in one way or the other. Orton concludes by pointing out that the greatest problem facing teachers in teaching about functions is the confusion over the use of the words 'mapping' and 'function'. He provokes the challenge,

“Is there anyone with sufficient authority in the teaching of mathematics to tell us what we must all use, and why ?” (p.49)

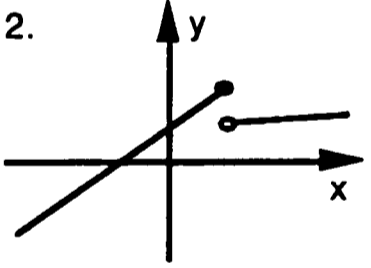
An early research article during the 1980s reporting some aspects of student understanding of the function concept was a paper by Dreyfus and Vinner (1982). In their research Dreyfus and Vinner asked a cross-section of 271 college students and 36 teachers in Israel a number of question about functions:

Does there exist a function whose graph is:

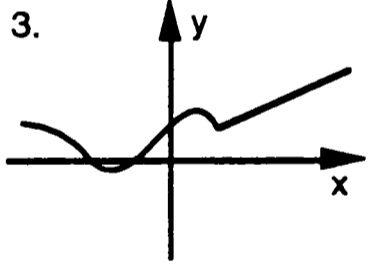
1.



2.



3.



4. Does there exist a function which assigns to every number different from zero its square and to 0 it assigns 1?

5. What in your opinion is a function?

Figure 1.1

They subdivided the college students into four groups, non-mathematics majors with low, intermediate and high levels of mathematical training and mathematics majors. The

following table shows the percentage of students whose responses were adjudged correct:

Mathematical Level:	Low	Intermediate	High	Math Majors	Teachers
Question: 1	55%	66%	64%	74%	97%
2	27%	48%	67%	86%	94%
3	36%	40%	53%	72%	94%
4	9%	22%	50%	60%	75%

Table 1.1: Student Responses to function questions

We can see that the percentages of correct answers improve with ability and experience, but non-mathematics majors in particular have a high percentage of incorrect responses.

The responses to question 5 on the notion of function included not only the standard definition, but also variants such as:

a correspondence between two variables,

a rule of correspondence,

a manipulation or operation (on one number to obtain another),

a formula, algebraic term, or equation,

a graph $y=f(x)$, etc.

These what we would called “prototypical” conceptions of function prove to be replicated in the groups of English and Malaysian students in this study (see chapter 4,5, and 7).

Similar studies were conducted by Barnes (1988) in Australia and Markovits *et al.* (1988) in the USA with school students and university students. Barnes asked a group of grade 11 school students and university students about various expressions for functions, such as whether the following define y as a function of x :

$$y=4,$$

$$x^2+y^2=1,$$

$$y = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 < x \leq 1 \\ 2-x & \text{if } x > 1 \end{cases}$$

A majority decided that the first did not, because the value of y did not depend on x . Many decided that the second *is* a function, presumably because it a circle which is familiar to them. Many had difficulty with the third because it appeared to define not one function but several. These results reinforce our hypothesis that students have prototypical ideas that functions must be of the form $y=f(x)$ where the right hand side is a single expression in x , which clearly shows the rule-based relationship between an independent and a dependent variable. It also shows that a function must be defined on a “continuous” domain of real numbers. We shall later see that this term “continuous” is rarely used in the mathematical sense, but often in the sense that the graph “continues” to extend over the full domain specified by the formula.

Markovits *et al.* (1988) pointed out that the definition of the function concept used in the New Mathematics causes problems because of the number of different components such as as domain, range and rule. The following question was asked to a group of high school students in the USA:

In the given coordinate system, draw the graph of a function such that the coordinates of each of the points A,B, [C, D, E, F] represent a pre-image and the corresponding image of the function:



The number of different such functions that can be drawn is -

- 0
- 1
- 2
- more than 2 but fewer than 10
- more than 10 but not infinite
- infinite.

Explain your answer.

Figure 1. 2

The first figure often evoked a straight line, allowing only one function because “two points can be connected by only one straight line”. The second caused difficulties like “If I draw a function such that all the points are on it, what will happen is for every x there will be two y and it will not be a function”.

Students’ conceptions of functions as prototypically linear would seem to be influenced by geometry which they learn simultaneously with algebra and also by the time they spend in the curriculum exclusively on linear functions (ibid p. 54).

In another study, Sfard (1987) administered a questionnaire to sixty 16 and 18-year olds, who were well acquainted with the notion of function and with its formal structural definition, in an attempt to find out whether they conceived of functions procedurally (as processes) or structurally (as objects). The result shows that the

majority of the pupils conceived the notion of function as a process rather than as a static structural construct (object). This result is similar to our result obtained in this study as shown in chapter 4 and 5 (the study in England) and in chapter 7 (the study in Malaysia).

In the following work, Sfard (1989) designed a study on functions in which students were taught initially by an operational (i.e. procedural as processes) approach that was gradually transformed into a structural (as objects) approach. Sfard wanted to examine whether a structural conception could be provoked in students by means of a teaching that adhered to such a sequence. Responses to a questionnaire administered at the end of the course showed that substantial progress toward structural conceptions had been achieved; nevertheless, she notes that, “our attempt to promote the structural conception cannot be regarded as fully successful” (p.158) and conjectures that reification (the sudden ability to see something familiar in a new light) is inherently so difficult that there may be students for whom the structural conception will remain practically out of reach. Sfard also emphasized that a lengthy period of experience is required before procedural conceptions can become transformed into structural conceptions.

Breidenbach *et al.* (1990) have distinguished four levels of function concept: prefunction, action, process, and object conceptions. A student with prefunction conception does not display very much understanding of function and is not able to perform the tasks which involve mathematical activities related to functions. An action conception is a repeatable mental or physical manipulation of objects involving, for example, the ability to put numbers into an algebraic expression and calculate it. A process conception of function involves a dynamic transformation of quantities according to some repeatable means so that, the same original quantity, will always produce the same transformed quantity. The student is able to imagine the transformation as a complete activity beginning with objects of some kind, doing something to these objects, and obtaining new objects as a result of what was done. A function is conceived of as an object if the student is able to perform actions on it, in

general, actions that transform it. An object conception is constructed by encapsulating a process.

1.3.4 Overview of some of the official statements concerning the importance of the concept of function in mathematics curriculum

The important role of the concept of function in mathematics curriculum has been advocated by various curriculum reports throughout the world especially during the “modern mathematics” era of the 1960s and 1970s. For example the report published by the Organization for European Economic Cooperation (O.E.E.C) in 1961 on “New Thinking in Mathematics” clearly suggested the fundamental role of the concept of function in the first cycle of the algebra course for pupils aged 11-15 years (OEEC 1961, pp. 1– 69).

In the USA, the president of the International Commission for the Study and Improvement of the Teaching of Mathematics, Choquet (1963) proposed a list of principles for the improvement of mathematics teaching in the American schools. One of the principles is that pupils should learn and construct examples of functions drawn from their everyday life (pp. 41– 44). In another report published in the USA in 1967 entitled “Cambridge Conference on Teacher Training” the authors said in relation to a chart which displayed the basic topics in the curriculum, together with their interrelations,

"Some of the most important items are deliberately omitted from the diagram, not because they are unimportant, but, on the contrary, because they should be nearly ubiquitous. Chief among these are the concept of FUNCTION and SET, which should be used throughout the development wherever natural examples and uses occur. By Grade 6, both the words function and set (and the ideas behind them) should be established firmly and correctly as natural parts of the pupil's mathematical language."
(p.98)

This emphasis on the role of the concept of function in the mathematics curriculum is persistently evident in the NCTM – STANDARDS (1989). This asserts that in grades 9–12, the mathematics curriculum should include the continued study of functions so that all students can:

- model real-world phenomena with a variety of functions

- represent and analyse relationships using tables, verbal rules, equations, and graphs
- translate among tabular, symbolic, and graphical representations of functions
- recognize that a variety of problem situations can be modelled by the same type of functions

(NCTM Standards 1989, p.154)

Although this may represent the “intended” curriculum, the data on the “implemented” curriculum and moreover what is learned by students were rarely available. This study will shed some limited light on this aspect.

In volume IV of “New trends in mathematics teaching”, prepared by the International Commission on Mathematical Instruction (ICMI) published by UNESCO, one of the authors, Engel (1979, p. 255) said that at the beginning of this century, an outstanding German mathematician, Felix Klein, initiated a reform of mathematics teaching. This reform movement adopted the slogan “functional thinking”. The reformers claimed that functional thinking must pervade all of mathematics teaching. What the students should have learned in his mathematics classes is thinking in term of functions. This reform has profoundly changed school mathematics throughout the world especially during the modern mathematics era of the sixties and seventies. One of the textbooks writes, Dieudonné (1969), for example, proposed that,

“The pupil must be taught to be aware of the absolute necessity of an axiomatic approach to mathematics. At the earliest possible moment, the pupil must get used to constant dealings with abstract concepts, one of the most difficult of which is the concept of a mapping... since in teaching these two points, we are dealing with the very cornerstones of the structure of modern mathematics, everything else taught during these early years ought to be made to take second place.” (p. 13)

In Malaysia, the importance of the function concept in the mathematics curriculum is clearly stated in the official statements produced by the Ministry of Education (Mathematics syllabus, DBP, 1975) or as reflected in the speech given by senior

ministry official, such as by the Director of Curriculum Development Centre (Asmah 1980). The role of the function concept is emphasised as one of the unifying concepts which integrate various branches of mathematics in the school curriculum.

1.4 Theoretical framework of the study

From the previous studies of the function concept, it is clear that various authors or researchers had contributed in their own ways to our understanding of the issues related to this concept. The present study adopts a rather unusual feature as compared to other previous studies with the fact that the main study is conducted in one country (Malaysia) as an extension of the preliminary study conducted in another country (England). Hence this study can be regarded as an exploratory, cross-sectional as well as cross-national study of the two different educational systems. The data obtained especially about the students' understanding of the function concept in secondary schools and university in England were compared with the similar data from Malaysian secondary schools and university students (chapter 7).

From the comparative study of this nature it is hoped that a better understanding and a useful insight into the problems of teaching and learning of the function concept can be ascertained and will stimulate the effort for improvement in the future. The data obtained from this study may also provide the curriculum developers, especially in Malaysia, with some informed advice on the implementation of the modern mathematics curriculum particularly with respect to the teaching and learning of the function concept in secondary schools. Following to some extent, the general research procedures of The Second International Mathematics Study of the International Association for the Evaluation of Educational Achievement which focuses on the intended, implemented, and attained curriculum (IEA, 1989), the present study will focus on four aspects of the curriculum; the *intended*, *perceived* (as interpreted by the teachers), *tested*, and *learned* curriculum with specific reference to the function concept. (Note that Malaysia,

due to some unknown reasons did not participate in the IEA study (Travers *et al.* 1989)).

However the underlying theoretical framework which guided the data collection, analysis of the results and its interpretation in this study is reflected in a current tendency of using metaphors from terms such as “students’ conceptions”, “alternative frameworks” etc. rather than the terms “students’ difficulties”, “error” or “misconceptions”. The focus of research has shifted from studies of students’ difficulties, errors or misconceptions to students’ knowledge which underlies the difficulties, and stresses the seeking for explanations for the origins of such problems (Balacheff 1990).

Since this study focuses on identifying and describing students’ conceptions of the function concept, the method based on the individual interview and the prepared questionnaire was regarded as the most appropriate means of obtaining such information.

In the early stage of investigating students’ conceptions (after the pretest in Coundon Court School and Beauchamp College) fifteen students from both schools were interviewed in order to cross-check their reasoning given in the questionnaire. After we were satisfied that their conceptions were actually reflected in the written reasoning, in the subsequent stage of the research (post-test), students were particularly asked to write down clearly their reasoning for each question in the space provided. This approach of conducting the study was repeated in Malaysia. Twenty students from Tun Fatimah School and Masai Secondary School were interviewed for a similar purpose as conducted in the pretest at Coundon Court School and Beauchamp College.

Looking from another perspective, to some extent, this study may also contribute to the concern of some educators such as Howson (1991), who says “Nowadays, we are very much aware of the distinction that must be made between the ‘intended’

curriculum as set out by governments and authorities and the curriculum 'implemented' by teachers. (That actually 'attained' by students is yet another matter).

1.5 Objectives of the study

The main objectives of this study are to:

1. Examine and identify the similarities and differences between the curriculum as visualized and specified by the curriculum developers, as perceived and taught by the teachers and as perceived and learned by the students-with particular reference to the function concept.
2. Examine how undergraduate students would respond on the same questionnaire as given to secondary school students with the purpose of finding out whether or not any particular conception is prevalent in higher education?
3. Examine whether or not the patterns of conceptions developed by English students hold for students in Malaysia.
4. Identify the possible factors (including attitudes and beliefs) that may influence the teaching approaches used by teachers.
5. Consider some implications of the findings for the improvement of teaching and learning of the function concept in secondary schools.

1.6 Summary and organization of the thesis

Chapter 1 provides an introduction and background to the study. It gives a review in the area of psychological study of the function concept. The emphasis is on the psychological background which leads to a strong interest in the concept and on the theoretical frameworks which underlie the reasons and justifications for procedures used in the study.

Chapter 2 provides an overview of the historical development of the function concept and examines the ways in which the concept of function is being used in the school textbooks.

Chapter 3 gives a review of theoretical issues concerning teaching and learning of mathematical concepts. Views on learning especially on the prototypical aspect contribute largely to the formulation of the questionnaires used in the study.

Chapter 4 provides a report of preliminary investigation on function prototypes.

Chapter 5 examines the effects of teaching on the function concept in two secondary schools in England.

Chapter 6 gives an overview of the current practice of mathematical education in Malaysia.

Chapter 7 describes the analysis of data in the main study in Malaysia and provides the discussion of the results obtained in four aspects of the curriculum; the intended curriculum, perceived curriculum, tested curriculum and learned curriculum.

Chapter 8 attempts to synthesize the results from all stages of the study. It also provides summary of major findings and discuss the results in relation to the main objectives of the study.

CHAPTER 2

THE FUNCTION CONCEPT

The discussion about the historical development of any mathematical concepts may take hundreds of pages. However the intention of this chapter is to give an overview of the historical development of the function concept and its uses in the school textbooks especially in the UK and USA which have direct relevance to the mathematics curriculum in Malaysia.

2.1 Overview of the historical development of the function concept

By studying the historical development of the function concept, we can fairly conclude that the emergence of the concept of function has been piecemeal. Many great mathematicians from the early eighteenth century to the middle of the twentieth century, in some way contributed to the clarification and generalization of this concept. According to Bell (1945), the idea of function, in one form or another, has been in existence for at least 4000 years from the days of the Babylonians. However an explicit awareness of this notion was evident in only about 300 years ago, since the time of Bernoulli (Ruthing 1984).

This discussion will be divided into two parts :

The first part discusses the different approaches to the development of the function concept as illustrated by the various definitions given by the mathematicians throughout the history.

The second part then focuses on the definitions of the function concept as adopted by the school textbooks, and this is followed by an analysis of the relative merits of different definitions and approaches to the concept from a pedagogical viewpoint.

2.2 Function definitions

The numerous definitions of the function concept given by the various mathematicians indicate that the concept of function has been a matter of great concern for some time in the history of mathematics. Boyer (1946) commented that “The development of the function concept has revolutionized mathematics in much the same way as did the nearly simultaneous rise of non-Euclidean geometry. It has transformed mathematics from a pure natural science – the queen of the sciences – into something vastly larger. It has established mathematics as the basis of all rigorous thinking – the logic of all possible relations.” In another book Boyer (1956) pointed out that in Greek mathematics the prototype of functions was evident in the use of proportions. He said that “this was somewhat equivalent to the modern use of equations as expressions of functional relationships, although far more restricted.” (p.5).

The advancement of scientific inquiry during the sixteen century provoked the development of the concept of function (see Klein 1968). The function concept was able to describe scientific phenomena which usually include other concepts such as temperature, growth, vibration, rate, acceleration, and so on, more accurately. The function concept also has attained a new significance and wider usage during the rapid development in mathematics itself especially in algebra during the 1600s.

From my own research to retrace the exact instance and route of development of the function concept, it seemed that there is no general consensus on the “when or where” the concept of function first emerged. However quite a number of authors such as Monna (1972) and Youschkevitch (1976) indicate that the function concept was first introduced by Leibniz which he calls ‘functio’. Leibniz’s concept of function was a generalization of y being a function of x . According to Youschkevitch (1976), the correspondence of Leibniz with Bernoulli during 1694 – 1698 actually traces how Leibniz intention to use a general term to represent arbitrary quantities dependent on some variable which then brought about the use of the term function in the sense of an

analytical expression (p.57). In other words he seemed to have a 'prototypical' view of the function concept based on formulae.

On the other hand some authors seemed to agree that the function concept in its new conception involving the notion of variable was first introduced by Bernoulli in 1718. According to Bernoulli a function is a quantity composed in some manner of a variable and any constants. For Euler, a quantity dependent on another, such that as the second changes, so does the first, is said to be a function.

This definition given by Euler indicates the idea of functional dependence. It is a central idea for mathematics as well as for science since so much of its study is concerned with investigating how one thing depends on another.

In 1837, Dirichlet proposed the concept of uniqueness in defining the notion of function,

If a variable y is related to a variable x so that whenever a numerical value is assigned to x , there is a rule according to which a unique value of y is determined, then y is said to be a function of the independent variable x .

The concept of function was made more precise by Dirichlet with the idea of the second quantity is uniquely determined from the first by some rule. In the search for further generality, the prominent mathematician Hardy in 1908 gave another definition,

All that is essential (to a function) is that there should be some relation between x and y such that to some values of x at any rate correspond values of y .

Hardy's further generalization, however, made the definition too wide and no clear distinction between function and relation. In other words Hardy's definition had ignored the idea of uniqueness developed by Dirichlet and the notion of dependence is somewhat vague. Furthermore this definition consequently admitted the concept of

“multi-valued functions” which then become part of mathematics literature for a considerable period of time.

In 1939, the French mathematicians writing under the pseudonym Bourbaki introduced the definition of function by using set-theory,

Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all $x \in E$, there exists a unique $y \in F$, which is in the given relation with x .

Bourbaki gave the name of function to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x ; y is said to be the value of the function at the element x , and the function is said to be determined by the given relation. Two equivalent functional relations determine the same function. In this way, Bourbaki actually gave both the usual formal definition of the function concept in term of functional relation and the procedural or process definition in terms of operation or rule.

The historical evidence of the evolution of the notion of function bears witness to the gradual changes of the procedural conceptions (or function as a dynamic process) from the time of Euler to structural conceptions (or function as a static concept or object) during the period of Bourbaki. In other words from the 18th century procedural conception by Euler in terms of independent and dependent variables to the 19th century elaboration by Dirichlet that emphasized the arbitrary rule of correspondence between real numbers to the 20th century structural conceptions by Bourbaki that defined function as a special kind of relation between two sets. And this definition by Bourbaki which is sometimes called the set-theoretic definition, exerts a major influence in the modern mathematics curriculum in the 1960s and 1970s.

2.3 Function concept in the school textbooks

In the “traditional” textbooks such as the book on Algebra by Durell (1930) which was widely used in Malaysia in the pre-modern mathematics era of the 1950s and 1960s (Asmah 1980), the concept of function is mainly represented by a formula or an algebraic expressions.

Any expression containing x , whose value can be found when the value of x is given, is called a function of x . Thus $7x$, $(3/4)x - 5$, $(2x - 1)/(x + 3)$, $x^3 - 5x$, etc, are all functions of x . The letter y is generally used to represent the function of x . (p.157)

This view of perceiving a function as a formula or an algebraic expression was shared by other authors such as Channon *et al.* (1959),

Any algebraic expression which involves the variable x (and no other variable) is called a function of x , and its value depends on the value of x . The symbol used is $f(x)$, which is read as ‘function of x ’; $f(2)$ means “the same expression with 2 is written instead of x ”, $f(-1)$ means “the same expression with -1 is written instead of x ”, and so on. (p. 152)

This persistent view of describing function as a mathematical expression in the algebraic sense was widespread. For example, in the seventh year book of the National Council of Teachers of Mathematics published in the USA in 1932, the concept of function was defined as follows :

Any mathematical expression containing a variable x , that has a definite value when a number is substituted for x , is a function of x .

From the quotations given above, we can clearly see that the function concept in the “traditional” textbooks is mainly confined to what we called the algebraic functions based on the used of formulas or expressions. It was assumed that the domain of the

function is to be a set of real numbers. The notation used to represent function and variable was only confined to $f(x)$ and x respectively.

The introduction of the idea of a function in the “modern” textbooks especially since about 1960, seems to follow a different approach. The function concept is introduced not as an isolated concept on its own, but rather as an idea which is related to other concepts previously defined. The order of presentation is usually as follows:

- i) the idea of a set
- ii) through the idea of ordered pair, the Cartesian product of two sets A and B
- iii) the concept of relation as a subset of $A \times B$.
- iv) the function concept defined as a particular kind of relation from A to B such that for every $a \in A$, there is $(a, b) \in f$, and $(a, c) \in f$ such that $b = c$.

The definition of function based on the formal use of set–theory is very popular in many modern mathematics textbooks in the 1960s and 1970s. Most of the textbooks in Malaysia seemed to follow the above sequence (see discussion in chapter 7). However as we have seen this is not the historical development of the concept although it may be a good logical development.

In the sixty–ninth yearbook of the National Society for the Study of Education, Buck (1970) introduces the definition of the function concept, based upon the terms *set*, *element*, and *Cartesian product* as follows :

Let A and B be sets, and let $A \times B$ denote the Cartesian product of A and B .

A subset f of $A \times B$ is a function if whenever (x_1, y_1) and (x_2, y_2) are elements of f and $x_1 = x_2$, then $y_1 = y_2$.

In the UK, the SMP Books were among the first “modern” mathematics texts which considered a function as a particular kind of relation,

*A relation is a connection between members of two sets or members of the same sets.
A mapping is a function if each member of the domain has only one image.*

(SMP Book 2, 1966, p. 158)

The modern mathematics textbooks produced by the Scottish Mathematics Group (SMG) further enhanced the notion of function as a kind of relation,

A function, or mapping, from a set A to a set B is a relation in which each element of A is related to exactly one element of B.

(Modern Mathematics for Schools Teacher's Book 5, 1973, p.50)

In the USA, one of its major modern mathematics projects, SMSG (School Mathematics Study Group) also defined the concept of function based on the formal use of sets,

Let A and B be sets and let there be given a rule which assigns exactly one member to B to each member of A. The rule, together with the set A, is said to be function and the set A is said to be its domain. The set of all members of B actually assigned to members of A by the rule is said to be the range of the function.

(SMSG, 1960)

The change in perceiving a function mainly as a formula or algebraic expression containing x in the “traditional” texts to a rule connecting two sets, and as a special type of relation in the “modern” texts was an indication of the priority by the mathematical community to make the definition of a function more precise.

This change was perhaps also due to the need to provide a strong mathematical foundation in which every concept should be defined in a logically satisfactory manner. This axiomatic development of mathematical concepts may be a good logical development although it may not reflect the historical development of the concepts, or be a suitable cognitive development.

In tracing the historical development of the function concept, we can see that the word “function” is used in various different ways. During the rapid development of the sciences since about the sixteenth century, the function concept is conceived as a relationship between variables. The physical scientists used the concept of function to describe many phenomena in nature in terms of the relationship between a certain quantity that changed with respect to another quantity.

For example, an Italian astronomer and physicist in the sixteenth century, Galileo, discovered the Law of Falling Bodies, $d = (1/2) gt^2$ where the distance of falling object is the function of the time taken. A more recent example of using the function concept in describing a relationship in the physical world was by Einstein in the early twentieth century. Einstein discovered that the amount of energy released when matter is changed into energy can be expressed by the formula $E = mc^2$, where c is the constant velocity of light.

Mathematicians on the other hand, were perhaps more concerned with the formalization of the subject. In the seventeenth century, such a relationship described by the scientists was given the name “function” by the German mathematician Leibniz (West *et al.* 1982). The concept then being popularized by a Swiss mathematician Euler in the eighteenth century by introducing the notation “ $f(x)$ ” for “function of x ” paved the way for expressing function as formulas or equation (Hellemans *et al.* 1988).

In the late nineteenth century, with the introduction of a theory of sets by a German mathematician Cantor, the concept of function is redefined in terms of sets (Bell, 1937). The function concept is then perceived formally as a set of ordered pairs satisfying a special property in which every first element is different. Probably due to the profound impact of Cantor’s theory of sets on mathematics, this definition of function which might be taken to be “a relation between two sets A and B in which each element of A is related to precisely one element in B” has become well established in mathematics terminology today.

A variety of ways of defining the function concept in the school textbooks probably reflects the fact that mathematicians or textbook authors have no agreed definition. We may conjecture that the definition adopted by each of them was likely to be coloured by their ‘prototypical’ conceptions of the function concept. It is also perhaps due to the complexity of the function concept itself, whereby it is not possible to include all their conceptions within one definition.

2.4 Pedagogical implications

After we have seen the historical development of the function concept and recognising the “well established” definition of the concept, we are then confronted with the issue of seeking the best possible way(s) of presenting this concept to the students. Should we follow the historical development of the concept, or is the best way to follow the logical development of the concept or should we concentrate on the cognitive development of the students in presenting the concept ? This delicate issue is quite impossible to solve at once. Perhaps some good advice together with the ongoing research evidence in recent years will be useful in this respect. Nevertheless we would anticipate that the prototypical conceptions developed by the mathematicians throughout history may have some similarity with the various ‘prototypical’ conceptions experienced by school pupils. Some evidence of this aspect will be discussed in chapters 4, 5, and 7.

CHAPTER 3

TEACHING AND LEARNING OF MATHEMATICAL CONCEPTS

Since the main discussion of this thesis was centred around a few major terms such as “teaching and learning of mathematical concepts”, it is necessary to explain in some details the meaning of these terms in the context of this thesis and their relationship to other viewpoints, previous and contemporary meanings. The argument will then build on the prototypical model of learning mathematical concepts. We argue that students construct their mathematical concepts by abstracting the ‘best’ prototypes which exemplify the concept (In the sense of Rosch, 1975). This ‘best’ idiosyncratic individuals’ prototype(s) will be used to determine whether other examples belong to the conceptual class membership. Our results in the preliminary and the study in England (see chapter 4 and 5) suggest that the prototypes that students develop for the function concept are ideas such as $y=x^2$, any “typical polynomial”, $y=1/x$, a sine curve, a relationship between two variables in which y varies with or is dependent on x , a “continuous graph” and so on. Other examples of functions will be determined on how closely each example corresponds to these individuals’ prototypes.

3.1 Changing views of teaching and learning

We can reasonably say that, in the past (before the influence of the theories and psychology of learning by a few noted scholars such as Piaget, Bruner, Ausubel and others, mathematics was often taught in a didactic manner emphasising instrumental understanding (in the sense of Skemp (1971)) rather than relational understanding. Thus the mastery of mechanical techniques, computational skills and manipulations and rote learning were often a common feature in the teaching of mathematics. Consequently mathematics was often not meaningful and learning it became a routine and drudgery. It is generally acknowledged that this scenario of mathematics teaching and learning contributed to the belief that mathematics is a dull, dry and *a difficult subject to teach and to learn* (Cockcroft, 1982).

With the rapid development of the theories and psychology of learning especially during the 1960s and 1970s, an attempt has been made to incorporate the new theories of conceptual development and educational psychology as well as new teaching aids to the teaching and learning of mathematics. There is a major shift during this period of the so called modern mathematics era in the emphasis from the “teacher teaching” to the “pupil learning”. One of the fundamental aims of the modern mathematics curriculum is to emphasise on the pupil activity and guided discovery which is believed to increase the enjoyment and understanding in the pupil’s learning process. Furthermore the emphasis is now on the greater understanding of concepts and their applications to different situations rather than only on the manipulative skills in a narrow sense. In other words the question of “How do students learn?” is now as important if not more important than a question as “How and what should be taught?”.

3.2 Learning theories and conceptual development

Many theories of learning have been put forward by various psychologists which indicate the possible factors or ways that facilitate conceptual development. There are at least two major theoretical frameworks which greatly influenced the educational community since the beginning of this century.

