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Abstract

It is well known that many existing models of forest age structure dynamics describe time dynamics for a local area. However, it is known that real forest areas have age structures that vary from one gap to another. Local gaps are integrated into a joint forest ecosystem by various seed dispersion mechanisms and by the penetration of roots. The work of Antonovsky et al. (1989) was used as a base model for a qualitative description of a spatially distributed mono-species mixed age forest. This phenomenological model employs a diffusion term corresponding to various processes of "young" tree dispersion. The model allows one to predict the possibility of the existence of a stationary or traveling forest boundary. From one side of the boundary, the modeled forest demonstrates an equilibrium state with non-zero age class densities, while from the other side, there are no trees of the studied type. As was shown, the model analyzes the changes in the behavior of the forest boundary caused by an increase in the tree mortality rate due to anthropogenic impacts (from acid rain, for example). The present paper is devoted to studying the ability of the forest to resist the internal "negative" forces of forest ecosystems and external impacts on changes in the standing forest boundary (on a given model level).

On the Stability Analysis of the Standing Forest Boundary

M.Ya. Antonovsky,
*E.A. Aponina,**
*Yu.A. Kuznetsov**

1 Introduction

In the previous paper (Antonovsky, Aponina and Kuznetsov, 1989), the following model

$$\begin{aligned} U_t &= pV - (V-1)^2U - sU + U_{xx} \\ V_t &= U - hV \end{aligned} \tag{1}$$

was introduced as a qualitative description of a spatially distributed mono-species mixed-age forest. Here U and V are (scaled) densities of the tress of "young" and "old" age classes respectively, p, s and h are positive parameters, $t \geq 0$ and $x \in \Re$. The nonlinear term in (1) reflects an assumption that there exists an optimal density of old trees under which the recruitment of young trees is maximal. The diffusion term represents various processes of tree dispersion and has a phenomenological character. The parameter values were considered for which system (1) without diffusion has two stable equilibria: trivial and nontrivial.

In the previous paper two types of stationary solutions of (1) were proved to exist: *standing* and *traveling* fronts. The front asymptotically connects the nontrivial forest state with the trivial one and is considered as the simplest mathematical models of the forest boundary. *Standing fronts* are nonuniform time-independent solutions of (1): $U(x,t) = U_0(x)$, $V(x,t) = V_0(x)$. Function $V_0(x)$ satisfies the following differential equation:

$$hV_0'' + pV_0 - h(V_0-1)^2V_0 - shV_0 = 0. \tag{2}$$

If $V_0(x)$ is known, then $U_0(x) = hV_0(x)$. It was shown that standing fronts exist if and only if a specific relation between the parameter values is imposed:

$$\rho = (s + 1/9)h. \tag{3}$$

For parameter values satisfying (3), equation (2) may be integrated analytically and function $V_0(x)$ can be written down explicitly:

$$V_0(x) = \frac{4}{3}\left(1 + \exp\left(-\frac{4}{3\sqrt{2}}x\right)\right)^{-1}. \tag{4}$$

Note that due to translation invariance of (1) functions $U_0(x+\delta)$ and $V_0(x+\delta)$ with any real δ would also represent a standing front.

The present paper is devoted to the problem of stability of the standing fronts. In other words, we analyze time behavior of small perturbations of the standing front as solutions of (1). The approach is based on ideas first presented by J.W. Evans (1972; 1975) in the series of his words on the stability of the nerve impulses in Hodgkin-Huxley equations.

We consider this paper a preliminary step in the analysis of traveling forest boundary stability.

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2 The Stability Problem

Consider first the evolution of *finite* perturbations $u(x,t)$ and $v(x,t)$ of the front $(U_0(x), V_0(x))$:

$$
\begin{aligned}
U(x,t) &= U_0(x) + u(x,t), \\
V(x,t) &= V_0(x) + v(x,t).
\end{aligned}
$$

The perturbations satisfy equations

$$
\begin{aligned}
u_t &= -\left[(V_0 - 1)^2 = s\right] u + [\rho - 2(V_0 - 1) U_0] v + u_{xx} - \\
&\quad -U_0 v^2 - 2(V_0 - 1) uv + uv^2 \\
v_t &= u - hv,
\end{aligned} \tag{5}
$$

where V_0 is given by (4), $U_0 = hV_0$ and ρ satisfies (3). System (5) defines a dynamical system in various vector-valued functional spaces. For example, it generates a smooth local semiflow in space $C_{\text{unif}}(\Re)$ of bounded and uniformly continuous functions on $\Re$ or in Sobolev space $W^1(\Re)$ (Henry, 1981). Let H be one of these two spaces.

