HOLOMORPHIC HULLS AND HOLOMORPHIC CONVEXITY

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1. *Introduction*

It is a well-known theorem of Hartogs that any function f holomorphic on a neighborhood of the boundary M of the unit ball B in $\mathbf{C}^n$, $n > 1$, can be continued analytically to the interior of B. This phenomenon can also occur locally. Consider, for example, a locally defined strictly pseudoconvex hypersurface M in $\mathbf{C}^2$, then it is well known that there is an open set U in $\mathbf{C}^2$ such that any function holomorphic on a neighborhood of M can be continued analytically to U (see, e.g., Lewy [7]). In each case we say that the open set to which we continue is contained in the holomorphic hull (or envelope of holomorphy) of the set M.

These examples, among others, have led to recent investigations by various people to create a theory of holomorphic hulls and holomorphic convexity of subsets of $\mathbf{C}^n$ (or of any complex manifold) of any dimension. The classical theory for domains in $\mathbf{C}^n$ (or spread over $\mathbf{C}^n$, or Stein manifolds) satisfactorily accounts for the case when the subsets of $\mathbf{C}^n$ are open. On the other hand, the theory for lower dimensional sets is still in its infancy, but seems capable of developing into a sufficiently ample theory to account for the fundamental examples. One restriction we can make is to require that the lower dimensional sets be *real submanifolds* of $\mathbf{C}^n$, as in the case of the two examples above.

In Section 2 we develop some definitions and formalism in which the basic results known so far can be easily expressed.

In Section 3 we state some results concerning global and local holomorphic convexity of real submanifolds of a complex manifold. The first result in this section deals with the case that the fibre dimension of $H(M)$, the holomorphic tangent bundle to M (see Section 2), is identically zero, and M is a compact real C^∞ submanifold of a complex manifold X. Under these conditions M is necessarily holomorphically convex, and, in fact, M can be expressed as $\{x \in X : \phi(x) = 0\}$, where ϕ is a strictly plurisubharmonic function defined near M. Our second result here is that a locally defined C^∞ submanifold M of $\mathbf{C}^n$ with a vanishing Levi form is locally holomorphically convex. For a "generic" type of manifold this condition is also necessary.

Section 4 deals with the problem of showing that a local submanifold is extendible (has a non trivial local holomorphic hull) under certain geometric conditions (namely that the Levi form be non-zero). Recently, Greenfield [*3*] has proved that generic real C^{∞} submanifolds of $\mathbf{C}^n$ with a non-vanishing Levi form are extendible to a submanifold of one higher dimension (Theorem 3.7). All of the techniques for proving extendibility of real submanifolds stem from Bishop's important paper [*2*]. In this section we give a brief outline of Bishop's technique for embedding analytic disks and show how it can be utilized in proving extension theorems. One needs a type of Hartogs' theorem for parametrized families of analytic disks in order to use Bishop's technique. One such result is proved in [*11*], and we indicate how it can be used in obtaining the desired analytic continuation.

In Section 5 we indicate some open problems which arise naturally from the course of events outlined in the previous sections.

2. *Preliminaries*

Let X be a complex manifold with structure sheaf $\mathcal{O}$. If K is compact in X, let $C(K)$ be the Banach algebra of continuous complex-valued functions on K with the maximum norm. Let $\mathcal{O}(K) = \Gamma(K, \mathcal{O})$ be the ring of sections of $\mathcal{O}$ on K, and let $A(K)$ be the closure of $\mathcal{O}(K)|K$ in $C(K)$. Let $E(K)$ be the spectrum (maximal ideal space) of the algebra $A(K)$ (see, e.g., Gunning and Rossi [*4*] for definitions used here).

We say that K is a *holomorphic set* in X (S_δ set) if K is the intersection of open Stein submanifolds of X, and we say that K is *holomorphically convex* if $K = E(K)$, identifying K with its image in $E(K)$ by the evaluation map. A holomorphic set is holomorphically convex (see [*9*]), but the converse is unknown, except that it is true in special cases.

Let $K \subset X$ be connected, and suppose K' is connected and $K' \supset K$. Then K *is extendible to* K' if the restriction map

$$r : \mathcal{O}(K') \to \mathcal{O}(K)$$

is surjective. K is *extendible* if there is such a $K' \supsetneq K$ such that K is extendible to K'.