During the 1920s until around the 1940s the perspectives on the learning process were dominated by the behaviourist theory of learning or sometimes called “black-box” theory of learning. The narrow focus only on observable behavioural objectives as advocated by a few major behaviourists such as Thorndike (associations), Skinner (stimulus-response conditioning), Pavlov (conditioned reflexes) and Gagné (hierarchical theory of learning) was regarded to be too limited for further practical use in mathematics education (Gilbert *et al.* 1983).

Following the decline of behaviourism, the dominant orientation in psychology began to change especially during the 1960s and 1970s to a cognitive paradigm with the emphasis on the internal cognitive processes (Romberg *et al.* 1986). The theories of Piaget (theory of cognitive development), Bruner (theory of instruction) and Ausubel (theory of meaningful learning) focused explicitly on the cognitive aspect and conceptual development. Built on some aspects of the cognitive theories the popular contemporary perspectives of learning are dominated by the constructivist theory of learning. The central idea of this theory is the belief that knowledge is constructed by each individual through an

active participation in the learning process (Osborne and Wittrock, 1983). Learning mathematics therefore is not simply a matter of absorbing new concepts in a passive manner, but involves students in developing constructing and modifying or restructuring their existing ideas.

3.3 The process of conceptual development

According to Bruner (1966) knowing is a process, not a product. It is acknowledged that the development of a concept in an individual is a slow and complex process (Lovell, 1966). Any concept that a person has is not formed and remain fixed but continues to change and develop. It does not usually develop suddenly into its final form. Indeed concepts may widen and deepen throughout one's life as long as the brain and mind remain active (Lovell, 1962). Moreover concepts are acquired idiosyncratically. It is an individual matter – once learned it becomes part of an individual cognitive structure.

The constructivist view of learning believes that pupils come to the classroom is often already holding their own ideas about any concept to be taught which often based on their previous experiences. These various pre-existing ideas are sometimes called preconceptions, naive conceptions, alternative frameworks, alternative conceptions, mini-theories and so on. During the process of teaching, the pupils may then modify them, adopt new ones, persistently hold the ideas or even abandon their pre-existing ideas (Hewson 1981, Posner *et al.* 1982, Driver *et al.* 1986).

There are relatively few influential theories developed by mathematicians or psychologists which generate a significant impact to our understanding of the nature of mathematical concepts and its conceptual development. The Dienes' theory of mathematical learning clearly supports the view that mathematics is a constructive activity for pupils using concrete apparatus. He suggested that children need to build or construct their own concepts from within rather than having those concepts imposed upon them.

Skemp (1979) developed a model of learning and intelligence which emphasises the process of conceptual learning and concept enlargement by reducing the cognitive strain and aiding ease of operation in thinking, particularly in being able to go beyond the data and in projecting ideas in new situations. Skemp (1971) also discussed the importance of examples and counter examples as a useful tool for learning the mathematical concepts. Another important idea

proposed by Skemp (1976) which was widely accepted as a major aim of mathematics teaching is the concept of relational understanding (knowing both how and why) rather than only instrumental understanding (knowing rules without reasons).

Skemp also expressed his concern on the way mathematics was being taught based mainly on the logical development of the subject rather than taking account of the psychological aspect of the learner. As he pointed out that mathematics teaching “gives only the end-product of mathematical discovery (that is, all you have to do is learn it), and fails to bring about in the learner those processes by which mathematical discoveries are made. It teaches mathematical thought, not mathematical thinking.”

In the constructivist tradition the main aim of mathematics teaching and learning can be viewed as the development of correct mathematical concepts as generally agreed by the mathematicians. This process is continuous, active and occurring as a result of mental constructions by the learner. In other words the knowledge and meaning of all mathematical concepts are actively constructed in the mind of the learner. From a constructivist point of view, students' conceptions (or even misconceptions) are never arbitrary or unreasonable. Both are crucially important for teaching and learning mathematics.

Education can be viewed as a process of producing change in a student's conceptions rather than simply accumulating new information within the student's mind. It becomes very important that mathematics educators attempt to identify the nature and depth of their students' conceptions of the main basic mathematical concepts rather than assuming that defining a new concept in a precise and unambiguous manner will lead to the student's complete understanding of the concept.

Another major research paradigm for the theory of conceptual learning in recent years is based upon the prototype theory (Lakoff, 1987). This approach postulates that learning can be facilitated by providing best examples, matched non-examples and by considering relationships among concepts along with the concept definition to allow the learners to conceive a clear prototype of the learned concept and clearly identify critical attributes and recall specific examples. The individual will build a complex of interconnected prototypes which enable them to test whether newly encountered examples are instances of the learned concept (Smith, 1988).

The question of how prototypical examples facilitate schema acquisition has been discussed in the recent research by a team of science educators in the USA (Dusch *et al.* 1990). They pointed out that the prototypicality of core concepts subsumed within schemata should help in the encoding and retrieval of appropriate knowledge. They also suggested that the integration of prototypical instances of concepts within the instructional materials will aid learners to develop appropriate schema to be applied in future situations.

Research by Tennyson and Cocchiarella (1986) found that presenting students with prototypes of mathematical concepts produces higher levels of concept acquisition than presenting them with definitions and descriptions of critical attributes of the target concepts. But in the long term the searching for the best prototypical examples for each mathematical concept will be a long and interesting research. Furthermore the preconceptions and even the misconceptions of mathematical concepts held by the students at various level need to be ascertained before any attempt to design the curricula which based on the psychological and cognitive approach of the student.

In the context of this thesis, the word “prototypical” is used in the sense Rosch uses it in her theory of human categorization (1977). Her experiments show that people categorize objects, not in set theoretical terms, but in terms of prototypes and family resemblances. For example, small flying singing birds, like sparrows, robins, etc., are prototypical birds. Chickens, ostriches, and penguins are birds but are not central members of the category, so they are non-prototypical birds. But they are birds nonetheless, because they bear sufficient family resemblances (in the sense of Wittgenstein, 1953) to the prototype; that is, they share enough of the relevant properties of the prototype to be classified by people as birds.

To illustrate another example in our everyday life: A prototypical chair, for us, has a well-defined back, seat, four legs, and (optionally) two armrests. But there are non-prototypical chairs as well: hanging chairs, barber chairs, etc. We understand the non-prototypical chairs as being chairs, not just on their own terms, but by virtue of their relation to a prototypical chair. These chairs in their different ways, are sufficiently close to the prototype. In other words, according to Rosch, categorization is primarily a means of comprehending the world, and as such it must serve that purpose in a more flexible manner than a category which is defined in terms of set theory. The notion of a set is characterized by inherent properties of the entities in the category and this does

not accord with the way people categorize things and experiences. Conceptual categories (e.g., chair and bird) are not rigidly fixed in terms of inherent properties of the objects themselves. Categories are defined for purposes of human understanding by prototypes and family resemblances to those prototypes.

In mathematics, because so many of the concepts that are important to us are either abstract or not clearly delineated in our experience, we need to get a grasp of them by means of other concepts which are clearer to us. This leads us to hypothesize that the way student understands a mathematical concept such as the concept of function is in terms of prototypes like 'function as a formula', 'function as a rule', 'function as an operation', etc. These prototypical conceptions form part of the student's conceptual structure that will influence further learning – some in a negative way, which may generate conflicts or errors and can become an obstacle to future learning. The results obtained in this study show that some of the prototypical conceptions of the function concept are highly persistent and resistant to change through teaching.

3.4 Toward a theory of prototypical learning of mathematical concepts

The notion of a prototype is an old idea. It can be traced back in the literature of sociology, anthropology and psychology at least from the middle of this century. Theorists from various disciplines argued that the nature can be conceptualized in terms of stereotypes (in philosophy), prototypes (in anthropology), frames (in Artificial Intelligence) and scripts (in psychology). The terminology differs but the underlying meanings are remarkably similar: a concept specifies the typical characteristics of members of the class ; it does not have necessary and sufficient conditions; and it does not have clear-cut boundaries (Johnson-Laird, 1988).

The notion of a prototype, however, has only recently been widely used in the emerging new study of cognitive science and applied mainly to the research on category representation (Lakoff, 1987). It is a new significance of an old idea. Since the concept of a prototype in its new perspective is not yet fully and universally operationalised and it is still in a process of evolution, it appears essential to develop it further and proceed towards its possible application in the fairly new discipline of mathematics education. The preliminary idea of "prototype examples" developed by the students in the understanding of the function concept has been suggested by Tall and Bakar (to appear). However before we proceed with the discussion, it is perhaps necessary first to make

clear in what sense the words concept, concept formation, and concept prototype are used in this thesis.

3.4.1 Concepts in mathematics

The term “concept” has been used in a variety of meanings. It has been a source of interest for many great thinkers in most disciplines throughout the human civilization. However, in general it is frequently used to describe an idea or general notion (Concise Oxford Dictionary, 1990). In the context of this thesis, I propose to use it in a sense which is generally reflected in the thinking of educationists and psychologists rather than to delve into the debate on the precise definition of “concept”.

To give a glimpse of the struggle to make the definition of a “concept” more precise, we quote a few examples from among the recent scholars.

Bruner (1971) defines a concept simply as “equivalences in things”. Lovell (1971) elaborates further, “By concept is meant any term that can be recognised as a recurrent feature of an individual’s thinking which stands for or represent, a class of experience, provided he can go over the mental actions from which the term was derived and anchor it in first hand experience or reality” (p.21).

Gagné (1970) defines a concept as a class of entities that have the same relevant or defining characteristics, and this definition has been used by others in a slightly different wordings, such as Klausmeier *et al.* (1974, p.4), “a concept is ordered information about the properties of one or more things – objects, events, or processes that enable any particular thing or class of things to be differentiated from, and also related to, other things or classes of things”.

Skemp (1971) on the other hand, was not specifically concerned with the actual definition of “concept” but he proposed a model of a hierarchy of concepts containing primary concepts and secondary concepts which depending on their origin or complexity. Primary concepts are those which are derived directly from sensory experience, that is, “concepts whose examples are objects or events in the outside world”. Secondary concepts or higher order concepts in turn, are concepts which cannot be derived directly from sensory experience but have to be abstracted from other concepts. Skemp claims in mathematics, even the simpler concepts are secondary such as the concept of “three” may represent for example a set of three pigs (where “pig” is a primary concept).

For the case of the function concept, if we concede to Skemp's idea, it may turn out to be a much more "higher-order" concept since it consists of many other concepts such as domain, range, rule which may be derived in turn from other concepts such as set, element, object, one-to-one and so on which seems to be difficult for everyone to agree which one may represent primary or secondary concepts.

Although there is clearly no general agreement over the definition of a "concept", hardly anyone would be opposed to the idea that concepts have to be learned, formed within the learner. Davis (1966) for example advocates that "Concept learning is probably the most important of all instances of learned behaviour".

For the purpose of this study, we will consider a "concept" to be an idea to which a symbol or name is then attached to allow the concept to be mentally manipulated. Although the symbol or name of a certain concept may be conventionalised, the meaning which it represents remains unique to the individual. Symbol(s) or name(s) for a certain concept are available to all, whereas a concept is an outcome of a mental construct idiosyncratic to each individual.

3.4.2 Concept formation in mathematics

A number of researchers have proposed psychological theories dealing specifically with the nature and problems of concept formation in mathematics. Among the most popular ideas were proposed by Piaget and Bruner which then became the basis of the constructivist theory and spiral approaches to the curriculum respectively. In recent years, several researchers introduced further ideas in an attempt to explain the process of concept formation and conceptual development in mathematics.

These many ideas proposed by mathematicians and psychologists clearly indicate the complexity and intricate process of conceptual formation and development faced by the learners. Although mathematicians seemed to agree at some stages the 'concept definition' for certain concept, but it is impossible for everyone to agree one common 'concept image' for the particular concept. Perhaps it is true to say that any person's concept image may vary in many different ways (In the sense of Tall and Vinner, 1981).

The term concept image used by Tall and Vinner (1981) to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes... As the concept image develops it need not be coherent at all times... (p.152). The portion of the concept image which is activated at a particular time is referred to as the evoked concept image. The term concept definition is for indicating “a form of words used to specify that concept” (ibid.)

The problem of concept formation in mathematics may be even greater, due to the fact that mathematics consists of many abstract concepts and also different types of concepts. The concept of function for instance cannot be observed in its entirety in the real world but relates to ideas developed by mathematicians throughout the centuries. The way of introducing it metaphorically as a “function machine” as adopted in many textbooks is perhaps understandable to the majority of students, but when it is introduced in terms of domain, range and rule together with the notation f or $f(x)$, it is much more difficult to get the necessary concrete experience of it. Furthermore many authors still confused as to whether f should represent a function or $f(x)$.

In a number of writings specifically about how knowledge is acquired (Piaget 1970, 1973) describes that meanings of certain objects come from the actions or activities which involve internal manipulations of objects. He develops the notion of constructive learning in the process of conceptual development where one accommodates cognitive structures to encompass new experience and one assimilates the new experience into previously formed cognitive structures.

In explaining the process of conceptual development, Bruner (1967) prefers to see the human mind as having evolved through three modes of representations:

- (a) enactively, that is, by set of actions
- (b) iconically, that is, by images or graphics,

and

- (c) symbolically, that is, by using symbols or by logical propositions.

In this sense the function concept can be represented enactively with the use of a function machine which the child may manipulate the inputs and outputs, iconically by using an arrow diagram to show the mapping of two sets of elements or by using a graph of certain function, and symbolically in terms of

notation f , $f(x)$ or algebraic equations with specified domain and range. This sequence of conceptual development enables the same concept to be represented spirally in an increasing degrees of abstraction as the pupil enters the higher stage of learning. However, in the discussions on communication in teaching and the evoking of concepts by symbols, Skemp (1971) says "*We may think we are communicating when we are not....*" Furthermore in the framework of constructivism, teachers cannot transmit their own constructs to their students' mind but only communicate necessary information to be used as a source for students' constructs.

Another important factor which is closely related to the process of conceptual development is the development of language. The emphasis on the precision of definition of modern mathematics ("a function is a set of ordered pairs such that....") and the use of a more technical symbolization such as $f:x \rightarrow y$ for $x \in \{1,2,3,\dots,10\}$ may caused great difficulty for the majority of students. Did we push too much precision in language and technical symbolization too soon? Perhaps we may seriously consider the reminder by some mathematicians such as Reys *et al.* (1984, p.43) "*Remember, however, that precision in mathematical language is a product of learning; it is not necessarily a tool for the learning of mathematics.*"

Sfard (1989) proposed a three-phased model of conceptual formation: interiorization, condensation, and reification. During the first phase, called interiorization, some process is performed on already familiar mathematical objects. The second phase, called condensation, is one in which the operation or process is squeezed into more manageable units. The third phase, reification, involves the sudden ability to see the process as an object in its own right.

The condensation phase lasts as long as a new entity is conceived only operationally. Whereas interiorization and condensation are lengthy sequences of gradual, quantitative rather than qualitative changes, reification seems to be a leap: A process solidifies into an object, a static structure. The new entity is detached from the process that produced it. Sfard gives an example for the case of the function concept, "When the concept of function is reified, the person can be really proficient in solving equations in which 'unknowns' are functions (differential and functional equations, equations with parameters), talk about general properties of different processes performed on functions (such as composition and inversion), and eventually see a function as a not necessarily computable set of ordered pairs."

3.4.3 Prototypical concepts in mathematics

According to Lakoff (1987), it was Eleanor Rosch who first developed what has since come to be called “the theory of prototypes and basic level categories”, or simply “prototype theory”. Within cognitive psychology, categorization has become a major field of study after Rosch made categorization an issue.

Although the work of Rosch was mainly in the field of linguistics, we would like to argue that this theory has strong relevant in mathematics. For example in mathematics, the concept of matrix is more likely to be represented as a 2×2 matrix rather than as a 1×1 matrix. The concept of function is more likely to be represented as a formula in the form of $y=f(x)$ rather than a constant function $y=c$. Rosch further argues that categories or concepts are developed around a prototype (central example) of the category. That is, from a prototype which bears a strong similarity to other class members, the student generalizes to other examples, and from a prototype which bears little similarity to members of other classes, the student discriminates non-examples.

To explain this theory further, Smith (1988) suggests that in the process of constructing knowledge, an individual usually forms a prototype which may have several implicit features, and uses that as a criterion if another example is similar enough to belong in the same category. On the more developed level the properties of the prototype are analysed in order to have necessary and sufficient conditions for the concept. Furthermore, concepts or categories are coded in memory as prototypes. Students, therefore learn and remember concepts through the best example which represents an average, central, or prototypical form of a concept.

Research on prototype learning provides evidence that concept definition plays a secondary role in concept learning. In other words, students rarely learn well from concept definition, and most often it is not encoded in memory. Students seem to first acquire concepts from clear cases or best examples and then recognize an overall similarity between a new example and a known example. In this research, we hypothesize that given concept definition of a function followed with limited examples of the procedural aspects of the concept, students developed prototypical conceptions of the function concept with internal inconsistencies. These prototype concepts for the function concept are persistently similar throughout secondary and university level.

In the constructivist perspective, knowledge is a construction by the student rather than being mapped onto the student by an external environment. Knowledge is not imposed onto the student from the outside, instead it is constructed by the student's activities in experiential contexts. The student has to make the abstractions in order to possess the concept. The process of abstraction is central in the formation of concepts.

At the end of this thesis, we will argue that the prototypical theory of learning together with the constructivist theory is a potential framework within which to examine the issues involved in any attempt to understand how student construct meanings of mathematical concepts. In other words, concept learning in mathematics is the process of prototype formation. In addition we will examine the implications of our findings to the teaching of functions and mathematical concepts in general.

CHAPTER 4

THE PRELIMINARY STUDY IN ENGLAND

4.1 Analysis of results of preliminary investigation

Following ideas of gathering evidence about student conceptions of functions in Vinner (1983) and Barnes (1988), we asked a group of twenty eight students (aged 16/17) at the end of their first year of study in a British sixth-form to:

Explain in a sentence or so what you think a function is. If you can give a definition of a function then do so.

They had studied the notion of a function as part of their course preparing for 16+ exams over a year previously and since then had studied the notion of function in the calculus but without any emphasis on the technical aspects of the domain, range and so on. None gave satisfactory definitions, but all gave explanations, including the following:

- *a function is like an equation which has variable inputs*
- *processes the inputted number from another number that is put in*
- *a machine that will put out a number from another number that is put in*
- *an expression that gives a range of answers with different values of x*
- *a form of equation describing a curve/path on a graph*
- *a way of describing a curve on a cartesian graph in terms of x and y coordinates.*
- *an order which plots a curve or straight line on a graph*
- *a mathematical command which can change a variable into a different value*

- *a set of instructions that you can put number through*
- *a process that numbers go through, treating them all the same to get an answer*
- *a process which can be performed on any number and is represented in algebraic form using x as a variable*
- *a series of calculations to determine a final answer, to which you have submitted a digit*
- *a term which will produce a sequence of numbers, when a random set of numbers is fed into the term*

It is pleasing to note the number of students who have some idea of the process aspect of a function – taking some kind of input and carrying out some procedure to produce an output. But not one reply mentions that the process can only be applied to a certain domain of inputs, or that it takes a range of values, despite the fact that these definitions had been given to them earlier in their studies. Note also the number of technical mathematical words, such as *term, sequence, series, set* and so on which are used with colloquial, rather than mathematical meanings. Here lies an inextricably difficult part of the human communication process for both students and teachers. With each of the responses above a teacher may empathise with what the students say and realize that it contains within it the grain of truth.

But can we be sure that what another human being says is what we think has been said, or even that the speaker has said what (s)he intended to say? It substantiates the difficulty enunciated by Malik (1980) that

teachers engaged in teaching the function concept face enormous difficulties in communicating this abstract concept in the classroom.

Graph as function

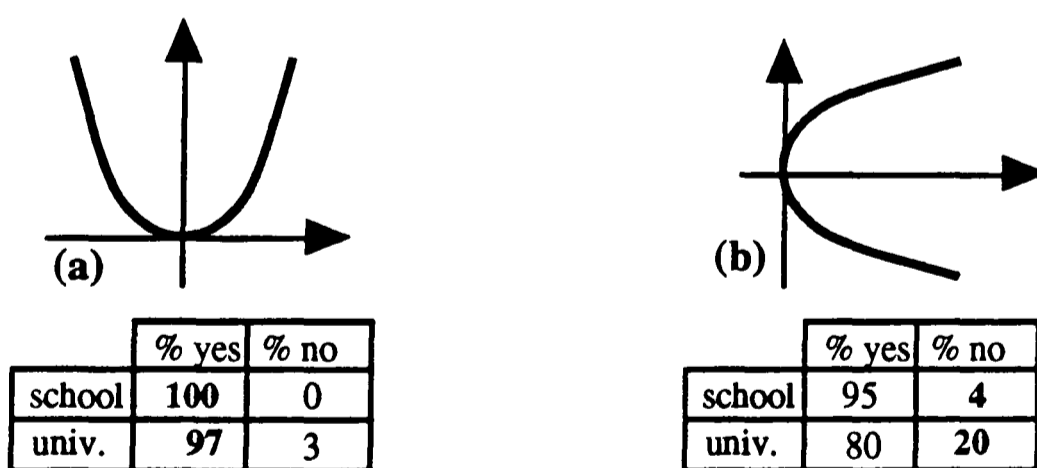
School mathematics is often intended to give students experiences of mathematical activities, rather than plumb the formal depths of logical meaning. The formalities may be mentioned, but they are not stressed because they do not appear to be appropriate until a student has a suitable richness of

experience, but the collection of activities inadvertently colours the meaning of the function concept with impressions that are different from the mathematical meaning which, in turn, can store up problems for later stages of development.

To investigate this, we asked the twenty eight sixth-formers mentioned earlier to state in a written questionnaire which of a given number of sketches could represent a function. The same questionnaire was given to one hundred and nine students in their first year of university prior to any university study of the function concept. The latter therefore represent the state of development of more able mathematics students at the end of their two years of sixth form study. It would be expected that these students would have a better idea of the function concept, and this was confirmed, but they still had aspects in their concept of function at variance with the formal definition. Students were given nine graphs, as shown below and asked.

Which of the following sketches could represent functions? tick one box of each case. Wherever you have said no, write a little explanation why by the diagram.

Here we show each graph followed by a table giving the percentage responses “yes” or “no” for each group. They do not always add up to 100% partly through rounding errors but also due to a small number of non-responses. The response which is more likely to be adjudged correct is given in bold face type. As we shall see, sometimes it is possible for the alternative response to be correct also...



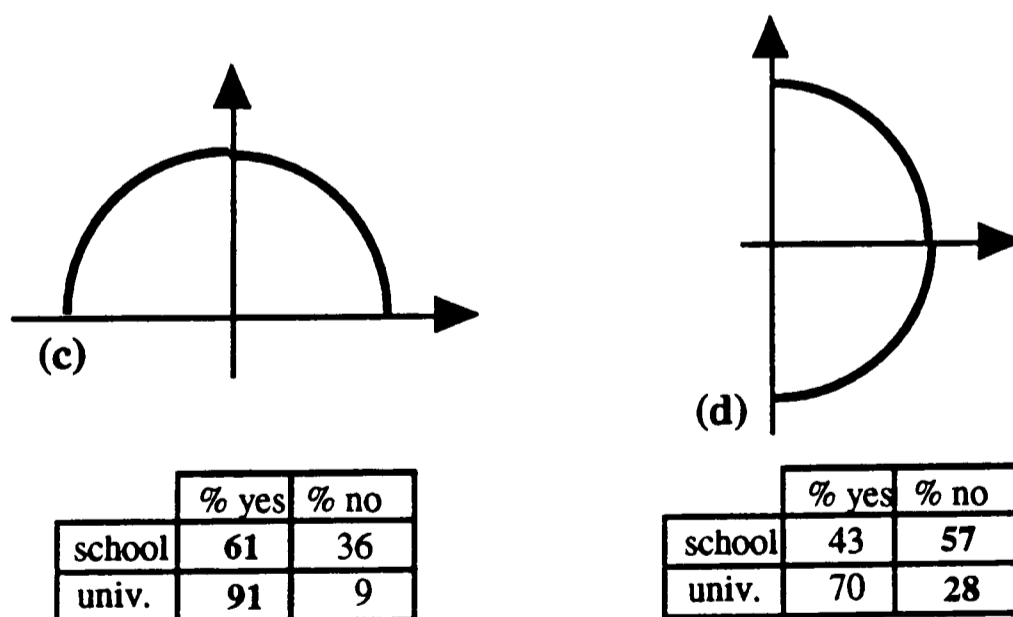
We see that virtually all students agreed that (a) is a function, with the vast majority asserting (b) is also. It was only after we asked this question that we realized that it was formulated in an ambiguous manner. It assumes the usual mathematical conventions – that the horizontal axis represent the independent variable and the vertical axis the dependent variable. But we did not say what we meant, although we think we meant what we said! There was no written

evidence that any school student considered (b) to represent “ x ” as a function of “ y ”. But two university students interpreted the graph in this light – one asserting “look at it a different way”, the other saying “ $f(y)=x$ ”. The increased percentage of university students suggesting (b) was not a function often did so with a comment equivalent to the fact that this “sometimes has two y ’s for each x ”.

A more simple explanation for so many students responding positively to both (a) and (b) is that the term “function” is usually associated with familiar graphs in the sixth form. Both graphs resonate with students’ mental prototypes for functions, so the students respond positively to them.

The single school student who apparently responded correctly to (b) gave no reason and failed to give consistent answers on the rest of the questions. Only one school pupil made any comment at all. He initially thought that (b) was not a function, saying “you have got two y -values for each x -value”, then changed his mind and crossed out his comment. It was as if he did remember the function definition, but then his thoughts were overwhelmed by more recent experiences of the function concept loosely linked to familiar graphical prototypes.

When the same question was asked in an analogous case using semicircles instead of parabolas, the responses were radically different



There is a drop to 61% of school pupils thinking figure (c) is a function and 57% now correctly respond that figure (d) is not. The drop in belief in figure (c) compared with (a) was accompanied with comments such as:

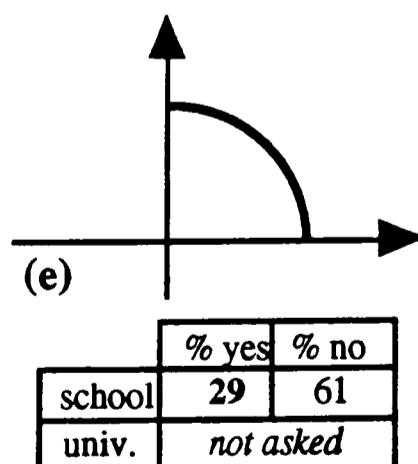
- *if function the graph would continue, not just stop*

- *stops dead, values are not limitless*
- *the lines would have to continue*
- *functions are usually continuous, needs a condition*
- *this could not apply to any value*

Here the word “continuous” does not seem to have its usual mathematical meaning, but the colloquial meaning of “continuing without a break”. Several of the explanations allude to ideas such as “continue, not just stop”, “stops dead”, “could not apply to any value”, which suggest that there is a feeling that functions should not be unnaturally curtailed. One student dotted in an extension of the graph to “continue” it for more values of x . This time there was no written evidence that any students were regarding x as a function of y , but this remains a possibility, certainly amongst the large number of positive university students.

The functions the students have handled in their course are polynomials, trigonometric functions, and their like, which naturally defined by a formula almost everywhere (except a few odd points where the expression may be undefined). Thus we may conjecture that their prototypes are “naturally defined everywhere the function is defined”, leading to apparent unease with “artificial” functions such as the top half of a circle.

The idea that a function should not be unnaturally curtailed is given more credence by the fact that only 29% of school pupils regarded (e) to be a function (this graph was not given in the university questionnaire).



Reasons for this included :

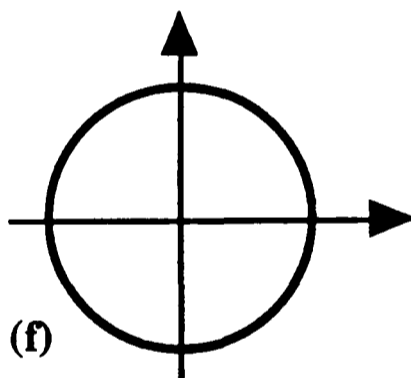
- *couldn't apply to any value*

- *if a function a graph would continue, not just stop*

again suggesting a sense of unease when the graph seemed arbitrarily restricted to a smaller domain. The school pupil's belief in a graph being a function through pictures (a), (c), (e) drops from 100% to 61% to 29% as the graph passes from parabola to semicircle to quadrant, becoming less familiar and restricted to a smaller and smaller domain. As one pupil wrote about the quadrant :

the graph is 'not complete'.

