POINTWISE SO_4 SYMMETRY OF THE BPST PSEUDOPARTICLE SOLUTION

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ABSTRACT

The BPST pseudoparticle solution is shown to be everywhere pointwise SO_4 symmetrical. It is further shown that on flat Euclidean space R_4, the only SU_2 gauge field that is everywhere SO_4 symmetrical is a BPST pseudoparticle solution.

I. INTRODUCTION

In a recent generalization[1] of the Dirac monopole to SU_2 gauge fields, it was found that the concept of "pointwise SO_4 symmetry" is useful. The meaning of this concept[2] can be explained in the following way:

Consider a four-dimensional manifold with a Riemannian geometry having $+ + + +$ signature. Let P be a point on the manifold. Consider an SU_2 gauge field with field strengths $(f^i_{\mu\nu})_P$ at the point P. Does $(f^i_{\mu\nu})_P$ serve to differentiate between the various directions from P? If it does not, we say the field has pointwise SO_4 symmetry at P. To be more precise, choose coordinates so that the metric at P is $g_{\mu\nu} = \delta_{\mu\nu}$. If any SO_4 rotation for the indices μ, ν in $(f^i_{\mu\nu})_P$ can be *compensated* for by a gauge transformation on the index i, the field has pointwise SO_4 symmetry at P.

In this paper I show that the pseudoparticle solution[3] of Belavin, Polyakov, Schwartz, and Tynpkin, to be called the BPST solution, is everywhere pointwise SO_4 symmetrical. I then show that the only gauge field (sourceless or not) on R_4, (i.e., on flat $+ + + +$ space), that is everywhere pointwise SO_4 symmetrical is the BPST solution.

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II. POINTWISE SO_4 SYMMETRY OF BPST SOLUTION

It was shown in the appendix of reference 1 that the following statements are identical

(a) $f^i_{\mu\nu}$ is pointwise SO_4 symmetrical at P, (also called orthogonal or regular at P),

(b) $f^i_{\mu\nu} f^{j\lambda\nu} = a^2 \delta^{ij} \delta^\lambda_\mu + a\epsilon^{ijk} f^{k\cdot\lambda}_\mu$ at P,

(c) $f^i_{\mu\nu} f^{j\lambda\nu} + f^j_{\mu\nu} f^{i\lambda\nu} = 2a^2 \delta^{ij} \delta^\lambda_\mu$ at P,

where a is a scalar function on the manifold. It is further easy to show from lemmas 1α, 1β and 4 of reference 1 that these statements are also identical to

(d) $f^i_{\mu\nu}$ is self dual or self antidual at P, and in a coordinate system for which $g_{\mu\nu} = \delta_{\mu\nu}$ at P,

$$\tilde{\mathcal{E}}\,\mathcal{E} = \tilde{\mathcal{H}}\,\mathcal{H} = a^2 \quad .$$

Theorem 1. The BPST solution is everywhere pointwise SO_4 symmetrical.

Proof: By a straightforward evaluation of the field strengths $\mathcal{E}$ and $\mathcal{H}$ for the BPST solution, we easily verify property (d) above. Hence the theorem is proved.

The square of the field strength, a^2, is easily computed to be

$$a^2 = \frac{16K^2}{[x^2 + K]^4} \quad (K > 0).$$

where $x^2 = x_1^2 + x_2^2 + x_3^2 + x_4^2$ and x_μ are the Cartesian coordinates. This function peaks at $x = 0$. Around a point P where $x \neq 0$, a SO_4 rotation of the whole field changes the magnitude of the field strength squared at most points. So the field is not SO_4 symmetrical at P. But it is pointwise SO_4 symmetrical at P in that if one considers only the value of $f^i_{\mu\nu}$ at P, then the rotation of the field strengths is equivalent to a gauge transformation of the original field strengths. Thus the value of $f^i_{\mu\nu}$ at P does not serve to choose an SO_4 frame around P.

Theorem 2. The BPST solution is the only SU_2 gauge field that is everywhere pointwise SO_4 symmetrical on R_4, the flat 4-dimensional Euclidean space.

