Application of the assumed mode method to the vibration analysis of rectangular plate structures in contact with fluid

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Abstract. *This investigation is related to the natural vibration analysis of bottom and vertical plate structures in contact with fluid using the assumed mode method. The procedure can consider thick plates and stiffened panels, where the Mindlin theory is adopted for a plate and Timoshenko beam theory for the stiffeners. The eigenvalue problem is formulated using Lagrange's equation of motion and taking into account potential and kinetic energies of a plate structure and fluid kinetic energy, respectively. Potential flow theory assumptions are adopted for the fluid and the effect of free surface waves is ignored. From the boundary conditions for the fluid and structure the fluid velocity potential is derived and it is utilized for the calculation of added mass using the assumed modes. Based on the developed theoretical background, the in-house software is developed and applied to the vibration analyses of plates and stiffened panels in contact with fluid with different domain sizes and various sets of boundary conditions. Comparisons with FEM-BEM numerical results show high accuracy of the proposed procedure.*

Key words: *bottom plate structures; vertical plate structures; vibration; potential flow; added mass; assumed mode method*

1. Introduction

Bare and stiffened plates are widely used in all engineering branches; aeronautical, civil, mechanical, naval, ocean, etc. Stiffening can be arranged to increase plate loading capacity, prevent buckling or even to alter its natural frequency. It is generally known that plates and stiffened panels in contact with fluid behave differently from the same structures in the air due to added mass effect, which significantly decreases

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natural frequencies. Vibration analysis of plates in contact with fluid has been an attractive topic for many years, and there are many papers dedicated to this issue [1],[2],[3].

Literature overviews on dynamic analysis of bottom and vertical plate structures in contact with fluid can be found in [3],[4],[5],[6]. Developed mathematical models can be classified into analytical ones [6],[7],[8], semi analytical models [9],[10],[11] and numerical models [12],[13],[14]. Nowadays, the finite element method (FEM) is probably the most advanced tool in practical engineering which can be reliably applied in both free and forced vibration analysis of plates and stiffened panels in contact with fluid having arbitrary set of boundary conditions, but at the same time it may be rather time-consuming in model generation and calculation execution. In this sense, in the early design stage when different topologies of plate structures are considered, it is preferable to use some simplified method.

Most of references in this field actually deal with thin (Kirchhoff) plate theories, and to the authors' knowledge there are only several studies dealing with the hydroelastic analysis of stiffened panels. Free vibration analyses of stiffened plates fully immersed and in contact with fluid, respectively, are done by transforming the structural part into orthotropic plate in [15]. The natural frequencies of vertical stiffened panels with thin plates and slender stiffeners in contact with water are reported in [16] and [17], and are obtained by using the energy method and expanding the velocity potential in water as a series of harmonic waves. Recently, a Rayleigh-Ritz based modal analysis of stiffened bottom plate in contact with finite fluid domain, neglecting the free surface waves and taking into account the effects of bending, transverse shear and rotary inertia in both the plate and stiffeners is presented [18].

In this paper, a solution based on the assumed mode method is examined. Theoretical contributions of the paper are presented in recent publications [3],[19],[20],[21], and here description of developed VAPS software is added with some additional numerical examples that confirm accuracy of the method. The applied concept is very similar with the Rayleigh-Ritz method, but instead of minimizing the energy functional, it opts to apply the Lagrange's equation of motion. Potential flow theory is adopted for fluid, and the velocity potential is utilized for the calculation of added mass to solve the coupled hydroelastic problem.

2. Problem statement

Vibration of bottom and vertical rectangular plates and stiffened panels in contact with an ideal and irrotational fluid, and having length a and width b is considered, Figure 1. The plate structure part is assumed to be elastic, while the other parts of the rectangular domain are treated as the rigid ones. Furthermore, the surface waves and hydrostatic pressure effects are neglected in this study.