Suppose now that M is a real C^{∞} submanifold of X with real tangent bundle $T(M)$. Let J be the canonical almost complex tensor given by the complex structure of X. Then J acts naturally on $T(M)$ considered as a real subbundle of $T(X)$. Define

$$H_x(M) = T_x(M) \cap JT_x(M),$$

the vector space of *holomorphic tangent vectors* at $x \in M$. $H_x(M)$ is a **C**-linear subspace of $T_x(X)$, and we define

$$h_x(M) = \dim_C H_x(M).$$

If $h_x(M)$ is constant ($= h(M)$) on a connected M, then

$$H(M) = \bigcup_{x \in M} H_x(M)$$

can be given the structure of a complex vector bundle over M with fibre isomorphic to $\mathbf{C}^{h(M)}$, the *holomorphic tangent bundle to M*.

If we consider the complexification $T(M) \otimes \mathbf{C}$ (tensor over $\mathbf{R}$), the almost complex tensor J induces a natural splitting (assuming a holomorphic tangent bundle exists on M)

$$H(M) \otimes \mathbf{C} = T^{1,0}(M) \oplus T^{0,1}(M),$$

where we can identify $T^{1,0}(M)$ with $H(M)$ and $T^{0,1}(M)$ with $\bar{H}(M)$, the conjugates of $H(M)$ (well defined in $H(M) \otimes \mathbf{C}$). We write simply

$$H(M) \otimes \mathbf{C} \cong H(M) \oplus \bar{H}(M).$$

Under these conditions we can define the *Levi form* at any point $x \in M$,

$$L_x(M)\colon\ H_x(M) \to T_x(M) \otimes \mathbf{C}/H_x(M) \oplus \bar{H}_x(M).$$

Let $s \in H_x(M)$, then there is a C^∞ section $Y \in \Gamma(H(M))$, such that $Y_x = s$. Let π_x be the natural projection

$$\pi_x\colon T_x(M) \otimes \mathbf{C} \to T_x(M) \otimes \mathbf{C}/H_x(M) \oplus \bar{H}_x(M).$$

Then set

$$L_x(M)(s) = \pi_x[Y, \bar{Y}]_x,$$

where the brackets denote the commutator of the vector fields Y and $\bar{Y}$. This definition is independent of the choice of Y (cf. [5]).

3. *Holomorphic Convexity of Differentiable Submanifolds*

In this section we want to study holomorphic convexity of real C^∞ submanifolds M embedded in a complex manifold X. We have first the following useful definition.

Definition 3.1. Let M be a real C^∞ submanifold of a complex manifold X. M is said to be *totally real* if $h_x(M) \equiv 0$ on M.

Examples of totally real submanifolds:

1) Let M be a real C^∞ submanifold of $\mathbf{R}^n$, where $\mathbf{R}^n$ is canonically embedded in its complexification $\mathbf{C}^n$, and thus M is a real C^∞ submanifold of $\mathbf{C}^n$. If $t \in T_x(M)$, then $Jt \perp \mathbf{R}^n$, but $t \in \mathbf{R}^n$, hence $T_x(M) \cap JT_x(M) = 0$, for any $x \in M$, and M is totally real.

2) Let M be a smooth (real) curve in $\mathbf{C}^n$. If $t \in T_x(M)$, then t and Jt are linearly independent since $\langle t, Jt \rangle = 0$ in the standard inner product of $\mathbf{R}^{2n} = \mathbf{C}^n$. Hence $Jt \notin T_x(M)$ since $T_x(M)$ is 1-dimensional. Thus M is totally real.

3) Let $M = \{z \in \mathbf{C}^n: |z_i| = 1,\ i = 1, \cdots, n\}$ be the standard torus in $\mathbf{C}^n$. Then by using simple arguments similar to those above it follows that M is totally real.

The following lemma is a useful tool concerning totally real submanifolds.

Lemma 3.1. *Let M be a compact totally real submanifold of a complex manifold X. Then there exists a neighborhood U of M and a strictly plurisubharmonic C^∞ function ϕ defined in U such that*

$$M = \{x \in U : \phi(x) = 0\}.$$

The proof of this is not hard. ϕ is first constructed locally, using a convenient choice of coordinates, and then the local functions are pieced together by using a partition of unity (see [*12*]). It follows from this lemma that a totally real compact submanifold is holomorphic, since there is a sequence $\varepsilon_j \to 0$ such that $U_j = \{x \in U:\ \phi(x) < \varepsilon_j\}$ are strongly pseudoconvex subdomains of X, and $\bigcap_{j=1}^{\infty} U_j = M$. But it is a well-known theorem of Grauert that each U_j is an open Stein manifold in X, and hence M is holomorphic (see [*4*]). Thus we have

Theorem 3.2. *If M is a totally real compact submanifold of X, then M is a holomorphic set, and consequently holomorphically convex.*

Actually, much more is true. Namely, one can prove that a totally real compact C^∞ submanifold M has the property that $A(M) = C(M)$, which is a much stronger result, and implies immediately that M is holomorphically convex (see [*8*] and the article by R. Nirenberg in these proceedings). Lemma 3.1 plays an important role in proving this stronger theorem.

