SOME ALGEBRAIC PROPERTIES OF THE IMAGE OF THE PERIOD MAPPING

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To Joseph

Let $\pi: X \to S$ be a proper smooth, algebraic map from one quasi-projective variety onto another. Let r be the fiber dimension of π, $s_0 \in S$ be a basepoint, $\pi_1(S, s_0)$ the fundamental group, and $\tilde{S}$ the universal cover of S. Then $\pi_1(S, s_0)$ acts on $\tilde{S}$ as covering transformations, and subsequently on the cohomology $H^k(\pi^{-1}(s_0), \mathbf{C})$, $(0 < k < r)$ by monodromy; thus we have $F^k \equiv \tilde{S} \times_{\pi_1(S, s_0)} H^k(\pi^{-1}(s_0), \mathbf{C})$ as a locally constant vector bundle over S. To each fiber $V_S \equiv \pi^{-1}(s)$ we can assign the Hodge filtration of $H^k(V_s, \mathbf{C})$

$$H^{k,0} \subseteq H^{k,0} + H^{k-1,1} \subset \cdots \subset H^{k,0} + H^{k-1,1} + \cdots + H^{0,k}.$$

It is a theorem of Griffiths [*1*] that the Hodge filtration varies holomorphically to give a filtration of F^k by holomorphic vector bundles $F^0 \subset F^1 \subset \cdots F^k$. We can further construct a domain whose points are flags $E^0 \subset E^1 \subset \cdots E^k$ where the dimension of E^m = the dimension of $H^{k,0} + \cdots + H^{k-m,m}$ and which satisfy various "period relations" [cf. *1*, *4*].

This domain D is the quotient of an algebraic group G by a compact subgroup; D can be considered to be an open subset of $X = G_{\mathbf{C}}/B$ where G_C is the complex form of the real group G and B is a parabolic subgroup with $V = G \cap B$. G arises as a group of matrices acting on $H^k(V_{s_0}, \mathbf{C})$ and the image Γ of $\pi_1(S, s_0)$ in $\mathrm{GL}(H^k(V_{s_0}, \mathbf{C}))$ acting by monodromy is contained in G. It is discrete and thus acts properly discontinuously on G/V to give $\Gamma \backslash G/V = \Gamma \backslash D$. Sending $s \in S$ to the flag given by the Hodge filtration is unique up to monodromy, and so we have a unique image in $\Gamma \backslash D$ and consequently a holomorphic map $p: S \to \Gamma \backslash D$. Γ is contained in an arithmetic group of G and for our purposes will be replaced by this arithmetic group without loss of generality.

Let $\bar{\pi}: \bar{X} \to \bar{S}$ be an extension of $\pi: X \to S$ where $\bar{X}$ and $\bar{S}$ are smooth compactifications of X and S respectively as projective varieties with normal crossings, which exist according to Hironaka.

Let $\Delta =$ the unit disc and $\Delta^* =$ the punctured unit disc. Let $s \in \bar{S} - S$ with a neighborhood U such that $U \cap S = \Delta^* \times \Delta^{n-1}$ where $n =$ dimension S. If the Picard-Lefschetz transformation from the above is of finite order, Griffiths [2] has shown p extends over s as a map into $\Gamma \backslash D$; further he has shown [4] that if the map has been thus extended, then it is proper.

We are interested in showing that the image of p is algebraic and p is an algebraic map. Now the mapping p satisfies one crucial property, the infinitesimal period relation [cf. 1, 2], which sometimes mitigates the fact that D is rarely Hermitian symmetric. There is a G invariant distribution H on the holomorphic tangent bundle of D, and the image of the tangent space of S under p belongs to H. There is a G invariant metric on D with holomorphic sectional curvature bounded below zero on H.

A map $p: x \to \Gamma \backslash D$ is said to be locally liftable if each point $x \in X$ has a neighborhood $U(x)$ such that p lifts to a map to D where

$$\begin{array}{ccc} U(x) & \to & D \\ & \searrow & \downarrow \\ & & \Gamma \backslash D \end{array}$$

commutes.

We can now state our result; I will sketch a proof, the full details of which will appear elsewhere.

Proposition. Let X be an algebraic curve and $\pi: x \to \Gamma \backslash D$ a locally liftable, horizontal, proper holomorphic map. The image is an algebraic curve and is rational.

Remark. The equivalence relation given by Hodge structures is, in this case, algebraic. We will actually prove more. The dimension of X can be anything and X can be an algebraic space as long as the dimension of the image is one; the map will then be rational. The dimension of X can always be reduced to the dimension of the image without loss of generality.

I have not been able to put an algebraic structure on the image when it has more than one dimension, but I can show that if such a structure exists so that π is algebraic, then it is unique. As these considerations are trivial in the one dimensional case, I will omit them.

Certain parts of the proof below generalize; the main difficulty in doing the general case comes from a lack of a generalization of a theorem of Schmid on the existence of fundamental domains containing the image of the period map.