Lagrange's equation can be applied to formulate the eigenvalue problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right)-\frac{\partial T}{\partial q_i}+\frac{\partial V}{\partial q_i}=0 \quad (1)$$

where V and T represent total system potential and kinetic energies, respectively, and q_i is generalized coordinate. In the case of bare plate in contact with fluid, one can write for system potential and kinetic energies [3]:

$$V = V_p, \quad T = T_p + T_w. \tag{2}$$

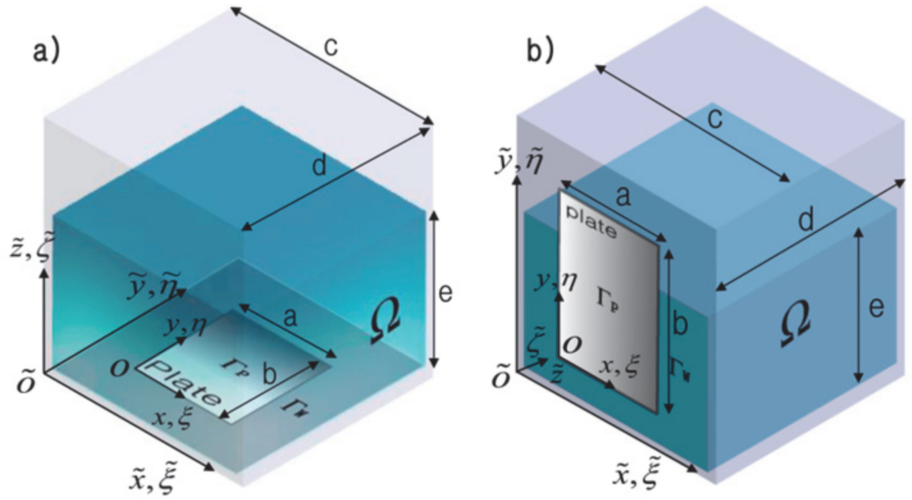

Figure 1. Plate structure in contact with fluid; a) bottom, b) vertical.

Similarly to that, following expressions are valid for stiffened panels in contact with fluid:

$$V = V_p + V_s, \quad T = T_p + T_s + T_w. \tag{3}$$

In the above formulae, V_p is the plate potential energy, V_s represents the potential energy of the attached stiffeners, T_p is the plate kinetic energy and T_s and T_w are the kinetic energies of the stiffeners and fluid, respectively.

The Mindlin thick plate theory which operates with three general displacements (potential functions), i.e. plate deflection w, and angles of cross-section rotation about the x and y axes, ψ_x and ψ_y, respectively, is adopted in the structural part of the mathematical model [23]. Free vibration analysis of plate structures in contact with fluid is performed by the assumed mode method [3],[19],[20],[24],[25], where lateral displacement and rotational angles are expressed by superposing the products of the orthogonal polynomials:

$$w(\xi,\eta,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} a_{mn}(t) X_m(\xi) Y_n(\eta) \tag{4}$$

$$\psi_\xi(\xi,\eta,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} b_{mn}(t) \Psi_m(\xi) Y_n(\eta) \tag{5}$$

$$\psi_\eta(\xi,\eta,t) = \sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}(t) X_m(\xi) \Phi_n(\eta) \tag{6}$$

where $\xi = x/a$ and $\eta = y/b$ are non-dimensional coordinates, $X_m(\xi)$, $Y_n(\eta)$, $\Psi_m(\xi)$ and $\Phi_n(\eta)$ are the orthogonal polynomials satisfying the arbitrary boundary conditions with respect to ξ and η, whose derivation is described in detail in [25]. M

and N represent the number of orthogonal polynomials used for the approximate solution in the ξ and η direction, respectively, and $a_{mn}(t)$, $b_{mn}(t)$ and $c_{mn}(t)$ are the influence coefficients of the orthogonal polynomials.

After substituting (2) or (3) into Lagrange's equation of motion (1), the following discrete matrix equation with 3xMxN degrees of freedom is obtained:

$$[M]\left\{\frac{\partial^2 q(t)}{\partial t^2}\right\}+[K]\{q(t)\}=0\,, \tag{7}$$

where

$$\{q(t)\}=\{a_{11}\ \dots\ a_{MN}\ b_{11}\ \dots\ b_{MN}\ c_{11}\ \dots\ c_{MN}\ \}^T\,, \tag{8}$$

Also [M] and [K] are the mass and the stiffness matrix, respectively. Their constitution for both stiffeners and plate is given in [20] and [25], respectively.

