MARKET AND COUNTERPARTY CREDIT RISK: SELECTED COMPUTATIONAL AND MANAGERIAL ASPECTS

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"No book can ever be finished. While working on it we learn just enough to find it immature the moment we turn away from it"

The Open Society and its Enemies
Karl R. Popper

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ABBREVIATIONS

ABS	Asset backed securities
AFME	Association for Financial Markets in Europe
ALM	Asset liability management
ASC	Accounting standards codification
ASRF	Asymptotic single risk factor
ASW	Asset swap
BCBS	Basel Committee on Banking Supervision
BCVA	Bilateral credit valuation adjustment
BIS	Bank of international settlement
Bps.	Basis points
C-CDS	Contingent credit default swap
CCR	Counterparty credit risk
CDF	Cumulative distribution function
CDO	Collateralized debt obligations
CDS	Credit default swap
CE	Current exposure
CEM	Current exposure method
CFT	Continuous Fourier transformation
CIR	Cox Ingersoll Ross
Corp.	Corporation
CRD	Capital Requirement Directive
CRR	Capital requirement regulation
CS01	Credit spread one basis point
CSA	Credit support annex
CVA	Credit valuation adjustment

DB	Defined benefit
DC	Defined contribution
DFT	Discrete Fourier transformation
Distrib.	Distribution
DJS	Direct jump to simulation
DV01	Dollar value [change through] one basis point [shift]
EAD	Exposure at default
EBA	European Banking Authority
ECB	European Central Bank
EE	Expected exposure
EffEE	Effective expected exposure
EffEPE	Effective expected positive exposure
EMIR	European Market Infrastructure Regulatory
EPE	Expected positive exposure
EU	European Union
EUR	Euro
FBA	Funding benefit adjustment
FCA	Funding cost adjustment
FFT	Fast Fourier transformation
FRFT	Fractional Fourier transformation
FTDCVA	First-to-default credit valuation adjustment
FVA	Funding valuation adjustment
FX	Foreign exchange
GMV	Gross market value
HGB	Handelsgesetzbuch ("German Gaap")
IDW RS HFA	IDW RS Hauptfachausschuss der Wirtschaftsprüfer
IDW RS	IDW Stellungnahmen zu Rechnungslegung

IDW	Institut der Wirtschaftsprüfer
IFRS	International Financial Reporting Standards
IMM	Internal model method
IR	Interest rates
IRB	Internal rating based approach
IRR	Interest rate risk
IRS	Interest rate swap
ISDA	International Swaps and Derivatives Association
ITM	In-the-money
KPI	Key performance indicator
LDI	Liability driven investment
LGD	Loss given default
LS	Least square
M	Month(s)
Mm	Million
MTA	Minimum transfer amount
MtM	Mark-to-market
NEE	Negative expected exposure
NPV	Net present value
OTC	Over the counter
OTM	Out-of-the-money
P&L	Profit and loss [statement]
PD	Probability of default
PDS	Price-dependent simulation
PnL	Profit and loss [statement]
PV01	Present value [change through] one basis point [shift]
RC	Regulatory capital

RMBS	Residential mortgage-backed securities
RR	Recovery rate
RW	Risk weight
RWA	Risk weighted asset
RWR	Right way risk
SA-CCR	Standard approach for measuring counterparty credit risk
SDE	Stochastic differential equation
SM	Standardized method
SPV	Special purpose vehicle
SSRD	Shifted square root diffusion
SSRJD	Shifted squared root (jump) diffusion
transform.	Transformation
UCVA	Unilateral credit valuation adjustment
UDVA	Unilateral debt valuation adjustment
USD	United States Dollar
US-GAAP	United States Generally Accepted Principles
VaR	Value at risk
WWA	Wrong way risk
WYSIATI	What you see is all there is
Y	Year(s)
ZCB	Zero coupon bond

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CHAPTER 1:
INTRODUCTION AND SUMMARY OF CONTRIBUTION

1.1. INTRODUCTORY REMARKS

While *interest rates* determine the costs for borrowing and lending money *credit spreads* denote the additional charge reflecting the fact that debtors are default-prone. Hence, both interest rates and credit spreads drive the costs and returns of everyday life items such as mortgages and saving plans. Both factors are of considerable significance for financial and non-financial institutions as well as for the general public as a whole. In the following thesis we study the modeling of interest rates and credit spreads. We also analyze the use of so-called *financial derivatives* to price and manage interest rate risk as well as credit risk, especially discussing the *counterparty credit risk* that derivatives themselves might exhibit.

Financial derivatives are assets whose value is derived from the value of another (underlying) asset. Take for example a call option on the stock of a company. Such an option gives the right to buy the stock in the future for a pre-defined price. The value of such an option is thus derived from the value of the company's equity.[1] Most prominent example of interest rate derivatives are interest rate swaps, contracts in which two parties agree on swapping future interest payments, e.g. floating rate against a fixed rate relating to a predefined notional amount. Financial derivatives take many forms and are "*limited only by the imagination of man*" (Berkshire Hathaway, 2002, p. 13). Standardized derivatives such as equity options are exchange-traded. The lion's share of financial derivatives is, however, less standardized and is traded bilaterally, i.e. over the counter (OTC). Following the financial crisis and the

[1] Notice that in turn the value of the company's equity can be interpreted as a call option on its assets and is thus *derived* from the value of the latter (see Merton, 1974).

resulting regulatory changes an increasing portion of financial derivatives is being traded through central counterparties (see ISDA, 2015).

The discourse around financial derivatives – especially the ones traded OTC – has been controversial at best. On the one hand, they are celebrated as financial innovations, allowing risk to be borne by those best positioned to do so. After all, derivatives such as interest rate swaps enable not only banks but also corporates and pension funds to offset unwanted risks, guaranteeing a certain level of financial performance.[2] On the other hand, derivatives have been associated with "*excessive and opaque risk-taking*" (BCBS, 2013) and even "*market manipulation*" (Stulz, 2010). They have drawn criticism of financial market "gurus" like Warren Buffer and George Soros, describing them as "*time bombs*", "*financial weapons of mass destruction*" and "*toxic instruments*".[3] While facilitating a more liquid transfer of risk derivatives introduce counterparty credit risk, because derivative traders can default on their claims. This is why the "web of linkages" (Stulz, 2010) derivatives build across financial institutions has made financial markets more fragile, considerably attributing to the financial crisis that began in 2007.

The role of derivatives in the financial crisis indeed led to fundamental adjustments in regulation of banks in particular and financial markets in general.[4] Still, since Warren Buffer made his scathing statements about OTC derivatives in 2002 they seem to have only gained in popularity at least until the financial crisis. Figure 1 illustrates how outstanding notional amounts of financial derivatives skyrocketed since the late 90s. This evolution mirrors an increased interest in reducing (hedging) financial risk (especially arising from interest rate changes) combined with an enhanced capability in speculation activities. This evolution has

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[2] See for example the analysis around credit default swaps (CDS) by Stulz (2010) or the publication of ISDA (2014b) on end user activity in the OTC market.

[3] The quotes are taken from the financial report of Berkshire Hathaway (2002) in which Charlie Munger and Warren Buffer explain their exit from the derivatives business, and from an article written by Soros (2009) in which he pleads for banning "naked" credit default swap (CDS).

[4] See the margin requirements for non-centrally cleared derivatives by BCBS (2013).

been accompanied by a new "science", trying to price derivatives and capture the dynamics of the underlying financial risk factors in mathematical terms. Computational finance has also been subject to a very controversial discourse. Financial models such as the Black-Scholes model for pricing options have helped researchers receive the "Nobel Prize"[5] in economic sciences, being praised for "*[paving] the way for economic valuations in many areas and [facilitating] more efficient risk management in society*".[6] On the other hand, others have deprived this research branch of any scientific notion, accusing it of "*charlatanism*" and in increasing system blindness, also referred to as "*model dope*".[7]

While thoroughly deep-diving into a range of different technical and managerial aspects especially around interest rates and credit spreads we aim on maintaining a bird's view with regards to the overall discourse. We especially intend to contribute to the discussion on derivatives between the poles of being "innovations" and "time bombs" on the one hand, and the discussion around the scientific notions and the value added offered by financial models on the other.

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[5] Officially referred to as Bank of Sweden Prize in Economic Sciences in Memory of Alfred Nobel.

[6] See the press release of the Royal Swedish Academy of Science (1997) for awarding the prize to Robert Merton and Myron Scholes. Comments in brackets [.] have been added by the author.

[7] See MacKenzie and Spears (2014) for a discussion on the role of copula models in pricing Credit Debt Obligations (CDO), analyzing the existence of "model dope". Mikosch (2005) discuss the use of copula functions. Haug and Taleb (2011) analyze the non-use of the Black-Scholed formula for equity options. See Taleb (1997) for a more general critic of derivative pricing.

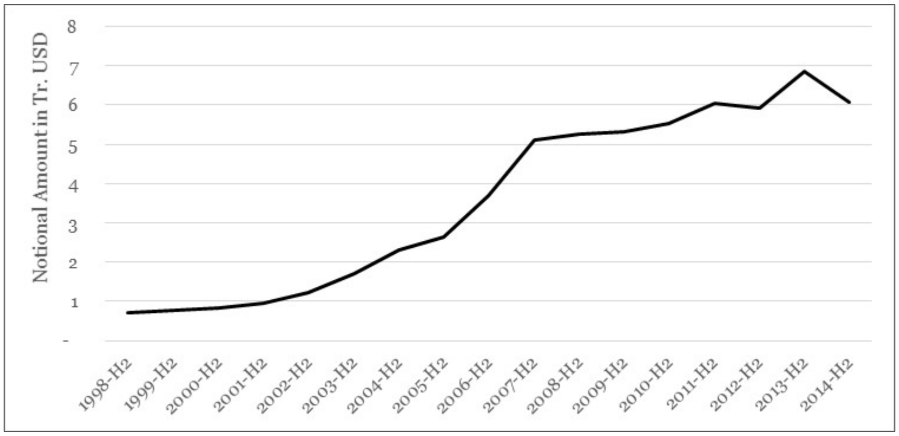

Figure 1: Outstanding Notional Amounts of Over the Counter (OTC) Derivatives

The figure illustrates the outstanding notional amount of OTC derivatives as given by the Bank of International Settlement (BIS) (www.BIS.org). The notional amount illustrated includes foreign exchange (FX) derivatives, commodity contracts, credit default swaps (CDS) as well as interest rate derivatives.[8]

We will start by looking into a classic example of an end user that uses derivatives to offset financial risks, analyzing how interest rate swaps can be used to immunize pension funds against interest rate changes without affecting the strategic asset allocation. We will elaborate on popular asset liability management (ALM) tools used in practice and will analyze the possible value added offered by stochastic interest rate models. Our analysis is narrowed to the presumably promising model offered by Cairns (2004), proclaiming to model interest rates realistically under the risk-neutral measure. For this purpose we develop a novel procedure to structure so-called overlays of interest rate swaps. We then analyze the hedge effectiveness offered by the overlays when the Cairns model is used to measure sensitivity, comparing its performance with a more conventional measure, the duration-based PV01 – also referred to as DV01 – metric.

[8] Notice that while notional amount is an appropriate indicator of popularity of OTC derivatives, netted exposure would be a more adequate metric in indicating risk as it considers netting or collateral agreements.

We will then turn to the counterparty credit risk exhibited by financial derivatives, giving a compact overview in modeling and managing credit valuation adjustment (CVA), a metric that has emerged as a standard method for pricing counterparty credit risk. We also discuss the regulatory and accounting landscape behind counterparty credit risk and CVA which also flows into a critical analysis of the prevailing discourse. Besides thoroughly discussing key technical and managerial aspects around counterparty credit risk and CVA we aim to achieve a novel overall evaluation of regulatory efforts on the one hand, and lobbying activities on the other. We also synthesize the implications for derivative traders, discussing the challenges around pricing CVA and the corresponding limits of risk-neutral valuation, especially when it comes to constructing adequate hedges.

Finally, we discuss the specific case of credit default swaps (CDS), financial instruments in which the splits between being an "innovation" and a "time bomb" is most evident. In this context we revisit the key characteristics of CDS contracts, and discuss approaches for modeling credit risk in general and credit spreads in particular. We show that while CDS can be used to mitigate (counterparty) credit risk they are not excluded from exhibiting such risk themselves. This is mostly evident if the credit quality of the protection seller and the reference entity are interdependent, i.e. if wrong way risk is present. For this purpose we revisit an approach offered by Brigo and Capponi (2010) to capture this feature. Besides decomposing the approach into its bits and pieces, and elaborating on aspects Brigo and Capponi (2010) left relatively open, we also offer a respective computational tune up for the model. Subsequently we run a critical evaluation of the Brigo and Capponi (2010) approach in particular, and CVA modeling in general. We analyze both the capability of such models in delivering an arbitrage-free framework as well as in its use for inter- and intra-organizational communication. Besides revealing key challenges, risks and limitations, we also aim on shedding light on possible benefits and insights offered by such approaches.

1.2. THESIS SUMMARY

The thesis is structured in two main parts. Part 1 is dedicated to a specific issue around interest rate risk management, while Part 2 deals with topics around counterparty credit risk management. In the following we give an abstract for each chapter.

Part 1 covers interest rate risk management, consisting of one chapter:

– **Chapter 2:** *Managing Interest Rate Risks of Pension Funds – An Application of the Cairns Model*:[9] Long-term portfolios consisting of asset and liabilities such as pension funds often exhibit significant sensitivities to changes in interest rates. Due to the separation of responsibilities and the otherwise unwanted complication, interest rate risk management of these portfolios is often done with a so-called derivative overlay (as part of asset liability management, ALM). The interest rate sensitivity is immunized by adding corresponding derivatives – mostly interest rate swaps – without affecting the strategic asset allocation.

  The use of stochastic models in this process is particular and ALM in general is limited. One of the main reasons behind this is the lack of respective approaches that combine arbitrage-free valuation with realistic modeling of short- and long-term interest rate dynamics, a gap the interest rate model of Cairns (2004) proclaims to address. We have therefore chosen to apply the Cairns model to the practical challenge of immunizing a pension fund against interest rate risk.

  We start by giving an overview on ALM in general and interest rate risk management in particular, revisiting the duration-based PV01 metric for measuring interest rate

[9] Chapter 2 is an adaptation of previous work of the author published in Balder and Schwake (2011). This means that some elaborations, especially in the computational part, are *identical*. Chapter 2 also uses materials already published by the author in Mahayni and Schwake (2013), especially regarding some of the exemplary calculations.

sensitivity. We describe the key mechanisms of interest rate swaps, elaborating on their use in ALM strategies.

After revisiting the two-factor version of the Cairns model and its main features, we derive respective model-based sensitivity measures. We subsequently discuss the use of the extended Kalman filter approach in calibrating the parameters of the model.

This is followed by a comparison of the hedge effectiveness offered by the Cairns model with the one given by the PV01 metric. For this purpose we introduce a novel rule-based and model-independent algorithm that immunizes pension fund-like portfolios against interest rate risk by structuring an overlay of appropriate swaps. The hedge effectiveness offered by both overlays is analyzed in a backtesting environment and in a Monte Carlo simulation scheme. While we do identify slight advantages offered by the Cairns model we are not able to justify its use through hedge effectiveness solely, especially if we bear the sophistication of its application in mind (compared with the PV01 approach). We, however, conclude that the Cairns model can offer an appropriate framework for analyzing investment strategies and facilitating respective discussions, especially because it can combine risk-neutrality with realistic modeling.

Part 2 covers counterparty credit risk management, consisting of the two following chapters:

– **Chapter 3:** *Pricing and Managing Counterparty Credit Risk in Theory and Practice*. It is at the latest since the financial crisis in general and the collapse of Lehman Brothers in particular that the "default-free scheme" has been finally falsified. It has become clear that derivative traders are default-prone and that the risk of them defaulting impacts derivative prices. The increased significance of counterparty credit risk has also drawn the attention of regulators, standard setters and auditors. Those have in turn further stressed the significance of the subject matter through more punitive rules and regulations. This pressure seems to have pushed derivative traders – especially financial institutions – to step up their procedures around pricing, managing and mitigating counterparty credit risk. Credit valuation adjustment (CVA) has emerged as a standard method for pricing counterparty credit risk. CVA can be interpreted as the cost of hedging the counterparty credit risk of the respective position. This introduces a new derivative instrument, usually referred to as contingent credit default swap (C-CDS) that, in return to a premium, insures the (stochastic) exposure at default. The pricing of a C-CDS (i.e. CVA valuation) usually turns out to be a much more elaborate task than pricing the default-free derivative itself.

Chapter 3 intends to give a compact overview in modeling and managing CVA, accompanied by a critical analysis of the prevailing discourse. This analysis is the heart of the chapter, which aims at exploring the challenges around CVA from different angles rather than offering a comprehensive description of all relevant aspects. It aims on achieving an overall evaluation of regulatory efforts on the one hand, and lobbying activities on the other.

After giving an overview on counterparty credit risk and CVA literature we study the quantification of CVA from a theoretically consistent perspective as well as based on a more practical approach. We then turn to the accounting and regulatory landscape, revisiting key requirements around counterparty credit risk and CVA risk. Subsequently, we give an overview on charging CVA and discuss its allocation across an organization in order to allow for adequate incentives. We also discuss the possibility to reduce counterparty credit risk, especially through collateralization and hedging activities.

Our analysis enables us to show how financial institutions are heavily driven by regulatory requirements that originally aimed to actually mirror how they "do business". We elaborate on how banks not only measure and manage CVA according to detailed requirements, but also on how such regulations considerably affect the way banks price their products.

We also show the limits of arbitrage-free valuation, especially when it comes to practical implementation of CVA pricing models or constructing adequate hedges. We conclude that clinging to use market implied parameters or elaborate models that need overcomplicated calibration without reflecting on their economic sense might not only imply mere model and valuation risks, but also significant financial risks at the latest when it comes to hedge CVA.

– **Chapter 4**: *Pricing Credit Default Swaps with Wrong Way Risk – Model Implementation and Computational Tune Up.*[10] One possibility for investors to mitigate counterparty credit risk is to buy protection in form of a credit default swap (CDS). If the credit quality of both protection seller and reference entity are positively interdependent, such a "protection" becomes questionable if not worthless. Brigo and Capponi (2010) were among the first to propose an arbitrage-free framework to price CDS considering this *wrong way risk* in conjunction with symmetric pricing.

  We start by describing the key mechanisms of CDS contracts, and giving an overview on the "competing" approaches for modeling credit risk in general and credit spreads in particular. We place the promising approach of Brigo and Capponi (2010) within the category of reduced form credit risk models that use copula functions to model default dependency. The approach is subsequently decomposed into its bits and pieces, allowing a discussion around its benefits and pitfalls. We also provide a step-by-step implementation guide, going into detail on aspects that Brigo and Capponi (2010) left open, especially the computation of the conditional survival probability of the reference entity. We illustrate how the fractional Fourier transformation (FRFT) can be used for this purpose, and propose a respective computational tune-up through a heuristic approximation. We then use a

[10] Chapter 4 is an adaptation of previous work of the author published in Grominski et al. (2012),). This means also that some elaborations, especially the computational part, are *identical*.

case study with real market data to demonstrate the use of the model and the insights it delivers.

Finally, we run a critical evaluation of the Brigo and Capponi (2010) approach in particular, and CVA modeling in general. We analyze both the capability of the model in delivering an arbitrage-free framework as well as in its use for inter- and intra-organizational communication. We show how the model can be used to facilitate discussions around the specific case of CVA for CDS contracts with considerable wrong way risk. We argue, however, that less elaborate models might display more appropriate solutions, avoiding over-complication and higher model risks, especially if wrong way risk is assumed to be less significant. After all, while Brigo and Capponi (2010) *do* offer a coherent and risk-neutral framework, they *do not* specify the needed duplication strategy. The lack of instruments to calibrate a risk-neutral correlation matrix and of a possibility to hedge own credit risk puts a question mark on the practicability of the model.

PART I:

INTEREST RATE RISK MANAGEMENT

CHAPTER 2:
INTEREST RATE RISK MANAGEMENT FOR PENSION FUNDS – AN APPLICATION OF THE CAIRNS MODEL

2.1. INTRODUCTION

Funded retirement arrangements like pension funds and life insurance products have extreme long-term claims, exceeding 50 or even 60 years. These liabilities are mostly financed by bonds with maturities way shorter on the asset side. Besides the duration gap produced by the lack of liquid long-term bonds, the providers and asset managers of such products have to deal with the risk of interest rates maturing in decades. Due to the separation of responsibilities and the otherwise unwanted complication, interest rate risk management of these portfolios is often done with a so-called derivative overlay (as part of asset liability management, ALM). The interest rate sensitivity is *immunized* by adding corresponding derivatives – mostly interest rate swaps – without affecting the strategic asset allocation.

A common practice in measuring interest rate sensitivity is using the PV01-approach. This duration-based metric measures interest rate risk as the change in value due to a shift of the interest rate curve. Based on a vector of PV01 metrics an interest rate swap overlay is usually introduced in order to (statically) hedge interest rate risks. Although this method might seem suspiciously easy for the academic world, it is quite wide spread under practitioners.

A key motive behind the limited use of theoretically more elaborate models (e.g. stochastic interest rate models) is the lack of respective approaches that combine arbitrage-free valuation with realistic modeling of short- *and* long-term interest rate dynamics. While celebrated interest rate models such as the ones introduced by Cox et al. (1985), Heath et al. (1992) or Miltersen et al. (1997) offer arbitrage-free valuation of derivatives, they are presumably less capable in modeling realistic dynamics, especially when it comes to long time horizons. On the other hand, actuarial models such the approaches of Wilkie (1995) or

Yakoubov et al. (1999) put more emphasis on modeling realistic dynamics but abandon the arbitrage-free framework.

In the meanwhile the risk-neutral model-family for interest rates introduced by Cairns (2004) simultaneously models short- and long-term interest rates. Adequately calibrated, the Cairns model proclaims to deliver realistic simulation of the whole term structure, making it especially promising for the case of pension funds. We have therefore chosen to put the model to the test by applying it to structuring swap overlays, and comparing it with the popular PV01-approach.

For this purpose we introduce interest rate sensitivity measures based on the two-factor version of the Cairns model. For the construction of the swap overlays we introduce a novel algorithm that is rule-based and model independent. Given an interest rate model, the algorithm decomposes the interest rate sensitivity of portfolios containing assets and liabilities. It then adds swaps to the portfolio in order to immunize it against interest rate risks. A subsequent linear optimization defines the optimal notional amounts needed to ensure interest rate risk is fully hedged. Using a realistic pension fund portfolio structure we compare the hedge effectiveness delivered by the Cairns model with the one offered by the PV01-approach. The comparison is done both in a backtest environment and using a Monte Carlo simulation scheme.

The remaining parts of Chapter 2 are structured as follows. Section 2.2 gives an overview of interest rate risk management as part of asset liability management (ALM), starting with respective definitions and a literature summary given in Subchapter 2.2.1. Being a key and popular measure of interest rate risk used in ALM strategies we dedicate Subchapter 2.2.2 to the PV01 metric. The basis mechanisms behind the use of interest rate swaps are revisited in Subchapter 2.2.3. Section 2.3 then moves to stochastic interest rate models, starting with a respective literature summary, given in Subchapter 2.3.1. Subchapter 2.3.2 then describes the main features of the chosen Cairns model for interest rates, while Subchapter 2.3.3 revisits the methodology chosen for estimating the model parameters (*extended Kalman filter*). In

Subchapter 2.3.4 we derive the sensitivity measures ("Cairns deltas") that will be used to quantify and manage interest rate risk. Section 2.4 introduces a novel procedure for structuring an interest rate swap overlay. In Section 2.5 we use a realistic use case to illustrate the application of the Cairns approach for modeling and managing interest rate risk within pension funds, comparing its performance with the one offered by the PV01 approach. After giving an overview on implementing and calibrating the model (given in Subchapters 2.5.1 and 2.5.2), we illustrate the structuring of an interest rate swap overlay in Subchapter 2.5.3. We test its performance using a backtest approach as well as a Monte Carlo simulation scheme, in Subchapter 2.5.4 and Subchapter 2.5.5., respectively. We conclude the analysis in Section 2.6.

Notice that this Chapter is an adaptation of previous work of the author published in Balder and Schwake (2011), and that some elaborations, especially the computational part, are identical. [11] This Chapter also uses materials already published by the author in Mahayni and Schwake (2013), especially some of the exemplary calculations. In order to avoid redundancy we will refrain from continuously referring to both papers.

[11] Balder (2014) also published an adopted version of Balder and Schwake (2011).

2.2. OVERVIEW ON ASSET LIABILITY MANAGEMENT

2.2.1. DEFINITIONS AND LITERATURE OVERVIEW

Asset liability management (ALM) refers to the simultaneous coordination of assets and liabilities, aiming on capturing the overarching risk profile. This rather broad term is usually reserved to the management of financial assets and liabilities, mostly relevant for financial institutions, insurance companies and pension schemes.[12] We will focus on the latter and will narrow our analysis to interest rate risk management. We acknowledge that ALM within pension funds might also cover further risk factors such as inflation, credit spreads and longevity risk.[13] Still, interest rate risk is presumed as most relevant, leading even to possible analogy between the terms ALM and interest rate risk management (IRR) in practice jargon (see for example Brick, 2014).

In the case of pension funds liabilities consist of pension payments to retired plan members or ones that will retire in the future.[14] These liabilities behave similarly to long-term bonds, exhibiting a (*relatively high*) sensitivity to interest rates. Assets held by pension funds are predominantly "fixed income" securities with additional investments in "riskier" markets

[12] The definition is inspired amongst others by Sodhi (2005) and Fabbozi et al. (2005). For pension funds one also uses the term liability driven investment, LDI (see Ryan, 2013).

[13] This short list of risk factors is not exhaustive. For credit spread modeling we refer the reader to Chapter 3 of this thesis. For modelling longevity risk see for example Mahayni and Steuten (2013).

[14] Pension payments are paid out according to a predefined procedure. One speaks of "defined contribution" (DC) plans, when the pension plan sponsor (e.g. a company) is *only* obliged to pay pre-determined contributions into the fund (e.g. periodically). All remaining investment risk is borne by the pension members (e.g. employees). Defined benefit (DB) plans refer, on the other hand, to pension schemes that contain (partially) guaranteed payments. Thus, strictly speaking our analysis is limited to DB pension funds. The definition of DC plans follows the one used in International Financial Reporting Standards (IFRS) given in IAS 19.7, while the definition of DB plans is derived by negation.

such as equities and real estate.[15] Still, one usually focuses on fixed income securities when it comes to interest rate risk, because of their relatively high portfolio portion and the ambiguous interest rate sensitivity of other asset classes.

In a broader sense ALM strategies offer the foundation for the optimal asset allocation of the core portfolio (see also Ryan, 2013). Such optimization schemes will need to consider the dynamics of both the liabilities and the assets in an integrated manner, providing answers with regards to future contributions and changes in the asset allocation (see Fabozzi et al., 2005). Prominent examples for such "full-fledged" solutions are given by Mulvey et al. (2000) as well as Gondzio and Kouwenberg (2001). The former proposes a stochastic planning model, aiming on maximizing the expected wealth while minimizing the risk of the pension fund collapsing. They make the case that borrowing (e.g. from the fund sponsor) can be optimal at certain situations. Gondzio and Kouwenberg (2001) analyze Dutch pension plans, offering a so-called decomposition-based algorithm that is implemented on a particular parallel computer.

A review of stochastic ALM models is given for example in Sodhi (2005), who covers models for banks, insurance companies as well as pension funds. He also notes that such approaches face methodological and computational challenges. First, they struggle with offering a methodologically sound framework to model all the different aspects (e.g. interest rates, mortality rates, salary variances etc.), ensuring consistency within the model as well as with financial theory. Second, considering a large number of different factors and scenarios implies substantial difficulty in solving the optimization problem. Gondzio and Kouwenberg (2001) for example report on modeling "*4,826,809 scenarios, 12,469,250 constraints and 24,938,502 variables*" stating that it "*is the largest stochastic linear program ever solved.*" There is one further considerable challenge which surrounds the role of models in decision

[15] The asset allocation differs significantly from region to region. According to Towers Watson (2015) Australian pension schemes invest more than 50% in equity, while the respective figure at Swiss plans is less than 30%.

making. As put by a fund manager quoted in Fabozzi et al. (2005), the complexity of such models makes them appear as a "*black box to the investment committee*".

These challenges give at least a partial explanation why "static" approaches are common in ALM strategies (Ryan, 2013). Most prominent examples, especially when it comes to managing interest rate risks, are "cash flow matching" and "duration matching". In the following we will give a short description of both techniques. For a more comprehensive review we refer the interested reader for example to Fabozzi (2000). Ryan (2013) gives an overview regarding the historical development of both methods and ALM for pension funds in general.

Cash flow matching – also referred to as dedication – is attributed amongst others to Leibowitz (1986). It follows a rather straightforward logic; one *dedicates* inflows to expected outflows. In its simplest form the technique seeks a set of fixed income securities with coupon and notional (re-)payments that mirror the projected withdrawals and pension disbursements. Most obvious advantage of this rather conservative approach is its ability to robustly mitigate interest rate risk *and* liquidity risk, while allowing for quite a simple asset allocation. Its most considerable pitfall is its absent feasibility. While pension liabilities have maturities that exceed 30 years, most liquid fixed income securities (e.g. sovereign bonds) have way shorter time to maturities (e.g. 10 years). Moreover, by prescribing the asset allocation, the technique limits any possibility of actively managing the assets.

In contrary to dedication, duration matching aims on *immunizing* the portfolio against interest rate changes through matching the sensitivities of assets and liabilities. The duration concept goes back to Macaulay (1938). Its use in managing interest risk is attributed amongst others to Redington (1952). More prominent examples of analyzing this static approach were given later, e.g. by Fisher and Weil (1971). Duration measures sensitivity to changes in the level of interest rates. It is a tractable and easily computed metric that can facilitate discussions around the overall interest rate risk of a given portfolio. However, it fails – per definition – to capture convexity effects, e.g. exhibited by a bond price when interest rate changes are more material. For this purpose, second order approximations, e.g. dollar convexity measures (see Fabbozi, 2000, pp. 68-77), have been proposed. More importantly duration-based approaches have a significant theoretical shortfall. Not considering possible non-parallel shifts (i.e. changes in the shape) of the term structure implies inconsistency with arbitrage-free valuation, meaning that duration delivers a misleading measurement of risk (see Ingersoll et al., 1978).[16] A possible solution was introduced by Chambers and Carlton (1988) and Ho (1992), proposing the use of duration vectors – also referred to as key durations, measuring the sensitivity to yields with differing maturities.

2.2.2. PV01 AS A MEASURE FOR INTEREST RATE SENSITIVITY

A particular key duration-based measure is PV01 – also referred to as DV01 (dollar value) – which measures sensitivity as the change of the present value (PV) due to a shift of the interest rate curve by 1 basis point (01). Sensitivity is thus measured as the valuation change – here of a zero bond – due to a parallel shift of the interest rate curve of 1 basis point. A zero bond with the notional amount $\ddot{N}$ has the following sensitivity to changes in the interest rate r

$$PV01 = \frac{1}{10.000} \cdot \frac{\partial P(t,T)}{\partial r(t,T)} \tag{2.1}$$

$$= \frac{1}{10.000} \cdot \frac{\partial\left(\ddot{N} e^{-r(t,T)(T-t)}\right)}{\partial r(t,T)}$$

$$= -10^4 \cdot (T-t) \cdot \ddot{N} \cdot P(t,T)$$

where $P(t,T)$ stands for the value at time t of the zero bond that matures in T. Notice that by decomposing the portfolio into a stream of cash flows one can estimate a vector of PV01 metrics. For this purpose the cash flow positions are decomposed into a series of zero

[16] This is due to the fact that one can construct “duplication portfolios”, generating structurally superior payoff profiles.

coupon bonds (ZCB), each with a notional value $\ddot{N}_i$ and maturity T_i. One then generates a PV01 metric for each ZCB. This allocates a sensitivity measure to each relevant maturity, capturing possible convexities in the spirit of Chambers and Carlton (1988).

This will be illustrated in the example given in Table 1 below. For this purpose we assume a (very) simplified portfolio, consisting of a single obligation with a face value of 50 million units, maturing in 5 years. This liability is financed by a single ZCB with a face value of 51 million units, maturing in 2 years. The vector of PV01 metrics (as well as the total sensitivity) is given in Table 1.

Time period $\boldsymbol{T_i}$	**Yield** $\boldsymbol{r(0,T_i)}$	**Discount factor** $\boldsymbol{P(0,T_i)}$	**Cash flows** $\boldsymbol{\ddot{N}_i}$	**Present Values** $\boldsymbol{(III \cdot IV)}$	**PV01**
		1.0000			
1	1.2%	0.9881			
2	1.3%	0.9745	51.000.000	49.691.090	-9.938
3	1.6%	0.9535			
4	1.7%	0.9348			
5	1.9%	0.9102	-50.000.000	-45.509.188	22.755
				Total PV01:	12.816

Table 1: Example for Computing a Vector of PV01 Metrics for a Single Liability

The table gives a step-by-step guide for PV01 computation using Equation (2.1). In the first column the (maturity) time periods are given. The respective (*exemplary*) sport rates (observed in time period 0) are given in column II. Using the sport rates the discount factors are computed and given in column III. The (decomposed) cash flows of the portfolio are given in column IV. The present value and the PV01 metric are calculated and given in columns V and VI, respectively.[17]

2.2.3. Hedging with Interest Rate Swaps: Basic Mechanisms

Interest rate risk can be managed via a variety of respective derivative instruments. Most prominent example is given by interest rate swaps (IRS). Further examples are floors, caps

[17] The example was first presented in Schwake and Mahayni (2013, p. 66).

and swaptions (see for example Brigo and Mercurio, 2006). One speaks of overlay management if derivatives are being used “on top”, i.e. without affecting the (active) asset allocation. In the following we will illustrate how interest rate swaps can be used in hedging interest rate risks of portfolios containing assets and liabilities. The objective of the overlay strategy is to seek swaps that allow the sensitivity of the asset side to be equal to the sensitivity of the liability side, i.e. to *immunize* the overall portfolio.

Interest rate swaps (IRS) refer to contracts in which counterparties agree to exchange interest payments in a predefined frequency (e.g. monthly) for an agreed upon time period. For example while one counterparty pays a fixed rate (in relation to a predefined notional amount) the other party pays in return a floating rate (e.g. based on 3 months-Euribor).[18] The compatibility of IRS in hedging is easily seen by duplicating the interest rate derivatives. The value at time t of a swap settling in t_0, maturing in t_n while paying a fixed rate c can be written as:

$$V^{swap}(t) = P(t, t_0) - \sum_{i=0}^{n} cP(t, t_i) - P(t, t_n), t \in [0, t_n]. \tag{2.2}$$

The swap can thus be duplicated in the following manner. The fixed leg is seen as a portfolio of zero coupon bonds with c as their face values. The floating leg is duplicated using a position in a bond maturing with the swap in t_n, and a contrariwise bond maturing at the end of the forward period in t_0. In a payer (receiver) swap a short (long) position is built up in t_0 and a long (short) position is built up in t_n.

The values of both separate legs can thus be written as

[18] Interest rate swaps could be constructed to allow for swapping fixed against fixed payments or floating against floating payments. Other examples on interest rate swaps (including illustrations) are given in Section 3.2.

$$V^{floating}(t) = P(t, t_0) - P(t, t_n), t \in [0, t_n] \tag{2.3}$$

$$V^{fix}(t) = \sum_{i=0}^{n} cP(t, t_i), \qquad t \in [0, t_n].$$

This means that the fixed leg will have sensitivities towards interest rate changes on every payment date of the swap. Yet the more significant sensitivities (for a given time period) will arise from the floating leg. A (spot) swap will demonstrate such sensitivity at its maturity, while forward swaps will have significant sensitivities at maturity as well as at their first fixing date.

These mechanisms will be illustrated in the following example which builds on the simplified framework we used in Subchapter 2.2.2 above. For this purpose we assume a (very) simplified portfolio, consisting of a single forward receiver swap. The first fixing date will be in 2 years, and the settlement date will take place in 3 years-times, maturing in 5 years from now. Table 2 summarizes the decomposed cash flow positions of the swap, using the duplication strategy discussed above (see Equation (2.3)).

Time period T_i	Yield $r(0,T_i)$	Discount factor $P(0,T_i)$	Cash flows $\ddot{N}_i$	Present Values $(III \cdot IV)$	PV01
		1.0000			
1	1.2%	0.9881			
2	1.3%	0.9745	-50,000,000	-48,716,754	9,743
3	1.6%	0.9535	1,149,101	1,095,247	-329
4	1.7%	0.9348	1,149,101	1,073,560	-429
5	1.9%	0.9102	51,149,101	46,513,608	-23,257
				Total PV01:	-14,271

Table 2: Example for Computing a Vector of PV01 Metrics for a Forward Receiver Swap

The table gives a step-by-step guide for PV01 computation using Equation (2.1). In the first column the (maturity) time periods are given in years. The respective (*exemplary*) yields (observed in time period 0) are given in column II. Using the yields the discount factors are computed and given in column III. The (decomposed) cash flows of an exemplary forward receiver swap are given in column IV according to the duplication strategy (see Equation (2.3)). The present value and the PV01 metrics are calculated and given in columns V and VI, respectively.[19]

By looking at the sensitivities exhibited by the single liability in the first part of the example (see Table 1) and the sensitivity metrics given by the receiver swap (see Table 2) it becomes clear why swaps are interesting in hedging interest rate sensitivities of portfolios with a duration gap. Although we have not defined a set of swaps that immunizes the portfolio against interest rate sensitivities the sensitivity structure offered by the explanatory swap basically mirrors the sensitivity structure of the underlying portfolio, already offering a (partial) hedge.

The mechanisms can be summarized as follows:

- Receiver (payer) swaps reduce (increase) the overall sensitivity to interest rate changes – also referred to as reducing (increasing) the duration.

[19] The example was first presented in Schwake and Mahayni (2013, p. 67).

- Forward receiver (payer) swaps increase (decrease) the sensitivity at the first settlement date (end of the forward period) while decreasing (increasing) the sensitivity towards the maturity time period of the swap

2.3. MODELING INTEREST RATE DYNAMICS

2.3.1. Remarks and Literature Overview

Due to the challenging characteristics of interest rate instruments numerous models have been developed in order to mathematically capture the dynamics of term structures (i.e. yield curves), following these key objectives:

a. Arbitrage-free and risk-neutral valuation of interest rate derivatives
b. Realistic modeling of interest rate changes (motivated by Figure 2 below), e.g.:
   i. Dynamics should fit observed historical data
   ii. Interest rates should be non-negative, yet possibly getting close to zero
   iii. Interest rates should be a mean reverting
   iv. Term structures should exhibit a variety of shapes (incl. non-parallel movements)
   v. Long periods with both relatively high and low interest rates should be possible

At the latest since the work of Black (1976) have option theory and the increasing need for derivatives pricing given this research branch a considerable push. Most of the celebrated models in this context focus on the valuation of short- and middle-term contingent claims, i.e. fulfilling objective (a) rather than objective (b). In the following we will give a short recap on prominent examples of stochastic models for interest rates. This summary is largely inspired by Brigo and Mercurio (2006) who offer a comprehensive and detailed description and analysis of all models, and to whom we refer the interested reader. For the use of interest rate models for pricing application see also Hull (2006), and Björk (2009). Andresen and Piterbarg (2010 a, b, c) present a more up-to-date and recent analysis of arbitrage-free models in three comprehensive volumes. For stochastic basis spread modeling refer to Mercurio and Xie (2012). For multi curve modeling and OIS discounting see Hull and White (2012b).

Prominent examples of so-called short rate models were given by Vasicek (1977), Rendleman and Bartter (1980), and Cox, Ingersoll and Ross (Cox et al., 1985). These approaches are motivated by the fact that the prices of zero coupon bonds are completely driven by the probabilistic dynamics of the instantaneous short rate.[20] Modeling the short rate in a risk-neutral manner will thus enable us to model the whole term structure. The most significant pitfall of the models mentioned is that the endogenous term structure delivered by the model is not confirmed by (exogenous) market implied ones. A solution has been proposed through arbitrage-free short rate models, offered for example by Hull and White (1990), Black et al. (1990) as well as Black and Karasinski (1991). Brigo and Mercurio (2006) also analyze the shifted CIR model (CIR++), offering an analytically tractable approach that fits to current market information.

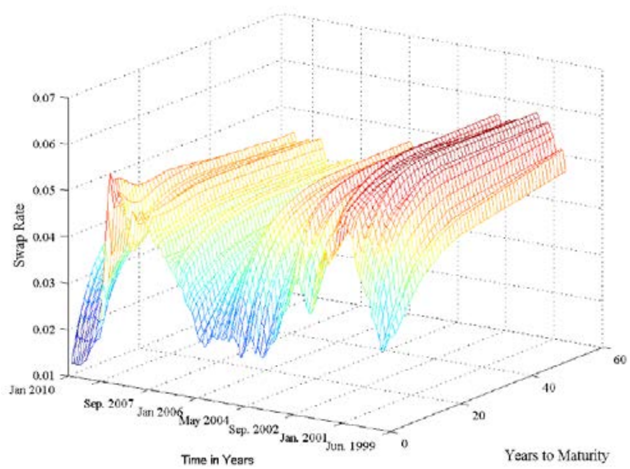

Figure 2: Historical Evolution of Swap Rates Term Structures

The figure illustrates historically observed evolution of the term-term structures of swap rates (Euribor)[21] between June 1999 and June 2010.

[20] In Appendix B we revisit the risk-neutral valuation paradigm offered by Harrison and Pliska (1983), showing how the dynamics of the instantaneous short rate explain the distribution of zero coupon bond prices.

[21] In line with the practice common during the years observed (especially prior to the financial crisis) the Euribor curve contains yields of varying tenors (e.g. 3 months, 6 months etc.).

Heath et al. (1992) proposed an arbitrage-free approach to model instantaneous forward rates. In contrast to short rate models, the approach is more general and aims to describe the dynamics of the whole yield curve directly. The model offers an adequate framework for studying the properties of arbitrage freedom, and has gained substantial interest in academic literature. The models of Brace et al. (1997) and Miltersen et al. (1997) focused on modeling observed (not theoretical) interest rates (e.g. Libor rate). Adding stochastic features to the volatility term as in Wu and Zhang (2002) or the introduction of a stochastic basis between tenor dependent term structures as in Mercurio (2010) are some of the main adaptations done in order to deliver more accurate risk-neutral pricing.

Although the above mentioned models partially satisfy some of the requirements around realistic modelling (e.g. mean reverting), their primary focus remains in achieving arbitrage-free valuation, offering pricing frameworks for derivatives. In a parallel stream, approaches of Wilkie (1995) and Yakoubov et al. (1999) have put more emphasis on the actuarial (risk oriented) interest of realistic modeling, i.e. focusing on objective (b) rather than objective (a). These time-discrete approaches are not designed for the valuation of interest rate instruments and their short-term risk management.

Cairns (2004) introduced a family of models that aims on satisfying both objectives, i.e. allowing for arbitrage-free valuation and realistic modeling of interest rates with diverse maturities. This makes the Cairn models promising with regards to managing interest rate risk in long-term pension funds. We have, therefore, decided to narrow our analysis to this approach, especially studying its ability in delivering a consistent framework for ALM as well as any value add it offers if compared to popular static approaches (i.e. duration-based PV01).

2.3.2. The Cairns Approach: Model Framework

Based on the framework of Flesaker and Hughston (1996), Cairns (2004) introduces a family of models that aim to satisfy both the arbitrage-free evaluation and the realistic modeling of interest rates with diverse maturities. In the following we will shortly revisit the approach and its main characteristics. For previous work on the family models offered by Cairns see also Lutz (2006) and Pfeiffer et al. (2010).

Cairns (2004) denotes the price at time t of the zero coupon bond $P(t,T)$, maturing in T as follows:

$$P(t,T) = \frac{\int_{T-t}^{\infty} H\big(u, X(t)\big)du}{\int_{0}^{\infty} H\big(u, X(t)\big)du}. \tag{2.4}$$

These bond prices are a function of the specified martingale family

$$H(u,x) = \exp\left[-\beta u + \sum_{i=1}^{n} \sigma_i X_i e^{-\alpha_i u} - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\rho_{ij}\sigma_i\sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i+\alpha_j)u}\right] \tag{2.5}$$

for some parameters β, $\alpha_i, \dots, \alpha_n, \sigma_i, \dots, \sigma_n$ and n correlated X_i factors with ρ_{ij} standing for the respective correlation coefficient. The n factors are a function of an Ornstein-Uhlenbeck process, driven by the same number of independent Brownian motions $\widehat{Z}_j$

$$dX_i(t) = \alpha_i\big(\mu_i - X(t)\big)dt + \sum_{j=1}^{n} c_{ij} d\widehat{Z}_j(t) \tag{2.6}$$

with μ standing for an additional constant parameter per factor. The matrix $C = (c_{ij})_{i;j=1}^{n}$ is defined in such a manner that CC' represents the correlation matrix for the processes, $X_1(t), \dots, X_n(t)$, $CC' = (\rho_{ij})_{i;j=1}^{n}$. The solution of the stochastic differential equation equals

$$X_i(t) = e^{-\alpha_i t} X_i(0) + \mu_i(1 - e^{-\alpha_i t}) + \sum_{j=1}^{n} c_{ij} \int_{o}^{t} e^{-\alpha_i(t-s)}\, d\widehat{Z}_j(s) \tag{2.7}$$

Given suitably parametrized values for the constants, β, α, σ, ρ, μ and an appropriate number of factors, Cairns (2004) shows that the model satisfies the following:

a) All interest rates are positive
b) All interest rates can get close to zero

c) The model is mean reverting

d) Long periods with both relatively high and low interest rates are possible

e) Par yields for long-term bonds should have realistic probabilities of reaching both high and low values

f) The model is preferably time homogeneous

g) The constant parameters in the model need no regular recalibration

While most of these characteristics have already been targeted by other models, points (d) and (e) are the unique ones that distinguish the Cairns-model at most. For the derivation of these characteristics the α-parameters are vital. These are the parameters that drive the mean-reverting Ornstein-Uhlenbeck processes. Given at least one rather low α-term will lead to one factor X being subject to long-term cycles, feeding through to long-term cycles in interest rates. Furthermore, such a parameter would allow par yields on long-term bonds to vary over a wide range. Meanwhile, Cairns (2004) shows that the β-term can be interpreted as a long-term forward interest rate.

In the following we will narrow the analysis to the two-factor version of the model, i.e. having X_1 and X_2 factors. As noted by Cairns (2004) two factors already enable the demonstration of key features of the multi-factor version.[22] Figure 1 illustrates the dynamics of the term structure under the use of the two-factor version. While Subfigure (a) demonstrates the ability of generating various changes with respect to the curvature of the term structure, Subfigure (b) illustrates the overall dynamics, incl. the ability of modeling relatively low as well as relatively high yield curves without re-calibrating the model.