Proof. Now the map $\pi: X \to \Gamma \backslash D$ factors through the desingularization Y of the image to give

$$\begin{array}{ccc} X & \searrow & \\ A\big\downarrow & & \Gamma\backslash D. \\ Y & \nearrow & \end{array}$$

A is a proper map and X is an algebraic curve. It can be shown directly, or indirectly using the fact that Y inherits a complete metric with curvature bounded below zero in which it has finite volume, that Y is an algebraic curve, and $X \xrightarrow{A} Y$ is a finitely sheeted, finitely branched covering map. Thus we have reduced to the situation of π a proper, horizontal, holomorphic, locally liftable immersion of an affine curve X.

Now $\pi(X)$ will be algebraic unless an infinite number of points ge identified. To see this, assume Z is an algebraic curve with only finitely many points identified; we can then clearly assume Z is a compact Riemann surface R with only a finite number of points identified. Pick a line bundle on R with a positive metric; now when identifying the points of R to get Z, also identify fibers by a linear map sending unit balls to unit balls. We now simply apply Grauert's generalization of the Kodaira Vanishing Theorem to give an embedding of a compact analytic space with positive line bundle into projective space.

X has a compactification $\bar{X}$ as a compact Riemann surface so that $\bar{X}-X$ consists of finitely many points. We must thus show that there do not exist sequences $\{x_n\}$, $\{y_n\}$ with $x_n \to x$ and $y_n \to y$ where $\{x,y\} \subseteq \bar{X}-X$ and $\pi(x_n) = \pi(y_n)$ with $x = y$ or $x \neq y$ and $x_n \neq y_n$.

This is done by a purely local lemma, which is the essential step in the proof.

Lemma. Let $\pi\colon \Delta^* \to \Gamma\backslash D$ be a holomorphic, horizontal, locally liftable map of Δ^*, the punctured disc, into $\Gamma\backslash D$. Let $\tilde{\pi}\colon H^+ \to D$ be a lifting of π to H^+, the upper half plane, into D and let $\tilde{\pi}(z+1) = T\tilde{\pi}(z)$ where the Picard-Lefschetz transformation $T \in \Gamma$ is of infinite order. The image of π for a smaller punctured disc is a nonsingular punctured disc. If ϕ_1, ϕ_2 are two maps as above from $\Delta^* \to \Gamma\backslash D$ with P.L. transformations T_1, T_2, and if $x_n \to 0$, $y_n \to 0$ so that $\phi_1(x_n) = \phi_2(y_n)$, then the images of ϕ_1 and ϕ_2 are the same. Furthermore, there exist integers m and n so that $T_1^m = T_2^n$.

Remark. We can assume π, ϕ_1, and ϕ_2 are generically one-to-one by the same reduction as before when we reduced to the case of an immerison.

If the images of ϕ_1, ϕ_2 were in a bounded domain, we could use the Riemann extension theorem to extend over 0 in a larger bounded domain and then ϕ_1, ϕ_2 would each have a disc for image with a sequence of points clustering at zero in common, and thus the same image. Likewise if π did not have a non-singular image on any smaller disc, it would have two disjoint sequences $x_n \to 0$, $y_n \to 0$ with $\pi(x_n) = \pi(y_n)$. But this could only be if it were so in a neighborhood of the x_n and y_n, contradicting the generic one-to-one property.

We will therefore construct a bounded domain in which the images of the punctured discs lie.

Proof. We will do the case for ϕ_1, ϕ_2 assumed different; the other case for a single π is easier and falls out of the ϕ_1, ϕ_2 considerations.

We choose liftings $\tilde{\phi}_1, \tilde{\phi}_2$ of ϕ_1, ϕ_2 which have the same limit point x on $\partial D \subset X$. We can find a Siegel set S for Γ in D so that $\phi_1(z), \phi_2(z)$, with $|\operatorname{Re} z| \leqq B$ for some $B > 0$, belong to S. This is by a result of Schmid [6]. Schmid also showed there is a $p \in D$ so that $\exp z \ln T(p)$ get close asymptotically to such a $\tilde{\phi}$. Here, I will not make "close" precise (see Figure 1).

So we have $\mathcal{O}_1(z) = \exp z \ln T_1(p_1)$, $\mathcal{O}_2(z) = \exp z \ln T_2(p_2)$; further because of the above result on Siegel sets we can assume $\{x_n\}$, $\{y_n\}$ have liftings $\{\tilde{x}_n\}$, $\{\tilde{y}_n\}$, $|\operatorname{Re} \tilde{x}_n| \leqq B, |\operatorname{Re} \tilde{y}_n| \leqq B$. The T_i are unipotent and therefore the closures of the orbits $\mathcal{O}_1, \mathcal{O}_2$ in X are projective lines with a common point $x \in \partial D$. Using the asymptotic estimates of Schmid and the fact that $\tilde{\phi}_1(\tilde{x}_n) = \tilde{\phi}_2(\tilde{y}_n)$ and $|\operatorname{Re} \tilde{x}_n| \leqq B, |\operatorname{Re} \tilde{y}_n| \leqq B$, we show that $\mathcal{O}_1(\tilde{x}_n), \mathcal{O}_2(\tilde{y}_n)$ get closer to one another in a neighborhood of x than is compatible with their being disjoint algebraic sets; they are thus the same. By further argument along the same lines we show $T_1^m = T_2^n$.