If one assumes harmonic vibrations, i.e.

$$\begin{gathered}\{q(t)\}=\{Q\ \}e^{j\omega t},\quad w(\xi,\eta,t)=W(\xi,\eta)e^{j\omega t},\\ \psi_\xi(\xi,\eta,t)=\Psi_\xi(\xi,\eta)e^{j\omega t},\quad \psi_\eta(\xi,\eta,t)=\Psi_\eta(\xi,\eta)e^{j\omega t}\end{gathered} \tag{9}$$

from equation (7) an eigenvalue problem is obtained, giving natural frequencies and eigenvectors of the system. In above equations j is the unit imaginary and ω represents the angular frequency. The corresponding mode shapes are then obtained from equations (4), (5) and (6). The orthogonal polynomials corresponding to the target model should be applied, and here the characteristic polynomials having the properties of Timoshenko beam functions which satisfy the specified boundary conditions are used [3],[19],[20],[21],[24],[25]:

$$\int_0^1\left(\rho A W_m W_n+\rho I\Psi_m\Psi_n\right)\mathrm{d}\zeta=\begin{cases}0, & m\neq n\\ 1, & m=n\end{cases} \tag{10}$$

where W and Ψ represent transverse displacement and rotational angle, respectively. Also, $\zeta = x/L$, where L represents beam length, ρ is the material density, A is the cross-sectional area and I is the moment of inertia of the corresponding beam.

Kinetic energy of the fluid is formulated within the hydrodynamic model, while the relevant expressions for plate/panel potential and kinetic energies are listed within the structural model section.

3. Hydrodynamic model

In this paper, the hydrodynamic model is explained only for the bottom plate case, while the one relevant for the vertical plate case is available in [3]. However, basic assumptions and equations are the same, and solved for different boundary conditions. The velocity potential $\phi(\tilde{x},\tilde{y},\tilde{z},t)$ satisfies the Laplace equation and the relevant boundary conditions in fluid domain Ω as shown in Figure 1a). Hence, one can write:

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial\tilde{x}^2}+\frac{\partial^2\phi}{\partial\tilde{y}^2}+\frac{\partial^2\phi}{\partial\tilde{z}^2}=0,\qquad \text{in } \Omega \tag{11}$$

$$\frac{\partial \phi}{\partial \tilde{x}} = 0, \qquad \text{on } \tilde{x} = 0,\ c \text{ (rigid wall)} \tag{12}$$

$$\frac{\partial \phi}{\partial \tilde{y}} = 0, \qquad \text{on } \tilde{y} = 0,\ d \text{ (rigid wall)} \tag{13}$$

$$\phi = 0, \qquad \text{on } \tilde{z} = e \text{ (no surface waves)} \tag{14}$$

$$-\left.\frac{\partial \phi}{\partial \tilde{z}}\right|_{\tilde{z}=0} = \begin{cases} \dfrac{\partial w}{\partial t} & \text{on wetted plate/panel surface } (\Gamma_P) \\ 0 & \text{on the other part } (\Gamma_W) \end{cases} \tag{15}$$

where $w = W(x,y)T(t)$ is the transverse deflection of plate structure. If we assume the solution of the Laplace equation, Eq. (11), in the following form:

$$\phi(\tilde{x},\tilde{y},\tilde{z},t) = X(\tilde{x})Y(\tilde{y})Z(\tilde{z})\dot{T}(t) \tag{16}$$

and applying the method of separation of variables considering boundary conditions (12) to (14), we obtain [4]:

$$\phi = d\dot{T}(t)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty} E_{pq}\cos(p\pi\tilde{\xi})\cos(q\pi\tilde{\eta})F_{pq}(\tilde{\zeta}) \tag{17}$$

where

$$\begin{aligned} F_{pq}(\tilde{\zeta}) &= \begin{cases} 1-\tilde{\varsigma} & \text{if } p = q = 0 \\ e^{C_{pq}\tilde{\varsigma}} - e^{C_{pq}(2-\tilde{\varsigma})} & \text{elsewhere} \end{cases} \\ C_{pq} &= \pi\beta\sqrt{(p/\lambda)^2 + q^2} \\ \lambda &= c/d, \quad \beta = e/d \end{aligned} \tag{18}$$