Proof: (1) According to Appendix A of reference 1, a field that is pointwise SO_4 symmetrical can be gauge transformed to the standard form, with only one parameter, a, its amplitude. For a field that is everywhere pointwise SO_4 symmetrical, the field strengths in the proper gauge are of the standard form (IA10) or (IA11) everywhere. The amplitude a is a function of the x's. We shall write these equations in the following form

$$f^i_{\mu\nu} = a\eta^i_{\mu\nu} \quad , \tag{1}$$

or

$$f^i_{\mu\nu} = a\bar{\eta}^i_{\mu\nu} \quad , \tag{2}$$

where η and $\bar{\eta}$ are the symbols introduced by 't Hooft[4]:

$$\eta^i_{\mu\nu} = \epsilon_{i\mu\nu4} + \delta_{\mu i}\delta_{\nu 4} - \delta_{\nu i}\delta_{\mu 4} \quad , \tag{3}$$

$$\bar{\eta}^i_{\mu\nu} = \epsilon_{i\mu\nu4} - \delta_{\mu i}\delta_{\nu 4} + \delta_{\nu i}\delta_{\mu 4} \quad . \tag{4}$$

For the self-dual case (1), the Bianchi identity becomes[5]

$$\eta^{\xi\mu\nu\lambda}(\eta^i_{\mu\nu}a_{,\lambda} - aC^i_{jk}\eta^j_{\mu\nu}b^k_\lambda) = 0 \quad . \tag{5}$$

Since $\eta^i_{\mu\nu}$ is self dual, this becomes

$$\eta^{i\xi\lambda}a_{,\lambda} - aC^i_{jk}\,\eta^{j\xi\lambda}b^k_\lambda = 0 \quad . \tag{6}$$

(2) Now define a matrix M and columns Δ and b by

$$<i\xi|M|k\lambda> = C^i_{jk}\,\eta^{j\xi\lambda} \quad , \tag{7}$$

$$<i\xi|\Delta> = \eta^{i\xi\lambda}a_{,\lambda} \quad , \tag{8}$$

$$<k\lambda|b> = b^k_\lambda \quad . \tag{9}$$

Eq. (6) becomes

$$\Delta - aMb = 0 \quad . \tag{10}$$

Using[4,1]

$$\eta^i_{\alpha\lambda}\eta^j_{\beta\lambda} = \epsilon^{ijk}\,\eta_{\alpha\beta} + \delta^{ij}\delta_{\alpha\beta} \tag{11}$$

we can prove

$$(M + 1)M = 2 \quad . \tag{12}$$

Thus

$$b = a^{-1}(M + 1)\Delta/2 \quad .$$

Or

$$b^i_\xi = -a_{,\mu}\eta_{\xi\mu}(2a)^{-1} \quad . \tag{13}$$

(3) Substituting this into the equation for $f^i_{\mu\nu}$ in terms of b^i_ξ and its derivatives, we obtain as necessary and sufficient conditions for (1):

$$A_{,\mu\nu} = \delta_{\mu\nu}B + A_{,\mu}A_{,\nu} \quad , \tag{14}$$

$$2a = -A_{,\mu\mu} - (A_{,\mu})^2 \quad , \tag{15}$$

where

$$A = \frac{1}{2} ln|a| \quad , \tag{16}$$

and B is a scalar function of the coordinates. Putting

$$G = \exp(-A) \quad , \tag{17}$$

(14) becomes

$$G_{,ij} = 0,(i \neq j) \quad ,$$

and

$$G_{,11} = G_{,22} = G_{,33} = G_{,44} \quad .$$

These equations can be integrated, giving

$$G = \alpha(x - c)^2 + \beta \tag{18}$$

where c_μ is a point in R_4, and α and β are numbers. Substitution into (10), (16), and (17) gives

$$a = \frac{\pm 1}{(\alpha(x - c)^2 + \beta)^2} \quad .$$

To avoid singularities in $f^i_{\mu\nu}$, a must remain finite. Thus $\alpha\beta \not< 0$. Eq. (15) gives then $4\alpha\beta = 1$, and we obtain, with $K = \beta\alpha^{-1} > 0$,

$$a = \frac{4K}{[(x-c)^2 + K]^2} \quad , \tag{19}$$

and

$$b^i_\xi = -a_{,\mu}\, \eta^i_{\xi\mu}(2a)^{-1} \quad . \tag{20}$$

These two equations give exactly the BPST solution.[3] (20) is precisely in the form of the Corrigan-Fairlie-Wilczek-'t Hooft Ansatz.[6]

(4) The proof for the antiself dual case is entirely similar.

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