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Preface

In the neighborhoods of attracting heteroclinic cycles, the time averages fail to converge for almost all initial conditions, but spiral closer and closer to the boundary of a polygon. This is shown by using a Poincaré-section argument.

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1. Introduction

In dynamical systems describing the interaction of n different "populations", heteroclinic cycles occur whenever the species of the population supersede each other in cyclic fashion.

A heteroclinic cycle is a cyclic arrangement of saddle equilibria connected by orbits which have one saddle point as α-limit and another one as ω-limit; such systems occur in ecology, genetics, chemical kinetics, game theory, etc. Generally, (continuous time) systems with a heteroclinic cycle are not structurally stable, as the saddle connections can be broken up by arbitrarily small perturbations. However, the heteroclinic cycles may be preserved if the perturbations respect some essential feature or symmetry of the system (for example if there exist invariant subspaces) (see e.g. Guckenheimer and Holmes (1988), Melbourne (1989), Melbourne, Chossat and Golubitsky (1989)).

For general Lotka–Volterra equations

$$\dot{x}_i = x_i f_i(\mathbf{x}) \quad (i = 1, \ldots, n) \quad \text{on } \mathbb{R}^n_+$$

or replicator equations

$$\dot{x}_i = x_i(f_i(\mathbf{x}) - \bar{f}(\mathbf{x})) \quad \text{with} \quad \bar{f}(\mathbf{x}) = \sum_{i=1}^{n} x_i f_i(\mathbf{x}) \quad \text{on the simplex } S_n,$$

which describe models as mentioned above, heteroclinic cycles occur as robust phenomena. The reason is that the boundary of the state space, which corresponds to the absence of one or several populations, is flow invariant.

May and Leonard (1975) give an example of a dynamical system with a heteroclinic cycle, which models competition between three species: Species 2 outcompetes species 1, but is beaten by species 3, which in turn is replaced by species 1.

Other simple examples are the "stone-scissors-paper" game and the "battle-of-the-sexes" in game dynamics; in chemical kinetics such behaviour occurs within the hypercycle (see Hofbauer and Sigmund (1988)).

Let us consider the behaviour of an orbit whose ω-limit is a heteroclinic cycle. If it approaches one of the saddle equilibria, it will linger there for a long period of time, then it skitters alongside the saddle connections to the next fixed point, where it will loiter for a much longer time, then journey to the next one, and so on. Since the orbit converges to the heteroclinic cycle, it comes closer and closer to the fixed points and consequently remains there for longer and longer periods. We will see that, if the saddles are hyperbolic, these sojourn times increase exponentially, while the times required to move from one saddle point to the next change very little. Asymptotically these times are those which the saddle connections require to move from the neighbourhood of one fixed point to the next. They can be neglected, in comparison with the times the orbit spends within these neighbourhoods.

Thus the dynamics is somewhat unpredictable: nearby orbits may leave the neighbourhoods at very different times and evolve quite out of phase. It is difficult to tell in advance near which saddle the orbit $\mathbf{x}(T)$ will linger at some large time T.

Mathematically, the state of the population will never attain any of the fixed points in which only one species is present. In biological systems or numerical simulations, however, the frequencies of the species cannot fall below a certain value (resp. small fluctuations can lead to extinction of some species), and the state will end up in one of the saddle points. It is impossible to predict which one. If by some oportunistic event (e.g. migration or mutation) one of the missing species is reintroduced again, the cycle continues and the population may end up in a different state.

We shall show that the time averages ($\frac{1}{T}\int_0^T \mathbf{x}(t)dt$ for continuous dynamical systems or $\frac{1}{N}\sum_{t=0}^{N-1}\mathbf{x}(t)$ for discrete ones) of orbits tending to the heteroclinic cycle

will not converge, but will spiral closer and closer to the boundary of a polygon. During the time the orbit is bogged down in the vicinity of one of the saddles, the time average, which was close to one of the corners of the polygon, moves towards the next saddle point. This behaviour seems to have been noticed first by E. C. Zeeman (unpublished); Hofbauer and Sigmund (1988) prove this result for the May–Leonard system. We use a more general method.

Let us consider as a simple example the "stone-scissors-paper" game with payoff-matrix

$$A = \begin{pmatrix} 0 & 1-\varepsilon & -1 \\ -1 & 0 & 1-\varepsilon \\ 1-\varepsilon & -1 & 0 \end{pmatrix}.$$

The state of the population at any time is given by a vector $\mathbf{x} = (x_1, x_2, x_3)$, where $x_i = x_i(t)$ denotes the frequency of individuals which play strategy i; the payoff a_{ij} for a player using strategy i against a j-player is given by A. We therefore have three available strategies which beat each other in cyclic fashion. If we assume that the rate of increase $\frac{\dot{x}_i}{x_i}$ of strategy i is given by $(A\mathbf{x})_i - \mathbf{x}A\mathbf{x}$ (the difference between its payoff and the average payoff of the population), we obtain the game dynamical equation

$$\dot{x}_i = x_i((A\mathbf{x})_i - \mathbf{x}A\mathbf{x})$$

on the simplex S_3. If $\varepsilon > 0$ the boundary of the simplex forms an attracting heteroclinic cycle, and the time averages of all orbits $t \mapsto \mathbf{x}(t)$ in the interior of S_3 (with exception of the interior fixed point) tend to the triangle spanned by

$$\mathbf{A}_1 = \frac{1}{(1-\varepsilon)^2 + (1-\varepsilon) + 1}(1, 1-\varepsilon, (1-\varepsilon)^2)$$

and corresponding points $\mathbf{A}_2$ and $\mathbf{A}_3$ (see Fig. 1). (Cf. section 4.)

Fig. 1

In section 2 we deal with two-dimensional ~~hyperbolic~~ heteroclinic cycles, in section 3 we apply these results to some examples, in section 4 we give a sketch for the case of higher dimensional systems, and finally, in section 5 we briefly sketch the case of discrete dynamical systems.

2. The two-dimensional case

Let $\dot{\mathbf{x}} = f(\mathbf{x})$ be a continuous dynamical system defined on a subset of $\mathbb{R}^2$ and let $\mathbf{F}_0, \ldots, \mathbf{F}_{n-1}$ be saddle equilibria which correspond to a heteroclinic cycle Γ.

Each saddle point $\mathbf{F}_i$ of Γ has a positive eigenvalue $\lambda_i > 0$ and a negative eigenvalue $-\mu_i < 0$. We say that Γ is repelling (resp. attracting) if it is the α- (resp. ω-) limit of any orbit which starts in a vicinity of Γ. Γ is repelling if $\rho := \prod_{i=0}^{n-1} \frac{\mu_i}{\lambda_i} < 1$ (i.e. the product of the "outgoing velocities" is greater than the product of the "incoming velocities"), and it is attracting if $\rho > 1$. (See e.g. Hofbauer and Sigmund (1988). It will also follow from the proof we give.)