In the above equations, beside already mentioned ξ and η, the following non-dimensional coordinates are introduced

$$\tilde{\xi} = \tilde{x}/c, \quad \tilde{\eta} = \tilde{y}/d, \quad \tilde{\zeta} = \tilde{z}/e\,. \tag{19}$$

Substituting equation (17) into (15) and applying the orthogonality property of the trigonometric functions and the Fourier series expansion, the coefficient E_{pq} is exactly expressed in the form of integral equations as follows:

$$E_{pq} = \frac{\varepsilon_{pq}}{Q_{pq}}\iint_{\Gamma_P} W(\xi,\eta)\cos(p\pi\tilde{\xi})\cos(q\pi\tilde{\eta})d\Gamma \tag{20}$$

where

$$\varepsilon_{pq} = \begin{cases} 1 & \text{if } p = q = 0 \\ 2 & \text{if } p = 0,\ q \neq 0 \text{ or } q = 0,\ p \neq 0 \\ 4 & \text{elsewhere.} \end{cases} \tag{21}$$

$$Q_{pq} = \begin{cases} 1 & \text{if } p = q = 0 \\ -C_{pq}(1 + e^{2C_{pq}}) & \text{elsewehere.} \end{cases} \tag{22}$$

Hence, the kinetic energy of the fluid, whose density is ρ_w, is expressed as follows:

$$T_w = \frac{1}{2}\rho_w \iiint_{\Omega} (\nabla\phi)^2\, d\Omega \tag{23}$$

where ∇ is the gradient operator. By applying Green's theorem to the previous equation and substituting into equation (3), the volume integral in the fluid domain can be transformed into a surface integral for the plate region Γ_P as follows:

$$\begin{aligned} T_w(t) &= -\frac{1}{2}\rho_w \iint_{\Gamma_P} \left(\phi \frac{\partial \phi}{\partial \tilde{z}} \right)\Bigg|_{\tilde{z}=0} d\Gamma \\ &= \frac{1}{2}\rho_w cde\dot{T}(t)^2 \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \left\{ \frac{\varepsilon_{pq}}{Q_{pq}} \bar{F}_{pq} \left(\iint_{\Gamma_P} W(\xi,\eta)\cos(p\pi\tilde{\xi})\cos(q\pi\tilde{\eta}) d\Gamma \right)^2 \right\} \end{aligned} \tag{24}$$

where

$$\bar{F}_{pq} = \begin{cases} 1, & \text{if } p = q = 0 \\ 1 - e^{2C_{pq}}, & \text{elsewhere.} \end{cases} \tag{25}$$

and transverse deflection $W(\xi,\eta)$ which expresses the hydroelastic effect can be approximated by shape functions of the considered plate.

4. Outline of the structural model

By taking into account the non-dimensional coordinates ξ and η, and introducing $\alpha = a/b$ and $S = kGh/D$, the following expressions are valid for the potential and kinetic energy of plate with length a and width b, having arbitrary edge constraints, Figure 2, [3],[19],[20],[21],[25]:

$$\begin{aligned} V_p = \frac{D}{2\alpha}\int_0^1\int_0^1 & \left[\left(\frac{\partial \psi_\xi}{\partial \xi} \right)^2 + \alpha^2 \left(\frac{\partial \psi_\eta}{\partial \eta} \right)^2 + 2\nu\alpha \frac{\partial \psi_\xi}{\partial \xi}\frac{\partial \psi_\eta}{\partial \eta} + \frac{1-\nu}{2}\left(\alpha \frac{\partial \psi_\xi}{\partial \eta} + \frac{\partial \psi_\eta}{\partial \xi} \right)^2 \right. \\ & \left. + S\left(\left(\frac{\partial w}{\partial \xi} - a\psi_\xi \right)^2 + \alpha^2 \left(\frac{\partial w}{\partial \eta} - b\psi_\eta \right)^2 \right) \right] \mathrm{d}\xi\mathrm{d}\eta \\ & + \int_0^1 \left[K_{Rx1}\psi_\xi^2(0,\eta) + SK_{Tx1}w^2(0,\eta) \right]\mathrm{d}\eta + \alpha^2 \int_0^1 \left[K_{Ry1}\psi_\eta^2(\xi,0) + SK_{Ty1}w^2(\xi,0) \right]\mathrm{d}\xi \\ & + \int_0^1 \left[K_{Rx2}\psi_\xi^2(1,\eta) + SK_{Tx2}w^2(1,\eta) \right]\mathrm{d}\eta + \alpha^2 \int_0^1 \left[K_{Ry2}\psi_\eta^2(\xi,1) + SK_{Ty2}w^2(\xi,1) \right]\mathrm{d}\xi, \end{aligned} \tag{26}$$