As is well known, stability of the front solution of (1) under small perturbations is determined by the spectrum $\sigma(L)$ of the linearization operator of (5) in H:

$$
L \begin{pmatrix} u \\ v \end{pmatrix} = A(x) \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} u_{xx} \\ 0 \end{pmatrix}, \tag{6}
$$

where

$$
A(x) = \begin{pmatrix} -(V_0(x) - 1)^2 - s & h(s + 1/9 - 2V_0(x)(V_0(x) - 1)) \\ 1 & -h \end{pmatrix}. \tag{7}
$$

Due to the translation invariance of the original equations, $\sigma(L)$ always has an eigenvalue $\lambda = 0$ with the eigenfunction

$$
\begin{pmatrix} U_0'(x) \\ V_0'(x) \end{pmatrix}.
$$

The front is stable if all the spectrum $\sigma(L)$, except one simple eigenvalue $\lambda = 0$, lies away from the imaginary axis in the left half-plane of the complex plane C. Note that in the case of stability perturbed around $(U_0(x), V_0(x))$ solutions of (1) may (and, generically, do) converge to a translated front $(U_0(x + \delta), V_0(x + \delta))$ with some nonzero δ. If there is a spectrum point in the right half-plane, then the front is unstable.

Thus we have to analyze spectrum $\sigma(L)$ of linear differential operator L is H define by (6). Operator L is not a self-adjoint and its coefficients depend upon x. However, the coefficients exponentially approach constant values as $x \to \pm\infty$. Generally, spectrum $\sigma(L)$ consists of the *essential spectrum* and *eigenvalues* of finite multiplicity (see, for example, Goldberg (1966) for exact definitions and details).

3 Essential Spectrum of the Linearization Operator

The right boundary of the essential spectrum of L is determined (see Henry, 1981) by the right boundary of the eigenvalues of the following matrices:

$$
M_\pm(k) = A_\pm - k^2 C, \tag{8}
$$

where $k \in \Re$,

$$
C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
$$

and $A_\pm$ are limit values of the matrix $A(x)$ given by (7) as $x \to \pm\infty$. We have

$$A_- = \begin{pmatrix} -(1+s) & h(s+1/9) \\ 1 & -h \end{pmatrix}$$

and

$$A_+ = \begin{pmatrix} -(1/9+s) & h(s-7/9) \\ 1 & -h \end{pmatrix}$$

Note that matrices $A_\pm$ coincide with the linearization matrices of system (1) without diffusion computed in its stable trivial and nontrivial equilibria and thus have eigenvalues with negative real parts.

Using simple 'continuous arguments,' we can show that eigenvalues $\lambda^\pm_{1,2}$ of matrices $M_\pm(k)$ have negative real parts for all $k \in \Re$. Let $s_\pm(k)$ and $\Delta_\pm(k)$ stand for the trace and the determinant of $M_\pm(k)$ given by (8). As is mentioned above, the eigenvalues of $M_\pm(0) = A_\pm$ lie in the left half-plane of C. The eigenvalues of $M_\pm(k)$ may, in principle, cross the imaginary axis only at point $\lambda = 0$ because if they are complex, then $2\Re\lambda^\pm_{1,2} = s_\pm(k) < 0$ for all $k \in \Re$. But matrices $M_\pm(d)$ have no zero eigenvalues because $\Delta_\pm(k) \neq 0$ for all $k \in \Re$. Therefore, the eigenvalues of $M_\pm(k)$ cannot cross the imaginary axis while k varies from $-\infty$ to $+\infty$ and thus locate in the left half-plane for all $k \in \Re$.

Hence, the essential spectrum of L lies in the left half-plane of C and does not effect the front stability.