Discussion afterwards revealed that the student thought of it as part of a circle, so it was not a function because it was not all drawn. To this student a function is a natural totality given by a formula, and it is essential to have it all not an unnaturally selected part. Although a quadrant of a circle (which is the graph of a function) is considered not to be a function by most pupils, the situation is reversed with a complete circle. Approximately two thirds of the students in school and university incorrectly considered the circle in figure (f) to be a function :



	% yes	% no
school	64	29
univ.	65	35

Those thinking it was not a function included two from school saying :

You can't work a function that goes back on itself

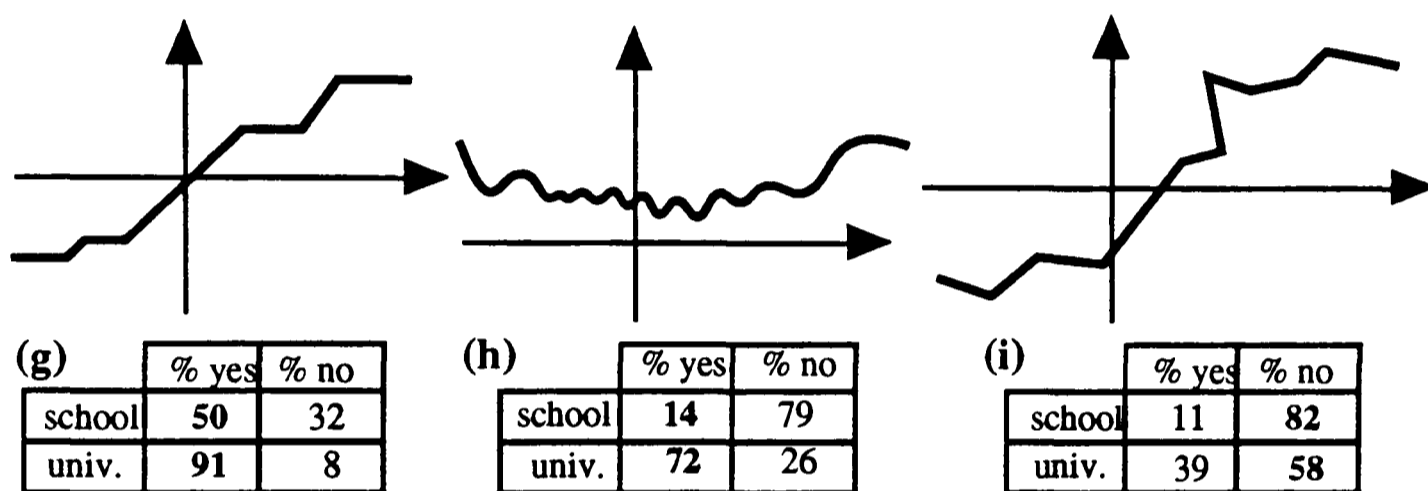
and

equation is $x^2 + y^2 = 25$

which implicitly – but not explicitly – suggests that y is not determined uniquely by x . Amongst the minority of university students who (correctly) thought it was not a function, most alluded to the idea that each value of x might be related to more than one value of y . The persistence of two thirds

of the students thinking a circle is a function once more suggests that familiarity with the graph evokes the function concept. This belief bears little relationship to most of the descriptions of a function given by the pupils in terms of processes. Another highly probable reason for so many pupils thinking that a circle is a function arises from the use of language in the mathematical classroom. Many of us still use the term “implicit function” (or “many-valued function”) to describe such a relationship, and the circle is a prototype example of this phenomenon.

The final three pictures presented to students – (g), (h) and (i) – presented even more conflict. they look strange, so none of them fit the students' mental collection of prototypes.



Theoretically, both (g) and (h) could satisfy the function definition, but (i) does not because there is a part of the graph where one value of x corresponds to more than one value of y .

In general the university students cope better with these more general curves. The fact that more school pupils seem successful with (i) is an illusion, due to the fact that they deny that (i) can be a graph because it looks unfamiliar, rather than because of any formal property of a function. Time and time again they respond that a graph cannot represent a function because it looks too irregular or because they cannot think of a formula to represent it :

(g) (h) (i) are not functions because: *graphs are usually smooth, either a straight line or curve, not a combination of the two, nor staggered, when dealing with a function*

no -because the lines above are totally random, non - uniform, these are absurd,

(h) is *too complicated to be defined as a function*

(h) is *totally irregular and couldn't be represented by a function*

(h) has *no regular pattern too difficult to be defined by a function*

(h) is not a function because *curves and straight lines don't mix*

(h) is *too irregular*

Even when (i) is correctly stated to be not a function, the reasons are often related to the irregularity of the pattern or the lack of a formula. Again we ask if the concept of "irregularity" of a function is actually taught. We think not. None of these graphs matches their mental collection of prototypes for the function concept. Because their experience is usually in terms of graphs given by a formula which tends to have a recognizable shape, their prototypes tend to be "given by a formula", have a "smooth" graph, seem "regular" and so on.

They therefore verbalize some of their perceived mismatches in their own words. Three school pupils do focus on the part of the graph where there are three y-values for each x-value :

- *here the curve goes back on itself*
- *this goes back on itself*
- *there is an irregular peak which could not be created from a function*

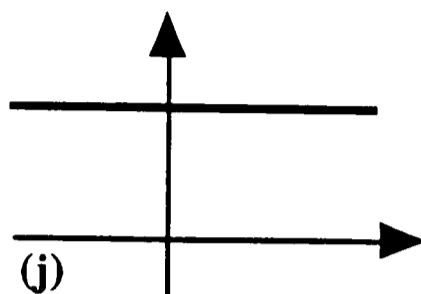
They are beginning to evoke the restriction that each x should have only one y. But they have not applied this test consistently in the earlier examples, and the definition of each function given by each of them does not mention this fact. For these three students a function is:

- *a mathematical command or identity*
- *an equation with a variable factor– tells us what happens to a variable factor, e.g. $f(x)=x+2$,*
- *the product of a series of numbers which the numbers must undergo*

Thus not one of the school pupils consistently evokes a coherent function concept. Only eight of the university students (7% of the total) gave a

consistent set of replies to all the graphs, with one further student giving consistent replies in which he allowed x to be a function of y as well as y to be a function of x .

One graph was given to the university students, but not to those at school (in lieu of graph (e) above)



	% yes	% no
school	<i>not given</i>	
univ.	55	44

Here almost half the students at university think that a constant is not a function. It appears that they are concerned that y is not a function of x , because y is independent of the value of x . Where do students pick up such ideas?

Algebraic expressions as functions

To look at the meaning of a function in terms of formulae (as in Barnes, 1988), we asked the university students to say which of a number of symbolic expressions or procedures could represent y as a function of x . Some of these were algebraic equivalents of the pictorial representations mentioned earlier. The responses are given in table 4.1. Thirty eight of the 109 students explicitly mentioned at least once in their response that, for each x there must be one y , or that the function must be “many-one” or equivalent. In addition to the total percentage of students responding yes or no, we include two extra columns (“% yes*” and “% no*”) representing the percentages of these 38 “more knowledgeable” students. The latter have, at some stage of their earlier career, encountered and now remember more technical aspects of the function concept and we would expect them to perform better. The rest, of course, may have discussed such technical aspects but do not evoke them explicitly in their response.

	University students (N=109)		Subset showing some technical knowledge (N=38)	
	% yes	% no	% yes*	% no*
(1) $y = x^2$	96	4	95	3
(2) $y = 4$	30	69	47	53
(3) $x^2 + y^2 = 1$	62	37	40	60
(4) $y = \frac{3}{x}$	91	9	84	16
(5) $xy = 5$	82	17	82	18
(6) $y = \pm \sqrt{4x-1}$	67	33	34	66
(7) $y = \begin{cases} 0 & \text{if } x \leq 0 \\ x & \text{if } 0 \leq x \leq 1 \\ 2-x & \text{if } x > 1 \end{cases}$	92	7	95	5
(8) $y = 0$ if x is a rational number	50	48	42	58
(9) $y = 0$ (if x is a rational number), $y = 1$ (if x is an irrational number).	75	22	79	21

Table 4.1

Once again the expression $y=x^2$ is almost universally regarded as a function, but the constant $y=4$ is not. As in Barnes (1988), a majority of all students consider the circle $x^2 + y^2 = 1$ to be a function. In each of the latter two cases those exhibiting a more technical knowledge perform better, but still only 47% think that $y=4$ is a function whilst 60% think that $x^2 + y^2 = 1$ is not.

Expressions (4) and (5) show that the majority of students see $y=3/x$, $xy=5$ as functions, the major obstacle for the first being that it is not defined for $x=0$, and for the second, not only is it not defined for $x=0$, but the expression is not considered as a function until it has been manipulated to get “ y as an expression involving x ”. The latter is a common prototype for a function.

Expression (6) shows that the majority of the students think that $y = \pm\sqrt{4x-1}$ is a function. This resonates with the “ y equals an expression in x ” prototype. The fact that y is not given uniquely is less significant for the majority, although the minority giving more technical responses show a marked improvement because they are consciously aware that a function must give (at most) one value of y for each value of x .

Expressions (7), (8) and (9) address the problem of defining functions differently on different sub-domains. These do not fit the prototypes familiar to most students. Even so, the correct response to (7) is remarkably high. Experience suggests that students whose function prototypes involve a single formula may consider expression (7) not as one, but as three different functions (Vinner 1983). In fact, no student made such a

comment, indeed, those failing to respond positively were more concerned that the printing of the inequality signs might be ambiguous. Perhaps it helps in this case that each formula on the subdomains is familiar and that the function is everywhere defined. Certainly the fact that (8) is not everywhere defined caused problems because

- *y is not defined for all x*
- *doesn't state what y is if x is not rational*
- *no definition of y if x is irrational*

The difficulties with (8) and (9) seem also due to the strangeness of these expressions and the fact that they do not fit the students' mental prototypes.

- *is not a function of x, there is no connection mathematically*
- *no real link with x, i.e. not actually applying a function to x, where the answer would be y*
- *y is not in proportion to x,*
- *no relation between x and y*
- *not continuous on the real number line*
- *y=0 is constant*
- *y doesn't change as x changes*

Conflicts with constant functions

Comparing student performance on the expression $y=4$ and the graph of $y=\text{constant}$, we find only 28% reply correctly in the affirmative to both. 41% respond negatively to both questions, 29% say the graph corresponds to a function but the algebraic expression does not, with only 3% the other way round (table 4. 2).

The percentages for the 38 students giving more technical responses are starred in brackets. Although the percentage of correct responses rises from 28% to 42% for these students, it is still only a minority.

Is $y=\text{const}$ a function?		algebra	
		% yes	% no
g r a p h	% yes	28 (42*)	29 (26*)
	% no	3 (5*)	41 (26*)

Table 4.2

There is evidence of conflict in a significant number of scripts, as students change their mind when realizing that the algebraic expression clearly does not involve x , but the graph seems more likely to be a function. One student who thought initially that $y=4$ was not a function, then wrote it as $y=4x^0$, hence obtaining "a formula involving x ". This may very well be related to the description of the relationship between x and y in terms of variables: that the dependent variable y varies as the independent variable x varies. The expression $y=4$ offends this prototype because y does not vary!

The circle as a function

Comparing the responses to the graphic and algebraic representations of a circle, we find that 52% erroneously regard both graph and expression as representing functions, 12% say "yes" to graph and "no" to expression, 10% say "no" to graph and "yes" to expression, and only 25% correctly say "no" to both (table 4.3). The more technical responses increase the percentage correct from 25% to 47% – still less than half.

Is a circle a function?		algebra	
		% yes	% no
g r a p h	% yes	52 (18*)	12 (24*)
	% no	10 (11*)	25 (47*)

Table 4.3

The position is worse when we consider which students give a correct response to both questions in algebraic and graphic modes:

Only 11% of all students assert both that $y=\text{constant}$ is a function and a circle is not. The percentage only increases to 29% among the more technical responses.

Thus, even among the most able students in the sixth form, the vast majority do not have a coherent concept of function at the end of their A-Level studies.

Reflections

Because the general function concept is difficult to discuss in full generality we take the pragmatic route of de-emphasizing theory and emphasizing technical experience. Attempts to teach the formal theory, as in the New Mathematics of the sixties, have proved unsuccessful. But the other side of the coin – teaching the concept through examples, as in the current curriculum – leads to mental prototypes which give erroneous impressions of the general idea of a function. Even amongst the students who receive some training in the notion of a function, only a small minority respond coherently and consistently. We have described some of the symptoms, but not the cure.

The function concept is an extremely complex idea whose wider ramifications took centuries to be made explicit. In the development of the individual student the full implications only become apparent over a period of several years. We therefore believe that there are bound to be conceptual obstacles as the concept matures in the mind. When the function concept is introduced initially, the examples and non-examples which become prototypes for the concept are naturally limited in various ways, producing conflicts with the formal definition. We can attempt to give more general experiences which will improve the situation, but we face a formidable, fundamental obstacle:

The learner cannot construct the abstract concept of function without concept inaction, and they cannot study examples of the function concept in action without developing prototype examples having built – in limitations that do not apply to the abstract concept.

The literature is littered with examples of failure to comprehend the full complexities of the function concept (Dreyfus & Vinner 1982, Vinner 1983, Even 1988, Markovits *et al.* 1988, Barnes 1988, Tall 1990). Clearly, if we are to make progress we must attempt to develop an approach which makes the prototypes developed by the students as appropriate as possible. One promising approach is the use of computer programming to encourage the student to construct functions as processes through programming the procedures which take an input and process it to give the

corresponding output. Successful steps have already been made in this direction (Breidenbach *et al.* 1990).

However, we should continue to be aware of the conflicts which will occur from time to time as the learner has new experiences of sophisticated mathematical concepts. It is awareness that mental reorganization to cope with increasing complexity is both difficult and necessary that will help us design more appropriate curricula in the future.

The results obtained in this preliminary investigation on “students’ mental prototypes” provide an impetus for observing an actual classroom teaching of functions in schools. With the advice of the supervisor, two secondary schools were selected for the next stage of study.

CHAPTER 5

THE MAIN STUDY IN ENGLAND

5.1 Observation of teaching approach in two English secondary schools

Introduction

One of the methods of investigating students' understanding of mathematics is to look into how the concepts have been taught and the activities they have engaged in which may have contributed to their understanding.

The description to be reported here arose out of an investigation into the classroom interactions in two secondary schools, that is, Coundon Court School, Coventry and Beauchamp College, Leicester. This investigation was intended as a main-study to explore general teaching-learning patterns and various ways of approaching the teaching of function.

The plan of the report is as follows:

The first section contains a general description of the syllabuses followed by the two schools. The second section of this report relates specifically to the teaching of function in both schools. The permission to document some important aspects of the teaching was given to me by the teachers involved with the understanding that it is only for the research purposes (personal communication, 15/6/89).

Finally, in the third section, I attempt to draw together the description presented in the first two sections and to present some conclusions and hypotheses for further investigation.

Syllabuses

Mathematics teachers in Coundon Court School follow the Joint Matriculation Board (JMB) syllabus. This syllabus specifies clearly the objectives of the course, knowledge and abilities to be tested, however without any indication or suggestion about the teaching and learning approach. This is left entirely for the teachers to decide. However there is a note on the preference of mathematical notation, terminology and conventions to be used in the A-Level examination.

In Beauchamp College, the mathematics course at A-Level follows the 'new' SMP syllabus which consists of six modules (i.e foundations, calculus, problem-solving, mechanics, function and statistics) to be covered in the lower sixth and another four modules (i.e. Pure Mathematics, statistics and mechanics) in the upper sixth. There is a school-based assessment of the first year work (20%) and differentiated examinations at the end of the second year (80%).

Teaching and learning of functions

The following is an account of my assessment of teaching and learning associated with the topic of functions at the sixth form level in these two secondary schools. This assessment is based on a number of classroom observations during the autumn 1989.

There is one lower sixth class with 16 students in Coundon Court School. Two senior mathematics teachers taught this class alternately on Monday and Wednesday, each of 45 minutes duration. In Beauchamp College, there are two lower sixth classes with 10 students in one class and 12 students in the other which were taught by the head of department and another senior teacher respectively.

All students in both schools have obtained at least grade C in GCSE in the previous year. Students in Coundon Court follow the JMB syllabus whereas students in Beauchamp College follow the SMP modules. In this report, first of all, I would like to

record that all the teachers involved were very helpful and willingly discussed with me after most of the lessons which I attended. I made a total of 8 observations in Coundon Court and another 8 observations in Beauchamp College (4 observations for each class).

In both schools I was, at the first visit with my supervisor, introduced to the class as “a research student from Malaysia who is going to sit in some of our classes”. I had an impression from the beginning and throughout the study that my presence in the classroom was not unusual as the students were used to having visitors in their classroom from time to time. Furthermore, the teaching and learning process did not seem to be effected by my presence.

Although all important teaching episodes were recorded, this description is not an effort to explain all of the occurrences in the classroom, but rather to focus on some explicit examples which contribute directly to the students’ understanding and the difficulties (if any) in learning the function concept experienced by the students. I was particularly interested in how the function concept was being developed in the classroom teaching especially at the beginning of the students’ conception of the function concept, rather than with its later application.

Before I proceed further, the seating arrangement in the classroom in both schools deserves mention. In Beauchamp College, the seating arrangement in the classroom is rather like a seminar room where the students sitting face to face. They can see, talk to and listen to each other even when each of them is engaged on their own particular task. The class climate is more friendly and relaxed. The teacher was free to move about and fully involved in interactions with individuals or small groups of students. This is not the case of Coundon Court which follows the usual or ‘formal’ seating arrangement in row and column. The students cannot see or talk to each other, they normally talk to or listen to the teacher.

Teaching strategies

(a) Coundon Court School

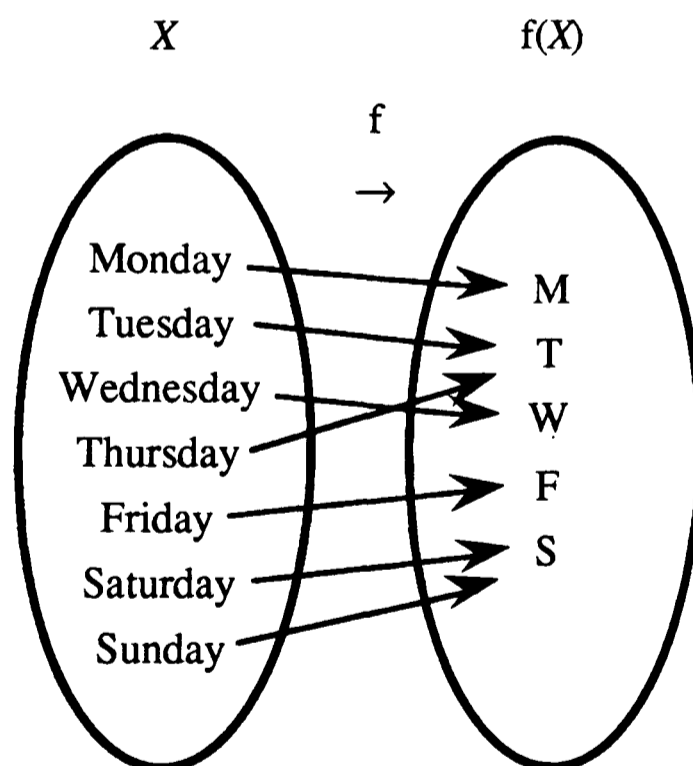
In Coundon Court, the teaching of the function concept starts with the teacher introducing the general idea of a function as :

A function f is a rule which assigns $f:x \rightarrow y$ and called 'f maps x to y '.

The teacher mentioned that another term used to describe a function is a mapping. It is synonymous. Students are told that the most preferable notation is $y=f(x)$, although they are also told that :

You are free to use any suitable notation as you wish.

This statement then followed by the example of a function f which maps each day of the week onto its initial letter:



Then a function f is described as a mapping from set X into set Y and denoted by $f:X \rightarrow Y$.

I think the teacher assumed that the concept of set, elements of sets and Venn Diagram had been learned and understood by all the students in his class. The students were then

told that the set X is called the domain of f and set Y is called the codomain. Here the teacher emphasized that by definition of a function:

Every element in the domain of a function has one and only one image in the codomain

Another term then introduced is the range of the function as the set $\{W,T,F,M,S\}$ which containing all the images under f and denoted by $f(x)$. The teacher also pointed out :

As you can see any element in the range is the image of one or more elements in the domain, for example, F is the image of Friday, but T is the image of both Tuesday and Thursday.

At this stage the students were mainly listening and writing down some notes considered necessary by them.

The teacher now considered three functions with special properties.

1. Let X be the set of natural numbers $\{0,1,2,3,\dots\}$ and Y be the set of single digits $\{0,1,2,\dots,8,9\}$. Let $f:X \rightarrow Y$ be the function which maps a natural number to its final digit.

The teacher used the overhead projector to tell the class with relatively little further explanation that this function is said 'to be a mapping of X onto Y '. The teacher also introduced another term :

A function which is onto, that is, if every elements of the codomain y is the image of at least one element of the domain x , then it can also be called surjective

A second example was then given to the class which concerned the mapping of the set of natural numbers into itself, defined by the function $g:x \rightarrow x$. Until this stage the teaching was mainly through an exposition by the teacher without much significant interaction between teacher and students.

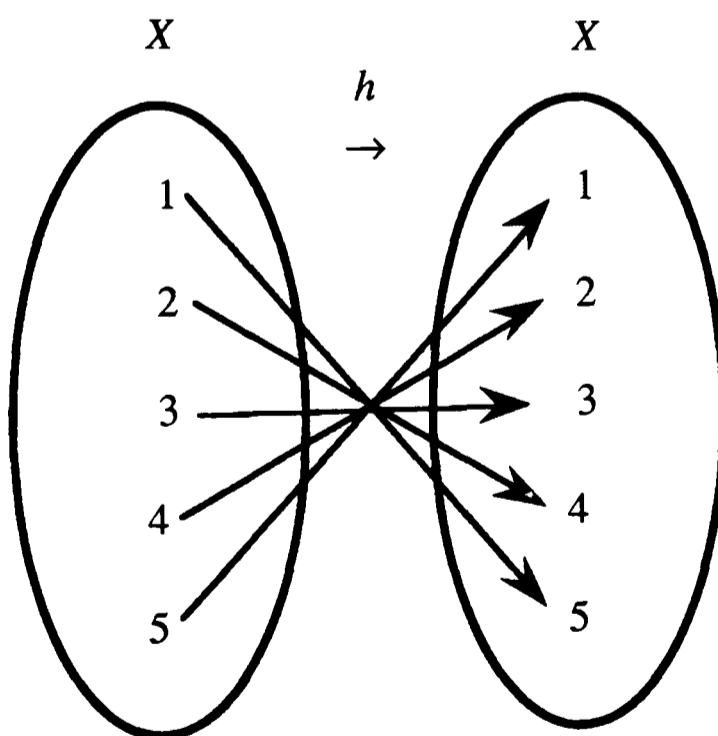
For the function g , students were told that the set is both the domain and the codomain, but the range of g is a subset of, $\{0, 1, 4, 9, 16, \dots\}$. In this case, every element of the range is the image of only one element of the domain. The teacher said that g is a mapping 'into' rather than 'onto' and this function is one-one which can also be called injective. Students were also told that a function which is not 'one-one' is called 'many-one'.

At this stage, the teacher asked the whole class, some recall-type questions :

What can be said about the function in example (1) and the function defined earlier on the set of days of the week. Is it one-one or many-one?

All students in the classroom seemed to be thinking for a few moments and some of them gave responses that both the functions are many-to-one.

Another example given shows a mapping from the set X to itself defined by $h:x \rightarrow 6-x$ where $X=\{1,2,3,4,5\}$.



The teacher asked : *What are the specific features of these two sets ?*

Some students' responses were : *Both are the same set X*

The range of the function is the set X itself

The teacher then summarised :

As shown in the diagram, every element of the range is the image of exactly one element of the domain. The function is said to be a one-one mapping of x onto itself and a function which is one-one correspondence between x and y is also called bijective.

At the end of this episode of whole-class teaching, blackboard exercises and some additional exercises displayed on an overhead projector were given to the students for them to practise finding (the range of a function, deciding whether a particular function f , (i) one-one (ii) many-one (iii) f maps X onto Y , as well as some new properties such as one-one correspondence and so on. This 'exercise' part of the lesson will enable the teacher to interact individually with the student or with a small group of the students in the classroom. The lesson ended by the teacher giving a few questions as homework.

Most of the lessons I have observed in Coundon Court followed almost the same pattern or perhaps the 'standard pattern' for a mathematics lesson: an introductory and a substantial exposition from the teacher after which the class works through an exercise in order to reinforce or to consolidate and practise the concepts and skills which has just been taught.

After completing the sub-topic on 'mappings and functions' the lesson then proceeds with 'the graph of a function' which involves the sketching of a graph of a function defined in a given interval and examining the general behaviour of a function.

(b) Beauchamp College

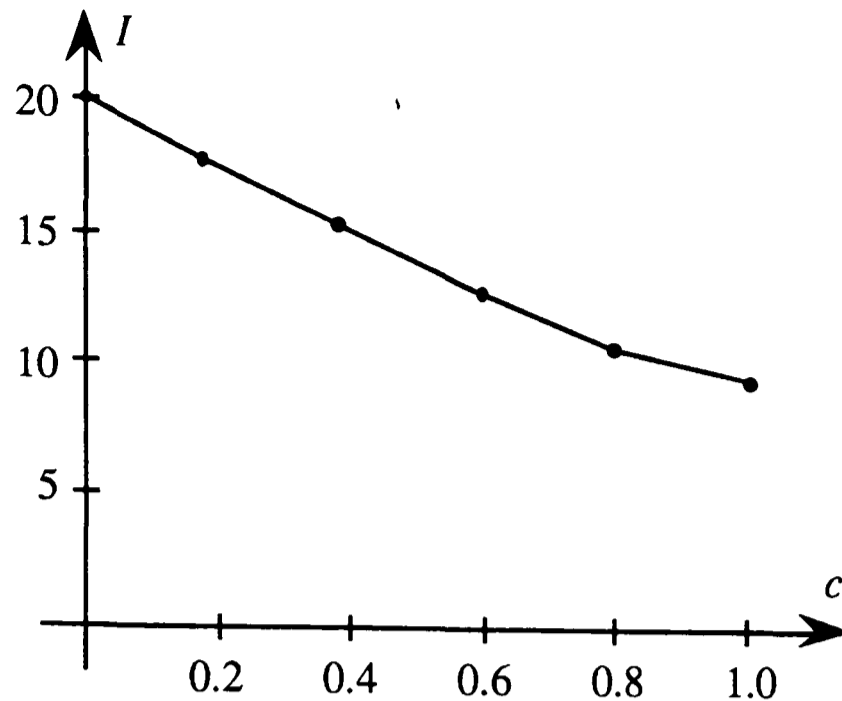
In Beauchamp College, on the other hand, mathematics teachers follow quite closely the prepared modules by the SMP for their mathematics course at A-Level. With respect to the teaching of function the students are not introduced to anything about the notion of set, relation or mapping, instead the teacher introduced the function concept by discussing the experiment performed by a scientist to test a scientific law called “Bear’s Law of absorption”. Since every student has their own text-module, the teacher ask them to refer to the diagram of an experiment while he explaining that :

A scientist performs an experiment to test Bear’s Law of Absorption. Light is shown through a coloured solution and the intensity of light energy is measured. She finds that if she varies concentration of the solution, her readings are as follows.

The task and the graph associated with the experiment were shown on the overhead projector.