We linearize the system in an arbitrarily small neighbourhood $U(\mathbf{F}_i)$ of the saddle equilibria and take local coordinates such that $\dot{x} = \lambda_i x$ and $\dot{y} = -\mu_i y$ hold approximately. Now, consider the cross-sections $S_i = \{(x, y) : y = 1\}$ "before" $\mathbf{F}_i$ and $S_i' = \{(x, y) : x = 1\}$ "after" $\mathbf{F}_i$ (see Fig. 2).

Fig. 2

Lemma 1. *Let $\mathbf{x}(t)$ be an orbit with initial point $\mathbf{x} = (x, 1)$, in the local coordinates in $U(\mathbf{F}_i)$, and t_i be the time spent in $U(\mathbf{F}_i)$. Let $\rho_i := \frac{\mu_{i-1}}{\lambda_i}$. Then*

$$\lim_{x \to 0} \frac{t_{i+1}}{t_i} = \rho_{i+1} .$$

REMARK: We always will count the indices of the saddles $\mathbf{F}_i$, of the eigenvalues, and of the ρ_i modulo n.

PROOF: An orbit which starts in $(x,1)$ crosses S_i' in $(1,y) = (xe^{\lambda_i t}, e^{-\mu_i t})$. Therefore the time the orbit spends between S_i and S_i' is $t = -\frac{1}{\lambda_i}\log x$. The transition map $\varphi_i : S_i \to S_i'$, $(x,1) \mapsto (1,y)$, is given by $\varphi_i(x) = x^{\frac{\mu_i}{\lambda_i}}$. (Since one coordinate is fixed, φ_i is a function in one variable.) The transition map $\psi_i : S_i' \to S_{i+1}$, $(1,y) \mapsto (x',1)$ can be expanded into a Taylor series and $\psi_i((1,0)) = (0,1)$. Thus we have in first order approximation $\psi_i(x) = a_i x$, for some positive constant a_i.

It is enough to consider $i = 0$.

$$t_1 = -\frac{1}{\lambda_1}\log \psi_0 \circ \varphi_0(x) = -\frac{1}{\lambda_1}\log a_0 x^{\frac{\mu_0}{\lambda_0}}$$

$$\Longrightarrow \quad \frac{t_1}{t_0} = \frac{\lambda_0}{\lambda_1} \cdot \frac{\log a_0 + \frac{\mu_0}{\lambda_0}\log x}{\log x} \to \frac{\mu_0}{\lambda_1} \quad \text{for } x \to 0. \qquad \square$$

REMARK 1: It is easy to see that the choice of the Poincaré-sections at $(x,1)$ resp. $(1,y)$ is no restriction of generality. We obtain the same result if we choose (x,η) and (η',y) for any $\eta, \eta' > 0$.

REMARK 2: For the Poincaré map $g = \psi_{n-1} \circ \varphi_{n-1} \circ \cdots \circ \psi_0 \circ \varphi_0 : S_0 \to S_0$ we obtain $g(x) = ax^\rho$, where

$$\rho = \prod_{i=1}^{n} \rho_i = \prod_{i=0}^{n-1} \frac{\mu_i}{\lambda_i} \quad \text{and} \quad a = a_{n-1} a_{n-2}^{\frac{\mu_{n-1}}{\lambda_{n-1}}} \cdots a_0^{\prod_{i=1}^{n-1} \frac{\mu_i}{\lambda_i}}$$

and

$$\frac{t_{i+n}}{t_i} \to \rho \quad \text{for } x \to 0.$$

Hence, asymptotically the sojourn times near the saddle points $\mathbf{F}_i$ increase exponentially with factor ρ.

(A determination of the precise form of the Poincaré map which also takes higher order terms into consideration can be found e.g. in Anosov and Arnold (1988).)

REMARK 3: Proof of Lemma 1 shows that the heteroclinic cycle is attracting if $\rho > 1$ and that it is repelling if $\rho < 1$.

Theorem 1. *Let* $\mathbf{x}(t)$ *be an orbit whose* ω*-limit is* Γ. *The accumulation points of the time average* $\frac{1}{T}\int_0^T \mathbf{x}(t)dt$ *form the boundary of the polygon* $\mathbf{A}_0 \ldots \mathbf{A}_{n-1}$ *where*

$$\begin{aligned}\mathbf{A}_i &= \frac{\mathbf{F}_{i+1} + \rho_{i+2}\mathbf{F}_{i+2} + \ldots + \rho_{i+2}\cdots\rho_{i+n}\mathbf{F}_i}{1 + \rho_{i+2} + \ldots + \rho_{i+2}\cdots\rho_{i+n}} \\ &= \frac{\lambda_{i+2}\cdots\lambda_{i+n}\mathbf{F}_{i+1} + \lambda_{i+3}\cdots\lambda_{i+n}\mu_{i+1}\mathbf{F}_{i+2} + \ldots + \mu_{i+1}\cdots\mu_{i+n-1}\mathbf{F}_{i+n}}{\lambda_{i+2}\cdots\lambda_{i+n} + \lambda_{i+3}\cdots\lambda_{i+n}\mu_{i+1} + \ldots + \mu_{i+1}\cdots\mu_{i+n-1}}.\end{aligned} \tag{1}$$

The points $\mathbf{A}_i$, $\mathbf{A}_{i+1}$, *and* $\mathbf{F}_{i+1}$ *are collinear.*

REMARK: Thus, asymptotically, the time averages move on a line from $\mathbf{A}_i$ to $\mathbf{A}_{i+1}$ in the direction of $\mathbf{F}_{i+1}$.

PROOF:

(i) Consider first a piecewise constant function $\mathbf{x}_t$ from $\mathbb{R}$ to $\mathbb{R}^2$ whose values alternate cyclically between the $\mathbf{F}_i$, with the property that if it remains at $\mathbf{F}_{i-1}$ for a time interval of length t_{i-1} then it next takes the value $\mathbf{F}_i$ for a span of time t_i, with $t_i = \rho_i t_{i-1}$. The jump to $\mathbf{F}_i$ takes place at time T_i. (While the indices of the points F_i, the eigenvalues and the ρ_i are counted cyclically, the indices of the times t_i and T_i are not.)