We want to consider now submanifolds M with well-defined holomorphic tangent bundles, i.e., $h_x(M) \equiv \text{const}$ on each component of M. We shall follow the terminology introduced by Greenfield [*3*].

Definition 3.2. Let M be a connected real C^∞ submanifold of a complex manifold X.

1) If $h_x(M) = \max(0, \dim_{\mathbf{R}} M - \dim_{\mathbf{C}} X)$, then M is *generic* at $x \in M$.
2) If $h_x(M)$ is constant on M, then M is called a *C-R submanifold* of X.
3) M is a *generic submanifold* of X if M is generic at each $x \in M$.

Remarks. 1) *C-R* submanifold refers to the fact that there are well-defined induced Cauchy-Riemann equations on M, which relate to some interesting unsolved problems.

2) A generic submanifold is automatically a C-R submanifold, and if M is generic at x, then M is generic near x.

Examples of generic submanifolds:

1) Let M be a real C^∞ hypersurface in a complex manifold X with $\dim_{\mathbf{C}} X = n$, $\dim_{\mathbf{R}} M = 2n-1$. Then M is a generic submanifold with $h_x(M) \equiv n-1$.

2) Let $M = \{z \in \mathbf{C}^3 \colon |z_1|^2 + |z_2|^2 = 1, \ |z_3| = 1\}$. Then $\dim_{\mathbf{R}} M = 4$, $h_x(M) \equiv 1$. This is a compact generic submanifold which is not totally real, and hence provides an example for Theorem 3.4 below.

3) A totally real submanifold of $\mathbf{C}^n$ is necessarily a generic submanifold.

Restricting ourselves to generic submanifolds of $\mathbf{C}^n$ we can state a converse to Theorem 3.2.

Theorem 3.3. *Let M be a compact generic submanifold of X. If M is holomorphically convex, then M is totally real.*

This theorem is presumably true without the genericity assumption, but the present techniques do not seem to carry over to the non-generic case. Namely, the above theorem is a trivial consequence of the following one, the proof of which uses strongly the assumption of genericity and is outlined in Section 4.

Theorem 3.4. *If M is a connected compact generic submanifold of a complex manifold X, and if $h(M) > 0$, then M is extendible. Moreover, M is extendible to a subset of X which contains a submanifold N of one higher real dimension than M.*

To obtain Theorem 3.3, we merely note that an extendible set cannot be holomorphic or holomorphically convex.

So for compact submanifolds, being totally real is "essentially" the geometric characterization of being holomorphically convex. However, locally holomorphically convex submanifolds do not have to be locally totally real, and here we use the Levi form as a geometric measure of local holomorphic convexity.

Theorem 3.5. *Let M be a C-R submanifold of an open set $U \subset \mathbf{C}^n$. Suppose $x \in M$. If $L_y(M) \equiv 0$, for y near x on M, then M is locally holomorphic, and hence locally holomorphically convex.*

This is proved geometrically, similar to the proof of Theorem 3.2, by constructing locally a plurisubharmonic function which vanishes only on M (this function is not strictly plurisubharmonic in general). This gives then a sequence of pseudoconvex domains whose intersection is a sufficiently small compact neighborhood of x in M. The solution to the Levi problem in $\mathbf{C}^n$ (pseudoconvex domains are domains of holomorphy, see [*6*]) then implies that M is locally holomorphically convex (see [*13*] for a complete proof).

Restricting ourselves to generic submanifolds again, we obtain a characterization of locally holomorphically convex submanifolds.

Theorem 3.6. *Let M be a generic submanifold of an open set $U \subset \mathbf{C}^n$, and let $x \in M$, then the following are equivalent:*

(i) *$L_y(M) \equiv 0$ for y near x on M.*

(ii) *M is locally holomorphic.*

(iii) *M is locally holomorphically convex.*

As we remarked in Section 2, it is unknown in general whether holomorphic sets and holomorphically convex sets are the same class of sets, but the above theorem shows that the localization of these concepts agree on the class of generic submanifolds.