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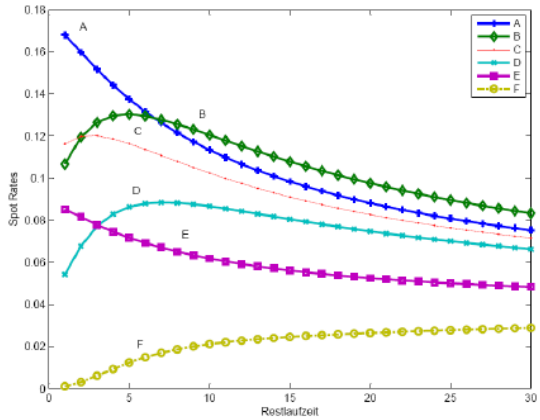
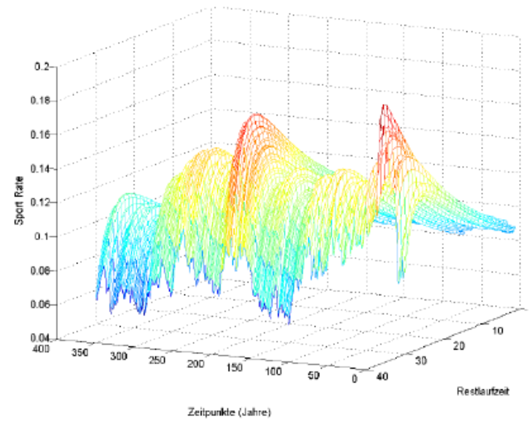
[22] Jamshidian and Zhu (1997) for example show that three factors explain 93% to 94% of interest rate dynamics. They also show that two factors already explain up to 91% of variations in the yield curve, while one factor will exhibit an explanatory power of 68% to 76%. Interestingly, Rebonato (1998) indicates that one factor models can already explain up to 92% of the dynamics, while two factors might capture 99.1% of the variations.

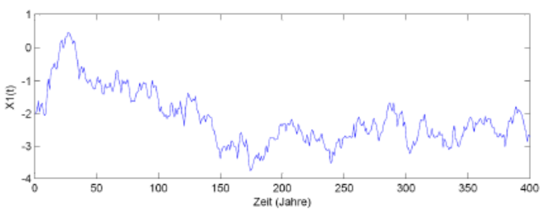
As discussed in Cairns (2004) the measure under which the term shown in Equation (2.4) is a martingale can either be interpreted as a real-world measure $\mathbb{P}$ or a risk-neutral measure $\mathbb{Q}$. Using the risk-neutral measure $\mathbb{Q}$ will allow endogenously generated prices to fit market prices, standing in line with no arbitrage theory. Theoretically speaking we should opt to use the risk-neutral measure $\mathbb{Q}$ as we are seeking to hedge interest rate risk, i.e. use exogenous price information. Our focus is however not limited to "pricing". We are interested in shedding light on the value added offered by the model in managing interest rate risk of long-term portfolios, emphasizing the need in generating realistic dynamics of interest rates. For this purpose we choose a historical calibration method (i.e. under the real-world measure $\mathbb{P}$) as will be discussed in Subchapter 2.3.3.



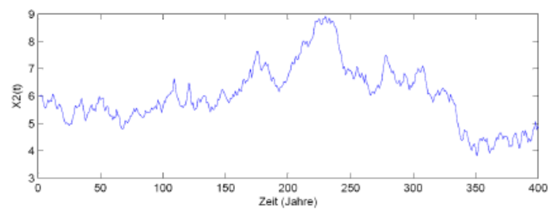
(a) Possible Term-Structures of Interest Rates



(b) Simulated Spot Rates



(c) Simulated State Variable X_1



(d) Simulated State Variable X_2

Figure 3: Simulation Results using the Cairns Model

Own calculation with two state variables using the following parameters: $\mu_1 = -2; \mu_2 = 6; \alpha_1 = 0.6; \alpha_2 = 0.06; \sigma_1 = 0.6; \sigma_2 = 0.4; \rho_{12} = -0.5; \beta = 0.04$. In (b) 400 paths of spot rates were simulated using the state variables given in (c) and (d). In (a) the state variables have the following values: A: $X(t) = (1,3)'$, B: $X(t) = (-1,5)'$, C: $X(t) = (0,3)'$, D: $X(t) = (-2,3)'$, E: $X(t) = (1,-1)'$, F: $X(t) = (-8,-4)'$.

2.3.3. MODEL IMPLEMENTATION

In the following we show how the parameters of the two-factor version of the Cairns model can be estimated. We subsequently display the robustness of the approach chosen.

We base the estimation of the parameters of the Cairns model on the so-called extended Kalman filter approach. For this purpose we adopt the algorithm described by Lutz (2006). The approach has also been analyzed for example by Babbs and Nowman (1999) to calibrate generalized Vasicek term structure models. Duan and Simonato (1999) have applied it to affine term structure models, and Chen and Scott (2003) have narrowed the analyses to Cox-Ingersoll-Ross models. In the following we will briefly revisit the main steps of the calibration algorithm. For a more comprehensive elaboration we refer the reader to Lutz (2006).

We first define a measure- and a transition formula. Let y stand for the q-number of spot rates available. Whereas each spot rate has a maturity of $m_i, = 1, \dots, q$. The model-value of these spot rates is given as

$$R(t, t+m_i) = -\frac{1}{m_i} log P(t,T) = \underbrace{\frac{1}{m_i} \log\left(\frac{\int_0^\infty H\big(u, X(t)\big)du}{\int_{m_i}^\infty H\big(u, X(t)\big)du}\right)}_{G_i(X(u);\theta)}. \tag{2.8}$$

The so-called measure formula describes the relation between the realized and the modeled rates is defined as

$$y_t = G(X(t), \theta) + \epsilon_t, \quad \epsilon_t \sim N\big(0, R(\theta)\big) \tag{2.9}$$

$$\text{with } G(X(t), \theta) = \begin{pmatrix} G_1(X(t), \theta) \\ \vdots \\ G_q(X(t), \theta) \end{pmatrix}$$

$$\text{and } R_t(\theta) = \left(\begin{bmatrix} \nu_1^2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \nu_q^2 \end{bmatrix}\right).$$

The vector θ contains $8 + q$ parameters $(\mu_1, \mu_2, \alpha_1, \alpha_2, \rho_{12}, \sigma_1, \sigma_2, \nu_1, \ldots, \nu_q)$ that need to be estimated. Because of the normality of $G(X(t), \theta)$ we first linearize the measure function. Using $\frac{\partial}{\partial x_i} H\big(u, X(t)\big) = H\big(u, X(t)\big)\sigma_i e^{-\alpha_i u}$, the gradient $\nabla_x = G_j(x;\theta) = \left(\frac{\partial}{\partial x_1} G_j(x;\theta), \frac{\partial}{\partial x_2} G_j(x;\theta)\right)$ is defined as

$$\frac{\partial}{\partial x_j} G_j(x;\theta) = \frac{1}{m_j}\left(\frac{\int_0^\infty H\big(u, X(t)\big)\sigma_i e^{-\alpha_i u} du}{\int_0^\infty H\big(u, X(t)\big) du} + \frac{\int_{m_j}^\infty H\big(u, X(t)\big)\sigma_i e^{-\alpha_i u} du}{\int_{m_j}^\infty H\big(u, X(t)\big) du}\right). \tag{2.10}$$

The linearized measuring equation can be written as

$$y_t \approx G\big(\hat{x}_{t|t-1};\theta\big) + D_{t|t-1}\big(X_t - \hat{x}_{t|t-1}\big) + \epsilon_t, \tag{2.11}$$

$$\text{with } D_{t|t-1} = \begin{pmatrix} \frac{\partial}{\partial x_1} G_1\big(\hat{x}_{t|t-1};\theta\big) & \frac{\partial}{\partial x_2} G_2\big(\hat{x}_{t|t-1};\theta\big) \\ \vdots & \vdots \\ \frac{\partial}{\partial x_1} G_q\big(\hat{x}_{t|t-1};\theta\big) & \frac{\partial}{\partial x_2} G_q\big(\hat{x}_{t|t-1};\theta\big) \end{pmatrix} \in \mathbb{R}^{q\times 2}.$$

We are interested in the process of two factors under the real-world measure $\mathbb{P}$. As stated by Lutz (2006), due to the time-discrete version of Equation (2.7) the two factors (X) have a multivariate normal distribution and can be written as

$$X\big(t + \Delta t | X(t)\big) = \underbrace{\begin{pmatrix} \mu_1(1 - e^{-\alpha_1 \Delta t}) \\ \mu_2(1 - e^{-\alpha_2 \Delta t}) \end{pmatrix}}_{=:F_0(\theta)} + \underbrace{\begin{pmatrix} e^{-\alpha_1 \Delta t} & 0 \\ 0 & e^{-\alpha_2 \Delta t} \end{pmatrix}}_{=:F_1(\theta)} \begin{pmatrix} X_1(t) \\ X_2(t) \end{pmatrix} + \eta(t) \tag{2.12}$$

$$= F_0(\theta) + F_1(\theta) X(t) + \eta(t)$$

$$\text{with } \eta(t) \sim N\big(0, Q(\theta)\big)$$

$$\text{and } Q(\theta) = \left(\frac{\rho_{ij}}{\alpha_i+\alpha_j}\left(1 - e^{-(\alpha_i+\alpha_j)\Delta t}\right)\right)^2_{i,j=1}.$$

Starting with a given parameter θ and an unconditional expected value for the risk factors $x_{0|0}$, and the covariance matrix for the factors $P_{0|0}$, the calculation steps are as follows:

i. The information in $t-1$ is used to calculate the ex-ante proxies for t as follows

$$\hat{x}_{t|t-1} = F_t^0(\theta) + F_t^1(\theta)\hat{x}_{t-1|t-1}, \tag{2.13}$$

$$P_{t|t-1} = F_t^1(\theta)P_{t-1|t-1}F_t^1(\theta) + Q_t. \tag{2.14}$$

ii. Given the new information y_t, the forecasting error ϵ_t and the covariance matrix $F_{t|t-1}$ can be estimated using the following equations, respectively

$$\epsilon_t = y_t - G(\hat{x}_{t|t-1};\theta), \text{ and} \tag{2.15}$$

$$F_{t|t-1} = D_{t|t-1}P_{t-1|t-1}D'_{t|t-1} + R_t. \tag{2.16}$$

iii. The proxies can then be corrected as follows:

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + \underbrace{P_{t-1|t-1}D'_{t|t-1}F_{t|t-1}^{-1}}_{K_t}\epsilon_t, \text{ and} \tag{2.17}$$

$$P_{t|t} = (I - K_t D_{t|t-1})P_{t|t-1}. \tag{2.18}$$

iv. The value of the log likelihood function is calculated in the following manner

$$\text{Log}L(y,\theta) = -\frac{1}{2}\sum_{t=1}^{T} n\log(2\pi) + \log\left|F_{t|t-1}\right| + \epsilon'_t F_{t|t-1}^{-1}\epsilon_t. \tag{2.19}$$

The steps i - iv are repeated until $\hat{\theta} = argmax\log L(y,\theta)$ is found, i.e. until the likelihood function is maximized.

Testing the Calibration Approach

In order to test the consistency of our calibration technique we ran an in-sample test. We calibrated the model parameters based on simulated interest rate curves. Given the starting values with lower and upper bounds as seen in Table 3 we simulated the spot yields with the following yearly maturities: 1-10, 20, 30, 40 and 50. The measuring error for each maturity $\epsilon_t \sim N(0, \upsilon^2)$ was simulated with $\upsilon = 0.001$. In each one of the 300 simulations, 25 monthly interest rate curves were generated. These curves where then used to calibrate the model (using the extended Kalman filter approach). The average of the calibrated parameters and their standard deviations are also given in Table 3. We conclude that although the calibration results show some room for improvement (especially with regards to μ_1 and μ_2), they still suffice for our purpose.[23]

	μ_1	μ_2	α_1	α_2	σ_1	σ_2	ρ_{12}	β	ν
Simulation input									
Parameters	-1.20	0.87	0.48	0.07	-0.62	0.43	0.44	0.06	0.001
Calibration input									
Start values	-2.00	-2.00	0.30	0.05	-0.60	0.50	0.40	0.04	0.001
Lower bound	-4.00	-4.00	0.20	0.02	-0.80	0.02	0.02	0.02	0.000001
Upper bound	10.00	10.00	0.80	0.20	0.80	0.60	0.60	0.08	0.01
Calibration results									
Mean	-1.56	2.17	0.38	0.15	-0.84	0.62	0.50	0.06	0.001
Standard deviation	3.09	1.49	0.11	0.02	0.15	0.13	0.09	0.00	0.00

Table 3: Results for Testing the Capability of the Kalman Filter Approach in Calibrating the Cairns Model

The table summarizes results of testing the applicability of calibrating the two-factor version of the Cairns model using the extended Kalman filter approach. For this purpose a Monte Carlo simulation was conducted based on the input parameters seen in the third row of the table. Based on the simulated term-structures the model was calibrated, starting with the values seen in the fifth

[23] Separate calculations have shown that μ_1 and μ_2 have a negligible impact on the sensitivity measure we will introduce. Balder (2014) for example ignores these parameters all together.

row while being restricted by the given lower and upper bounds. The results of the calibration (mean and respective standard deviation) are given per parameter in the last two rows of the table.

2.3.4. MEASURING INTEREST RATE SENSITIVITY

In the following we propose a possible sensitivity measure using the Cairns model. The objective is ultimately to use this measure in a respective immunization strategy in the spirit of delta hedging as for example discussed in Jarrow and Turnbull (1994). In contrast to the short rate or market models described above, the Cairns model is not able to represent bond prices as a function of an (instantaneous) interest rate. This eliminates the possibility of estimating the sensitivity of bond prices with respect to interest rates. We therefore offer the derivative with respect to the risk factors X_i as an alternative and label it as Cairn delta Δ_i (in short, delta).

Using the basic dynamics for the Cairns model seen in Equation (2.4) gives the following first derivative with respect to factor X_i

$$\Delta_i = \frac{\partial P\big(t,T,X(t)\big)}{\partial X_i} = \tag{2.20}$$

$$\frac{\left(\int_{T-t}^{\infty} \sigma_i e^{-\alpha_i u} H\big(u,X(t)\big)du\right)\left(\int_0^{\infty} H\big(u,X(t)\big)du\right)}{\left(\int_0^{\infty} H\big(u,X(t)\big)du\right)^2} - \frac{\left(\int_0^{\infty} \sigma_i e^{-\alpha_i u} H\big(u,X(t)\big)du\right)\left(\int_{T-t}^{\infty} H\big(u,X(t)\big)du\right)}{\left(\int_0^{\infty} H\big(u,X(t)\big)du\right)^2}.$$

Figure 2 shows the usual course of the delta measure as a function of time to maturity, allowing for an additional "zoom", illustrating the sensitivities in the short-term in more detail. In the case of the two-factor version of the model, X_1 and X_2 drive the bond prices that are in return a function of the mean-reverting parameters α_1 and α_2, respectively.

Notice that the delta-values first rise (absolutely), reaching a "maturity level". This level seems to vary from one factor to another. At some point in time the sensitivity measures converge to zero. It becomes apparent that the second risk factor captures longer time cycles, feeding ultimately into the dynamics of long-term interest rates, affirming the observation made by Cairns (2004) that one relatively low value for the parameter α_i (in this case $\alpha_2 = 0.03$) will allow for this feature. This means for example that bonds with maturities of 10 years are way more sensitive to changes in X_2 than in X_1.

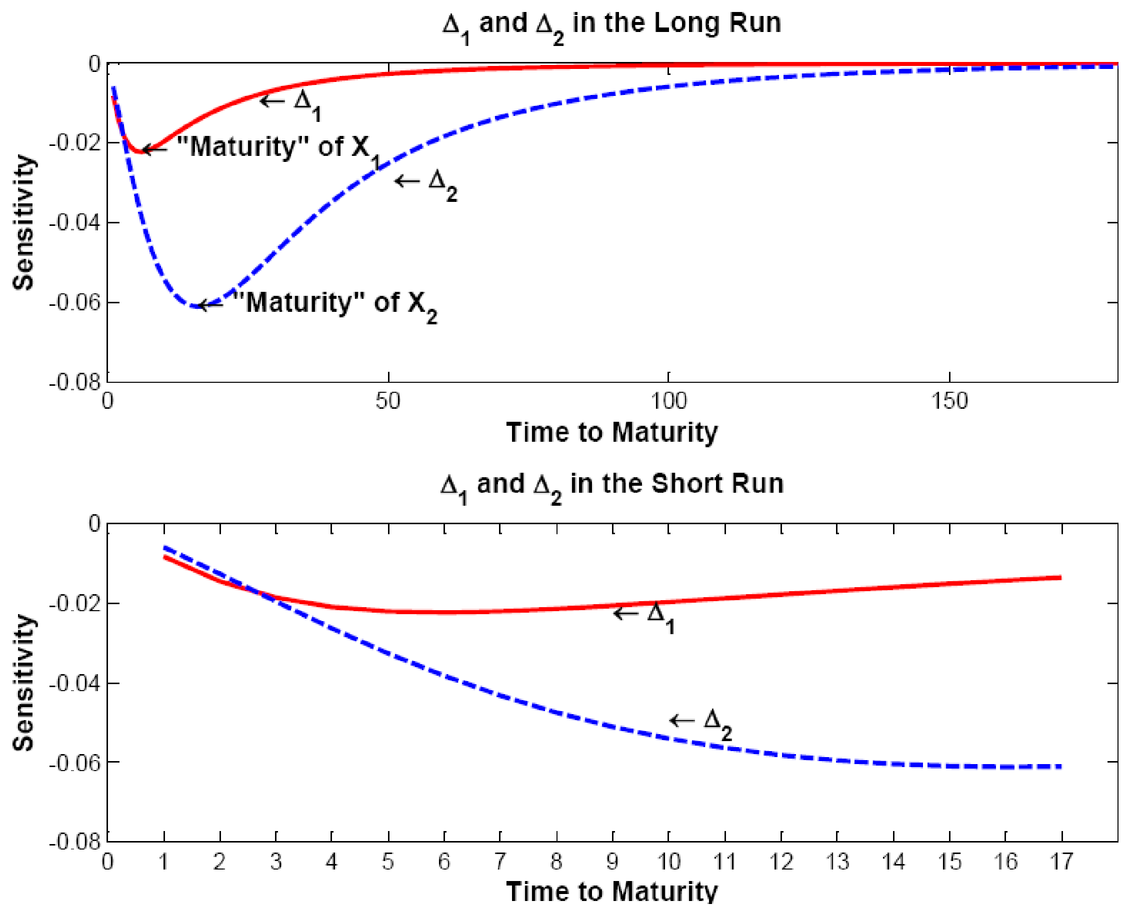


Figure 4: Sensitivity Measures of Bond Prices Using the Cairns Model as a Function of Time to Maturity

Using the two-factor version of the Cairns model the figures display the course of the first derivative of bond prices with respect to X_2 and X_1. The remaining (constant) factors of the model are: $\mu_1 = -2.12, \mu_2, = 8.89$; $\alpha_1 = 0.44$; $\alpha_2 = 0.03$; $\sigma_1 = 0.43$; $\sigma_2 = 0.60$; $\rho_{12} = -0.59$; $\beta = 0.03$.

2.4. PROCEDURE FOR STRUCTURING AN OPTIMAL SWAP-OVERLAY

In the following we will illustrate how a set of interest rate swaps is defined in order to immunize a pension-fund-like portfolio, i.e. how to structure a swap overlay that minimizes the volatility of the "funding-status" (present value of the assets / present value of the liabilities). We assume that the pension fund is closed, meaning that no additional contributions are expected, i.e. inflows stem only from held assets.

We start by decomposing the portfolio into its netted cash flows. These cash flow positions can be seen as a set of zero coupon bonds (ZCB) that need to be hedged individually, e.g. if we have 50 yearly cash flows we will need 50 hedge instruments to achieve an optimal hedge. For each ZCB we derive the interest rate sensitivity $\Delta_{i,m}^{p}$ as follows

$$\Delta_{i,m}^{p} = \Delta_{i,m} \cdot CF_m, \quad m \in [1, \dots, M] \tag{2.21}$$

whereas M is the time of maturity of the portfolio and CF_m stands for the netted cash flow at time period m. In our example $\Delta_{i,m}$ is calculated using Equation (2.20), where i stands for the risk factor considered (i.e. either 1 or 2 as we are applying the two-factor version of the Cairns model). One can of course replace $\Delta_{i,m}^{p}$ through the PV01 measure given in Equation (2.1).

In line with common practice we aggregate the cash flow sensitivities into buckets, i.e. we reduce the number of ZCB and thus the number of hedge instruments needed. The theoretical rationale behind buckets would be that interest rate sensitivities of neighboring time periods (especially with long-term maturities) exhibit a relatively high correlation, e.g. yields with time to maturity between 10 and 15 years will be subject to similar dynamics. The practical rationale would be that *liquid* interest rate swaps are available for a set of maturities, predefining the bucket structure.

This means that for each bucket j the sensitivities are calculated in the following manner

$$\Delta_{i,j}^{bucket} = \sum_{beg_j}^{end_j} \Delta_{i,m} \cdot CF_m, \quad j \in [1, \dots, p], i \in [1,2] \tag{2.22}$$

with the vectors beg_j and end_j containing the starting and ending time periods of the p time buckets considered.

In Algorithm 1 below we give the pseudo code to define the swaps needed for the overlay, given a portfolio cash flow $\widehat{CF}$, i.e. the code defines the number of swaps, their settlement dates, maturity dates and respective swap rates.[24] The index $n_i, i \in [1, \dots, p]$ takes on the value 1 once the bucket $i$has been immunized and 0 otherwise.

For each chosen interest rate swap j and time period i we define s_{ij} as follows

$$s_{ij} = \begin{cases} 1 & t_i = \text{first settlement date} \\ c & t_i = \text{payment date prior maturity} \\ (1+c) & t_i = \text{maturity} \\ 0 & \text{elsewhere.} \end{cases} \tag{2.23}$$

For each bucket h and each swap l (and delta k) we then define

$$s_{hj}^{b,\Delta_k} = \sum_{i=beg_l}^{end_l} \Delta_{i,k} \cdot s_{ij}, l \in= [1, \dots m], h \in \left[1, \dots, \dot{N}^{bucket}\right], k \in [1,2]. \tag{2.24}$$

The final step is solving the following linear equation and finding the optimal values of the vector $\ddot{N}_{swap,j} \in (-\infty, +\infty)$, containing the notional values of the swaps:

[24] The algorithm is inspired by Bemmann (2008). It, however, differs in its criteria for choosing the relevant buckets and the respective swaps. More importantly, it does not remain in the stepwise optimization, but uses a linear optimization for to achieve the hedge.

$$\begin{pmatrix} s_{11}^{b,\Delta_1} & s_{12}^{b,\Delta_1} & \cdots & s_{1m}^{b,\Delta_1} \\ s_{21}^{b,\Delta_1} & s_{22}^{b,\Delta_1} & \cdots & s_{2m}^{b,\Delta_1} \\ \vdots & \vdots & \ddots & \vdots \\ s_{p1}^{b,\Delta_1} & s_{p2}^{b,\Delta_1} & \cdots & s_{pm}^{b,\Delta_1} \\ s_{11}^{b,\Delta_2} & s_{12}^{b,\Delta_2} & \cdots & s_{1m}^{b,\Delta_2} \\ s_{21}^{b,\Delta_2} & s_{22}^{b,\Delta_2} & \cdots & s_{2m}^{b,\Delta_2} \\ \vdots & \vdots & \ddots & \vdots \\ s_{p1}^{b,\Delta_2} & s_{p2}^{b,\Delta_2} & \cdots & s_{pm}^{b,\Delta_2} \end{pmatrix} \begin{pmatrix} \ddot{N}_{swap,1} \\ \ddot{N}_{swap,2} \\ \vdots \\ \ddot{N}_{swap,m} \end{pmatrix} = \begin{pmatrix} -\Delta_{1,1}^{b} \\ -\Delta_{1,2}^{b} \\ \vdots \\ -\Delta_{1,p}^{b} \\ -\Delta_{2,1}^{b} \\ -\Delta_{2,2}^{b} \\ \vdots \\ -\Delta_{2,p}^{b} \end{pmatrix}. \tag{2.25}$$

Algorithm 1 : An algorithm for choosing $\boldsymbol{p}$ swaps to net interest rate sensitivity

Calculate $\hat{\Delta}_{\mathbf{1}}$ and $\hat{\Delta}_{\mathbf{2}}$ (e.g. as in Equation **(2.20))**

for $\boldsymbol{i} = \mathbf{1}$ to $\boldsymbol{p}$ **do**

  for $\boldsymbol{i} = \mathbf{1}$ to $\boldsymbol{p}$ **do**

    if $\boldsymbol{n_i} = \mathbf{0}$ **then** *(with $\boldsymbol{n_i}, \boldsymbol{i} \in [\mathbf{1}, \dots, \boldsymbol{p}]$ taking on the value 1 once bucket $\boldsymbol{i}$ has been netted, 0 otherwise.)*

      Calculate the delta values for each bucket to be netted

$$\hat{\Delta}_{\mathbf{1}}(\boldsymbol{i}) = \Delta_{\mathbf{1},\boldsymbol{i}}^{\boldsymbol{bucket}} = \sum_{\boldsymbol{beg_i}}^{\boldsymbol{end_i}} \Delta_{\mathbf{1},\boldsymbol{m}}^{\boldsymbol{port}} \cdot \boldsymbol{CF_m}$$

$$\hat{\Delta}_{2}(\boldsymbol{i}) = \Delta_{2,\boldsymbol{i}}^{\boldsymbol{bucket}} = \sum_{\boldsymbol{beg_i}}^{\boldsymbol{end_i}} \Delta_{\mathbf{2},\boldsymbol{m}}^{\boldsymbol{port}} \cdot \boldsymbol{CF_m}$$

    end if

  end for

  Calculate $\Delta_{\boldsymbol{min}}^{\boldsymbol{j}} = \min(\hat{\Delta}_{\mathbf{1}})$ **and** $\Delta_{\boldsymbol{max}}^{\boldsymbol{k}} = \max(\hat{\Delta}_{\mathbf{1}})$

  if $\Delta_{\boldsymbol{min}}^{\boldsymbol{j}} < \mathbf{0}$ and $\Delta_{\boldsymbol{max}}^{\boldsymbol{j}} > \mathbf{0}$ then

    Construct an appropriate forward swap

    set settlement date: $\boldsymbol{t_0^i} = \boldsymbol{beg_j}$

    set maturity date: $\boldsymbol{t_n^i} = \boldsymbol{beg_k}$

    set swap rate: $\boldsymbol{c} = \frac{\boldsymbol{P}(\boldsymbol{t},\boldsymbol{t_0^i}) - \boldsymbol{P}(\boldsymbol{t},\boldsymbol{t_n^i})}{\sum_{\boldsymbol{z}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{P}(\boldsymbol{t},\boldsymbol{t_z})}$

  else

    $\Delta_{\boldsymbol{net}}^{\boldsymbol{l}} = \boldsymbol{max}\left(\left|\Delta_{\boldsymbol{min}}^{\boldsymbol{i}}\right|, \left|\Delta_{\boldsymbol{max}}^{\boldsymbol{i}}\right|\right)$

    Build an appropriate spot swap

    set settlement date: $\boldsymbol{t_0^i} = \mathbf{0}$

    set maturity date: $\boldsymbol{t_n^i} = \boldsymbol{beg_l}$

    set swap rate: $\boldsymbol{c} = \frac{\mathbf{1} - \boldsymbol{P}(\boldsymbol{t},\boldsymbol{t_n^i})}{\sum_{\boldsymbol{z}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{P}(\boldsymbol{t},\boldsymbol{t_z})}$

  end if

end for

Given p buckets - each containing one time period - the swap overlay of p swaps would basically mirror the cash flows of the portfolio, netting the payments and thus hedging all interest rate risks. Using such an overlay the portfolio would be perfectly "immune" to interest rate risks.

While in the example given above the procedure is constructed to hedge *two* sensitivity parameters, the algorithm can be generalized to hedge l factors. In particular this means that we can apply the algorithm to (*one*) PV01 measure in order to define a respective optimal overlay.

2.5. CASE STUDY

2.5.1. REMARKS

In the following we will illustrate the use of the algorithm proposed in the previous Section 2.4 to structure an interest rate swap overlay for an exemplary (but realistic) pension fund. The case study will allow the analysis of possible value added offered by using model-based – instead of duration-based – sensitivities in hedging. Besides examining the hedge effectiveness of both approaches, our analysis aims on shedding light on further challenges and advantaged offered by applying the two-factor version of the Cairn model in managing interest rate risk.

We base our analysis on a realistic pension-fund structure, leaning on the portfolio illustrated by Bemmann (2008). The pension fund has a typical liability structure that is assumed to be given (e.g. using the base case scenario of previous actuarial modeling). The pension fund is assumed to be fixed and observed in the year 2000. While the bulk of the liabilities will mature in 2010 to 2030 years, a substantial portion of the liabilities exhibits a maturity between 30 and 50 years. The net present value of liabilities as seen in the year 2000 is € 536.50 mm. The portfolio is funded by fixed income securities (bonds) that exhibit a net present value of € 464.69 mm in year 2000. We assume that the majority of the assets will mature in the next 10 years, leading to a significant duration gap. We further assume that the pension fund does not accept any additional contributions, i.e. inflows stem only from held assets. Finally, we introduce a cash account with € 71.82 mm, initializing the funding status at 1.

2.5.2. PREPARATIONS – MODEL CALIBRATION

We start by calibrating the parameters of the Cairns model. Again, we acknowledge the need to calibrate the model to fit market implied information (i.e. under the risk-neutral measure $\mathbb{Q}$), especially when it comes to pricing and hedging. We have, however, deliberately decided to base the calibration on historical data to test the ability of the model in offering realistic dynamics of interest rates. For this purpose we apply the extended Kalman filter approach described in Subchapter 2.3.3 to 10 years of historical data from the European swap market as illustrated in Figure 2 above. In particular the calibration uses monthly swap rate curves from June 2000 to June 2010. In each curve the yields of the following 20 yearly maturities are drawn: 1-12, 15, 20, 25, 30, 35, 40, 45, and 50. The following Table 4 summarizes the calibration results per parameter, also exhibiting the maximum log likelihood value achieved

	μ_1	μ_2	α_1	α_2	σ_1	σ_2	ρ_{12}	β	$\log L$
Calibrated parameters	-2.12	8.98	0.44	0.03	0.43	0.60	-0.59	0.03	16,677

Table 4: Calibration Results of the Cairns Model using Historical Interest Rate Curves

The table summarizes the estimations of the constant parameters of the two-factor version of the Cairns model. The estimation has been done using monthly Euribor curves seen in Figure 2. The calibration method is based on the extended Kalman filter approach.

Recall that the extended Kalman filter approach assumes normally distributed measuring error for each maturity $\epsilon_t \sim N(0, \upsilon^2)$. In our case this means that in addition to the eight parameters seen in Table 4 the calibration approach also delivers 20 standard deviation values $\upsilon_i, i \in \{1-12, 15, 20, 25, 30, 35, 40, 45, 50\}$ as seen in Figure 5. It becomes apparent that the two-factor model is especially able to deliver a good-fit for the dynamics of yields with yearly maturities between 9 and 15. For short-term yields as well as for maturities exceeding 30 years, the calibration seems to deliver somewhat less accurate results. This might result from the fact that the calibrated model integrates only two risk factors. Three and more factors might minimize the measuring error.

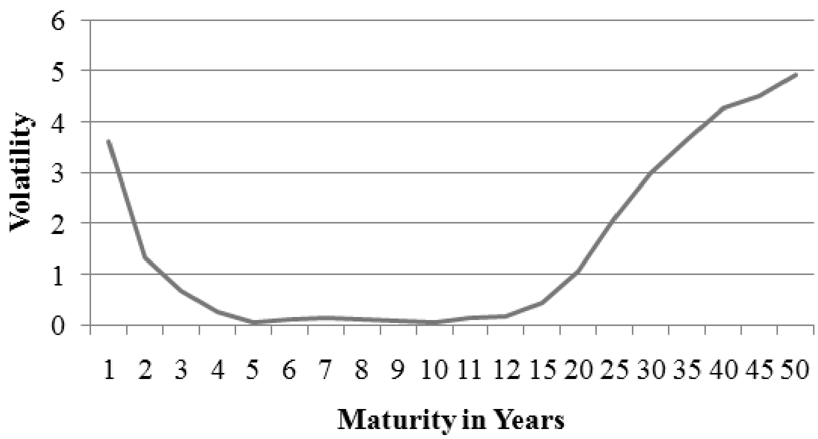


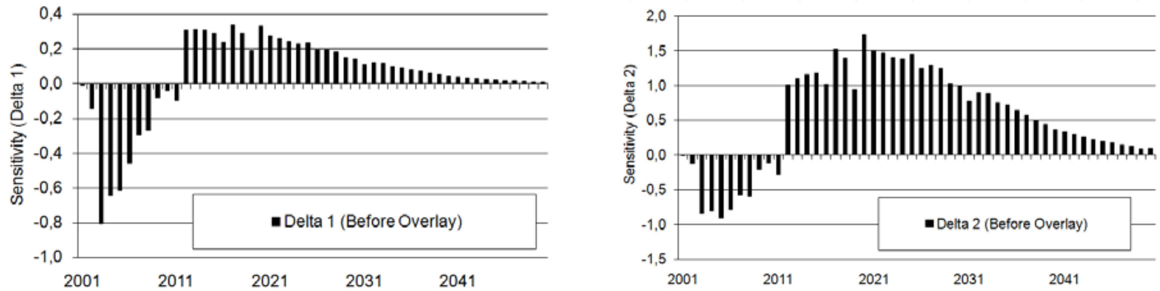
Figure 5: Volatility of the Measuring Error Used in the Calibration of the Cairns Model

The graph illustrates the estimated standard deviation (volatility) υ of the measuring errors $\epsilon_t \sim N(0, \upsilon^2)$ per maturity t while calibrating the two-factor version of the Cairns model to historical data (Euribor curves from June 2000-June 2010).

2.5.3. Structuring an Optimal Swap-Overlay

Following the procedure described in Section 2.4 we start by decomposing the portfolio into netted *(yearly)* cash flow positions (that can be interpreted as separate ZCB with differing notional amounts and maturities). We then apply the model-based sensitivity measures given in Equation (2.20) to the ZCB positions, measured in Δ_1 and Δ_2. The result is shown in Figure 6. In parallel we also apply the PV01 measure given in Equation (2.1), accordingly.

Figure 6 demonstrates the different dependencies, the two deltas have on time to maturity. Both Δ_1 and Δ_2 exhibit decreasing sensitivity parameters with increasing time to maturity. This stands in line with the higher volatilities of shorter interest rates (made visible in Figure 2). In the years following 2010, Δ_2 seems to be significantly larger than Δ_1, showing the responsibility of the second risk factor for the longer-term interest rate dynamics. The figures resemble the common course of sensitivities, pension funds usually exhibit.



a) Delta 1 of the Portfolio (I) b) Delta 2 of the Portfolio (I)

Figure 6: The Course of Interest Rate Sensitivity of an Exemplary Pension Fund (I)

The figures illustrate Δ_1 (delta 1) and Δ_2 (delta 2) exhibited by the netted cash flows of the exemplary pension fund.

In line with the procedure described in the previous section we assume a pre-defined bucket structure, given in Table 5. In practice such a structure would be a function of the liquidity of the swap market, i.e. building buckets around the time horizons for which spot and forward swaps can be easily traded. Further considerations of possibly higher volatility of short-term yields might play a further role in defining the buckets. One must, however, acknowledge that the final structure will not be free of arbitrariness.

Bucket Number	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
Begin	2001	2002	2003	2006	2011	2016	2021	2031	2041
End	2001	2002	2005	2010	2015	2020	2030	2040	2050

Table 5: Predefined Bucket Structure

The table summarizes the exemplary (and simplified) bucket structure to be used in the case study. 9 buckets with a decreasing granularity are assumed. While the first two buckets consist only of one year of cash flow each, the 9th bucket entails 9 years.

Given the yearly sensitivities and the bucket structure we design two swap overlays; one based on the Cairns deltas, and one based on the PV01 measure. For this purpose we run Algorithm 1 twice to define the needed swaps for each overlay. The respective notional amounts are found by running the linear optimization scheme, following Equation (2.25).

Table 6 summarizes the chosen swaps if the model-based deltas are used. As expected the number of swaps equals the predefined number of buckets. The total notional amount of the swap overlay is € 440.38 mm. In comparison, using the PV01 measure would deliver a notional amount of € 419.35 mm. The smaller notional amount is due to the fact that the PV01-approach hedges only one sensitivity measure (not two risk factors).[25]

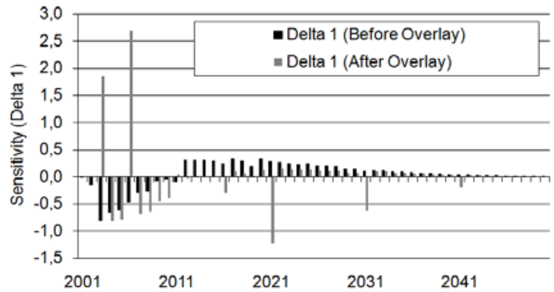
Swap	**Life time**	**Receiver/Payer**	**Notional (€ mm)**	**Swap Rate**
Swap I	2003-2021	Receiver	142.24	6.31%
Swap II	2006-2031	Receiver	149.20	6.36%
Swap III	2021-2041	Receiver	20.99	6.02%
Swap IV	2000-2011	Payer	24.67	5.92%
Swap V	2000-2031	Payer	42.88	6.12%
Swap VI	2000-2041	Receiver	33.62	6.10%
Swap VII	2000-2002	Payer	8.52	5.34%
Swap VIII	2000-2001	Receiver	0.07	5.04%
Swap IX	2000-2016	Receiver	18.20	6.05%

Table 6: Swap Overlay Structure Using the Cairns Model

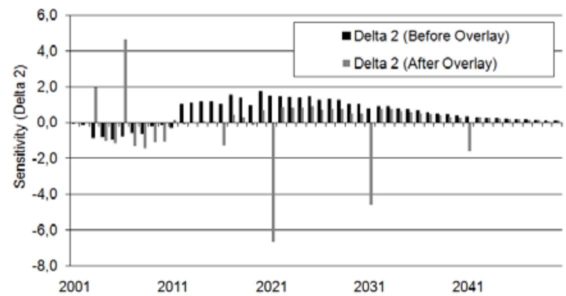
The table summarizes the swaps chosen to immunize the pension fund against interest rate risks. Their settlement date, maturity date (life time) and swap rate are defined by Algorithm 1. The notional amounts are an output of the linear optimization given in Equation (2.25) which also defines the feature "receiver" or "payer" according to the sign (-/+) of the amount chosen.

The plotted gray bars in the Figure 7(a) and Figure 7(b) show the sensitivities remaining after the introduction of the overlay. These figures show the extreme sensitivities that are built up on the level of the payment dates. After all, the algorithm treats sensitivity on bucket level.

[25] In order to match the cash flows the portfolio, an overlay of 50 swaps would be needed. Using the algorithm described in the previous chapter such an overlay would have a total notional amount of € 589.67 mm.



c) Delta 1 of the Portfolio (II)



d) Delta 2 of the Portfolio (II)

Figure 7: The Course of Interest Rate Sensitivity of an Exemplary Pension Fund (II)

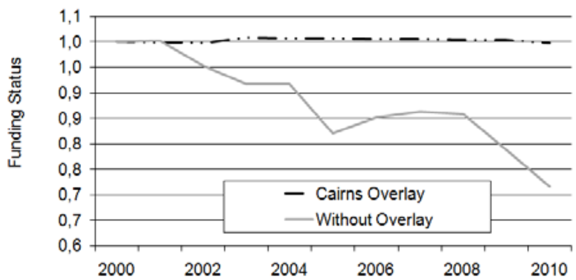
The figures exhibit Δ_1 (delta 1) and Δ_2 (delta 2) before and after the hedge on the level of the payment dates. On this level extreme sensitivities might be built (like the negative bar seen in the year 2021) while on bucket level the sensitivities are hedged.

2.5.4. HEDGE EFFECTIVENESS – BACKTEST APPROACH

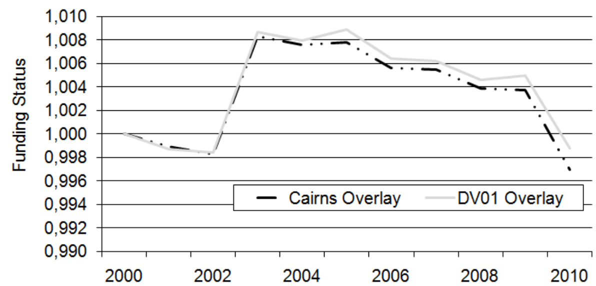
Following the approach given in Bemmann (2008) we "backtest" the effectiveness of the hedging strategy, i.e. we retrospectively test the effectiveness delivered from June 2000 to June 2010.

The backtesting approach works as follows. We evaluate the assets and liabilities using the respective curve at each subsequent period, using an annual frequency (starting with June 2001). We assume no re-investments, i.e. all inflows (interest payments from bonds and swaps and notional repayments) are transferred into the cash account. The cash account is used to settle all outflows (interest payments and pensions). Throughout the backtesting period the swap overlay is assumed not to have been re-adjusted.

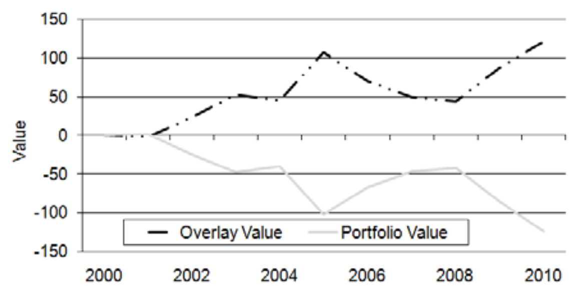
Notice that we base our backtesting approach on the same sample used to calibrate the parameters of the Cairns model, i.e. allowing for an *in-sample* test of the internal consistency of the model. We acknowledge that a proper *out-of-sample* analysis would be needed to test the prediction power of the model. Yet, we are more interested in testing the capabilities of the model in capturing realistic dynamics and in consistently being able to measure the respective sensitivities, leaving analysis regarding the prediction power of the model for future work.



a) Funding Status with and without an Overlay



b) Funding Status with a Cairns- and a PV01 Overlay



c) Portfolio Value and Overlay Value

Figure 8: Backtesting Results for a Swap-Overlay Structure

The figures display the results of backtesting the structured swap overlay. Subfigure (a) illustrates the funding status with and without an overlay. Subfigure (b) compares the hedge effectiveness delivered by using the Cairn deltas on the one hand and the PV01 metric on the other. Subfigure (c) illustrates the change in the value of the underlying portfolio (consisting of assets and liabilities) in comparison with the change in the value of the swap overlay.

The results of the backtest are given in Figure 8. The underlying portfolio seems to have been relatively vulnerable, especially to the more recent decline in interest rates (seen in Figure 2). This development would have influenced the funding status of the portfolio massively as seen in Figure 8(a). The funding status of the non-hedged portfolio would have reached approximately 0.7. It becomes evident that both overlay structures (by either using the Cairns deltas or by using the PV01 approach) deliver a similar and effective hedge, stabilizing the funding status around 1. This is possible because the changes in the value of the overlay mirror the changes in the underlying portfolio, seen in Figure 8(c).[26]

We conclude that the backtest approach is able to confirm the internal consistency of the model and to ensure the robustness of the offered algorithm.

2.5.5. HEDGE EFFECTIVENESS – MONTE CARLO SIMULATION APPROACH

In a final step we use a Monte Carlo simulation scheme to analyze the dynamics of the underlying portfolio and effectiveness of the overlay under a complete distribution function.

For this purpose we simulate 5.000 values for two risk factors (X_1 and X_2), generating discount factors with maturities 1 - 50 years. The simulation is based on the parameters calibrated and given in the Table 4 above. Per scenario we evaluate the underlying portfolio and the swap overlay (which is assumed to be left unadjusted).

Table 7 summarizes the distribution parameters of the funding status for the strategies in comparison. While the average and median values deliver no differences, the hedge effectiveness becomes evident through the comparison of the minimum and maximum values. While a non-hedged portfolio exhibits a funding status ranging from 0.74 to 1.49, the PV01-overlay would shrink that range to 0.9562 - 1.0132. The Cairns-overlay would then minimize the range to 0.9587 - 1.0090. This is also apparent when comparing the short fall probabilities. Using the target funding status of 0.98 the Cairns-overlay decreases the probability from 77% to 7%, while the PV01-overlay still has a short probability of 10%.

[26] We acknowledge that asset managers would probably readjust such static hedges frequently (e.g. on a monthly basis). Still, our static approach is able to deliver a quite robust hedge so that we at this part did not see the necessity for a dynamic hedging strategy.