4 Eigenvalues of the Linearization Operator

Eigenvalues and eigenfunctions of L are determined by the eigenproblem:

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix},$$

where bounded functions $u = u(x)$, $v = v(x)$ should not be identical zeros. The problem can be rewritten as the following system of equations:

$$u_{xx} - \left[(V_0 - 1)^2 + s\right] u + h\left[s + 1/9 - 2V_0(V_0 - 1)\right] v = \lambda u$$

$$u - hv = \lambda v.$$

For $\lambda \neq -h$ we get from the second equation

$$v = \frac{u}{\lambda + h}$$

and obtain one scalar linear differential equation for u:

$$u_{xx} = \Phi_\lambda(x) u, \tag{9}$$

where

$$\Phi_\lambda(x) = \lambda + s - \frac{h}{\lambda + h}(s + 1/9) + [V_0(x) - 1]^2 + \frac{2h}{\lambda + h} V_0(x)\left[V_0(x) - 1\right]. \tag{10}$$

Function $\Phi_\lambda(x)$ defined by (10) is asymptotically constant for large $|x|$:

$$\Phi^-_\lambda = \Phi_\lambda(-\infty) = \lambda + s + 1 - \frac{h}{\lambda + h}(s + 1/9),$$

$$\Phi^-_\lambda = \Phi_\lambda(-\infty) = \lambda + s + 1/9 - \frac{h}{\lambda + h}(s - 7/9),$$

Equation (9) depends upon λ as a parameter. Those values of λ for which (9) has bounded nonzero solution $u(x)$ correspond to the eigenvalues of operator L. It is clear now that eigenvalues of L are real.

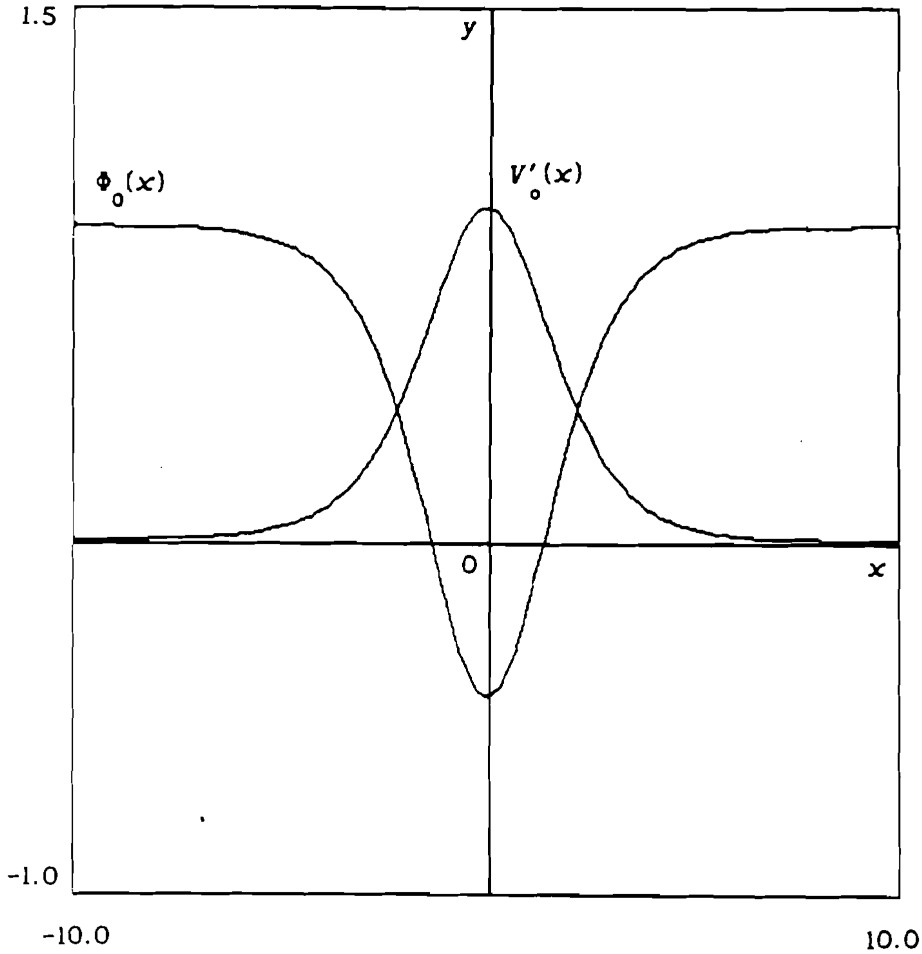

Figure 1: Potential $|Phi_0(x)$ and eigenfunction $V_0(x)$ corresponding to $E = 0$ in Schrödinger equation (11).