$$T_p = \frac{\rho ab}{2}\int_0^1\int_0^1 \left[h\left(\frac{\partial w}{\partial t} \right)^2 + \frac{h^3}{12}\left(\frac{\partial \psi_\xi}{\partial t} \right)^2 + \frac{h^3}{12}\left(\frac{\partial \psi_\eta}{\partial t} \right)^2 \right] \mathrm{d}\xi\mathrm{d}\eta \tag{27}$$

D represents plate flexural rigidity $D = Eh^3/\left(12\left(1-\nu^2\right)\right)$, while E and $G = E/\left(2\left(1+\nu\right)\right)$ are Young's modulus and shear modulus, respectively. Further, h is plate thickness, k is shear coefficient, while ν is Poisson's ratio. $K_{Tx1} = \left(k_{Tx1}a/kGh\right)$, $K_{Tx2} = \left(k_{Tx2}a/kGh\right)$, $K_{Ty1} = \left(k_{Ty1}b/kGh\right)$ and $K_{Ty2} = \left(k_{Ty2}b/kGh\right)$ are non-dimensional stiffness at $x = 0$, $x = a$, $y = 0$ and $y = b$, respectively, and correspond to the translational spring constants per unit length k_{Tx1}, k_{Tx2}, k_{Ty1} and k_{Ty2}. In the same manner, $K_{Rx1} = \left(k_{Rx1}a / D\right)$, $K_{Rx2} = \left(k_{Rx2}a / D\right)$, $K_{Ry1} = \left(k_{Ry1}b / D\right)$ and $K_{Ry2} = \left(k_{Ry2}b / D\right)$ correspond to the rotational spring constants per unit length k_{Rx1} , k_{Rx2} , k_{Ry1} and k_{Ry2} , respectively.

The effect of the framing is taken into account by adding its potential and kinetic energy to the corresponding energies of the bare plate. The expressions for potential and kinetic energies of a number of stiffeners placed in a longitudinal direction are presented in [3],[20],[21].

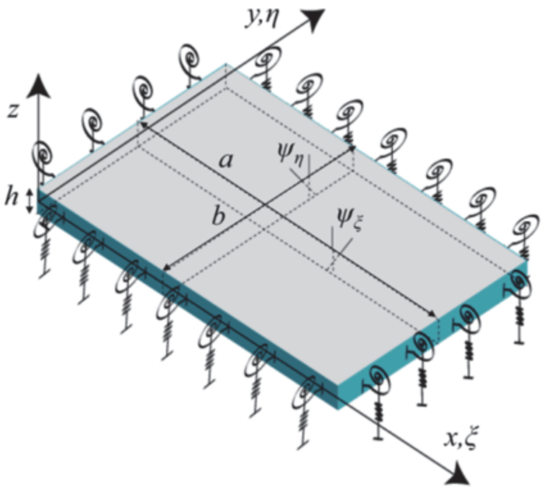

Figure 2. Thick rectangular plate with arbitrary boundary conditions.

Based on the above described numerical procedure, VAPS (Vibration Analysis of Plate Systems) software is developed, Figure 3.