The time average of $\mathbf{x}_t$ over one full cycle starting at T_i is given by

$$\mathbf{A}_{i-1} = \frac{\mathbf{F}_i + \rho_{i+1}\mathbf{F}_{i+1} + \ldots + \rho_{i+1}\cdots\rho_{i+n-1}\mathbf{F}_{i+n-1}}{1 + \rho_{i+1} + \ldots + \rho_{i+1}\cdots\rho_{i+n-1}}.$$

A straightforward computation shows that

$$\mathbf{A}_i = \frac{[\rho_{i+1} + \ldots + \rho_{i+1}\cdots\rho_{i+n} + (1-\rho)]\mathbf{A}_{i-1} + (\rho - 1)\mathbf{F}_i}{\rho_{i+1} + \ldots + \rho_{i+1}\cdots\rho_{i+n}}. \tag{2}$$

We have

$$\lim_{\substack{k\to\infty \\ k\equiv i \pmod n}} \frac{1}{T_k}\int_0^{T_k} \mathbf{x}_t dt = \mathbf{A}_{i-1},$$

since this average consists of full cycles (up to an initial phase which does not affect the limit). If $T \in [T_k, T_{k+1})$ (for $k \equiv i \pmod n$), then

$$\frac{1}{T}\int_0^T \mathbf{x}_t dt = \frac{T_k}{T}\left(\frac{1}{T_k}\int_0^{T_k} \mathbf{x}_t dt\right) + \frac{T - T_k}{T}\left(\frac{1}{T - T_k}\int_{T_k}^T \mathbf{x}_t dt\right).$$

The time average in the second term is just $\mathbf{F}_i$. The time average of the first summation term converges to $\mathbf{A}_{i-1}$. For sufficiently large T (i.e. k) the time average is arbitrarily close to a convex combination of $\mathbf{A}_{i-1}$ and $\mathbf{F}_i$. For $T = T_k$ this yields $\mathbf{A}_{i-1}$, for $T = T_{k+1}$ it yields in the limit $\mathbf{A}_i$: Since $k = i + nm$ for some suitable $m \in \mathbb{N}$ ($m \to \infty$ as $k \to \infty$) we can write

$$T_k = T_i + \frac{\rho^m - 1}{\rho - 1}(\rho_i + \rho_i\rho_{i+1} + \cdots + \rho_i \cdots \rho_{i+n-1})t_{i-1}$$

and

$$T_{k+1} - T_k = \rho^m \rho_i t_{i-1}.$$

Hence, for $m \to \infty$ we obtain (2).

Thus every possible time limit lies on the polygon spanned by the $\mathbf{A}_i$. Conversely, any given point on the segment from $\mathbf{A}_{i-1}$ to $\mathbf{A}_i$ occurs as limit point for a suitable sequence of time averages.

(ii) Let us consider next a piecewise constant function with the same properties as above, except that $t_i = \rho_i t_{i-1}$ is replaced by $\frac{t_i}{t_{i-1}} \to \rho_i$ for $i \to \infty$ (we recall that the index of ρ_i is taken modulo n). The same argument as in (i) applies, with the exception that now the time averages over a full cycle starting at T_0 *converge* to $\mathbf{A}_{i-1}$ as given by (1).

(iii) Next let us consider a function $\mathbf{x}_t$ which is no longer piecewise constant. It still, as in (ii), takes on the value $\mathbf{F}_i$ for intervals of lenght t_i, but now the translation from one $\mathbf{F}_i$ to the next does not occur as a jump, but as a continuous movement during some time interval of uniformly bounded length. Since the union of these transition intervals is of density zero, this does not affect the accumulation points of the time averages of $\mathbf{x}_t$, they are still given by the polygon as above.

(iv) Finally, we note that the behaviour of the time average of the orbit $\mathbf{x}(t)$ is asymptotically the same as that of $\mathbf{x}_t$: For any $\varepsilon > 0$ we can find neighbourhoods $U_\varepsilon(\mathbf{F}_i)$ of diameter $< \varepsilon$ into which we can introduce local coordinates as

in Lemma 1. The times the orbit $\mathbf{x}(t)$ needs to switch from $U_\varepsilon(\mathbf{F}_i)$ to $U_\varepsilon(\mathbf{F}_{i+1})$ converge to the time needed by the saddle-connections to cross from one neighbourhood to the next, hence they are uniformly bounded. The two functions $\mathbf{x}(t)$ and $\mathbf{x}_t$ differ only in that, while $\mathbf{x}(t)$ is in $U_\varepsilon(\mathbf{F}_i)$, $\mathbf{x}_t$ is precisely at $\mathbf{F}_i$. Therefore the time averages differ only by at most ε, which can be chosen arbitrarily small.

□

REMARK: $\mathbf{A}_i \to \mathbf{A}_{i+1}$ for $\rho \to 1$; that is, the polygon degenerates to a single point in this case.

If $\rho = 1$, it is not possible to say generally if the heteroclinic cycle is attracting or repelling. If it is attracting, the sojourn times near the saddles do not grow exponentially, but linearly.

We see by Lemma 1 that the Poincaré map is approximately given by the linear map $g(x) = ax$, and therefore the ratio of the time intervals $\frac{t_{i+n}}{t_i}$ converges to 1. We have

$$t_{i+n} - t_i \to -\frac{1}{\lambda_i} \log a \quad \text{as } i \to \infty.$$

For Lotka–Volterra and replicator equations the time averages for orbits which tend to the inner equilibrium or are periodic, converge to the inner equilibrium. Theorem 1 shows that the time averages converge to this equilibrium also for $\rho = 1$, even if the heteroclinic cycle is attracting.

We have discussed the "generic" case that all saddle equilibria of an attracting heteroclinic cycle are hyperbolic. By a heuristic argument it seems plausible that the formulas for the degenerate case can be obtained by simple passage to the limit. We show that (1) still holds, by way of example, for the case that one of the "outgoing" eigenvalues of the saddle points, say λ_0, is zero.

As in the generic case we can linearize the vector field in neighbourhoods of the saddle $\mathbf{F}_i$ for $i = 1, \ldots, n-1$. In $U(\mathbf{F}_0)$ we can take local coordinates such

that $\dot{x} = ax^2 + bxy$ (we recall that $\{x = 0\}$ is invariant) and $\dot{y} = -\mu_0 y$ hold approximately. (In the first equation we consider terms of second order, since the corresponding eigenvalue is zero. a, b are appropriate constants.) If we neglect higher order terms we can consider an equation of the form $\dot{x} = \nu x^2$, $\dot{y} = -\mu_0 y$, which we obtain by a differentiable coordinate transformation of the form $\mathbf{x} \to \mathbf{x} + h(\mathbf{x})$, where $h(\mathbf{x})$ is a homogeneous vector polynomial. (This is a special case of a theorem of Poincaré–Dulac, see e.g. Arnold (1983).)