A priori boundary for the eigenvalues Equation (9) can be considered a one-dimensional Schrödinger equation with a parameter-dependent potential $\Phi_\lambda(x)$:

$$- u'' + \Phi_\lambda(x)u = Eu, \tag{11}$$

where 'energy $E \equiv 0$'. The eigenproblem can now be formulated as follows: find those values of real parameter λ for which the potential $\Phi_\lambda(x)$ has isolated nondegenerate zero energy level.

For each fixed pair of parameters (s, h) there is a value $\lambda_{\max}$ such that for $\lambda > \lambda_{\max}$ the potential $\Phi_\lambda(x) > 0$ for all $x \in \Re$. This means that there are not eigenvalues of L with $\lambda > \lambda_{\max}$. The value $\lambda_{\max}$ can be easily estimated numerically by plotting the graph of $\Phi_\lambda(x)$.

Note that for $\lambda = 0$ potential (10) has an interval of negative values and the corresponding zero-energy eigenfunction is $V_0'(x)$, where $V_0(x)$ is given by (3). Potential $\Phi_0(x)$ and its zero-energy eigenfunction are presented in Fig. 1.

Numerical algorithm for eigenvalue computation Therefore in order to determine the stability of the front for given fixed values of (s, h) one has to investigate the interval $(0, \lambda_{\max})$. In what follows, we will construct a (discontinuous) function $D(\lambda)$, zeros of which correspond to the eigenvalues of L and will present an algorithm for its numerical computation. In fact, to establish stability, one can only check the absence of signed changes of $D(\lambda)$ within the defined interval. Equation (9) can be rewritten as a linear two-dimensional non-autonomous system

$$y' = B(x)y, \tag{12}$$

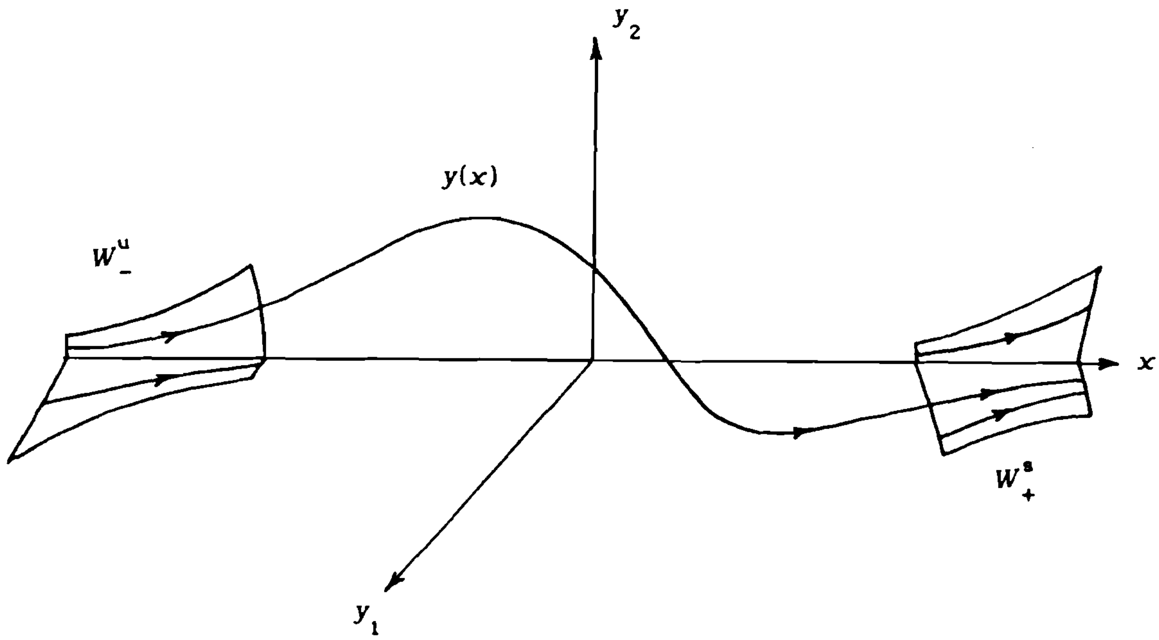

Figure 2: Any bounded solution $y(x)$ of (12) should belong to the intersection of W^u_- and W^s_+.