Concentration c (mgcm^{-3})	0	0.2	0.4	0.6	0.8	1.0
Intensity I	20.0	17.4	15.2	13.2	11.5	10.0

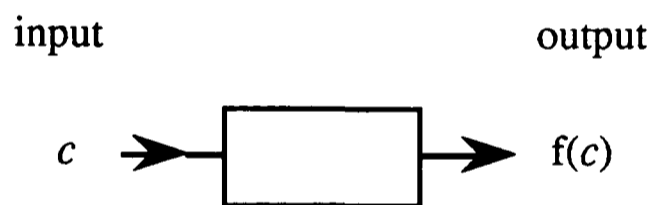
Then the graph shows :



Students were told that the two variables I and c are related by the mathematical expression or formula $I=20/2^c$. The teacher said that :

We can use this formula to calculate the volume of I for any given value of c , or for a given value of I , we can solve the resulting equation and find a value for c

Another diagram is shown to illustrate the meaning of a function f .



The teacher stressed that it is often useful to view an expression from a different point of view and consider it as a device which gives an output for any given input. In other words, if the scientist inputs any value of c into her formula she will get a corresponding output which tells us the values of I . Students were then asked to compare the notation $f(c)=20/2^c$ and $I=20/2^c$.

What are the advantages and disadvantages of each notation? This question generated quite a lot of interest among the students, it is reflected in some discussion among themselves before a few of them gives responses as :

"It is clear to see the relation of c with I in $I=20/2^c$ ",

"We can find directly the value of f at any value of c such as $f(1)=20, f(2)=5$ and so on in $f(c)=20/2^c$ ".

From here, the teacher emphasized the dependence of c by the use of function notation $I=f(c)$ where $f(c)=20/2^c$. The students then led to the definition of a function which consists of two statements :

the rule: which tells you how values of the function are assigned or calculated, and the domain: which tells you the set of values to which the rule may be applied.

The teacher then proceeded to explain further that :

When a function is written down, both the rule and the domain should be given.

Most of the exercises given as a classwork and as a homework were taken from the SMP text-module. Some of the lessons in this college were conducted in the computing room. Every student had access to one micro computer. After a few lessons on the introduction to function concept, subsequent work is on 'investigating curves' by using the graph plotter software designed by D.O. Tall of Warwick University specifically for SMP 16-19 course module. The exercises using a microcomputer had enabled the students to become familiar with such basic functions :

$$y=x, y=x^2, y=x^3, y=x^4, y=x^{1/2}, y=x^{1/3}, y=1/x, y=1/x^2, y=\sin x,$$

$$y=\cos x, y=\tan x, y=\log x, y=3x, y=(1/2)x, y=|x| \text{ and } y=\text{int}(x).$$

It has also provided further reinforcement of the function concept as well as for the students to become familiar with their graphs. Furthermore, this exercise enabled the students to investigate the geometrical properties of particular functions and observe the interrelationships between geometry and algebra.

It is interesting to see the lesson conducted in the computing room which shows a major shift from normal classroom 'chalk and talk' teaching. The lesson usually starts

with a little introductory exposition by the teacher and is then followed by the practical activity or 'hands-on' activity using the microcomputer which in many instances became more meaningful and of a greater interest to the students and perhaps they are likely to understand the concepts better.

When the teacher went round the class and I also took the opportunity to become a "participant observer", it appeared that most of students volunteered to explain and discuss their answers with the teacher. The students seemed to be happy and enjoyed the lesson because they had an opportunity to find out the answers for themselves.

Discussion and conclusion

It is evident that from the classroom observations in these two schools, that the function concept has been taught and developed by two quite different approaches.

In Coundon Court, the teaching emphasized content which formalises the set language, mathematical terms and notations involved. The teachers preferred to develop the concept in a more formal approach, using a rather technical language which enriched their mathematical vocabulary and provided a good basis for further study in mathematics, although in a way it lost the basic idea of functional relationship during the initial stage of developing a function concept. However the concept has been taught in accordance with the requirement of the JMB syllabus and for the preparation for A-Level examinations in pure mathematics. Some evidence of the students' familiarity with the technical terms such as injective, subjective, and bijective will be given in the following section.

In Beauchamp College, which follows the SMP 16-19 syllabus, there is some evidence of the change in approach to mathematics teaching and learning embodied in the new SMP 16-19 scheme. This school is one of the three in Leicestershire with teachers active in the ATM (Association of Teachers of Mathematics) who are used to a more

interactive style of teaching and applied to join the SMP as a group to carry through their convictions into the A-Level course.

The implementation of any mathematics course depends mainly on the teachers. The methods of teaching adopted by the individual teacher were very dependent on the teacher's own experiences, attitudes, beliefs and values. The difference in philosophy between Coundon Court and Beauchamp College was clearly reflected by the Deputy Head of Beauchamp College, who also taught mathematics to the sixth form when shown the proposed test questionnaire, he explained that

our students in pre-16 are no longer taught to differentiate between mapping, relation or function – indeed it is not much stressed at A-Level. In the past when we did teach modern maths, our students would have scored highly on such a test..

A similar response was given by the Head of Mathematics at Beauchamp College when asked about his preference of teaching the topic of function

the terms such as injective, surjective, bijective etc. should be in the domain of university mathematics and not school mathematics.

The emphasis on function as a rule is clearly demonstrated in the teaching at Beauchamp College. The teacher gave particular emphasis to establish the concept from the notion of one quantity depending on or varying with another quantity. The teacher also tried to develop the idea of function by showing the example from real life and related to the actual scientific experiment. The use of the function machine helps to establish the concept of one number (or input) generating another number (or output) and each input being uniquely associated with one output. Furthermore, 'function' is introduced as a mathematical term not only associated with 'formula' or 'expression' but rather stresses the functional relationship involved. Generally, teachers in Beauchamp College practise the philosophy of the SMP which has been designed with the basic assumption that students' active participation in learning will provide

opportunities for deeper understanding of mathematical concepts. The teachers had actively use microcomputers as a tool for teaching mathematical functions.

5.2 Analysis of results in the pilot study

The following is the analysis of the results from the pre-test and post-test on function given to a group of sixth formers in the above mentioned schools. The pre-test was administered to the students in Coundon Court Secondary School and Beauchamp College during the autumn term 1989, before any work on function had been introduced in their lesson. The same test was repeated in the post-test to the same group of students in both schools after they had finished the relevant work on function in the spring term 1990. The number of students involved in this pilot study is sixteen and twenty two at Coundon Court and Beauchamp College respectively.

The aim was that results from pre-test and post-test should be used to assess whether significant improvements were made by the students towards the development of their knowledge of the function concept.

A detailed analysis of the students responses in every questions is given as follows:

Questions 1, 2 and 3 are analysed separately from questions 4, 5 and 6 due to the difference in nature of the questions involved.

Question 1 asks students to describe what is happening to the left hand side numbers to get the numbers on the right hand side :

$$1 \longrightarrow 4$$

$$2 \longrightarrow 5$$

$$3 \longrightarrow 6$$

This question is equivalent to level 3 of attainment target 6 in the U.K National Curriculum proved to be too easy and straightforward. All students in both schools gave the correct answer whether in pre-test and post-test although with different forms and wordings, such as :

add on 3,

adding 3,

addition of 3 etc.

This is a warm-up question in order to give confidence to the students but could also cause an interesting response such as think of function as a process.

Question 2 asks students to find the values of $f(4)$, $f(1/2)$, $f(-7)$, $f(10)$, $f(-9)$ and $f(h)$ from a given function f such that $f(x) = 2x+3$. Almost all students gave the correct answers. This shows that their skills of algebraic manipulation were good.

The performance of the students in question 3 to question 6 in the pre-test and post-test are given in the following tables and will be analysed in the rest of this chapter.

Question 3 asks students to describe their conception of function. Students responses are too diverse and some are ambiguous. This probably reflects the variety of approaches in which the function concept has been taught and introduced in the textbooks. The students' answers can be divided into two categories which will be called the '*static*' and '*dynamic*' conceptions. Although none gave formal definitions but all gave descriptions which can be categorized into one of the following:

Static conception: formula, equation, expression, rule

Dynamic conception: operation / process

as shown in the table 5.1 with the percentage of each categories given by the students in both schools in the pre -test and post-test.

concept prototype	Pre-test		Post-test	
	Coundon Court	Beauchamp College	Coundon Court	Beauchamp College

Static	(% of responses)			
formula	12	14	20	6
equation	19	23	0	31
expression	6	14	12	6
rule	25	9	30	19

Dynamic				
process/operation	37	29	30	37

none of the above	1	11	8	1
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Table 5.1

All of the student responses were not classified as correct or incorrect, instead classifications were made according to whether the response contains the words ‘formula’, ‘equation’, ‘expression’, ‘rule’ or ‘process/operation’ or other words which we considered as similar meanings to each of those words. Since this is a very subjective question, great care has been taken in analysing the responses. However our experience in the preliminary investigation (see chapter 4) proves to be useful in categorizing the students’ responses in this main study. All of the students responses were evaluated twice with the help of the supervisor in order to assure reliability in the classifying procedure. Nevertheless we realized that some small variations of interpretations may occur with other researchers.

Some examples of the answers for each category are as follows:

Formula :

“it is finding an answer for a problem by using *formula* (usually)”

“function is a *formula* into which values of x are put”,

“It is a sort of *formula* for variables to be calculated”.

Equation :

“It is a sort of *equation* instead of writing $2x+3$ for example you can write $f(x)$ and substitute the value into the *equation*”,

“function is an *equation* which you draw”,

“mathematical function is an *equation* where you can substitute a number in and find the answer at the end”.

Expression :

“A function shows y as an *expression* of x ”,

“A mathematical *expression* which takes place on a statement to change”

“A function is an *expression* where the outcome is dependent on the value of a variable e.g. $2x+4$ function of x where x is a variable”.

Rule :

“A function is a *rule* which is applied to one set of numbers to get another set (does not always have to be a number)”

Process :

“The mapping of the function is a *process* of carrying out a formula i.e $y=x+1$ into set, graph or table form”,

“The *process* you have to follow using x . You only get an answer for each x ”,

“function is a *process* such that when applied to a value x only one possible value x is obtained”.

Operation :

“An *operation* on something to achieve a desire result”,

“*Operation* on x ”,

“An *operation* performed on a number”,

“A function is an *operation* which is made to any algebraic expression to give another expression”.

It is important to note that a significant percentage of students in both schools have a significant grasp of the process aspect of function taking some kind of input and carrying out a certain procedure to produce an output. Behind the static conceptions of functions, we believe, lies the idea of function as an object and behind the dynamic conceptions lies the idea of function as a process.

On the basis of the answers given in this question we can say that most of the students have not had an explicit definition of function so they respond by evoking their experiences related to the word ‘function’ usually by giving examples from what has been taught or from their textbooks. It also shows how difficult it is for the students to master the coherent and meaningful conception of function, especially to master the full formal set up in terms of domain and range, even after many hours studying the function concept in their course.

Algebraic equations for the function

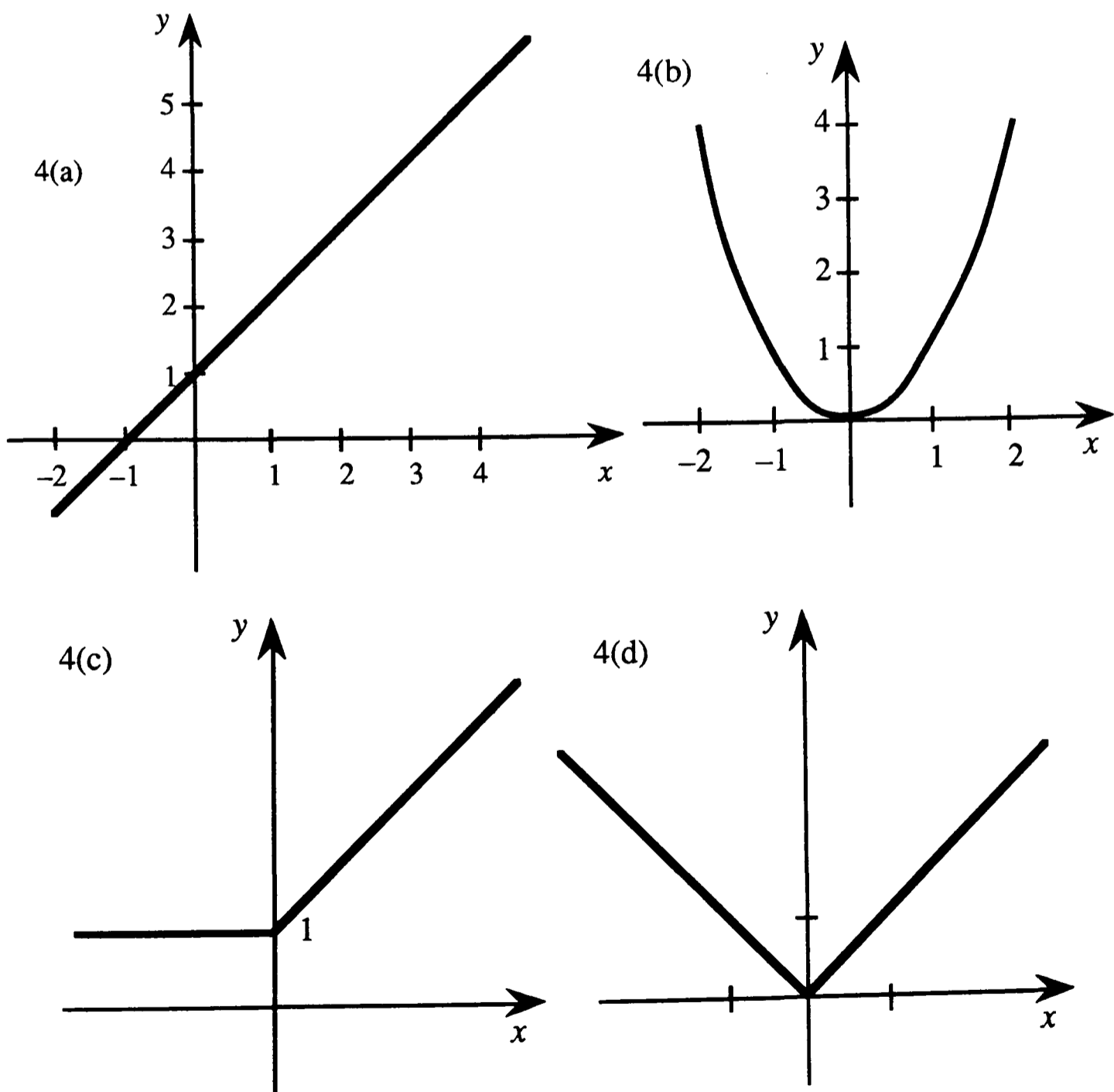
To investigate the students’ ability to find an algebraic equation for a function, I asked the sixth formers mentioned earlier to find an algebraic equation for the function given

by the four associated graphs. The following table shows the percentage of correct answers given in the pre-test and post-test.

	Pre-test		Post-test	
	Coundon Court	Beauchamp College	Coundon Court	Beauchamp College
4(a)	100	82	100	88
4(b)	88	95	90	100
4(c)	0	0	0	0
4(d)	13	36	60	44

(% of correct responses)

Table 5.2



The results show that most of the students answer correctly to 4(a) and 4(b) which resonate with their function prototypes. But no students have experience of symbolism to express a function such as that of 4(c). The prototypes of functions are given by a single formula. In 4(d) there is a single formula $y = |x|$ and, though less familiar, this shows considerably greater success than 4(c).

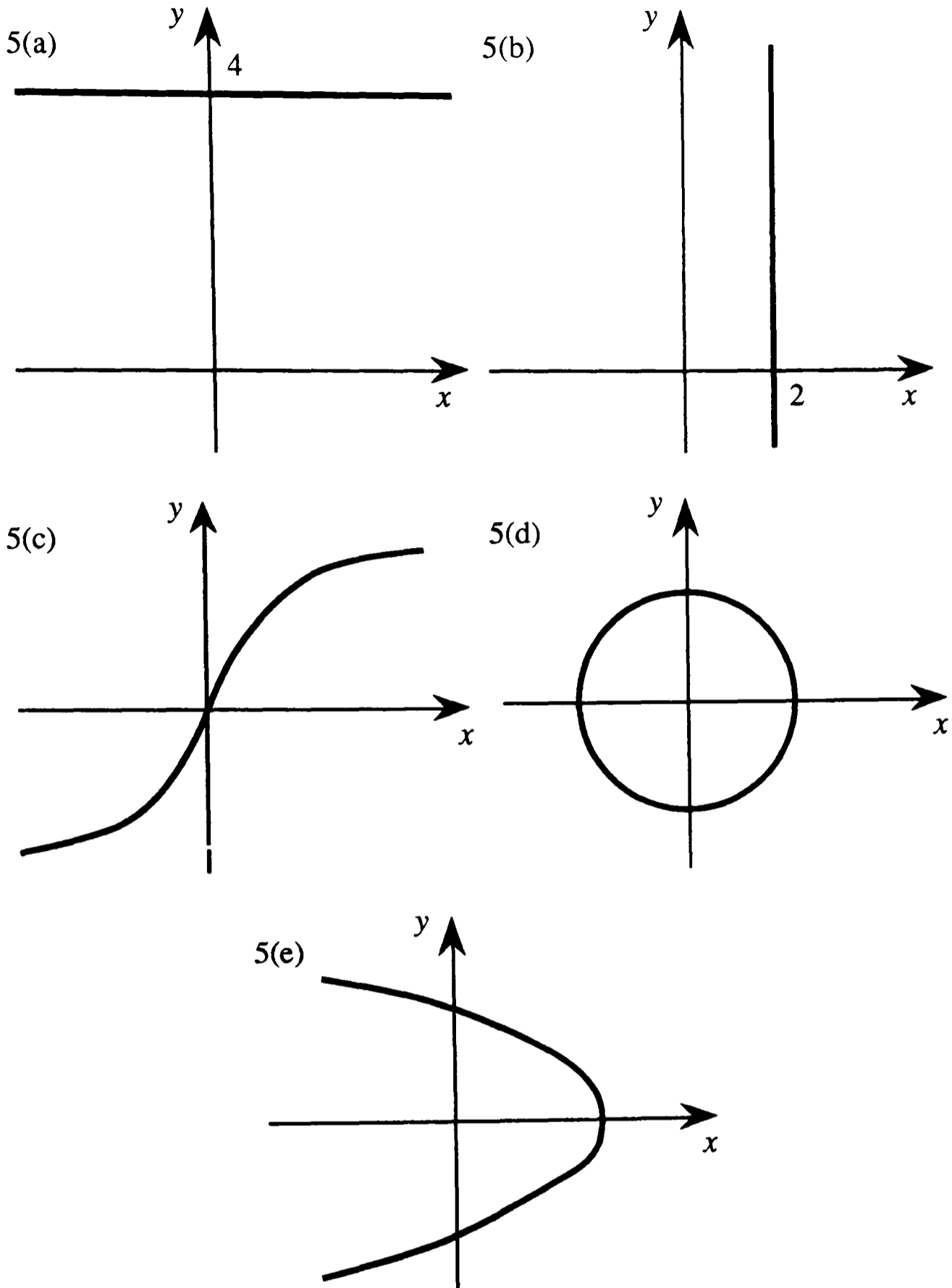
Graphs as functions

From five sketches given, students were asked to identify which of these sketches could represent a function. The responses from the pre-test and post-test are as follows:

The percentage of responses do not always add up to 100% partly due to the rounding errors and also to a small number of non responses. The response which is more likely to be adjudged correct is given in bold face type.

	Coundon Court				Beauchamp College			
	Pre-test		Post-test		Pre-test		Post-test	
	% Yes	% No	% Yes	% No	% Yes	% No	% Yes	% No
5(a)	13	88	0	90	50	45	56	44
5(b)	19	75	20	50	50	45	13	81
5(c)	100	0	90	10	86	14	88	12
5(d)	38	56	90	0	23	73	38	56
5(e)	69	31	60	40	45	45	25	75

Table 5.3



Students in Beauchamp College performed generally better in all questions than students in Coundon Court.

None of the students in Coundon Court gave a correct answer to question 5(a) about a constant function in the post-test. This is mainly due to the prototypical image created

by the definition given earlier in the course which clearly emphasises the importance of variables x and y in defining the function concept.

“as $y = 7$, no x given”

“because y is constant and cannot be written in terms of x ”

“ y always 4, not a function of x ”

“ y is constant at 4 and so unrelated to x ”

“ y has a constant value regardless of the value of x ”

“ $y = 4$, y is not a variable”

“ y is not changing”

“ y is not affected by x , it is always 4”

“because y doesn't change”

“ y does not change with changing x values”

“ y is constant, need variable”

“ y is constant”

The response on question 5(d) about a circle shows that 90% of the students in Coundon Court had a strong prototypical image of a circle as a function since its consist of variables x and y . This reflected in the reasons given such as,

“because x and y are changing and y can be written in terms of x ”

“because there are x and y values”

“for changing values of x there is a corresponding y value connected by a complex equation such as $x^2 + y^2 = 1$ ”

“y changing in accordance with x”

“both x and y are varying”

“The two are linked (by a complex equation) but each value of x still gives specific value for y”

“x and y vary”

Algebraic expressions as functions

To investigate the students' understanding of a function in terms of formulae. I also asked the students to identify which of a number of algebraic expressions could represent y as a function of x . The responses are given in the following table :

	Coundon Court				Beauchamp College			
	Pre-test		Post-test		Pre-test		Post-test	
	% Yes	% No	% Yes	% No	% Yes	% No	% Yes	% No
6(a) $y=-x+1$	94	6	100	0	82	18	87	13
6(b) $y=7$	0	100	0	100	9	91	13	87
6(c) $y=x^2+1$	94	6	100	6	77	23	88	12
6(d) $y^2=x$	56	38	60	40	55	27	69	31
6(e) $x^2+y^2=4$	38	62	70	30	41	64	50	50

Table 5.4

The responses for 6(b) confirmed our hypothesis that students, prototypical conceptions of function are strongly related to the idea that a function must be an expression relating x and y . Many students clearly denied $y=7$ is a function because

“y is not expressed in a equation with x, it is a fixed value”

“because x is not stated”

"because there is no x variable"

"x is not involved in the equation"

"no value of x, y is always 7"

"no x in equation"

"at (b) there is no defined relation between y and x. Everywhere else there is one"

"it has not got a value of x"

"y is fixed and not a function of x"

"y is not at all related or linked to x"

" there is no x in the equation, therefore y cannot be a function of x"

" doesn't include x in the expression"

The results of the pre-test and post-test in both schools demonstrated that these prototypical conceptions proved to be pervasive, stable, and often resistant to change.

Understanding of technical terms

I asked a group of sixteen sixth-formers (aged 16/17) in Coundon Court at the end of their relevant work on functions to :

Explain in a sentence or so what you understand of the terms injective, surjective and bijective.

They had studied these notions as part of their course on functions over a term previously. Although few students did not give any response, some of the answers given included the following :

injective :

a one to one function

this is where one number maps onto only one number

where a value of x has one set value for $f(x)$, and no more (i.e. you get only one answer, example $f(x)=2x$.

surjective :

an onto function

is when a function is onto

a function which maps the number in X onto at least one value of Y

one value of y has more than one corresponding values of x ; example first letter of days of the weeks.

bijective:

when a function is both one to one and 'onto'

a one- one and onto, where one number x maps exactly onto one value of y and every element of y is the image of at least one element of x .

where the function is both one to one and onto i.e the x value can have either one corresponding value and others (i.e both surjective and injective): example $y=x$.

It is therefore important to note that:

approximately 90% of students in Coundon Court provide clear evidence of their familiarity with technical set-theoretic terms and yet their conceptions of functions did not match with that familiarity.

This accords with the comment of the Beauchamp College teachers who claimed that they could teach these terms for the students to score highly on tests if necessary. It

shows that even when formal definitions are emphasised at the outset, if the experiences which follow are of a different nature, they may soon dominate the concept image.

University students' responses

The same questionnaire was also given to 98 undergraduate students at Warwick University who follow the four year BA(QTS) course. The questionnaire was administered during the Autumn term 1989.

The following table gives the categories of responses to question 3:

	Pre-test		Post-test	
concept prototype	Year 1	Year 2	Year 3	Year 4
	N=55	N=16	N=15	N=12
Static	(% of responses)			
formula	0	0	7	0
equation	2	0	7	0
expression	11	12	0	0
rule	15	19	7	17
Dynamic				
process/operation	38	44	53	42
none of the above	13	19	13	17
no response	20	6	13	25

Table 5.5

The results show that at the university level, students generally have a strong conception of function as a process or operation. Some of the following responses may prove this claim.

"The process you have to follow using x . You only get on answer for each x ."

“The procedure you take a value through to get the one you want”

“A function is a process such that when applied to a value x , only one possible value x' is obtained”

“A function is a process by which a number is related to another number.”

“An operation performed on a number”

“A process performed on a number”

“A function is a process of mapping one number onto another.”

“A process that involves changing a value, say y , in relation to another value, say x , to give a result.”

“A process which you can do to certain values of x which will give you certain values of y . A operation can only work when you have one value of y for each x .”

It is important to note that from the above responses, it is clear that students had strong conceptions of a function as a process. Students' conceptual images of a function were mainly in terms of functional relationships between variables x and y .

The responses to question 4, 5, and 6 are as follows:

	% of correct responses			
	Year 1	Year 2	Year 3	Year 4
	N=55	N=16	N=15	N=12
4(a)	91	100	93	92
4(b)	93	100	67	100
4(c)	7	44	27	42
4(d)	67	75	33	83

Table 5.6

The results of 4(c) show that less than 50% of students had an experience of symbolism to express such a function. This clearly indicates that the prototypes of functions for most of the university students are those given by a single formula.

	Year 1		Year 2		Year 3		Year 4	
	% Yes	% No	% Yes	% No	% Yes	% No	% Yes	% No
5(a)	33	67	56	44	27	73	33	67
5(b)	18	82	25	75	20	80	0	100
5(c)	93	2	100	0	93	7	100	0
5(d)	45	47	50	50	67	27	67	33
5(e)	56	40	56	38	73	13	58	25

Table 5.7

Although the university students had been exposed to more types of functions throughout their course, the increased exposure to the function concept apparently did not lead to correct conceptions or to alleviate wrong prototypical conceptions. The reasons given for denying a constant function $y = 4$ as a function include:

“y is independent of x”

“because the value of y cannot change”

“y is not dependent on x”

“x does not enter into it”

“equation doesn't involve x, $y = 4$ ”

“function does not vary with y”

“y doesn't vary with x”

“x is unaffected by y”

“value of x is not affected by y ”

“ y does not vary”

“no equation linking x , y ”

“ $y=4$, y has no function of x ”

“ $y=4$, nothing to do with x ”

“states for y and not x ”

“does not use both x and y axes”

“ $y=4$, hence x not in equation”

“unless $y=0x+4$ ”

As we can clearly observe, the reasons given for $y=4$ is not a function were centred around the belief that there must be some kind of relationships between x and y in order to be a function.

The responses to question 5(d) about whether a circle is a function also provide further evidence for our hypothesis that students develop prototypes for the function concept based on properties of familiar examples. Students apparently test whether something is a function or not through ‘prototype matching’ with their familiar examples. The responses “Yes” in this case, often accompanied with the following reasons:

“yes, it is a formula”

“formula of circle”

“ y can be expressed in terms of x , so is a function of x ”

“yes, because involves x and y ”

"x varies with y"

"y alters as x alters"

"so the line depends on x and y"

"y is directly effected by x"

"y value is dependent on x values"

"involves both x and y"

"passes through both x and y axes"

"any value x gives value of y"

"have to use x to find y"

"the equation contains both x and y"

Some did not provide any reasons, instead provides the answer such as,

"yes, because I have seen functions like this before"

"no, because we were told in a lecture"

The responses to question 6 concerning the algebraic expressions as functions are as follows:

	Year 1		Year 2		Year 3		Year 4	
	% Yes	% No	% Yes	% No	% Yes	% No	% Yes	% No
6(a) $y=-x+1$	95	4	100	0	87	13	100	0
6(b) $y=7$	16	78	50	50	13	80	25	75
6(c) $y=x^2+1$	84	16	88	0	93	7	100	0
6(d) $y^2=x$	56	40	50	50	87	13	67	33
6(e) $x^2+y^2=4$	38	56	50	50	73	20	75	25

Table 5.8

The responses to question 6(b), again show that regardless of the age or the level of education, prototypical conception that a constant function is not a function was prevalent. Some of the responses of the university students appeared to be more sophisticated - included more technical terms - than those of the school students. By answering "No" they used the terms such as

" y is not dependent on x but always equal to 7 for all value of x",

" this is not a function as there is only one undetermined variable involved"

" y is unreliant upon x"

" no, as there are no values for x"

" because x is not involved "

" y is not dependent on x"

" not a function as $y=7$ whatever the value of x"

" as x is not involved in the equation, $y=7$, 7 is the only answer"

" contains no x"

" y independent of x"

" x does not enter in the equation"

" no relation with x"

" this is not a function as there is only one undetermined variable involved. It needs to be of the form $y= f(x)$ "

" No x! "

" does not include x in equation"

“No x value given here”

“only one without x in it, the others can be written as functions”

It should be noted that the reasons given for a constant function not being a function were remarkably similar in either its graphical (5(a)) and algebraic forms (6(b)).