Swap	Cairns overlay	PV01 overlay	No overlay
Minimum	0.9587	0.9562	0.7417
Maximum	1.0090	1.0132	1.4866
Median	0.9966	0.9958	1.1236
Average	0.9939	0.9933	1.1336
Standard Deviation	0.0077	0.0085	0.1559
Prob.(coverage < 1)	77%	74%	20%
Prob.(coverage < 0.98)	7%	10%	17%
Prob.(coverage < 0.96)	0%	0%	13%

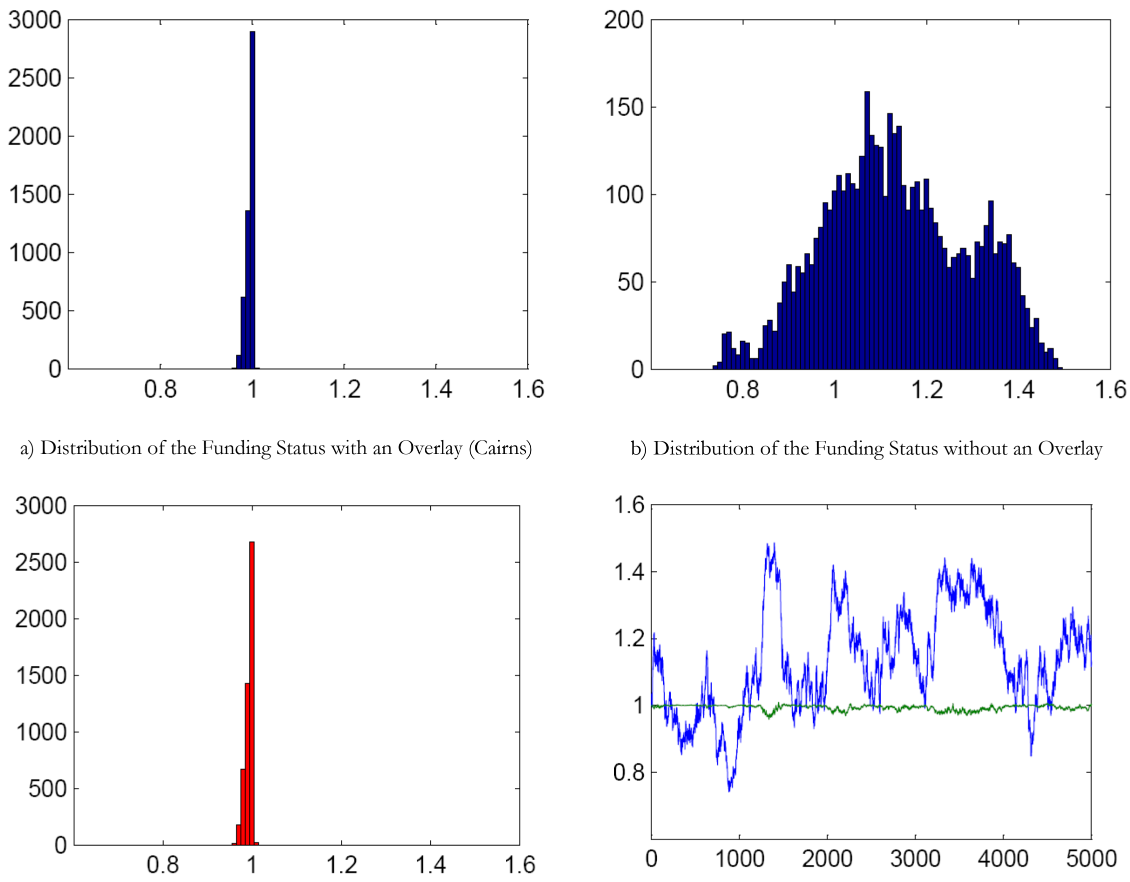
Table 7: Distribution Parameters of the Simulated Funding Status in Comparison

The table illustrates the results delivered by the Monte Carlo simulation scheme, summarizing the different distribution metrics for the three strategies: 1) structuring an overlay based on Cairns deltas; 2) structuring an overlay based on PV01 measures, 3) leaving the underlying portfolio without a hedge. The overlay is structured according to the procedure given in Subchapter 2.4. The simulation is based on the parameters calibrated and given in the Table 4 above.

The delivered results are illustrated in Figure 9. Subfigures 9(a), 9(b) and 9(c) show the histograms of the different strategies, strengthening the stated results. SubFigure 9(d) plots simulated funding status of the non-hedged portfolio and hedged portfolio in comparison, showing the hedge efficiency and the avoided volatility.

The results delivered reconfirm that the algorithm offered is model consistent. This means that if the model parameters are accurately calibrated, the demonstrated strategy would deliver a plausible hedge. Although quite insignificant, immunizing against two factors (deltas) would deliver slightly better results than hedging a vector of PV01 metrics.

More importantly, having a model in place allows portfolio managers to run a Monte Carlo simulation. Even if the simulation scheme is not fully integrated in the optimization procedure portfolio managers can leverage on such approaches to "*open up possibilities […] to reason on, to be aware of fat-tail risk, risky events*" (Fabozzi et al., 2005). Monte Carlo simulation thus offers a tool-kit for portfolio managers to analyze risks, offering a framework to discuss risks and respective mitigation actions.



a) Distribution of the Funding Status with an Overlay (Cairns)

b) Distribution of the Funding Status without an Overlay

c) Distribution of the Funding Status with an Overlay (PV01)

d) Hedged and Unhedged Funding Status in Comparison

Figure 9: Monte Carlo Simulation Results for a Swap-Overlay

The figures illustrate the results delivered by the Monte Carlo simulation scheme: Subfigures (a), (b) and (c) show the histograms of the different strategies, strengthening the stated results. Subfigure (d) plots resulting funding status of the non-hedged portfolio and hedged portfolio. The overlay is structured according to the procedure given in Subchapter 2.4. The simulation is based on the parameters calibrated and given in the Table 4 above.

2.6. CONCLUDING REMARKS

We have started by going through the spectrum of ALM strategies in general and interest rate risk management of pension funds in particular, concluding that the practical use of stochastic models is limited. Static approaches such as the duration-based PV01 measure seem to be structurally preferred by asset managers despite their theoretical pitfalls. A key motive behind the limited use of stochastic models is the lack of respective approaches that combine arbitrage-free valuation with realistic modeling of short- *and* long-term interest rate dynamics, a gap Cairns (2004) proclaims to address. We have therefore chosen to apply the Cairns model to the practical challenge of immunizing a pension fund against interest rate risk, analyzing its ability in actually closing that gap.

After revisiting the two-factor version of the Cairns model and its main features, we derived respective sensitivity measures (deltas). We then discussed the use of the extended Kalman filter approach in calibrating the parameters of the model. By going through these steps we hoped to shed light on the relevant complexities and implementation challenges.

At the heart of our analysis was the comparison of the hedge effectiveness offered by the Cairns model with the one given by the popular PV01 metric. For this purpose we introduced a rule-based and model-independent algorithm that immunizes pension fund-like portfolios against interest rate risk by structuring an overlay of appropriate swaps. Using a realistic example of a pension fund we subsequently ran the algorithm twice, structuring two possible overlays, one using model-based sensitivities (Cairns overlay) and one using the PV01 measures (PV01 overlay).

First, the hedge effectiveness offered by both overlays was analyzed in a backtesting environment in which both overlays exhibited similarly satisfying results. Second, a Monte Carlo simulation was applied. By observing a wider range of respective scenarios the simulation scheme was able to identify the *slight* advantage offered by the Cairns model. The use of the model can thus not be motivated by higher hedge effectiveness, especially if we bear the sophistication of its application in mind (compared with the PV01 approach).

As made apparent by the Monte Carlo simulation using a model-based approach offers further advantages as summarized in the following statement cited in Fabozzi et al. (2005):

> *"[I]t allows one to simulate dynamic investment strategies. It gives you the distributions, the confidence levels.*
>
> *By reducing uncertainty, modeling allows better decision making. Using powerful modeling tools, management can analyze scenarios and observe, through computer simulations, the future consequences of decisions."*

Having that said, the Cairns model offers a possible framework for such analysis, facilitating discussions around investment and especially because it can combine risk-neutrality with realistic modeling.

PART II:

COUNTERPARTY CREDIT RISK MANAGEMENT

CHAPTER 3:
PRICING AND MANAGING COUNTERPARTY CREDIT RISK IN THEORY AND PRACTICE

3.1. INTRODUCTION

In the previous chapter we looked into modeling and pricing interest rate derivatives, implicitly assuming that no further risk factors are material. We have in particular assumed, either that the counterparties dealing with such OTC derivatives are default-free or that their default will have no *considerable* effect on the derivative's value. Our assumption seems to be aligned with the prevailing view, at least prior to the financial crisis. This becomes apparent when looking at prominent textbooks in financial risk modeling and derivatives pricing such as Hull (2006), Björk (2009), Rebonato (2002) or vastly cited publications such as Miltersen et al. (1997) or Brace et al. (1997) – just to name a few – in which counterparty credit risk is either mentioned as by-the-way-issue or is missing completely.

Although from a risk management and a regulatory perspective there seems to have been – at least to a certain degree – an awareness for the inherent credit risk in OTC derivatives, e.g. the regulatory framework Basel II required financial institutions to capitalize the default risk inherent in their OTC portfolios, the incorporation of this element in pricing theory has been sporadic at best.

It is at the latest since the financial crisis in general and the collapse of Lehman Brothers in particular that the "default-free scheme" has been finally falsified. It has become clear that derivative traders – also AAA-rated investment banks – are default-prone and that their default considerably impacts prices. Major financial institutions claim to have started pricing and managing counterparty credit risk on a systematic basis before the crisis.[27] Still, it was the credit deterioration and the overall spread volatility during the financial crisis that finally inaugurated counterparty credit risk as a significant risk factor that must be incorporated in pricing.[28] This has been empirically confirmed for example by Arora et al. (2012) and becomes apparent if we look at the increasing number of publications that are dedicated to modeling, pricing and managing counterparty credit risk such as Brigo and Pallavicini (2008), Gregory (2009), Brigo and Capponi (2010) or Lipton and Sepp (2009) and textbooks such Cesari et al. (2009), Canabarro (2010), Gregory (2012), or Brigo et al. (2013a). This list is non exhaustive and can be extended if necessary.

The significance of counterparty credit risk has also drawn the attention of the regulators, standard setters and auditors, who in return further stressed the significance of the subject matter. Being material, counterparty credit risk has to be incorporated in fair value measurement, directly affecting profit and loss (P&L) statements and their volatility. According to the more recent regulatory framework Basel III, financial institutions are henceforth required to capitalize this additional volatility, significantly increasing the minimal regulatory capital needed.

The standard method for pricing counterparty credit risk is through a separate so-called credit valuation adjustment (CVA). CVA can be interpreted as the cost of hedging the counterparty credit risk of the respective position. This introduces a new derivative instrument, usually referred to as contingent credit default swap (C-CDS) that, in return to a premium, insures the (stochastic) exposure at default. The pricing of a C-CDS (i.e. CVA valuation) usually turns out to be a much more elaborate task than pricing the default-free derivative itself. After all, CVA depends not only on expected exposure – which is by itself sufficiently complex – but also on credit risk and the on the interconnection between both.

[27] Cesari et al. (2009) who worked for a major European Investment Bank claim for example to have started with a systematic modeling of counterparty credit risk in 2005.

[28] See similar arguments for instance in Gregory (2009) or Brigo and Capponi (2010).

This means that closed-form solutions for CVA remain an exception. In addition, portfolios containing different classes of derivatives (e.g. interest rate and equity derivatives) underline the fact that C-CDS is probably one of the most complicated derivatives to price, especially if further aspects such as netting and collateral are considered. Bear in mind that derivative traders are not only interested in valuing CVA, but also in hedging and managing it.

The higher awareness and the regulatory pressure seem to have pushed derivatives traders to step up the mitigation of counterparty credit risk. Being the most efficient way of reducing counterparty credit risk, collateral agreements, seem to have become more popular.[29] Counterparties are not only entering new collateral agreements, they are also amending existing ones by reducing thresholds and adjusting the types of eligible collateral. By amending such agreements the inherent counterparty credit risk – and thus CVA (as well as further exposure-dependent adjustments) changes, adjusting in return the overall value of the respective OTC derivatives portfolio.[30] This underlines the necessity of models to compute the effects of such amendments in order to reach mutual agreements and fair compensations.

The following Chapter 3 intends to give a compact overview in modeling and managing CVA, always accompanied by a critical analysis of the prevailing discourse. This analysis is the heart of the chapter, which aims on exploring the challenges around CVA from different angles rather than offering a comprehensive description of all relevant aspects. For a more

[29] See the margin surveys conducted annually by the International Swap and Derivatives Association (ISDA), for example ISDA (2014a), especially pp. 6-7.

[30] Notice that further factors, especially funding issues have to be considered as main drivers in such an analysis as well. Due to the fact that uncollateralized exposure has to be funded (e.g. due to the fact that the hedge portfolio, i.e. the duplication strategy of the default-free derivative, is fully collateralized) and the fact that funding costs of financial institutions have significantly increased, there is a similar on-going discussion on incorporating funding effects through a separate metric, funding valuation adjustment (FVA). Due to the complexity of the issue we will continue our analysis by focusing on counterparty credit risk and will touch on FVA only if necessary. For further reading on funding of OTC derivatives and FVA see for example Morini and Prampolini (2011), Hull and White (2012a) or Burgard and Kjaer (2013) as well as Pallavicini et al. (2011).

comprehensive description on all aspects around CVA we refer the reader to Gregory (2012). For a more technical focus on modelling counterparty credit risk we refer the reader to Cesari et al. (2009) or Brigo et al. (2013a), all of which have inspired the elaborations and analysis throughout the chapter.

Section 3.2 starts with an overview of key terms and definitions, linking these to relevant previous research and literature. Section 3.3 then lays the foundations for quantifying CVA, starting with the general pricing framework in Subchapter 3.3.1, and then moving to a more practical approach in Subchapter 3.3.2. Subchapter 3.3.3 takes us through a step-by-step approach to estimate the expected exposure profile, and Subchapter 3.3.4 elaborates on the techniques of estimating the probabilities of default. Section 3.4 describes and analyzes the regulatory and accounting requirements behind the CVA discourse, starting with describing the regulatory landscape in Subchapter 3.4.1. A critical analysis of relevant accounting standards is then given in Subchapter 3.4.2 whilst Subchapter 3.4.3 focuses on relevant regulatory requirements. A conclusion of the analysis regarding counterparty credit risk and CVA regulation is given in Subchapter 3.4.4. Section 3.5 moves to discuss and analyze key aspects in managing CVA risk. Subchapter 3.5.1 starts with looking into pricing CVA and analyzes the role and effectiveness of central units that manage CVA, also referred to as CVA desks. Key aspects around mitigating CVA risk through collateralization and hedging are discussed critically in Subchapter 3.5.2 and Subchapter 3.5.3, respectively. Subchapter 3.5.4 gives a concluding statement with regards to managing and mitigating CVA risk.

3.2. PRELIMINARIES, DEFINITIONS AND LITERATURE OVERVIEW

Counterparty Credit Risk

Given two debt securities (bonds) from two issuers that have differing credit quality – and all other things held equal – it is apparent that an investor will pay less for the bond of the riskier issuer. This difference is associated with a (higher) risk premium the investor demands. After all, there is a higher probability that the investor will face (credit) losses. Consentingly the credit insurance – and thus the hedge strategy – will also be more expensive. In an analogous manner it is obvious that an investor will pay less (or offer worse conditions) for a derivative that is traded over the counter (OTC) if the counterparty has a higher risk of defaulting. Notice that counterparty credit risk refers only to OTC derivatives, and does not apply to exchange-traded derivatives.

Despite the obvious similarities, counterparty credit risk differs from credit risk mainly in that the value that can potentially be lost is stochastic due to underlying market risk factors driving the derivative value, and in it being bilateral. This means that both counterparties are default prone and that the value of many derivatives (e.g. interest rate swaps) can be either positive or negative. In the following we will describe the motivation behind it.

We will assume an investor and a counterparty trading an OTC derivative. The following Table 8 summarizes the effects due to a default of either the investor or the counterparty. A default of the counterparty will affect the investor only if the net present value (NPV) from the investor's perspective is positive. If the NPV from the investor's perspective is negative and the counterparty defaults there will be no effect as the former will still be able to meet the obligations to the latter. Analogously if the investor defaults before the counterparty while the NPV from the latter's perspective is positive, the latter will incur a loss. The bilateral aspect comes into play by interpreting the losses the counterparty incurs as gains of the investor and vice versa.

	NPV of the Derivative at Default	
	Positive (Asset)	**Negative (Liability)**
Counterparty defaults before the investor	Loss for the investor	-
Investor defaults before the counterparty	-	Loss for the counterparty

Table 8: Default-Dependent Losses in OTC Derivatives Transactions

The matrix summarizes the effects due to the default of either the investor or the counterparty, depending on the net present value (NPV) of the OTC derivative being either positive or negative from the Investor's perspective.

The losses incurred by either the investor or the counterparty should not be interpreted as mere book value impairments. The direct impact such mechanisms can display is better illustrated if such losses are interpreted in terms of replacement costs for possible hedge positions.

For illustration purposes we assume a bank dealing two simplified interest rate swaps (IRS) with two counterparties as seen in Figure 10. Both positions are assumed, again for simplification reasons, to be uncollateralized. The bank pays a fixed rate to counterparty A while receiving a floating rate (three-month Euribor). This position (payer swap) is (perfectly) hedged through an opposite position (receiver swap) with counterparty B, in which the bank pays the same floating rate and receives the same fixed one.

Let us assume that at a given time period the receiver swap has a positive value. If counterparty B defaults in that time period the bank will lose its hedge. In order to obtain a new hedge for its position with counterparty A, the bank will need to enter into an identical receiver swap with a new counterparty C that will in return demand a fair compensation.

Assuming sterile conditions, counterparty C will demand at least the value of the (default-free) receiver swap.[31] The loss incurred by the bank is absorbed to a certain extent by the recovery rate the bank is – under normal conditions – expected to receive from the bankruptcy assets of counterparty B.

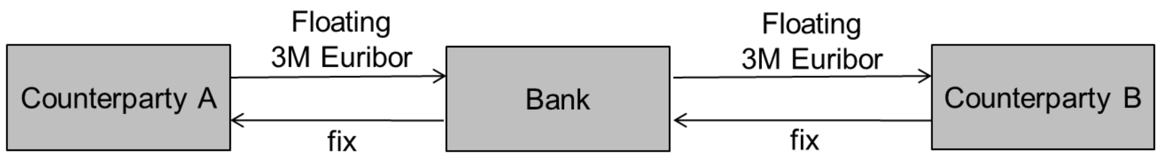


Figure 10: Perfect Hedge Position to Illustrate the Effects of Replacement Risk

The following figure illustrates a simplified perfect hedge. While dealing an payer interest rate swap with counterparty A the bank closes an opposite receiver swap with counterparty B.

Pricing Counterparty Credit Risk – The Unilateral Case

In order for the investor to price the counterparty credit risk arising from the OTC derivative with the counterparty, she needs to look at the scenarios in which she observes (relevant) defaults of the counterparty, to estimate the expected exposure, and to allocate a (risk-neutral) conditional probability to these defaults with a respective recovery rate.

Thus, for pricing of counterparty credit risk the current value of the derivative is not sufficient. The investor needs an estimate of what can be lost in the future (given a default of the counterparty). Expected exposure estimation has thus to take the following into consideration; possible evolution of relevant risk factors (e.g. interest rate curves in case of

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[31] Notice that this remains a simplified illustration as in reality the bank will need some time to find a new counterparty while the value of the swap is stochastic, meaning that the losses of the bank can even be greater. In a real market environment counterparties will price further aspects as funding and of course counterparty credit risk if the derivative is not collateralized. We will return to this topic when we discuss the estimation of expected exposure, especially the assumptions around the close-out amount in Subchapter 3.3.2.

IRS), the specific terms of the deal (e.g. amortization structure of the IRS) as well as possible interconnection between the underlying market risk factors and the probability of the counterparty defaulting.

CVA is usually referred to as unilateral when the own credit risk is being ignored, i.e. when the counterparty doing the computation is assumed to be default-free. For the investor to compute a unilateral CVA (UCVA) means to observe only the scenarios in which the counterparty might default while the NPV is positive from her perspective.

Sorensen and Bollier (1994) were among the first to explore pricing UCVA as also discussed by Cherubini (2005), Brigo and Mercurio (2006) as well as Arvanitis and Gregory (2001). These works focused mainly on counterparty credit risk for one IRS, while netting aspects were subsequently introduced for example by Brigo and Masetti (2005). The main results being that the expected exposure can be approximated using swaptions with different maturities.

Taking possible interdependencies between credit risk and interest rates, Brigo and Pallavicini (2008) modelled UCVA using Monte Carlo simulation methods. A main product of this work is the consideration of wrong way risk (WWR), i.e. the possibility to be negatively affected from counterparty credit risk and market risk simultaneously.

In order to better understand wrong way risk we will assume interest rates (e.g. three-month Euribor) and credit spreads of the counterparty to be positively correlated. If the investor has a long interest rates position with the counterparty (e.g. a payer swap), she will be "hit twice" if credit spreads and exposure rise simultaneously. Put in other words, the risk of the counterparty defaulting is highest when the investor needs her most. A further example, commonly used in illustrating wrong way risk is for the investor to buy a put option (over the counter) on the stock of the counterparty *from* the counterparty. The positive interdependence between the put price (i.e. exposure) and the default risk of the counterparty are most evident. Consistently one speaks of right way risk (RWR) if exposure and credit spreads are conversely interdependent.

Brigo and Bakkar (2009) discussed pricing counterparty credit unilaterally for energy-commodities. Brigo and Chourdakis (2009) explored pricing it for energy-commodities, and Brigo et al. (2011b) delivered a framework for equity products.

DVA and Bilateral CVA

As already elaborated in the previous chapter, assuming a counterparty to be default-free – in our case that being the investor – is definitely nontrivial. In order for the (counterparty credit-) risk adjusted price to be symmetric, both counterparties have to consider their own credit risk while computing CVA. Looking at Table 8, we see that if the investor defaults while owing the counterparty, the former will reap a profit. This might seem counterintuitive in terms of making a profit through one's own default. But from a shareholders' perspective not having to repay an obligation of one's own firm (in the course of filed bankruptcy) is indeed beneficial. Still, it remains in dispute whether an institution should price a profit it will be making only by defaulting, especially because that implies a gain from possible own-credit deterioration.

Accounting standard setters seem to advocate the consideration of own credit risk as required by IFRS 13 in combination with IFRS 9 as well as ASC 820 (previous FAS 157). From a regulatory (prudent) perspective (under Basel II and Basel III), financial institutions are however required to neglect own credit risk for regulatory capital computation purposes.[32]

The metric for taking one's own counterparty credit risk is usually referred to as Debt Valuation Adjustment (DVA). Laterally reverse to UCVA we thus get a unilateral DVA

[32] We refer the interest reader to Subchapter 3.4.2 and Subchapter 3.4.3 of this work for more elaborate discussion around DVA and relative accounting and regulatory requirements.

(UDVA). A DVA from the investor's perspective is hence UCVA from the counterparty's perspective. For example in their UCVA model (also incorporating wrong way risk and right way risk) Hull and White (2011) discuss the analogous adaptation for DVA purses.

Adjusting the UCVA by a DVA (both from one counterparty's perspective) gathers up to a so-called bilateral CVA (BCVA), defined as follows[33]

$$\mathrm{BCVA} = \mathrm{UCVA} - \mathrm{UDVA}. \tag{3.1}$$

Notice that BCVA can display either a positive or a negative adjustment, depending on exposure expectations and on whose credit riskiness is higher, the investor or the counterparty.

First-to-Default CVA

The main fallback of the bilateral CVA is that possible interdependencies between the credit risk of the investor and the counterparty is not accounted for. Already implicitly explored by Duffie and Huang (1996) an explicit modeling of what we will refer to as *first-to-default CVA* was introduced by Brigo and Capponi (2010).[34] Pricing CVA for credit default swaps (CDS), Brigo and Capponi (2010) took possible correlations between the credit risk of the involved counterparties into consideration. This approach has subsequently been extended to consider netting and collateral by Brigo et al. (2013b).[35] First-to-default CVA has also been underlined by Gregory (2009) or Brigo and Morini (2011).[36]

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[33] See for instance Gregory (2009) or Albanese et al. (2013).

[34] In his PhD thesis Capponi (2009) elaborates on some of the main results that were later published in Brigo and Capponi (2010).

[35] Chapter 4 of this thesis is dedicated to modeling the framework of Brigo and Capponi (2010), offering a step-by-step modeling approach and a computational tune-up.

[36] What we refer to as first-to-default CVA is referred to Brigo and Capponi (2010) or Gregory (2009) as bilateral CVA. The term first-to-default CVA stems from Albanese et al. (2013).

Analogously to BCVA the first-to-default CVA – also abbreviated as FTDCVA – can either be positive or negative. Besides exposure expectations and the credit riskiness the latter depends on the interdependence (i.e. correlation or contingency) between the probabilities of default of both counterparties.

As mentioned by Albanese et al. (2013) first-to-default CVA has some theoretical pitfalls. The metric decreases if the credit of the computing counterparty deteriorates, almost vanishing at default. Compared to unilateral CVA a more significant shortcoming of the first-to-default CVA is its unhedgebility. After all, hedging a first-to-default CVA involves hedging DVA, an almost impossible task, dependent on selling one's own credit risk.[37]

The main fallback is however a practical one. Modeling a unilateral CVA is by itself a sophisticated task. Adding possible interdependencies between the credit risks of both counterparties that have to be modelled (and calibrated) will make things even more complicated (if not unfeasible).

As mentioned, from a regulatory perspective financial institutions are required to base their equity computations on a UCVA while accounting standards specifically require the consideration of DVA. The bilateral CVA framework seems in this respect to offer a practical compromise in which DVA can be "switched off and on", depending on the respective computation purpose.

[37] For a more elaborate discussion around pricing DVA see Subchapter 3.4.2.

3.3. COMPUTING CREDIT VALUATION ADJUSTMENT

3.3.1. GENERAL PRICING FRAMEWORK FOR CVA

Before moving to the more practical (unilateral) CVA and DVA we will in the following start with the more general pricing framework based on the so-called first-to-default CVA as for example introduced in Brigo and Capponi (2010).[38]

For this purpose we will return to the two defined parties, dealing an OTC derivative; the investor (name "0"), and a counterparty (name "2").[39] The computations are always done from the investor's perspective. We define $\mathbb{E}_t^{\mathbb{Q}}\{\pi(t,T)\}$ as the counterparty credit risk-free value at time t of an OTC derivative maturing at time T (under the risk-neutral measure $\mathbb{Q}$). The aim is to define a separate metric that adjusts this term for (bilateral) counterparty credit risk.

We are interested in observing the scenarios in which the default of the counterparty implies a loss to the investor and vice versa. Each party will bear a loss due to the default of the other side only if the value of the derivative is positive from its perspective, i.e. if the derivative is in the money (ITM). Put in other words, the default of a counterparty is relevant if the derivative is a liability held by the counterparty.

Let τ_0 and τ_2 stand for the default time periods of the investor and the counterparty, respectively. For the sake of academic completeness we will define the probability space and the different relevant variables as used by Brigo and Capponi (2009). The computations are assumed to be conducted in the probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$. Again, $\mathbb{Q}$ is the risk-neutral measure, and $\mathcal{G}_t$ is a filtration driving the whole market. $\mathcal{F}_t$ is a further subfilteration

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[38] As mentioned, Grominski et al. (2012) have already published main results presented in this thesis.

[39] Notice that name 1 is reserved for a possible reference entity, e.g. relevant in a credit default swap (CDS).

standing for all observable market quantities exept for default events, hence $\mathcal{F}_t \subseteq \mathcal{G}_t := \mathcal{F}_t \vee \mathcal{H}_t$.[40] $\mathcal{H}_t$ stands for the subfilteration standing only for all default events. The stopping time is defined by the time period of the first default, i.e.

$$\tau = \{\tau_0, \tau_2\}. \tag{3.2}$$

Given a default event, i.e. τ is a stopping time of $\mathcal{F}_t$, then the stopped filtration is given by

$$\mathcal{F}_\tau = \sigma(\mathcal{F}_t \cup \{t < \tau\}, t \geq 0). \tag{3.3}$$

If τ is a stopping time of $\mathcal{G}_t$, then the stopped filteration is given by

$$\mathcal{G}_\tau = \sigma(\mathcal{G}_t \cup \{t < \tau\}, t \geq 0). \tag{3.4}$$

Following Brigo and Capponi (2009) we define the following mutually exclusive and collectively exhaustive scenarios

$$\begin{aligned} \mathrm{A} &= \{\tau_0 \leq \tau_2 \leq \mathrm{T}\} & \mathrm{B} &= \{\tau_0 \leq \mathrm{T} \leq \tau_2\} \\ \mathrm{C} &= \{\tau_2 \leq \tau_0 \leq \mathrm{T}\} & \mathrm{D} &= \{\tau_2 \leq T \leq \tau_0\} \\ \mathrm{E} &= \{\mathrm{T} \leq \tau_0 \leq \tau_2\} & \mathrm{F} &= \{\mathrm{T} \leq \tau_2 \leq \tau_0\}. \end{aligned} \tag{3.5}$$

A stands for the scenarios in which the investor defaults before the counterparty which defaults before the maturity of the derivative. C stands for similar scenarios in which the counterparty defaults first. D (B) stands for the scenarios in which the counterparty (investor) defaults before maturity, while the investor (counterparty) outlives the derivative. In E and F both counterparties outlive the derivative. Notice that a simultaneous default is excluded, i.e. it is assumed that the investor and the counterparty will not default at the same time, technically formulated as

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[40] Brigo and Capponi (2009) define $\mathcal{H}_t = \sigma(\{\tau_0 \leq u\} \vee \{\tau_1 \leq u\} \vee \{\tau_2 \leq u\}: u \leq t)$, thus capturing the default events of the counterparties and the reference portfolio. For further elaboration on their specific models see also Chapter 4 of this thesis.

$$\mathbb{Q}(\tau_0 = \tau_2) = 0. \tag{3.6}$$

Brigo et al. (2013a, p. 281) argue that this assumption is verified by most models, referring the reader to one prominent exception, the multivariate exponential distribution used by Marshall and Olkin (1967). For the introduction of simultaneous defaults in CVA pricing see also Gregory (2009).

We define LGD_0 and LGD_2 as the loss given default ratios for the investor and the counterparty, respectively. Notice that $LGD = 1 - RR$ where RR stands for the recovery rate. Let $NPV(t)$ be the net present value (NPV) of the derivative at time period t and $C(t)$ the respective collateral amount received, we define the net exposure E at time t as follows

$$E(t) = max\{NPV(t) - C(t), 0\}. \tag{3.7}$$

Analogously we define negative exposure NE as follows

$$NE(t) = min\{NPV(t) - C(t), 0\}. \tag{3.8}$$

Let $P(t,T)$ stand for the discounting factor for time period T at time t. Brigo and Capponi (2009) deliver a proof that the counterparty credit risk-adjusted value of the derivative can be defined as follows

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}}\{\hat{\pi}(t,T)\} &= \mathbb{E}_t^{\mathbb{Q}}\{\pi(t,T)\} && \text{(S.1)} \quad (3.9) \\ &- \mathbb{E}_t^{\mathbb{Q}}\{LGD_2 1_{C\cup D} P(t,\tau_2) E(\tau_2)\} && \text{(S.2)} \\ &+ \mathbb{E}_t^{\mathbb{Q}}\{LGD_0 1_{A\cup B} P(t,\tau_0) NE(\tau_0)\}. && \text{(S.3)} \end{aligned}$$

Bearing in mind that all quantities are given from the investor's perspective Equation (3.9) clarifies that the counterparty credit risk-adjusted derivative value is a function of three complementing terms.

- As mentioned above (S.1) stands for the risk-neutral contract value when no relevant defaults occur. In that case the risk-neutral value is driven by the discounted contractual cash flows.
- (S.2) stands for the correction term, taking relevant defaults of the counterparty into consideration, i.e. standing for the CVA term. The term can also be explained as the risk-neutral expectation for losses, put in terms of discounted and LGD-adjusted positive exposure if the counterparty defaults.
- Analogously (S.3) stands for the correction term, taking relevant defaults of the investor into consideration, i.e. standing for the DVA term. Notice that while S.2 incorporates scenarios in which the exposure is positive from the investor's perspective, S.3 takes only scenarios into consideration in which the exposure is positive from the counterparty's perspective.

The first-to-default CVA can be written as follows[41]

$$\begin{aligned} FTDCVA \;=\; & \mathbb{E}_t^{\mathbb{Q}}\{LGD_2 1_{C\cup D} P(t,\tau_2) E(\tau_2)\} \\ & -\mathbb{E}_t^{\mathbb{Q}}\{LGD_0 1_{A\cup B} P(t,\tau_0) NE(\tau_0)\}. \end{aligned} \tag{3.10}$$

As stated by Brigo and Capponi (2010) it becomes apparent that the valuation of CVA involves the valuation of a short position in a call option (the DVA term) and of a long position in a put option (the CVA term). Both options refer to the remaining exposure and have a strike of zero.

Equation (3.10) is the value of a hedge portfolio, offering a perfect protection against counterparty credit risk arising from both the counterparty and the investor. Notice that for the investor such a strategy would imply not only buying a protection on the exposure but

[41] Notice that Brigo and Capponi (2009) refer to the term as the general bilateral credit valuation adjustment. We on the other hand stay with the term first-to-default CVA, standing in line the elaborations given in Subchapter 3.2.

also selling protection on her own credit risk, quite a complicated task if not an impossible one. We will analyze the implication of DVA in general and its hedging in particular in Subchapter 3.4.2.

Using the same notation for FTDCVA we will in the following define the unilateral CVA. For this purpose we assume the investor to be credit risk free. We are thus only interested in the scenarios in which $\tau_2 \leq T$. The unilateral CVA can thus be written as follows:

$$UCVA = \mathbb{E}_t^{\mathbb{Q}}\{LGD_2 1_{\tau_2 \leq T} P(t, \tau_2) E(\tau_2)\}. \tag{3.11}$$

The price process of UCVA can also be given through the following risk the risk-neutral expectation of the credit losses given in Equation (3.11).[42]

$$UCVA(t) = \mathrm{LGD}_2 \int_t^T \mathbb{E}_u^{\mathbb{Q}}\{P(t,u))E(u)|\tau_2 = u\,\}d\mathbb{Q}(\tau_2 \leq u). \tag{3.12}$$

where $\mathbb{Q}(\tau_2 \leq t)$ stands for the risk-neutral default probability of the counterparty. As for instance noted by Cesari et al. (2009, p. 217) Equation (3.12) represents the price process of a C-CDS.[43] The value of such a C-CDS is the function of the risk-neutral probability of default and the respective recovery as well as the exposure.

3.3.2. PRACTICAL PRICING FRAMEWORK FOR CVA

Looking at Equation (3.12) it becomes apparent that the exposure is conditional on the default event and the survival of the counterparty, i.e. exposure and credit risk are interdependent. As stated above interdependence between exposure and credit risk is

[42] Pykhtin and Zhu (2007) were among the first to use similar notation (in their case of unilateral CVA).

[43] They note that the value of the C-CDS at time τ_2 equals the exposure $E(\tau_2)$. One could however argue that the value has to take the recovery into consideration, i.e. equaling $\mathrm{LGD}_2 E(\tau_2)$.

relevant, because of possible right-way/wrong-way risk. Modeling interdependence between credit and exposure as well as between the credit risks of the counterparties involves however elaborate Monte Carlo simulation schemes.[44] It also presumes the possibility to estimate the (risk-neutral) parameters of such interdependency using market information. It is worth stating that assuming no right-way/wrong-way risk at least for interest rates and foreign exchange derivatives seems to be an acceptable assumption as noted for example by Pykhtin and Zhu (2007).[45] One can of course think of a variety of examples where right-way/wrong-way risk would play a significant role also for interest rate and foreign exchange derivatives, especially ones referring to exotic currencies.[46] Still, the majority of derivatives traders today either lack the capabilities to do so or they find the implied complexity to be unjustified. The financial institutions that do model right-way/wrong-way risk seem to be doing it in form of an additional analysis at best.[47]

In order to examine the widespread methodology for pricing CVA we will therefore introduce the assumption of no right-way/wrong-way risk as for instance done by Pykhtin and Zhu (2007) or Gregory (2012). Doing so allows us to define UCVA as follows:

$$UCVA(t) = \mathrm{LGD}_2 \int_t^T \mathbb{E}_u^{\mathbb{Q}}\{P(t,u)E(u)\} d\mathbb{Q}(\tau_2 \leq u). \tag{3.13}$$

Still, solving Equation (3.13) analytically is limited to a rather small number of simple examples such as stand-alone European options. Computing exposure on the counterparty

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[44] See the example given in Chapter 4. See also Cesari et al. (2009), p. 224.

[45] See also Gregory (2012), pp. 242-263.

[46] Assume for example an American bank dealing a cross currency swap with the German state agency (Finanzagentur) in which the former borrows an amount in USD in exchange for an amount in EUR. The bank will then pay 3M Libor and receive three-month Euribor. At maturity the counterparties will exchange the borrowed notionals. The bank might assume that deterioration of Germany's credit quality will go along with a depreciation of the Euro. In our example this would mean that the riskier Germany is, the less it will have to pay, i.e. the exposure exhibits a right way risk as it decreases with increasing credit risk.

[47] See for example the survey done by Deloitte and Solum (2013).

level, taking netting and collateral into consideration, will always require simulation approaches. Assuming the simulation is done in the time steps $\{t_k\}_{k=1}^{N}$ UCVA can be defined using the following time-descrete manner

$$UCVA(t) = \mathrm{LGD}_2 \sum_{i=1}^{N} \mathbb{E}_u^{\mathbb{Q}}\{P(t,t_k)E(t_k)\}\, \mathbb{Q}(t_{k-1} < \tau_2 \leq t_k). \tag{3.14}$$

Analogously DVA could be computed in the following manner

$$DVA(t) = \mathrm{LGD}_0 \sum_{i=1}^{N} \mathbb{E}_t^{\mathbb{Q}}\{P(t,t_k)NE(t_k)\}\, \mathbb{Q}(t_{k-1} < \tau_0 \leq t_k). \tag{3.15}$$

In the same manner we are also able to define the (contingent) bilateral CVA, i.e.

$$\begin{aligned} BCVA(t) \quad &= \mathrm{LGD}_2 \sum_{i=1}^{N} \mathbb{E}_u^{\mathbb{Q}}\{P(t,t_k)E(t_k)\}\, \mathbb{Q}(t_{k-1} < \tau_0, t_{k-1} < \tau_2 \leq t_k) \\ &- \mathrm{LGD}_0 \sum_{i=1}^{N} \mathbb{E}_t^{\mathbb{Q}}\{P(t,t_k)NE(t_k)\}\, \mathbb{Q}(t_{k-1} < \tau_2, t_{k-1} < \tau_0 \leq t_k). \end{aligned} \tag{3.16}$$

Notice that the CVA term of Equation (3.16) takes the survival probability of the investor into consideration while the DVA term depends on the survival probability of the counterparty. After all, the investor will suffer a loss from the default of the counterparty if she will not default until then, and vice versa.

The discrete manner of computing CVA with no right-way/wrong-way risk as given in Equation (3.14) offers a variety of very convenient aspects, especially the ability of modeling the necessary factors, i.e. expected exposure and credit, in two separate blocks. In the following we will elaborate on the steps needed to model the expected exposure (Subchapter 3.3.3) and estimating the credit risk (Subchapter 3.3.4), i.e. the probability of default and the respective recovery.

3.3.3. ESTIMATING EXPECTED EXPOSURE

We are interested in estimating the exposure that the investor is expected to have towards a given counterparty. For this purpose we need an approach for modeling the evolution of the value of the portfolio, taking possible netting and collateral agreements into account.

Again, we assume no correlation between the credit risks of the counterparty and the investor on the one hand, and the risk factors driving the exposure on the other. A possible interdependency between the credit risk of the investor and the counterparty is also excluded. This practical approach enables us to examine the modeling of expected exposure in a separate manner.

The proposed framework can be split in the following main steps:[48]

- **Step 1:** Generation of scenarios for price factors
- **Step 2:** Revaluation of instruments
- **Step 3:** Netting set aggregation and collateral adjustment
- **Step 4:** Definition of expected exposure profiles

Notice that a set of future dates $\{t_k\}_{k=1}^{N}$ in which the portfolio needs to be revaluated is pre-required. In practice this decision is a payoff between computational power on the one hand and precision on the other. While daily valuation will capture margin calls and possible jumps in the value of the portfolio it might exhibit computational and technical challenges. A possible solution is given by choosing a set of future dates with decreasing granularity, e.g. starting with one month of daily revaluation, then moving to 11 months of monthly revaluation, followed by quarterly revaluations etc. More sophisticated approaches define the future dates based on the portfolio at hand, e.g. revaluation dates depend on the cash flows

[48] Comparable computation steps were given for example by Canabarro and Duffie (2003) or Pykhtin and Zhu (2007).

of the derivatives and the margin calls of the respective collateral agreement. This way exposure changes are captured while managing possible computational and technical challenges. In Figure 11 for example 16 generic simulation steps illustrate the revaluation dates.

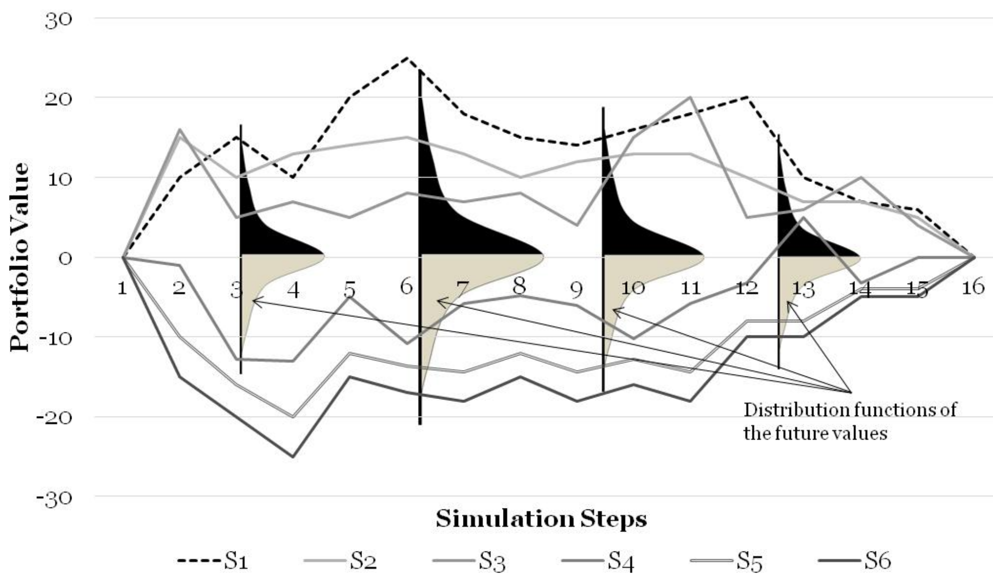

Figure 11: Framework for Exposure Simulation

The figure illustrates the general framework for estimating expected exposure. The lines stand for the estimated values of the portfolio as a function of a simulated price factor path (e.g. the value of an interest rate swap in dependence of the simulated interest rate curve). At every future date the distribution of possible portfolio values is simulated. For UCVA purposes we are only interested in the positive values. On the other hand, the expected exposure for a given time period is defined as the average of the positive values. DVA is a function of the expected negative value. The expected negative exposure is defined as the average of the negative values for a given date.

Step 1: Generation of Scenarios for Price Factors

The first block is the generation of possible scenarios for the factors driving the value of the portfolio at the given set of future dates $\{t_k\}_{k=1}^{N}$. Interest rate derivatives will for example be driven by reference rates such as Eonia, three-month Euribor or 12-month Libor. Other examples of price factors are foreign exchange rates, stock prices, commodity prices or inflation indexes.

Pykhtin and Zhu (2007) distinguish between two general methods for generating price factor scenarios. They refer to the first method as Price-Dependent Simulation (PDS) and to the second as Direct Jump to Simulation Date (DJS). Within the PDS method the price factors simulated for a future date t_i would depend on the simulated results in the previous future date t_{i-1}. In the DJS method they are only dependent on the results given for the simulation date, i.e. t_0. Although both methods should bring identical price factor distributions for a given future date the valuation of path-dependent products such as American or Bermudian options might however imply path-dependent simulation of price factors, implying that for CVA purposes PDS approaches are more adequate.

In line with derivative pricing CVA computation is based on risk-neutral valuation. This means for example that for the generation of exposure of interest rate derivatives (e.g. interest rate swap) arbitrage-free interest rate models are needed.[49] A prominent example used for this purposes is the Hull White interest rate model. We will apply the one factor version of this model in an example below. This model describes the dynamics of the short term interest rate using the following stochastic differential equation (SDE):

$$dr_t = (\theta(t) - ar_t)dt + \sigma dW_t \tag{3.17}$$

where r_t stands for the short rate at time period t. $\kappa, \theta, a, \sigma, u$ are the model parameters, W_t is a standard Brownian motion. This means that for each future date a set of short-term interest rates (e.g. 10.000) needs to be simulated in order to generate the same number of

[49] For an overview on risk-neutral models for interest rate risk see Subchapter 2.3.1.

respective interest rate curves. In Figure 11 for example six different sets of path-dependent risk factors are given at each future date.

A further complication is the simulation of correlated market risk factors, e.g. generation of interdependent interest rates and foreign exchange rates. While interdependency can be modelled through simplified approaches, the market lacks sufficient instruments to allow for its risk-neutral calibration. Interdependence can for example be modelled through introducing respective correlation coefficients driving the diffusion process. Due to the lack of instruments to calibrate the needed correlation matrix some practitioners refer to historical values for approximations.[50]

Step 2: Revaluation of Instruments

Given the sets of generated risk factors for each future date the instruments are subsequently revaluated, producing respective distribution of derivative values. For interest rate swaps for example a pricing model is needed that re-estimates the remaining cash flows and the respective value of the instrument at each time step and scenario. In the example illustrated in Figure 11 the instrument needs to be revaluated in each one of the 16 time steps and for each one of the 6 scenarios, i.e. $6 \times 16 = 96$. In a more realistic example of a portfolio consisting of 50 positions then need to be revaluated 50 times, given 10.000 scenarios it would imply 25.000.000 revaluations. This makes it clear that in large portfolios of financial institutions with numerous counterparties and instruments this number can easily explode, and needs to be managed carefully.

Having adequate pricing models for each instrument in the portfolio can be a very challenging task, especially if the portfolio contains exotic derivatives that require Monte Carlo pricing techniques (e.g. Bermudian swaptions etc.). After all, this requires a draw of realizations for the relevant risk factors (outer step) and further (inner steps) to re-valuate the

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[50] See for example the survey on counterparty credit risk conducted by Deloitte & Solum (2013), esp. p. 25.

instrument, conditional on the drawn risk factor distribution. Gordy and Juneja (2008) or more recent work of Broadie et al. (2011) demonstrate possible solutions for nested Monte Carlo simulations. American Monte Carlo methods as described in Longstaff and Schwartz (2001) or Glasserman and Yu (2002) provide a further possible solution if closed-form pricing solutions are not available. These methods exploit future simulations in approximating the (expected) exposure at a given revaluation time period, avoiding the inner step simulation.

In practice exotic derivatives tend to be part of a bigger portfolio that is mainly driven by plain vanilla instruments (e.g. interest rate swaps). In such cases approximations tend to suffice, e.g. Bermudian swaptions can be proxied through European swaptions as proposed by Gregory (2012, p. 164) or even through more crude add-on-based estimates.

In addition, the value of an instrument might depend on past events (i.e. on the simulation path prior to the revaluation date) as for barrier options or callable swaps for example. This underlines the importance of using path-dependent simulation approaches as proposed by Pykhtin and Zhu (2007).[51]

Step 3: Netting Set Aggregation and Collateral Adjustment

If the instruments are part of a netting set their values at each time step and scenario need to be summed up, producing an aggregated distribution of the value of the portfolio at each time step. Let $NPV(i, s, t)$ stand for the net present value of instrument i, in scenrio s at time period t. Assuming that the instruments $i = 1, \dots, I$ are all part of the same netting set the aggregated value for a given combination of a scenario and a time period is defined as follows

[51] Pykhtin and Zhu (2007), p. 19 also refer to Lomibau and Zhu (2005) for possible solutions also relevant for Direct Jump to Simulation Date (DJS) approaches.

$$NPV_P(s,t) = \sum_{i=1}^{I} NPV(i,s,t). \tag{3.18}$$

In addition, a possible collateral agreement needs to be considered. Let $C(s,t)$ stand for the value of the collateral posted or received at a given scenario and time period, the exposure of the netting set if defined as follows

$$E(s,t) = max\{NPV_P(s,t) - C(s,t), 0\}. \tag{3.19}$$

This means that if the NPV is negative and the investor posts a collateral amount to the counterparty that exceeds the absolute value of the NPV, the investor will have a positive exposure (due to overcollateralization).