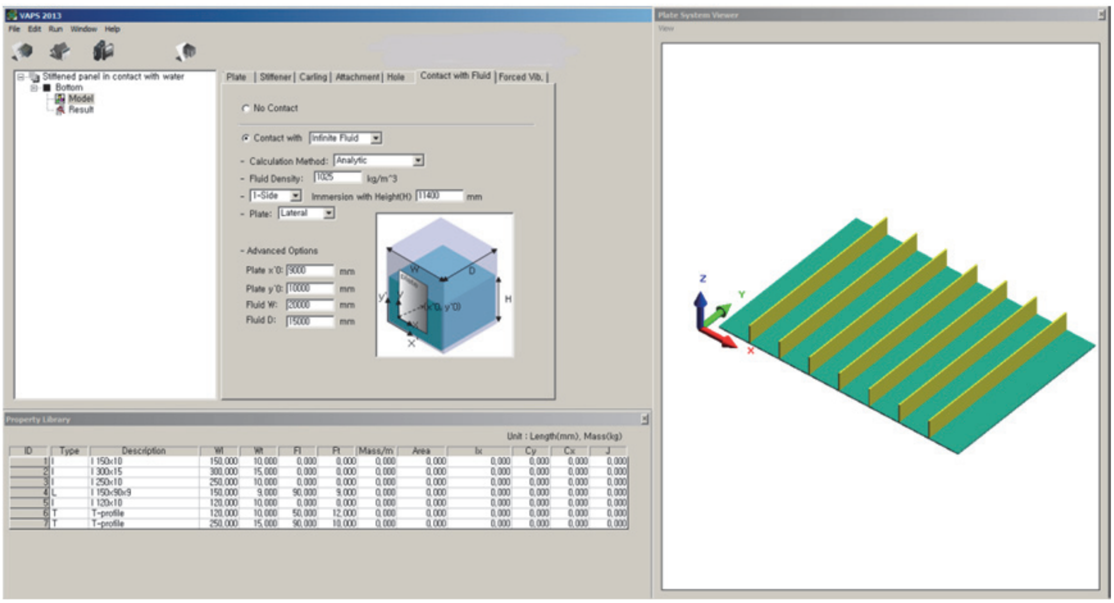

Figure 3. Model preparation in VAPS software.

Its current version is applicable for the dry and wet natural vibration analysis of thin and thick rectangular plates and stiffened panels with and without cutouts, respectively, and with all possible combinations of boundary conditions. VAPS is executed using Windows OS and has its own graphical user interface which enables almost instantaneous model preparation and shows 3-dimensional graphic model of plate structures.

5. Results

The natural response of bottom and vertical plate structures in contact with fluid on one side is analysed using the developed in-house code, based on the presented theoretical background. Material properties and calculation data are specified in Table 1. The validation is done through comparisons with FEM-BEM results obtained by NASTRAN software [26], which combines the finite element method for structure and boundary element method for the fluid. The results obtained by the developed code are designated with PM (Present Method). The number of polynomials in longitudinal and transverse direction is set following the convergence test presented in [21].

Table 1. Material properties and calculation data

Item	*Symbol*	*Value*
Young's modulus	E	$2.1\text{x}10^{11}$ N/m^2
Plate material density	ρ_s	7850 kg/m^3
Fluid density	ρ_w	1025 kg/m^3
Poisson's ratio	ν	0.3
Shear correction factor	k	5/6
No. of polynomials in ξ direction	M	13
No. of polynomials in η direction	N	13
No. of trigonometrical series in $\tilde{\xi}$ direction	P	100
No. of trigonometrical series in $\tilde{\eta}$ direction	Q	100

5.1. Vibration of bare plates in contact with fluid

Wet natural vibrations of bottom and vertical bare plates with length and width 2.0 m and 1.4 m, respectively, are analysed. The plate relative thickness yields h/b=0.05. The FEM-BEM calculation models and are shown in Figure 4. Natural frequencies for bottom and vertical plates having CCCC and CCFC boundary conditions, respectively, are listed in Table 2, where very good agreement with FEM-BEM solutions can be noticed.