To question 6(e) about a circle, we found the following reasons for “Yes”,

“y is found depending a value of x”

“would do if rearranged to $y^2 = 4 - x^2$ ”

“can be expressed as x”

“y value is again dependent on value of x”

“y is obtained depending on the value of x”

“x varies with y”

“does involve x”

“involve both y and x”

“can be written in terms of x”

“to know y you have to know value of x”

“since, $y^2 = 4 - x^2$ ”

The results of this pilot study revealed that students at various levels had a very strong prototype of a function as an algebraic expression involving x and y.

CHAPTER 6

MATHEMATICS EDUCATION IN MALAYSIA

6.1 Modern Mathematics in Malaysian Secondary Schools

A brief historical overview is perhaps necessary to give a reader some ideas of how the reformation of mathematics curriculum taken place in Malaysia in the early seventies and its relation with other countries.

It was widely acknowledged that the world wide mathematics curriculum reform took place particularly in the United States and followed by other countries were very much influenced by the launching of the Sputnik by Russia in 1957. The Americans believed that in order to maintain U.S technological advancement there must be a massive improvement in their mathematics and science curriculum. Within a few years most of the schools in the U.S had embarked on the “New Mathematics” programme.

In the U.K on the other hand, it is generally agreed that the starting of the School Mathematics Project (SMP) in 1961 was considered to mark the beginning of the introduction of “Modern Mathematics” programme in schools. The SMP has become one of the largest and the most widespread Modern Mathematics project in Britain (see Watson, 1976, for further discussion).

Many of the modern mathematics curricula in the developing countries originated either from Britain or the USA. Malaysia, with an education system based mainly on that of Britain since the colonial days, chose for obvious reasons the curriculum materials from the U.K rather than from anywhere else. Furthermore by studying modern mathematics similar to that in the U.K, it will enable Malaysian students to prepare themselves for external examinations to enter overseas universities such those as in the UK.

Starting in 1970, the modern mathematics curriculum based on the UK projects was introduced in stages in secondary schools in Malaysia. The Scottish Mathematics Group

(SMG) materials were chosen as a basis for the Modern Mathematics curriculum in the lower secondary schools and the Schools Mathematics Project materials were chosen as a basis for the Modern Mathematics curriculum in the upper secondary schools (CDC report (1974), SMP Ten Years report, Thwaites (1972)).

In relation to the teaching of function concept, the SMG had this to say,

The language of sets, inequations as well as equations, the meaning of a variable and the idea of function are emphasized. The concept of mapping is introduced and used where most appropriate. The algebra has less emphasis on manipulation and more understanding of mappings. The modern idea of a function $f:x \rightarrow f(x)$ as a set of ordered pairs or as the operation taking a variable in a number system to another variable in the number system is more important (p. 337).

In the SMP, one of the suggested modifications to the mathematics curriculum in school was

A wider conception of function should be instilled, and many examples of functions and relations which are not obviously mathematical should be introduced (ibid, p.355).

6.2 Some of the problems and issues pertaining to mathematics education in Malaysia

It is perhaps useful to give the reader some picture of the previous and current scenario of mathematics education in Malaysia. Many of the ideas are mainly based on the experiences of the author himself as the former secondary school teacher and currently as a lecturer in mathematics education at the UTM.

From a brief historical overview of the reform of the secondary mathematics curriculum discussed in the previous section, the reader may reflect on the problems of curriculum adaptation especially experienced by mathematics teachers in Malaysia in the process of implementing it in the schools.

6.2.1 Problems of Curriculum Adaptation

The Modern Mathematics programme was implemented in stages starting with 26 secondary schools at the form one level (age 12+) in 1970, and culminated in a complete changeover by 1976 (See CDC Report 1974). Although some of the immediate problems that surfaced from the adaptation of the foreign curricula such as the preparation of trained teachers and curriculum materials were recognised but the problems that surfaced in the later stages of implementation have proved to be and continue to be more difficult to solve.

From my own perceptions through the years of implementing the curriculum, there were at least three major problems and constraints arising from curriculum adaptation namely the conceptual, the pedagogical, and the psychological problems which took shape and coherence only in the wake of curriculum implementation.

1. Conceptual problems

The modern mathematics curricula were developed and very much influenced by the work of mathematicians in the universities. As professional mathematicians they were often concerned with the logical development of the subject rather than the cognitive aspect of the learner. The main priority is to present mathematics as a system of conceptual knowledge based on clear definitions of mathematical concepts. For example according to Tall (1990),

"In the new math there was a valiant attempt to build the function concept from a formal definition in terms of the cartesian product of sets A and B: Let A and B be sets, and let $A \times B$ denote the cartesian product of A and B. A subset f of $A \times B$ is a function if whenever (x_1, y_1) and (x_2, y_2) are elements of f and $x_1 = x_2$, then $y_1 = y_2$. However, there is much empirical evidence to show that, though this definition is an excellent mathematical foundation, it may not be a good cognitive root."

In other words the logical foundations of the subject are not the same as psychological ones: the logical foundation which is often concerned with the precise definitions of mathematical concepts may be suitable for a formal development or to exhibit the structure of the subject but it is not necessarily suitable as a basis for learning. In this sense Tall (1990) suggests that a cognitive root is a concept embedded in an approach which may build on concepts familiar to the student and which are also suitable for future learning. Tall claimed that an appropriate cognitive root of the calculus is the notion of “local straightness” which underlies the concept of differentiation. A student who understands it can look at the graph of a function and be able to mentally “zoom in” and has an insight to the process of changing gradient of the curve (see Tall (1985): Supergraph).

Although much of the content of modern mathematics is regarded as universally relevant, the adapted curriculum was intended and developed initially for students of a different cultural and educational background. The development of curriculum was not preceded by a study of the development of basic mathematical concepts among children. It is not surprising, therefore, to see that the level of work required in the new syllabus was often beyond the level of the student's conceptual development.

The research evidence also revealed (see, for example, Hart 1981) the existence of a serious mismatch between the cognitive demand of the British secondary science and mathematics curricula and the level of cognitive development among the pupils. Furthermore the aims and objectives of the SMP O-level course was intended for pupils in Britain from the top 25 per cent of the ability range (Thwaites 1972). In other words the intention of the SMP which is to provide a course for the high ability students in Britain may not be suitable for the entire student population in Malaysia.

2. Pedagogical problems

Since the Malaysian educational system is highly centralized, the involvement of teachers in the development of new curricula was often negligible. The teachers generally felt that the new curriculum is imposed upon them by the central agency and they were expected to familiarise with the content, language and notation of modern mathematics and to become leaders “encouraging active participation, discussion,...” and to emphasise “understanding of concepts rather than rote learning” (DBP 1975).

It is generally acknowledged that not all mathematics teachers were involved in some kind of formal in-service course for teaching modern mathematics. Furthermore many of the older generation of teachers had never been exposed or experienced themselves with the proper training of modern curricula. Virtually all the heads of mathematics departments were exposed in some details of the implementation aspects of the curricula in the in-service course, but how the policy was to be understood by the teachers in schools will depend mainly on the way the heads of departments disseminate information about the curriculum in their respective schools (Maimunah *et al.* 1991). It is often the case that teachers did not have enough confidence or experience to implement the teaching methods as required by the proposed curricula. Some of the evidence of this aspect will be given in section 7.3.

Another major constraint of implementing the curriculum is due to the problems of the heavily examination-oriented system of Malaysian education. The general tendency of teachers is to complete the syllabus and to prepare students for public examinations. It is not surprising, therefore, to see many teachers often choose to emphasize the mastery of computational or manipulative skills rather than to stress the understanding of the mathematical concepts. Furthermore, as we can see, for instance, the examination questions on the function concept prepared by the Examination Syndicate for many years place very little emphasis on understanding or comprehension of the concept.

3. Psychological problems

One of the basic assumptions made in the adaptation of Western science and mathematics curricula by the developing countries is that the curriculum is equally suitable for all pupils irrespective of the cultural and educational background of the country involved. This was clearly stated by one of the science advisers from the Scottish Education Department, A.W. Jeffrey who was from 1966-67, attached to the Malaysian Ministry of Education that the curriculum "... equally suitable for Perth Scotland, Perth Australia, Perth Ontario, Parit Buntar in Malaysia" (p.254). The rationale is that what is good for developed countries ought to be good for developing countries. However this is a questionable premise.

Perhaps the most misleading comparison of whatever "Perth" with Parit Buntar is due to the fact that Parit Buntar is a very small town in a remote area of northern Peninsular Malaysia with obvious disadvantages in terms of social, cultural, technological and educational backgrounds as compared to any "Perth".

By tradition, teaching and learning approaches in Malaysia have always been teacher-oriented with the student passively receiving knowledge. A strong psychological resistance from the teachers and students to the discovery and inquiry approach of the adopted curricula was recently reported by Tan (1991). Long accustomed to the traditional teacher - student relationship often based on the authoritarian style normally ran counter to the inquiry approach and more democratic style of the adopted curricula.

6.3 Recent developments of mathematics curriculum

It is clear that there are some great unforeseen problems associated with the adaptation of a foreign mathematics curriculum as the implementation stage progresses. In 1979 a report of Cabinet Committee to review the implementation of the National Education Policy was published with one of the important recommendations to restructure the entire school curriculum so that it reflected a Malaysian identity and background.

Consequently the Integrated Secondary School Curriculum (Kurikulum Bersepadu Sekolah Menengah or KBSM) was designed and ready for implementation by 1989.

This new curriculum has been designed by local curriculum planners and educationists who presumably are more aware of the local issues and aspirations of the people who will be using it. At the time of writing, the KBSM has only reached the third year (Form Three) level and there is very little to say about its effectiveness at the moment. However as has been written by many mathematics educators in recent years, unless the design of mathematics curricula is based on the psychological or cognitive approach of the learner, the crucial problems of mathematics education still remain to be solved. This study will provide some limited evidence of the psychological aspect of learning mathematics especially with specific reference to the function concept.

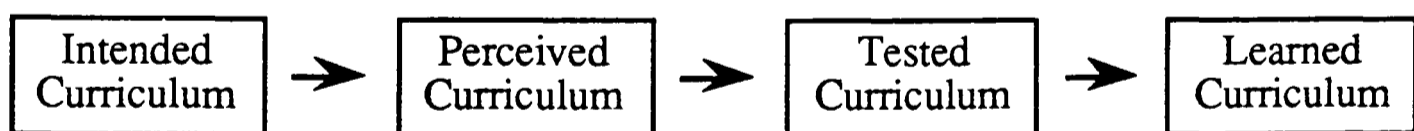
CHAPTER 7

THE STUDY IN MALAYSIA

7.1 Plan for main study in Malaysia

From our preliminary investigation, we hypothesize that students (in this case a group of English A-level students) develop prototypes for the function concept in much the same way as they develop prototypes for other concepts. Investigations which later involve a sample of first year university students also reveal significant misconceptions concerning the function concept. We then hypothesise that this would conflict with the intended curriculum given in terms of a formal definition.

This preliminary study prompted me to proceed to the second stage, that is, the main study, conducted in two secondary schools in England and then planning for the study in Malaysia. An extension of the study in Malaysia focused on the function concept as intended, perceived, tested and learned by the students.



The *intended* curriculum is reflected in official mathematics syllabuses, curriculum guides, and textbooks published, or approved, by the Ministry of Education. The *perceived* curriculum concerns how the teacher interprets and eventually puts into practice the intended curriculum. The *tested* curriculum focuses on how the intended curriculum is translated into the assessment procedures. The *learned* curriculum provides evidence on what students have learned as measured by tests and questionnaires.

The study in Malaysia:

By the middle of April 1990, the sponsor, UTM, had been informed and asked for permission to conduct the study in Malaysia. At the same time, permission was sought from the Ministry of Education to conduct the research in secondary schools.

After obtaining the approval from the ministry (see approval letter, appendix 4), the permission was then sought again from the Departments Education of three states involved in the study. These three states, Kedah (Northern Malaysia), Selangor (Central Malaysia) and Johor (Southern Malaysia) were chosen in order to make the samples of students, teachers and schools large enough and representative, so that a more meaningful generalization of the results can be obtained. The location of schools (rural and urban), types of schools (boarding and public), teachers' background (qualifications and experiences) were considered in the selection of sample (see list of schools participated in the study, appendix 6).

The Conduct of the Study

The conduct of this research proceeded in three stages:

7.1.1 Analysis of the intended curriculum

The information and materials for the first part of the study were acquired through two major sources, namely, consulting the primary resources (mathematics syllabus, textbooks, teachers' guides, past year examination questions, report on in-service courses) and undertaking interviews with three Ministry curriculum officers.

Detailed analysis of the primary resources (see 7.2), gave me a very clear idea of the official curriculum as prescribed by the ministry particularly on the teaching of function concept. These two initial tasks served as an important guide for conducting the second stage of the study, that is, the visiting of teachers and schools.

7.1.2 Conducting the study in secondary schools

All the headmasters and head of mathematics departments in the schools participating in the study were approached. The purposes and methods of investigation to be used were explained in detail.

In order to test the validity of the questionnaire, it was first given to a group of 45 students in A. B. Yassin Secondary School in Johor Bahru, a week before the actual study at other schools. At the same time the questions to be posed in the interview with teachers were tested with two senior mathematics teachers at that school.

In order to complement the answers given in the questionnaire, a few students in every school were selected for a short interviews. Similarly every head of mathematics department was interviewed to complement their answers given in the questionnaire. Some of the interviews with the teachers were tape recorded while others were not. This inconsistency seemed to be unavoidable because some teachers were rather uneasy about expressing their views freely in front of a tape recorder. In either case, I noted down as far as possible all the important points which were raised by the teachers, some of which were not given in detail in the questionnaire.

In all these tasks the information and data obtained were very useful as the data for the perceived curriculum and the learned curriculum (see full discussion in sections 7.3 and 7.5 respectively).

In these two stages, I did notice that this study was very much facilitated by the nature of my job with the University of Technology Malaysia, a university which is involved in the training of mathematics teachers in Malaysia. My relationships with quite a number of ministry officials through seminars, discussions and various meetings particularly on curriculum matters (as I am also one of the co-authors of a mathematics textbook for secondary level; *Matematik KBSM Tingkatan 1*, DBP, 1989) and the fact

that some of the teachers involved in this study were my former students at the University of Technology Malaysia certainly facilitated this research to a great extent.

7.1.3 Conducting study at the University of Technology Malaysia

A similar study as conducted at Warwick University during the pilot study were extended to UTM with a sample of students following the Bachelor of Science with Education (majoring in computer in education). The results of this group as compared to the Warwick's group are discussed in section 7.5.

7.2 THE INTENDED CURRICULUM

7.2.1 General background of the mathematics syllabus and textbooks in Malaysian secondary schools

After the decision by the Ministry of Education to reform the mathematics education in secondary schools in Malaysia in 1970 (see CDC report 1974 for details of events led to the introduction of Modern Mathematics into Malaysian secondary schools, p.7), the Scottish Mathematics Group (SMG) books were adapted for use as the "Modern Mathematics for Malaysian Schools" in form 1, 2 and 3 (lower secondary level, age 12+) and the School Mathematics Project (SMP) materials for form 4 and 5 (upper secondary level, age 15+) pupils.

The fundamental reason for the choice of SMG materials for lower secondary level was that they were not radically different from traditional mathematics previously used in Malaysia (CDC 1974, p. 8). In fact, only about one-third of the topics in the SMG syllabus were new, the other two-thirds are still traditional. This was planned so that the change over to Modern Mathematics did not lead to many serious problems in the future.

The choice of the SMP over SMG at the upper secondary level was because the SMP had a new series of books meant mainly for the comprehensive schools which are rather similar to the Malaysian secondary schools (CDC, 1974). Furthermore the SMP had a complete range of texts up to form 6 which would be useful should the modern mathematics programme be extended to that level. A series of two books under the title “*Modern Mathematics for Malaysia*” for upper secondary schools was mainly an adaptation from the SMP Books X, Y, and Z. These books are published in Bahasa Malaysia and in English by the Ministry of Education:

Hisab Moden untuk Malaysia Tingkatan 4

Modern Mathematics for Malaysia Form 4

Hisab Moden untuk Malaysia Tingkatan 5

Modern Mathematics for Malaysia Form 5

Accompanying each text-book is a Teachers’ Guide which contains, for each chapter, aims, notes, methods of teaching, solutions and hints to the problems. These books are distributed free to all form 4 and 5 mathematics teachers in Malaysia. Beside the above series, other books on Modern Mathematics have been approved by the Text-Book Bureau of the Ministry of Education for use in secondary schools includes:

Modern Mathematics, Books 1, 2, 3, 4, and 5

by Cheah *et al.* 1973

Pupils who took Modern Mathematics in form 4 and 5 eventually took the Alternative C (Modern Mathematics) papers set by the Cambridge Examination Syndicate for the Malaysian Certificate of Education (MCE) at 16+, which is equivalent to GCE Ordinary Level in Britain.

7.2.2 An analysis of the curriculum materials with special reference to the function concept

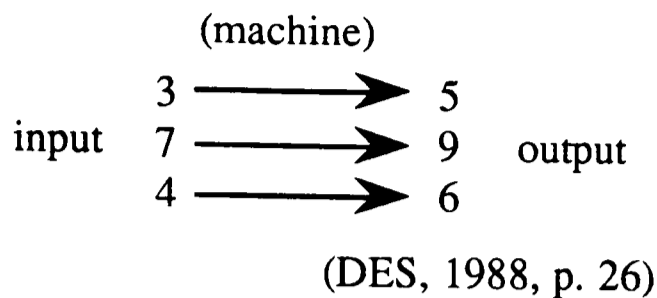
One of the most discernible features in a modern mathematics curriculum is the attempt to organise the mathematical contents into an integrated body of knowledge on the basis of some fundamental and unifying concepts. These central unifying concepts usually include the basic notions of sets, relations and functions which are sometimes identified as the “integrating themes” that relate various topics in mathematics.

Some readers including the author himself may remember that during the pre-Modern Mathematics era, the concept of function was formally taught at the university level. Nowadays, the concept of function (implicitly or explicitly) is dominant throughout secondary mathematics and is stressed throughout the work in algebra, trigonometry, calculus and transformation geometry.

Perhaps this was the main influence of the concept of the “spiral curriculum” suggested by Bruner (1963) that the basic and central ideas of any subject should be revised continually, broadening and deepening as the learning proceeds. Essentially, Bruner proposed that “our schools may be wasting precious years by postponing the teaching of many important subjects on the ground that they are too difficult ... the foundations of any subject may be taught to anybody at any stage in some form” (Bruner 1963, p. 12). This spiral curriculum was justified on two reasons. Firstly, it provided a structure for the subject and secondly, as a result, learning was made easier (CDC, 1974 p. 3).

In form two (age 13+), once the language of sets has been taught, the idea of function is introduced from the notion of relation as a set of ordered pairs, illustrated by arrow diagrams. The concept of function is then presented as a special kind of relation. The use of the mapping arrow (\rightarrow) helps to establish the idea of one set of numbers generating another set of numbers. This approach also has been used in the new National Curriculum in the UK to introduce the concept of function to the primary school pupils (level 3, algebra target 2). For example pupils were asked to describe

what is happening to the left hand numbers (inputs) to get the numbers on the right hand side (outputs) and then give the answer to “What is the function?”



The method of introducing the function concept as a function machine is very popular in many school textbooks even to the higher level such as in the new SMP-module for sixth forms in Britain.

The way of representing functions (or synonymously regarded as mappings) of the form $x \rightarrow f(x)$ and the idea of one to one correspondence and simple Cartesian graphs is treated in the second chapter of the textbook “*Modern Mathematics for Malaysian Schools*” (Cheah *et al.* 1973, 2nd Edition).

In Book 3 (for form 3, age 14+), the concept of logarithm (Chapter 8) is then introduced through the idea of one to one correspondence with some explanation regarding the functional property of logarithm. In the following chapter on trigonometric functions of sine, cosine and tangent, after the graphs of the three functions in the range of 0° to 90° have been shown, then the attention is drawn to the functional property that for each angle θ , there is one and only one corresponding value for each $\sin \theta$, $\cos \theta$ and $\tan \theta$. The functional property of sine and cosine has been emphasised in the statement “We say that the sine and cosine functions (mappings) map the set of angle sizes to the set of real numbers” (ibid, p.110).

The topic of transformation geometry is introduced in various stages during form 2 and 3. The idea of reflection appears in chapter 12, Book 2; rotation in chapter 21, Book 2 and translation in chapter 2, Book 3. In these chapters, the words such as mappings and images are used quite frequently in order to show the explicit relation between any of the geometrical transformations to the generic idea of function. In other words, the

later work in form 2 and 3 provide further reinforcement of the concept of function developed earlier in the chapter Relations, Mappings and Graphs (Chapter 2, Book 2).

The way the notions of set, relation and function are developed throughout the secondary mathematics reflect the official ideal of the curriculum developers that in the modern mathematics course “there is a conscious effort to teach the nature of mathematics by the inclusion of structure and pattern that occur in different branches of mathematics” (Khan, 1975, p. 7). It is also the intention of the curriculum developers that the use of the set language and the notions of relation and function will provide a foundation for more elaborate teaching of mathematical structure in later years. In the CDC report (1974) there is a clear statement that “Modern Mathematics also provides a good foundation for pupils who will pursue a mathematical or a mathematical-based course in higher education” (p.4).

The “spiral” approach as a way of progressing and reinforcing the concept of function in differing and more sophisticated problems is highlighted in the upper secondary curriculum (form 4 and 5). In Book 4, chapter 5, the topic on function continues with special reference to the detailed investigations of various kind of relations and functions, the differences between functions and relations and their common properties, the emphasis on the meaning of domain and range and the function notation $f:A \rightarrow B$, the formalization of the definition of function, the identification of examples and non-examples of function and the various representations of function.

The formal definition of function concept adopted in the text is,

“A function (mapping) is an assignment between two sets, A and B, in which each element of A is assigned a unique element in B”(*Modern Mathematics for Malaysia*, Book 4, p. 79).

Since modern mathematics emphasizes precise language and exact expression, it is thought that the function concept should be introduced via the language of sets (Khan

1975). In Book 5, the study of functions concentrates on the graphical representation of simple linear and quadratic functions, composite functions and the inverses of functions. The main emphasis is for students to be able to plot Cartesian coordinates to represent simple function mappings such as $y = x+1$, $y = 1/x$, $y = x^2$, $y = mx$ and $y = ax^2 + bx + c$ and know the form and properties of such graphs (*Modern Mathematics for Malaysia*, Book 5, Chapter 4 and 5).

Another important feature in the form five work on function is the need for students to understand and interpret properties and behaviour of a function which describes real life situations such as travel graphs, conversion graphs, growth and decay graphs and be able to interpret the meaning of the area under a curve for the axes that represent certain quantities. It is also clear that the links between algebraic equations, graphs and tables of values is constantly being reinforced.

The study of functions and graphs continues in the sixth form with the main aim is to show the interrelationships between algebra and geometry and to study the geometrical properties of particular functions (see MPM mathematics syllabus, 1980). Emphasis is placed on graph sketching and the knowledge of the effect of simple transformations as represented by $y = af(x)$, $y = f(ax)$, $y = f(x)+a$, $y = f(x-a)$. In relation to this, the study by Dreyfus and Eisenberg (1987) found that students have considerable difficulties relating the algebra of transformations (such shifts $f(x) \rightarrow f(x)+k$, $f(x) \rightarrow f(x+k)$ and stretches $f(x) \rightarrow kf(x)$, $f(x) \rightarrow f(kx)$) to their corresponding graphical representations. Of these, the transformations $f(x) \rightarrow f(x+k)$ and $f(x) \rightarrow f(kx)$ naturally proved to be the most difficult.

A dominant feature at this level is the development of rigorous skills of finding the inverses of functions and work on the compositions of functions such as finding the domain and range of fg . The use of more technical terms and notations such as codomain, mapped into, onto is also clearly evident.

At the university level, particularly in the UTM, the idea of function is included in a wider introductory course for first year Calculus and the applications and properties of various kind of functions is being stressed as the students proceed on to the second and third year course. Usually the courses put a strong emphasis on using concepts and techniques to solve problems in wider contexts. The main concern is the development of general processes and strategies so that the students be able to apply their knowledge in a more flexible and confident manner (UTM prospectus – Mathematics syllabus for BSc Ed course, 1988).

7.2.3 Some comments on the Intended curriculum with specific reference to the function concept

As we have seen the sequence of development in the Modern Mathematics curriculum especially in Malaysia progressed through the notion of “set” as the basic idea, “relations” between sets and “functions” as special types of relation. This is perhaps the usual logical progression of the subject as advocated by many mathematicians (as appears in many mathematics textbooks, for example see the book “Sets, Relations and Functions” by James F.Gray, 1962). However, this axiomatic approach has been questioned by a few recent mathematicians such as Skemp (1971) as “... It teaches mathematical thought, not mathematical thinking.” Tall (1986) suggested “... it is an excellent mathematical foundation, it may not be a good cognitive root.” and Sierpinska (1988) pointed out “... it is a didactical error – an antididactical inversion .”

In traditional mathematics the function concept is generally related to the notion of dependence, describing how one thing depends on another. In other words, the function concept is an abstraction from many real life situations in which two changing quantities are related. In the introduction to the function concept as appears in the textbooks from form 2 to form 5, there is hardly any strong evidence that this viewpoint is adopted. Most of the examples and exercises given to the students are

mainly limited to numerical examples and finding the value of $f(x)$ for certain values of x from a given function f (see for example, Book 4, p. 87).

In other topics such as matrices, transformation geometry and trigonometry, the function concept which underlies many of its methods has not been highlighted. It seems that the function concept has been treated in isolation from other topics; for example in the use of matrix methods for solving simultaneous linear equations, it is not seen as a kind of geometrical transformation or function in two dimensional space which transform the points of the domain into specific points of the codomain

$$\begin{array}{l} ax + by = c \\ dx + ey = f \end{array} \longrightarrow \begin{pmatrix} a & b \\ d & e \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} c \\ f \end{pmatrix}$$

7.3 THE PERCEIVED CURRICULUM

In the previous section, we have discussed and analysed the content and objectives of the modern mathematics curriculum in lower and upper secondary schools in Malaysia as proclaimed by the Ministry of Education. In this chapter the discussion will be focused on the way in which content and objectives with respect to the teaching of the function concept are understood or perceived by the teachers and how these might differ from the intentions of the curriculum as discussed earlier.

The following analysis gives a summary of responses based on the questionnaire given to a group of 80 secondary mathematics teachers from 24 secondary schools throughout Peninsular Malaysia; 16 schools from the state of Johor, 5 schools from the state of Selangor and 3 schools from the state of Kedah which represented the Southern, Central and Northern part of Malaysia respectively. The questionnaire is supplemented by the interviews with 11 senior mathematics teachers (mostly are the Head of Departments) in the selected schools within the sample.

The teachers' responses are discussed under the following headings :

1. The function concept in the school curriculum
2. Teaching approaches
3. Algorithmic skills versus conceptual understanding
4. Effects of public examination
5. Difficulties in teaching the function concept

7.3.1 The function concept in the school curriculum

The first question posed to the teachers is “In modern mathematics the concept of function has been introduced at the very early stage in school curriculum . What do you think the *most* important reason for this ?”