This pre-requires the modelling of the collateral for each time step, or at least the most significant properties of the collateral agreement at hand. Such agreements are usually specified in an annex to the master service agreement used. A Credit Support Annex (CSA) is for example part of the most commonly used ISDA master agreement as published by the International Swaps and Derivatives Association. Main properties of a CSA that are usually modelled are the following: [52]

a. Type of eligible collateral, e.g. cash, sovereign bonds, equity etc.
b. Threshold amount, i.e. the amount which the portfolio value needs to exceed before an exchange of collateral takes place
c. Minimum transfer amount (MTA), i.e. the amount which the collateral to be exchanged needs to exceed before an exchange takes place
d. Margin periods, i.e. the frequency of collateral exchanges (e.g. daily, weekly etc.)

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[52] Further properties are the rounding method of the exchanged amount or the interest rate paid on the collateral posted (e.g. Eonia).

Netting effects are crucial in exposure simulation, especially if an institution trades contrary positions (e.g. payer swaps and receiver swaps) with the same counterparty and within one netting set as becomes visible when looking at Table 9.

Table 9 illustrates the effects of netting and collateral using the data published by the Bank for International Settlements (BIS) and ISDA. According to these estimates netting agreements reduce the worldwide OTC derivative exposure by more than 80%. Considering netting and collateral agreements reduces the exposure by even more than 90-95%. In Subchapter 3.5.2 below we will discuss the effectiveness of collateralization as a technique to mitigate CVA.

In USD trillions	**2007**	**2008**	**2009**	**2010**	**2011**	**2012**
Gross market value (GMV)	15.8	35.3	21.5	21.3	27.3	24.7
Gross credit exposure (after netting)	3.3	5.0	3.5	3.5	3.9	3.6
% of gross market value	20.6%	14.2%	16.3%	16.3%	14.3%	14.6%
Gross credit exposure (after netting and collateral)	1.1	1.7	1.1	1.1	1.1	1.1
% of gross market value	7.2%	4.8%	5.1%	4.9%	4.1%	4.4%

Table 9: Gross Credit Exposure after Netting

The table illustrates the effect of netting through survey data as published by ISDA and BIS.[53] Gross market value stands for the market value of all outstanding contracts before netting. Gross credit exposure considers netting benefits.

Recall that the exposure amount we are estimating is the amount the non-defaulted counterparty will demand from the defaulted counterparty at the time of default, also referred to as the closeout amount. Looking at common literature around CVA pricing it is interesting to see that the close-out amount is assumed to be counterparty credit risk-free.[54] Brigo and Morini (2011) argue that the legal (ISDA) documentation suggests that the non-

[53] Refer to the OTC market analysis, ISDA (2013).

[54] See for example Gregory (2009) or Brigo and Capponi (2009, 2010).

defaulted counterparty should actually price its own-counterparty credit risk, i.e. DVA.[55] This introduces a recursive problem. The expected exposure amount is dependent on future DVA amounts, which in return depend on the expected exposure amount. Iterative approaches to solve the problem are discussed also in Gregory and German (2012). Acknowledging the need for such approaches (to ensure a more realistic illustration) we will in the following retain the assumption of risk-free closeout amount for simplicity reasons, in line with common literature and practice.

Step 4: Definition of Expected Exposure Profiles

For CVA purposes we are interested in the (risk-neutral) expectation of what the investor can lose given a default of the counterparty. In mathematical terms we need to define the following expected exposure (EE) for each re-valuation time period

$$EE(t) = \mathbb{E}_t^{\mathbb{Q}}[E(s,t)] \tag{3.20}$$

where $E(s,t)$ is the net exposure at time t for a given scenario s, given in Equation (3.19). $\mathbb{E}_t^{\mathbb{Q}}$ stands for risk neutral expectation at time t. Looking at Figure 11 this means that we need to define the average of the positive values of the distribution at each one of the 16 time steps. Notice that for DVA purposes the mirroring exposure profile needs to be estimated. The negative expected exposure (NEE) stands for the risk-neutral expectation of what the counterparty can lose if the investor defaults, i.e. the average of the negative values of the distribution.

Figure 12 exhibits a typical expected exposure profile for an interest rate swap and is given for illustration purposes only. The typical profile increase is a function of the drift captured

[55] Brigo and Morini (2010) and Brigo and Morini (2011) do not limit the aspect to be a modeling question, and discuss the unwanted implications on creditors if the closeout amount is credit risk-free.

in current forward rates whereas the decrease is a function of the amortizing value of the swap as less and less cash flows remain to be exchanged.

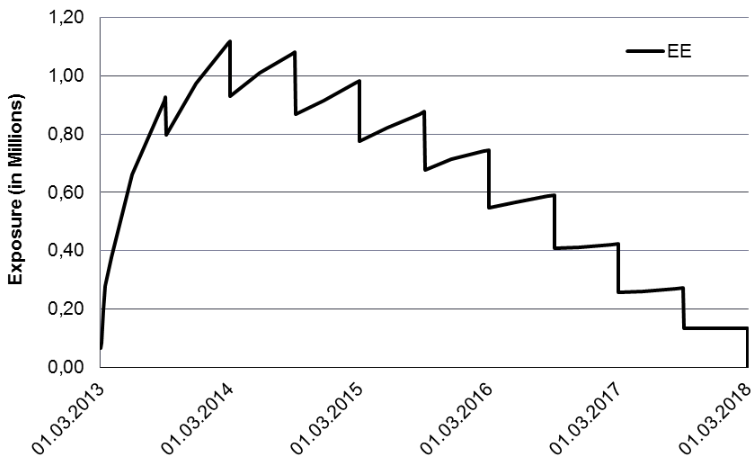

Figure 12: Expected Exposure Profile of an Interest Rate Swap

The straight line in the figure illustrates the expected exposure (EE) profile an exemplary interest rate swap with semiannual payments with the following properties: Reference rate 6-month Euribor; effective date is March 1, 2013, maturing in five years. Notional is 100.000.000 € and swap rate is 0.9%. The exposure profile was estimated using a Hull-White 1 factor interest rate model (own calculation for illustration purposes only).

3.3.4. Estimating Default Probabilities and Recovery Rates

In this subchapter we move to discuss practical approaches for estimating the marginal default probabilities $\mathbb{Q}(t_{k-1} < \tau_0, t_{k-1} < \tau_2 \leq t_k)$ and recovery rates $(1 - \mathtt{LGD}_2)$ needed for CVA valuation as given in Equation (3.16) above.

Recall that CVA has been defined as a financial instrument which is priced under the risk-neutral measure $\mathbb{Q}$. This means that – in contradiction to credit risk models – CVA pricing cannot be based on historical probabilities of default. In order to align with arbitrage-free valuation we need to use probabilities of default as implied by credit-sensitive instruments. There is a variety of such instruments with respective quotes in the market, e.g. single-name credit default swaps (CDS), index CDS, asset swaps (ASW) or bonds. Whereas credit default swaps (CDS) can be considered as the most straightforward ones.[56] Also from a theoretical perspective CDS spreads have the cleanest isolation of credit risk, enabling the estimation of default risk and the respective implied risk premium.

It is worth mentioning that also yields of default-able bonds are a standard source for measuring implied default probability. Notice, however, that besides credit risk, bond spreads may incorporate a significant portion of further aspects such as liquidity. For empirical analysis on bonds spreads and credit risk see for example Longstaff et al. (2005). For the relationship between bond spreads and CDS spreads see for example Hull et al. (2004). Credit linked notes and Asset swaps are further quite popular instruments that are worth being mentioned.[57]

Notice that historical and market implied probabilities differ systematically. The former represent an actual assessment under the real world-measure of an entity defaulting while market implied probabilities reflect current market quotes and associated hedge costs.[58]

Following the notations given by Brigo et al. (2013a, esp. pp. 66-70) we will in the following discuss how CDS can be used to estimate risk-neutral probabilities of survival (and of default) in a model-independent manner.[59] A more elaborate discussion on key CDS characteristics is given in Subchapter 4.2.1 of this thesis.

For this purpose we assume a standard CDS contract with inception time T_a, and maturity time T_b. The protection buyer pays a premium R, e.g. regular payments of a (credit) spread

[56] As for example also noted by Schönbucher (2005), p. 15.

[57] For an overview and description of a variety of credit dependent derivatives see for example Schönbucher (2005), especially pp. 8-50.

[58] For studies on the differences between risk-neutral and real-world probabilities of default see for example Altman (1989), Hull et al. (2005) and Giesecke et al. (2010).

[59] For a proof of the discussed we refer the reader to Brigo at al. (2013a), pp. 66-70.

times a given protection amount. If the reference entity defaults within the lifetime of the contract the protection seller will compensate the buyer with a respective settlement payment (i.e. loss given default (LGD) times the protected amount). Notice that the valuation of such a contract might seem mathematically rather straightforward. After all, we only need to estimate the expected cash flows for each leg. The complication comes through the fact that the cash flows are credit-sensitive per se. The premiums need to be paid only if the entity has survived until that time period while the protection amount needs to be paid only if the entity defaults within the time period. The cash flows need thus to be weighted with probabilities of survival and default, respectively.

The mid-market premium seen at inception time $R_{a.b}^{MID}(0)$ ensures a fair valuation of the contract, i.e. that the premium leg value equals the protection leg value. This means that the CDS contract has a value of zero, i.e. $CDS_{a,b}(0, R_{a.b}^{MID}(0), LGD) = 0$.

Given the market premiums $R_{a.b}^{MID}(0)$ for a set of different maturities, e.g. $T_b = 1y,\ 2\text{y}, \dots\ 5\text{y}$ and $T_a\ =\ 0$, an assumption for the LGD underlying the quotes as well as the current (risk-free) discount function $P(0,\cdot)$ and the given survival probabilities $\mathbb{Q}(\tau >\cdot)$ – with · standing for the maturities of the used instruments – can be stripped by solving the following equation:

$$ProtecLeg_{0,b}(LGD, P(0,\cdot), \mathbb{Q}(\tau >\cdot)) = PremiumLeg_{0,b}\big(R; P(0,\cdot), \mathbb{Q}(\tau >\cdot)\big) \tag{3.21}$$

whereas the protection leg and the premium leg are valued as follows:[60]

$$ProtecLeg_{a,b}\big(LGD; P(0,\cdot), \mathbb{Q}(\tau >\cdot)\big) \coloneqq -LGD \int_{T_a}^{T_c} P(0,t) d_t \boxed{\mathbb{Q}(\tau \geq t)}, \tag{3.22}$$

$$PremiumLeg_{0,b}\big(R; P(0,\cdot), \mathbb{Q}(\tau >\cdot)\big) \coloneqq \tag{3.23}$$

[60] As also noted by Brigo et al. (2013a), p. 68, given a discretization time step that is small enough the given integrals can be approximated numerically through by summations of Riemann-Stieltjes sums.

$$R\left\{-\int_{T_a}^{T_c} P(0,t)\left(t-T_{\beta(t)-1}\right)d_t \boxed{\mathbb{Q}(\tau \geq t)} + \sum_{i=a+1}^{b} P(0,T_i)\alpha_i \boxed{\mathbb{Q}(\tau \geq T_i)}\right\}.$$

Recall that we have not dropped the assumption of independency between exposure and credit, and in this case explicitly between interest rates and default.

To strip the probabilities from the above equations a step-wise approach is needed. We start with $T_b = 1y$ in order to find the market implied survival probabilities $\{\mathbb{Q}(\tau \geq t), t \leq 1y\}$. The results are then used as input for $T_b = 2y$, then moving to $T_b = 3y$ and so on and so forth, finally estimating the implied survival probabilities $\{\mathbb{Q}(\tau \geq t), t \in (1y, 2y, 3y, 4y, 5y)\}$.

	Example A			**Example B**		
Maturity (in years)	CDS spreads (in basis points)	Survival probability	Marginal default probability	CDS spreads (in basis points)	Survival probability	Marginal default probability
1	5	0.9992%	0.08%	126	0.9793%	2.07%
2	7	0.9975%	0.25%	139	0.9549%	4.51%
3	10	0.9949%	0.51%	151	0.9273%	7.27%
4	12	0.9917%	0.83%	162	0.8969%	10.31%
5	13	0.9887%	1.11%	173	0.8640%	13.60%

Table 10: Stripping Default Probabilities from Quoted CDS Spreads

The table above illustrates the implied survival probabilities and the marginal default probabilities using the CDS spreads quoted in the market. The CDS spreads used represent the average spreads quoted for the following financial institutions: Deutsche Bank, Santander Bank, Barclays, UniCredit, and Citi (with a recovery rate of 40%). Example A and Example B stand for the averages based on the 7th of July 2006, and the 5th of October 2010, respectively.

Table 10 summarizes the stripped survival probabilities of exemplary quoted CDS spreads following the above given method. The marginal default probabilities needed for CVA valuation are subsequently defined through the difference between the respective survival probabilities. The examples given in Table 10 are based on real market data, illustrating the increase in market implied default probabilities in particular, and in the significance and level of CVA in general following the financial crisis.

After having discussed practical approaches to estimate expected exposure and default probabilities we turn to recovery rates (or loss given default, LGD) which are needed twofold to price CVA: (1) indirectly in order to strip probabilities of default; and (b) directly as seen in Equation (3.16). The recovery rate stands for the percentage of par value that investors will receive after a default event of the reference entity.

Evidence suggests that collateral and thus recovery rates are stochastic, and jointly distributed with probabilities of default, especially in times of distress. This means that adequate credit risk modelling should rely on stochastic models for recovery rates as for example discussed in Altman (2006), Bilal and Singh (2012) as well as Li (2009) for the specific case of CVA pricing. Market convention, however, seems to rely on a more practical approach, assuming a constant recovery rate. The CDS data quotes given for example by the data provider Markit assume a constant recovery rate of 40%, i.e. LGD of 60%.[61] The use of a constant recovery rate is not limited to practice and is also wide-spread in academic and research work as seen in Brigo and Chourdakis (2009) or Lipton and Sepp (2009).

In any case, it indeed seems consistent to use the same assumption for recovery rate twice, i.e. for stripping default probabilities and directly for CVA valuation. This would in fact also limit the effect of LGD on CVA due to a cancellation effect, as referred to by Gregory (2012, p. 252). On the one hand, the higher (lower) the LGD assumed the lower (higher) the default probabilities stripped and the CVA computed. On the other hand, CVA is a direct function of LGD, i.e. a higher LGD increases CVA and vice versa.

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[61] See for example Brigo et al. (2013a), pp. 185-186 or Bilal and Singh (2012).

3.4. ACCOUNTING AND REGULATORY BACKGROUND OF COUNTERPARTY CREDIT RISK

3.4.1. REMARKS

Counterparty credit risk in general and CVA in particular seem to be highly driven by regulatory and accounting standards.[62] From a regulatory perspective financial institutions are faced with requirements with respect to measuring and capitalizing counterparty credit and (unilateral) CVA volatility. In addition, international accounting standards require the consideration of (bilateral) CVA in fair value measurement, affecting not only financial institutions but all OTC derivative traders such as corporates that use such derivatives for example to hedge their interest rate and foreign exchange positions. In fact such non-financial institutions seem to be concerned with possibly being impaired twice. On the one hand accounting standards require them to compute CVA and DVA, facing them with methodological and technical challenges. In addition, banks will try and rollout their higher requirements through higher margins. Watt (2011) cites the Head of Finance at Lufthansa, one of the major German non-financials, saying the following:

> *"We think the new [CVA capital] charge will have a big impact on our business. Long-term derivatives will be a problem. For some of these trades, we understand the cost will be so high that we will have to accept the risk of sharp movements in whatever we are hedging against, like floating interest rates or oil prices. That means our profit and loss figures will be more volatile"*

As will be displayed, the accounting and regulatory requirements can be seen as quite straightforward from a theoretical perspective. Still, as seen from the above given quote there seems to be a controversial discussion around their diverse practical implications. An

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[62] This view is shared for example by Albanese et al. (2013).

overview on counterparty credit risk cannot leave out an elaboration on these requirements because they seem to influence the discourse on risk management and CVA pricing.

In the following two subchapters we will give an overview of the main accounting and regulatory requirements, focusing on international accounting standards (IFRS and US-GAAP) and the Basel III framework.[63] In Subchapter 3.4.4 we summarize our analysis of the regulatory efforts, commenting on whether the fear of the cited Lufthansa manager is justified.

3.4.2. Accounting Background

CVA as Part of an Exit Price

Derivatives have to be measured and disclosed according to the fair value measurement both under the international financial accounting standards (IFRS) and under US-GAAP as specified in IFRS 13 and ASC 815. Both standards refer to fair value as an "exit price" and require the use of a "fair value hierarchy". In the following we will give a recap for both terms.

According to IFRS 13.9 or ASC 820-10-20 exit price is defined as the price that would be received for an asset or the price that would be paid to transfer a liability. As specified in IFRS 13.72 and ASC 820-10-35 exit prices have to be estimated according to a fair value hierarchy which categorizes pricing parameters along three levels. If possible one must use quoted prices for identical assets and liabilities in active markets, these categorized as *level 1* parameters. For example the liquid stock price should be used to evaluate a respective stock position. If such quotes are not available one must refer to *level 2* parameters, i.e. direct or indirect input parameters and standard pricing methods. The value of interest rate swaps is

[63] For a further reading on the regulatory framework around CVA and counterparty credit risk see for example Schwake et al. (2011) who also give an overview of the specific requirements under the German GAAP (Handelsgesetzbuch, HGB).

for instance not directly quoted. Bootstrapping of interest rate curves (such as three-month Euribor or Eonia) from market information is needed and the valuation is subsequently done using a discounting cashflow method. Lastly *level 3* parameters consist of unobservable parameters that need to be estimated, e.g. for the valuation of an asset backed security (ABS) one needs to estimate a default probability and severity of the respective pool of loans. Such parameters are usually not quoted and have to be estimated using historical statistics, peer group comparisons or expert estimations.

Taking counterparty credit risk into consideration in the fair value of an OTC derivative can in general be motivated twofold. For one, being a significant risk factor counterparty credit risk will be priced in by third parties, making CVA part of an exit price. In addition, both US-GAAP and IFRS have specific requirements to take non-performance risk into consideration as specified for example in ASC 80-10 and IFRS 13.42. Notice that this implies the consideration of own credit risk in derivative pricing as specifically required by IFRS 13.42:

> *"The fair value of a liability reflects the effect of non-performance risk. Non-performance risk includes, but may not be limited to an entity's own credit risk (as defined in IFRS 7: Financial Instruments: Disclosures)."*

It goes without saying that there are circumstances in which CVA or DVA for a given entity or for parts of its OTC derivatives portfolio are concluded as not material. The consideration of CVA or DVA would then not be obligatory.[64] This might for example be the case if collateral agreements with daily posting are in place and the counterparties involved exhibit high credit quality. This could also be the case for corporates with relatively

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[64] See also Schwake et al. (2011), pp. 293-294.

small derivatives portfolio.[65] In any case materiality is entity-specific and has usually to be validated and agreed upon with the auditor.

Estimation of Credit Risk

As mentioned in previous subchapters – especially Subchapter 3.3.3 – one of the drivers of CVA is the credit risk of the counterparty, i.e. the risk of default and changes in the credit quality.[66] According to the "fair value hierarchy" this has to be estimated using as much market data as possible. If such data is not available, approximations and indirect data can be used. Historical and statistical estimation or expert opinions can only be used if all other alternatives are not available.

Following the "fair value hierarchy" one should thus use credit spreads that are directly traceable to the respective counterparty. This means that – if available – single name CDS spreads should be used. This aligns with what we have discussed in Subchapter 3.3.3 above.[67] If such spreads are not available entities should map these to other tradable CDS spreads (e.g. categorized by rating, geography or industry). According to this logic internally estimated spreads should hence only be used if mapping procedures are either impossible or economically meaningless. In order to deliver market implied valuation (i.e. staying in line with duplicating strategy or hedge strategy valuation) but also in order to stay aligned with the "fair value hierarchy" this spread has to contain – besides default risk – an estimate for a market risk premium.

Despite the fact that these requirements are quite straightforward from an academic and theoretical point of view they seem to imply significant challenges for many entities. After

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[65] It is worth mentioning that both IFRS 13 and ASC 820 explicitly allow the computation of CVA on counterparty level (in order to take netting and collateral aspects into consideration).

[66] An overview on credit risk modeling is given in Subchapter 4.2.2.

[67] In its comment on fair value measurement, the Institute of Public Auditors in Germany, names CDS spreads as sole example for market source of credit risk estimation (see IDW RS HFA 47, paragraph 102).

all, for a large part of entities there are no liquid CDS spreads, and the use of mapping procedures seems arbitrary, lacking economic reason.[68] Credit risk of small and medium sized corporations (e.g. a local German corporation with 30 employees in the construction business) can hardly be mapped to index CDS, containing large and international corporations. A further reason why corporations might not want to use market implied information is their apparent higher volatility, a reason that cannot, however, be cleansed from balance sheet manipulation. It can furthermore be argued that if hedging of counterparty credit risk is impossible then market implied valuation is not needed (also referred to as warehouse book valuation). In that case CVA is interpreted as a metric for expected credit loss. Still, risk-neutral valuation is based on duplication strategies, irrespective of whether hedging strategies are truly in place. In addition, from an accounting point of view, market implied valuation can be motivated not only by being market standard but because of its limited manipulability if compared with entity specific information. Moreover, it facilitates a better comparability of financial statements of differing entities.

In a recent comprehensive assessment the European Central Bank (ECB) challenged valuation and provisioning methodologies of European banks. The assessment revealed a gab of € 3.1 billion to CVA calculation., resulting in a 27% increase in the respective metric for the banks that were part of the assessment. Most of this adjustment was driven by an incorrect use of historical – instead of market implied – probabilities of default.[69]

It is worth mentioning that the standardization of risk-neutral CVA valuation, especially in the market of counterparties with no liquid CDS spreads which largely intersects with the non-financial market in which no collateral is daily posted is very difficult, because arbitrage

[68] For empirical findings on the use of internal spreads see for example Schwake et al. (2011) or Deloitte and Solum (2013), p. 26. For an attempt to use internal data with market implied information see for example Knoth and Schulz (2010).

[69] See ECB (2014), pp. 98-100.

opportunities are rather excluded and the counterparties do not have to agree on the value of the portfolio on daily basis.

It can be concluded that marked implied default probabilities seem to have become more popular.[70] This trend cannot however be regarded independently from the specifications of IFRS 13 and subsequent higher focus of audit firms and supervisors on the subject matter.

Accounting for Debt Valuation Adjustment (DVA)

A further topic that has quite controversially been discussed is DVA. The accounting standard requirements are actually rather clear with regard to this topic. Entities *have* to consider their own credit risk, i.e. CVA has to be computed on a bilateral basis. Still, studies show that the consideration of DVA for accounting purposes but also in front-office pricing is not as straightforward as the standard setters might have had in mind.[71] DVA seems to be presumed as unintuitive, implying a variety of unwanted challenges. In the following we will go through the main issues around DVA, extending and commenting the discussion given by Gregory (2009).

First, by taking DVA into account, the evaluating entity assumes a gain from its own default. This is indeed questionable from the entity's perspective because such gains are irrelevant and unrealizable. From a valuation perspective, a bilateral CVA might allocate a higher value to a derivatives portfolio than its "default-free" value (also referred to as marked-to-market value, MtM). On the other hand, for the shareholders not having to redeem obligations (in that case derivative obligations) *does* exhibit a profit when one's company is at default. Taking DVA into account would thus make sense if the shareholders are viewed as the main stakeholders of financial reporting. Even if some counterparties claim not to price a DVA

[70] See for example Schwake et al. (2011), pp. 295-298 or the more recent analysis done by EBA (2015).

[71] For the consideration of DVA in pricing and accounting purposes see the study done by Deloitte and Solum (2013), especially p. 29. Schwake et al. (2011) have collected information from publicly available information on DVA, focusing on its consideration for accounting purposes. For DVA in pricing see 3.5.1.

component the fact that they are pricing a CVA component plainly means that someone else is symmetrically (willingly or not, knowingly or not) pricing DVA. Arora et al. (2012) find evidence that the higher the credit risk of a dealer, the lower is the price he can charge for selling a CDS. This implies that these dealers are (willingly or not, knowingly or not) taking their own credit into consideration.

Gregory (2009) does argue that from an entity's perspective realizing DVA is possible if the entity is near default, and brings in some examples from the insurance business. In such cases the counterparties would readily unwound the portfolios and pay-out the entity a respective compensation. We argue that the realization of DVA is not limited to such extreme cases. Counterparties can for example also agree on raising the collateralization, e.g. through a two way collateral agreement with minimal thresholds or through central clearing. These amendments would trigger a change in the value of the bilateral CVA. The counterparty that benefits more from the amendment (through a relief in its CVA position) would thereby compensate the other party, allowing it to (at least partially) realize DVA.

Second, accounting for DVA implies that the financial results are subject for changes in one's own credit risk, hence making a profit from own credit deterioration. The following newspaper quotes on the financial reports of leading investment banks expose how unintuitive this effect is:

> *"Citigroup benefited from a paper gain of nearly $ 2 billion, reflecting a sharp increase in the perceived riskiness of its debt […]. That contributed about one-third of its pre-tax operating profit […]"*[72]

> *"Debit valuation adjustments […] helped Citigroup Inc. post a $ 3.77 billion profit Monday even as its revenue fell. And J.P. Morgan Chase & Co. included a $1.9 billion pre-tax benefit from debt valuation adjustments in its investment bank when it posted third-quarter*

[72] See Dash (2011).

> *earnings last Thursday. And this morning, Bank of America Corp. reported booking a $ 1.7 billion gain due to the accounting rule."*[73]

On the other hand one can argue that these effects are transparent, and can thus be correctly interpreted by analysts and investors. Furthermore, one could also argue that if the derivatives portfolios are unwound or better collateralized at the time period of the financial reports the entity will indeed receive higher compensations.

We conclude that the unintended effect of "making money" with one's own default is rather a critical issue for *management* accounting, less so for *financial* accounting. This is especially relevant for derivatives traders with relatively large OTC portfolios. In these cases trading departments should not be able to make profits (and hence pay higher bonuses to their employees) due to a higher DVA. This can be reached for example either by not considering DVA in management accounting or by designating DVA to a special unit (e.g. CVA desk) or by taking DVA into account only on a bank-wide level.

Third, from a risk management perspective accounting for DVA contradicts the principle of prudency, especially if the bilateral CVA metric is either zero or negative, implying no or negative counterparty credit risk.[74] The regulator is however aware of these effects. Financial institutions are required therefore to use unilateral CVA for determining regulatory capital. Also the capital charge for CVA risk, i.e. CVA volatility accounts only for the volatility driven by the unilateral CVA.

Fourth, as noted by Albanese et al. (2013) or Gregory (2009), a main problem of bilateral CVA is it being unhedgeable. For one, the unilateral CVA part has to be hedged only as long

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[73] See Burne (2011).

[74] This might be the case if the DVA component is larger than the UCVA component, delivering a negative BCVA. Notice that this is not only a function of credit risk. If both counterparties have similar credit risk but the expected negative exposure is significantly larger than the expected positive exposure the DVA component would be larger, delivering a total negative adjustment.

as the entity doing the computation is solvent. More complex is the fact that DVA hedging implies selling a protection on one's own default, quite an impossible task. Gregory (2009) mentions beta hedging as a possible approach. The company might sell protection on comparable counterparties (e.g. from one's peer-group). However, this strategy will deliver a proxy-hedge for credit spread volatility at best. In a worst case scenario it might also imply dramatic losses if the credit risk of such comparable counterparties deteriorates or if they even default on their obligations while the own entity is still solvent.

There is actually a further – and less discussed alternative – for hedging one's own credit risk. This possibility is however limited to a handful of entities, whose single name CDS is part of a liquid CDS index.[75] If for example a single name CDS referenced to an entity (e.g. Deutsche Bank) is part of a CDS index (e.g. iTraxx Europe Senior Financials) that company would theoretically be able to sell its own credit risk. The company could sell the index and buy protection on all the remaining counterparties in the index. It has however to be stressed that this hedge strategy would still be very (and probably prohibitively) expensive, especially if re-hedge costs are considered.

Fifth, and probably the most controversial and sophisticated aspect both from a theoretical as well as from a practical perspective is the overlap between DVA and expected funding benefits. In the following we will synthesize the key takeaways that are relevant for our analysis of the overall discourse on DVA, but will refrain from giving a comprehensive overview, especially with regards to a possible misalignment between pricing funding and financial theory. For this purpose we refer the interested reader to Morini and Prampolini (2011), Pallavacini et al. (2011), Hull and White (2012a) or Burgard and Kjaer (2013).[76]

[75] The author thanks R. O. for this intuitive idea, raised in a discussion on CVA and DVA.

[76] On the controversy around funding of derivatives while also touching the overlap between DVA and funding see for example Cameron (2014). For a high level overview on FVA see also Fries et al. (2013).

Funding costs (benefits) are associated with *uncollateralized* positive (negative) exposure. While positive exposure needs to be funded, negative exposure is interpreted as a liability, i.e. a funding source. This can be illustrated through the following example. Assume a bank is trading an uncollateralized interest rate swap with a client. The bank hedges the interest rate risk arising from this swap through a mirroring position, traded with a bank. The only difference between the two swaps is that while the client derivative is *not* collateralized the hedge position *is*. Now if the client position is positive (negative) the hedge position will be negative (positive). In the first case the bank will need to post collateral to the hedging counterparty while not receiving any from the client. The bank will thus need to fund an amount equal to the positive exposure. In the second case the bank will receive collateral from the hedge counterparty and will not need to post any to the client. The bank will thus receive a funding amount equal to the negative exposure.[77]

A funding cost adjustment (FCA) considers expected funding costs arising from expected positive exposure, i.e. depending on the same exposure profile like UCVA and the assumed funding spread. On the other hand, funding benefit adjustment (FBA) considers expected funding benefits arising from expected negative exposure, i.e. depending on the same exposure profile as DVA and an assumed funding spread. It becomes clear that an institution that prices FBA and DVA might be double counting, especially if the funding spread and the credit spread (used for DVA valuation) contain similar information. Morini and Prampolini (2011) point out that funding benefit should be based only on the CDS-bond-basis, excluding possible double counting per definition. In practice one also sees traders that price FBA while being reluctant in pricing DVA. In a survey done by Deloitte and Solum (2013, p. 39) we find the following conclusion: "*[…] banks are increasingly seeing*

[77] One can argue that the above given motivation is flowed because the value of a derivative cannot be driven by the funding structure of the trading institution. Recall, however, that arbitrage-free valuation is based on the possibility to structure a duplication portfolio. In practice the hedge of an uncollateralized trade is collateralized. This implies additional funding considerations to achieve a duplication. See also Burgard and Kjaer (2013).

DVA as a funding benefit and not as a benefit in the event they default. We note that whilst a bank may consider CVA + symmetric funding [FCA + FBA] to be relevant, they may refer to the funding benefit as DVA."

We conclude that although DVA has a variety of pitfalls and challenges that have to be taken into account its consideration is not a mere theoretical and accounting aspect. If some market participants are pricing CVA then other market participants must (willingly or not, knowingly or not) be taking their own credit risk into consideration, i.e. pricing (at least to a certain extent) DVA.

3.4.3. REGULATORY BACKGROUND

Remarks

In December 2010 the Basel Committee on Banking Supervision (BCBS) published a document containing its revised standards under the title "Basel III: A Global Regulatory Framework for more Resilient Banks and Banking Systems". Through the new basis for international lawmaking, the BCBS intended to deliver a (quick) response to the financial crisis, taking lessons-learned from recent events into consideration. Basel III offers major amendments to the previous Basel II[78] accord in general and with respect to counterparty credit risk and CVA in particular.

The regulatory requirements for capitalizing counterparty credit risk, in terms of default risk, have been amended on the one hand to be more prudent (e.g. stressed exposure etc.). On the other hand Basel III offers financial institutions an incentive to clear their OTC derivatives through a central counterparty (CCP), assigning significantly lower risk weights for such exposures. This is aligned with further regulatory efforts to shift the OTC market towards central clearing as codified in the US-American Dodd-Frank Act and in the European Regulation EMIR.[79]

Basel III has in particular introduced a new capital charge for CVA risk, i.e. CVA volatility. It has been correctly anticipated that this new charge will increase the capital requirement for counterparty credit risk considerably. Since the publication of the first draft of Basel III in December 2009, CVA capital charge seems to have been dominating the discourse on counterparty credit risk and CVA pricing, raising the attention of the top management not only of financial institutions but also of non-financial corporations, followed by vehement

[78] Basel II is the common reference to the document of the BCBS "International Convergence of Capital Measurement and Capital Standards", published in 2006.

[79] Dodd-Frank Act stands for the Dodd-Frank Wall Street Reform and Consumer Protection Act signed into federal law on July 21, 2010. EMIR stands for European Market Infrastructure Regulatory interred into force on August 16, 2012.

critique and organized lobbyism.[80] At least in Europe, this opposition seems to have been effective. The latest package of the Capital Requirement Directive (CRD IV) which transposed Basel III into EU law, introduces exemptions from the CVA capital charge for non-financial counterparties, a clear advantage to European institutions, when compared to north American banks for instance.[81]

In the following we will display the very main regulatory requirements around counterparty credit risk and CVA risk (the emphasis is on *very main*). The following elaboration is in no means an exhaustive description of all the respective requirements.[82] We will especially not elaborate on country specific amendments (e.g. through an endorsement process of the framework). The aim of the following elaboration is rather to pinpoint the relevant issues in order to demonstrate the interplay between the regulatory requirements, accounting issues and the discourse around counterparty credit risk and CVA, trying in particular to understand whether the fear of the cited Lufthansa manager is justified.

Measuring and Capitalizing (Counterparty) Credit Risk

As noted, banks have already under Basel II been required to capitalize and measure the default risk inherent in their derivatives portfolios, i.e. (counterparty) credit risk. For this purpose, the supervisor prescribes a loan-equivalent technique, requiring the capitalization of the risk-weighted exposure at default, computed on a netting-set level. Banks must thus hold sufficient capital against their risk weighted assets (RWA), defined as follows:

$$\mathrm{RWA} = \mathrm{RW} \cdot \mathrm{EAD}. \tag{3.24}$$

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[80] The first draft for the Basel III accord was published by the BCBS in December 2009 in a document titled "Strengthening the Resilience of the Banking Sector". For the discussion on the CVA capital charge within the first Basel III draft see for example Pengelly (2010). For the CVA capital charge in its last version see for example Rebonato et al. (2010).

[81] CRD IV entered into force on July 17, 2013.

[82] For a more extensive description see for example Gregory (2012), pp. 371-402.

RW stands for the risk-weight of the position, depending in general on the credit risk of the counterparty and the maturity of the portfolio at hand. EAD is the exposure at default to a specific counterparty with a one year horizon, depending in general on the market risk inherent in the derivatives portfolio, taking possible collateralization effects into consideration.

The Basel framework offers three main approaches for assessing the risk-weight of a position; (1) the standardized approach, (2) the internal rating-based approach (IRBA) and (3) the advanced IRBA. Under the first approach banks are offered rating-based grids of risk-weights for each class of counterparties (e.g. sovereigns, financial institutions etc.). IRBA, on the other hand, prescribes the use of a model-based formula to define the risk weight. Following BCBS (2005) the risk weight RW can be defined as

$$\mathrm{RW} = \mathrm{LGD}\cdot\left[N\left(\frac{N^{-1}(PD)-\sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}}\right)-PD\right]\cdot MF(M,PD)\cdot 12.5 \tag{3.25}$$

with LGD standing for the loss given default, i.e. the proportion of the exposure that will not be recovered. $N(\cdot)$ is the standard normal cumulative distribution function. PD is the probability of default (in one year) and ρ is the correlation factor between the obligor and the market factor. MF is a so-called maturity factor, a function of the maturity M for several values of PD.

Under IRBA banks will be allowed to estimate the default probability, subject to meeting predefined conditions and explicit supervisory approval. Banks operating under advanced IRBA will be allowed to estimate LGD and exposure at default (EAD) as will be explained below in more detail.

Assuming a minimum capital ratio of 8% – as required under Basel II – the minimum regulatory capital requirement RC is given as follows

$$\mathrm{RC} = \mathrm{LGD} \cdot \left[N\left(\frac{N^{-1}(PD) - \sqrt{\rho}N^{-1}(0.999)}{\sqrt{1-\rho}} \right) - PD \right] \cdot MF(M, PD) \cdot EAD. \tag{3.26}$$

Equations (3.25) and (3.26) find their theoretical foundation in the Asymptotic Single Risk Factor (ASRF) model of portfolio credit risk, developed amongst others by Gordy (2003).[83] In the following, we will not elaborate extensively on this model and will refer the interested reader to the cited works and to the explanatory note given in BCBS (2005). Relevant for our analysis on counterparty credit risk and CVA is the main objective of the formula: Assuming a large homogenous and granular pool of obligors with a normal distributed default rate, the formula aims on approximating the (expected and unexpected) credit loss that will be exceeded with a small probability of 0.1%. The maturity adjustment is expected to capture the higher risk given by longer maturities for example due to a higher probability of credit deterioration, i.e. migration.

Hence, the above given formula delivers a value at risk (VaR) measure for credit risk with a confidence level of 99.9% and a time horizon of one year. This stands in line with the proclaimed objective behind the Basel framework to capitalize a peek loss, a balance between prudency and solvency on the one hand and economic efficiency on the other.

The Basel Committee argues that by fixing the confidence level to 99.9% "*an institution is expected to suffer losses that exceed its level of tier 1 and tier 2 capital on average once in a thousand years",* adding that "*[t]his confidence level might seem rather high.*" The Committee advocates this *high* level as "*[protecting] against estimation errors, that might inevitably occur from banks' internal [...] estimations, as well as other model uncertainties.*"[84] The financial crisis offers a hindsight in which these elaborations appear rather inadequate, if not even naive. Basel III has addressed this issue by

[83] See also Merton (1974), Schönbucher (2005) and Vasicek (2002). For an overview on credit risk modeling see Subchapter 4.2.2.

[84] See BCBS (2005), especially p. 11.

prescribing an evolution for the minimum capital requirement, raising it up to 14.5% by 2018.[85] Future will tell whether this increase suffices.

Estimating Regulatory Exposure (EAD)

A challenging task within the exercise of capitalizing counterparty credit risk is estimating the amount at stake, i.e. the exposure at default (EAD) from Equation (3.24). We have described the challenges involved in estimating expected exposure in Subchapter 3.3.3. In the following we will focus on unique aspects relevant for regulatory purposes while revisiting some notations in order to allow for a more fluent elaboration.

The supervisor prescribes three possible alternatives for this purpose: (1) the current exposure method (CEM), (2) the standardized method (SM) and (3) the internal model method (IMM).[86] The use of SM seems however to be rather rare so that we will in the following solely focus on the significantly more popular approaches, CEM and IMM.[87]

The Internal Model Method (IMM)

The IMM approach is based on the use of Monte Carlo schemes to estimate the (future) value of the derivatives portfolio, i.e. through the simulation of possible evolutions of the relevant risk factors in a forward looking manner.[88] The exposure E inherent in a portfolio of OTC derivatives at a given time period t is defined as follows

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[85] We refer the reader to critical analysis of the VaR measure found for example in Beder (1995) or Taleb (1997), pp. 445-453.

[86] Notice that at the time of writing these lines BCBS has introduced a new standardized approach for measuring counterparty credit risk exposures (SA-CCR). The SA-CCR is supposed to be more risk sensitive, taking collateral and netting effects better into consideration. Because banks are still not required to adopt the approach we will keep it uncommented and refer the interested reader to BCBS (2014).

[87] According to an international survey on counterparty credit risk only one participating institution stated to use SM whilst no participant planned to use SM in the future, see Deloitte and Solum (2013), p. 8.

[88] For a more detailed description of exposure simulation we refer the reader to Subchapter 3.3.3.

$$E(t) = max\{NPV(t) - C(t), 0\} \tag{3.27}$$

where NPV stands for the net present value of a given netting set, i.e. the (default-free) value or the marked-to-market (MtM) metric of the respective derivatives. C is the value of the collateral posted or received. This means that if the NPV is negative and the bank posts a collateral amount to the counterparty that exceeds the absolute value of the NPV, the bank will have a positive exposure (due to overcollateralization). Notice that the IMM approach enables the full consideration of netting and collateral effects.

After producing possible exposure profiles for a given time period set $\{t_k\}$ the expected exposure EE at time period t_k is given as follows

$$EE(t_k) = \mathbb{E}^{\mathbb{P}}[E(t_k)] \tag{3.28}$$

Where E is the net exposure at time t given in Equation (3.27) and $\mathbb{E}^{\mathbb{P}}$ denoting the expectation under the probability (real-world) measure $\mathbb{P}$. It is worth mentioning that the exposure models used for regulatory purposes are supposed under Basel III to be calibrated to historical stress periods, allowing for more prudent exposure estimation (also deviating from the risk-neutral measure $\mathbb{Q}$ used in CVA valuation). In order to assign yet more prudency to the computation the regulator introduces a further exposure metric, the effective expected exposure $EffEE$, defined as the maximum expected exposure until the date of calculation, given as follows

$$EffEE(t_k) = \max_{t_i \in [0, t_k]}\{EE(t_i)\}, \tag{3.29}$$

Focusing on a time horizon of one year (H), the effective expected positive exposure $EffEPE$ is defined as the average of the $EffEPE$ given in the following equation

$$EffEPE = \frac{1}{H}\sum_{t<H} EffEE(t_i)\Delta t_i \tag{3.30}$$

where $\Delta t_i \equiv t_i - t_{i-1}$. The regulatory exposure at default EAD is finally computed as a product of the effective expected positive exposure and a multiplier alpha α, i.e.

$$EAD = \alpha\, EffEPE. \tag{3.31}$$

The aim of the α factor is to consider the fact that the portfolios are not granular and that the interdependencies between the exposure profiles and credit risk of the counterparties can be disadvantageous. The α factor can thus be seen as a surcharge for concentration risk and general wrong way risk. The regulator sets the default value of the α factor at 1.4 while allowing institutions to calibrate their own values.[89]

The Current Exposure Method (CEM)

The CEM offers an alternative proxy approach for institutions that do not have (or do not want) the ability to use Monte Carlo schemes to estimate their exposures. The regulatory exposure at default EAD is defined as the sum of the current exposure (CE) of a given netting set, taking collateral into consideration, i.e. the exposure E at time period 0 from Equation (3.27), and a prescribed addon factor. EAD is given by:

$$\mathrm{EAD} = \mathrm{CE} + \mathrm{Addon} \tag{3.32}$$

where Addon is an adjustment, given as a percentage of the notional amount, depending on the underlying asset class and the maturity of the derivatives portfolio as given in Table 11 below. In contrast to the internal model approach CEM allows only for a limited consideration of netting effects, using 60% of the current netting benefit. Notice in addition that while IMM might assign a zero exposure to a netting set, CEM, being significantly more prudent, will usually deliver higher exposure metrics. If for example the current value of the

[89] The regulator sets a floor of 1.2 if own calibration methods are used. The own calibration of alpha seems to be quite rare, see Deloitte and Solum (2013). For measuring and analyzing the alpha factor see for example Cespedes et al. (2010).

derivative portfolio is significantly negative, i.e. the portfolio is deep out-of-the-money (OTM), the simulation models might produce no positive exposure for the next one year horizon, possibly delivering an effective exposure of zero. Under the CEM this is rather excluded as the pre-described addon factors will always be positive.

Residual Maturity	**Interest Rate**	**FX and Gold**	**Equity**	**Precious Metal**	**Other Commodities**
$\boldsymbol{rM} \leq 1$	0%	1%	6%	7%	10%
$1 < \boldsymbol{rM} < 5$	0.5%	5%	8%	7%	12%
$\boldsymbol{rM} \geq 5$	1.5%	7.5%	10%	8%	15%

Table 11: Addon Factors Used within the CEM Approach

The table above summarizes the regulatory predetermined addon factors for computing the potential future exposure according to the current exposure method (CEM). The addon factors are given a percentage of the notional amount of the derivative position. rM stands for residual maturity in years.

Concluding Remarks on Capitalizing Default Risk

It can be concluded that both IMM as well as CEM aim on assigning derivatives a loan-like risk metric with a horizon of one year, (theoretically) allowing a similar treatment of counterparty credit risk inherent in OTC derivatives and credit risk, arising from loan (debt) instruments.

Notice, however, that the overall minimum regulatory capital requirement for default risk captures – or at least aims on capturing – further relevant risks. Migration risk (incremental credit risk) is captured through a maturity factor. Market risk is addressed by allowing for asset class dependent addon factors (under the CEM) or an explicit risk-factor dependent simulation (under IMM). Concentration risk and wrong way risk are supposed to be captured by the alpha α factor.

The capital requirement measure is on no account a pricing metric as it is not based on arbitrage-free or risk-neutral aspects, solely quantifying the risk inherent in the next year. It can rather be qualified as a risk metric, based on historically estimated parameters that are supposed to be prudent, not necessarily economically plausible.

CVA Capital Charge – Remarks

Adding a price adjustment to incorporate counterparty credit risk, e.g. in form of a CVA metric, introduces additional price sensitivities. CVA depends on the credit quality of the counterparty, usually quantified in terms of credit spread sensitivity (e.g. CS01). If for instance the credit quality deteriorates and spreads widen, CVA will increase, leading to value losses. Being exposure dependent, CVA exhibits also market risk sensitivity (e.g. DV01). Decreasing interest rates will for instance boost the value of receiver swaps as well as their expected exposure, subsequently enhancing CVA. In addition, CVA would also exhibit a sensitivity to the correlation between credit spreads and the relevant risk factors.

The Basel Committee reports that during the financial crises two-third of the credit losses incurred by financial institutions were driven by the overall deteriorating credit quality and the increase of credit spreads (see BCBS, 2011). Losses incurred through counterparties actually defaulting and filing bankruptcy accounted only for one-third of the credit losses reported. This drove the Basel Committee to require for an additional capital charge. The CVA capital charge aims on capitalizing CVA volatility arising from credit spread sensitivity. Notice that the regulator (uncommented-wise) left further sensitivities of CVA, i.e. DV01 and gamma, not accounted for.

Basel III offers two alternatives for measuring CVA risk, a standard and an advanced method. In the following we will give a brief account of both approaches.