Now consider cross-sections $S_i = \{(x, 1)\}$ and $S_i' = \{(1, y)\}$ as in Lemma 1.

Lemma 1 still holds with $\rho_n = \infty$.

PROOF: The transition map $\varphi_0 : S_0 \to S_0'$ is now given by

$$(x, 1) \mapsto \left(\frac{1}{\frac{1}{x} - \nu t}, e^{-\mu_0 t} \right) = (1, y),$$

so that

$$t = \frac{1}{\nu} \cdot \frac{1 - x}{x} \quad \text{and} \quad \varphi_0(x) = e^{-\frac{\mu_0}{\nu} \cdot \frac{1-x}{x}}.$$

This yields $\psi_0 \circ \varphi_0(x) = a_0 e^{-\frac{\mu_0}{\nu} \cdot \frac{1-x}{x}}$ (for an appropriate positive constant a_0).

We have

$$\frac{t_1}{t_0} = -\frac{1}{\lambda_1} \left(\frac{\nu x}{1 - x} \log a_0 - \mu_0 \right) \to \frac{\mu_0}{\lambda_1} = \rho_1 \quad \text{as } x \to 0.$$

Further,

$$\frac{t_n}{t_{n-1}} = -\frac{\lambda_{n-1}}{\nu} \cdot \frac{1 - \psi_{n-1} \circ \varphi_{n-1}(x)}{\log x \cdot \psi_{n-1} \circ \varphi_{n-1}(x)} \to \infty \quad \text{as } x \to 0,$$

since $\psi_{n-1} \circ \varphi_{n-1}(x) = a_{n-1} x^{\frac{\mu_{n-1}}{\lambda_{n-1}}}$, as in the generic case. □

Hence *Theorem 1 holds also for* $\lambda_0 = 0$, *that is*

$$\mathbf{A}_0 = \mathbf{F}_0$$

$$\mathbf{A}_1 = \frac{\mathbf{F}_0 + \rho_1 \mathbf{F}_1}{1 + \rho_1}$$

$$\vdots$$

$$\mathbf{A}_{n-1} = \frac{\mathbf{F}_0 + \rho_1 \mathbf{F}_1 + \cdots + \rho_1 \cdots \rho_{n-1} \mathbf{F}_{n-1}}{1 + \cdots \rho_1 \cdots \rho_{n-1}}.$$

3. Examples

Example 1

In the "general stone-scissors-paper" game, described by the equation

$$\dot{x}_i = x_i[(Ax)_i - \mathbf{x}Ax] \quad \text{on } S_3$$

with

$$A = \begin{pmatrix} 0 & a_3 & -b_2 \\ -b_1 & 0 & a_2 \\ a_1 & -b_3 & 0 \end{pmatrix} \qquad (a_i, b_i > 0),$$

the heteroclinic cycle Γ: $\mathbf{F}_1 = (1,0,0) \to \mathbf{F}_2 = (0,0,1) \to \mathbf{F}_3 = (0,1,0) \to \mathbf{F}_1$ is attracting iff $b_1 b_2 b_3 > a_1 a_2 a_3$ (a_i and $-b_i$ are the eigenvalues at $\mathbf{F}_i$). The time averages approach the triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ with

$$\mathbf{A}_1 = \frac{1}{a_1a_3 + a_1b_2 + b_2b_3}(b_2b_3, a_1b_2, a_1a_3)$$
$$\mathbf{A}_2 = \frac{1}{a_1a_2 + a_2b_3 + b_1b_3}(a_2b_3, a_1a_2, b_1b_3)$$
$$\mathbf{A}_3 = \frac{1}{a_2a_3 + a_3b_1 + b_1b_2}(a_2a_3, b_1b_2, a_3b_1)\,.$$

(See Fig. 1.)

Example 2

The equation

$$\dot{x} = x(1-x)(a + bx + cy)$$
$$\dot{y} = y(1-y)(d + ex + fy)$$

on $[0,1] \times [0,1]$ describes the dynamics of asymmetric games between two populations interacting with themselves and with each other, with two strategies for each (see Schuster et al. (1981)).

The boundary of the state space forms a heteroclinic cycle $\mathbf{F}_1 = (0,0) \to \mathbf{F}_2 = (1,0) \to \mathbf{F}_3 = (1,1) \to \mathbf{F}_4 = (0,1) \to \mathbf{F}_1$ if the expressions $\lambda_1 = a$, $\lambda_2 = d + e$,

$\lambda_3 = -(a+b+c)$, $\lambda_4 = -(d+f)$, $\mu_1 = -d$, $\mu_2 = a+b$, $\mu_3 = d+e+f$, $\mu_4 = -(a+c)$ are all positive (resp. all negative; then the cycle runs the other way round). λ_i and $-\mu_i$ are the eigenvalues of the saddle equilibria. If $\prod \lambda_i > \prod \mu_i$, the cycle is attracting, and the time averages spiral to the quadrangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4$ (see Fig. 3),

$$\mathbf{A}_1 = \frac{1}{\lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\mu_2 + \lambda_1\mu_2\mu_3 + \mu_2\mu_3\mu_4} \cdot (\lambda_1\lambda_4(\lambda_3+\mu_2), \lambda_1\mu_2(\lambda_4+\mu_3))$$

$$\mathbf{A}_2 = \frac{1}{\lambda_4\lambda_1\lambda_2 + \lambda_1\lambda_2\mu_3 + \lambda_2\mu_3\mu_4 + \mu_3\mu_4\mu_1} \cdot (\lambda_4\lambda_1\lambda_2 + \mu_3\mu_4\mu_1, \lambda_1\lambda_2(\lambda_4+\mu_3))$$

$$\mathbf{A}_3 = \frac{1}{\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\mu_4 + \lambda_3\mu_4\mu_1 + \mu_4\mu_1\mu_2} \cdot (\mu_4\mu_1(\lambda_3+\mu_2), \lambda_1\lambda_2\lambda_3 + \mu_4\mu_1\mu_2)$$

$$\mathbf{A}_4 = \frac{1}{\lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\mu_1 + \lambda_4\mu_1\mu_2 + \mu_1\mu_2\mu_3} \cdot (\lambda_4\mu_1(\lambda_3+\mu_2), \mu_1\mu_2(\lambda_4+\mu_3)).$$

Fig. 3

4. Time averages for higher dimensional attracting heteroclinic cycles

In higher dimensions it is difficult to describe heteroclinic cycles generally. We will discuss the situation for the May Leonard system (1975) and for the simplex S_4.