The purpose of this question is to examine teachers’ perceptions of the role of the function concept in the modern mathematics curriculum and from here to identify the differences between the role of function concept as specified in the intended curriculum and that perceived by the teachers. The details of teachers’ responses to the above question which we consider reflect the role of the function concept as perceived by teachers can be classified as follows:

The most important reason given	% of responses
a) basic concept in mathematics	29
b) applicable in other topics	44
c) related to everyday life	5
d) preparation for higher studies	11
e) did not agree with the early introduction of the function concept in the school curriculum	3
f) importance because the Minister of Education says so – I am not involved in the planning of the curriculum	1
g) did not have any comment	6

Table 7.1

A significant percentage of teachers (29%) seemed to regard the function concept as a basic concept in the mathematics curriculum and another 44% were concerned with the role of function concept in other topics. This implicitly suggests it has a unifying role, but such a role was never explicitly mentioned in their responses.

None of them considered the specific role of function concept as a unifying theme integrating various branches of mathematics curriculum as specified clearly in the intended curriculum (see CDC report 1974, p. 3).

Two respondents who did not agree with the early introduction of the function concept in the school curriculum gave the reason that it is a difficult and abstract concept for lower secondary pupils to grasp.

7.3.2 Teaching approaches

It is generally agreed that whatever the curriculum, its effectiveness depends upon the knowledge and skills of the teachers who implement it. However for a given syllabus, different teachers may teach it in a variety of ways. This is perhaps one of the reasons why the curriculum developers especially in the centralized education system such as Malaysia thought that detailing the mathematics curriculum in clear behavioural terms is very important (see the preface in every chapter in teachers' guides, CDC 1975). Teachers are expected to follow closely the prescribed syllabus and textbooks prepared by the Ministry or some other recommended textbooks approved by the Ministry (such as Cheah *et al.* 1975, Tan W.S. 1979).

In question 2, we asked teachers to give their preference of teaching approaches and the reasons for choosing the particular approach in their teaching "Please indicate your preference in the teaching of the function concept and give your reasons:

- a) teaching should begin by giving the formal definition of function and then followed by the examples and non-examples

- b) teaching should begin by introducing some examples (especially related to real life problems) which precede the formal definition.”

The teachers’ responses are as follows :

preference of teaching approaches	% responses
approach (a)	31
approach (b)	69

Table 7.2

From the results obtained, the majority of teachers in the sample (69%) preferred the second approach of teaching which emphasises the importance of some real life examples related to function concept before the formal definition is given. This basic philosophy of modern mathematics teaching preferred by the teachers to some extent reflects the understanding of teachers of the “guided discovery ” approach as envisaged by the intended curriculum.

However there is quite a substantial percentage of teachers (31%) who did not follow the recommended teaching approach that

“the work in the classroom must be related to other fields of knowledge to the pupils immediate experiences and to other activities in which he is engaged”

(DBP, 3rd Edition 1975 p. 1)

The following quotations indicate some of the reasons given by the teachers for preferring a specific approach in their teaching:

Among the reasons for preferring approach (a) are as follows;

“A mathematical concept can be understood through its definition. Examples can be compared with the definition and students can remember them.”

“Teaching can be more systematic and students can easily understand.”

“For students to understand it fully, the concept must start from its foundation.”

“Because it is an abstract concept .”

“After the students understand the definition of the concept, they will be able to differentiate the examples or non examples of the concept.”

“It is only through the understanding of the concept that further learning can be enhanced.”

Among the reasons given for preferring approach (b) were the following:

“To make the learning more attractive, because it relates with the students’ everyday experiences.”

“By giving the definition, students will not be able to understand and it may confuse them.”

“It is better to teach initially with the concrete examples before proceeding to the abstract definition of the concept.”

“Because the examples can help students understand the concept.”

“Usually the students do not have an interest in the mathematical definition.”

7.3.3 Algorithmic skills versus conceptual understanding

In question 3, teachers were asked about the main emphasis in their teaching of the function concept.

- a) emphasize the formation and understanding of the function concept, for instance, at the end of the lesson, students should be able to

describe the meaning or give the correct definition and also should be able to differentiate the examples of functions from non examples of functions.

- b) emphasize the mastery of algorithmic and problem solving skills related to the function concept .

The results obtained are as follows :

preference of teaching approaches	% responses
approach (a)	30
approach (b)	55
both (a) and (b)	15

Table 7.3

The official document (CDC 1974, p.3) states “In modern mathematics the stress is on the understanding of concepts and their applications to different situations rather than on manipulative skills”.

Although the intended curriculum emphasises conceptual understanding, in the perceived curriculum, only 45% of the teachers follow this approach.

Most of the teachers who give a strong emphasis on the mastery of algorithmic skills mentioned clearly that the main reason for their choice is due to the requirement of preparing students for the MCE examination. However in the interviews, one senior mathematics teacher puts it to me that “we can change the emphasis from algorithmic skills to conceptual understanding provided that the examination questions reflect the need for conceptual understanding” (personal communication, 20/8/90).

Among the reasons given by the teachers for emphasising method (a) are as follows

“If the students can understand the concept, it is more likely that they can do the manipulations.”

“It is better to understand the concept first. This can help the students to solve various problems including non-routine problems. Furthermore they can understand the method used.”

“to make the students more critical and analytical in their thinking”

Among the reasons given for emphasising method (b) includes:

“manipulative skills are more important than the understanding of concept.”

“because MCE examination place more stress on the algorithmic skills rather than the understanding of concept.”

“to satisfy the need of the parents in getting their children to pass the examination.”

7.3.4 The effects of public examination

In question 4, we asked for teachers’ opinions on whether or not they think that

“If the student can answer correctly the following questions involving the function concept, then can we say that the student has understood the concept of function ? Please gives your reasons.”

a) Given $f(x) = c/x^2$ and $f(2) = 5$, find the value of i) c ii) $f(5)$

(MCE, 1978, Q. 20, paper 1)

b) Given $f(p) = p^2 - 3p + 4$, find the values of p such that $f(p) = 4$

(MCE, 1985, Q.1, paper 2)

The results obtained are as follows:

	% responses
Yes	40
No	60

Table 7.4

Among the reasons given by the teachers who said “yes” includes :

“If they do not understand it, its difficult for them to solve such a problem.”

“Because if they can solve the problems, then they have acquired the concept of domain and range.”

“Because they know that for every value of x they obtain only one value of $f(x)$.”

On the other hand those who said “no” gives the reasons such as:

“It only involves algebraic manipulations and not the definition of function.”

“These questions did not require the understanding of the concept.”

“they only solve it mechanically without being able to understand the concept.”

Almost all questions on the function concept previously appeared in the MCE examination are similar to the above questions used in the questionnaire (see a sample of MCE questions given in section 7.4 on the tested curriculum). From the results obtained, it is clear that quite a substantial percentage of teachers (60%) did not agree that the questions on function concept require any conceptual understanding on part of the students.

In other words *the examination questions did not reflect the aims and objectives of the intended curriculum which strongly stress the understanding of mathematical concepts.*

7.3.5 Difficulties in teaching the function concept

In question 5, we asked about the difficulties or problems experienced by the teachers in teaching of function concept in schools. The responses given can be classified as follows :

	% responses
It is an abstract concept	50
difficult to apply the concept in other problems	13
students' low ability and weak in mathematics	16
no problem/ no comment	21

Table 7.5

It is clear from the results obtained in this study that a significant proportion of teachers (50%) consider the function concept as too abstract and difficult for students in secondary schools to understand. This argument has been highlighted in many recent research studies on the problems and difficulties in learning function concept at various levels (see for example Even 1988, Breidenbach *et al.* (1990), Vinner *et al.* 1989).

7.4 THE TESTED CURRICULUM

The term “tested curriculum” is used here to denote the knowledge and understanding which were assessed in the public examination such as in the MCE examination in Malaysia. The main objective in this chapter is to examine closely “to what extent the questions particularly related the function concept in the MCE examination reflected the aims and objectives of the intended curriculum?” The discussion will be based mainly on the analysis of the previous examination questions on the function concept as appeared in paper 1 and paper 2 of the examination.

7.4.1 An analysis of examination questions

The following questions on the function concept were selected from a sample of the previous MCE examinations as appeared in paper 1 and paper 2 from 1972 to 1986.

Given that $f(x) = (x-1)^2(x+3) - 30$, find the values of (a) $f(3)$ (b) $f(4)$

(c) Estimate the value of x for $f(x) = 0$.

(MCE 1972, paper 1, no. 30)

$f(x)$ is defined by $f(x) = bx^2 + cx$, b and c are constants.

Given that $f(1) = 8$ and $f(2) = 22$, calculate the values of b and c .

Hence, solve the equation $f(x) = 2$.

(MCE 1972, paper 2, no. 1)

Given $f(x) = 2x^2 - 11x + k$ and $f(3) = 0$, find the value of k and $f(-1)$

(MCE 1974, paper 1, no. 12)

Given $f(x) = (2x - 3)(x + k)$ and $f(3) = 30$, find the value of k . Hence find the values of x for which $f(x) = 0$.

(MCE 1976, paper 2, no. 2)

Given $f(x) = c/x^2$ and $f(2) = 5$, find the value of (a) c (b) $f(5)$

(MCE 1978, paper 1, no. 20)

Given the function $f(x) = px^3 + 2$ where p is a constant and $f(1) = 0$, find $f(1/2)$

(MCE 1980, paper 2, no. 1)

Function $f(x) = tx^2 - kx$ where $f(1) = 3$ and $f(-1) = 5$. Find the value of t and k , and hence find $f(x)/x$

(MCE 1982, paper 2, no. 1)

Given that the function $f(x) = px^2 + p^2x - p$ and $f(2) = 5$, find the values of p

(MCE 1985, paper 2, no. 1)

Given that $f(p) = p^2 - 3p + 4$, calculate the values of p for which $f(p) = 4$

(MCE 1986, paper 1, no. 23)

From a sample of questions given above, it is clear that the questions on the function concept are very popular, appeared every year in both paper 1 and paper 2 of the examination. For students who can detect the pattern of questions, it is fairly easy to speculate the type and nature of questions on the function concept in the examination. There are at least two main features about the nature of questions on the function concept prevalent in the MCE examination from 1972 to 1986:

- (i) Most of the questions on the function concept are related to the knowledge and skills of solving linear or quadratic equations.
- (ii) Although the questions in paper 1 are normally shorter than questions in paper 2, the nature of the questions is remarkably similar.

Students also can easily notice that for many years (1972, 1980, 1982, 1985), questions on functions were the first to be asked in the paper 2 of the examination.

The strong connection between the function concept and the process of solving algebraic equations probably contributes to a significant percentage of the students' responses which regard "function as an equation" or "function as an expression" (see next section). Another important fact about the nature of the questions on function in the MCE examination is that there is no apparent link between the work on function with the graph of a function.

This is also perhaps one of the reasons that the students' conceptions of function were not linked to any kind of graph of a function as indicated in table 7.6 (next section). The close relation between the function concept and the graph of a function is reflected in the sequence of topics as appeared in the intended curriculum (the topic on functions

is followed by the topic on graphs in form 5, but the questions on functions are totally separated from the question on graphs in the examination.

There is a clear tendency to repeat the same types of questions every year in the examination. This trend might not reveal many positive indications about the level or quality of teaching and learning in schools. It perhaps shows that the subject is static and has not kept pace or is not in tune with the latest developments of mathematics as portrayed in the intended curriculum.

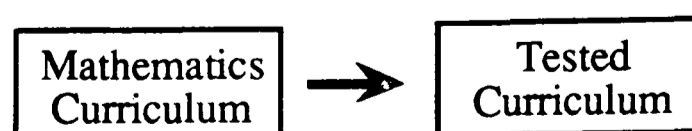
From the sample of previous questions given above, it gives strong evidence that most of the questions on the function concept were only focused on assessing students' computational skills or instrumental understanding (in the sense of Skemp) rather than on students' conceptual understanding or relational understanding of the function concept.

The repetition of routine function problems is not in accordance with the statement in the official syllabus (CDC 1975) that,

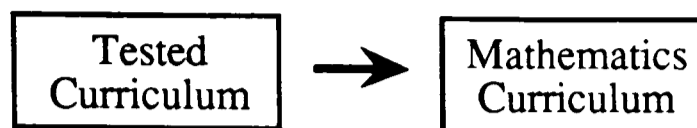
"... emphasis should be placed on real understanding rather than on knowledge of, or endless training in the techniques of mathematical operations." (p.1)

7.4.2 Influence of examination on the teaching and learning

In an ideal situation, the logical way or the guiding principle of curriculum development is to decide what to assess would be to begin with what we want the students to learn. In other words examinations should follow, not determine, the curriculum; after the intended curriculum has been established, then the examination or the tested curriculum as we referred to it here, should be designed to test what had been envisaged in the intended curriculum.



In reality, however, at least in the Malaysian situation, we find the reverse process is occurring. The assessment already exists and is clearly implied by the examination questions. This is more likely to exert a strong influence on the curriculum.



In the education system which is mainly dominated by examinations, the content of the examination will strongly dictate what needs to be taught and learned in schools. The evidence of the teachers' responses as discussed in the previous chapter indicates that in many cases, the teachers were more concerned with the preparation of the students to pass the examination especially the terminal examination such as the MCE.

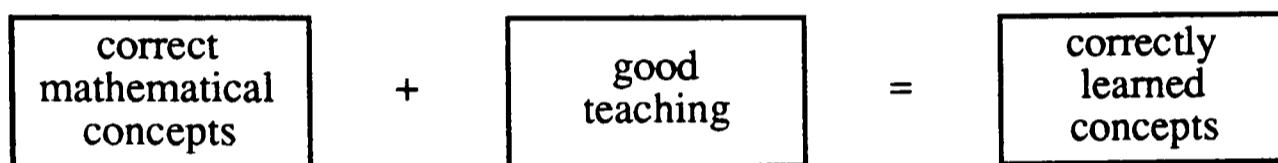
As indicated in the previous chapter, a substantial percentage (55%) of teachers in the sample of study do not seem to follow the recommended guidelines as envisaged by the Ministry but many of them view examinations as their main priority. They are moulding their curriculum or teaching to fit what the MCE examination tests — “teaching to test”. The past years' questions on the function concept did not require one to understand the meaning of the concept. As one senior mathematics teacher claims that “Although the function concept has been taught since the age of 14 (form 2), many students do only the mechanical part of the concept without gaining an insight into the concept of function itself” (personal communication, 18/8/90). Moreover the function concept has been presented as an isolated concept and skills to be mastered on its own which contradict the notion of mathematics as a unified subject which should be taught as an integrated whole (see CDC 1975).

It is also widely felt that the teachers are under strong pressure to produce the highest possible marks for mathematics in the examination as the results will be published and compared among schools in the country. Since the examination results have high status, it is used to compare students' relative performance in a norm referenced manner, teachers are therefore generally obliged to parents as stated by the teacher “to

satisfy the need of the parents in getting their children to pass the examination” (see section 7.3.3).

7.5 THE LEARNED CURRICULUM

As teachers, perhaps we sometimes concur with the belief “I teach, therefore the students learns” as we might hope and suppose. Each “chunk” of mathematical knowledge is given to the student in a neat logical structure. It is an approach that has often been the basis from which the intended curriculum is translated into the actual classroom practices. In this way, teaching and learning are often seen as reciprocal processes. Teaching is assumed to be followed automatically by learning. If the teacher has taught the class about “function”, then the class has learnt it.



It is known that this is not so.

In this chapter we will present some empirical evidence that what a student learns is not necessarily a mirror image of that being presented by the teacher. Even with many undergraduate students, the last box in the diagram (correctly learnt concepts) may not follow. Despite being exposed to the teaching of the function concept from the age of 12 or 13 years, many students at all levels still did not grasp this basic concept in the way intended. Instead what we would call the “prototypical conceptions” formed by the students at the lower secondary stage were not being developed further at their later stages. And a high percentage of the final year undergraduate students lost completely the essence of the concept ; perhaps “when you never use, you may lose.”

7.5.1 Prototypical conceptions of the function concept

The main purpose of the questionnaire used in this study (see appendix) was to examine some of the conceptions that secondary schools and undergraduate students in Malaysia have for the concept of function. The questionnaire was administered, in Malay, to a total of 408 secondary schools students in the Peninsular Malaysia and a group of 184 undergraduate students at the UTM.

Before the subjects were asked: “Can you explain in a sentence or so what you think a function is ? If you can give a definition of a function then please do so”, they were asked to solve five questions on function previously appeared in the MCE examinations. The questions were selected at random from the 1974 to 1986 examinations consisting some from paper 1 and some from paper 2. Questions appeared in paper 2 intended to be slightly harder than questions in paper 1 (see section 7.4). These questions were used as a warming up exercise and for developing their confidence and to get a “feel” on what a function is.

On the basis of an analysis of 100 randomly chosen transcripts, we can firmly conclude that virtually all students from form 4 to the undergraduate satisfactorily answered all these five questions :

1. Given $f(x) = 2x^2 - 11x + k$ and $f(3) = 0$, find the value of k and hence $f(-1)$
(MCE 1974, paper 1, no 12)
2. Given $f(x) = c/x^2$ and $f(2) = 5$, find the value of i) c ii) $f(5)$
(MCE 1978, paper 1, no 20)
3. Given a function $f(x) = px^3 + 2$ where p is a constant and $f(1) = 0$, find $f(1/2)$
(MCE 1980, paper 2, no 1)
4. Given a function $f(x) = (p + x^2)/x$ and $f(2) = 5$, find the value of p . Hence find the value of x , other than 2 such that $f(x) = 5$
(MCE 1984, paper 2, no 2)

5. Given $f(p) = p^2 - 3p + 4$, find the values of p such that $f(p)=4$.

(MCE 1986, paper 1, no 23)

After analysing 100 transcripts as mentioned above, I found out that since more than 95% of the students in the sample answered these questions correctly, I then decided to continue analysing question 6. My action is also based on the findings of the Ministry's Examination Report published yearly which normally gave a favourable comments on students performance in answering questions on functions. (See for example, Laporan Prestasi SPM (SPM/MCE Performance Report) 1978, 1980 and 1984).

Question 6 was designed specifically to examine the students' prototypical conceptions of the function concept. As has been mentioned in the earlier chapter (see section 7.2 on the intended curriculum), the secondary schools students had studied the notion of a function since the beginning of their form two (age 13) and they continue to use functions especially in the calculus throughout the A-level and followed through the university level. The formal definition in terms of set theory emphasising the technical aspects of domain, range and rule was given in form 4 (age 15/16).

The prototypical conceptions may be categorized into two main categories: static and dynamic conceptions (similar to categories used in the pilot study in chapter 4) as shown in the following table:

Approximate age at year's end	Secondary school pupils				University students		
	15	16	17	18	19	20	21
Level of education	F4	F5	F6(L)	F6(U)	Y1	Y2	Y3
Number of subjects	123	118	87	80	58	49	77

Concept prototype	% of responses						
Static							
equation	7	14	25	22	5	16	0
formula	0	2	2	1	0	0	1
expression	10	8	24	24	46	14	0
rule	35	24	21	13	10	20	1
Dynamic							
process/operation	4	8	9	20	2	4	1
none of the above	14	11	9	4	10	22	6
don't know	6	2	1	1	10	4	5
no response	24	31	9	15	17	22	87

Table 7.6 : Distribution of students' prototypical conceptions of function

An analysis of the transcripts revealed the following significant evidence:

- *None* of the respondents gave a satisfactory definition of the function in a form similar to that given to them earlier by their teachers or textbooks. Students seem to have many bits and pieces of information about the function concept, but they lacked a meaningful and coherent view of it.
- of those who responded, *all* of them gave explanations which reflected personal prototypical conceptions of functions.
- 87% of the final year students failed to give any response.

There is a clear difference between Malaysian subjects and their English counterparts in terms of the types of prototypical image of the function concept. Malaysian students seemed to hold strong prototypical images of a function as an equation, expression, or rule (in the 'static' category). On the other hand English students generally had a strong prototype of function as a process or operation (in the 'dynamic category'). This perhaps due to the different approaches in introducing the function concept in schools in England and Malaysia. Some examples of the answers for each category are as follows:

Formula :

“a *formula* with varying quantities”

“a *formula* showing a relation between two types of variables, for example x and y ”

Equation :

“an *equation* which can be substitute by any values of x in order to get corresponding values of $f(x)$ ”

“function is an *equation* which can be solved”

“function is an *equation* consisting of x and $f(x)$ ”

“an *equation* with variables “

“a mathematical *equation* with $f(x)$ and x ”

“*equation* of the form $y = mx + c$ ”

“*equation* relating one variable with another variable, such as $f(x)=x^2$ ”

“an *equation* of the form $y= ax^2 +bx + c$ or others”

Expression :

“function is an *expression* showing the relationships between variables”

“an *expression* with some unknown values”

“an *expression* of the form $f(x)=kx$ where k is a constant and x is a variables”

“it is a *relation* between x and y ”

“is a mathematical *expression* consisting of variable x , for example $f(x)=x^2 +1$ ”

rule :

“is *rule* which maps a value x to $f(x)$ ”

“function is a *mapping* of one value x with one value of y ”

Process/Operation :

“function is a useful *operation* where by a number can be substitute to obtain an answer.”

“a *procedure* which transform an input number to an output number”

An examination of the responses given by the students at all levels in Malaysia indicate that prototypical conceptions of a function evoked by these students were in many cases reflected its similarity with the prototypical conceptions (such as function as a formula, equation, expression, rule, and process or operation) developed by students in our study in the UK (see chapter 4, and 5).

One of the striking results obtained in this study which caused great concern to the mathematics educators especially at the UTM is the fact that 87% of year 3, the undergraduate students, gave no response at all and 5% more responded that they don't know. This group of students, however, ~~were~~ cooperated fully in answering the questionnaire as clearly evident from their answers of the earlier five questions on the 'procedural' skills.

In the current practice of teaching and examination, these prototypical conceptions, in most cases, are unnoticed by the teachers and the students. For those at the university level, the function concept is used implicitly in terms of its application in other branches of mathematics or in physical science subjects. This situation may contribute to the fact that most of the students especially in the third year are completely unable to describe what the function is.

7.5.2 What is and is not a function

This questionnaire (see appendix 3) given to Malaysian students is an amended version of the first (used in the pilot study in England). This was necessary in order to reflect the overall contexts in which the function concept had been taught in Malaysian schools (see section 7.2 on the intended curriculum).

The responses to question 7 are given in table 7.7.

However as stated earlier in chapter 4 and 5, some of the total percentages do not add up to 100% partly because of rounding errors and also due to a small number of non-responses. The response which is more likely to be adjudged correct is given in bold face type. The results which have direct relevant to the hypothesis of this study will be highlighted.

	Form 4		Form 5		Form 6L		Form 6U		Year 1		Year 2		Year 3	
	N=123		N=118		N=87		N=80		N=58		N=49		N=77	
	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No	Yes	No
7(a)	90	9	96	4	99	1	98	1	97	0	98	0	36	1
7(b)	84	16	86	13	97	3	91	5	95	2	92	6	32	5
7(c)	77	20	83	14	93	7	81	19	88	7	86	10	32	1
7(d)	50	46	63	31	55	45	70	23	9	86	16	80	8	25
7(e)	29	61	34	54	33	62	10	85	74	22	76	18	19	10
7(f)	26	63	32	56	16	75	5	86	14	78	10	86	10	17
7(g)	48	38	53	32	73	27	75	18	91	3	82	6	23	5
7(h)	18	63	15	68	13	84	51	44	8	80	7	78	10	82
7(i)	50	33	54	31	51	44	56	26	10	79	12	51	13	12
7(j)	28	54	33	38	40	55	53	28	7	79	10	76	6	14
7(k)	37	35	39	35	62	32	56	23	70	19	76	6	12	3
7(l)	28	42	45	28	55	31	58	23	60	26	39	41	6	8

Table 7.7

The responses to question 7(e) show that a substantial percentage especially among secondary school students said that constant function $y=2$ is not a function due to the following reasons,

"no, because it did not involve x"

"no, because it is not an equation"

"graph of a function should be a curve"

"because value of y or f(x) is constant"

"because it contain only one value"

"no opposite variable"

"because not a quadratic equation"

“ did not give a relation between x and y”

“ because it has only one variable”

“ f(x) is constant”

“ x varies without any effect on y”

This data consistently shows that students in both samples in England and Malaysia, at least for those involved in this study, had a strong prototypical conception that a function must be a formula or an equation involving x and y . The university students, especially of the first year and second year performed better on this question. However the conclusion cannot be generalized for the third year group, since a large percentage of them did not respond to this question – indeed to the other question as well.

We have evidence to show that some of the correct answers given to question 7(f) were actually produced by incorrect prototypical conceptions of function such as,

“no, because there is no y to link with x”

“no, because it is a straight line”

“no, because x didn't change”

The responses to question 7(j) about whether a circle is a function exhibit many interesting reasons, such as,

Yes, because

“values of x and y changes in certain way”

“it is of the form $f(x)=x^2+y^2$ ”

“it is a graph of a function”

“ it can be written as $y^2= 1-x^2$ “

“ because values of x and y are related”

“ x and y varies in such a way satisfy this equation”

“ show a relation of two variables “

From the analysis of responses, it is clear that students at various levels developed a limited conception and possess a strong inflexible prototype of function as a formula or as an algebraic expression consisting x and y . Students at the university level did not seem to develop more flexible prototypes as they gained more exposure to the notion of function in their course.

CHAPTER 8

CONCLUSIONS, DISCUSSION AND RECOMMENDATIONS

8.1 Introduction

As stated earlier, this study surveyed and analysed data relating to four aspects of the mathematics curriculum, namely the intended curriculum, the perceived curriculum, the tested curriculum and the learned curriculum with specific reference to the teaching and learning of the function concept in Malaysia. The main aim of the study is to evaluate all major components of a curriculum in order to understand the impact and the effectiveness of the modern mathematics curriculum which has been in progress during the last two decades.

The data used in this study were collected in Malaysia during the academic year 1989–90. The sample consisted of 80 secondary mathematics teachers and 408 pupils from 24 secondary schools throughout the Peninsular Malaysia. A total of 184 undergraduate students at the UTM also participated in the study.

8.2 Summary of major findings

The major findings of the study may be summarized in four separate sections as follows:

8.2.1 Results pertaining to the analysis of the intended curriculum

- i) The Modern Mathematics curriculum in Malaysia was mainly based on the work of the Scottish Mathematics Group (SMG) and the School Mathematics Project (SMP). The Scottish and SMP texts are being translated into Bahasa Malaysia with very little adaptation other than at the superficial level of place-names and currencies. This is perhaps the

easiest and quickest way to set curriculum change or reform in motion. It is also may be the most practical way of up-dating the curriculum in a situation with limited resources and experience in such work.

- ii) The “top down” approach to curriculum development in Malaysia has contributed to the feeling of “non-ownership” of the mathematics curriculum by teachers. This is also one of the results of the study on curriculum implementation in Malaysia few years ago by Zanzali (1987).
- iii) The curriculum is designed logically to build on the function concept as a general foundation of other areas of the mathematics curriculum. However this conflicts with the cognitive development of the students and their development of ‘prototypical’ conceptions.

8.2.2 Results pertaining to the perceived curriculum

The following results on the teachers’ interpretation on the teaching of the concept of function are of interest :

- i) In introducing the concept of function, nearly 70% of the teachers in the sample tend to emphasize the procedural skills rather than the conceptual understanding as envisaged by the intended curriculum.
- ii) The importance of the mastery of algorithmic and procedural skills related to the function concept is mainly in accordance with the perception of a large percentage (55%) of teachers who considered their main task is to prepare students for the success in the public examination.

8.2.3 Results associated with the data on the tested curriculum

The most important feature obtained from the analysis of the previous MCE examination papers is that the types of questions related to the function concept do not adequately test the students understanding of the function concept in a broader sense but rather a mechanistic or procedural aspect of the concept. In other words there is no consistency between the intended curriculum and the tested curriculum. Both are going in their own different directions.

8.2.4 Results related to the learned curriculum

The important findings are as follows:

- i) This research evidence suggests that for one mathematical concept such as for the function concept, students across ages developed what we would call the “idiosyncratic personal prototypical conceptions” during the secondary and tertiary education.
- ii) A large percentage (87%) of third year university students were unable to provide any explanation to the meaning of the function concept. This clearly shows that students have not necessarily understood the meaning of the concept after so many years of study.