CVA Capital Charge – Standard Method

The standard method aims on capturing the losses due to credit spread sensitivity using a closed-form solution approach.[90] The approach addresses banks using less sophisticated approaches to estimate their exposure (e.g. CEM). The CVA capital charge K for the bank-wide portfolio is measured using the following predefined formula:

$$K = 2.33\sqrt{H}\sqrt{\left[0.5\cdot\left(\sum_{i=1}^{n} w_i\left(EAD_i M_i - B_i^h M_i^h\right)\right) - w_i B_i^h M_i^h\right]^2 + 0.75\sum_{i=1}^{n} w_i^2\left(EAD_i M_i - B_i^h M_i^h\right)^2} \tag{3.33}$$

with H being the one-year time horizon in units of a year (i.e. 1 year). n stands for the number of netting sets. EAD_i is the exposure at default for netting set i as computed for example using CEM. M_i stands the effective maturity of the netting set in units of a year. w_i is the weight assigned to the credit risk of the counterparty as predefined according to the following table:

Credit quality steps	1	2	3	4	5	6
Weight $\boldsymbol{w_i}$	0.7%	0.8%	1.0%	2.0%	3.0%	10.0%

Table 12: Weights Used in the Standard Approach for Calculating CVA Capital Charge

The table above summarizes the weight used in the standard formula for calculating CVA capital charge as given by CRR IV, Article 384.

[90] See also CRR IV, Article 382.

Notice that the regulator explicitly allows the recognition of credit hedges. B_i in the formula above stands for the notional of a credit derivative, e.g. single name or index CDS, protecting from the default of counterparty i.[91]

Pykhtin (2012) shows that the Basel III equation for CVA capital charge is the closed-form solution of the value at risk based on a one-factor credit spread model with conservative assumptions. Instead of using market implied volatilities for credit spreads the equation relies on (conservative) constant risk weights.

CVA Capital Charge – Advanced Method

Under the advanced method the CVA capital charge is measured using a Monte Carlo scheme in which the credit spreads for each counterparty are simulated and the CVA VaR is subsequently computed, based on the following prescribed formula:[92]

$$CVA = LGD_{MKT}\sum_{i=1}^{T} Max\left(0; exp\left(-\frac{s_{i-1}t_{i-1}}{LGD_{MKT}}\right) - exp\left(-\frac{s_i t_i}{LGD_{MKT}}\right)\right)\cdot\left(\frac{EE_{i-1}P_{i-1} + EE_i P_i}{2}\right) \tag{3.34}$$

with LGD_{MKT} standing for the loss given default ratio, based on market implied information. T is the maturity of the netting set and t_i stands for the respective future time period. P_i is the discount factor at time period i. EE_i is the expected exposure, considering possible collateral and netting agreements, and based on the stressed calibration method as elaborated above for EPE purposes. s_i is the simulated credit spread at future time period i. Basel III explicitly requires the use of market implied data to estimate the credit spreads, prescribing a hierarchy of sources and a respective mapping approach. If possible banks should rely on CDS spreads referencing to the counterparty. If such spreads are not available they are

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[91] Eligible hedges according to CRD IV, Article 386 are (a) single-name CDS or other equivalent hedging instruments referencing the counterparty directly; (b) index CDS.

[92] See CRD IV, Article 383.

expected to develop a mapping procedure based on rating, industry and region of counterparties.[93]

CVA is thus being modeled as a function of the same components as given in the practical Equation (3.16) for CVA pricing: (1) LGD is clearly visible and is based on market implied information, (2) the first part of the summation is the derived default probability, and (3) stands for the discounted expected exposure as average between each two future time steps. The VaR measure is computed using a 99% confidence level and 10-day horizon (in addition to a predefined multiplier of three).

Banks are allowed to incorporate possible hedge positions that reduce credit sensitivity (e.g. CS01). This means that credit hedges such as single name CDS can be modeled to reduce the expected exposure profile, and thus the CVA VaR computed. Hedges aiming to reduce other sensitivities, e.g. interest rate sensitivities, are on the other hand not considered as eligible.

3.4.4. Evaluation Summary

As with other regulatory capital requirements the approaches described above exhibit certain shortcomings and inconsistencies. After all, regulatory capital requirements can be seen as following contradicting objectives. On the one hand they are supposed to display theoretical soundness that is presumably aligned with the way "banks do business". On the other hand they are supposedly prudent, ensuring banks are sufficiently capitalized for extreme cases. In the following we give a very short summary of the criticism against CVA capital charge, and

[93] "Whenever the credit default swap spread of the counterparty is available, this must be used. Whenever such a credit default swap spread is not available, the bank must use a proxy spread that is appropriate based on the rating, industry and region of the counterparty." CRD IV, Article 383, Paragraph 1.

will subsequently conclude with an own assessment of the requirements and their consequences.[94]

Banks have criticized the misalignment of the approaches with how they calculate CVA, i.e. with how "they do business"

- Aiming to be theoretically sound the advanced approach (in line with accounting requirements for fair value measurement) prescribes the use of market implied parameters to estimate LGD and default probabilities (CDS spreads). Recall that many banks have relied their valuation on historical and rating-based parameters. Agnostic of the reasons – which have already been discussed in Subchapter 3.4.2 – banks seem to advocate the use of theoretically inconsistent approaches, especially contradicting the risk-neutral valuation scheme.

CVA capital charge has also been criticized for its prudency:

- The advanced approach prescribes the use of stressed expected exposure measures, i.e. the use of data from historical periods of distress to calibrate the exposure models. This clearly deviates from risk-neutral valuation, which is behind the use of market implied data. In the meanwhile the standard method prescribes the use of EAD which is per se a conservative measure, standing for a different metric than the exposure measures used for CVA calculation.[95]
- In contrary to accounting standards the regulatory capital approaches focus only on the unilateral CVA. Excluding DVA means that a "natural hedge" for the bilateral CVA is not considered. After all, the credit spreads of the bank and its counterparty might display a similar dependency to a common market factor. If CVA increases

[94] For a more comprehensive overview of the industry's criticism against the CVA capital charge methods we refer the reader to Gregory (2012), pp. 393-396.

[95] The conservative assumptions used in the standardized approach have also been discussed, especially regarding the use of risk weights instead of market implied CDS volatilities.

due to a general market deterioration DVA will also increase, absorbing – at least to some extent – some of the CVA volatility.

- Both the standard and advanced approaches consider credit-sensitive hedges as eligible. Holding a single name CDS referencing to the counterparty will for example reduce the exposure and thus the CVA capital charge. Other hedges, e.g. interest rate or FX hedges aiming to reduce the volatility of the CVA due to exposure changes are, however, excluded.

Given CVA is a volatile number with significant impact on the P&L of a bank it is straightforward that regulators are now requiring its capitalization. The more interesting aspect about CVA capital charge and the associated discourse is that it gives an almost school-book-like example of how regulatory-driven financial institutions actually are. It also shows the tautology behind many of the discussions around regulatory requirements.

Yes, it can be argued that CVA is a volatile number that needs to be capitalized. Yet, it seems that most banks started accounting for CVA in their financial statements after the introduction of the regulatory requirements described above. Recent studies *do* show that banks "*have progressively converged in reflecting the cost of the credit risk of their counterparties in the fair value of derivatives using market implied data based on CDS spreads and proxy spreads in the vast majority of cases*."[96] But as also acknowledged by the regulators "*[t]his convergence is the result of industry practice, as well as a consequence of the implementation […]of IFRS 13 and the Basel CVA framework*." One could argue that the supply with regulatory capital requirements has produced its own demand through insisting on using (volatile) marked implied data that increased the volatility of banks' P&L statements. After all, it is only fair to note that risk-neutral and arbitrage-free valuation presume traded hedging instruments that allow for a "duplication strategy". As discussed above this is hardly the case for CVA, especially when it is material at most. Clinging to use market implied parameters without reflecting on their economic sense and

[96] EBA (2015), p. 8.

most importantly on their consequences risks financial statements to aspire for something they cannot deliver per se.[97]

If we, however, agree that financial statements should give a "true and fair view" then – as it applies for all other plain-vanilla and exotic derivatives – considering CVA and measuring it according to the fair value hierarchy is straightforward. Bearing in mind that the disclosed CVA measures are a proxy at best, their bare existence increases the value of the financial statements as it points to a significant risk, reducing at least some of the opaqueness surrounding counterparty credit risk. After all, stakeholders that haven't been aware of CVA might make decisions without considering all relevant risks.[98]

Evidence indeed shows that banks have started to charge significantly more for uncollateralized derivatives as a reaction to the new regulatory paradigm. A comparative study coordinated by the Association for Financial Markets in Europe (AFME) and cited by EBA (2015) claims for example that the costs associated with regulatory capital for counterparties with a rating of BB have doubled. This indeed reveals that the lobby-like pressure coming from the industry was not justified, and that the concerns articulated by the Lufthansa manager quoted above hit the spot. A CVA capital charge would increase hedging costs, possibly making them too punitive for counterparties with no access to liquid collateral.

This also explains why the EU lawmakers have reacted to the pressure by allowing to exempt transactions with non-financial counterparties such as Lufthansa.[99] Interestingly, however –

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[97] The increased use of CDS spreads for CVA valuation has indeed increased the demand for CDS in the market which in return increased the level of CVA as shown for example by Bilal and Singh (2012).

[98] Psychologists refer to the fact that humans tend to jump to conclusions based on limited evidence as WYSIATI, which stands for what you see is all there is, see Kahneman (2011), especially pp. 85-88.

[99] The CRR exempts the following: (I) Transaction with non-financial counterparties below the EMIR clearing threshold, as codified in CRR Article 382(4)(a); (II) Transactions between clearing members and clients in the context of indirect clearing when the clearing member is acting as an intermediary between the client and a qualifying central counterparty - CRR Article 382(3); (III) Intragroup transactions - CRR Article 382(4)(b);

as EBA (2015) seemingly acknowledges – "*accounting CVA is usually [already] reflected in the price of derivative contracts*". The increased charges are thus only a function of the bank's decision "*to also pass on the capital costs associated with the regulatory CVA risk charge to their counterparties*". In the following we will take a second glance at this rather daring hypothesis that seems to pass through without sufficient scrutiny.

Indeed, the CVA capital charge formulas are not equal to the formula used to price CVA and supposedly also used to measure it for accounting purposes, as illustrated by Equation (3.16). Still, both formulas are definitely not miles away from each other (especially under the advanced approach). A bank that incorporates the (accounting) CVA will be double-counting for a series of factors if it then fully incorporates the CVA capital charge on top. Think of the (accounting) CVA as the hedge cost for counterparty credit risk. If we charge these costs to the client then we should theoretically be able to hedge the exposure-sensitivity as well as the credit-sensitivity of CVA. Hedging the latter is crucial as it will minimize the regulatory capital requirement both under the standard and the advanced methods. The remaining requirements will result from the prudency of the regulatory requirements, especially the use of stressed exposure measures etc.

CVA capital charge will thus indeed lead to an increase in hedge costs. It is, however, questionable whether that increase will be as significant as the evidence shows. It can rather be concluded that the evidence discovers that banks have not been pricing for CVA adequately in the past. This might be driven by the competitive market, cross-selling aspects or the fact that people selling these product plainly did not think systematically about counterparty credit risk, because no metric pointed to it.

The exemption rule will thus only help the industry if banks systematically misprice their products, i.e. charging less for them than the hedge costs they imply. Interestingly banks also

(IV) Transactions with pension funds - CRR Article 382(4)(c) and CRR Article 482; and (V) Transactions with sovereign counterparties - CRR Article 382(4)(d) pledged for these exemptions which might speak for a strong belief in cross-selling aspects and the increased opaqueness they bring along. It might also speak for the fact that the risk-neutral and arbitrage-free perspective – for right or for wrong – did not yet penetrate the executive management levels of financial institutions.

If the above analysis can be synthesized into one key takeaway it is that financial institutions are heavily driven by regulatory requirements. They do not only measure and manage their risks according to detailed requirements they also trade and price according to such requirements that originally aimed to actually mirror how banks "do business". Bearing in mind that regulatory requirements should set minimum standards for all financial institutions it remains questionable if the detailed level of such requirements is beneficial. Especially when it comes to topics like CVA where there is a lack of theoretical soundness and practical experience it would be more beneficial to let a larger number of entities experiment than having one entity prescribing everything in detail to the rest.

3.5. MANAGING AND MITIGATING CREDIT VALUATION ADJUSTMENT

3.5.1. MANAGING AND PRICING CVA

The previous subsections revealed the technical challenges around modeling CVA. They also shed light on the evolving regulatory and accounting landscape. Agnostic of whether the regulatory requirements increased the awareness of CVA or whether they are only a reaction of its increased importance, derivative traders - especially banks - are faced with the challenges of systematically incorporating counterparty credit risk into pricing. This implies techniques for charging CVA to the client but also to allocate it across the organization in order to allow for adequate incentives. In this subchapter we will give an overview of such techniques, evaluating their effectiveness from the perspective of the management of a financial institution as well as their counterparties.

The technical challenges around CVA, especially the discussions around estimating default probabilities, revealed how shaky (if not even unscientific) CVA modeling can be. Lacking theoretically sound valuation schemes at all times, traders - as they probably always do - cannot base their pricing solely on mathematical considerations. Not incorporating CVA into pricing can indeed lead to adverse selection in Akerlof's (1970) sense as the bank will not only face a problem of not covering costs through default, it will also draw a higher proportion of counterparties with relatively high credit risk. Still, it may remain a well-taken management decision to *invest* in order for example to increase market shares in a specific segment or in order to reap cross-selling benefits from other products. In the following we will discuss key issues around incorporating CVA into pricing. We will show that even when it seems highly technical, the need for strategic and tactical considerations will prevail as all decisions will be taken under a significant proportion of uncertainty.

Incremental CVA

As already discussed in previous subchapters exposure and thus CVA are calculated on netting set level. If the instruments within a netting set do not exhibit offsetting effects, e.g. if the derivatives are identical, then CVA on a netting set level will equal the sum of the individual CVA metrics. If, however, the instruments do exhibit offsetting effects, e.g. a netting set with payer and receiver swaps, the overall *unilateral* CVA will be smaller than the sum of the individual instruments.[100]

In case of the unilateral CVA the relation can be formalized as follows:

$$CVA_P \leq \sum_{i=1}^{n} CVA_i \tag{3.35}$$

with CVA_P standing for CVA on a netting-set (portfolio) level. n is the number of instruments within the netting set, and CVA_i is the CVA of the instrument i. This means that pricing unilateral CVA on a standalone basis could be punitive if a netting set is in place. Pricing a so-called incremental CVA allows the consideration of such offsetting effects. Estimating the incremental CVA of a new trade includes the quantification of CVA on a netting-set level before and after the trade has been introduced. The following equation formalizes this relation

$$CVA_i^{incremental} = CVA_{P+i} - CVA_P \tag{3.36}$$

where $CVA_i^{incremental}$ stands for the incremental CVA of instrument i. CVA_{P+i} stands for CVA of the netting set if instrument i included, and CVA_P stands for the CVA on netting set if instrument i is not included.

[100] Notice that the effect on the *bilateral* CVA is not that intuitive, because the final metric is not only a function of the expected positive exposure but also of the expected negative exposure, and thus DVA. See also Gregory (2012), pp. 177-178.

From a pricing perspective charging for the incremental CVA seems the theoretically right thing to do. A transaction gets charged with the CVA amount it contributes. This does, however, imply certain challenges that will be discussed in the following.

First, incremental CVA is not additive, i.e. it is not possibly to allocate the portfolio CVA metric to the different trades.[101] This implies possible inconsistencies when simultaneously trading a number of derivatives with the same counterparty while being interested in having an incremental CVA metric for each trade. This pitfall seems more relevant for CVA allocation, i.e. post pricing.

In addition, pricing incremental CVA implies that the unilateral CVA of a new trade does not have to be positive. If a trade exhibits extreme offsetting effects, reducing the expected exposure of a netting set, its incremental CVA would be negative.[102] Assuming both counterparties are equally informed and sophisticated a negative CVA would be an obvious contractual prerequisite. OTC derivatives are, however, not always traded between equally informed and sophisticated counterparties. One can question the willingness of a trader to share a negative CVA with an uninformed counterparty, e.g. a buy-side (client) without the technical capabilities. Also from a management perspective one can argue that flooring CVA, e.g. by zero, would avoid too progressive a pricing, especially relevant for non-collateralized client portfolios.

A further aspect of incremental CVA is its dependency on the order in which transactions are traded. Think for example of a netting set with numerous derivatives, e.g. receiver and payer swaps. The counterparties now plan to trade two mirroring instruments. The first instrument (trade A) is a receiver swap (e.g. semi-annual payments of three-month Euribor,

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[101] See also Pykhtin and Rosen (2010).

[102] As discussed above bilateral CVA does not have to be positive per definition, e.g. if the counterparty doing the calculation has the worse credit quality or if the expected negative exposure is larger than the expected positive exposure.

receiving a fixed spread of 100 basis points etc.). The second instrument (trade B) mirrors the first, i.e. payer swap (semi-annual payments of a fixed spread of 100 basis points, receiving three-month Euribor etc.). The incremental CVA of the individual trades A and B will differ significantly, depending on the sequence in which they are traded (e.g. A-B or B-A). This simplified example illustrates that using incremental CVA can in fact be counterintuitive, especially if one is able to "anticipate" future transactions with the counterparty. This makes it also clear why traders might not charge for CVA as they expect further transactions (with offsetting netting effects) to be traded with the respective counterparties. The fact that incremental CVA is sequence-dependent raises a further issue which is relevant for banks in which the different trading desks are internally charged for CVA (see below). A trader might "optimize" the order in which he deals to minimize internal charges.

In addition, incremental CVA implies further significant technical challenges. CVA calculation as such requires sophisticated technical capabilities (e.g. estimation of expected exposure, taking netting and collateral agreements into consideration). Calculating an incremental CVA raises the bar significantly as the calculation needs to happen twice (with and without the deal) and fast to allow for usual OTC derivative trading time slots. This makes it worthwhile to consider the "use case" of CVA pricing. Yes, CVA might remain significant even if collateral agreements are in place, especially if derivatives with significant jump risk are being traded.[103] Still, CVA is most significant if no collateral agreements are in place and if the credit risk of the counterparties is asymmetric. This is usually the case when banks trade with (buy-side) clients (e.g. non-financial counterparties) in contrary to trading with other financial institutions. Interestingly, one might argue that in such cases the netting sets will tend be rather one-sided as such counterparties trade derivatives with banks in order to hedge similar positions (e.g. hedging interest rate risks of floating loans). The "use case"

[103] See Subchapter 3.5.2 for further elaborations on this.

of CVA pricing does not argue against incremental CVA. It does, however, put it into proportion, facilitating a more adequate discussion around the importance and urgency of implementing the respective technical capabilities. This explains why derivative traders rely also on approximations for CVA pricing such as lookup tables with grids of predefined CVA values, e.g. depending on the maturity and credit quality.[104]

Pricing DVA

As noted in Subchapter 3.4.2 above, if CVA is priced then some counterparties are (willingly or not, knowingly or not) pricing DVA. This indeed sheds a different light on the discussion around how counterintuitive it presumably is to price DVA. After all, the unilateral CVA, calculated by one counterparty is the DVA of the other counterparty, independently of whether one can monetize DVA or not. Still, it seems hard to detach the discussion around pricing CVA and DVA from the *pricer* at hand, especially if the counterparties can be classified as price setters or price takers, e.g. as is the case when financial institutions trade with clients from industries other than financial services. While two banks trading would accept and expect CVA *and* DVA to be priced in order to allow for fair contractual terms and symmetrical pricing, a bank trading with a client (e.g. non-financial services corporation) might be reluctant to price in its own counterparty credit risk, because of a variety of reasons. (a) The trader might argue that his bank's credit risk is negligible if compared with the credit risk of the client. This can indeed be the case in many situations, making DVA seem like theoretical (over-) complication with no real practical use; (b) The trader might see himself as a service provider (and not as a counterparty on equal terms); (c) A reputational issue might arise by "admitting" that the bank is default-prone in front of a client; (d) Lastly DVA simply reduces the profit margin, leaving no incentive for the trader to price it in the first place. The latter argument (d) might seem banal, but it is an important indication that

[104] See also Gregory (2012), p. 411.

the discourse around DVA is not a pure scientific search for truth. Profit considerations cannot be excluded, especially if the debate is led by people from the industry.

Interestingly – and as also noted in Subchapter 3.4.2 – evidence does show that an increasing number of banks are pricing DVA while some interpret it as a funding benefit adjustment.[105] Independently of whether this interpretation is based on sound theoretical considerations, the evidence does point to the fact that CVA on a unilateral basis has become more scares and harder to advocate for.

Central Management and Pricing – CVA Desk

Acknowledging the complexity and uniqueness of the topic banks usually dedicate specialized organizational units to price and manage CVA, usually referred to as CVA desks.[106] The main objective of these units is to offer a counterparty credit risk (CCR) hedge to the individual desks. For this reason the trading desks pass on the CVA amount (they charged the counterparty for) to the CVA desk as illustrated in Figure 13.

Figure 13: Illustration of the Role of a CVA Desk in Hedging and Pricing

The figure illustrates the CVA charging process. The trading desk (e.g. interest rate swap desk) passes on a CVA charge for a trade. In return all counterparty credit risk inherent by that trade, i.e. default risk and CVA volatility is borne by the CVA desk which usually has the mandate to (partially) hedge default risk and CVA volatility.

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[105] For evidence on pricing DVA, and its increased interpretation as funding benefit adjustment see for example Deloitte & Solum (2013), pp. 38-39.

[106] See for example EBA (2015), pp. 22-23 or Deloitte & Solum (2013), p. 23.

The CCR hedge between the different desks and the CVA desk can take the form of an internal contingent CDS (C-CDS) contract (discussed in Section 3.1 above). A perfect CCR hedge means that the individual desks suffer no losses if the counterparty defaults nor do they need to deal with additional volatility as a function of CVA changes. If the hedge is based on a bilateral CVA it would also imply that individual trading desks do not profit (nor do they lose) from deterioration (improvement) in the credit quality of their banks, avoiding non-plausible P&L effects on trading desk level.

In addition, these units are supposed to ensure the following: (a) Complex decisions around pricing and mitigating CVA (as well as DVA) are taken by employees with adequate training and professional knowhow; (b) Possible diversification (netting) effects are considered. This becomes especially relevant when considering effects within a multi-asset netting set; and (c) Expertise to develop and run adequate modeling capabilities for the different CVA components, e.g. expected exposure, collateral agreements, probabilities of default, wrong-way-risk etc.

An essential question that needs to be discussed is whether CVA desks should operate as profit generating units or as a cost center. After all, it seems intuitive for a trading department to want to "make money" from CVA, and to incentivize the CVA desk to operate accordingly. Doing that, the banks might, however, run the risk of incentivizing the CVA desk to generate profits at the expense of the other trading desks or to increase the risk positions, in misalignment with the overall trading strategy, e.g. through hedging CVA risk of other banks or other proprietary (prop) trading positions. It might therefore seem more promising to define the CVA desk as a so-called utility function. Gregory (2012, pp. 409-410) uses the term to indicate that "*the mandate of the CVA desk is to have a flat PnL*", while commenting that "*a zero PnL mandate on annual basis is still not ideal*" as it might give rise to wrong incentives. In addition he points to several measures that can reduce CVA without automatically increasing the risk.

In order to ensure that the CVA desk is acting according to the overall strategy of the bank, a set of adequate quantitative key performance indicators (KPIs) needs to be defined, in alignment with its overall mandate. This indeed depends highly on the size and structure of the OTC derivative portfolio at hand and the overall strategy of the bank, and can thus hardly be discussed in general terms.[107] In the following we give several examples of such KPIs for illustration purposes only:

- Quantitative KPIs: CVA volatility vs. target, hedge effectiveness, changes in CVA regulatory capital requirements (RWA volatility), workout recovery etc.
- Qualitative KPIs: Quality of CSAs, evaluation of hedging strategies, feedback from other trading desks regarding pricing, consulting etc.

3.5.2. Mitigating CVA – Collateral Agreements

Besides netting agreements an obvious way of reducing counterparty credit risk is through collateralization. For this purpose the counterparties agree on posting collateral whenever their position is out-of-the-money, i.e. when the derivative portfolio displays a liability from their perspective. This reduces the net exposure of the counterparty with in-the-money positions as it receives a respective collateral amount as formalized in Equation (3.7) and Equation (3.8) above.[108]

Due to the stochastic nature of derivative values – incl. the fact that they can be either positive or negative – the counterparties need to agree on respective terms for posting and receiving collateral. Such terms are usually specified in an annex to the master service agreement used. The ISDA Credit Support Annex (CSA) is the most popular format for

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[107] In Subchapter 3.5.2 and Subchapter 3.5.3 we discuss possible mitigation strategies, possibly run by a CVA desk.

[108] As also shown in Equation (3.7) and Equation (3.8), p. 40, overcollateralization will lead to increased counterparty credit risk for the party posting the collateral.

such agreements.[109] We will therefore in the following refer to CSAs and collateral agreements synonymously.

As illustrated in Table 9 above, ISDA estimates that collateral agreements in combination with netting agreements reduce the overall OTC exposure by not less than 95%, leaving less than 5% of the gross market exposure uncollateralized – which is anyhow 1.1 trillion USD.[110] This can be seen as a testimony of the high popularity of netting and collateral agreements and their effectiveness in reducing exposure. The popularity of collateral agreements is most evident within the financial services sector as seen in Table 13. Almost every derivative traded between banks seems to be collateralized. On the other hand, counterparties outside of the financial services sector seem still to struggle with collateralizing their derivatives, maybe due to limited access to liquid assets that are considered eligible.

Counterparty type	**CSA**	**No CSA**
Banks and security firms	95.5%	4.5%
Hedge funds	94.1%	5.9%
Pension funds	75.3%	24.7%
Non-financial institutions	28.6%	71.4%
Government-sponsored entities / government agencies	42.4%	57.6%

Table 13: Percentage of Active Bilateral Derivative Collateral Agreements by Counterparty Type

The table illustrates the percentage of collateral agreements (CSAs) by counterparty type as of December 31, 2014. Source of the table is the market survey of ISDA (2015).

[109] For statistics on the topic see for example the 2015 margin survey of ISDA (2015), e.g. p. 10.
[110] See Table 9. p. 48 above.

Product type	CSA	No CSA
Commodity derivatives	59.1%	40.9%
Credit derivatives	97.0%	3.0%
Equity derivatives	91.3%	8.7%
Fixed-income derivatives	88.9%	11.1%
Foreign exchange derivatives	73.0%	27.0%

Table 14: Bilateral Derivative Collateral Transactions by Product Type

The table illustrates the percentage of collateral agreements (CSAs) by product type as of December 31, 2014. Source of the table is the market survey of ISDA (2015).

As for example shown by Brigo et al. (2013b) or Lipton and Shelton (2012) collateral agreements can reduce the expected exposure, and can thus have a mitigating effect on counterparty credit risk and CVA. Still, a differentiated analysis is needed in order to understand which risks remain despite having a CSA in place, and which new challenges collateral agreements introduce. For this purpose we revisit and analyze the key characteristics of CSAs (introduced shortly in Subchapter 3.3.3):[111]

i. *Collateral type*: The CSA defines the eligible type of collateral each counterparty needs to post, e.g. cash, government bonds, corporate bonds etc. Cash can be seen as the least risky collateral type.[112] Still, an FX risk may arise if the currency of the collateral differs from the currency of the derivatives. If, in addition, the CSA allows for cash to be posted in more than one currency, e.g. EUR and USD a cheapest to deliver option is introduced. Such an option might exhibit a considerable value which needs to be taken into account as for example shown by Fujii and Takahashi (2011).

[111] See also Cesari et al. (2009), pp. 190-191.

[112] According to ISDA (2015) cash (especially USD and EUR) makes out around 75% of the collateral received and delivered worldwide. The second most popular type are government securities (especially of the United States and member states of the European Union) who make ~18% of the collateral delivered.

Collateral becomes riskier if we turn to securities such as bonds (especially with rather risky issuers) or even equities. To account for such risks counterparties might agree on so-called haircuts.[113] Posting securities such as bonds might imply additional wrong-way risk that needs to be analyzed and considered. This is especially relevant if the credit risk of the counterparty (e.g. state-owned bank), posting the collateral, is highly correlated with the credit risk of the issuer (e.g. government bond).

ii. *Threshold*: A threshold stands for the level underneath which collateral will not be called. If the portfolio value exceeds the threshold collateral will be exchanged to cover the surplus exposure. The threshold amount will remain uncollateralized.

Counterparties might agree on thresholds to reduce operational costs, but from a CVA perspective they can be significant and need to be considered.[114] Thresholds can also be defined asymmetrically, i.e. the counterparties will have differing amounts. This is especially relevant if one counterparty has an "unlimited" threshold as this implies that it will never post collateral. In that case one can also refer to the CSA as being one-sided.

From a modelling perspective thresholds can be relatively challenging if they can change, e.g. if a rating downgrade triggers a lower threshold.[115]

iii. *Minimum transfer amount (MTA)*: MTA is the amount which the collateral to be exchanged needs to exceed before an exchange takes place. Because collateral can only be exchanged in blocks that exceed the MTA it also resembles an uncollateralized amount.

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[113] See also Gregory (2012), pp. 68-69.

[114] See for example Cesari et al. (2009), especially pp. 190-194 or Gregory (2012), pp. 261-262.

[115] According to ISDA (2015), especially p. 22, the most popular threshold methodology is based on credit ratings.

iv. *Margining frequency*: The CSA also defines the frequency in which collateral is exchanged (e.g. daily, weekly etc.). The bigger the time difference between each collateral call the bigger the risk of the exposure changing, i.e. the bigger the risk of having undercollateralized exposure.

This risk is of course also a function of the type of derivatives traded. Derivatives with jump-like changes will have a higher probability of causing significant under collateralization as shown by Brigo et al. (2011a) for CDS, especially with high wrong-way risk. This can be seen as one factor driving the relatively high collateral coverage of credit derivatives (97% CSA) as seen in Table 14.[116]

Daily margining – which can be considered as the least risky frequency – is especially common among institutions with large OTC portfolios. This seems to remain a challenge for institutions with smaller OTC portfolios that lack the operational capabilities.[117]

Besides the contractual margining frequency one has to assess the risk that arises when the counterparty defaults due to the time it takes to liquidate the portfolio, i.e. close-out risk which as noted for example by Cesari et al. (2009, p. 195) need to be taken into account when estimating CVA. For this purpose one assumes that default can happen at any time, and adjusts the frequency in which collateral is called by a so-called margin period of risk, i.e. the expected time it takes to complete the close-out procedures after the counterparty has defaulted.[118]

Estimating the margin period of risk is not trivial and involves a number of operational and legal aspects. Among others one has to estimate the time it takes to re-valuate the positions, to receive the collateral (incl. disputes and settlement risk),

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[116] One might argue in addition that a further factor is the type of counterparty trading the different derivative types, i.e. credit derivatives are mostly traded by financial institutions etc.

[117] ISDA (2015), p. 26, reports that especially among large portfolios (greater than 5,000 trades) daily reconciliation is the norm (87.1%) and expects this to be further driven by Dodd-Frank and EMIR regulations.

[118] See for example Brigo et al. (2013a), p. 313.

and to liquidate the positions and structure new hedges.[119] Institutions can also rely on historical figures or regulatory requirements.[120]

v. *Re-hypothecation*: A further aspect that needs to be considered is whether the collateral taker has an unrestricted right to lend and sell the collateral under a "repo" or re-hypothecate it. Re-hypothecation is especially relevant for non-cash collateral agreements, introducing a new credit risk for the counterparty that is posting the collateral. After all, the collateral giver is now exposed to the risk of an additional party (e.g. buyer of collateral) defaulting. Brigo et al. (2013a, p. 335) show that re-hypothecation can have ludicrous consequences, making collateral agreements riskier than having no collateral at all.

It can be concluded that collateral agreements can be an efficient tool to mitigate counterparty credit risk. Having a collateral agreement in place does not, however, automatically imply full-collateralization. CVA pricing and counterparty credit risk management need to consider the above mentioned CSA characteristics in order to adequately estimate future exposure. Even if the CSA is structured to enable full-collateralization (i.e. cash collateral, zero MTA, zero (bilateral) thresholds with daily frequency) the remaining close-out risk implies that the counterparty credit risk is not eliminated. While on a trade level such risk might indeed be negligible this conclusion might differ on a portfolio level, especially given a large OTC portfolio with risky counterparties.

3.5.3. Mitigating CVA – Hedging Strategies

An institution can also decide to hedge its positions to mitigate counterpart credit risk and reduce the CVA volatilities inherent in its books. In the following we will touch upon the

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[119] See for example Gregory (2012), pp. 146-148.

[120] According to Gregory (2012), p. 147 (footnote 23), it took market participants around 5-10 business days to complete close-out after Lehman Brothers defaulted. Basel III defines margin period of risk up to 20 days. See for example CRV IV, Article 285.

key issues around CVA hedging, analyzing the challenges in light of the regulatory framework as well as the consequences it can have on the market as a whole.

A perfect hedge for CVA can be achieved through a single-name contingent credit default swap (C-CDS), i.e. a CDS in which the protected amount equals the (stochastic) exposure at default of a predefined counterparty. After all, CVA can be defined as the funds required to duplicate a C-CDS, i.e. CVA is the value of such a contract.[121] For this reason – as shortly discussed in Subchapter 3.5.1 – internal deals between a possible CVA desk and the different trading desks usually take the form of a C-CDS. Accordingly, a single-name C-CDS is considered an eligible credit hedge that would reduce the regulatory CVA capital charge.[122] Still, the use of C-CDS to hedge CVA externally is rather limited.[123] Gregory (2012, pp. 347-387) points out two obvious risks that C-CDS contracts might exhibit, possibly explaining their relative low popularity:

a. *Legal risk*. C-CDS contracts need to reference to all the relevant trade economics of all deals within the protected netting set (e.g. day-count-conventions, reference rates, maturity dates etc.). Otherwise the protection seller and buyer will possibly run into disputes with regards to the exposure amount at default. Notice that the legal challenges increase if the netting set is adjusted regularly, e.g. through new trades and novation of older ones. Such legal disputes might be indeed more manageable within an institution which explains the feasibility to use C-CDS internally.
b. *Joint default risk*. The protection buyer runs a possible wrong way risk if the credit risk of the protection seller and the reference entity are correlated. The quality of the credit of the protection seller needs not only to be better than the reference entity,

[121] As shown in Subchapters 3.1 and 3.3.1.

[122] See also Subchapter 3.4.3 for CVA capital charge and eligible hedges.

[123] In the survey conducted by EBA (2015) only one bank stated that it is uses C-CDS hedging. See also Deloitte and Solum (2013), p. 33.

but also to display a rather low interdependency with it. Otherwise the protection seller might default exactly when the protection buyer needs him most. This type of wrong way risk is, however, not limited to C-CDS contracts and might be also inherent in standard CDS contracts as well. We will dedicate a separate Chapter 4 to discuss wrong way risk within CDS.

In addition to the above, C-CDS contracts display an enormous practical challenge for protection sellers, because – as already noted by Cesari et al. (2009, p. 220) – a duplicating strategy might be prohibitively expensive, and the hedging instruments could be illiquid, if not inexistent. The statement might seem tautological, but it does mirror the challenges of every institution seeking to hedge its CVA risk. After all, CVA exhibits sensitivities to a very large number of factors that might also exhibit interdependencies between themselves. Using a standard taxonomy of CVA sensitivities we will discuss key aspects around respective hedging strategies:[124]

a. *Credit sensitivities*. CVA is sensitive to changes in the credit quality of a counterparty. This includes CVA sensitivity to CDS spreads (*delta*), delta sensitivity to CDS spreads (*gamma*), and CVA sensitivity to changes in the volatility of CDS spreads (*vega*).

A standard instrument to hedge these risks is a single name CDS. A perfect static hedge using single name CDS would replicate the expected exposure profile, offering a protection at each time point equaling the expected amount to be lost given a default of the counterparty. As the exposure profile might change over time such a static hedge would need rebalancing, implying respective additional costs. Alternatively one might chose to keep the approximation of the exposure profile rather gross or even decide to hedge only the average expected exposure (also referred to as expected positive exposure, EPE), leaving parts of the exposure (willingly) unhedged. Such a hedge would still reduce the P&L volatility, also offering

[124] For more comprehensive elaborations on CVA hedging we refer the reader to Gregory (2012), pp. 339-369.

a partial hedge if the counterparty defaults. As illustrated in the example given in Figure 14 below there is a payoff between an adequate credit protection and hedging costs.[125]

If a single name CDS is not available or illiquid an institution might chose to hedge the credit component using index CDS (e.g. iTraxx Europe Senior Financials). Such a proxy will however not offer protection against the idiosyncratic credit risk of the counterparty. It will offer a sensitivity hedge if the credit risk of the counterparty and the index at hand display a high correlation. It does not, however, offer any protection if the counterparty *does* default (if the index does not include a CDS referencing to the specific counterparty).

From a regulatory perspective both single name CDS and index CDS are considered eligible hedges. Having an eligible hedge in place reduces the exposure, subsequently decreasing the CVA capital charge both under the standard and advanced methods as seen in Equations (3.33) and (3.34), respectively. Notice, however, that both the advanced and the standard method use rather prudent exposure methods as has been discussed in Subchapter 3.4.4 above. This leads to a structural misalignment between the regulatory CVA capital charge on one hand and with how CVA is computed for financial accounting on the other. From a hedging perspective this implies a further payoff that needs to be considered as both metrics demand differing hedging strategies. For example – as has been described above – the standard approach is based on a prudent exposure measure, exposure at default (EAD) which demands an economic over-hedge in order to eliminate the CVA capital charge completely.

Evidence indicates that hedging CVA risk with CDS might lead to unintended negative consequences.[126] An increased use of CDS – especially in times of crises – might cause a widening of CDS spreads. From the perspective of the institutions

[125] For other possibilities to determine an optimal static hedge refer also to Cesari et al. (2009), pp. 220-221.
[126] See also Carver (2011), Carver (2013) or Gregory (2012), pp. 360-361.

seeking to hedge their risk a widening of CDS spreads actually increases the level of CVA, possibly triggering a loop of higher demands to hedge CVA and higher hedging costs. From the perspective of debtors, widening credit spreads imply higher debt issuance costs that in time of crisis might even be prohibitively high.[127] Avoiding the negative consequences of such pro-cyclicality might have been one of the reasons the European implementation of Basel III (CRD IV) exempts transactions with sovereign counterparties from the CVA capital charge.

In addition, buying CDS to hedge counterparty credit will lead to an increase of "naked" CDS positions in the market, i.e. situations in which the protection buyer does not hold the original risk in form of debt securities (e.g. bonds) in his portfolio. "Naked" positions are associated with pure speculation, and claims to ban them found resonance in respective legislations, e.g. in Germany and in the EU.[128] This means that an investor interested in hedging the counterparty credit risk inherent in a contract with an Italian sovereign will need to have an Italian government bond in his portfolio. While the ban does seem to follow intended market regulation it does contradict the Basel III accord which incentives the use of CDS for hedging derivative (not only bond) exposure.

b. *Exposure sensitivities.* CVA is sensitive to the same factors that underlie the counterparty credit risk-free value of the derivative. Given changes in the underlying market factors the exposure profile will change, triggering a change in the CVA metric as well. For example for the CVA metric of a standard interest rate swap the respective sensitivities would include one to interest rate changes (*delta*) and a delta sensitivity to interest rate changes (*gamma*) as well as sensitivity to changes in the

[127] Bilal and Singh (2012) look at the link between CVA hedging and sovereign debt issuance in the European "periphery" (e.g. Greece, Italy etc.).

[128] See for example Augustin (2014).

volatility of interest rates (*vega*).[129] In that case the expected exposure profile can be duplicated by a series of swaptions as shown by Sorensen and Bollier (1994).

A netting set would, however, usually include differing swaps (with differing reference rates) as well as other derivative products (e.g. FX options, FX swaps etc.). This sheds light on the complexity of deriving all relevant CVA sensitivities, including the cross-gamma sensitivities between the different market factors (e.g. correlation between three-month Euribor and USD/EUR exchange rate). Let alone finding liquid hedging instruments. Theoretically such hedges can also increase the overall CVA risk, depending on the quality of the counterparty with which they are traded, especially if no collateral agreements are in place.

Whilst hedging against exposure sensitivity might reduce the CVA volatility, banks are disincentivized to do so from a regulatory perspective. Hedges aiming to reduce the exposure sensitivity are not considered as eligible, neither under the standard nor under the advanced CVA capital charge approach. As seen in Equations (3.33) and (3.34) none of the approaches is able to capture such hedges as mitigants. Interestingly larger banks report that they do hedge for exposure sensitivity[130], aiming on reducing the P&L volatility, accepting the regulatory disadvantages.

c. *Cross gammas*. CVA displays a sensitivity to the interdependency between the exposure and the credit quality of the counterparty, also referred to as cross gamma. This brings us back to the concept of wrong way risk (and right way risk). As will be discussed in the next Chapter 4 quantifying wrong way risk is rather complicated. Let alone finding appropriate instruments to hedge it. As noted by Gregory (2012, p.

[129] For delta hedging of interest rate risk see also Chapter 2 of this thesis.
[130] See for example EBA (2015), pp. 88-91.

360) in most cases hedging wrong way risk will only be partial at best. In specific cases strategies can be constructed.[131]

Cesari et al. (2009, p. 223) argue that the difficulties around hedging wrong way risk are mitigated somewhat by the fact that the cross gamma is rarely the key risk driver. On the other hand Brigo and Capponi (2010) or Hull and White (2011) discuss cases in which wrong way risk *is* a key driver behind CVA risk.[132]

d. *Own credit*. Bilateral CVA exhibits a sensitivity to changes in the credit quality of the institution, i.e. DVA. We have discussed the risks and challenges around hedging DVA in Subchapter 3.4.2 above.

It can be concluded that while hedging can significantly help in mitigating CVA risk, an institution cannot rely on it solely due to a variety of significant challenges. CVA of a given netting set might exhibit sensitivities to a very large number of factors, incl. cross-gammas. This implies technical challenges in measuring sensitivities that are accompanied with significant model risks (e.g. measuring right way risk). Even if the sensitivities are measured adequate hedging instruments with sufficient liquidity need to be available (e.g. single name CDS). Hedging CVA has thus also to rely on proxies and alternative approaches. An adequately manned CVA desk becomes crucial in finding such solutions. Consider for example a bank that is interested in reducing its (net) exposure to a peer and is not able to find adequate hedging instruments. CVA managers will need to "think out-of-the-box" in order to find alternative solutions, e.g. borrowing money (e.g. through a loan) from the other bank as it functions similarly to collateral etc.

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[131] Gregory (2012), p. 360 gives the example of an institution trading a cross currency swap with a sovereign. While the institution pays USD the sovereign pays in the local currency of the institution. By buying a local currency CDS and selling a USD CDS the institution ensures that the FX rate at default will be hedged.

[132] See also Gregory (2012), pp. 358-359.

In addition, banks need to consider the structural misalignment between regulatory CVA capital charge on the one hand and accounting standards on the other. The challenges will differ from one institution to the other, highly depending on the OTC portfolio at hand as well as the overall trading strategy of the bank.

For some institutions CVA hedging might indeed be unfeasible. Consider for example a locally active tier 2 bank, offering OTC derivatives to its clients (mostly mid-sized local companies with no liquid single name CDS). Most of the client OTC book is left uncollateralized so that the counterparty credit risk is not insignificant. The bank hedges its market risk positions (e.g. interest rate risk) through respective transactions with other banks (usually a handful of tier 1 banks with bigger portfolios). These transactions are adequately collateralized (with daily margining, no threshold etc.) so that the counterparty credit risk is rather limited, and no hedging is needed. Still, the bank can be interested in hedging the CVA risk inherent in its client book. With respect to credit sensitivity the bank can only hedge some generic or sector risk, based on index CDS (as no single name CDSs are available). There is, however, a good possibility that none of its counterparties can be economically mapped to a respective index. This leaves the bank only with the possibility of hedging the market risk component of CVA. Such a strategy might seem expensive if it only reduces the CVA volatility, having no mitigating effect on the regulatory capital requirements.

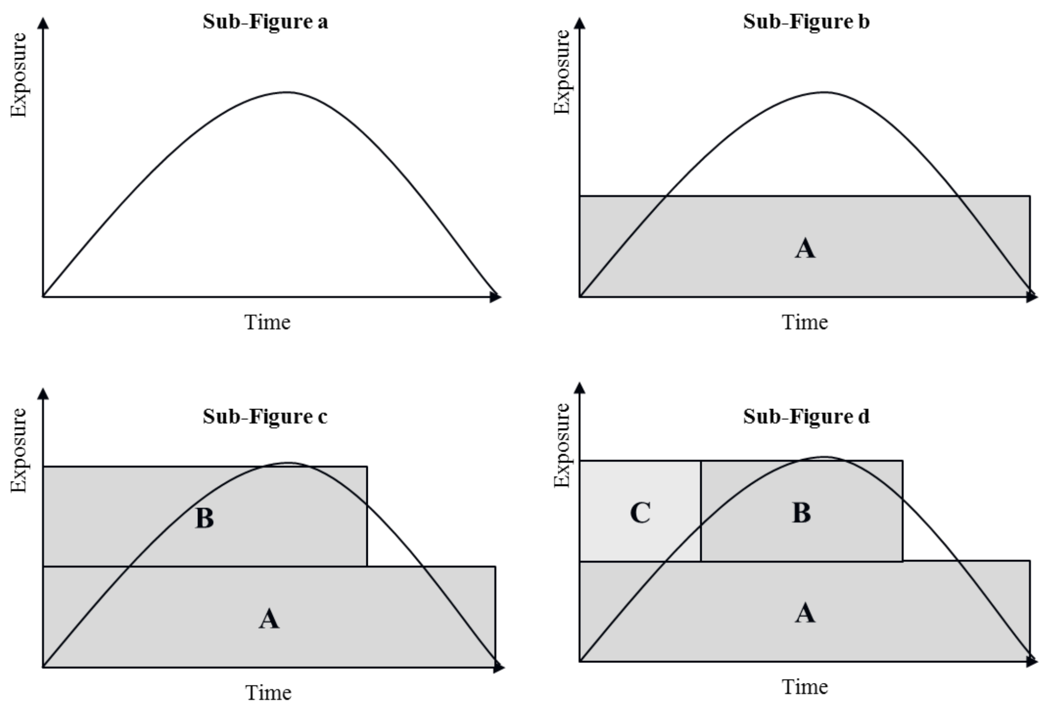

Figure 14: Practical Hedging Strategies for CVA Credit Sensitivity

Subfigure (a) illustrates an exemplary uncollateralized expected exposure profile (e.g. of an interest rate swap book) following Bree et al. (2011). Subfigure (b) illustrates the case of hedging the average expected exposure profile, also referred to as expected positive exposure (EPE). The shaded square "A" stands for the CDS contract with a notional amount equaling the EPE, and a maturity equaling the maturity of the netting set. It becomes clear that at the begging and at the end of the life time of the netting set the CDS contract exhibits an over-hedge while in between a substantial exposure is left not hedged for. A better proxy for the exposure profile is possible through buying one further CDS contract "B" and selling another "C", as seen in Sub-Figure (c) and (d), respectively. It becomes clear that the better the approximation of the exposure profile is, the more expensive the hedge strategy is. This is crucial because the expected exposure profile might change, demanding rebalancing of the CDS positions as well.