(a) The system of May and Leonard

The Lotka–Volterra equation

$$\dot{x}_1 = x_1(1 - x_1 - \alpha x_2 - \beta x_3)$$

$$\dot{x}_2 = x_2(1 - \beta x_1 - x_2 - \alpha x_3)$$

$$\dot{x}_3 = x_3(1 - \alpha x_2 - \beta x_2 - x_3)$$

on $\mathbb{R}^3_+$ describes a competition between three species.

The equilibria $\mathbf{F}_1 = (1,0,0)$, $\mathbf{F}_2 = (0,1,0)$ and $\mathbf{F}_3 = (0,0,1)$ form an attracting heteroclinic cycle, if $0 < \beta < 1 < \alpha$ and $\alpha + \beta > 2$ (see May and Leonard (1975)).

The eigenvalues of the saddle equilibria "in direction of the heteroclinic cycle" are $\lambda := 1 - \beta$ and $-\mu := -(\alpha - 1)$; -1 is the "transversal" eigenvalue.

If we linearize the system at $\mathbf{F}_i$, we obtain

$$\dot{x} = -x \qquad \dot{y} = \lambda y \qquad \dot{z} = -\mu z.$$

Analogously to the two-dimensional case, we take cross-sections $S_i = \{(x, y, 1)\}$ "before" and $S_i' = \{(x, 1, z)\}$ "after" $\mathbf{F}_i$. The transition map $\varphi_i : S_i \to S_i'$ for the linearized system is given by $\varphi_i(x, y, 1) = (xe^{-t}, ye^{\lambda t}, e^{-\mu t}) = (xy^{\frac{1}{\lambda}}, 1, y^{\frac{\mu}{\lambda}})$, and $t = -\frac{1}{\lambda} \log y$ is the time the orbit spends between S_i and S_i'. The transition map $\psi_i : S_i' \to S_{i+1}$ maps $(x, 1, 0)$ to $(x', 0, 1)$, hence it can be approximated by $\psi_i(x, 1, z) = (\bar{f}_i(x, z), a_i z, 1)$, where a_i is an appropriate positive constant and $\bar{f}_i(x, z)$ some differentiable function (see Fig. 4).

Fig. 4

Therefore

$$\psi_i \circ \varphi_i(x, y, 1) = (f_i(x, y), a_i y^{\frac{\mu}{\lambda}}, 1).$$

The situation is similar to the two-dimensional case and Theorem 1 holds. Thus the time averages for orbits in $\operatorname{int} \mathbb{R}^3_+$ converging to this cycle (all except those on the diagonal) tend to the boundary of the triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$, where

$$\mathbf{A}_i = \frac{\lambda^2 \mathbf{F}_{i+1} + \lambda\mu \mathbf{F}_{i+2} + \mu^2 \mathbf{F}_i}{\lambda^2 + \lambda\mu + \mu^2}$$

(the indices of the $\mathbf{F}_i$ are counted modulo 3), which lies on the simplex S_3. That is

$$\mathbf{A}_1 = \frac{1}{(1-\beta)^2 + (1-\beta)(\alpha-1) + (\alpha-1)^2} \cdot \left((\alpha-1)^2, (1-\beta)^2, (\alpha-1)(1-\beta)\right)$$

$$\mathbf{A}_2 = \frac{1}{(1-\beta)^2 + (1-\beta)(\alpha-1) + (\alpha-1)^2} \cdot \left((\alpha-1)(1-\beta), (\alpha-1)^2, (1-\beta)^2\right)$$

$$\mathbf{A}_3 = \frac{1}{(1-\beta)^2 + (1-\beta)(\alpha-1) + (\alpha-1)^2} \cdot \left((1-\beta)^2, (\alpha-1)(1-\beta), (\alpha-1)^2\right).$$

The "transversal" eigenvalues do not affect the time averages.

This result was shown by Hofbauer and Sigmund (1988), by a different method.

An analogous situation yields the system

$$\dot{x} = x(\ell + ax^2 + by^2 + cz^2)$$

$$\dot{y} = y(\ell + cx^2 + ay^2 + bz^2)$$

$$\dot{z} = z(\ell + bx^2 + cy^2 + az^2),$$

which was studied by Guckenheimer and Holmes (1988). The trajectories $\mathbf{F}_1 = (0, \sqrt{-\frac{1}{a}}, 0) \to \mathbf{F}_2 = (\sqrt{-\frac{1}{a}}, 0, 0) \to \mathbf{F}_3 = (0, 0, \sqrt{-\frac{1}{a}}) \to \mathbf{F}_1$ form an attracting heteroclinic cycle if $\ell = 1$, $a + b + c = -1$, $\frac{1}{3} < a < 0$, $c < a < b < 0$. The eigenvalues "in direction of the cycle" are $\lambda = \frac{1}{a}(a - b)$ and $-\mu = -\frac{1}{a}(a - c)$ and therefore the time averages for almost all orbits spiral to the triangle $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$,

$$\mathbf{A}_1 = \frac{\sqrt{-\frac{1}{a}}}{(a-b)^2 + (a-b)(a-c) + (a-c)^2} \cdot \left((a-b)^2, (a-c)^2, (a-b)(a-c)\right)$$

$$\mathbf{A}_2 = \frac{\sqrt{-\frac{1}{a}}}{(a-b)^2 + (a-b)(a-c) + (a-c)^2} \cdot \left((a-c)^2, (a-b)(a-c), (a-b)^2\right)$$

$$\mathbf{A}_3 = \frac{\sqrt{-\frac{1}{a}}}{(a-b)^2 + (a-b)(a-c) + (a-c)^2} \cdot \left((a-b)(a-c), (a-b)^2, (a-c)^2\right).$$

(b) Attracting heteroclinic cycles on S_4

The previous section may suggest that formula (1) also holds in higher dimensions, that is, the set of accumulation points of the time averages only depends on the "incoming" and "outgoing" eigenvalues at each saddle and are not influenced by the "transversal" eigenvalues. However, the following example shows that this needs not always be true.

Consider a dynamical system defined on the simplex S_4 where the corners of the simplex $\mathbf{F}_1 = (1, 0, 0, 0), \ldots, \mathbf{F}_4 = (0, 0, 0, 1)$ and the edges $\mathbf{F}_i\mathbf{F}_{i+1}$ form an

attracting heteroclinic cycle and the "faces" $\{x_i = 0\}$ are invariant. Further, let there be a positive eigenvalue λ_i and two negative eigenvalues $-\mu_i$ and $-\sigma_i$ at every vertex $\mathbf{F}_i$ (λ_i and $-\mu_i$ are the eigenvalues "along" the cycle, $-\sigma_i$ is the "transversal" eigenvalue). (We count the indices modulo 4.)