with 'time' x, where

$$y = \begin{pmatrix} u \\ w \end{pmatrix}, B(x) = \begin{pmatrix} 0 & 1 \\ \Phi_\Lambda(x) & 0 \end{pmatrix}.$$

System (12) is 'asympototically autonomous' for large $|x|$. Consider asymtotic matrices

$$B^\pm = B(\pm\infty).$$

Each matrix $B^\pm$ has one positive and one negative eigenvalue. Hence in the extended phase space of (12) with coordinates (x, y), there is a two-dimensional invariant manifold W^u_- composed by all solutions of (12) exponentially vanishing as $x \to -\infty$. The intersections of these manifolds with a plane $x =$ const are asymtotically (for $x \to \pm\infty$) tangent to the eigenvectors of $B^\pm$

$$V^-_\lambda = \begin{pmatrix} 1 \\ \left(\Phi^-_\lambda\right)^{1/2} \end{pmatrix} \tag{13}$$

and

$$V^+_\lambda = \begin{pmatrix} 1 \\ \left(\Phi^-_\lambda\right)^{1/2} \end{pmatrix}$$

respectively. Function $y(x)$ is a bounded solution of (12) if and only if it belongs to the intersection of the manifolds W^u_- and W^s_+ (see Fig. 2).

The solutions of (12) depend upon parameter λ. Let $y_0 = y_0(\lambda, x)$ be a solution of (12) vanishing as $x \to -\infty : y_0 \in W^u_-$. The following ideas of J.W. Evans (1972; 1975), introduce a scalar function $\psi = \psi(\lambda, x)$ by the expression

$$\psi(\lambda, x) = \langle z_0(\lambda, x), W^+_\lambda \rangle,$$

where

$$z_0(\lambda, x) = \frac{y_0(\lambda, x)}{\|y_0(\lambda, x)\|},$$

W_λ^+ is an eigenvector of the transpose to B^+ matrix corresponding to its positive eigenvalue:

$$W_\lambda^+ = \begin{pmatrix} (\Phi_\lambda^+)^{1/2} \\ 1 \end{pmatrix},$$

and $< \cdot , \cdot >$ denotes the standard scalar product in $\Re^2$; $\| \cdot \| =< \cdot , \cdot >^{1/2}$. Consider the asymptotic value of ψ (which exists for all λ):

$$D(\lambda) = \psi(\lambda, +\infty). \tag{14}$$

If $D(\lambda) = 0$ then solution $y_0(\lambda, x)$ is also vanishing as $x \to +\infty$ and thus for this value of λ, equation (9) has a nontrivial bounded solution defined by the first component of $y_0(\lambda, x)$. Therefore the point λ of the sign change of $D(\lambda)$ is an eigenvalue of the operator L.

Note that norm $y_0(\lambda, x)$ of exponentially increases as $x \to \infty$ for almost all λ. Fortunately, it is possible to compute normalized solution $z_0(\lambda, x)$ without computation of $y_0(\lambda, x)$ by integration of (12). We can utilize the following property: if $\|z_0(\lambda, x_0)\| = 1$ for some initial $x = x_0$, then $z_0(\lambda, x)$ satisfies the equation

$$z_0' = B(x)z_0 - \langle z_0, B(x)z_0 \rangle z_0. \tag{15}$$

For numerical computations of $D(\lambda)$ we can choose $x_0 = -T$, where $T > 0$ is sufficiently large and use (normalized) vector V_λ^- defined by (13) as the initial value for (15). The value of corresponding solution $z_0(\lambda, x)$ of (15) for $x = T$ can then be used to evaluate $D(\lambda)$ defined by (14).

A numerical example The proposed algorithm was implemented as a program for TraX (Interactive ODE Simulator) developed at the Research Computing Center of the USSR Academy of Sciences by Dr. V. Levitin.