3.5.4. Evaluation Summary

The level of sophistication but also the mere feasibility of managing counterparty credit risk and CVA depends highly on the institution at hand, especially the OTC derivatives strategy it follows and the respective financial and technical capabilities it aspires. The bigger and more complicated the OTC portfolio, the higher the inherent counterparty credit risk possibly is. This means also that it becomes more worthwhile to invest in technical and management infrastructures (e.g. CVA computation engine, CVA desk etc.) on the one hand. On the other hand it also implies a widening of the spectrum of possible mitigation strategies, e.g. introducing the possibility of hedging credit risk sensitivity due to a higher probability of having counterparties with liquid CDS.

Still, counterparty credit risk and CVA aspects, incl. pricing and management issues are relevant for every institution trading OTC derivatives, independent of its portfolio size and complexity. This becomes apparent if regulatory requirements and accounting standards are considered, especially the ones affecting financial institutions. As the below given example of Bayer indicates, having elaborated capabilities to measure and manage counterpart credit risk and CVA is not limited to banks. Head of corporate financial controlling at Bayer is cited by a press release of a company offering IT solutions for counterparty credit risk and CVA, saying:

> *"To help mitigate risks, Bayer decided to utilize the same risk management methodologies used by some of the largest and most advanced global banks. [The IT solutions] enable Bayer to determine, monitor and steer our counterparty risk accurately and consistently across our counterparties. With the automated […] solution in particular, we are now able to calculate*

> *Monte Carlo-based CVA and DVA and attribute the results accurately to the trade level, even within netting sets, helping us to comply with regulations such as IFRS 13.*[133]

Beyond complying with rules and regulations every institution trading OTC derivatives should be aware of counterparty credit risk and CVA simply because if it does not, it might get "punished" by the market. After all, many of the topics we have discussed in the previous subchapters indicate much opaqueness around pricing CVA. This usually benefits the more sophisticated and capable counterparties. By having minimal capabilities to price and manage – or just to comprehend – CVA issues the less sophisticated and capable counterparties limit their vulnerability. If for example a larger institution (e.g. tier 1 investment bank) approaches a (buy-side) client (e.g. health care company), and offers to amend existing collateral agreements, e.g. lowering the thresholds. The client is well advised to investigate the effects such an amendment would have on the expected exposure profile, and thus CVA (and DVA), before agreeing.

A key takeaway from our discussion around CVA is that it reveals the limits of risk-neutral valuation, especially when it comes to practical implementation of pricing models or constructing adequate hedges. Clinging to use market implied parameters or elaborate models that need overcomplicated calibration (e.g. first-to-default CVA models) without reflecting on their economic sense might not only imply mere model and valuation risks, but also significant financial risks at the latest when it comes to hedging CVA. While elaborate and theory-based modeling is crucial to allow for intersubjective discussions around factors driving CVA and subsequent consequences, every CVA manager will need to tailor his management tools to his portfolio, focusing mainly on not being "arbitraged".

[133] See Sungard (2013). In the press release Bayer is defined as "[…] a global enterprise with core competencies in the fields of health care, agriculture and high-tech materials."

CHAPTER 4:
PRICING CREDIT DEFAULT SWAPS WITH WRONG WAY RISK – MODEL IMPLEMENTATION AND COMPUTATIONAL TUNE UP

4.1. INTRODUCTION AND LITERATURE OVERVIEW

One possibility for investors to mitigate counterparty credit risk is to buy protection in form of a credit default swap (CDS) as discussed in the previous Chapter 3, esp. Subchapter 3.5.3. In return for a premium, investors buying CDS contracts, receive protection for a certain notional amount of debt of a predefined reference entity. CDS contracts can thus be used to hedge the risk of the counterparty defaulting or its credit quality deteriorating.

We have, however, also extensively discussed the fact that *all* over-the-counter (OTC) derivatives exhibit counterparty credit risk, and there is no reason why CDS contracts should be an exception.[134] After all, protection buyers now have an exposure to a new counterparty with non-zero probability of default. Protection buyers can still benefit if the credit quality of the protection seller is superior to that of the reference entity, especially if both counterparties agree on posting collateral. If, however, the credit quality of both protection seller and reference entity are positively interdependent, such a "protection" becomes questionable if not worthless, since the protection seller might default exactly when needed most.

This inherent wrong-way risk and the systemic risk accompanied by it are key reasons why CDS contracts have been identified by some as contributors to the financial crisis, analyzed for example by Mirochnik (2010) or Stulz (2010).[135] In his article "Buffet's 'time bomb' goes

[134] For the definition of counterparty credit risk see for example Section 3.2 above.

[135] Note that both cited papers do not use the term wrong way risk, which originates from counterparty credit risk literature, e.g. Cesari et al. (2009).

off on Wall Street" Kelleher (2008) cites an interviewee, explaining the reasons behind the crisis as follows:

> *"This was supposedly a way to hedge risk,"* [...] *"I'm sure their predictive models* [financial institutions] *were right as far as the risk of the things they were insuring against. But what they didn't factor in was the risk that the sellers of this protection wouldn't pay... That's what we're seeing now."*

Although the opinion of the interviewee around the predictive power of financial modeling seems inappropriately flattering, the comment around the lack of consideration of possible wrong way risk does hit the spot, especially when it comes to the fad of buying credit protection against structured securities as collateralized debt obligations (CDOs) from monoline insurers, i.e. insurance companies that traditionally provide coverage for one insurance line.[136]

Monoline insurers were common underwriters of such CDSs in the years leading to the subprime and financial crisis.[137] According to ISDA (2011) by 2008 monoline insurers wrote protections "*on tens of billions of dollars*" on CDOs on residential mortgage-backed securities (RMBS), primarily referencing to subprime loans. This made them extremely sensitive to a deterioration in the mortgage market. Still, monoline insurers had top ratings (mostly AAA), granting guaranteed structured notes an almost credit-risk-free status even though monoline insurers successfully resisted collateralization.[138]

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[136] Much of the information given in this Subchapter is backed by evidence from ISDA (2011). See also Brunnermeier (2009) for a qualitative analysis of the situation that led to the financial crisis also elaborating on the role of monoline insurance companies. Notice that as discussed in Jaffee (2006) monoline insurers focus on one product line due to legal restrictions. This means, paradoxically, that the significant concentration risk borne by monoliners can be attributed to regulation.

[137] For more on the subprime crises and asset-backed securities see for example Longstaff (2010).

[138] See ISDA (2011) according to which some monoliners did agree on posting collateral if downgraded.

Amid the plummeting of market prices for CDOs on RMBS in 2007 the investment community seemed to have finally found the catch. Rating agencies started downgrading monoline insurers while others had to file for bankruptcy. Market prices started accounting for significant credit valuation adjustments (CVA) due to the deteriorating of the credit quality of both reference entity and the protection seller. The significant CVA figures that went through the profit and loss statements of the banks increased the demand for modeling approaches that incorporate wrong way risk.[139] Banks, hedge funds, monoline insurers, asset managers, auditors, regulators – to name a few – were in the need for such models to compute CVA consistently with the new prevailing view. In the following we will go through some of the most prominent examples that tried to deliver appropriate solutions, motivating our focus on Brigo and Capponi (2010).

In the spirits of Merton's structural approach Blanchet-Scalliet and Patras (2008) derive closed-form solutions for counterparty credit on a CDS. Yet, due to the focus on analytically solvable dynamics their model cannot be fit to CDS spread quotes. Lipton and Sepp (2009) present a multi-dimensional jump-diffusion structural model that can be calibrated to current market information. The model demonstrates the significance of default correlation and credit spread volatility while pricing CVA on CDS.

Leung and Kwok (2005) examine bilateral counterparty credit risk using a reduced form framework with deterministic functions and contagious defaults. Based on the shifted squared root (jump) diffusion (SSRJD) – previously introduced in Brigo and Alfonsi (2005) – Brigo and Chourdakis (2009) introduce a unilateral CVA model with stochastic credit spreads. They adopt stochastic intensity models for the default events, connected via a Gaussian copula function. Brigo and Capponi (2010) generalize this approach. Assuming all three entities are subject to interdependent default risk, they explicitly model what we have

[139] In its survey of 12 international investment banks ISDA (2011) attributed CVA charges of more than $ 50 billion to monoline exposure.

referred to as first-to-default CVA (FTDCVA).[140] This means that the model can be used to analyze wrong-way risk as well as the presumable inconsistencies around pricing own credit risk, i.e. DVA. We have therefore chosen to revisit this model and analyze it in detail.

Further model extensions were offered for example by Assefa et al. (2009) who introduce netting and collateral agreements. Using a Markov chain copula Crepey et al. (2010) model wrong way risk with possible simultaneous defaults. Li (2009) examine the effect of stochastic recovery on CVA of CDS.

As mentioned, the following Chapter 4 is dedicated to analyze the approach offered by Brigo and Capponi (2010). When appropriate we will also refer to the more comprehensive elaborations and calculations given in Brigo and Capponi (2009) or Capponi (2009). We introduce a step-by-step implementation guide and offer a computation tune up for the most elaborate part of the computation, i.e. generation of the default probability structure of the reference entity, conditional on the default of either the investor or the counterparty. For this purpose Brigo and Capponi (2010) suggest the use of the Fractional Fourier Transformation (FRFT) technique. The heuristic approach we suggest is based on an *easy-to-implement* lognormal approximation. Throughout a variety of examples we show that this approximation delivers rather robust and satisfying results, while requiring less computational power and less excessive implementation. A case study based on real market data demonstrates the use of the model and the insights it delivers. In addition, the codes (written in R) used for the main functions of the algorithm are given in Appendix D.

The descriptive part is followed by a comprehensive critical evaluation of the Brigo and Capponi (2010) approach in particular, and CVA modeling in general. We analyze both the capability of the model in delivering an arbitrage-free framework as well as in its use for inter- and intra-organizational communication. A specific feature of interest in this respect is

[140] See also Subchapter 3.2 above.

the use of a copula function to model interdependency, a formula accused of "killing Wall Street".[141] Besides revealing key challenges, risks and limitations of the model, the analysis will also point to possible benefits and insights offered by such approaches.

The remaining of Chapter 4 is structured as follows. In Subchapter 4.2 we start with a short introduction to CDS contracts, followed by an overview on structural and reduced form credit modeling, explicitly elaborating on their use also for CVA modeling, given in Subchapter 4.2.2. Subchapter 4.2.3 is dedicated to the use of (Gaussian) copula functions to model default dependency. A detailed description of the model is given in Section 4.3. While Subchapter 4.3.1 revisits the first-to-default CVA for CDS, Subchapter 4.3.2 gives an overview of the algorithm. Subchapter 4.3.3. discusses the approach used to model interdependent defaults, e.g. stochastic intensity approach in conjunction with a Gaussian copula. In Subchapter 4.3.4 we revisit the computation of the conditional value of the CDS given a default. We give a detailed guide for estimating the needed partial derivatives of the Gaussian copula as well as the cumulative distribution function (CDF) using the method suggested by Brigo and Capponi (2010), i.e. fractional Fourier transformation (FRFT) technique. In the subsequent Subchapter 4.3.5 we offer a heuristic approach based on a log-normal approximation of the CDF, reducing the complicity of the implementation and speeding up the computation. Numerical examples are delivered both for understanding the mechanism of the model in general and to back our heuristic approach in particular. The case study given in Section 4.4 returns to our introductory example of monoline insurers selling protection on structured notes. An evaluation of the approach is discussed in Section 4.5. We conclude the chapter in Section 4.6.

Notice that the Chapter is an adaptation of previous work of the author published in Grominski et al. (2012), and that some elaborations, especially in the computational part, are

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[141] See for example MacKenzie and Spears (2014).

identical. In order to avoid redundancy we will, however, refrain from continuously referring to Grominski et al. (2012).

4.2. PRELIMINARIES AND DEFINITIONS

4.2.1. Credit Default Swaps

A credit default swap (CDS) transfers credit risk from a protection buyer to a protection seller.[142] One can differentiate between a single name CDS and an index CDS. While single name CDSs offer a protection of a certain notional amount of debt of *one* specified reference entity, index CDSs are linked (obviously) to an index with *numerous* entities. We will in the following focus on single name CDS, and will henceforth omit "single name" for simplicity reasons.

In a CDS contract the protection buyer pays an upfront or periodic premium. In return the protection seller grants the buyer a settlement payment based on predefined procedures if the reference entity defaults during the life time of the contract. The documentation of a CDS will thus include a reference debt (e.g. a specific bond) and a reference entity (e.g. a legal entity such as a corporate, a special purpose vehicle (SPV) or a sovereign). In addition, the documentation will need to define relevant credit events that trigger a default, e.g. bankruptcy of obligor, distressed restructuring, delinquent payments (90 days past due).

[142] The following summary on CDS is largely inspired by the more comprehensive elaboration given Augustin (2014) and Gregory (2012), especially pp. 211-224, to which we refer the interested reader.

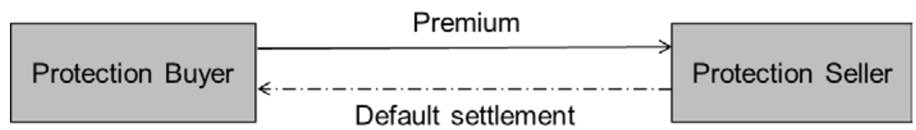

Figure 15: Visual Illustration of a Single Name CDS Contract

The protection buyer pays the protection seller a premium (upfront and/or periodically) in exchange for a settlement payment due when the reference entity suffers a credit event (e.g. default).

Given a credit event a CDS may have a "physically" or a "cash" settlement. In the following we will describe the difference between both possibilities, exploring the advantages and limitations:

- *Physical settlement*. One speaks of a physical settlement when the protection buyer transforms debt securities issued by the reference entity (with notional amount equal to the protection amount) to the protection seller. The protection seller compensated the buyer with the full notional amount in cash. The method seems rather straightforward at first sight, limiting the need to involve third parties and to use elaborate evaluation schemes. It does however introduce a "cheapest-to-deliver" risk for the protection seller. Given a physical settlement the protection buyer will chose to transform the cheapest bond available, e.g. the most illiquid. As a consequence CDS contracts might restrict the use of securities that can be transformed given a restructuring event, e.g. "Modified Restructuring" omits the possibility of delivering securities with a remaining maturity exceeding 30 months.

  In addition it also increases the risk of "delivery squeeze", leading to an increase in the prices of the reference bonds. Physical settlement meets it limitations especially when the protection amount sold in the market is larger than the notional amount of traded debt (due to "naked" CDS positions, see also Subchapter 3.5.3). Gupta and Sundaram (2012) refer to Summe and Mengle (2006) who for example report that when Delphi Corporation filed bankruptcy in 2005 the amount protected was estimated to be around $28 billion while the notional amount in deliverable bonds was merely $2 billion.

- *Cash settlement.* As no securities are delivered under a cash settlement the protection seller will only need to compensate the protection buyer for the incurred loss, i.e. the difference between the par value and the recovery value. While introducing a possible solution for the delivery squeeze, cash settlement imply a process for determining the recovery value such as auctions.[143]

The use of credit default swap (CDS) is controversial at best, as analyzed for example by Stulz (2010). While praised as innovative solutions that transform risk to those best suitable to bear it, they have also been associated with the financial crisis, e.g. due to "naked" CDS positions taken for pure speculation. Prominent figures such as the investor Warren Buffer have referred to them as "time bombs" and others such as hedge fund manager George Soros pleaded to ban them completely.[144] In order to limit the effect of speculation on CDS and the associated cost of issuing debt, the EU banned the use of "naked" CDS for respective sovereign reference entities (see also Subsection 3.5.3). In the meanwhile the regulatory framework Basel III recognizes CDS as eligible collateral. Basel III as well as international accounting standards demand the use CDS spreads for counterparty credit risk and CVA measurement (see also Section 3.4).

One might be aware of the possible limitations and market distortions, CDS contracts cause, and still be interested in using the information they imply. Especially from a no-arbitrage theory perspective CDS (spreads) are crucial as they determine respective prices and conditions under which one excludes being arbitraged, agnostic of whether the market turns out to be "mispriced". In the subsequent Subchapter we will revisit two approaches for

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[143] Gupta and Sundaram (2012) analyze the CDS auctions and possible informative biases. They also describe in detail the auction process designed by ISDA and auction administrators CreditEx and Markit.

[144] See Berkshire Hathaway (2002) and Soros (2009).

modelling credit, elaborating on the use CDS spreads to generate risk-neutral probabilities of default.

4.2.2. Modeling Credit Risk – Structural vs. Reduced Form Models

Two broad approaches have emerged in literature to model credit risk: structural (or firm-value) models, based on the work of Black and Scholes (1973) and Merton (1974), and reduced form – also referred to as intensity or hazard rate models given a suitable context – models that go back to Jarrow and Turnbull (1992) and Duffie and Singleton (1999). In the following we will give a short overview of the "*two competing methodologies*" (Bielecki and Rutkowski 2001, p. 26), linking both to CVA literature and the explored model approach introduced by Brigo and Capponi (2010). The description given below is inspired by Bielecki and Rutkowski (2001), Brigo et al. (2013a, esp. pp. 48-86) and Jarrow et al. (2003) to which we refer the interested reader for more comprehensive elaborations.

Structural models

Structural models relate default to an economic variable (e.g. firm value). A default is triggered when the economic variable crosses a pre-defined threshold (or barrier). This methodology goes back to the model introduced by Merton (1974) in which firm assets follow a lognormal distribution and the firm is assumed to be financed both by equity and debt (in form of a finite zero coupon bond). The model postulates that the firm defaults if the notional amount of its debt exceeds the value of its assets (only) at the time of maturity. The equity value can thus be viewed as a European call option on all assets of the firm. The option matures with the zero bond and the strike price equals its face value. The Merton approach shows two kern elements of this category of models: a) Key issue within this methodology are modelling assumptions around the evolution of the firm's value and the firm's capital structure. For this reason they are referred to as "structural" or "firm value" models, and b) Structural models usually base these assertions on option pricing models, using these also to calibrate probabilities of default.

Black and Cox (1976) offered a notable extension to the Merton approach by introducing continuous default barrier, i.e. relaxing the assumption that default can happen only at debt maturity. Many extensions and generalizations have been offered. Prominent examples that introduced stochastic interest rates were Shimko et al. (1993), Nielsen et al. (1993) and Longstaff and Schwartz (1995). More complex debt structures were modelled for example by Leland (1998) and Tauren (1999).

The implied default probability of these models for a company with a high credit quality in the short-term is close to zero, especially contrasting high near term CDS spreads. For this purpose further extensions were introduced for example by Hull and White (2001), incorporating curvilinear barrier. Jumps in a firm's value were introduced for example by Lipton (2002) and Sepp (2004, 2006).

Structural models are seen as elegant, because they are motivated with economic reason, allowing investors' expectations of firm's future performance to be considered. Moreover, structural models can be seen as forward looking with evidence of explanatory power as for example discussed in Arora et al. (2005). In addition, they imply an arbitrage relationship between equity and debt which can be informative when it comes to analyze relative pricing of respective financial instruments. Indeed structural models have found their way into the center of "*mainstream*" risk management. As discussed in Jarrow and Turnbull (2000) they lay the foundations for prominent risk management services such as Moody's KMV and CreditMetrics. More importantly – through the asymptotic single risk factor (ASRF) approach discussed in Subchapter 3.4.3 – they display the backbone of the Basel regulation framework, clearly influencing measurement and capitalization of credit risk.

As noted above a variety of CVA structural models, considering wrong-way risk have been introduced, e.g. Blanchet-Scalliet and Patras (2008), Lipton and Sepp (2009) and Lipton and Savenscu (2013).

As shown by Jarrow and Protter (2004) structural approaches have a restrictive assumption, implying that modelers have "*continuous and detailed information about all of the firm's assets and liabilities*." More importantly they show that the modelers and firm's managers (and regulators in the case of commercial banks) would all have to hold the complete information about default time and expected recovery, meaning also that the default time is *always* predictable. Moreover, from a pricing perspective structural models are still regarded as limited, because they generally provide a poor fit of market information, particularly underpricing short term securities as for example discussed in Jarrow et al. (2003) or Capponi (2009). Moreover, as shown for example by Brigo et al. (2013a, p. 65 or p. 80) structural models can be challenging when it comes to practical calibration, especially when the credit risk of more than three entities is involved.

Reduced form models

In contrary to structural models reduced form models do not proclaim any economic rationale behind default which is not triggered by observable variables but is given through an exogenously driven jump process, i.e. a Poisson process with stochastic or deterministic intensity. Modelling default through an exogenous process frees reduced form approaches also from assumptions around firm's assets and its capital structure. It also implies that default becomes unpredictable.

Main motivation behind intensity models is their suitability to model credit spreads and the easiness in which they can be calibrated to CDS quotes. Prominent examples of reduced form models were given by Jarrow and Turnbull (1992), Madan and Unal (1998), Duffie and Singleton (1999), Hull and White (2001) as well as Brigo and Alfonsi (2005). This list is not exhaustive and can be easily extended.

Following the notations given by Brigo et al. (2013a, esp. pp. 66-77) we will reintroduce the main assumptions and features behind reduced form models. For this purpose we assume that default time τ is the first jump of Poisson process. Given no default has occurred yet, the (risk-neutral) probability $\mathbb{Q}$ of defaulting in the next dt is

$$\mathbb{Q}(\tau \in [t, t+dt] | \tau > t, \text{market info up to } t) = \lambda(t)dt. \tag{4.1}$$

with $\lambda(t)dt$ standing for intensity or hazard rate which – for simplicity reasons - is assumed to be strictly positive. It can be seen as an ad-hoc combination of market and financial variables. The intensity function can be fit to market data (e.g. CDS spreads), and is the exogenous force related to the dynamics of the firm at hand. The cumulative intensity function Λ is given as

$$\Lambda(t) := \int_0^t \lambda(u)du. \tag{4.2}$$

Recall that the transformation of the jump time τ of a Poisson process according to its own cumulated intensity Λ gives an exponential random variable, i.e.

$$\Lambda(\tau) =: \xi \backsim \textbf{standard exponential random variable with mean 1} \tag{4.3}$$

with ξ being independent of all other variables, e.g. interest rates, equities, and other (stochastic) intensities etc. Notice that exactly this independence and stochasticity of ξ are what allow reduced form models to assume incomplete markets, produce unpredictable defaults and allow for instantaneous credit spreads to be different from zero.

Inverting the Equation (4.3) leads to

$$\tau =: \Lambda^{-1}(\xi) \tag{4.4}$$

which illustrates why ξ are also referred to as *default triggers*. The survival probability at time period t is given as follows

$$\mathbb{Q}\{\tau > t\} = \mathbb{Q}\{\Lambda(\tau) > \Lambda(t)\} = \mathbb{Q}\{\xi > \Lambda(t)\} = \mathbb{E}^{\mathbb{Q}}\left\{e^{-\int_0^t \lambda(u)du}\right\} \tag{4.5}$$

because the cumulative distribution function (CDF) of a standard exponential random variable is given by $\mathbb{Q}\{\xi \geq \text{x}\} = e^{-x}$.

Looking at Equation (4.5) it becomes clear that the survival probability is just the price of a zero coupon bond while the stochastic intensity process replaces the stochastic interest rate process, usually noted with r. This allows us to interpret intensities as instantaneous credit spreads. Now assuming that the intensity λ is constant then pricing formula for a bond is simplified to

$$P(t,T) = e^{-(r+\lambda)(T-t)}, \tag{4.6}$$

displaying the similarity between the intensity (or hazard rate) and the credit spread more clearly. Analogously practitioners make an intensive use of the following formula

$$\lambda = \frac{R_{0.b}^{MID}(0)}{LGD} \tag{4.7}$$

with $R_{0.b}^{MID}(0)$ standing for the mid-market spread of a CDS, running from time period 0 to b. Despite the fact that this formula is an approximation at best it is popular among practitioners, especially for quick plausibility checks. After all, it is very simple as it does not require any extensive calculation or assumptions, e.g. regarding interest rates dynamics. Moreover, it intuitively relates CDS spreads to probabilities of default.[145]

It is of course more realistic to assume credit spreads – and thus intensities – to be volatile. As noted survival probabilities can be interpreted as zero coupon bond prices. This implies that stochastic interest rate models can be used to model intensity. Recall that the intensity process is strictly positive, excluding some interest rate models (e.g. Gaussian models). Flexibility and analytical tractability makes Cox Ingersoll Ross (CIR) processes especially appealing in this case. Their suitability is studied for example by Brigo and Alfonsi (2005), Brigo and Cousot (2006) or Brigo and El-Bachir (2008). Also Brigo and Capponi (2009, 2010) model intensity using a shifted CIR process (i.e. with a drift term to ensure an exact fit of the term structure) as will be elaborated in more detail below. In principle, the CIR

[145] For the proof of this equation we refer the interest reader to Brigo et al. (2013a), pp. 70-71.

process they use incorporates an additional jump term that Brigo and Capponi (2010), however, omit in the implementation part of their paper.

The fact that intensity models lack an economic rationale can be seen as a weakness. Moreover, Arora et al. (2005) argue that the flexibility of intensity models to fit observable data makes them more prone to focus on in-sample fitting properties while displaying poor out-of-sample predictive ability. While this argument might be worth analyzing from a credit risk management or rating perspective, it is rather irrelevant for arbitrage-free pricing. In contrary, the intuitiveness and tractability in which reduced form approaches presumably depict market implied information makes it worthwhile to analyze their application to pricing, incl. arbitrage free CVA modeling.

4.2.3. Copula Functions – Modeling Multiname Defaults

Modeling default dependency is central when it comes to bilateral CVA, especially in the case of CDS with wrong way risk. For this purpose reduced form approaches are usually enriched with copula functions.[146] A prominent examples is given by Schönbucher and Schubert (2000).[147] Copulas introduce a very general manner to model dependence of random variables. Theoretically there is an infinite number of possible copula functions. Due to the scarcity of data on default interdependence Schönbucher (2005), however, advices to use more convenient low-parametric families of copula functions. One such function is the Gaussian copula used in the approach of Brigo and Capponi (2010). Other examples include the t-copula and the Archimedean copulae.[148]

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[146] See also Brigo et al. (2013a), especially pp. 78-86, that have inspired the elaboration on Copula functions. For alternative approaches to model correlated defaults (e.g. correlated intensity functions) see also Duffie and Singleton (2003), especially pp. 229-249.

[147] The use of copulas to model dependency in general is attributed to Li (2000) as well as Frey and McNeil (2003).

[148] See also McNeil et al. (2005), especially pp. 184-237.

In the following, we provide a definition for copula functions in general and the Gaussian copula in particular. In addition, we provide the procedure for generating triggers of default on the basis of a Gaussian copula.

Following Schönbucher (2005), a copula function can be defined as follows

Definition 1 *A function $C : [0,1]^I \rightarrow [0,1]$ is a copula if there are uniform random variables $U_1, \dots, U_I$ with values in $[0,1]$ while C is their joint distribution function. C has uniform marginal distributions, meaning for all $i \leq I$; $u_i \in [0,1]$ there is*

$$C(1, \dots, 1, u_i, 1, \dots, 1) = u_i. \tag{4.8}$$

According to Sklar's Theorem as given below, any multivariate distribution function F can defined as a copula.

Theorem 1 (Sklar) *$X_1, \dots X_I$ are random variables with the following marginal distribution functions $F_1, F_2, \dots, F_I$. If their joint distribution function is F, then there exists a copula (I-dimensional) C such that for all $(x_1, \dots, x_I) \in \mathbb{R}$:*

$$F(x_1, \dots, x_I) = C\big(F_1(x_1), F_2(x_2), \dots, F_I(x_I)\big). \tag{4.9}$$

This implies that C is the distribution function of $\big(F_1(x_1), F_2(x_2), \dots, F_I(x_I)\big)$. Whenever $F_1, F_2, \dots, F_I$ are continuous C will be unique. If not, C will be uniquely determined on $RanF_1 \times \cdots \times RanF_I$. Whereas $RanF_i$ stands for the range of $F_I (i = 1, \dots, I)$.[149]

Definition 2 (Gaussian copula) *Let $X_1, \dots X_I$ be random normal distributed variables with the means $\mu_1, \dots \mu_I$, the standard deviations $\sigma_1, \dots \sigma_I$ and the correlation matrix R. By definition the distribution function $C_R(u_0, u_1, u_2)$ of the standard normal distributed variables U_i is a so-called Gaussian copula. Whereas U_i are*

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[149] See Schönbucher (2005), pp. 326-333.

$$U_i = \Phi\left(\frac{X_1 - \mu_1}{\sigma_I}\right), i \leq I \tag{4.10}$$

and

$$C_R(u_0, u_1, u_2) = \mathrm{Q}(U_0 \leq u_0, U_1 \leq u_1, U_2 \leq u_2) \tag{4.11}$$

where $\Phi(\cdot)$ *stands for the cumulative univariate standard normal distribution function.*

In the following we give a procedure for generating triggers of default for three parties on the basis of a Gaussian copula:

1. Generate a matrix Z^m, containing of three vectors each with n independent standard normal distributed variables $Z_1^v = (z_1^1, \dots, z_n^1), Z_2^v = (z_1^2, \dots, z_n^2)$ and $Z_3^v = (z_1^3, \dots, z_n^3)$
2. Define a matrix $A^m = Chol(R)^{-1} \cdot Z^m$, with $Chol(R)^{-1}$ standing for the Cholesky decomposition of the correlation matrix between the independent uniforms R
3. Define $U^m = \Phi(A^m)$ (item-wise)
4. Use U^m to define the default triggers $\xi^m = -\log(1 - U^m)$ (item-wise)

Despite their popularity copula models are seen as controversial, even being blamed for causing the financial crisis[150] Their main disadvantage is the lack of a robust methodology to estimate the copula function, especially being short of feasible and logical (market implied or historical) data to calibrate the correlation matrix. Indeed one can argue that "*the rationale for their applications is murky*" (Mikosch, 2005).

They are considered to be superior to linear correlation that is not fit to model dependence between variables that are not jointly instantaneous Gaussian shocks. More importantly, both advocates and opponents underline the mathematical convenience copula functions offer. They allow separate modeling of the individual (marginal) credit risk on the one hand,

[150] See MacKenzie and Spears (2014).

and the modeling of the interdependence (joint) structure on the other. Moreover, especially the Gaussian copula function is popular because of its easiness in simulating correlated normal distributed variables through the Cholesky decomposition.

4.3. THE FRAMEWORK OF THE MODEL

4.3.1. FIRST-TO-DEFAULT CREDIT VALUATION ADJUSTMENT FOR CREDIT DEFAULT SWAPS

Based on the general pricing framework for CVA introduced in Subchapter 3.3.1 we will in the following specify the relevant pricing framework for the case of CDS contracts. For the sake of completion we will recall key elements of the general framework. For a more comprehensive description we refer the reader to the respective Subchapter 3.3.1 above.

Analogous to the general framework we define an investor (name "0") and a counterparty (name "2"), dealing a CDS. Note that these definitions are agnostic of which party is selling and which party is the buying protection. In addition, we introduce a reference entity (name "1") to whose default the CDS contract is linked. Notice that if the reference entity is assumed to be default-free name "1" can be removed.

Let τ_0, τ_1 and τ_2 stand for the default time periods of the investor, the reference entity and the counterparty, respectively. Analogous to the general framework the computations are assumed to be conducted in the probability space $(\Omega, \mathcal{G}, \mathcal{G}_t, \mathbb{Q})$.[151] In line with the elaborations given in Subchapter 3.3.1 we define the counterparty credit risk-free CDS value at inception time period as follows

$$\mathrm{CDS}_{a,b}(0, \mathrm{S}_1, \mathrm{LGD}_1) = \mathbb{E}_t^{\mathbb{Q}}\{\pi(0, T)\} \tag{4.12}$$

with LGD_1 standing for the assumed (constant) loss given default. We assume a periodic premium S_1 is paid in the time interval $[T_a; T_b]$. We also assume deterministic interest rates

[151] As stated in Subchapter 3.3.1 $\mathbb{Q}$ stands for the risk-neutral measure, and $\mathcal{G}_t$ is a filtration driving the whole market. $\mathcal{F}_t$ is a further subfilteration standing for all observable market quantities except for default events, hence $\mathcal{F}_t \subseteq \mathcal{G}_t \coloneqq \mathcal{F}_t \vee \mathcal{H}_t$. $\mathcal{H}_t$ stands for the subfilteration standing only for all default events.

which lead to independence between default events and interest rates. Ignoring the default probability of the seller and the buyer for the time being, the model-independent value of the CDS from the perspective of the receiver (protection seller) at inception is given as follows

$$\begin{aligned}\mathrm{CDS}_{a,b}(0, \mathrm{S}_1, \mathrm{LGD}_1) = S_1 \Bigg[-\int_{T_a}^{T_b} P(0,t)\big(t - T_{\gamma(t)-1}\big)\, d\,\boxed{\mathbb{Q}(\tau_1 > t)} \\ + \sum_{i=a+1}^{b} \alpha_i P(0,T) \boxed{\mathbb{Q}(\tau_i > T_i)} \Bigg] + LGD_1 \left[\int_{T_a}^{T_b} P(0,t) d \boxed{\mathbb{Q}(\tau_1 > t)} \right]\end{aligned} \tag{4.13}$$

whereas α_i represents the time elapsing between payment period t_{i-1} and t_i, measured in years. $\gamma(t)$ is the next payment time period after t. $P(t;\, x)$ is the (deterministic) discount factor. $\mathbb{Q}(\tau_1 > t)$ stands for the survival probability of the reference entity, i.e. the probability that the reference entity defaults only after time period t. The term in the first pair of brackets represents the expected amount (of premiums) the receiver will collect in case the reference entity defaults during the life time of the CDS and in case it does not. The term in the second pair of brackets is the expected value (LGD weighted), the receiver will have to pay, if the reference entity defaults. Notice that the first and last highlighted terms stand for the partial derivatives of the default (not the survival) probability. In order to make this more intuitive let $F(t) = \mathbb{Q}(\tau_1 > t) \Leftrightarrow 1 - F(t) = \mathbb{Q}(\tau_1 \leq t)$, the partial derivative of both functions is thus $\frac{\partial F(t)}{\partial t} = f(t) \Leftrightarrow \partial\big(1 - F(t)\big)/\partial t = -f(t)$, respectively.

Now let $NPV(t)$ be the net present value (NPV) of the CDS at time period T_j, with $\mathrm{T}_a < \mathrm{T}_j < T_b$, denoted as follows

$$NPV\big(\mathrm{T}_j, \mathrm{T}_b\big)\,) = \mathrm{CDS}_{a,b}(\mathrm{T}_j, \mathrm{S}_1, \mathrm{LGD}_1). \tag{4.14}$$

We combine Equation (4.13) with Equation (4.14), leading to

$$\begin{aligned}\mathrm{CDS}_{a,b}\left(\mathrm{T}_j, \mathrm{S}_1, \mathrm{LGD}_1\right) &= 1_{\tau_1 > T_j} \overline{CDS}_{a,b}\left(\mathrm{T}_j, \mathrm{S}_1, \mathrm{LGD}_1\right) \\ &\coloneqq 1_{\tau_1 > T_j} \left\{ S_1 \left[-\int_{\max(T_a, T_j)}^{T_b} P(T_j, t)(t - T_{\gamma(t)-1}) d\mathbb{Q}\left(\tau_1 > t | \mathcal{G}_{\mathrm{T}_j}\right) \right.\right. \\ &\left. + \sum_{i=\max(a,j)+1}^{b} \alpha_i P(T_j, T_i) \mathbb{Q}\left(\tau_1 > T_i | \mathcal{G}_{T_j}\right) \right] \\ &\left. + LGD_1 \left[\int_{\max(T_a, T_j)}^{T_b} P(T_j, t) d\mathbb{Q}\left(\tau_1 > t | \mathcal{G}_{\mathrm{T}_j}\right) \right] \right\}\end{aligned} \tag{4.15}$$

which stands for the residual value of the CDS contract at time T_j, conditional on the available information at T_j, especially with regards to credit information, incl. default events of the three entities. Note that $1_{\tau_1 > T_j}$ ensures the reference entity has not defaulted yet. Plugging Equation (4.15) with Equation (3.10) we receive the first-to-default CVA for specific case of CDS contracts[152]

$$\begin{aligned}FTDCVA - \mathrm{CDS}_{a,b}(\mathrm{t}, S_1, \mathrm{LGD}_1) = \; & LGD_2 \, \mathbb{E}_t \left\{ 1_{C \cup D} P(t, \tau_2) \left[1_{\tau_1 > \tau_2} \overline{CDS}_{a,b}(\tau_2, S_1, LGD_1) \right]^+ \right\} \\ & - LGD_0 \mathbb{E}_t \left\{ 1_{A \cup B} P(t, \tau_0) \left[-1_{\tau_1 > \tau_0} \overline{CDS}_{a,b}(\tau_0, S_1, LGD_1) \right]^+ \right\}\end{aligned} \tag{4.16}$$

with $\mathrm{A} = \{\tau_0 \leq \tau_2 \leq \mathrm{T}\}$, $\mathrm{B} = \{\tau_0 \leq \mathrm{T} \leq \tau_2\}$, $\mathrm{C} = \{\tau_2 \leq \tau_0 \leq \mathrm{T}\}$, and $\mathrm{D} = \{\tau_2 \leq T \leq \tau_0\}$.

In the first right term an adjustment for the default probability of the counterparty is undertaken. The events (C and D) – in which the counterparty defaults before the other two parties – are considered. The condition term $\tau_1 > \tau_2$ makes sure that only such default events are considered, in which the reference entity outlives the counterparty. The sign (+) implies that the CDS value at τ_2 is positive from the perspective of the investor, meaning that the CDS contract is a liability of the counterparty. In the second term on the right hand a second adjustment is taken to consider the counterparty credit risk of the investor, i.e. the events (A

[152] For a proof see Capponi (2009), p. 67.

and B) in which the investor defaults before the other two parties. The condition $\tau_1 > \tau_0$ makes sure that only such default events are considered, in which the reference entity outlives the investor. The sign (+) in combination with (-) before the indicator implies that the CDS value at τ_0 is negative from the perspective of the investor, i.e. positive from the perspective of the counterparty that then suffers a respective loss.

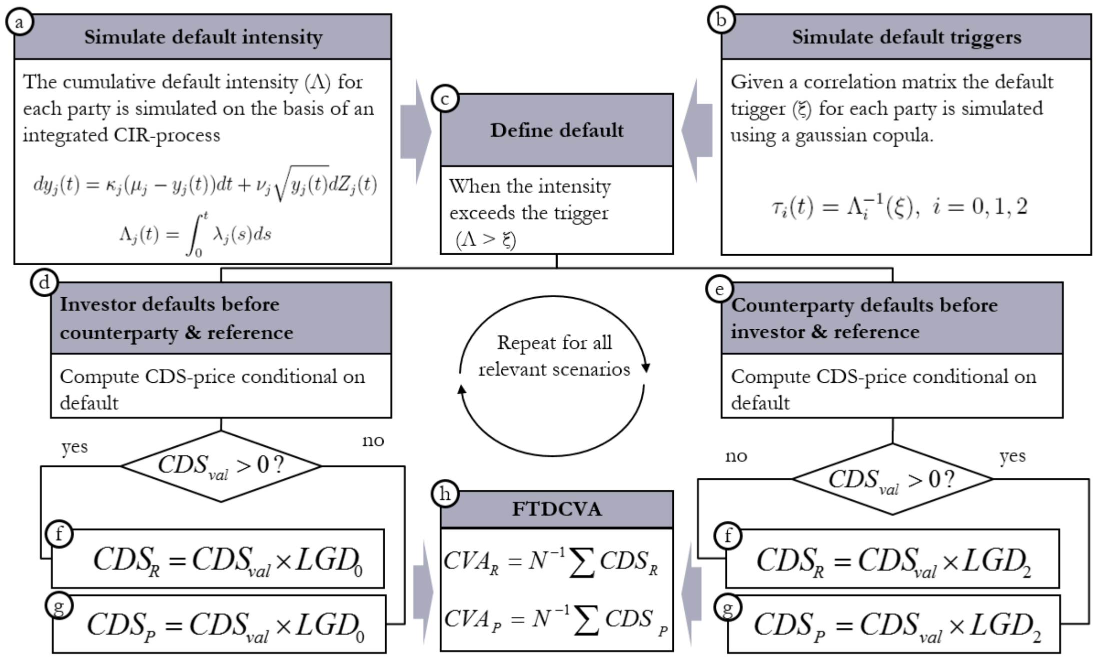


Figure 16: Illustration of the Algorithm for Computing FTDCVA for CDS

The figure gives an overview of the algorithm proposed by Brigo and Capponi (2010): a) Three independent default intensity processes are simulated. b) Based on a Gaussian copula and a given correlations default triggers are simulated. c) Defaults are generated by comparing the cumulated intensity for each entity with the respective simulated trigger. d-g) The algorithm considers relevant scenarios and executes all subsequent computations from the perspective of the investor that can either be a receiver (R) or a payer (P). Consider the case in which the investor is the payer, buying protection from the counterparty. The investor will suffer a loss if the counterparty defaults (e) and the CDS value is positive (g). If, however, the investor defaults first (d) he will “gain” from his own default is the residual value is negative.

4.3.2. AN OVERVIEW OF THE ALGORITHM

The task of the algorithm proposed by Brigo and Capponi (2010) is to compute Equation (4.16). It assumes the investor can be either the payer or the receiver of the CDS contract, delivering two FTDCVA metrics, CVA_P and CVA_R, respectively. In the following we will give a high level description of the numerical approach, while more detail on the various items will follow in subsequent sections. An illustrative overview is also given in Figure 16.

The risk-neutral probabilities of default as seen in Equation (4.16) are considered numerically through a Monte Carlo scheme, simulating default time periods of the three entities. For this purpose Brigo and Capponi (2010) rely on an intensity approach, simulating three independent shifted CIR processes. In the meanwhile a Gaussian copula function models the interdependencies between the defaults, simulating default triggers for each entity. A default occurs when the integrated CIR process of one the parties exceeds the value of the respective trigger.

The algorithm then considers only relevant defaults, i.e. defaults either of the investor or of the counterparty, given that the reference entity survived. For these cases the value the surviving entity loses given a default is computed, i.e. the LGD_1-weighted conditional residual value of CDS seen in Equation (4.15). Averaging these terms according to the logic given in Figure 16 delivers the FTDCVA metrics.

For this purpose the conditional survival probabilities of the reference entity are needed, i.e. conditional on the set of information given a default of the counterparty (τ_0) or the investor (τ_2), defined as

$$1_{C\cup D} 1_{\tau_1>\tau_2} \mathbb{Q}(\tau_1 > t | \mathcal{G}_{\tau_2}) \tag{4.17}$$

and

$$1_{A\cup B} 1_{\tau_1>\tau_0} \mathbb{Q}(\tau_1 > t | \mathcal{G}_{\tau_0}), \tag{4.18}$$

respectively.

As shown by Brigo and Capponi (2009) the probabilities given in Equations (4.17) and (4.18) can be computed in a closed-form manner using a Fourier transformation as will be shown in a subsequent subchapter. Technically speaking this is actually the most elaborate part of the model, and we contribute a greater part of this paper to explain the needed steps in detail. We will also introduce a heuristic tune up, based on a log-normal approximation.

4.3.3. MODELING INTERDEPENDENT DEFAULTS

In the following we will first take the reader through the steps needed to model defaults via an intensity process. Subsequently we will elaborate on the use of Gaussian copulas in modeling interdependencies between the respective processes.

Brigo and Capponi (2010) use a stochastic intensity model that mirrors the shifted interest rate CIR model – also referred to as CIR++ – examined for example by Brigo and Mercurio (2006). Instead of modelling the instantaneous short rate the reduced form approach models the instantaneous intensity that can also be interpreted as the instantaneous default probability. The model incorporates a shift term that allows for exact calibration of CDS quotes. Although Brigo and Capponi (2010) introduce jumps in the intensity process formally, they subsequently drop the term – possibly due to technical challenges – and refer the reader to future work. We therefore, will also focus on intensities without jumps, i.e. using the shifted squared root diffusion (SSRD), previously examined by Brigo and Alfonsi (2005).

Within the SSRD approach the stochastic intensity for each party j is given by

$$\lambda_j(t) = y_j(t) + \psi_j(t, \beta), t \geq 0, j = 0,1,2 \tag{4.19}$$

with ψ_j standing for the shift term, i.e. a deterministic function guaranteeing that the modeled survival probabilities equal the market implied terms. The dynamics of the stochastic term y_j are defined as follows

$$dy_j(t) = \kappa_j \left(\mu_j - y_j(t)\right) dt + \nu_j \sqrt{y_j(t)} dZ_j(t), \qquad j = 0,1,2 \tag{4.20}$$

whereas Z_j is a Brownian motion process under the risk-neutral measure $\mathbb{Q}$. The CIR process of each entity is represented by the vector $\beta_j = \left(\kappa_j, \mu_j, \nu_j, y(0)\right)$, consisting of positive constants. The integrated processes are then given by

$$\Lambda_j(t) = \int_0^t \lambda_j(s)ds\,,\; Y_j(t) = \int_0^t y_j(s)ds\,,\; \Psi_j(t, \beta_t) = \int_0^t \psi_j\left(s, \beta_j\right)ds. \tag{4.21}$$

Model implied survival probabilities are defined as follows

$$\begin{aligned} \mathbb{Q}(\tau_i > t) &:= \mathbb{E}^{\mathbb{Q}}\left[e^{-Y_i(t)}\right] \\ &:= P^{CIR}(0, t, \beta_i). \end{aligned} \tag{4.22}$$

$P^{CIR}(0, t, \beta_i)$ resembles the zero coupon price within the CIR approach, solved analytically by the following equation

$$P^{CIR}(0, t, \beta_i) = A(t, T, \beta_i) \exp\{-B(t, T, \beta_i)\} \tag{4.23}$$

where

$$h = \sqrt{\kappa + 2\sigma^2}$$

$$\begin{aligned} A(t, T, \beta_i) &= \left[\frac{2hexp(\kappa + h)(T - t)/2}{2h + (\kappa + h)(\exp(T - t)\, h - 1)}\right]^{\frac{2\kappa\Theta}{\sigma^2}} \\ B(t, T, \beta_i) &= \frac{2(exp(T - t)h - 1)}{2h + (\kappa + h)(\exp(T - t)\, h - 1)}. \end{aligned} \tag{4.24}$$

Calibration of the parameters can be obtained by comparing the model implied survival probabilities $P^{CIR}(0, t, \beta_i)$ and market implied probabilities $Q(\tau_i > t)_{market}$ as will be shown in the case study below. For stripping market implied survival probabilities from CDS spreads please see also Subchapter 3.3.4 above.