Such situations may occur in replicator equations (see e.g. Hofbauer and Sigmund (1988), Gaunersdorfer et al. (1990)). We first discuss a result from Hofbauer (1987), see also Hofbauer and Sigmund (1988):

The linearization at $\mathbf{F}_i$ yields

$$\dot{x} = \lambda_i x \qquad \dot{y} = -\mu_i y \qquad \dot{z} = -\sigma_i z.$$

We take the cross-sections as in the previous example, and the transition maps are given by $\varphi_i(x,1,z) = (1, x^{\frac{\mu_i}{\lambda_i}}, z x^{\frac{\sigma_i}{\lambda_i}})$ resp. $\psi_i(1,y,z) = (a_i z, 1, b_i y)$, since the faces of S_4 are invariant (see Fig. 5).

Fig. 5

Then $\psi_i \circ \varphi_i(x,1,z) = (a_i z x^{\frac{\sigma_i}{\lambda_i}}, 1, b_i x^{\frac{\mu_i}{\lambda_i}})$ (a_i and b_i are appropriate constants), and the Poincaré map is of the form $g(x,z) = (Ax^\alpha z^\beta, Bx^\gamma z^\delta)$ (A, B, α, β, γ, δ are positive constants, depending on the eigenvalues λ_i, $-\mu_i$ and $-\sigma_i$).

It is convenient to change the coordinates to $y_1 := -\log x$ and $y_2 := -\log z$. Then the transition map can be written in the form $\psi_i \circ \varphi_i : \mathbf{y} \mapsto P_i \mathbf{y} + \mathbf{q}_i$ where

$$P_i = \begin{pmatrix} \frac{\sigma_i}{\lambda_i} & 1 \\ \frac{\mu_i}{\lambda_i} & 0 \end{pmatrix}$$

and $\mathbf{q}_i$ depends on a_i, b_i, λ_i, μ_i, σ_i.

The Poincaré map $S_1 \to S_1$ transforms to $\mathbf{y} \mapsto P\mathbf{y} + \mathbf{q}$ with

$$P = \prod_{i=1}^{4} P_i = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} -\log A \\ -\log B \end{pmatrix}$$

and the heteroclinic cycle Γ corresponds to the limit $y_1, y_2 \to \infty$.

As all entries of P are positive, P has a dominant eigenvalue $\rho > 0$ and has positive left and right eigenvalues $\mathbf{u}$ and $\mathbf{v}$, by the theorem of Perron–Frobenius.

If $\rho \neq 1$ we can ignore the translation term $\mathbf{q}$ (by replacing $\mathbf{y}$ by $\mathbf{y} + \hat{\mathbf{y}}$, where $\hat{\mathbf{y}}$ is the fixed point of g). If $\rho > 1$ we have $\rho^{-k} P^k \mathbf{y} \to \mathbf{u}$ as $k \to \infty$ for all $\mathbf{y} > 0$, hence Γ is attracting. For $\rho < 1$ all orbits of $\mathbf{y} \to P\mathbf{y}$ tend to 0 and Γ is repelling.

As $\rho^{-k} P^k \mathbf{y} \to \mathbf{u}$ (for $k \to \infty$), there is a h_k for every k, $\lim_{k\to\infty} h_k = 0$, such that $(P^k \mathbf{y})_1 = \rho^k (u_1 + h_k)$. We define t_i as in section 1. The ratio of the time intervals an orbit spends within a neighbourhood of $\mathbf{F}_1$ converges to

$$\lim_{\substack{k\to\infty \\ k\equiv 1}} \frac{t_{k+4}}{t_k} = \lim_{k\to\infty} \frac{(P^{k+1}\mathbf{y})_1}{(P^k \mathbf{y})_1} = \lim_{k\to\infty} \frac{\rho^{k+1}(u_1 + h_{k+1})}{\rho^k (u_1 + h_k)} = \rho.$$

That is, the length of the time intervals the orbit remains "near" the saddles asymptotically grows exponentially with factor ρ (cf. the planar case).

Further we have

$$\begin{aligned} \lim_{y_1, y_2 \to \infty} \frac{t_{i+1}}{t_i} &= \lim_{y_1, y_2 \to \infty} \frac{\lambda_i}{\lambda_{i+1}} \cdot \frac{(\psi_i \circ \varphi_i(\mathbf{y}))_1}{y_1} \\ &= \frac{\sigma_i}{\lambda_{i+1}} + \frac{u_2}{u_1} = \frac{\sigma_i}{\lambda_{i+1}} + \frac{\rho - \alpha(i)}{\beta(i)}, \end{aligned}$$

where $\alpha(i)$ and $\beta(i)$ are entries of the matrix $P(i) = \prod_{j=i}^{i+3} P_j$.

In what follows we restrict ourselves to the symmetric case $\lambda_i =: \lambda$, $\mu_i := \mu$, and $\sigma_i := \sigma$ for all i. Then $\rho = \omega^4$, where ω is the dominant eigenvalue of P_i, i.e. $\omega^2 - \frac{\sigma}{\lambda}\omega - \frac{\mu}{\lambda} = 0$ and $\mu = \omega(\lambda\omega - \sigma)$. Further $\alpha = \frac{\mu^2}{\lambda^2} + 3\frac{\mu\sigma^2}{\lambda^3} + \frac{\sigma^4}{\lambda^4}$ and $\beta = 2\frac{\mu\sigma}{\lambda^2} + \frac{\sigma^3}{\lambda^3}$.

(It is easy to see, that ω resp. ρ are > 1 iff $\lambda < \mu + \sigma$.)

Thus the accumulation points of the time averages of orbits that tend to the heteroclinic cycle form the boundary of the "three-dimensional quadrangle" $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\mathbf{A}_4$ with

$$\mathbf{A}_i = \frac{\mathbf{F}_{i+1} + \omega\mathbf{F}_{i+2} + \omega^2\mathbf{F}_{i+3} + \omega^3\mathbf{F}_i}{1 + \omega + \omega^2 + \omega^3}.$$

We see that, analogously to the planar case, the polygon degenerates to a point for $\rho \to 1$ (i.e. $\omega \to 1$).

5. Difference equations

Now let us consider an analogous situation for a discrete dynamical system $\mathbf{x} \mapsto f(\mathbf{x})$, where f is a differentiable function.

Again we linearize the system near the saddle equilibria $\mathbf{F}_i$ and take local coordinates such that $x \mapsto \lambda_i x$, $y \mapsto \mu_i y$ ($\lambda_i > 1, 0 < \mu_i < 1$) approximates the dynamics near $\mathbf{F}_i$ (where λ_i and μ_i are the eigenvalues of the linearization).