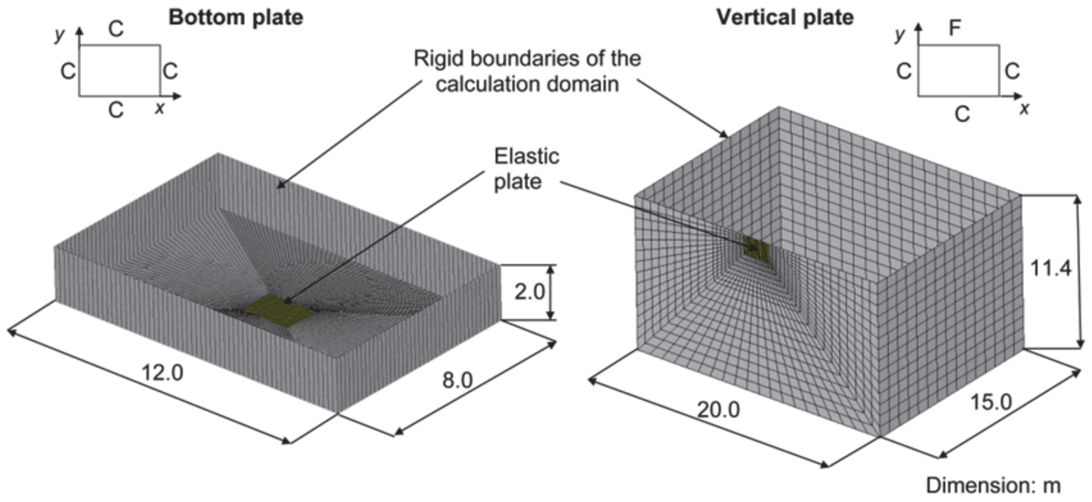

Figure 4. Bottom and vertical plates in contact with fluid (FEM-BEM calculation models).

Table 2. Natural frequencies of bottom and vertical bare plates in contact with fluid, f_i (Hz)

Mode no.	*Plate structure*					
	Bottom plate (CCCC)			*Vertical plate (CCFC)*		
	PM	*FEM-BEM*	*Diff., %*	*PM*	*FEM-BEM*	*Diff., %*
1	171.68	171.94	-0.15	91.19	90.89	0.33
2	307.00	305.59	0.46	214.41	212.28	0.81
3	470.67	471.47	-0.17	237.12	235.53	0.68

5.2. Vibration of stiffened plates in contact with fluid

The assumed mode method is also applied to longitudinally stiffened bottom and side plates of rectangular container, Figure 5. In both cases, clamped boundary conditions (CCCC) are assumed.

Natural vibration analysis is performed for different filling levels and natural frequencies obtained according to the proposed method and FEM-BEM solutions, respectively, are presented in Figure 6, where also very good agreement, as in case of bare plates, is obvious.

In addition, mode shapes of bottom stiffened plate of rectangular container are shown in Figure 7. Very similar patterns are obtained by the assumed mode method and general FE tool.

The obtained results, both in case of bare and stiffened plates in contact with different fluid domains, confirm that developed method gives very close results to those obtained by the commercial software NASTRAN [26].

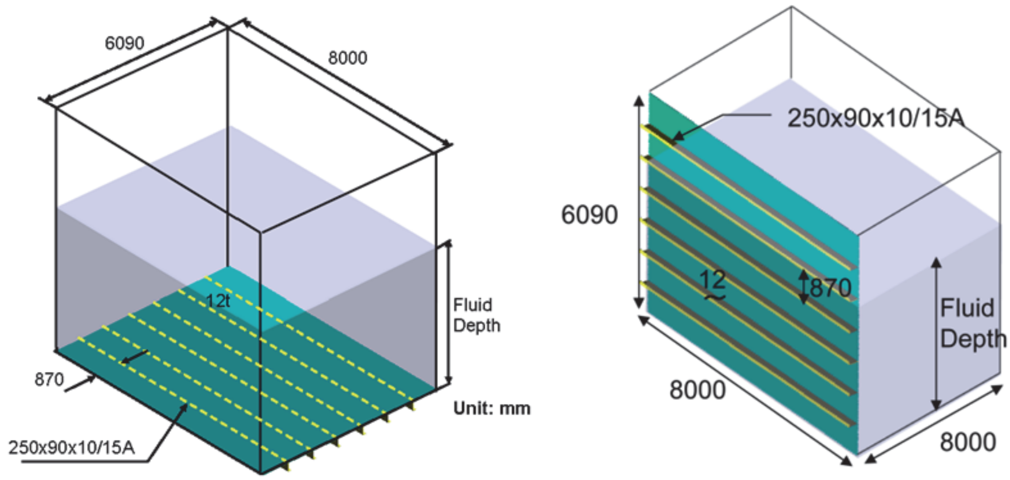

Figure 5. Longitudinally stiffened bottom and side plates of rectangular container in contact with fluid.