Finally we turn to the shift (deterministic) term of the intensity process given Equation (4.19). Its integrated process $\Psi_j(t, \beta_t)$ already introduced in Equation (4.21) is defined as follows

$$\Psi_j(t, \beta_t) \; = log\left(\frac{P^{CIR}(0, t, \beta_i)}{\mathbb{Q}(\tau_i > t)_{market}}\right), \tag{4.25}$$

taking the differences between model-implied and market implied survival probabilities into consideration, ensuring that the model values are consistent with observed market data.

For the simulation of the intensity dynamics of each party, Brigo and Capponi (2009) offer the following formula

$$\mathrm{y}(t) = \frac{\nu^2\left(1 - e^{-\kappa(t-u)}\right)}{4\kappa}\chi'_d\left(\frac{4\kappa e^{-\kappa(t-u)}}{\nu^2\left(1 - e^{-\kappa(t-u)}\right)}y(u)\right) \tag{4.26}$$

with

$$d = \frac{4\kappa\mu}{\nu^2} \tag{4.27}$$

whereas $\chi'_d(\mathrm{o})$ stands for a non-central chi-square random variable with d degrees of freedom. o is the non-centrality parameter. Using the trapezoidal rule, the integrated process $Y(t)$ can then be defined through the following approximation

$$Y(t) = \int_0^t y(s)ds = \frac{1}{2\delta_t}\sum_{z=1}^{t} y(z-1) + y(z) \tag{4.28}$$

with δ_t is a predefined discretization parameter.

Let $\Lambda_\mathrm{i}(\mathrm{t})$ stand be the integrated stochastic process for the default intensity of party i. We define ξ_i as the default trigger of party i. The default time period of the same party can be given as such

$$\tau_i(\mathrm{t}) = \Lambda_i^{-1}(\xi_i),\ \mathrm{i} = 0{,}1{,}2. \tag{4.29}$$

Again, the three CIR processes are independent. The introduction of interdependencies between the default times of the three parties are generated on the basis of a Gaussian copula that determines the correlated default triggers.

The default triggers are defined as exponential random variables with the following uniforms

$$\mathrm{U}_i = 1 - \exp\{-\xi_i\}. \tag{4.30}$$

The uniforms from Equation (4.30) are correlated through a trivariate Gaussian copula function

$$\mathrm{C}_R(u_0, u_1, u_2) = \mathbb{Q}(U_0 < u_0, U_1 < u_1, U_2 < u_2). \tag{4.31}$$

For the simulation of these default triggers we thus only need the following correlation matrix as an input

$$\mathrm{R} = \begin{pmatrix} 1 & \rho_{j,h} & \rho_{j,i} \\ \rho_{j,h} & 1 & \rho_{i,j} \\ \rho_{j,i} & \rho_{i,j} & 1 \end{pmatrix} \tag{4.32}$$

whereas $\rho_{j,i}$ stands for the correlation coefficient between the default triggers of the parties j and i.

Remarks

The short elaboration given above illustrates the theoretical and practical advantages the SSRD approach has. The dynamics of the process are well researched, with relatively straightforward approaches to implement fast simulation procedures. Moreover, the CIR framework offers analytical (closed-form) solutions for bonds that can be applied for survival probabilities, facilitating easy calibration schemes.

The convenience of the calibration is supported by the fact that interdependency is modelled separately through a copula function. However, the estimation of the correlation matrix given in Equation (4.32) is indeed a key weak spot of the copula approach and of the CVA model. After all, we lack natural methodologies and data for their calibration, implying unsolved challenges in hedging the CVA metric produced as will be discussed in Section 4.5 below.

4.3.4. MODELING THE CONDITIONAL EXPECTED EXPOSURE

Conditional Copula Values

As already stated, the missing information in order to compute the conditional value of the CDS contract - as seen in Equation (4.15) - are the conditional survival probabilities given in Equations (4.17) and (4.18). Following Brigo and Capponi (2009), we define $F_{\Lambda_i(t)}$ as the cumulative distribution function of the cumulative (shifted) intensity of the CIR processses.

Brigo and Capponi (2009) show that the (missing) survival probabilities can be computed as follows

$$1_{C\cup D}1_{\tau_1>\tau_2}\mathbb{Q}(\tau_1 > t|\mathcal{G}_{\tau_2}) = \tag{4.33}$$
$$1_{\tau_2\leq T}1_{\tau_2\leq\tau_0}\left(1_{\bar{A}} + 1_{\tau_2<t}1_{\tau_1\geq\tau_2}\int_{\bar{U}_{1,2}}^{1} F_{\Lambda_1(t)-\Lambda_1(\tau_2)}(-\log(1-u_1) - \Lambda_1(\tau_2))dC_{1|0,2}(u_1;U_2)\right)$$

and

$$1_{A\cup B}1_{\tau_1>\tau_0}\mathbb{Q}(\tau_1 > t|\mathcal{G}_{\tau_0}) = \tag{4.34}$$
$$1_{\tau_0\leq T}1_{\tau_0\leq\tau_2}\left(1_{\bar{B}} + 1_{\tau_0<t}1_{\tau_1\geq\tau_0}\int_{\bar{U}_{1,2}}^{1} F_{\Lambda_1(t)-\Lambda_1(\tau_0)}(-\log(1-u_1) - \Lambda_1(\tau_0))dC_{1|2,0}(u_1;U_0)\right)$$

where

$$\bar{U}_{i,j} = 1 - \exp\{-\Lambda_i(\tau_j)\} \tag{4.35}$$

and the scenarios $\bar{A}$ and $\bar{B}$ are defined as $\bar{\mathrm{A}} = \{t \leq \tau_2 \leq \tau_1\}$ and $\bar{\mathrm{B}} = \{\mathrm{t} \leq \tau_0 \leq \tau_1\}$, respectively. Following Brigo and Capponi (2009), the conditional copula values used in equations (4.34) and (4.35) are given as

$$C_{1|0,2}(u_1; U_2) = \tag{4.36}$$
$$\frac{\frac{\partial C_{1,2}(u_1,u_1)}{\partial u_2}|_{u_2=U_2} - \frac{\partial C(\bar{U}_{0,2},u_1,u_1)}{\partial u_2}|_{u_2=U_2} - \frac{\partial C_{1,2}(\bar{U}_{1,2},u_1)}{\partial u_2}|_{u_2=U_2} + \frac{\partial C(\bar{U}_{0,2},\bar{U}_{1,2},u_2)}{\partial u_2}|_{u_2=U_2}}{1 - \frac{\partial C_{0,2}(\bar{U}_{0,2},u_2)}{\partial u_2}|_{u_2=U_2} - \frac{\partial C_{1,2}(\bar{U}_{1,2},u_2)}{\partial u_2}|_{u_2=U_2} + \frac{C(\bar{U}_{0,2},\bar{U}_{1,2},u_2)}{\partial u_2}|_{u_2=U_2}}$$

and

$$C_{1|2,0}(u_1; U_0) = \tag{4.37}$$
$$\frac{\frac{\partial C_{0,1}(u_0,u_1)}{\partial u_0}|_{u_0=U_0} - \frac{\partial C(u_0,u_1,\bar{U}_{2,0})}{\partial u_0}|_{u_0=U_0} - \frac{\partial C_{0,1}(u_0,\bar{U}_{1,0})}{\partial u_0}|_{u_0=U_0} + \frac{\partial C(u_1,\bar{U}_{1,0},\bar{U}_{2,0})}{\partial u_0}|_{u_0=U_0}}{1 - \frac{\partial C_{0,2}(u_0,\bar{U}_{2,0})}{\partial u_0}|_{u_0=U_0} - \frac{\partial C_{0,1}(\bar{U}_{1,2},u_2)}{\partial u_0}|_{u_0=U_0} + \frac{C(u_0,\bar{U}_{1,0},\bar{U}_{2,0})}{\partial u_0}|_{u_0=U_0}},$$

respectively. $C_{0,1}$ is the bivariate copula connecting the default time of i and j, while C denotes the trivariate copula connecting the default time of all three parties. We revisit the proof of Equations (4.37) in Appendix E, p. 201.

In order to make the computation of Equation (4.33) and Equation (4.34) more clear, we shall distinguish between the following three different terms needed:

Term 1: $\frac{\partial C_{i,j}(u_i,u_j)}{\partial u_j}|_{u_j=U_j}$

Term 2: $\frac{\partial C(u_h,u_i,u_j)}{\partial u_j}|_{u_i=U_j}$

Term 3: $F_{\Lambda_i(t)}$

The derivation all three terms will be described in the following.

Partial Derivatives of the Gaussian Copula (Term 1 and Term 2)

Due to the specifications of the Gaussian copula, Terms 1 and 2 can be calculated analytically. Term 1 is the partial derivative of the bivariate Gaussian copula. It is given by Schönbucher (2005) and recited by Capponi (2009), and can be computed as follows:

$$\frac{\partial C_{i,j}(u_i, u_j)}{\partial u_j}|_{u_j=U_j} = P^*(U_i < u_i | U_j) = \Phi\left(\frac{\Phi^{-1}(u_i) - \rho_{i,j}\Phi^{-1}(u_j)}{\sqrt{1-\rho_{i,j}^2}}\right) \quad (4.38)$$

whereas $\rho_{i,j}$ denotes the correlation coefficient between i and j, and $\Phi(\cdot)$ stands for the cumulative univariate standard normal distribution function.

Term 2 is the partial derivative of a trivariate Gaussian copula, and is the bivariate density of i and h, given j is thus given by

$$\begin{aligned} \frac{\partial C(u_h, u_i, u_j)}{\partial u_j}|_{u_i=U_j} &= P^*(U_i < u_i | U_j) \\ &= \Phi_{0,R^*(\Sigma)}(\bar{u}_h, \bar{u}_i) \end{aligned} \quad (4.39)$$

where $\Phi_{0,R}$ stands for the cumulative bivariate standard normal distribution function, and $R^*(\Sigma)$ is the correlation matrix given by the covariance matrix Σ of the variables $\bar{u}_h$ and, $\bar{u}_i$ which are defined below.

The aim is to write $\bar{u}_h$ and $\bar{u}_i$ in dependence of u_i and u_h while u_j is given. For this purpose, we first compute the Cholesky decomposition of the correlation matrix between the independent uniforms R as given in Equation (4.32) as follows

$$D^m = \begin{pmatrix} 1 & D_{1,2} & D_{1,3} \\ 0 & D_{2,2} & D_{2,3} \\ 0 & 0 & D_{3,3} \end{pmatrix} = Chol\begin{pmatrix} 1 & \rho_{j,h} & \rho_{j,i} \\ \rho_{j,h} & 1 & \rho_{i,j} \\ \rho_{j,i} & \rho_{i,j} & 1 \end{pmatrix}. \quad (4.40)$$

Notice that $D_{1,2} = \rho_{j,h}$ and $D_{1,3} = \rho_{j,i}$. Given independent values for u_i, u_h and u_j the dependent values for $\bar{u}_i$, $\bar{u}_h$ and $\bar{u}_j$ can be computed as follows:

$$\begin{pmatrix} \bar{u}_i \\ \bar{u}_h \\ \bar{u}_j \end{pmatrix} = \begin{pmatrix} 1 & D_{1,2} & D_{1,3} \\ 0 & D_{2,2} & D_{2,3} \\ 0 & 0 & D_{3,3} \end{pmatrix}^T \cdot \begin{pmatrix} u_i \\ u_h \\ u_j \end{pmatrix}. \tag{4.41}$$

Obviously $\bar{u}_i$ is given by u_i. The dependent uniforms follow normal distributions given by

$$\bar{u}_h = D_{1,2} \cdot u_i + D_{1,2} \cdot u_h \qquad \sim N(D_{1,2} \cdot u_i, \left(D_{2,2}\right)^2 \tag{4.42}$$

$$\bar{u}_j = D_{1,3} \cdot u_i + D_{2,3} \cdot u_h + D_{3,3} \cdot u_j \qquad \sim N(D_{1,3} \cdot u_i, \left(D_{1,3}\right)^2 + \left(D_{3,3}\right)^2.$$

The covariance of both variables is given by

$$cov\left(D_{2,2} \cdot u_h, D_{2,3} \cdot u_h + D_{3,3} \cdot u_i\right) = \tag{4.43}$$

$$cov\left(D_{2,2} \cdot u_h, D_{2,3} \cdot u_h\right) + cov\left(D_{2,2} \cdot u_h, D_{3,3} \cdot u_i\right)$$

$$= D_{2,2} \cdot D_{2,3}.$$

Considering the variance given in Equation (4.42) the conditional covariance matrix we are seeking can be written as follows

$$\Sigma = \begin{pmatrix} (D_{2,2})^2 & D_{2,2} \cdot D_{2,3} \\ D_{2,2} \cdot D_{2,3} & (D_{2,3})^2 + (D_{3,3})^2 \end{pmatrix}. \tag{4.44}$$

The bivariate distribution of the uniforms u_h, and u_i conditional on u_j is given by

$$\binom{u_j}{u_i} | u_j = U_j \Big) \sim \mathrm{N}\left(\begin{pmatrix} u_j \cdot D_{1,2} \\ u_j \cdot D_{1,3} \end{pmatrix}, \begin{pmatrix} (D_{2,2})^2 & D_{2,2} \cdot D_{2,3} \\ D_{2,2} \cdot D_{2,3} & (D_{2,3})^2 + (D_{3,3})^2 \end{pmatrix}\right). \tag{4.45}$$

CDF using Fourier Transformation (Term 3)

The third term $F_{\Lambda_i(t)}$ is the cumulative distribution function of the integrated CIR process. Brigo and Capponi (2009) state that this can be done by using the Fourier transformation by inverting the characteristic function of the integrated CIR process.

In the following we will describe how this can be done. As noted by Carr et al. (2003), the characteristic function for $\Lambda_i(t)$ is well known and is given by

$$f(t) = E[\exp\{iu\Lambda(t)\}] = \emptyset(u, t, y(0), \kappa, \lambda) \tag{4.46}$$

$$= a(t, u)exp\{b(t, u), y(0)\}$$

where

$$a(t, u) = \frac{\exp\left\{\frac{\kappa^2 \eta t}{\lambda^2}\right\}}{\left(\cosh\left(\frac{\gamma t}{2}\right) + \frac{\kappa}{\gamma}\sinh\left(\frac{\gamma t}{2}\right)\right)^{2k\eta/\lambda^2}} \tag{4.47}$$

$$b(t, u) = \frac{2iu}{\kappa + \gamma\coth(\frac{\gamma t}{2})}$$

$$\gamma = \sqrt{\kappa - 2\lambda^2 iu}.$$

The imaginary unit is denoted by i. The aim is thus to compute the continuous Fourier transform (CFT), which is defined as follows

$$CFT[f](\omega) = \int_{-\infty}^{\infty} e^{-it\omega} f(t)dt \tag{4.48}$$

where ω and t are the spaces of the transformed and the original function, respectively. Following Bailey and Swarztrauber (1993) and Chourdakis (2005), we will transform the characteristic discretization of the CFT by the fractional fast Fourier transformation (FRFT).

This is a technical adjustment of the fast Fourier transformation (FFT), it again being a fast implementation of the discrete Fourier transformation (DFT). The DFT is defined as

$$DFT[f_j](\omega) = \sum_{j=0}^{n-1} e^{-2\pi i \frac{jk}{n}}, \quad f_j = f(\omega_k) \tag{4.49}$$

and the inverse is given by

$$DFT^{-1}[\hat{f}_j](t_j) = \frac{1}{n} \sum_{j=0}^{n-1} e^{2\pi i \frac{jk}{n}}\, \hat{f}_j, \qquad \hat{f}_j = \hat{f}(\omega_k). \tag{4.50}$$

The fractional Fourier transform (FRFT) is defined as

$$FRFT(f_j, \alpha)(\omega_k) = \sum_{j=0}^{n-1} e^{-2i\pi k\alpha} f(t_j) \tag{4.51}$$

$$\left(e^{-i\pi k^2 \alpha}\right) DFT^{-1}\left[DFT[x_j] DFT[y_j]\right](\omega_k), 0 \le k \le n$$

where

$$x = \left(\left(f_j e^{-i\pi j^2 \alpha}\right)_{j=0}^{n-1}, (0)_{j=n}^{2n}\right) \tag{4.52}$$

$$y = \left(\left(e^{i\pi j^2 \alpha}\right)_{j=0}^{n-1}, \left(e^{i\pi (2n-j)^2 \alpha}\right)_{j=n}^{2n}\right).$$

Our aim is thus to compute the cumulative distribution function (CDF) through the FRFT as seen in Equation (4.51). In order to deliver a good approximation one has to define appropriate ranges for the respective spaces (t and ω, being the spaces for x_j and y_j, respectively).[153] In order to compute the radius of the ω-space, we offer a heuristic approach

---

[153] We denote with $\tilde{\delta}$ the grid size of the input vector x_j, and with $\tilde{\Lambda}$ the grid size of y_j. Due to the parameter α, the FRFT allows an independent choice for both the grid sizes, where $\alpha = \tilde{\Lambda}\,\tilde{\delta}$. In the continuous fast Fourier transformation, the inverse relation between the grid sizes ($\tilde{\Lambda}\,\tilde{\delta} = 2\pi n$) leads to inefficiencies in the calculations as denoted by Chourdakis (2005).

on the basis of the first and second moments of the CIR process derived by Dufresne (2001). Given a time period t, time to maturity $T-t$, and the CIR parameters $y(t), \kappa, \mu, \nu$ these moments are given by

$$E^*(Y) = \frac{y(t)}{-\kappa} - \frac{\kappa\mu}{\kappa^2} - (T-t)\frac{\kappa\mu}{-\kappa} + e^{-(T-t)\kappa}\left[\frac{y(t)}{-\kappa} + \frac{\kappa\mu}{-\kappa^2}\right], \tag{4.53}$$

$$E^*(Y^2) = -\frac{y(t)\nu^2}{-\kappa^3} - \frac{5\kappa\mu\nu^2}{-2\kappa^4} - (T-t)\frac{\kappa\mu\nu^2}{-\kappa^3} + e^{-(T-t)\kappa}\left[\frac{2\kappa\mu\nu^2}{-\kappa^4} - (T-t)\left(\frac{2y(t)\nu^2}{-\kappa^2} + \frac{2\kappa\mu\nu^2}{-\kappa^3}\right)\right] + e^{-2(T-t)\kappa}\left[\frac{y(t)\nu^2}{-\kappa^5} + \frac{\kappa\mu\nu^2}{-2\kappa^6}\right].$$

The radius ω is subsequently given by

$$\omega = E^*(Y) + \epsilon\sqrt{E^*(Y^2) - E^*(Y)^2} \tag{4.54}$$

for a predefined parameters ϵ.[154]

In the case of the t-space we offer an iterative estimation algorithm, screening the convergence of the characteristic function outside the predefined radius. An example for a respective code is given in Appendix D.

Figure 17 displays the functions involved in the procedure of estimating the cumulative distribution function (CDF) through the FRFT. While the monotone increasing straight greed line stands for the CDF computed by the FRFT, the lognormal like red line is the normalized density function of the integrated CIR-Process. The normed characteristic function of the CIR process is plotted as a blue wavy line.

154 In our calculations, we have found that this parameter can be assumed to be approximately 10. In general the parameter can be derived from Chebischev inequality.

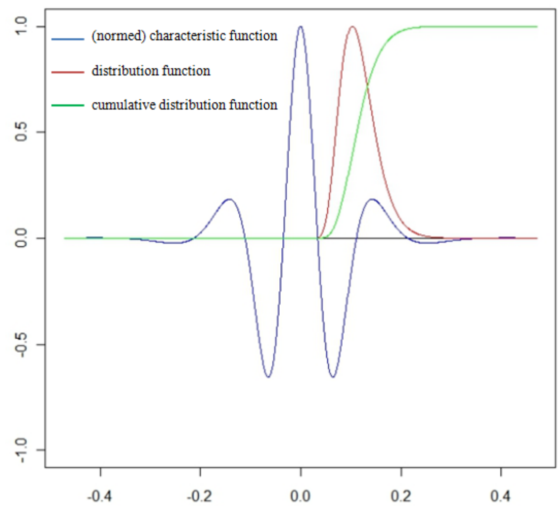


Figure 17: Results of the Fractional Fast Fourier Transformation

The figure gives a visual example for the results of the FRFT. The monotone increasing straight green line stands for the CDF computed by the FRFT. The lognormal-like red shape is the output of the transformation before summing up the results. The (normed) characteristic function of the CIR process and the input needed for the FRFT is plotted through a blue wavy line.

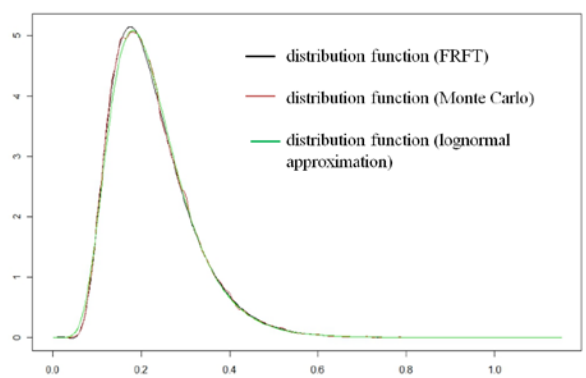


Figure 18: Lognormal Distribution vs. FRFT Approach

The figure gives a visual example for the fit offered by the lognormal approximation of the distribution function of the CIR process. The somewhat wavy red line stands for the distribution function computed through the Monte Carlo scheme. The green line is the distribution function if lognormal-approximation is used. The black line stands for the distribution function when the FRFT technique is used.

4.3.5. COMPUTATIONAL TUNE UP AND NUMERICAL EXAMPLES

The Fourier transformation approach, proposed to compute the cumulative distribution function of the integrated CIR process can be seen as theoretically sound. It can, however, turn to be costly in terms of implementation as well as in terms of computational power. Especially when it comes to pricing, faster approaches are welcome to ensure trading feasibility. In addition, due to the needed definitions of the ranges, i.e. Equation (4.52), the FRFT-approach might demand an extensive parametrization process. This becomes very relevant, when the CIR parameters of the involved parties vary significantly or if a high number of parties is involved (e.g. in a portfolio of CDS).

Motivated by the visual results seen in Figure 17, we offer an approximation for the CDF by a shifted lognormal distribution, matching the first three moments of the distribution function. For this purpose, one has only to compute the central moments in order to compute the cumulative distribution function. Whilst the first two moments are given in Equation (4.53) we illustrate – for completion reasons – the estimation of the third moment.

Given a time period t, time to maturity $T-t$, and the CIR parameters $y(t), \kappa, \mu, \nu$, the third moment is given by

$$\begin{aligned}
E^*(Y^3) = &-\frac{3y(t)\nu^3}{-\kappa^5} - \frac{11\kappa\mu\nu^4}{-\kappa^6} - (T-t)\frac{3\kappa\mu\nu^4}{-\kappa^5}\cdots \\
&+ e^{-(T-t)\kappa}\left[-\frac{3y(t)\nu^4}{-2\kappa^5} - \frac{15\kappa\mu\nu^4}{-2\kappa^6} - (T-t)\left(\frac{3y(t)\nu^4}{-\kappa^4 + \frac{9\kappa\mu\nu^4}{-\kappa^5}}\right)\cdots\right. \\
&\left. + t^2\left(\frac{3y(t)\nu^4}{-\kappa^5} + \frac{3\kappa\mu\nu^4}{-2\kappa^6}\right)\right]\cdots \\
&+ e^{-2(T-t)\kappa}\left[\frac{3y(t)\nu^4}{-2\kappa^5} + \frac{3\kappa\mu\nu^4}{-\kappa^6} - (T-t)\left(\frac{6y(t)\nu^4}{-\kappa^4 + \frac{3\kappa\mu\nu^4}{-\kappa^5}}\right)\right]\cdots \\
&+ e^{-3(T-t)\kappa}\left[\frac{3y(t)\nu^4}{-2\kappa^5} + \frac{\kappa\mu\nu^4}{-2\kappa^6}\right].
\end{aligned} \tag{4.55}$$

In order to demonstrate the goodness of fit, offered by the heuristic approach we conducted a series of exemplary numerical exercises, comparing the results produces using the Fourier transformation with those produced using the log-normal approximation. In a second step we compare further examples with the results given by Capponi (2009). We based the parameterization of the CIR process on the ones given in Capponi (2009), pricing the first-to-default CVA (FTDCVA) for a five-year CDS. The results are summarized in Table 15, Table 16 and Table 17. The delivered CVA metrics are in basis points, given from the perspective of the investor (either protection payer or protection seller).

Table 15 summarizes the results of the first-to-default CVA if the investor is the receiver (protection seller) under a variety of correlation matrixes and volatility terms for the CIR process of the reference entity ν_1. Table 16 summarizes the first-to-default CVA results if the investor is the premium payer (protection buyer), following the structure of Table 15, i.e. summarizing the results of the first-to-default CVA under a variety of correlation matrixes and volatility terms for the CIR process of the reference entity ν_1. Table 17 tests further possible combinations of the correlation coefficients, and compares our results with these delivered by Capponi (2009).

Concerning the scenarios, in which the investor is the receiver (protection seller) the results of both approaches seem to be quite stable (s. Table 15). The results resemble the ones delivered by Capponi (2009, p. 73). Notice, however, that he uses a different volatility term for the counterparty, explaining some of the deviations we have when the investor is the payer (protection buyer). In that case our results display higher uncertainty as seen in Table 16 and Table 17, and made apparent by the standard error terms.[155] This leads to some deviations, especially in some more (numerically seen) challenging scenarios (e.g. when the correlation coefficient is 0.99). Still, one can conclude that in total the deviations between the

155 In such cases it seems that the results are quite dependent on the discretization parameters of the cumulative distribution function (CDF).

three approaches seem to lie in an acceptable range, especially if the standard errors are considered.

Beyond exhibiting the fitness of both the Fourier transformation technique implemented as well as the log-normal approximation these numerical examples shed light on the intuitive mechanisms of the algorithm as well as on its limits.

Table 15 shows that if the investor is the protection seller the FTDCVA metric decreases with an increase in the correlation between the reference entity and the counterparty, reaching near-zero terms. This can be explained by the pattern already discussed in Schönbucher and Schubert (2000). If the counterparty defaults before the reference entity while having a negative (positive) correlation the survival probability of the reference entity will increase (decrease). Now if the survival probability of the reference entity increases (decreases) the conditional value of the CDS – from the perspective of the protection seller – will increase (decrease), implying higher (lower) adjustments.

A contrary pattern is observed when the investor buys protection as seen in Table 16. The higher the correlation between the reference entity and the counterparty, the higher the adjustment. Table 17 shows that there is one exception to this rule, i.e. the case in which the reference entity and counterparty are almost perfectly correlated, i.e. $\rho_{1,2} = 0.99$, in which the adjustment decreases significantly. Because of the high correlations the default triggers are almost identical. If the counterparty defaults before the investor, the residual value of the CDS will be relatively low, because of the high probability that the reference entity would have had defaulted already. Brigo and Chourdakis (2009) classify the pattern as "*somewhat reasonable*", showing that it loses some of its significance if the reference entity becomes riskier. It does, however, imply some inconsistency, especially when it comes to pricing wrong-way risk. After all, wrong way risk – as seen in the monoline example – can also be accompanied by presumably riskless references. In cases of extreme wrong way risk the model will deliver rather low (if not negligible) CVA values.

Table 17 also shows that the influence of correlation stops being as straightforward when other interdependencies are modelled as well, e.g. when the investor and the reference entity also display a default correlation. In these cases the probability of the counterparty defaulting first is reduced through default contagion between the investor and the reference entity.

The volatility of the reference entity is one further item that is captured by the model. This is apparent from the last correlation combination given in Table 16. The volatility of the reference entity increases the value of the CDS contract – also the conditional value given a default – which in return induces a higher adjustment.

Correlation matrix ($\boldsymbol{\rho_{0,1}}, \boldsymbol{\rho_{0,2}}, \boldsymbol{\rho_{1,2}}$)	$\boldsymbol{\nu_1}$: **Implied CDS volatility**	**0.01** **1.5%**	**0.1** **15%**	**0.2** **28%**	**0.3** **37%**	**0.4** **42%**	**0.5** **42%**
(0, 0, -0.9)	Lognormal distribution	25 (1)	25 (1)	25 (1)	27 (1)	26 (1)	24 (1)
	Fourier transformation	26 (1)	26 (1)	26 (1)	28 (1)	27 (2)	25 (1)
(0, 0, -0.6)	Lognormal distribution	21 (1)	25 (1)	24 (1)	24 (1)	24 (1)	23 (1)
	Fourier transformation	25 (1)	26 (2)	25 (1)	25 (1)	24 (1)	24 (1)
(0, 0, -0.2)	Lognormal distribution	6 (1)	9 (1)	9 (1)	9 (1)	7 (1)	6 (1)
	Fourier transformation	9 (1)	12 (1)	11 (1)	11 (1)	8 (1)	12 (1)
(0, 0, 0)	Lognormal distribution	0 (0)	2 (0)	1 (0)	1 (0)	0 (0)	0 (0)
	Fourier transformation	0 (0)	2 (0)	2 (0)	1 (0)	1 (0)	6 (1)
(0, 0, 0.2)	Lognormal distribution	0 (0)	0 (0)	1 (0)	0 (0)	0 (0)	0 (0)
	Fourier transformation	-0 (0)	1 (0)	1 (0)	0 (0)	0 (0)	4 (0)
(0, 0, 0.6)	Lognormal distribution	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
	Fourier transformation	-0 (0)	0 (0)	0 (0)	-0 (0)	-0 (0)	2 (0)
(0, 0, 0.9)	Lognormal distribution	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
	Fourier transformation	-0 (0)	-0 (0)	-0 (0)	-0 (0)	-0 (0)	1 (0)

Table 15: Comparison of the Model Results Using the Analytical Approximation and FRFT (Part I)

The table summarizes the FTDCVA in basis points of the CDS receiver as computed using the lognormal distribution approximation and the Fast Fourier Transformation (FRFT) approach. The numbers in brackets stand for the standard errors. The investor (receiver) has the following parameters: $y_0 = 0.0001, \kappa_0 = 0.9\ \mu_0 = 0.001, \nu_0 = 0.01, \mathrm{LGD}_0 = 0.6$. The reference entity has the following parameters: $y_1 = 0.01, \kappa_1 = 0.8, \mu_1 = 0.02, \mathrm{LGD}_1 = 0.7$. The counterparty (payer) has the following parameters: $y_2 = 0.03, \kappa_2 = 0.5\ \mu_2 = 0.05, \nu_2 = 0.5, \mathrm{LGD}_2 = 0.65$. The interest free rate r is set to be 0.03.

Correlation matrix ($\boldsymbol{\rho_{0,1}}, \boldsymbol{\rho_{0,2}}, \boldsymbol{\rho_{1,2}}$)	$\boldsymbol{\nu_1}$: Implied CDS volatility	**0.01** **1.5%**	**0.1** **15%**	**0.2** **28%**	**0.3** **37%**	**0.4** **42%**	**0.5** **42%**
(0, 0, -0.9)	Lognormal distribution	0 (0)	-0 (0)	-0 (0)	-0 (0)	-0 (0)	-0 (0)
	Fourier transformation	0 (0)	-0 (2)	-0 (0)	-0 (0)	-0 (0)	-0 (1)
(0, 0, -0.6)	Lognormal distribution	-0 (0)	-0 (0)	0 (0)	0 (0)	0 (0)	0 (0)
	Fourier transformation	-0 (0)	0 (2)	0 (0)	0 (0)	-0 (0)	-0 (0)
(0, 0, -0.2)	Lognormal distribution	0 (0)	1 (0)	2 (0)	2 (0)	2 (0)	3 (0)
	Fourier transformation	0 (0)	1 (0)	2 (0)	1 (0)	2 (0)	2 (1)
(0, 0, 0)	Lognormal distribution	9 (0)	8 (1)	9 (1)	9 (1)	9 (1)	13 (1)
	Fourier transformation	3 (0)	6 (0)	6 (1)	7 (1)	8 (1)	9 (1)
(0, 0, 0.2)	Lognormal distribution	23 (2)	26 (2)	21 (2)	22 (2)	27 (2)	26 (2)
	Fourier transformation	22 (1)	22 (2)	18 (1)	20 (1)	26 (2)	19 (1)
(0, 0, 0.6)	Lognormal distribution	60 (5)	58 (5)	57 (5)	62 (5)	62 (5)	67 (5)
	Fourier transformation	55 (4)	55 (4)	53 (4)	59 (4)	61 (5)	54 (4)
(0, 0, 0.9)	Lognormal distribution	59 (6)	68 (6)	67 (6)	71 (6)	86 (7)	94 (7)
	Fourier transformation	71 (7)	67 (6)	71 (6)	70 (6)	88 (7)	83 (7)

Table 16: Comparison of the Model Results Using the Analytical Approximation and FRFT (Part II)

The table summarizes the FTDCVA in basis points of the CDS payer as computed using the lognormal distribution approximation and the Fast Fourier Transformation (FRFT) approach. The numbers in brackets stand for the standard errors. The investor (payer) has the following parameters: $y_0 = 0.0001, \kappa_0 = 0.9, \mu_0 = 0.001, \nu_0 = 0.01, LGD_0 = 0.6$. The reference entity has the following parameters: $y_1 = 0.01, \kappa_1 = 0.8, \mu_1 = 0.02, LGD_1 = 0.7$. The counterparty (receiver) has the following parameters: $y_2 = 0.03, \kappa_2 = 0.5, \mu_2 = 0.05, \nu_2 = 0.2, LGD_2 = 0.65$. The interest free rate r is set to be 0.03.

$(\rho_{0,1}, \rho_{0,2}, \rho_{1,2})$		FTDCVA	$(\rho_{0,1}, \rho_{0,2}, \rho_{1,2})$		FTDCVA
(0, 0, 0)	Lognormal distrib.	5 (0)	(0, 0.9, 0)	Lognormal distrib.	6 (0)
	Fourier transform.	6 (0)		Fourier transform.	6 (0)
	Capponi (2009)	6 (0)		Capponi (2009)	5 (0)
(0, 0, 0.1)	Lognormal distrib.	12 (1)	(0, 0.99, 0)	Lognormal distrib.	5 (0)
	Fourier transform.	12 (1)		Fourier transform.	6 (0)
	Capponi (2009)	15 (1)		Capponi (2009)	5 (0)
(0, 0, 0.3)	Lognormal distrib.	26 (2)	(0, 0.5, 0.5)	Lognormal distrib.	48 (4)
	Fourier transform.	30 (2)		Fourier transform.	46 (4)
	Capponi (2009)	37 (2)		Capponi (2009)	55 (4)
(0, 0, 0.6)	Lognormal distrib.	64 (5)	(0, 0.2, 0.9)	Lognormal distrib.	61 (6)
	Fourier transform.	52 (4)		Fourier transform.	53 (6)
	Capponi (2009)	73 (4)		Capponi (2009)	73 (6)
(0, 0, 0.9)	Lognormal distrib.	51 (6)	(0, 0.9, 0.2)	Lognormal distrib.	22 (2)
	Fourier transform.	68 (6)		Fourier transform.	22 (2)
	Capponi (2009)	75 (6)		Capponi (2009)	21 (2)
(0, 0, 0.99)	Lognormal distrib.	13	(0.5, 0.5, 0)	Lognormal distrib.	5 (0)
	Fourier transform.	15		Fourier transform.	5 (0)
	Capponi (2009)	25		Capponi (2009)	7 (0)
(0, 0.1, 0)	Lognormal distrib.	5 (0)	(0.2, 0.9, 0)	Lognormal distrib.	6 (0)
	Fourier transform.	5 (0)		Fourier transform.	5 (0)
	Capponi (2009)	6 (0)		Capponi (2009)	6 (0)
(0, 0.6, 0)	Lognormal distrib.	6 (0)	(0.8, 0.5, 0.2)	Lognormal distrib.	21 (2)
	Fourier transform.	6 (0)		Fourier transform.	23 (2)
	Capponi (2009)	6 (0)		Capponi (2009)	26 (1)
(0.2, 0.2, 0.2)	Lognormal distrib.	21 (2)	(0.5, 0.5, 0.5)	Lognormal distrib.	52 (4)
	Fourier transform.	25 (2)		Fourier transform.	43 (3)
	Capponi (2009)	25 (1)		Capponi (2009)	58 (4)

Table 17: Comparison of Analytical Approximation with FRFT and Capponi (2009) (Part III)

The table summarizes the FTDCVA in basis points of the CDS payer as computed using the lognormal distribution approximation and the Fast Fourier Transformation (FRFT) approach in comparison with the results published by Capponi (2009). The numbers in brackets stand for the standard errors. The investor (payer) has the following parameters: $y_0 = 0.0001, \kappa_0 = 0.9, \mu_0 = 0.001, \nu_0 = 0.1, LGD_0 = 0.6$. The reference entity has the following parameters: $y_1 = 0.01, \kappa_1 = 0.8, \mu_1 = 0.02, \nu_1 = 0.1, LGD_1 = 0.7$. The counterparty (receiver) has the following parameters: $y_2 = 0.03, \kappa_2 = 0.5, \mu_2 = 0.05, \nu_2 = 0.1, LGD_2 = 0.65$. The interest free rate r is set to be 0.03.

4.4. CASE STUDY

In the following we would like to return to our introductory example in which a financial institution buys protection in form of a CDS from a monoline insurance in order to hedge the credit risk arising from (one unit) senior structured note (RMBS). We assume a 5 year contract without a collateral agreement.

The case study will describe needed computational steps in more detail, especially with respect to calibrating the models to market data. It will also explore the insights offered by the model as well as its limits. In order to capture the dynamics caused by the financial crisis we will compute the first-to-default CVA based on real market data from 2006 as well as 2010. We especially expect to gain insights regarding the increasing significance of CVA in general and wrong way risk in particular.

The investor is defined as an average bank. For this purpose we estimated the average of the CDS spreads of five leading banks as seen in Table 18.[156] As an exemplary monoline insurer we chose Assured Corp., displaying the CDS spreads seen in Table 18. For the RMBS note we assume a constant CDS spread of 30 bps. and 400 bps. in the years 2006 and 2010, respectively. The significant shift in the curve of CDS spreads underlines the deteriorating credibility of all the parties involved due to the financial crisis.

Following the procedure described in Subchapter 3.3.4 we estimate the implied default probabilities (PD) of all involved parties based on the assumed CDS spreads. We assume the recovery rate to be 40% (i.e. LGD to be 60%), standing in line with market practice (see also Subsection 3.3.4). In order to limit the effect of the LGD assumption we use the same LGD in order to strip the market implied survival probabilities as well as within the CVA calculation.

[156] We used the same set of names in Subchapter 3.3.4 when we discussed calibration of default probabilities.

The CDS spreads imply the probabilities of defaults illustrated in Figure 19. The figure shows the drastic increase of default probabilities the market attributed (even to large and international) banks. While the market implied 5 year PD of an average bank was practically negligible before the crises, it exceeded 10% in 2010 as seen in Subfigure (b). Subfigure (a) shows how the market attributed almost credit risk-free status to monoline insurance companies before the crises, standing in line with their top rating. This changed dramatically in the course of the subprime crises. The situation in the monoline insurance market in general and the idiosyncratic risk of Assured Corp. in particular seem to have considerably increased the market implied probability of default across all maturities by 2010.

Using the closed-form solutions for survival probability given in Equation (4.23) and the market implied survival probabilities we subsequently calibrate the CIR parameters using a straight-forward least square (LS) method. The calibration results are listed in Table 19. The increased risk of all parties is especially visible through the surge in the volatility term ν.

Figure 20 displays a key advantage of stochastic intensity models in general and the SSRD approach in particular. The model-based survival probabilities fit the market implied ones across the whole term structure in a very satisfying manner.

We face the key shortfall of the CVA model however when estimating the correlation structure. As noted above we lack natural appropriate methodologies. For this reason we follow Cesari et al. (2009, p. 222) and study a range of possibilities instead of using one particular correlation matrix. Some scenarios seem plausible (e.g. the first example given in Table 20). Others are less plausible, but helpful to understand the mechanism of the model (e.g. the last scenario given in Table 20).

We then run the algorithm illustrated in Figure 16, delivering first-to-default CVA metrics for the years 2006 and 2010. The calculation of the conditional value of the CDS needed within the algorithm is based on the proposed lognormal approximation (for the cumulative distribution function, CDF). Note that we keep the maturity of the CDS contract (5 years) unchanged to exclude duration effects.

07.07.2006	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y
Banks:								
Deutsche Bank				10	12.8	13.8	16.2	23.2
Santander Bank				11	12.6	13.2	16	21
Barclays Capital				7	9.1	10	11.8	16
UniCredit				13.2	15.8	16.7	22	27
Citi		5	7.4	9.5	11.1	12.5	18	21
Banks' average		5	7.4	10.4	12.3	13.2	16.8	21.64
Monoline Insurer:								
Assured Corp.				18	21.6	26	32	41
07.07.2006	6M	1Y	2Y	3Y	4Y	5Y	7Y	10Y
Banks:								
Deutsche Bank	111.3	110.2	125.8	139.7	154.8	163.7	166.4	168.8
Santander Bank	151.1	149.7	161.3	174.1	186.6	194.1	188.6	192.2
Barclays Capital	110.0	109.1	124.2	138.4	150.4	159.8	164.7	168.4
UniCredit	124.7	123.6	140	153.6	165.1	171.9	176.5	174.3
Citi	138.2	136.9	142	147.5	153.5	176.0	159.0	158.1
Banks' average	127.1	125.9	138.7	150.7	162.1	173.1	171.0	172.4
Monoline Insurer:								
Assured Corp.	846.3	867.7	888.3	888.5	887.3	914	858.5	826.5

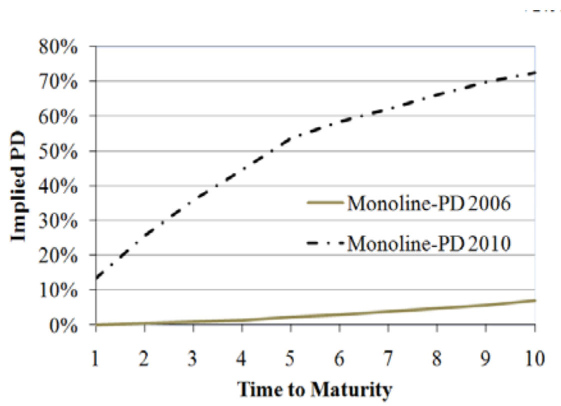
Table 18: CDS Spreads of Five Leading Banks and One Monoline Insurer

The table summarizes the CDS spreads (in basis points) of five leading banks and one exemplary insurance company in the year 2006 and 2010, respectively. Source is Bloomberg.

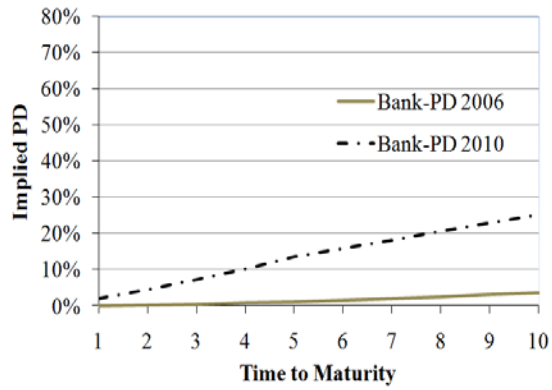
2006	y_0	κ	μ	ν
Bank	0.0003	0.0312	0.0268	0.0942
Reference Entity	0.0057	0.2000	0.0044	0.0000
Monoliner	0.0000	0.1426	0.0152	0.0050
2010	y_0	κ	μ	ν
Bank	0.0025	2.1462	0.0312	0.5000
Reference Entity	0.0719	35.2474	0.0664	0.0040
Monoliner	0.1522	0.0014	0.0009	0.1008

Table 19: Calibrated CIR Parameters for the Three Entities

The table summarizes the calibration results of the CIR process (intensity process) for the three entities, Bank (name "0"), Reference Entity (name "1"), and Monoliner (name "2").



e) Market implied PDs for an exemplary monoline insurance company



f) Market implied PDs for an exemplary bank

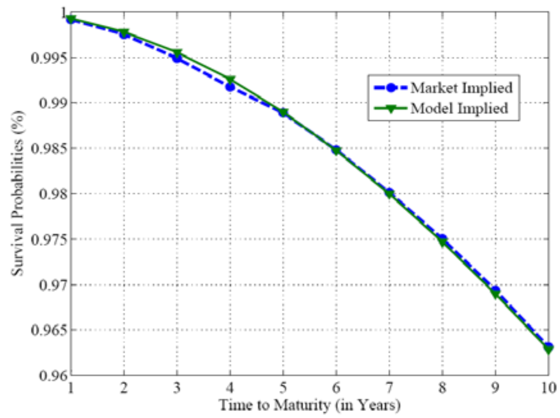
Figure 19: Market Implied Probabilities of Default for an Exemplary Bank

The figures display the market implied probabilities of default (PDs) calibrated using the CDS spreads provided in Table 18.

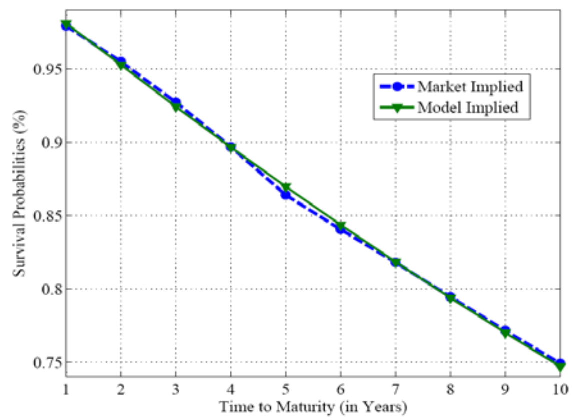
Correlation matrix ($\boldsymbol{\rho_{0,1}}, \boldsymbol{\rho_{0,2}}, \boldsymbol{\rho_{1,2}}$)	**FTDCVA (2006)**	**FTDCVA (2010)**	**CVA (2010) (% of CDS value)**
(0, 0.5, 0.5)	5 (0)	179 (4)	10%
(0, 0.2, 0.9)	17 (2)	267 (4)	16%
(0, 0.9, 0.2)	21 (3)	330 (9)	19%
(0.5, 0.5, 0)	7 (1)	198 (5)	11%
(0.2, 0.9, 0)	10 (1)	220 (5)	13%

Table 20: First-to-default CVA Results

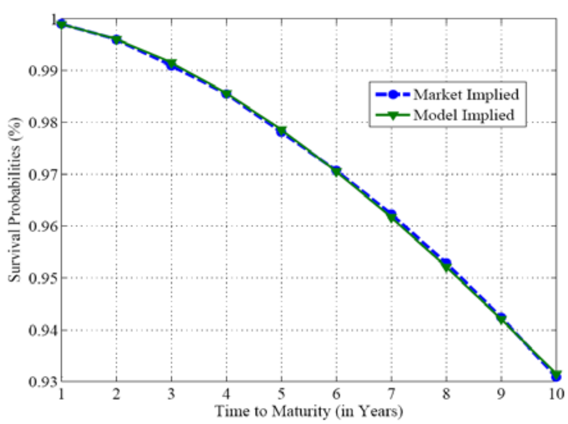
The table summarizes the FTDCVA in basis points of the CDS payer as computed using the lognormal distribution approximation. The numbers in brackets stand for the standard errors. The CDS has a maturity of 5 years and a CDS-spread of 30 bps. The last column summons the adjustments w.r.t the value of the CDS in 2010 (i.e. 1726 bps.). The investor (payer) is an average bank, the reference entity is a structured note and the counterparty (receiver) is a monoline insurer. All are assumed to have an LGD of 0.6. The CIR parameters of the three parties are given in Table 19.