Let $S_i = \{(x, \eta(x))\}$ ($\eta(x) \in (\eta - \delta, \eta + \delta)$ for some small $\delta > 0$) and $S_i' = \{(\eta(y), y)\}$ cross-sections "before" and "after" $\mathbf{F}_i$ (generally, we cannot take a straight line for S_i (resp. S_i'), but a curve lying in a small strip around the line $x = \eta$ resp. $y = \eta$). Without loss of generality we set $\eta = 1$. For the transition map $\varphi_i : S_i \to S_i'$ we get $(x, 1) \mapsto (\lambda_i^{k_i} x \approx 1, \mu_i^{k_i})$. This yields $k_i = -\frac{\log x}{\log \lambda_i} + h_i \in \mathbb{N}$ ($h_i = h_i(x) \in [0, \frac{1}{2}]$) and therefore $\varphi_i(x) = \mu_i^{-\frac{\log x}{\log \lambda_i} + h_i}$. We approximate the transition map $\psi_i : S_i' \to S_{i+1}$ by a linear map $\psi_i(x) = a_i x$, since a point $(x, 0)$ is mapped by a differentiable map to $(0, y)$, as in the continuous case.

This yields

$$\psi_i \circ \varphi_i(x) = a_i \mu_1^{-\frac{\log x}{\log \lambda_i} + h_i} =: \bar{a}_i \mu_i^{-\frac{\log x}{\log \lambda_i}}.$$

Hence the ratio of the sojourn times near the saddles is given by

$$\lim_{x \to 0} \frac{k_{i+1}}{k_i} = \frac{|\log \mu_i|}{\log \lambda_{i+1}}.$$

This yields formulas for the time averages that are analogous to the continuous case, except that the eigenvalues are replaced by their logarithms.

Example

Consider the discrete "stone-scissors-paper" game (c.f. Example 1)

$$x_i' = x_i(x_i + a_1 x_{i+1} + a_2 x_{i+2})/\Phi, \qquad i = 1,2,3,$$

where $a_i \geq 0$ and $\Phi = \sum_i x_i^2 + (a_1 + a_2)\sum_{i,j} x_i x_j$. The fixed points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ correspond to an attracting heteroclinic cycle, if $a_1 a_2 < 1$ (Hofbauer (1984)). The eigenvalues in direction of the cycle are a_1 and a_2, therefore the time averages spiral to the boundary of the triangle $\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3$ with

$$\mathbf{A}_1 = \frac{1}{|\log a_1|^2 + |\log a_1 \cdot \log a_2| + |\log a_2|^2} \cdot \left(|\log a_1|^2, |\log a_2|^2, |\log a_1 \cdot \log a_2|\right)$$

$$\mathbf{A}_2 = \frac{1}{|\log a_1|^2 + |\log a_1 \cdot \log a_2| + |\log a_2|^2} \cdot \left(|\log a_1 \cdot \log a_2|, |\log a_1|^2, |\log a_2|^2\right)$$

$$\mathbf{A}_3 = \frac{1}{|\log a_1|^2 + |\log a_1 \cdot \log a_2| + |\log a_2|^2} \cdot \left(|\log a_2|^2, |\log a_1 \cdot \log a_2|, |\log a_1|^2\right).$$

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Figure captions

Fig. 1: The accumulation points of the time average for the "stone-scissors-paper" game.

Fig. 2: Cross-sections for the linearized 2 dimensional system.

Fig. 3: The accumulation points of the time average for Example 2. (a) the generic case; (b) $\lambda_1 = 0$; (c) $\lambda_1 = \lambda_2 = 0$; (d) $\lambda_1 = \lambda_2 = 0$; (e) $\lambda_1 = \lambda_2 = \lambda_3 = 0$.

Fig. 4: Cross-sections for the linearized May Leonard system.

Fig. 5: A heteroclinic cycle on the simplex S_4.