8.3 Discussion

From this study, we found four sets of results regarding the teaching and learning of the function concept in secondary schools in Malaysia. The following discussion focuses on some selected results of the major findings listed in the previous section.

8.3.1 Question concerning the suitability of an adaptation of foreign curricula to the Malaysian situation.

The results revealed in this study points to the fact that there is a clear existence of serious mismatch between all four components of a curriculum (intended curriculum, perceived curriculum, tested curriculum and learned curriculum) in Malaysia. This revelation leads one to question the roots of the problems. In my opinion it must be closely related to the intended curriculum which in reality is on 'top' of these four curriculum levels.

The basic question is "Does a curriculum innovation based on ideas developed elsewhere (SMP and SMG) fit well cross-culturally with typical levels of resourcing, teacher competencies, pupils' styles of learning and cultural contexts in Malaysia?" Other pertinent questions need to be addressed includes "Is the level of cognitive development among the British secondary pupils compatible with that of Malaysian secondary pupils?" "Is the learning by inquiry or discovery approach as advocated by the developers of the SMG and SMP really suitable to the Malaysian situation?". With regards to the discovery learning, Orton (1987) claimed that "No research evidence was available which conclusively proved that discovery learning was superior to expository learning in terms of long-term learning gains" (p.86).

Zanzali (1987) has also proposed the importance of research study to determine the suitability of the inquiry and discovery method in the modern mathematics programmes in Malaysia. Wilson (1981) in the discussion of mathematics education in its cultural contact reported that in some developing countries, the open-ended investigations are not encouraged by the teachers due to the cultural reasons such as the belief that the children should not think for themselves, especially the girls (p. 110).

From the written record available leading to the adaptation of the SMG and SMP syllabuses, there was hardly any research evidence that those questions were being addressed by curriculum developers in the Ministry. Looking into the historical

evidence of the mathematics curriculum development in Malaysia, we can see that mathematics is often thought of as a universal subject; a definition and example used in the UK is also used in Malaysia. There is a serious set-back, if this universality of mathematics fails to extend to the teaching and learning of mathematics which normally takes place within a particular social and cultural context.

In recent years there has been a growing awareness of the importance of the cultural context on the teaching and learning of mathematics (e.g. Bishop 1985, D'Ambrosio 1986). The importance of having a meaningful background to the problems and examples used in mathematics teaching has been emphasized in several of such research studies. Furthermore the background knowledge including the appropriate level of language used in the textbooks should be suitable for the pupils in a particular country.

It is generally agreed that the modern mathematics curriculum apparently involves a high degree of accuracy and precision in its presentation. It also requires a higher reading ability and the mastery of many technical terms and concepts. In the initial stage of translation from the adapted materials, it seems that the term "function" has been translated to at least three Malay terms; "rangkap", "pemetaan" and "fungsi". These terms have been used by teachers and in textbooks for some time until quite recently only the word "fungsi" being accepted as the right translation for the word "function" (KBSM 1989). The problem is also further compounded by the poor or incorrect translations made by some translators. For example the statement in one of the textbooks (Balraj 1970) "While every mapping is a relation, not every relation is a mapping" was translated as "Walaupun tiap-tiap pemetaan itu adalah suatu perkaitan, akan tetapi tiap-tiap perkaitan bukanlah pemetaan" which if translated back in English would mean "while every mapping is a relation, every relation is not a mapping." (see further discussion in Liew 1977).

8.3.2 Questions related to the social and cultural norms

We have said before that the principles and practices of mathematics teaching and learning should not be separated from the social and cultural context. It is well known, in more general terms, that authoritarian classroom methods are normally common practice in Malaysia where the general society itself is authoritarian. New methods of teaching mathematics which includes inquiry–discovery approach, investigations and discussions usually do not readily flourish in the majority of classrooms. Students normally do not speak until they are spoken to, and in many instances they are expected to answer, but not to ask questions.

Reflecting on this scenario, it is very difficult to see how the approach that is recommended to the study of mathematics “Pupils should be encouraged whenever possible to carry out their own investigations to discover for themselves techniques and results. . . ” (DBP 1975) can easily be achieved.

8.3.3 Comments concerning the perceived curriculum

In Malaysia, as we have said earlier, all mathematics teachers operate under the same centralized education system in which a uniform intended curriculum is practised all over the country. Furthermore the teachers use the same textbooks, either the ones produced or approved by the Ministry of Education as the main teaching materials and source of ideas. However as we have seen, this does not mean that teachers interpret or perceive the intentions of the curriculum in the same way. Some educators would simply assume that “It is well known that teachers throughout the world do not slavishly (or even unslavishly) follow their national curriculum. What is intended by those who draw up national curricula is never implemented in all classrooms. ” (Howson 1991).

But in my opinion, it is absolutely essential for those who are concerned with the improvement of mathematics education should not take that assumption for granted but

the most important thing is to find out the underlying reasons for such discrepancy. On a small scale, this study has provided some “insight” into this aspect at least relevant to the Malaysian scenario.

8.3.4 Question related to the examination procedures

The evidence so far concerning the nature of the examination questions points to the fact that pupils’ performances are not being assessed in relation to the curriculum aims and objectives. As we have seen the questions on functions were mainly test procedural skills and techniques rather than the concept itself or any open ended problem-solving questions. In this sense we can say that the aims and objectives of the intended curriculum were not being translated into the examination objectives.

It also can be argued that there is no point in having a curriculum reform such as in modern mathematics in terms of aims and objectives if there is an examination system that ignores them or imposes quite different objectives on teachers as well as pupils.

It did not motivate teachers and pupils to work towards the achievement of those aims and objectives stipulated in the intended curriculum. The fact of the matter is that if we are not prepared to reform the examination system than we are also unable to reform the educational process itself.

8.3.5 Comments concerning the learned curriculum

As teachers or mathematics educators we are usually concerned as to whether this new curriculum, when taught in actual classrooms, achieves the intended results. This last component, the curriculum as learned by students is very important since it represents the “end-product” of the intended curriculum. In this study we have tried to find out what mathematical knowledge and skills concerning the concept of function the students have learned and to what extent.

In general terms, we found that what is learned by students at various levels bears little relation to the intended curriculum. It is also evident that much that is taught is not retained especially for students at the higher level.

This study therefore brings some new evidence on students' conceptions to complement what has already been noticed by several researchers, especially those who believed in the constructivist view of learning that place students' conceptions in a profoundly central position when studying the conceptual development of mathematical concepts (e.g. Glasersfeld, 1981, Steffe, 1988). Although in the preliminary stage, the present study extended some of other researchers works (Vinner, 1983; Barnes, 1988), and we are certainly not the first to point out the importance of recognizing students' conceptions and utilizing them to promote better teaching. However, the results of the pilot study (including classroom observations) certainly provide some new 'insight' to the teaching and learning of functions in schools. Furthermore the extension of the study in Malaysia has been conducted in an entirely new dimension as compared to the previous research. The focus of the study is not only on students' conceptions or difficulties, but more importantly, it is seen in relation with other aspects of the curriculum.

Specifically in this study, we claimed that our main contribution was in exposing serious misconception about the *functional relation* (functions are those in which y is defined as an explicit expression of x) and the *function concept* developed by students both in the UK and Malaysia. As a result, this points to the necessity of re-examining the present process of teaching and learning of functions at all levels.

It is also hoped that this study will help university lecturers, school teachers and others concerned with the teaching of mathematics in general to be aware of the nature of their students' conceptions; whatever the reasons underlying these conceptions might be, attention may be directed as to how they could be improved.

8.4 Implications of the study and some recommendations for the improvements of mathematics education in Malaysia

The effectiveness of any curriculum innovation and its implementation should be assessed and informed by systematic research findings. The intention of this study was to analyse and describe four aspects of the curriculum simultaneously;

- 1) The expected or official content; the *intended* curriculum,
- 2) The teachers' priorities in teaching; the *perceived* curriculum,
- 3) The examination materials; the *tested* curriculum,
- 4) The students understanding of the mathematical concept; the *learned* curriculum,

with specific reference to the function concept.

The findings of this study have several implications for mathematics education in Malaysia. Some of the most important implications together with suggestions for the improvement of mathematics education in general, are as follows:

- 1) The lack of congruity between these four components of the curriculum might be significantly improved if the mismatch between the nature of assessment and the goals of the curriculum as revealed in this study is properly taken into consideration by the curriculum developers and the examiners. This is because the style of examination questions has more direct impact on what goes on in the classrooms than any other aspect of the curriculum. The problems of coordination need to be resolved between the curriculum developers in the Curriculum Development Centre (CDC) who designed the curriculum and the examiners in the Examination Syndicate who are responsible for setting the examination questions.

- 2) The main feature of curriculum development in Malaysia is that the mode of decision-making is highly centralised. The intended curriculum is determined at the national level; the official syllabuses are gazetted and every school is expected to follow them (Singh. T, 1981). In other words, the innovation was initiated by the Ministry and detailed framework for implementation was also prepared centrally and it was then handed over to the teachers to be realized in the classroom.

As we have seen, at least two major problems surfaced during the implementation of the new curriculum; first, the lack of commitment and a sense of “ownership” of the curriculum by the teachers. Secondly, the lack of understanding of the philosophy and demands of the new curriculum such as the unfamiliarity of the role of the function concept as an “integrating theme” in modern mathematics and also the difficulties of accommodating an inquiry–discovery approach as advocated for its teaching and learning.

While it is desirable to consider teacher participation in curriculum development, in reality it is impossible to involve all mathematics teachers across the country in designing the curriculum. It may be more realistic if representative participation among teachers is considered in the planning of the curriculum.

- 3) The present study has revealed the deep conceptual problems related to the function concept which most student experience. This outcome of learning or what we have called the learned curriculum has rarely been assessed by the public examination such as the MCE examination which is summative in nature and designed to measure mostly the skills or competencies a student has mastered. This situation may be improved if some part of the examination is designed as formative assessment which can be used during the course to identify the

difficulties experienced by the students and to provide them with appropriate guidance.

Furthermore in the long term, the process of curriculum development should be based on a genuine understanding of the nature and problems of the learning process which involves the understanding of the students' cognitive development and difficulties. This challenge has been put forward by Tall (1986) when he wrote "The curriculum builder and teacher must be aware of the cognitive obstacles that may occur and cause cognitive conflict when the context is broadened and the learner moves to the next stage."

Although the philosophical and conceptual aspects of modern mathematics programmes may be acceptable but in the final analysis, the implementation and the students' learning which matters most.

- 4) This study also provides some indication of the need of integration between the curriculum, pedagogical and evaluation aspects. The current practice in Malaysia whereby the Curriculum Development Centre mainly prepare the curriculum specifications, Examination Syndicate concentrates on the assessments and Teachers Training Division, Colleges and Universities train the teachers should be looked into. In fact, some scholars in Malaysia voice their concerns of a lack of coordination between curriculum planners and implementers in the different division of the Ministry (Fatimah, NST 30/3/87).

It is hoped that this study not only provides yet another set of data and information regarding the mathematics programmes in Malaysia but it has sensitized into a more concrete proposal which will enable the theory and practise, ideal and reality meet in unity. Because we believe effort in improving the curriculum without a specific empirical evidence is ill-spent.

8.5 Limitations of the study

Among the limitations identified in this study were:

- 1) The absence of the researcher in the actual classroom teaching and learning of the function concept in the Malaysian classrooms was one of the weaknesses of this study. However, it is our belief that whatever teachers' philosophy as written down on the questionnaires and expressed in the interview is likely to be put into practice in their actual teaching in the classroom.
- 2) As a highly subjective mode of enquiry, particularly the interview aspect, the bias inherent in the interpretation was appreciated and thus its findings were considered with special caution. The responses included in this thesis are only those given in very clear terms and statements whose validity can be tested by other researchers. This problem compounded further by almost all responses being given in Malay, requiring translation by the researcher.
- 3) The limited number of teachers selected randomly in this study may not represent the views and philosophy of the entire teachers population in Malaysia. Furthermore, it could be argued that the teaching of one concept may not represent the general approaches or styles of teaching other concepts. The function concept, however, has a special nature – as a subtle and a good logical foundation rather than a good cognitive root for cognitive development.

8.6 Suggestions for further research

The process of curriculum implementation follows a clear progression from an intended curriculum to a learned curriculum. Since this study represents only a small scale

research involving one topic in the curriculum, there is much room for further research in this area. Because of this limitation, it is improper to make generalised statements concerning its results. However this research may set the beginning for further serious investigations involving different topics and larger samples of teachers and students throughout Malaysia. Further research on the 'use' of functions in other subjects may also be conducted. Students' conceptions of function especially in physics, where most of its concepts can be expressed as functions (Omasta *et al.*, 1988) might be of special interest.

The cross-age nature of this study may limit the type of conclusions that can be drawn from the data. A longitudinal study would provide greater insight into the students' conceptual development and their conceptual obstacles. Further investigations involving analyses of the effect of teaching on shifts in student conceptions are also necessary.

In the light of a very wide spectrum of curriculum continuum from planning and designing the intended curriculum through to the curriculum as learned by the learner, the research in this area may focus on various aspects which will contribute to the total picture of the mathematics curriculum. For example, the research may concentrate on an analysis of the numerous problems associated with putting curriculum plans into action, such as the question of teachers' preparedness, the suitability of imported curricula, evaluation of curriculum materials, conceptual demand of the intended curriculum, cognitive development of the students and so on.

Another related area of research is that of specifying an intended curriculum in such a way that it could be implemented without neglecting the opportunities for creativity and initiative by teachers and students (Cockcroft 1982). This surely needs a comprehensive and coordinated efforts from many mathematics educators in the country.

REFERENCES

- Asiah A.S., 1983: Perkembangan Kurikulum Matematik Sekolah Di Malaysia Sejak Zaman Penjajah (Curriculum Development of School Mathematics in Malaysia since the Colonial Times), *Berita Matematik*, pp.2–14.
- Ausubel D.P., 1968: *Educational Psychology: A Cognitive View*. New York: Holt, Rhinehart and Winston.
- Balacheff N., 1990: "Future Perspectives for Research in the Psychology of Mathematics Education", *Mathematics and Cognition*, ICMI Study Series, Cambridge University Press, pp.135–148.
- Bakar M.N. and Tall D.O. 1991: "Students' Mental Prototypes for Functions and Graphs", *Proceeding of PME–XV*, Assisi, Italy, Vol 1, pp.104–111.
- Balraj A.T, Cheah T.H, Tan B.T, and Yusuf M.D. 1970: *Modern Mathematics for Malaysian Schools*. Books 1–3.Pilot Edition. Oxford University Press. Kuala Lumpur.
- Barnes M., 1988: "Understanding the Function Concept: Some Results of Interviews with Secondary and Tertiary Students", *Research on Mathematics Education in Australia*, 24–33.
- Bell E.T., 1937: *Men of Mathematics*. New York: Simon and Schuster.
- Bell E.T., 1945: *The development of Mathematics*, 2nd Ed. London
- Berneveld G.V., and Krabbendam H.(Eds), 1982: *Conference on Functions Report 1*. Enschede, The Netherlands.
- Bishop A., 1985: "The social construction of meaning – a significant development for mathematics education?", *For the Learning of Mathematics*, 5(1), pp. 24–28.
- Boyer C. B., 1946: *The history of calculus and its conceptual development*, New York: Dover Publications.
- Boyer C.B., 1946: Proportion, equation, function. Three steps in the development of a concept. *Scripta Mathematica*, vol 12. pp. 5–13.
- Boyer C.B., 1956. *History of Analytic Geometry*. New York.
- Breidenbach D., Dubinsky E., Hawks J., and Nichols D., 1990. "Development of the process concept of function", *Purdue Conference on Functions*.
- Bruner J.S., 1963: " Observations on the Learning of Mathematics." *Science Education News*, April.
- Bruner J.S., 1966: *The Process of Education* . Cambridge, Mass.: Harvard University Press.
- Bruner J.S., 1966: *Toward a Theory of Instruction..* Cambridge, Mass.: The Belknap Press of Harvard University Press.

- Bruner J.S., 1971: "The Process of Education Revisited." *Phi Delta Kappan* 53,pp. 18–21.
- Buck R.C., 1970: "Functions". In E.G. Begle (Ed.), *The sixty-ninth yearbook of the National Society for the Study of Education. Part I: Mathematics Education*. Chicago, IL: National Society for the Study of Education.
- Cambridge Conference on Teacher Training, 1967: *Goals for Mathematics Education for Elementary School Teachers*, Boston, p.98.
- Channon and Mc Leish Smith. 1959: *General Mathematics*. Book 4.
- Cheah T.H, Tan B.T., Khor, G.K., and Yusof M.N., 1973–75. *Modern Mathematics for Malaysian Schools*, Kuala Lumpur, Oxford University Press.
- Choquet G., 1963: *What is Modern Mathematics?* Reading: Educational Explorers Ltd. pp. 41–44.
- Cockroft W.H., 1982: *Mathematics Counts*. London: HMSO.
- D' Ambrosio 1986: Socio – cultural bases for mathematical education. *Proceedings of the fifth International Congress on Mathematical Education*, Boston.
- Davis C.1966: "A note on two basic forms of concepts and concept learning", *Journal of Psychology*, 62.
- Davis R.B. 1982: "Teaching the concept of function: methods and reasons". *Proceedings of the workshop on functions, organized by the Foundation for Curriculum Development.*, Enschede, The Netherlands.
- Davis R.B., 1984: *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*. Croom–Helm, London.
- Dienes Z.P., 1971: *Building up Mathematics* (Fourth Edition) London: Hutchinson Educational Ltd.
- Dieudonne J., 1969: *Linear Algebra and Geometry*. London, Kershaw Publishing Co.
- Dreyfus T. and Eisenberg T., 1982: "Intuitive functional concepts: a Baseline Study on Intuitions". *Journal for Research in Mathematical Education*, 6, 2, 18–24.
- Dreyfus T. and Eisenberg T., 1983: "The function concept in college students: linearity, smoothness and periodicity", *Focus on Learning Problems in Mathematics*, 5, pp. 119–132.
- Dreyfus T. and Vinner S., 1982: "Some aspects of the function concept in college students and junior high school teachers", *Proceedings of the Sixth International Conference for the Psychology of Mathematics Education*, Antwerp, 12–17.
- Dreyfus T. and Vinner S., 1989: "Images and Definitions for the Concept of Function". *Journal for Research in Mathematics Education*, 20(4), 356–366.
- Driver R. and Oldham V., 1986: "A constructivist approach to curriculum development", *Studies in Science Education*, 13, 105–122.
- Dubinsky E., Hawks J. and Nichols D., 1989: "Development of the process conception of function by Pre–service Teachers in a Discrete Mathematics Course". *Proceedings of PME–XIII*, Paris, Vol 1, pp. 291–298.

- Durell C.V., 1930: *New Algebra for Schools*. London: Bell.
- Durell C.V., 1936: *General Arithmetic for Schools*. London: Bell.
- Durell C.V., 1971: *Certificate Mathematics*, Vol 2. London: Bell.
- Duschl R.A., Hamilton R, and Grandy R.E, 1990: "Psychology and epistemology: match or mismatch when applied to science education ?" *Int.Journal of Science Education*, vol.12, no 3.
- Department of Education and Science (DES) 1989: *Mathematics in the National Curriculum*, HMSO, London.
- Engel A., 1979: *New trends in mathematics teaching*, Vol 4, prepared by the International Commission on Mathematical Instruction (ICMI), UNESCO.p. 255.
- Even R., 1988: "Prospective secondary mathematics teachers' knowledge and understanding about mathematical function", Unpublished Ph.D thesis, Michigan State University.
- Even R., 1990: "Subject matter knowledge for teaching and the case of functions". *Educational Studies in Mathematics*, 21, 521–544.
- Gagné R.M. 1970: *The Condition of Learning*. New York: Holt, Rinehart and Winston.
- Gilbert J.K. and Watts D.M., 1983: Concepts, misconceptions and alternative conceptions: changing perspectives in science education. *Studies in Science Education*. 10, pp.61–98.
- Hart K.M., 1981: *Children's Understanding of Mathematics 11–16*. CSMS Project. John Murry, London.
- Hellemens A. and Bryan Bunch 1988: *The Timetables of Science: A Chronology of the Most Important People and Events in the History of Science*. Sidgwick & Jackson. London.
- Hewson P.W., 1981: A conceptual change approach to learning science. *European Journal of Science Education*, 3, 383–396.
- Hirst K., 1980: "Functions, Formulae and Graphs". *Mathematics Teaching*, Sept.
- Howson G., 1991: *National Curricula in Mathematics*, The Mathematical Association, Southampton, Great Britain.
- Howson G. 1991: Mathematics Education: a historical view. *Impact of Science on Society*, 160, pp. 303–313.
- Janvier C., 1983: "Teaching the concept of function". *Mathematical Education for Teaching*, vol 4, no 2.
- Johnson –Laird P.N., 1988: *The Computer And The Mind: An Introduction To Cognitive Science*. Cambridge University Press.
- Khan W. 1974: Pelaksanaan Ilmu Hisab Moden di Semenanjung Malaysia – Peringkat Menengah Rendah. (Implementation of Modern Mathematics in Peninsular

- Malaysia – lower Secondary level). *Ministry of Education: Seminar Pelajaran Sains dan Hisab. Dis 1973*. Kuala Lumpur.
- Karplus R., 1979: "Continuous functions: students' viewpoint". *European Journal of Science Education*, vol 1. pp.379–415.
- Klausmeier J.H., Ghatala E.S., and Frayer D.A. 1974. *Conceptual learning and development – a cognitive view*. N.Y. Academic Press.
- Klein J., 1962: *Greek mathematical thought and the origin of algebra*. Cambridge, MA: MIT press.
- Klien M., 1968: *Mathematical Thought from Ancient to Modern Times*, Oxford University Press.
- Kleiner I., 1989: Evolution of the function concept: A brief survey. *The College Mathematics Journal*, 20(4), 282–300.
- Krause E.F., 1984: Functions and Equations in High School Algebra. *School Science and Mathematics*, 84(2), pp. 136–156.
- Lakoff G., 1987: *Women, fire and other dangerous things*. Chicago: The University of Chicago Press.
- Lovell K., 1966. *The Growth of Basic Mathematical and Scientific Concepts in Children*, London: University of London Press.
- Lovell K. 1971: Some aspects of the growth of the concept of function. In *Piagetian Cognitive Development Research and Mathematical Education*, N.C.T.M. Washington D.C. pp.12–33.
- Leinhardt G, Zaslavsky O, Stein M., 1990: Functions, Graphs, Graphing: Tasks, Learning and Teaching, *Review of Educational Research*, 60, 11–64.
- Liew S.T. 1977: A Critical Review of Malaysia's Modern Mathematics Programme at the Lower Secondary School Level, Unpublished M.A. Dissertation, University of Leeds.
- Malaysia, Curriculum Development Centre (CDC), 1974: *A brief guide to Modern Mathematics in Malaysian Secondary Schools*, Kuala Lumpur.
- Malik M.A. 1980: Historical and pedagogical aspects of the definition of function, *Int.J. Math.Ed. Sci. Technol.* 11, 4, 489-92.
- Mamona J.C. 1987: Students' Interpretation of Some Concepts of Mathematical Analysis. Unpublished Ph.D Thesis, University of Southampton.
- Modern Mathematics for Malaysia (Form 4 text and Teachers' Guide), Eastern University Press, 1973.
- Mathematics: *Syllabus for Secondary Schools*, Dewan Bahasa dan Pustaka (DBP), Kuala Lumpur, 1975.
- Maimunah S., and Keith M. Lewin, 1991: Curriculum Development in Malaysia. In *Curriculum Development in East Asia*, Colin Marsh and Paul Morris (Eds.), The Palmer Press, 1991, pp. 224–247.

Markovits Z., Eylon B. & Bruckheimer M., 1986: Functions today and yesterday, *For the Learning of Mathematics*, 6, 18–24.

Markovits Z., Eylon B. & Bruckheimer M., 1988: Difficulties Students have with the Function Concept, *The Ideas of Algebra*, K-12, N.C.T.M. 1988 Yearbook, 43-60.

Monna A.F. 1972/73: The Concept of Function in the 19th and 20th Centuries, in Particular with Regard to the Discussion Between Baire, Borel and Lebesgue, *Arch. Hist. Ex. Sci*, 9, 57–84.

National Council of Teachers of Mathematics (NCTM), 1989: *Curriculum and Evaluation Standards for School Mathematics*, Author, Reston VA.

New Straits Times (NST). *KBSR yet to live up to its promise*, 30/3/87, Kuala Lumpur.

Omasta E, and Lunetta V.N., 1988: Exploring Functions: A strategy for Teaching Physics Concepts and Problem– Solving. *Science Education*, 72, 5, pp. 625–636.

Orton A., 1970: A cross–sectional study of the development of the mathematical concept of a function in secondary schoolchildren of average and above average ability. Unpublished M.Ed. dissertation, University of Leeds.

Orton A., 1971: Teaching about functions in the secondary school, *Mathematics Teaching*, 57, 45–49.

Orton A., 1987: *Learning Mathematics: issues, theory and classroom practice*. Cassell education Ltd. London.

Organisation for European Economic Cooperation, 1961: *Synopses for Modern Secondary School Mathematics*. Paris, O.E.E.C. pp. 1–69.

Osborne R.J. and Wittrock M.C., 1983: Learning Science: A generative process. *Science Education*. 67, 4, 489– 508.

Piaget J., 1970: *Science of Education and the Psychology of the Child*. New York: Orion Press.

Piaget J., 1973: "Comments on Mathematical Education." In A.G. Howson (ed.), *Developments in Mathematical Education*, Cambridge University Press, London.

Piaget J., Grize J. B., Szeminska A. and Bang, V. (1977): *Epistemology and Psychology of Functions* (J. Castellanos & V. Anderson, trans.), Reidel, Dordrecht (original published 1968).

Posner, G.J., Strike K.A, Hewson P.W., and Gertzog W.A. 1982: "Accommodation of a scientific conception: Towards a theory of conceptual change", *Science Education*, 66, 211–227.

Read C.B., 1969: "The treatment of the concept of function." *School Science and mathematics*, vol 69, pp. 695–696.

Reys E.R., Marilyn N. Suydam, and Mary M. Lindquist, 1984: *Helping children learn mathematics*, Prentice Hall, Inc.

Romberg T.A. and Carpenter T.P., 1986: "Research on Teaching and Learning Mathematics: Two Disciplines of Scientific Inquiry". In *Handbook of Research on Teaching*, Wittrock M.C. (Ed.), 3rd Edition, Macmillan, New York

- Rosch E. and Mervis C.B., 1975.: "Family resemblances: Studies in the internal structure of categories". *Cognitive Psychology*, 7, 573–605.
- Rosch E., 1978: "Principles of Categorization". In E.Rosch and B.Lloyd (Eds), *Cognition and Categorization*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.
- Ruthing D., 1984: "Some Definitions of the Concept of Function from Bernoulli to Bourbaki," *Mathematics Intelligencer*, 6,4, 72–77.
- Schwarz B. & Bruckheimer M., 1988: "Representations of functions and analogies", *P.M.E. XII, Hungary*, 552–559.
- Scottish Mathematics Group (SMG), 1971–1975: *Modern Mathematics for Schools*, (Books 1–9), Chambers and Blackie.
- School Mathematics Project (SMP), 1969–1974: *Mathematics* (Books 1–7, A–H, X,Y,Z). Cambridge University Press.
- Sfard A., 1987: "Two conceptions of mathematical notions: Operational and structural". *Proceedings of PME– XI*, Montreal, Vol.3, pp. 162–169.
- Sfard A., 1989: "Transition from operational to structural conception: The notion of function revisited". *Proceedings of PME– XIII*, Paris, Vol.3, pp. 151–158.
- Shuard H, and Neill H., 1977: *From Graphs to Calculus. The Mathematics Curriculum*, Blackie.
- Sierpiska A., 1988: "Epistemological remarks on functions", *P.M.E. XII, Hungary*, 568–575.
- Singh T., 1981: "What mathematics is currently taught in Malaysian secondary Schools", *Proceedings of 2nd. South East Asia Conference of Mathematical Education*, pp.65–73.
- Smith E.E., 1988: "Concepts and thought". In *The Psychology of Human Thought*, editors Sternberg R.J and Smith E.E., Cambridge University Press, 19–49.
- Skemp R.R., 1971: *The Psychology of Learning Mathematics*, Pelican, London.
- Skemp R.R., 1976: "Relational understanding and instrumental understanding", *Mathematics Teaching*, 77, 22–26.
- Skemp R.R., 1979: *Intelligence, Learning and Action..* Wiley.
- Steffe L.P., Cobb P., and von Glasersfeld E., 1988: *Constructions of arithmetical meanings and strategies*. New York: Springer–Verlag.
- Swan M., 1982: "The teaching of functions and graphs". *Proceedings of the workshop on Functions*, The Netherlands.
- Tall D.O. and Vinner S., 1981: "Concept image and concept definition in mathematics with particular reference to limits and continuity", *Educational Studies in Mathematics*, 12, pp.151–169.
- Tall D.O. and Vinner S., 1981: "Concept Image and Concept Definition in Mathematics With Particular Reference to Limits and Continuity." *Educational Studies in Mathematics*, 81, pp. 151–169.