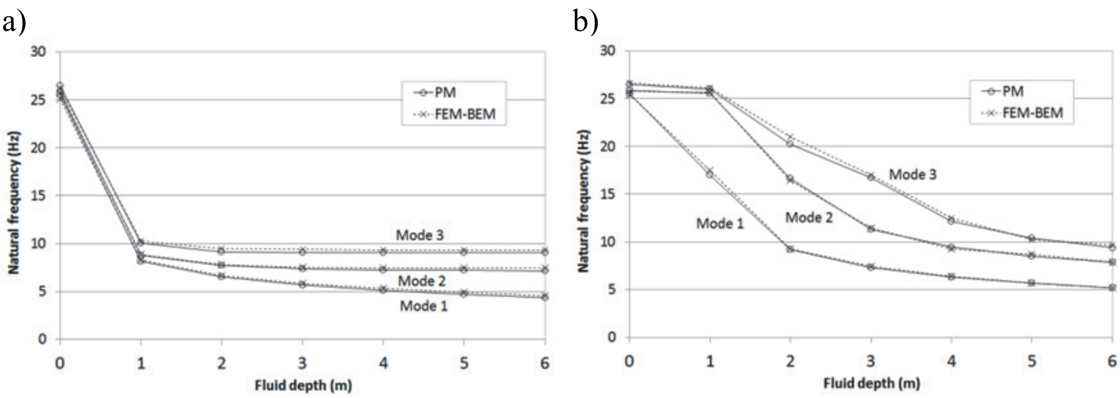

Figure 6. Natural frequencies of a stiffened plate of rectangular container for different fluid depths; a) bottom, b) side.

6. Conclusion

The problem of vibrating plates in contact with fluid is investigated for a long time and many rather complex solutions are offered in the relevant literature. In this paper, a simplified solution based on the assumed mode method for bottom and vertical bare/stiffened plate structures in contact with fluid is examined. Mindlin (thick) plate theory is adopted for plate and Timoshenko beam theory for the attached stiffeners. Potential and kinetic energies of rectangular plate structure and kinetic energy of the fluid, respectively, were formulated and used to derive the eigenvalue problem based on Lagrange's equation of motion. Illustrative numerical examples analysing the natural response of plates and stiffened panels in contact with different fluid domains are provided. Comparisons with FEM-BEM results confirmed high accuracy of the developed procedure, in spite of the relative simplicity of mathematical formulation. As a next step, the presented theory can be extended to the free or forced vibration problems of orthotropic and composite plates in contact with fluid which are a subject of investigation nowadays.

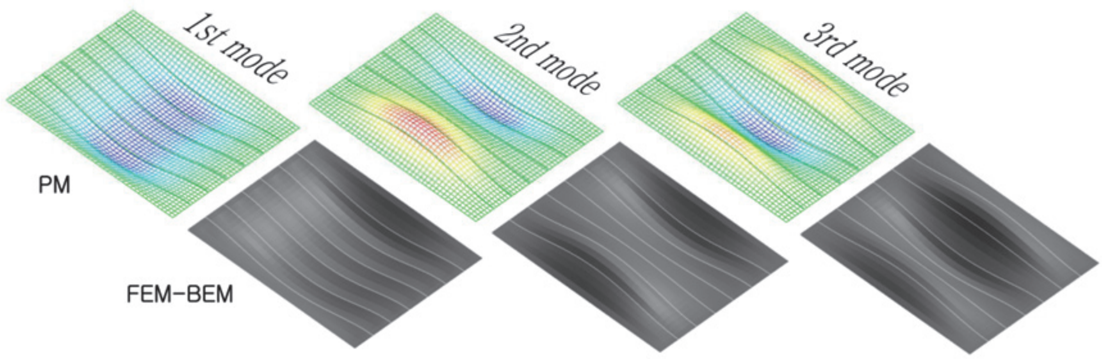

Figure 7. Mode shapes of bottom stiffened plate.

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