Fix $s = h = 1$. The results of evaluating function $\psi(\lambda, x)$ for $\lambda_1 = -0.01$ and $\lambda_2 = 0.01$ are presented in Fig. 3. They indicate the presence of an eigenvalue (of course, it is $\lambda = 0$!) of L within the interval (λ_1, λ_2). For these parameter values $\lambda_{max} = 0.193\ldots$ (see Fig. 4 for corresponding potential $\Phi_\lambda(x)$). Computations of $\psi(\lambda, x)$ for many values of λ within interval $(0, \lambda_{max})$ demonstrate the absence of the sign change of $D(\lambda)$ (see Fig. 5) which means the absence of eigenvalues of L with positive real parts and thus the stability of the corresponding standing front solution.

5 Summary

In this paper, we have considered the stability problem for the standing boundary of non-even age spatially distributed forest under spatial perturbation of its age structure near the boundary. The age dynamics was assumed to be described by a simple reaction-diffusion model (1).

We have derived linear equations governing the time evolution of small perturbations of the boundary and have analyzed their spectrum. It has been shown analytically that the essential spectrum of the corresponding linear differential operator always lies in the left half-plane of the complex plane and therefore does not affect stability. A numerical procedure for the location of the eigenvalues of the operator has been developed and applied to the equations (1). We have found that for certain parameter values, there are no eigenvalues with positive real parts, and hence the boundary is stable.

The proposed procedure can be extended to the analysis of the stability of the traveling forest boundary with minor changes.