- Tall D.O., 1985: *Supergraph*, (software for BBC compatible computers), Glentop, London.
- Tall D.O., 1990: "The transition to advanced mathematical thinking: functions, limits, infinity and proof", *Handbook on Research in Mathematics Education*, N.C.T.M. (to appear)
- Tall D.O., and Bakar M.N.,: "Students' Mental Prototypes for Functions and Graphs. *International Journal of Mathematical Education in Science and Technology*". (to appear– accepted for publication 20/2/91).
- Tan S.B., 1991: "The development of secondary school science curriculum in Malaysia". *Science Education*, 75, 2, pp. 243–250.
- Tan W.S., 1979: *Mathematik Moden Menengah Terbaru* (Books 1–5). Tropical Press.
- Tennyson R. and Cocchiarella M.,1986: "An empirically based instructional design theory for teaching concepts". *Review of Educational Research*, Vol 56, pp. 40-71.
- Tennyson R.D., Younger J. and Suebsonthi P., 1983: "Concept learning by children using instructional presentation forms for prototype formation and classification – skill development". *Journal of Educational Psychology*, 75, 2, 280 – 291.
- Thorndike E.L., 1923: *The psychology of algebra*, New York: Macmillan Co.
- Thorndike E.L., 1932:*The Fundamentals of Learning*. New York: Teachers College Press.
- Thomas H.L., 1969: *An Analysis of Stages in the Attainment of a Concept of a Function*. Unpublished Ph.D Thesis. University of Columbia.
- Thomas H.L., 1975: "The Concept of Function". In Roskopf, M.F. (Ed.), *Children's mathematical concepts*, New York: Teachers College Press.
- Travers K.J. and Westbury I., 1989: *The IEA Study of Mathematics 1: Analysis of mathematics curricula*. Pergamon Press.
- Thwaites B., 1972: *The School Mathematics Project: The first ten years*. Cambridge University Press. p.164.
- Vinner S., 1983: "Concept definition, concept image and the notion of function", *The Int.J.Math.Ed.Sci.Technol.* 14, 293-305.
- Vinner S. and Dreyfus T., 1989 : "Images and definitions for the concept of function". *Journal for Research in Mathematics Education*, 20, 356–366.
- Von Glasersfeld E., 1981: "The concept of adaptation and viability in a radical constructivist theory of knowledge". In I.E. Sigel, D.M. Brodzinsky, and R.M. Golinkoff, *New Directions in Piagetian Theory and Practice*. Lawrence Erlbaum Associates, New Jersey, pp. 87–95.
- Watson F.R., 1976: *Developments in Mathematics Teaching*. London, Open Books.
- West B.H., Griesbach E.N., Taylor J.D. and Taylor L.T., 1982: *Encyclopaedia of Mathematics*. Prentice Hall.

Wilson B., 1981. "Mathematics Education in its Cultural Context". *Proceedings of 2nd S.E.A. Conference on Math. Educ.* Kuala Lumpur.

Wittgenstein 1953: *Philosophical Investigations*. New York.

Youschkevitch A.P., 1976: "The Concept of Function up to the Middle of the 19th Century". *Arch. Hist. Ex. Sci.*, 37–85.

Zanzali A.A., 1987: "The Malaysian Mathematics Program: A case study of the difference between design intention and classroom implementation", Unpublished Ph.D Dissertation, University of Wisconsin–Madison, USA.

FUNCTION QUESTIONNAIRE

Name:

School/College:

Thank you for participating in this study. This is not a mathematics test. It is simply an investigation to find out your understanding of the function concept and its related ideas. The results would be used only for the purpose of research and not in any way used to assess your achievement in school. Please answer ALL the questions as best as you can. You may show your working on this paper.

1. Describe what is happening to the left hand side numbers to get the numbers on the right hand side

$$\begin{array}{l} 1 \rightarrow 4 \\ 2 \rightarrow 5 \\ 3 \rightarrow 6 \end{array}$$

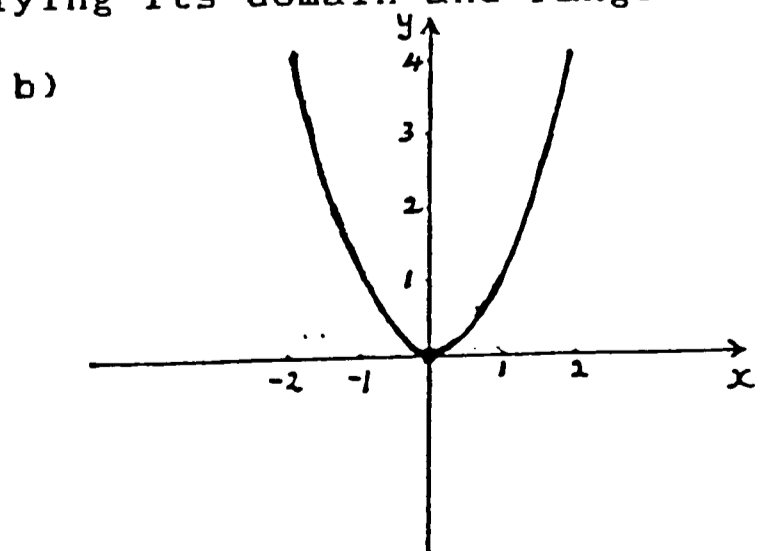
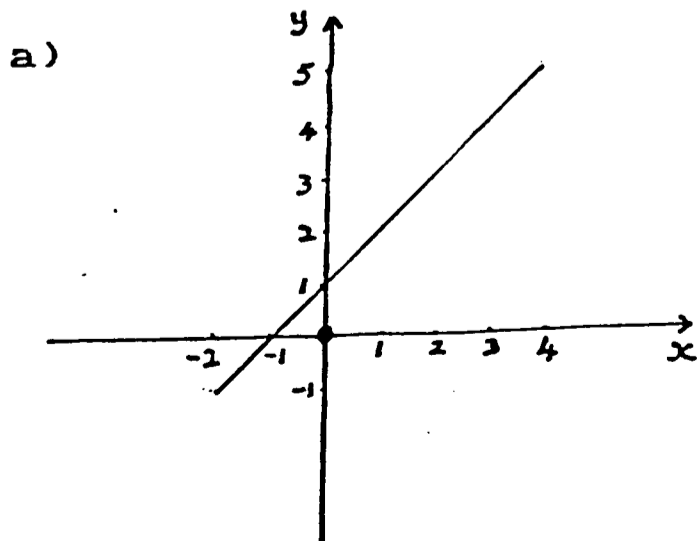
Ans:

2. Given that $f(x) = 2x + 3$, complete the following:

$$\begin{array}{lll} f(4) = \dots & f(1/2) = \dots & f(-7) = \dots \\ f(\dots) = 23 & f(\dots) = -15 & f(h) = \dots \end{array}$$

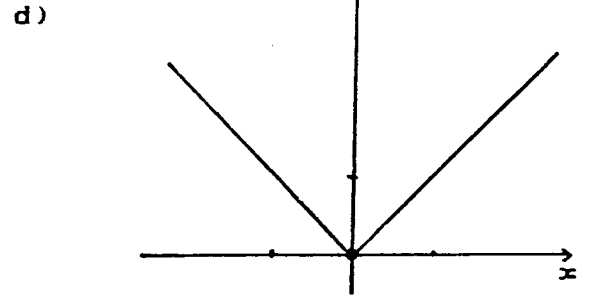
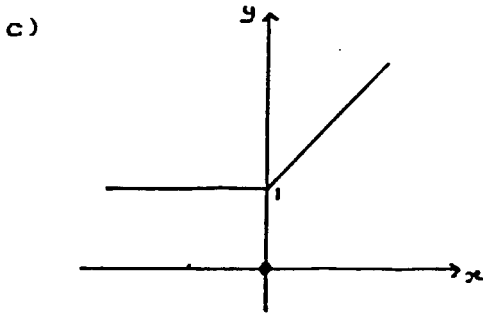
3. Say in your own words (if you can) what you think a mathematical function is:

4. Find an algebraic equation for the function shown in each of the following graphs. Specifying its domain and range:



equation:
domain :
range :

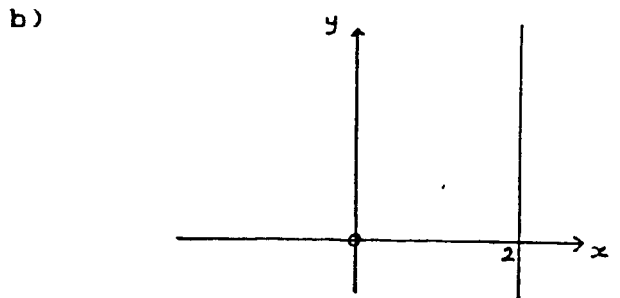
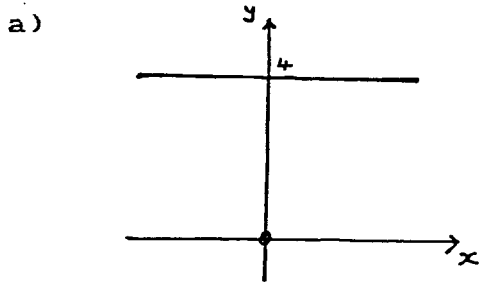
equation:
domain :
range :



equation:
domain :
range :

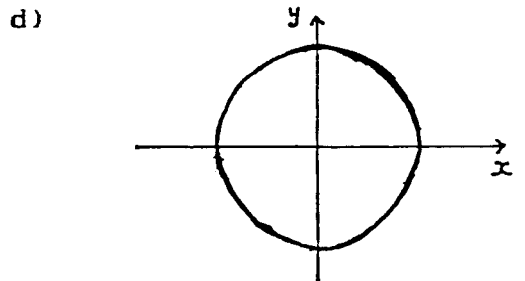
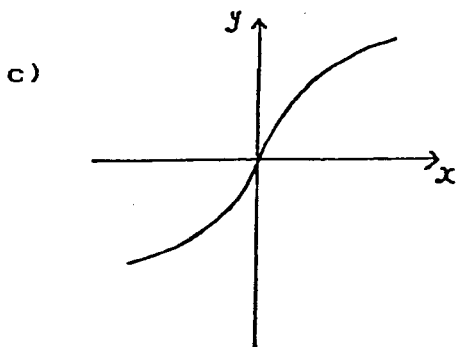
equation:
domain :
range :

5. Which of the following sketches could represent y as a function of x ? Underline Yes or No in each case and give a reason why.



Yes/No
reason:

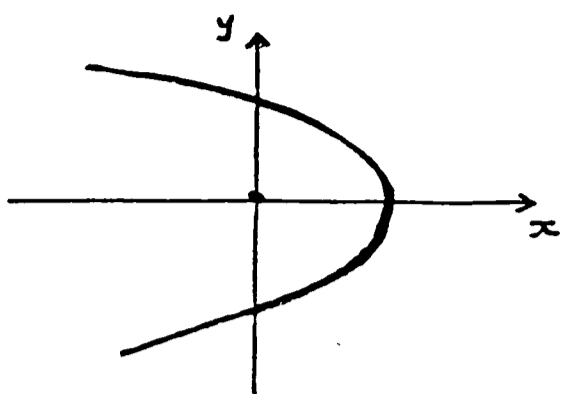
Yes/No
reason:



Yes/No
reason:

Yes/No
reason:

e)



Yes/No
reason:

6. Which of the following expressions define y as a function of x ? Please tick the appropriate equation(s) and give your reason.

a) $y = -x+1$

b) $y = 7$

c) $y = x^2+1$

d) $y^2 = x$

e) $x^2+y^2 = 4$

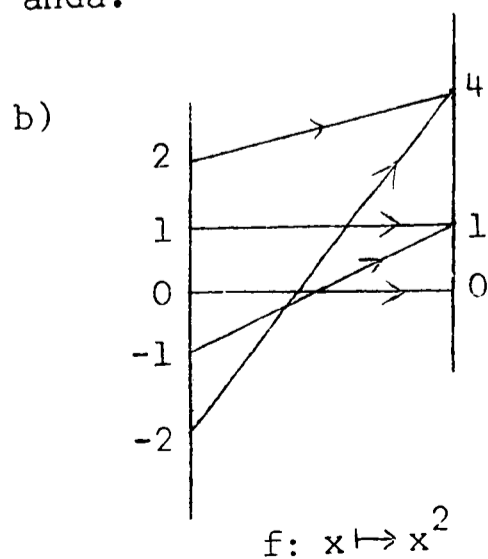
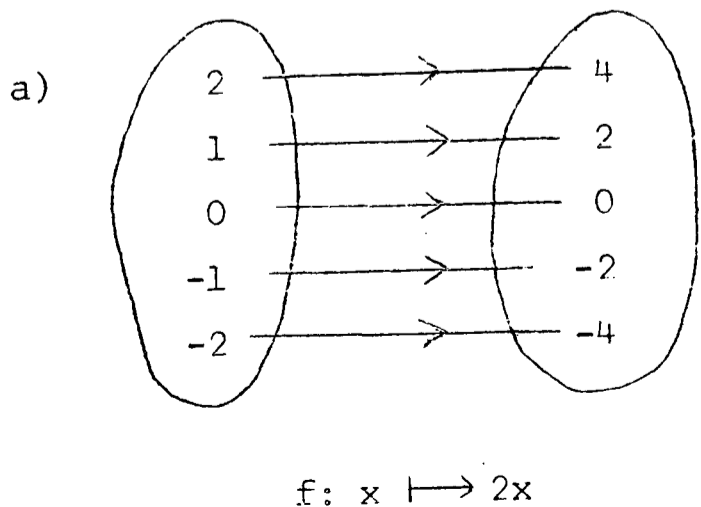
Thanks again for your participation and cooperation

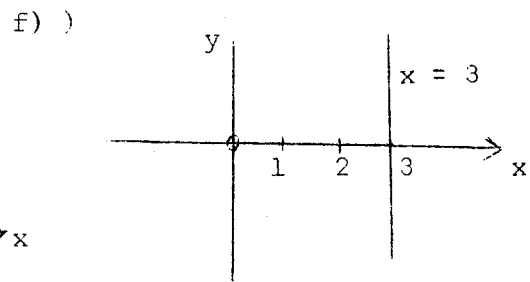
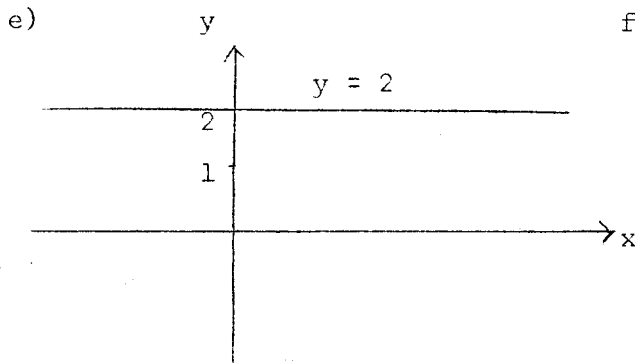
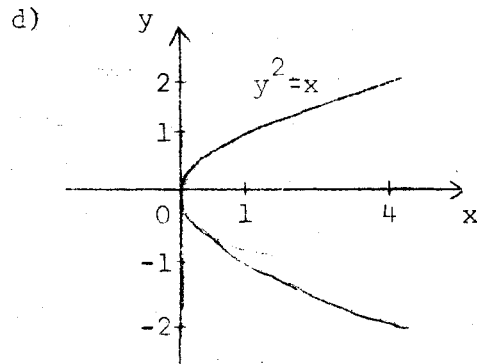
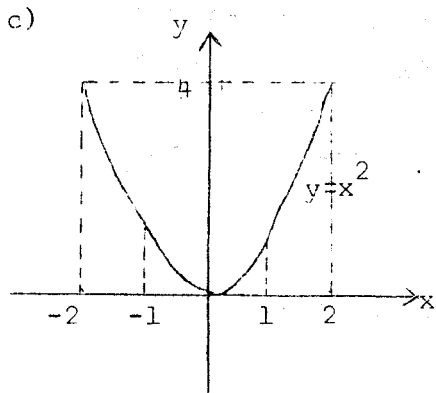
SOAL-SELIDIK DENGAN PELAJAR

Perhatian Untuk Pelajar:

Terima kasih kepada anda kerana mengambil bahagian dalam soal-selidik ini. Ini bukanlah satu ujian matematik untuk menilai prestasi anda di sekolah tetapi merupakan satu soal-selidik bagi tujuan penyelidikan semata-mata. Semua jawapan anda akan dirahsiakan. Sila jawab SEMUA soalan berikut dengan sebaik-baiknya.

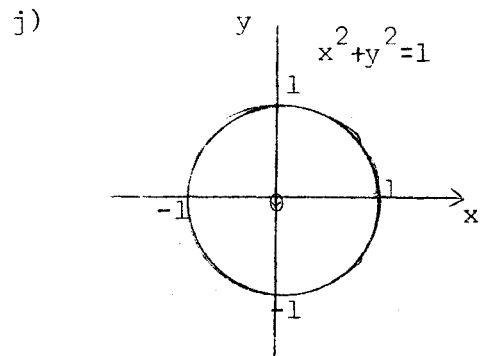
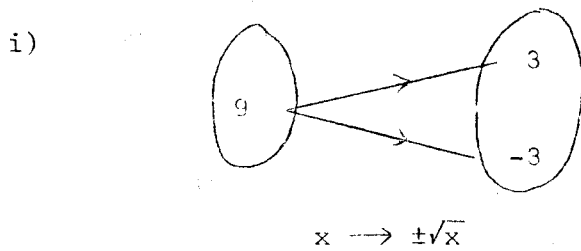
1. Diberi $f(x) = 2x^2 - 11x + k$ dan $f(3) = 0$, cari nilai k dan seterusnya $f(-1)$.
2. Diberi $f(x) = \frac{c}{x^2}$ dan $f(2) = 5$, carikan nilai bagi
 - i) c
 - ii) $f(5)$
3. Diberi fungsi $f(x) = px^3 + 2$ di mana p ialah pemalar dan $f(1) = 0$, carikan $f(\frac{1}{2})$.
4. Diberi bahawa fungsi $f(x) = \frac{p + x^2}{x}$ dan $f(2) = 5$, carikan nilai p .
Oleh yang demikian, carikan nilai x selain daripada 2 dengan keadaan $f(x) = 5$.
5. Diberi bahawa $f(p) = p^2 - 3p + 4$, hitung nilai-nilai bagi p dengan keadaan $f(p) = 4$.
6. Setelah menjawab soalan-soalan di atas, dapatkah anda terangkan apakah yang anda faham dengan konsep fungsi? Jika boleh berikan definisi bagi fungsi.
7. Nyatakan samada yang berikut merupakan satu fungsi atau bukan fungsi. Beri sebab kepada jawapan anda.





g) $\{(1, 2), (2, 3), (3, 4)\}$

h) $\{(1, 1), (1, 2), (2, 3)\}$



k) $y = x^2 + 1$ bagi $x \in \{-2, -1, 0, 1, 2\}$

l) Luas bulatan, $L = \pi j^2$ bagi jejari j yang positif.

Terima kasih.

Function Questionnaire

Thank you for participating in this study. This is not a mathematics test. It is simply an investigation to find out your understanding of the function concept and its related ideas. The results would be used only for the purpose of research and not in any way used to assess your achievement in school. Please answer ALL questions as best as you can.

1. Given $f(x) = 2x^2 - 11x + k$ and $f(3) = 0$, find the value of k and hence $f(-1)$.
2. Given $f(x) = c/x^2$ and $f(2) = 5$, find the value of i) c ii) $f(5)$.
3. Given that $f(x) = px^3 + 2$ where p is a constant and $f(1) = 0$, find $f(1/2)$.
4. Given a function $f(x) = (p + x^2)/x$ and $f(2) = 5$, find the value of p . Hence find the value of x , other than 2 such that $f(x) = 5$.
5. Given $f(p) = p^2 - 3p + 4$, find the values of p such that $f(p) = 4$.
6. After answering the above questions, can you explain in a sentence or so what you think a function is? If you can give a definition of a function then please do so.
7. Which of the following could represent functions? Give reason for your answer. (see 7(a), 7(b), 7(c), 7(d), 7(e), 7(f), 7(g), 7(h), 7(i), and 7(k) in Malay version). 7(l) Area for a circle $L = \pi j^2$ for j positive.

Thank you

UNIVERSITI TEKNOLOGI MALAYSIA

SOAL SELIDIK DENGAN GURU

Latarbelakang : Kelayakan Maktab/Universiti

Pengalaman :¹⁷..... tahun

Perhatian untuk guru:

Ini adalah satu soal-selidik untuk mengraji pendapat dan pandangan guru serta pendekatan yang biasa digunakan bagi mengajar konsep fungsi kepada murid-murid di sekolah. Segala maklumat yang diperolehi akan digunakan untuk tujuan penyelidikan semata-mata. Sila jawab SEMUA soalan berikut. Di atas kerjasama cikgu saya dahului dengan berbilang-banyak terima kasih.

1. Dalam kurikulum matematik moden sekarang konsep fungsi telah diperkenalkan lebih awal dalam sukatan pelajaran. Apakah pandangan cikgu mengenai sebab-sebabnya?

2. Semasa memperkenalkan konsep fungsi kepada murid-murid di sekolah, terdapat sekurang-kurangnya dua pendekatan yang boleh digunakan:
 - i) Memulakan pengajaran dengan memberikan definisi, diikuti dengan contoh-contoh fungsi dan bukan contoh-contoh fungsi terutama yang berkaitan dengan pengetahuan dan pengalaman harian atau sebenar bagi murid dan seterusnya murid menyelesaikan masalah-masalah yang melibatkan fungsi.
 - ii) Memulakan pengajaran dengan memperkenalkan contoh-contoh dan bukan contoh-contoh bagi fungsi terutama yang berkaitan dengan pengetahuan dan pengalaman harian atau sebenar bagi murid sebelum diberi definisi bagi konsep fungsi dan seterusnya murid menyelesaikan masalah-masalah yang melibatkan fungsi.

Pendekatan manakah yang lebih digemari oleh cikgu? Beri sebab-sebabnya. Jika pendekatan (i) digunakan, terangkan dengan lebih lanjut definisi yang digunakan.

Questionnaire for Teachers

Qualification: College/ University

Experience : years

Please answer ALL questions:

1. In modern mathematics the concept of function has been introduced at the very early stage in school curriculum. What do you think the most important reason for this?

2. Please indicate your preference in the teaching of the function concept and give your reasons:

- a) teaching should begin by giving the formal definition of function and then followed by the examples and non – examples.
- b) teaching should begin by introducing some examples (especially related to real life problems) which precede the formal definition.

3. What is the main emphasis in your teaching:

- a) the formation and understanding of the function concept, for instance at the end of the lesson, students should be able to describe the meaning or give the correct definition and also should be able to differentiate the examples of functions from non examples of functions.
- b) the mastery of algorithmic and problem solving skills related to the function concept.

4. If the student can answer correctly the following questions involving the function concept, can we say that the student has understood the concept of function? Please give your reasons.

a) Given $f(x) = c/x^2$ and $f(2) = 5$, find the value of i) c ii) $f(5)$

b) Given $f(p) = p^2 - 3p + 4$, find the values of p such that $f(p) = 4$.

5. Based on your own experiences, what is the main problem in teaching the function concept? Please explain your answer.

Thank you very much for your cooperation.

COUNDON COURT SCHOOL AND COMMUNITY COLLEGE
Northbrook Road, Coventry, CV6 2AJ

Head: David Kershaw, M.A., J.P. Telephone: 0203 33-5121/2/3/4

Our Ref:
Your Ref:



Md. Nor Bakar,
Department of Science Education
University of Warwick
Coventry
CV4 7AL

Dear Nor,

Thank you for your letter of 19th Septemebr 1989.
I am pleased to inform you that we can make 10.30 a.m. on
Monday, 2nd. October available for you to provide our Lower
Sixth pupils with your questionnaire.

Perhaps we can then work out the most convenient time
for individual interviews as a result of your visit.

Yours sincerely,



E. Bathgate
Head of Mathematics

Md Nor Bakar,
28 Dysart Close,
Coventry
CV1 5FF

THE BEAUCHAMP COLLEGE

Ridge Way, Oadby, Leicester LE2 5TP. Tel: Leicester (0533) 715809

Principal: Maureen Cruickshank M.A. (Oxon), M.Ed.

Dear Mr Bakar,

Thank you for your letter of 26th March 1990. You are very welcome to come in on Wednesday 4th April 1990 at 11.00 a.m. to post-test the group of students you pre-tested.

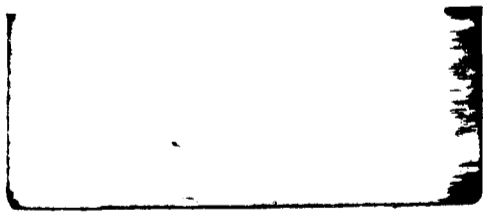
Please report to the reception area on arrival; I look forward to seeing you again.

Yours Sincerely,



C. Caban
Head of Faculty of Mathematics

Vice Principals: Grazyna Kennerley B.Sc., Mike Vybiral B.Sc., Ray Westwood M.A. (Cantab) M.Ed.



Ruj: JPK(PPSG).03-15/1(71)

Tarikh: 9 Oktober, 1990
20 Rabiulawal, 1411

Eneik Md. Noor bin Bakar,
Kampung Tebing,
Hosba,
06000 Jitra, Kedah Darulaman.

Tuan/Puan,

Kebenaran Monjalankan Kajian Di Sekolah-Sekolah:
**"Students' Conceptions of Fractions and Graph -
A Cross Cultural Study in England and Malaysia"**

Merujuk kepada surat tuan/puan bil. _____

bertarikh 8 Oktober 1990, saya diarah memaklumkan bahawa kebenaran adalah diberi bagi tuan/puan menjalankan kajian yang tersebut di atas di sekolah-sekolah berikut dalam Negeri Kedah Darulaman pada bulan Okt. - Dis. 90 serta memilih sampel-sampel murid dan/atau guru yang dikehendaki itu:

- 1 (1) **St. Tanki Abdul Malik, Alor Setar,**
- (2) **Kelel Sultan Abdul Hamid, Alor Setar,**
- 1/1 (3) **St. Sultanah Bahiyah, Alor Setar.**

Jumlah: 3 buah sekolah sahaja.

2. Selanjutnya tuan/puan bolehlah berhubung terus dengan Pengetua/Guru Besar tiap-tiap sekolah berkenaan untuk menetapkan tarikh lawatan dan untuk lain-lain keperluan tuan/puan.

Sekian dimaklumkan.

'BERKHIDMAT UNTUK NEGARA'

Saya yang menurut perintah,

(Comm)
(Mohd. Kamil bin Haji Mustafa)
Ketua Penolong Pengarah,
Unit Perhubungan & Pendaftaran,
b.p. Pengarah Pendidikan Kedah.

s.k. Pegawai Pendidikan Daerah,

Kota Setar.

/Pengetua/Guru Besar Sekolah-Sekolah yang berkenaan.

Tuan/Puan diingatkan supaya mengambil tindakan seperti yang diarahkan dalam para 2.4 Surat Pekeliling Ikhtisas Bil.1/84.

~~Jabatan Pendidikan Kedah dan Terengganu, UEM, Alor Setar, Johor.~~