We can now assume by normalizing that $\phi_1, \phi_2 \colon \Delta^* \to \Gamma \backslash D$ have the same P.L. transformation and we have liftings $\tilde{\phi}_1, \tilde{\phi}_2$ with a common asymptotic orbit $\exp z \ln T(p)$.

We now show we can find a polydisc $\Delta(p)$ around p, transversal to $\exp z \ln Tp$ such that the map $\exp z \ln T(r)$ for $(z, r) \in H^+ \times \Delta(p)$ is a one-to-one embedding into X. We further show the image B contains the images of $\tilde{\phi}_1, \tilde{\phi}_2$.

We then map down to $\{T^j\} \backslash D$; $B \cap D$ has for image in $\{T^j\} \backslash D$ a bounded domain containing the images of $\tilde{\phi}_1, \tilde{\phi}_2\colon\ H^+ \to D \to \{T^j\} \backslash D$. We are now, by the remark about the Riemann extension theorem, done.

Open question (Griffiths): Suppose $S \xrightarrow{\pi} \Gamma \backslash D$ has been proved to be algebraic, so that $\pi(S)$ is a quasi-projective variety and π is rational. Suppose further that S is defined over a field $K \supseteq Q$ with $[K, Q] < \infty$. Then is the same true of the whole situation $S \to \pi(S)$? This is closely related to the most basic question in Hodge structures: Which points of $\Gamma \backslash D$ come rom algebraic varieties?

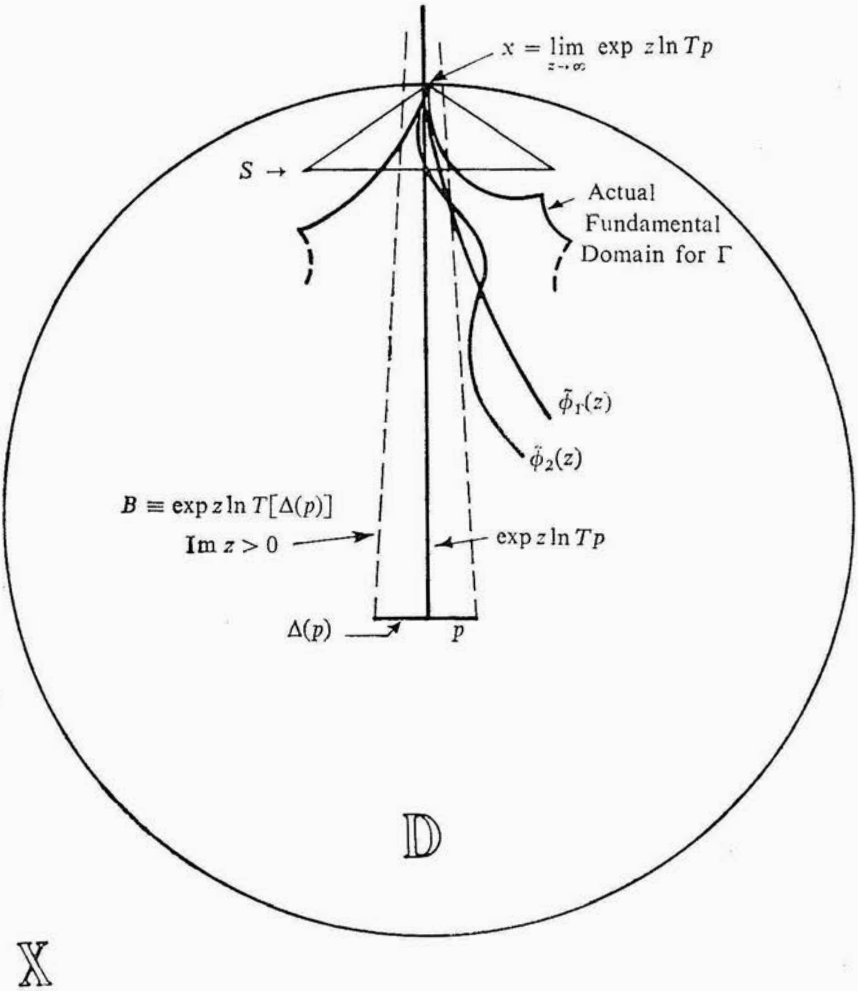

Figure 1

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In the above it was assumed that if A_1 and A_2 were horizontal locally liftable holomorphic maps from the punctured disc to $\Gamma \backslash D$ then there exists a Siegel set, S, for Γ in D such that both A_1 and A_2 have liftings $\tilde{A}_1$ and $\tilde{A}_2$ which take a Siegel set of the upper half plane into S.

Schmid actually proves a weaker result and therefore, though the above is probably true, it must at present be thrown into the hypotheses of the proposition.

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