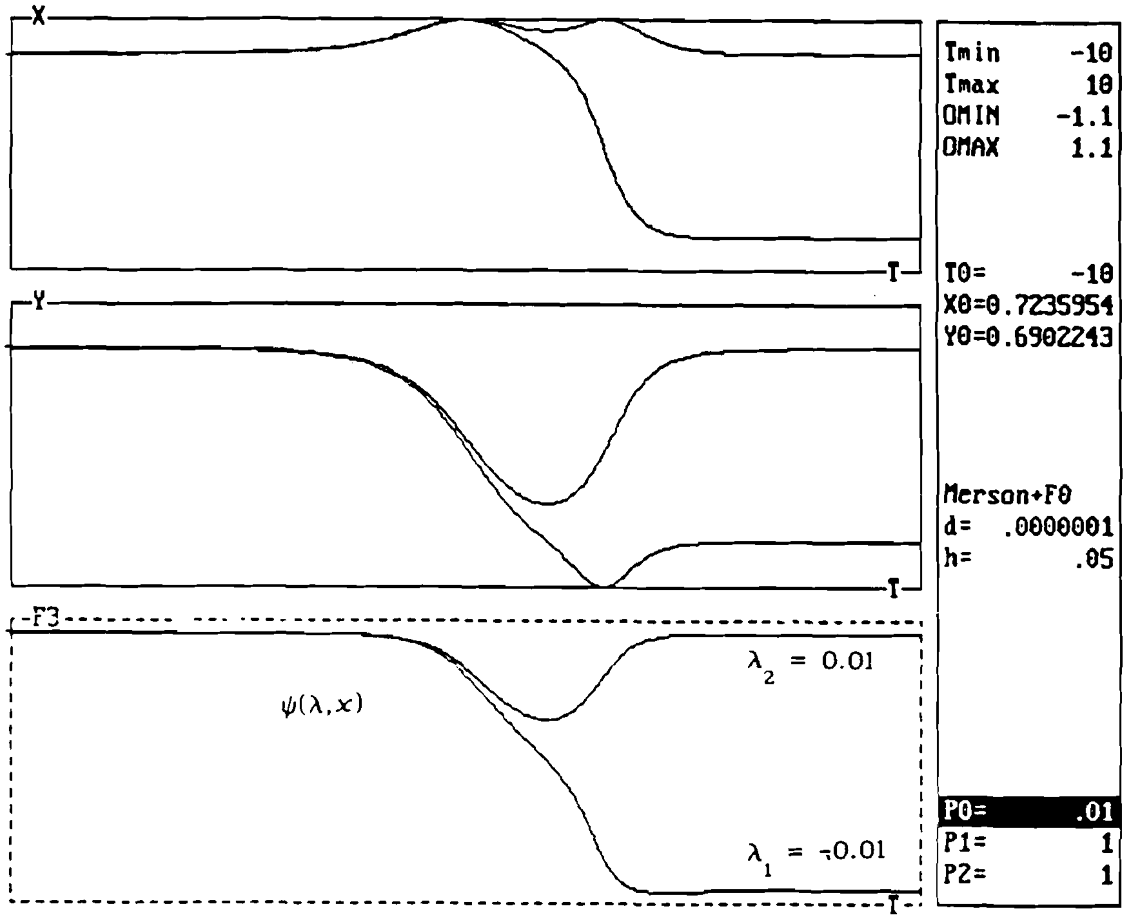

Figure 3: Function $\psi(\lambda, x)$ behavior of $\lambda_1 = -0.01$ and $\lambda_2 = 0.01$: $T \equiv x$, $X \equiv z_1$, $Y \equiv z_2$, $F3 \equiv \psi(\lambda, x)$, $p0 \equiv \lambda$, $pl \equiv s$, $p2 \equiv h$.