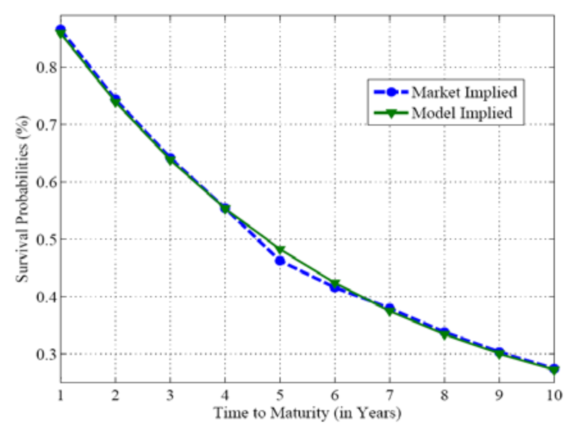
a) Survival Probabilities of the Bank in 2006



b) Survival Probabilities of the Bank in 2010



c) Survival Probabilities of the Monoliner in 2006



d) Survival Probabilities of the Monoliner in 2010

Figure 20: Market Implied Survival Probabilities vs. Model Implied Survival Probabilities

The figures display the market implied survival probabilities in comparison with the model implied survival probabilities of the average bank and an exemplary monoline insurer in 2006 and 2010, demonstrating the calibration of the CIR processes to market data.

The results shown in Table 20 are to be interpreted as follows. If the bank had bought a protection from a monoline insurer in 2006, it would have had to adjust the value of the CDS by 5 to 21 basis points, depending on the correlation matrix assumed. Assuming a correlation structure similar to the one in the first example might have already been conservative, and the CVA metric delivered is still rather negligible. This is due to the low CDS spreads observed (and the respective implied survival probabilities). Note that the initial value of the CDS in 2006 is zero.

The picture changes significantly in 2010. In the meanwhile the counterparty credit risk-free value of the CDS increased from 0 to 1726 bps. due to the increased risk of the reference entity, this being a key driver behind the increase in the CVA metrics across the correlation scenarios. A further driver is of course the increased risk of the counterparty. This becomes evident if we look at the second correlation scenario which seems plausible for 2010. In this case the bank would need to adjust the value of the CDS by approximately 16%. Notice that for the monoline insurer the CVA increase (e.g. from 5 to 267 bps.) would imply a profit due to the worsening of own credit risk.

4.5. CRITICAL EVALUATION

In the following we evaluate the model offered by Brigo and Capponi (2010), focusing on the capabilities of the approach in delivering its proclaimed target, i.e. "arbitrage-free framework" to price symmetric CVA for CDS. After discussing the theoretical and technical strengths and pitfalls we turn to discuss the practical "use test" of the model. Whilst some issues are only relevant for the specific approach at hand many arguments have a broader scope, aiming on intensity models, the use of (Gaussian) copula functions and derivative pricing models in general.

In principle the framework *does* capture default dependency, delivering a theoretically coherent and symmetric first-to-default CVA, i.e. incorporating the credit risk of both the investor and the counterparty. By modeling the default dependency through a copula function the approach allows for separate modeling of credit spread dynamics on the one hand and the default dependency on the other. This can be seen as an advantage from a practicality perspective, less so if mathematical rigor is central. After all, the calibration of a function with two or more parameters should not be done sequentially but simultaneously.

Credit spread dynamics are modelled through a standard stochastic intensity approach. Incorporating jumps would have facilitated the calibration of higher (but possible) implied volatilities as discussed in Brigo and El-Bachir (2008). Still, the chosen (and implemented) SSRD approach is not only mathematically convenient, it also captures market implied survival probabilities in a very satisfying manner.

Default is defined when the cumulative intensity process exceeds a respective default trigger. A Gaussian copula captures the dependency of the default triggers of the investor, the counterparty and the reference entity. Whereas the possibility of joint default is excluded. This assumption does not, however, seem that unrealistic, especially if the discretization is subsequently done in relatively small time steps (e.g. trading days). The Gaussian copula introduces convenient analytical tractability to the model, e.g. with respect to computing partial derivatives etc. The statistical rigor of using Gaussian copulas to model the dependency of default triggers remains controversial (see Mikosch, 2005). In addition to assuming constant default correlation, it fails to capture the "tails" of the distribution, systematically underestimating the conditional probability of default, which is exactly the proclaimed target of the "CVA with wrong-way risk" model. Moreover, as discussed above in Subchapter 4.3.5 the model tends to underestimate CVA in extreme scenarios when correlation between the reference entity and the counterparty is close to one, especially if the reference entity is relatively less risky.

Besides default risk the approach assumes deterministic interest rates and constant recovery rates. While the small role interest rates play in this case might support the negligence of their stochasticity, this does not apply to recovery rates. Li (2009) shows that although not as important as default correlation the volatility of recovery rates can drive CVA significantly. Modeling stochastic recovery rates will, however, introduce a further layer of complication and uncertainties, especially with regards to calibrating the recovery rate volatility.

A more relevant missing characteristic is collateralization. As seen in Table 14 credit derivatives are almost always collateralized. It needs, however, to be noted that especially in infamous examples of CDS contracts with wrong way risk (e.g. monoline insurers selling CDS on RMBS) no collateral agreements were put in place. Moreover, the framework *has* been extended by Brigo et al. (2013b) to model collateralization, allowing also for re-hypothecation.

More importantly while Brigo and Capponi (2010) *do* offer a risk-neutral framework to model CVA, they do *not* specify the needed duplication strategy. This can be pointed out as the key pitfall of the model, questioning the applicability and practicability of the model. First, we lack market quotes to calibrate a risk-neutral correlation matrix. Second, the duplication strategy would imply hedging own credit risk. As discussed in Subchapter 3.4.2 hedging DVA is already an almost impossible task. Let alone hedging the sensitivity of a first-to-default CVA to changes in own credit risk, which involves hedging sensitivities of conditional exposure values given a default of a counterparty and possible contagion effects.

Thus, although theoretically coherent the model exhibits significant pitfalls that limit its practical use, especially when it comes to pricing and hedging. Following the terminology of Taleb and Martin (2012) a trader hedging his positions based on the model presented would resemble a pilot flying according to the map of a different territory. Bearing in mind that models will principally deviate from reality, the impossibility of structuring a duplicating strategy calls the whole approach into question. It can be argued that using such "flawed" models ("wrong maps") might even increase the risk as it induces overconfidence – also referred to as "model-dope" (MacKenzie and Spears, 2014) – and allows for "charlatanism" (Taleb, 2012).[157] This links into what Hayek (1942) calls "scientism", i.e. the illusion of science. In their analysis of the Gaussian copula and its role in the financial crisis MacKenzie and Spears (2014) do confirm the risk of models being – "gamed" and exploited. They find, however, evidence that questions the existence of blind confidence in models, concluding that "model dope" notions should be treated as forms of "othering".[158] More importantly, they point to the coordinating role financial models play, facilitating communication within and between organizations, e.g. "*providing a shared yardstick that enabled accountants and auditors to determine whether a valuation was correct and risk managers to assess whether a position was properly hedged […]*".

Returning to the model at hand this means that while bearing the risks of using the model in mind, we need to analyze its use as a communication tool. Because the model is based on a standard modeling approach (e.g. reduced form approach in conjunction with Gaussian copula) while preserving theoretical coherence it can provide a framework to discuss the

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[157] See also Taleb (2012), pp. 238-240, Taleb (2007), p. xxv or Taleb (1997), pp. 445-453.

[158] MacKenzie and Spears (2014) cite amongst others the following research papers that question the notion of model dope: Beunza and Stark (2010) and Svetlova (2012). Notice that also Haug and Taleb (2011) argue that traders are not "blind" to the pitfalls of the Black-Scholes formula.

dynamics of wrong-way risk but also of DVA pricing. Such a discussion is not only of academic nature, but can contribute relevant insights to the regulatory and accounting discourse, revealing the significance of wrong way risk (especially amid the financial crisis) and shedding light on the controversies of accounting for DVA (see previous Section 4.4). Yet, it is exactly this "theoretical coherence" that limits the use of the model for inter- and intra-organizational communication. Simpler approaches (as the one discussed in Subchapter 3.3.2) in which the different building blocks (esp. exposure profile) are derived and presented in a more straightforward manner seem more appropriate for this purpose. Still, the model at hand can facilitate discussions around the specific (and rare) case of CVA for a netting set consisting only of one CDS contract (with not collateralization). It can for example offer an inter-subjective framework for discussions between banks on the one hand, and auditors and supervisors on the other, e.g. "*providing a shared yardstick*" to determine whether they have correctly adjusted for the counterparty risk arising from the credit protections they bought from monoline insurers, referencing to RMBS notes.

4.6. CONCLUDING REMARKS

We chose to focus on the implementation and analysis of the model proposed by Brigo and Capponi (2010), because it was one of the first CVA models promising an arbitrage-free framework for symmetric pricing in conjunction with dependency between probabilities of default and exposure.

First, we laid the grounds for the analysis by revisiting the mechanisms behind CDS contracts. We then placed the approach offered by Brigo and Capponi (2010) within the category of reduced-form credit risk models that use (Gaussian) copula functions to model default dependency. Subsequently, we decomposed the model which allowed a thorough analysis of the benefits and pitfalls of the model. In addition, we provided a step-by-step implementation guide, especially going into detail into the aspects that Brigo and Capponi (2010) left relatively open – the computation of the survival probability of the reference entity conditional on the default of either the counterparty or the investor. It has been illustrated in detail how the fractional Fourier transform (FRFT) can be used for this purpose. We also proposed a computational tune-up through a heuristic approximation, which reduces the complexity of the elaborate implementation and speeds-up the computation, while delivering satisfying pricing results.

We synthesize that while Brigo and Capponi (2010) *do* offer a coherent and risk-neutral framework, they do *not* specify the needed duplication strategy. The lack of instruments to calibrate a risk-neutral correlation matrix and possibility to hedge own credit risk puts the arbitrage-freedom of the model into question. In addition, we note that the theoretical coherence of the model actually limits its use for inter- and intra-organizational communication, and suggest that simpler approaches are more appropriate as they facilitate discussions around the different building blocks (e.g. exposure profile).

This does not deprive the model from the possibility of facilitating discussions around the specific case of CVA for CDS contracts (with no collateralization). After all, the model is able to explain the high adjustments needed for CDS on structured notes that were bought from monoline insurance, i.e. meeting the demand following the financial crisis.

CHAPTER 5: OVERALL CONCLUSION

This thesis can be placed within the literature on market and counterparty credit risk, contributing along the following three dimensions:

- *Interest rate risk management.* We gave an overview on asset liability management (ALM) in general and interest rate risk management in particular. In that respect we offered a novel procedure for structuring swap overlays for pensions funds, allowing for optimal hedging of interest rate risk without affecting the strategic asset allocation. We also extended the analysis of the Cairns (2004) stochastic interest rate model. Besides deriving respective model-based sensitivity measures (Cairns deltas), we applied the two-factor version of the model to the practical application of ALM for pension funds, analyzing its strengths and weaknesses when it comes to long-term contracts.

- *Pricing and managing counterparty credit risk.* We offered a compact overview on counterparty credit risk and credit valuation adjustment (CVA), running unique analyses around valuation, relevant accounting and regulatory requirements as well as pricing and mitigation. We illustrated how the CVA capital charge shows the tautology behind many of the discussions around regulatory requirements. It reveals how financial institutions are heavily driven by regulatory requirements that originally aimed to actually mirror the way banks "do business". In addition, we showed that the discourse around CVA and counterparty credit risk cannot be seen as a "pure scientific search for truth" as it is dominated by lobby-like argumentations. We agreed for example that the regulatory CVA capital charge will indeed lead to an increase in hedge costs. We questioned, however, whether the increase will be as significant as the evidence shows. We argued rather that banks have not been pricing for CVA adequately in the past. This might be due to the fact that banks did not think systematically about counterparty credit risk, because no metric pointed to it.

- *CVA modeling and wrong way risk*. We gave an overview on credit risk modeling in general and credit spreads in particular, revisiting the CVA for CDS model introduced by Brigo and Capponi (2010). We offered a step-by-step implementation guide, elaborating on the parts Brigo and Capponi (2010) left open. We especially offered a computational tune-up, and demonstrated its robustness across a variety of scenarios. After illustrating the use of the model using a realistic case study, we ran a novel analysis of the Brigo and Capponi (2010) model in particular and CVA modeling in general.

With regards to the overall discourse around financial derivatives we were able to illustrate the value added by using swaps to hedge interest rate risk of pension funds. We showed that financial derivatives can offer considerable benefits, allowing risk to be borne by the ones most fit to do so. By immunizing their portfolios against changes in interest rates, fund managers pursue the noble objective of ensuring the funding of pension payments. The discussion around counterparty credit risk, however, reveals a range of risks and challenges derivatives imply. This means especially that while structuring a swap overlay fund managers will need to consider counterparty credit risk, possibly pricing (bilateral) CVA. Most importantly they will need to accommodate to the changing needs of their counterparties, i.e. banks that are under regulatory pressure to collateralize their derivative exposure. For pension fund managers full collateralization involves funding considerable amounts of liquid assets, possibly affecting the strategic asset allocation and thus questioning the benefits offered by derivatives in the first place.

A similarly ambivalent evaluation is evident if we consider the models to price financial derivatives and to capture the dynamics of the underlying risk factors. In the case of the Cairns model we found an approach that *can* exhibit benefits, especially in modeling short- as well as long-term interest rates, facilitating founded discussions around investment discussions. However, by focusing on "realistic" modeling in times of considerable (and stochastic) basis spreads in combination with negative interest rates the model does also seem superseded.

Also in the case of Brigo and Capponi (2010) we were able to show that the model can be used for inter- and intra-organizational communication, albeit restricted to rather limited application. We concluded that while Brigo and Capponi (2010) *do* offer a coherent and risk-neutral framework, they do *not* specify the needed duplication strategy. The lack of instruments to calibrate a risk-neutral correlation matrix and possibility to hedge own credit risk puts a question mark on the applicability and practicability of the model. A similar tone is echoed in our discussion around pricing and managing CVA in general as we reveal the limits of arbitrage-free valuation, especially when it comes to the practical implementation of pricing models or constructing adequate hedges. Clinging to use elaborate models that need overcomplicated calibration without reflecting on their economic sense might not only imply mere model and valuation risks, but also significant financial risks at the latest when it comes to hedging.

We conclude that computational finance has been able to offer novel innovations, accompanied by elaborate mathematics that facilitate respective inter- and intra-organizational communication. Still, the unlimited complication of the real world in general and financial markets in particular has continuously challenged these efforts fundamentally. This reminds us of how new this field of research is, having yet a long journey ahead of it. In the meantime a considerable amount of skepticism and humility will do both academics and practitioners well. An attitude this thesis cannot and does not want to free itself from. After all, as put by Popper (1945, p. 249):

> *"You may be right and I may be wrong, and by an effort, we may get nearer to the truth"*

APPENDIX A: THE FRAMEWORK OF FLESAKER AND HUGHSTON

In the following we revisit the framework offered by Flesaker and Hughston (1996) for pricing zero coupon bonds, and elaborate on some extensions given by Cairns (2004).

The general positive-interest rate framework of Flesaker and Hughston (1996) models bond prices with

$$P(t,T) = \frac{\int_{T-t}^{\infty} H\big(u, X(t)\big)du}{\int_{0}^{\infty} H\big(u, X(t)\big)du} \tag{A.1}$$

for some function $H(u,x)$. In order to proof the above let $P(a,b)$ be the price at time a of a zero coupon bond that matures at time b (b > a). At maturity the bond price equals 1 unit. This price function is further differentiable at time b. Set $P(a,T)$ as the pricing measure (numeraire) so that

$$\frac{P(a,b)}{P(a,T)} = N(a,b). \tag{A.2}$$

It can be shown that $N(a,b)$ is a martingale. Because $P(a,a) = 1 \Rightarrow \frac{1}{P(a,T)} = N(a,a)$. Equation (A.2) can be transformed to

$$P(a,b) = P(a,T)N(a,b) = \frac{1}{N(a,a)} = \frac{N(a,b)}{N(a,a)}. \tag{A.3}$$

The ratio of two bond prices is given as

$$\frac{P(a,c)}{P(a,b)} = \frac{P(a,T)N(a,c)}{P(a,b)} = N(a,c)\frac{1}{N(a,b)} = \frac{N(a,c)}{N(a,b)}. \tag{A.4}$$

If for $c > b, \frac{P(a,c)}{P(a,b)} < 1$ then positive interest rates are obtained. This is reached if $N(a,c) < N(a,b)$, meaning that the derivative of this martingale (which is a martingale in itself) is subject to $\frac{\partial N(a,b)}{\partial b} < 0$.

There exists a family of martingales $M(a,b)$ that for $0 \leq a \leq b \leq s$ has the following characteristics:

1. $M(a,s) = E_a[M(b,s)]$
2. $M(a,s) > 0$
3. $M(0,s) = 1$
4. $\lim_{s\to\infty} M(0,s) = 1$

Using this family of martingales we define:

$$\frac{\partial N(a,b)}{\partial b} = \frac{\partial N(0,b)}{\partial b} M(a,b). \tag{A.5}$$

$N(a,T)$ being 1 and $N(0,b) = \frac{P(0,b)}{P(0,T)}$ then:

$$N(a,b) = 1 - \frac{1}{P(0,t)} \int_b^T \frac{\partial P(0,s)}{\partial s} M(a,s)ds. \tag{A.6}$$

Then by Equation (A.3) we have:

$$P(a,b) = \frac{P(0,t) - \int_b^T \frac{\partial P(0,s)}{\partial s} M(a,s)ds}{P(0,t) - \int_a^T \frac{\partial P(0,s)}{\partial s} M(a,s)ds}. \tag{A.7}$$

By taking the maturity of the numeraire to infinity we obtain:

$$P(a,b) = \frac{\int_b^\infty \frac{\partial P(0,s)}{\partial s} M(a,s)ds}{\int_a^\infty \frac{\partial P(0,s)}{\partial s} M(a,s)ds}. \tag{A.8}$$

Cairns (2004) defines $\phi(s) = \frac{\partial P(0,s)}{\partial s}$s) so that:

$$P(t,T) = \frac{\int_b^\infty \phi(s) M(a,s)ds}{\int_a^\infty \frac{\partial P(0,s)}{\partial s} M(a,s)ds}. \tag{A.9}$$

Cairns (2004) defines $M(t,T)$ by the following assumptions:

$$dM(t,T) = M(t,T)\sum_{i=1}^{n} \sigma_i(t,T) d\hat{Y}_i(t) \tag{A.10}$$

where $d\hat{Y}_i(t) = Cd\hat{Z}(t), \hat{Y}(0) = 0,$ and $\hat{Z}_1(t), \dots, \hat{Z}_n(t)$ are n independent Brownian motions under measure $\hat{P}$. C is calculated as a matrix with $c_{i,j}$ such that $CC' = \left(\rho_{i,j}\right)_{i,j=1}^{n}$ is the instantaneous correlation matrix between the generated Brownian motions $\hat{Y}_1(t), \dots, \hat{Y}_n(t)$ under the measure $\hat{P}$. Equation (A.10) can thus be written as:

$$dM(t,T) = M(t,T)\sum_{i=1}^{n} \sigma_i(t,T) \sum_{n}^{j=1} c_{i,j} d\, Z_i(t) \tag{A.11}$$

and

$$dlogM(t,T) = \sigma_i(t,T)'\hat{Y}(t) - \tfrac{1}{2}\textstyle\sum_{i,j=1}^{n} \sigma_i(t,T)\sigma_j(t,T)\, \underbrace{d\langle \hat{Y}_i(t), \hat{Y}_j(t)\rangle}_{d\rho_{i,j}}. \tag{A.12}$$

Defining $\sigma_i(t,T) = \sigma_i e^{-\alpha_i(T-t)}$ then:

$$dlogM(t,T) = \sum_{i=1}^{n} \sigma_j \int_0^t e^{-\alpha_i(T-s)}\, d\hat{Y}_i(t) - \frac{1}{2}\sum_{i,j=1}^{n} \rho_{i,j}\sigma_i\sigma_j \int_0^t e^{-(\alpha_i+\alpha_j)(T-s)}\, ds \tag{A.13}$$

or

$$dlogM(t,T) = \sum_{i=1}^{n} e^{-\alpha_i(T-t)} \underbrace{\hat{X}_i(t)}_{\int_0^t e^{-\alpha_i(T-s)} d\hat{Y}_i(t)} - \frac{1}{2}\sum_{i,j=1}^{n} \rho_{i,j}\sigma_i\sigma_j \int_0^t e^{-(\alpha_i+\alpha_j)(T-s)}\, ds$$

with $\hat{X}_i(t) = \int_0^t e^{-\alpha_i(T-s)} d\hat{Y}_i(t)$. So an Ornstein-Uhlenbeck process with $\hat{X}_i(t) = 0$ and $d\hat{X}_i(t) = -\alpha_i \hat{X}_i dt + \hat{Y}_i(t)$ is introduced as the risk generator.

In addition to Flesaker and Hughston (1996), Cairns (2004) further defines $\phi(s)$ from Equation (A.9) as:

$$\phi(s) = \phi e - \beta s + \sum_{i=1}^{n} \sigma_i \hat{x}_i e^{-\alpha_i s} - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\rho_{i,j}\sigma_i\sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i+\alpha_j)s} \tag{A.14}$$

for the parameters $\phi, \beta, \hat{x}_1, \dots, \hat{x}_n$. Then, for $t < s$:

$$\phi(s)M(t,s) = \phi e - \beta s + \sum_{i=1}^{n} \sigma_i e^{-\alpha_i(s-t)} X_i - \frac{1}{2}\sum_{i,j=1}^{n} \frac{\rho_{i,j}\sigma_i\sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i+\alpha_j)(s-t)} \tag{A.15}$$

whereas X_i is an Ornstein-Uhlenbeck process under the measure $\hat{P}$. with $X_i(0) = \hat{x}_i, X_i = \hat{x}_i e^{-\alpha_i t + \hat{X}_i(t)}$ and $dX_i(t) = -\alpha_i X_i(t)dt + d\hat{Y}_i(t)$ or:

$$dX_i(t) = \alpha_i X_i(t)dt + d\textstyle\sum_{j=1}^{n} c_{ij}\, d\hat{Z}_j(t). \tag{A.16}$$

Now if:

$$A(t,T) = \int_T^\infty \phi(s)M(t,s)ds = \phi e^{-\beta t} \int_{T-t}^\infty H\big(u, X(t)\big)du \tag{A.17}$$

where

$$H(u,x) = e - \beta u + \sum_{i=1}^{n} \sigma_i x_i e^{-\alpha_i u} - \frac{1}{2} \sum_{i,j=1}^{n} \frac{\rho_{i,j}\sigma_i\sigma_j}{\alpha_i + \alpha_j} e^{-(\alpha_i+\alpha_j)u} \tag{A.18}$$

which returns us back to Equation (A.1).

APPENDIX B: RISK-NEUTRAL VALUATION PARADIGM OF HARRISON AND PLISKA

Following the terminology of Brigo and Mercurio (2006) we will shortly recall the risk-neutral valuation paradigm offered by Harrison and Pliska (1983).

Each derivative with a stochastic payment at a future time T has a unique price at time period t ($t < T$) under the risk-neutral expectation $\mathbb{E}_t^{\mathbb{Q}}$. Given r as the risk free instantaneous discount rate, the risk-neutral expectation can be formalized as follows:

$$\mathbb{E}_t^{\mathbb{Q}}\left[\underbrace{exp\left(-\int_t^T r_s\, ds\right)}_{Stochastic\ discount\ factor}\ Payoff(Asset)_T\right]. \tag{A.19}$$

Hence all underlying assets must have a deterministic (and risk free) drift rate as an expected return. In order to compute the current value of uncertain payoffs one has to build the mean out of expectations that are discounted at the relevant (risk free) rate. The groundbreaking aspect about option pricing theory is that the real growth rate of the underlying asset (e.g. return on a stock) is not needed for pricing the derivative. This implies that in order to valuate a derivative instrument (e.g. interest rate swap) two investors do not have to have the same expectations on the future growth rate of the underlying (e.g. growth of the reference interest rate).

Assume a zero coupon bond (ZCB) with no credit risk that pays one unit of currency at time T. The value of the ZCB at time t can thus be written as follows:

$$P(t,T) = \mathbb{E}_t^{\mathbb{Q}}\left[exp\left(-\int_t^T r_s\, ds\ 1\right)\right],\ P(T,T) = 1. \tag{A.20}$$

The remaining interest rates needed for the following discussion on models can be expressed using ZCB prices.

The interbank reference spot Libor/Euribor rate with a maturity of T at time t can be expressed as:[159]

$$L(t,T) = \frac{1 - P(t,T)}{(T-t)P(t,T)}. \tag{A.21}$$

The forward Libor rate at time t, expiry T_{i-1} and maturity T_i can be computed as follows:

$$F_i(t) = \frac{1}{T_i - T_{i-1}}\left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right). \tag{A.22}$$

The periodic fixed swap rate of an interest rate swap referencing to the Libor rate with the tenor structure T_α, $T_{\alpha+1}$, …, T_β is given by:

$$s_{\alpha,\beta}(t) \coloneqq \frac{P(t,T_\alpha) - P\left(t,T_\beta\right)}{\sum_{i=\alpha+1}^{\beta}(T_i - T_{i-1})P(t,T_i)}\left(\frac{P(t,T_{i-1})}{P(t,T_i)} - 1\right). \tag{A.23}$$

Notice that swap rates are market rates with observable quotes in the market. While spot and forward Libor rates as well as swap rates can be stripped from ZCB, the price of ZCB is a function of the expected dynamics of the short rate.

[159] Notice that we are working within the so-called "single curve" paradigm. We are assuming no default risk of the bond issuer. Within this paradigm the fixing frequency of a reference rate does not have an influence on the rate, implying for example that 12-month Euribor and 3-month Euribor must be equal (e.g. on a per annum basis). Since the financial crises begging in 2008 this assumption, however, has been greatly falsified.

APPENDIX C: REVISITING THE PROOF FOR THE CONDITIONAL SURVIVAL FUNCTION

In the following we give the proof of the survival probability formula used in the this paper as given by Brigo and Capponi (2009) and Capponi (2009) while elaborating on some issues in a more detailed manner.

Proposition. *The conditional survival of the Reference Entity is given by:*

$$1_{C\cup D}1_{\tau_1>\tau_2}\mathbb{Q}\left(\tau_1 > t|\mathcal{G}_{\tau_2}\right) = \tag{A.24}$$

$$1_{\tau_2\leq T}1_{\tau_2\leq\tau_0}\left(1_{\bar{A}} + 1_{\tau_2<t}1_{\tau_1\geq\tau_2}\int_{\bar{U}_{1,2}}^{1} F_{\Lambda_1(t)-\Lambda_1(\tau_2)}\left(-\log(1-u_1)-\Lambda_1(\tau_2)\right)dC_{1|0,2}(u_1;U_2)\right)$$

where

$$C_{1|0,2}(u_1;U_2) = \tag{A.25}$$

$$\frac{\frac{\partial C_{1,2}(u_1,u_1)}{\partial u_2}|_{u_2=U_2} - \frac{\partial C\left(\bar{U}_{0,2},u_1,u_1\right)}{\partial u_2}|_{u_2=U_2} - \frac{\partial C_{1,2}\left(\bar{U}_{1,2},u_1\right)}{\partial u_2}|_{u_2=U_2} + \frac{\partial C\left(\bar{U}_{0,2},\bar{U}_{1,2},u_2\right)}{\partial u_2}|_{u_2=U_2}}{1 - \frac{\partial C_{0,2}\left(\bar{U}_{0,2},u_2\right)}{\partial u_2}|_{u_2=U_2} - \frac{\partial C_{1,2}\left(\bar{U}_{1,2},u_2\right)}{\partial u_2}|_{u_2=U_2} + \frac{C\left(\bar{U}_{0,2},\bar{U}_{1,2},u_2\right)}{\partial u_2}|_{u_2=U_2}}.$$

Again, this is the survival probability of the reference entity, conditional on the default of the counterparty (given $\tau = \tau_2$). The term $\mathbb{Q}\left(\tau_1 > t|\mathcal{G}_{\tau_2}\right)$ stands for the risk-neutral probability that the reference entity outlives the time period of computation t, given all information available when τ_2 becomes known. $1_{C\cup D}$ takes on the value 1 when the counterparty defaults before the investor and the reference entity. $1_{\tau_1>\tau_2}$ ensures the consideration of the scenarios in which the default of the reference entity exceeds the default of the counterparty.

Proof. The term for the survival probability can be given as

$$1_{C\cup D}1_{\tau_1>\tau_2}\mathbb{Q}\big(\tau_1>t|\mathcal{G}_{\tau_2}\big)=1_{\tau_2\leq T}1_{\tau_2\leq\tau_0}\big(1_{t<\tau_2<\tau_1}+1_{\tau_2<t}1_{\tau_1\geq\tau_2}\mathbb{E}\big[\mathbb{Q}\big(\Lambda_1(t)<\xi_1|\mathcal{G}_{\tau_2},\xi_1\big)|\mathcal{G}_{\tau_2}\big]\big). \tag{A.26}$$

The term outside of the brackets means that we are only interested in the scenarios in which the counterparty defaults before the maturity of the contract and before the investor defaults ($1_{C\cup D}{\sim}1_{\tau_2\leq T}1_{\tau_2\leq\tau_0}$). Inside the brackets we differentiate between two terms. The first term stands for the scenarios in which the default of the reference entity exceeds the default of the counterparty and the time period of computation ($1_{t<\tau_2<\tau_1}$). We define this scenario as $\bar{A}$. The second term inside the brackets stands for the survival probability of the reference entity (i.e. the risk-neutral probability that the barrier exceeds the value of the intensity process) conditional on the scenarios, in which the counterparty has defaulted before the time period of the computation ($1_{\tau_2<t}$) and before the investor ($1_{\tau_1\geq\tau_2}$). After solving the brackets, we insert the conditions into the risk-neutral probability calculations as follows:

$$1_{C\cup D}1_{\tau_1>\tau_2}\mathbb{Q}\big(\tau_1>t|\mathcal{G}_{\tau_2}\big)=1_{\tau_2\leq T}1_{\tau_2\leq\tau_0}1_{\bar{A}}+\mathbb{E}\big[1_{\tau_2<t}1_{\tau_1\geq\tau_2}1_{\tau_2\leq\tau_0}\mathbb{Q}\big(\Lambda_1(t)<\xi_1|\mathcal{G}_{\tau_2},\xi_1\big)|\mathcal{G}_{\tau_2}\big]. \tag{A.27}$$

Notice that $\Lambda_1(t)<\xi_1$ is the same as $\Lambda_1(t)-\Lambda_2(\tau_2)<\xi_1-\Lambda_2(\tau_2)$. Once we condition on the known terms ξ_1 and $\mathcal{G}_{\tau_2}$, the term $\xi_1-\Lambda_2(\tau_2)$ becomes also known. We introduce $F_{\Lambda_1(t)-\Lambda_1(\tau_2)}$ as the cumulative distribution function of the integrated CIR process $\Lambda_1(t)-\Lambda_1(\tau_2)$. We then rewrite Equation (A.27) into

$$1_{\bar{A}}+1_{\tau_2<t}1_{\tau_1\geq\tau_2}\mathbb{E}\big[F_{\Lambda_1(t)-\Lambda_1(\tau_2)}\big(-\log(1-U_i)-\Lambda_1(\tau_2)\big)|\mathcal{G}_{\tau_2},\{\xi_1>\Lambda_1(\tau_2)\},\{\xi_0>\Lambda_1(\tau_2)\}\big]. \tag{A.28}$$

In the next step we denote $\bar{U}_{i,j}=1-\exp\big(\Lambda_i(j)\big)$, where $i;\, j\,=\,0,1,2$ stand for the three entities. This would in return imply that $\xi_1-\Lambda_1(\tau_2)$ equals $-\log\big(1-\bar{U}_{i,j}\big)=\Lambda_1(\tau_2)$. Equation (A.28) can then be rewritten as:

$$1_{\bar{A}} + 1_{\tau_2<t} 1_{\tau_1 \geq \tau_2} \mathbb{E}\left[F_{\Lambda_1(t)-\Lambda_1(\tau_2)}\left(-\log(1-U_1) - \Lambda_1(\tau_2)\right) | \mathcal{G}_{\tau_2}, \{U_1 > \Lambda_1(\bar{U}_{1,2})\}, \{U_0 > \bar{U}_{0,2}\}\right]. \qquad \text{(A.29)}$$

The risk-neutral probability term can be rewritten using the appropriate integral of the cumulative distribution function, multiplied with the marginal distribution. The right term of Equation (A.29) can be written as follows:

$$\begin{aligned} &1_{\tau_2<t} 1_{\tau_1 \geq \tau_2} \int_0^1 F_{\Lambda_1(t)-\Lambda_1(\tau_2)} \left(-\log(1-U_1)\right) \cdots \\ &\qquad - \left(\Lambda_1(\tau_2)\right) dQ(U_1 < u_1) | \mathcal{G}_{\tau_2}, \{U_1 > \bar{U}_{1,2}\}, \{U_0 > \bar{U}_{0,2}\} = \end{aligned} \qquad (4.56)$$

$$\begin{aligned} &1_{\tau_2<t} 1_{\tau_1 \geq \tau_2} \int_{\bar{U}_{1,2}}^1 F_{\Lambda_1(t)-\Lambda_1(\tau_2)} \left(-\log(1-U_1)\right) \cdots \\ &\qquad - \left(\Lambda_1(\tau_2)\right) dQ(U_1 < u_1) | \mathcal{G}_{\tau_2}, \{U_1 > \bar{U}_{1,2}\}, \{U_0 > \bar{U}_{0,2}\}. \end{aligned}$$

APPENDIX D: FIRST-TO-DEFAULT CVA FOR CDS – IMPLEMENTATION IN R

In the following we have added snapshots of the code (written in R) for the main functions needed in the computation the model proposed.

```r
# CIR parameters
  CIRpar0    <- c(y00, mu0, kap0, vega0,(4 * kap0 * mu0)/(vega0*vega0))
  CIRpar1    <- c(y01, mu1, kap1, vega1,(4 * kap1 * mu1)/(vega1*vega1))
  CIRpar2    <- c(y02, mu2, kap2, vega2,(4 * kap2 * mu2)/(vega2*vega2))

if (Generate_new_y == 1) {
for (j in 1:3) {
# generate for each party N CIR processes
y[,j,1:N] <-   sde.sim(t0 = 0, T = Tmax, X0=CIRpar[j,1], N = length(ii)-1, M = N,
               theta=c(CIRpar[j,2]*CIRpar[j,3],CIRpar[j,3],CIRpar[j,4]),
               rcdist=rcCIR, method="cdist")
# Shift-Term
Phi[,j]     <- log(PCIR_fun(CIRpar[j,], ii) / calc_sp3(CIRpar_org,j,ii))
diff_y      <- apply(y,c(2,3),diff)
# Cum. CIR process:
Y[-1,,] <- apply((y[-length(y[,1,1]),,]+diff_y/2)*delta_y, c(2,3), cumsum)
# Cum. CIR process + Shift-term
LAMBDA[,j,1:N]   <- Y[,j,1:N] + Phi[,j]       # Cum. CIR process + shift
}
}

# define as default time scenarios
for (j in 1:3) {
  for (k in 1:N)
   {
   default      <- min(as.integer(which(LAMBDA[,j,k] > Zeta[j,k])))
   taus[k,j]    <- delta_y * (default-1)
   }
   taus[taus[,j]==Inf,j] <- Tmax
  }
```

Figure 21: R Code for the Generation of CIR processes and Default Scenarios

The figure displays the R-code for the simulation of the CIR processes and the correlated default scenarios.

```r
################################### MAIN LOOP ###################################
# For the number of default scenarios (N) calc. the CVA
for (i in 1:N) {
Y_with_T      <-  cbind(c(time_periods), ...
                  c(Y[,1,i]), c(Y[,2,i]), c(Y[,3,i]))
y_with_T      <-  cbind(c(time_periods), ...
                  c(y[,1,i]), c(y[,2,i]), c(y[,3,i]))
Lambda_wT     <-  cbind(c(time_periods), c(LAMBDA[,1,i]), ...
                  c(LAMBDA[,2,i]), c(LAMBDA[,3,i]))
# tau is a point in time with max. two decimal digits
    tau2 <- round(taus[i,3],rnd)
    tau0 <- round(taus[i,1],rnd)
# I: couterpary defaults before investor and underlying and contract maturity
 if (taus[i,3] < taus[i,1] && Tgamma_fun(taus[i,3],Pymnt_seq,Tgamma) < Tb){
  if (taus[i,2] > taus[i,3]) {
   U_1_ind       <- 1- exp(-Y_with_T[Y_with_T[,1]==Tgamma_fun(taus[i,3],...
                     Pymnt_seq,Tgamma),3] -
                     Phi_wT[Phi_wT[,1]==Tgamma_fun(taus[i,3],Pymnt_seq,Tgamma),3])
   U_nind_ind    <- 1- exp(-Y_with_T[Y_with_T[,1]==Tgamma_fun(taus[i,3],...
                     Pymnt_seq,Tgamma),2] -
                     Phi_wT[Phi_wT[,1]==Tgamma_fun(taus[i,3],Pymnt_seq,Tgamma),2])
   U_ind         <- 1 - exp(-Y_with_T[Y_with_T[,1]==Tgamma_fun(taus[i,3],...
                     Pymnt_seq,Tgamma),4] -
                     Phi_wT[Phi_wT[,1]==Tgamma_fun(taus[i,3],Pymnt_seq,Tgamma),4])
   BR_CVA_2[i,]  <- CDSAjust(Tgamma_fun(taus[i,3],Pymnt_seq,Tgamma),S1,LGD1,2,
                     U_1_ind, U_nind_ind, U_ind)
  ind_calc<<-c(ind_calc,i)
  cum_U_1_ind<<-c(cum_U_1_ind, U_1_ind)
 }
   }
# II: investor defaults before counterparty and underlying and contract maturity
 if (taus[i,1] < taus[i,3] && Tgamma_fun(taus[i,1],Pymnt_seq,Tgamma) < Tb){
  if (taus[i,2] > taus[i,1]) {
   U_1_ind       <- 1- exp(-Y_with_T[Y_with_T[,1]==Tgamma_fun(taus[i,1],...
                     Pymnt_seq,Tgamma),3] -
                     Phi_wT[Phi_wT[,1]==Tgamma_fun(taus[i,1],Pymnt_seq,Tgamma),3])
   U_nind_ind    <- 1- exp(-Y_with_T[Y_with_T[,1]==Tgamma_fun(taus[i,1],...
                     Pymnt_seq,Tgamma),4] -
                     Phi_wT[Phi_wT[,1]==Tgamma_fun(taus[i,1],Pymnt_seq,Tgamma),4])
   U_ind         <- 1 - exp(-Y_with_T[Y_with_T[,1]==Tgamma_fun(taus[i,1],...
                     Pymnt_seq,Tgamma),2] -
                     Phi_wT[Phi_wT[,1]==Tgamma_fun(taus[i,1],Pymnt_seq,Tgamma),2])
   BR_CVA_0[i,]  <- CDSAjust(Tgamma_fun(taus[i,1],Pymnt_seq,Tgamma),S1,LGD1,0,
                     U_1_ind, U_nind_ind, U_ind)
  }
   }
}
```

Figure 22: R Code for the Main Function of the Algorithm

The figure displays the R-code for the main loop of the algorithm and for computing the conditional uniforms.

```r
ComputeProb <- function(t_ind, T_ind, P_ind, U_1_ind, U_nind_ind, U_ind,phi_ind)
  {
  # given the information about the time of default and the relevant uniforms
  # this function computes the survival probability of the reference entity
  # using the lognormal approximation

    x_max_1    <- getRADIUS_W(y_with_T[y_with_T[,1]==T_ind,3], ...
                  -CIRpar1[3],CIRpar1[2]*CIRpar1[3],CIRpar1[4],t_ind-T_ind)
    x_max <-max(c(x_max_1,x_max_2))
    x_ind<-seq(x_min_1,x_max,length.out=Nr_runs_k)

    pk <- calculateCDF_an(x_ind,y_with_T[y_with_T[,1]==T_ind,3], ...
          CIRpar1[3],  CIRpar1[2],  CIRpar1[4], t_ind-T_ind)

    uk2              <- 1 - exp(-x_ind-phi_ind)
    uk               <- uk2 [ uk2 > U_1_ind ]
    repU_nind_ind    <- rep(U_nind_ind,length(x_ind))
    repU_ind         <- rep(U_ind,length(x_ind))
if (P_ind == 2) {
fk <-(ParDerCop2(uk,U_ind,corr23) - ParDerCop3_2(U_ind,uk,repU_nind_ind,A_2,D_2)
   - ParDerCop2(U_1_ind,U_ind,corr23) + ParDerCop3_2(U_ind,U_1_ind,...
   U_nind_ind,A_2, D_2)) / ((1- ParDerCop2(U_nind_ind,U_ind,corr13)) -
   ParDerCop2(U_1_ind,U_ind,corr23) + ParDerCop3_2(U_ind,U_1_ind,...
   U_nind_ind,A_2, D_2))

                 } else {
fk <-(ParDerCop2(uk,U_ind,corr12)-ParDerCop3_0(U_ind,uk,repU_nind_ind,A_0, D_0)
   - ParDerCop2(U_1_ind,U_ind,corr12) + ParDerCop3_0(U_ind,U_1_ind,...
   U_nind_ind,A_0, D_0)) / ((1- ParDerCop2(U_nind_ind,U_ind,corr13)) -
   ParDerCop2(U_1_ind,U_ind,corr12) + ParDerCop3_0(U_ind,U_1_ind,...
   U_nind_ind,A_0, D_0))
                 }
   pk_trnc<-pk[uk2>U_1_ind]
   delta_fk<-diff(fk)
   Q <- sum((pk_trnc[-length(pk_trnc)]) * delta_fk)
   sum_fk<<-c(sum_fk,sum(diff(fk)))
   return (Q)
  }
```

Figure 23: R Code for Computing Probabilities at Default

The figure displays the R-code for computing the survival probabilities of the reference entity at default of either the investor or the counterparty.

```
######################### trivariate Copula ####################################

 # Given the values     u1, u2, and u3,
 # the corr. coeff.     cor12 cor13
 # and the cov matrix   covm (between u2 and u3)
 # this function calcs pp
 # as the cond. prob. given u1

ParDerCop3_0 <- function(u1, u2, u3, cor12, cor13){
      x_hat <- qnorm(u2) - qnorm(u1) * cor12
      y_hat <- qnorm(u3) - qnorm(u1) * cor13
      pp<-x_hat
     for (i_pmt in 1:length(x_hat)) {
      pp[i_pmt]<-pmnorm(x=c(x_hat[i_pmt],y_hat[i_pmt]),varcov=covm_0)
        }
     return(pp)
    }
ParDerCop3_2 <- function(u1, u2, u3, cor12, cor13){
      x_hat <- qnorm(u2) - qnorm(u1) * cor12
      y_hat <- qnorm(u3) - qnorm(u1) * cor13
      pp<-x_hat
      for (i_pmt in 1:length(x_hat))  {
      pp[i_pmt]<-pmnorm(c(x_hat[i_pmt],y_hat[i_pmt]),varcov=covm_2)
      }
     return(pp)
    }

```

Figure 24: R Code for Computing the Copula Terms Needed

The figure displays the R-code for computing the trivariate copula term seen in Equations (4.33) and (4.34).

```r
characteristic_func <- function(zeta, x0, alpha, bet, gamm, t) {
  Z <- sqrt(alpha^2-2*gamm^2*zeta*1i)
  term1 <- exp(-alpha*t/2) / ( cosh(Z*t/2) - alpha/Z*sinh(Z*t/2))
  term2 <- 1i*zeta*x0/Z * 2*sinh(Z*t/2) / ( cosh(Z*t/2) - alpha/Z*sinh(Z*t/2))
  return( term1^(2*bet/gamm^2) * exp(term2))
}

# *** FRFT ***
frft <- function(x, alpha) {
  L <- length(x)
  y <- c( x * exp(-1i*pi*(0:(L-1))^2*alpha), rep(0+0i, L))
  z <- c(     exp(+1i*pi*(0:(L-1))^2*alpha), exp(1i*pi*(L:1)^2*alpha))
  RET <- fft(fft(y)*fft(z), inverse=TRUE)/(2*L)
  return( exp(-1i*pi*(0:(L-1))^2*alpha) * RET[1:L])
}

getRADIUS_T <- function(x0, alpha, bet, gamm, t) {
  a_seq <- c( -2^19/2^seq(1:64), rev(2^19/2^seq(1:64)))
  epsilon   <- 1e-3
  abs_ <- abs(Re(characteristic_func(a_seq, x0, alpha, bet, gamm, t)))
  index1 <- which( abs_/max(abs_) > epsilon)
  index2 <- which( rev(abs_)/max(abs_) > epsilon)
  arange    <- max( abs(a_seq[max(index1[1]-2,1)]),
                    abs(rev(a_seq)[max(index2[1]-2,1)]))
  by_value  <- arange/2^14
  abs_ <- abs(Re(characteristic_func(seq(-arange,arange-by_value, by=by_value),
  x0, alpha, bet, gamm, t)))
  index1 <- which( abs_/max(abs_) > epsilon)
  index2 <- which( rev(abs_)/max(abs_) > epsilon)
  return( min(arange-by_value*(index1[1]-1),
  arange-by_value-by_value*(index2[1]-1)))
}
```

Figure 25: The Code for Computing the Fractional Fourier Transform

The figure displays the R-code for computing the characteristic function, the fractional Fourier Transform as well as heuristic used for the parameterization of the radius t.

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