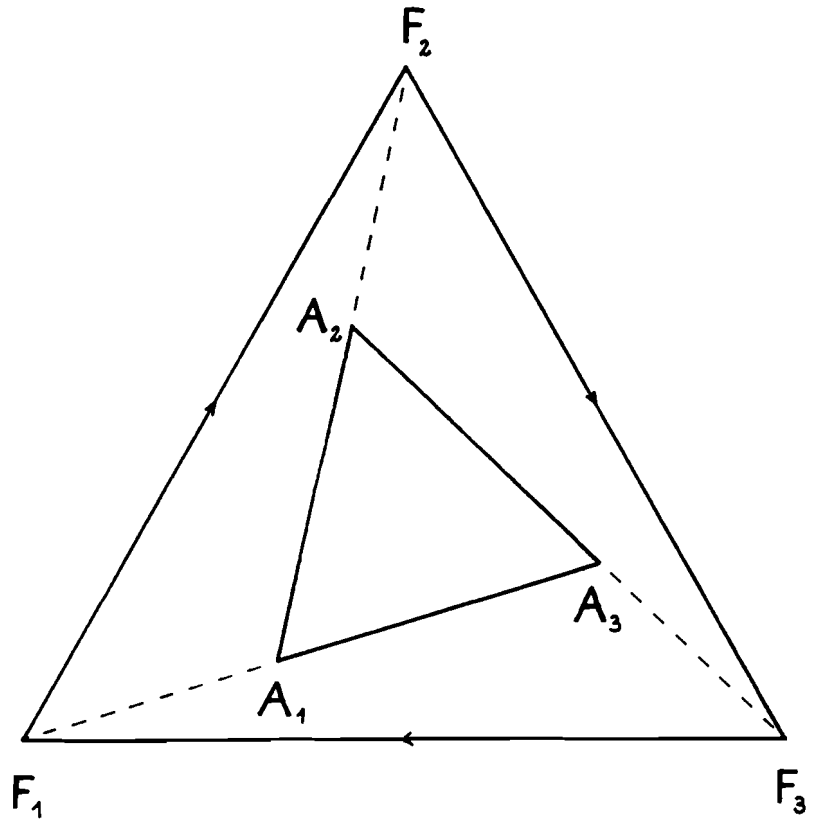

Fig. 1

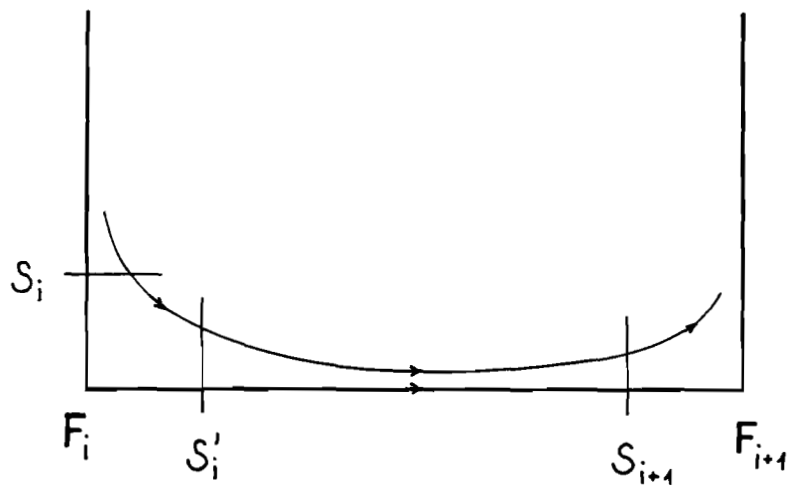

Fig. 2

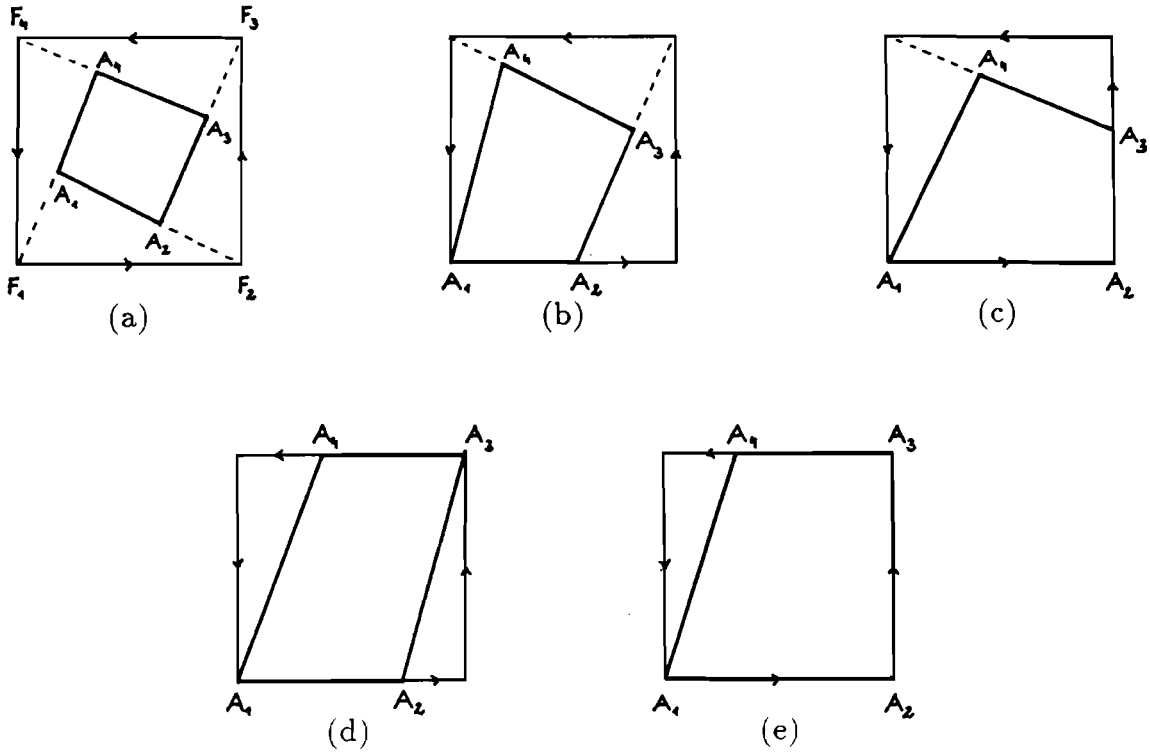
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Fig. 3

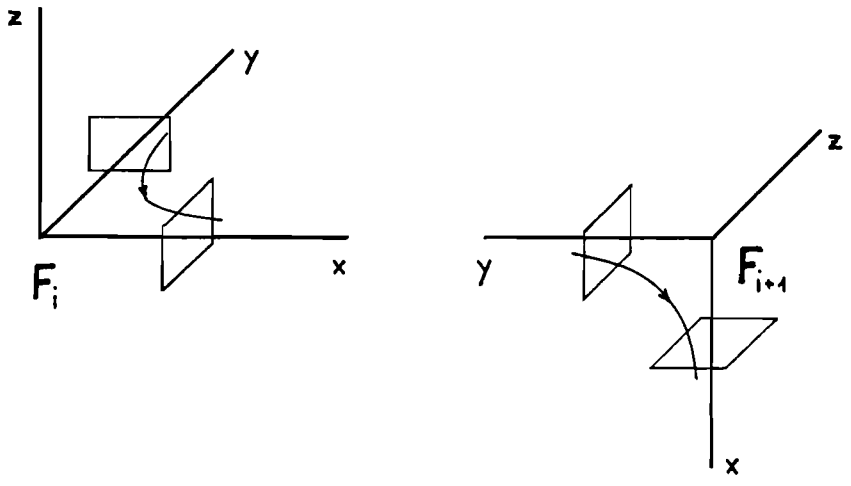

Fig. 4

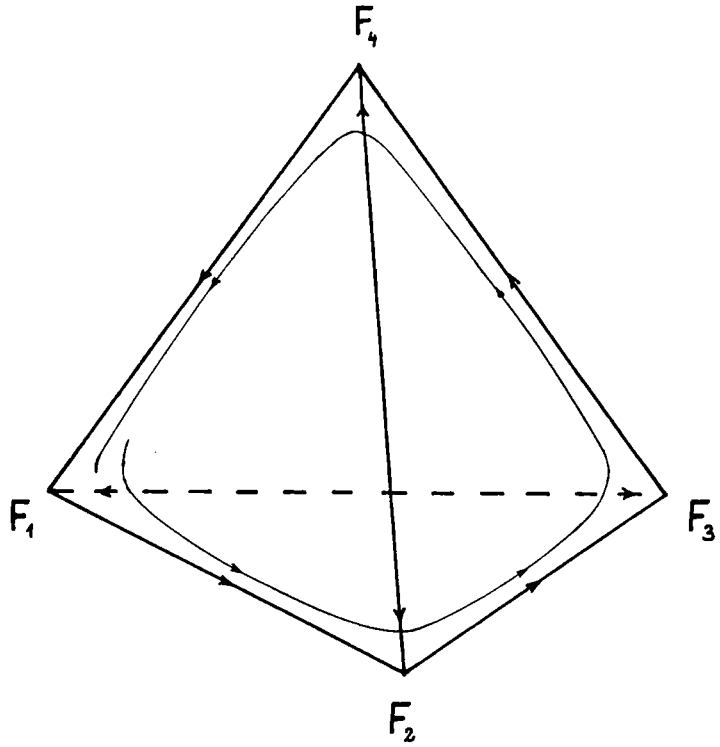

Fig. 5