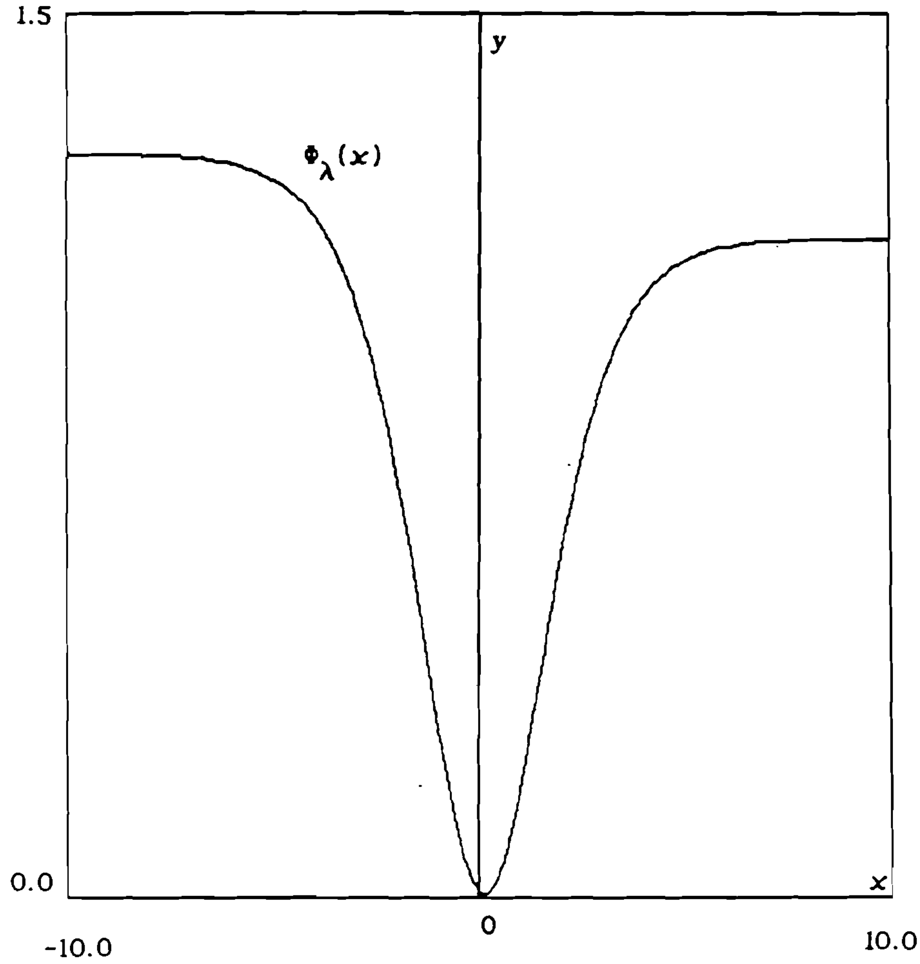

Figure 4: Nonnegative potential $\Phi_\lambda(x)$ for $s = h = 1$ and

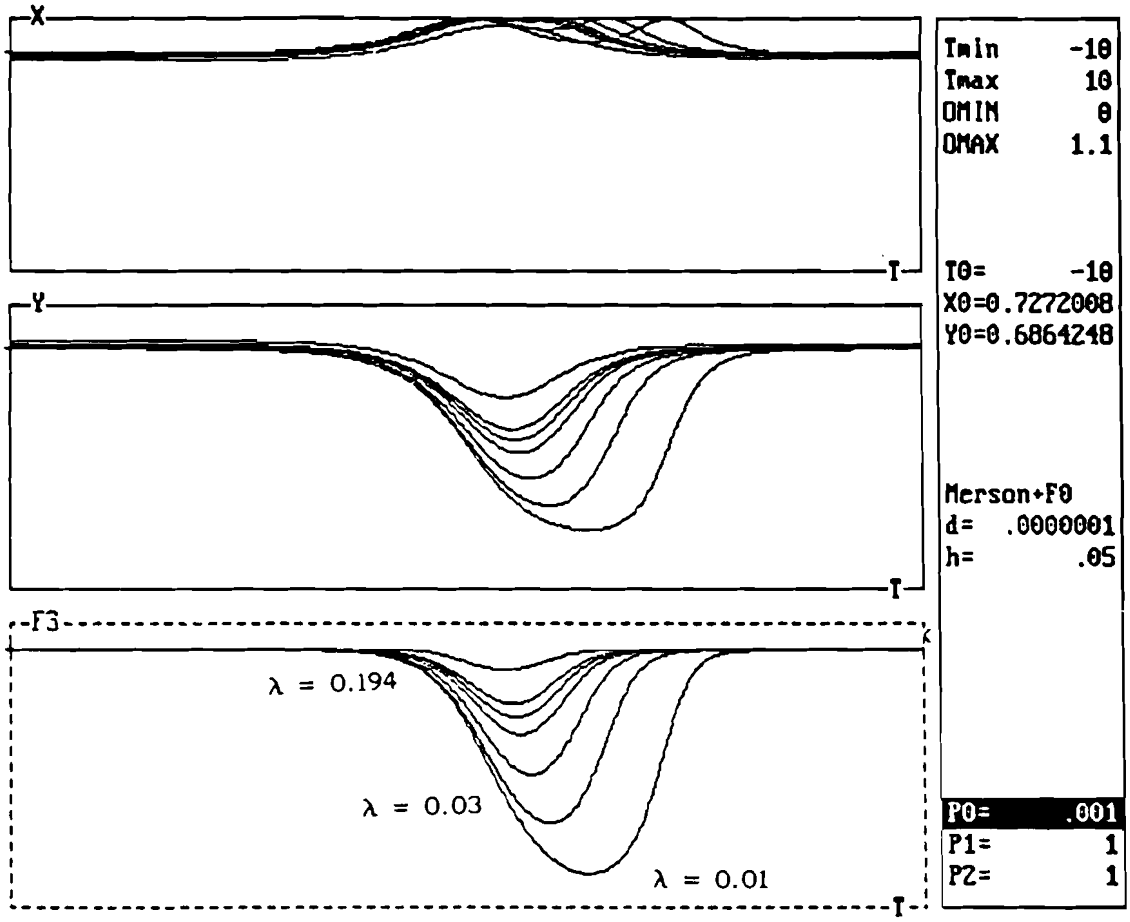

Figure 5: Family of functions $\psi(\lambda, x)$ for various and $s = h = 1$. The absence of the sign change means stability of the standing front. The notations are the same as in Fig. 3.

References

Antonovsky, M.Ya., Aponina, E.M., and Kuznetsov, Yu.A. (1989) Spatial-temporal structure of mixed-age forest boundary: The simplest mathematical model. WP-89-54. International Institute for Applied Systems Analysis, Laxenburg, Austria.

Evans. J.W. (1972) Nerve axon equations, III: Stability of the nerve impulse. *Indiana Univ. Math. J.*, **22**, 577–593.

Evans, J.W. (1975) Nerve axon equations, IV: The Stable and the unstable impulse. *Indiana Univ. Math. J.*, **24**, 1169–1190.

Goldberg, S. Unbounded Linear Operators: Theory and Applications. McGraw-Hill, New York, 1966.

Henry, D. (1981) Geometric Theory of Semilinear Parabolic Equations. Lecture Notes in Mathematics, **80**, Springer-Verlag, Berlin, Heidelberg, New York.