The above theorem is a consequence of the following result.

Theorem 3.7. *Let M be a generic submanifold of an open set $U \subset \mathbf{C}^n$, $n > 1$. If $L_x(M) \neq 0$ at $x \in M$, then M is locally extendible at x. Moreover, M is extendible to a submanifold of $\mathbf{C}^n$ of at least one higher real dimension.*

The proof of this theorem is due to Greenfield [*3*]. A special case was proved in [*13*], and the first part of the theorem was stated in [*11*], but an incorrect proof was given.

4. *Holomorphic Hulls and Extendibility of Submanifolds*

The object of this section is to discuss the local extendibility of submanifolds of $\mathbf{C}^n$ under appropriate geometric hypotheses. To carry out the analytic continuation involved we use a type of Kontinuitätssatz with a parametrized family of analytic disks.

Let $\Delta = \{\zeta \in \mathbf{C}: |\zeta| < 1\}$ be the open unit disk.

Definition 4.1. Let T be a q-dimensional C^∞ manifold, and let

$$F: \bar{\Delta} \times T \to \mathbf{C}^n$$

be a continuous map, where $n > 1$, $1 \leqq q \leqq 2n - 2$. Suppose $F\big|\Delta \times t$ is a holomorphic map for each fixed $t \in T$, then F is a *continuous family of analytic disks* in $\mathbf{C}^n$ parametrized by T.

Theorem 4.1. *Suppose $\pi_0(T) = \pi_1(T) = 0$. Let $F: \bar{\Delta} \times T \to \mathbf{C}^n$ be a continuous family of analytic disks such that for some $t_0 \in T$, $F(\bar{\Delta} \times t_0)$ is a point in $\mathbf{C}^n$, then $F(\partial\Delta \times T)$ is extendible to $F(\bar{\Delta} \times T)$.*

This theorem is proved (in a slightly more general case) in [*11*]. The proof depends upon the classical Cartan-Thullen Theorem and a form of the Kontinuitätssatz due to Behnke and Sommer (see [*11*] for references).

If we want to show that a certain submanifold $M \subset \mathbf{C}^n$ is extendible we can then try to find a family of analytic disks in $\mathbf{C}^n$ whose boundaries lie on M, whose interiors do not lie entirely in M, and such that for some value of the parameter the analytic disks shrink to a point. This is what Bishop's construction [*2*] allows us to do at a generic point of a submanifold of $\mathbf{C}^n$. Whether the interiors of such a family lie on M or not depends upon the geometric hypothesis at hand (the Levi form).

Suppose M is a submanifold of $\mathbf{C}^n$ which is generic at $x \in M$. Assume (without loss of generality) that x is the origin in $\mathbf{C}^n$. If M is totally real $(h_0(M) = 0)$ near 0, then it follows from Theorem 3.2 that M is locally holomorphically convex at 0. To obtain a local extension of M near 0, we must then assume that $h_0(M) > 0$.

Suppose $\dim_{\mathbf{R}} M = k$, then M is defined near 0 by local coordinates

$$\psi : \mathbf{R}^k \to \mathbf{C}^n$$
$$\psi(\mathbf{R}^k) = M \cap U$$

where U is open in $\mathbf{C}^n$, and $\psi(0) = 0 \in U$. Using the techniques originated by Bishop (see [*2*], [*11*], [*13*], [*10*], [*3*]), one can construct an appropriate family of analytic disks. Namely, let I be a closed interval, and I^l denotes the l-fold Cartesian product of I. We find continuous maps f, F to obtain the following commutative diagram

$$\begin{array}{ccc} \mathbf{R}^k & \xrightarrow{\ \psi\ } & \mathbf{C}^n \\ \uparrow f & & \uparrow F \\ \partial\Delta \times I^{k-1} & \xrightarrow{\ i\ } & \bar{\Delta} \times I^{k-1} \end{array},$$

where i is the natural injection, and such that F is a continuous family of analytic disks parametrized by I^{k-1}, and such that $F(\bar{\Delta} \times t_0) = 0$, for some $t_0 \in I^{k-1}$. We solve for f, F in the following manner. For each fixed $t \in I^{k-1}$,

$\psi \cdot f$ is to be the boundary value of a holomorphic map of the unit disk into $\mathbf{C}^n$. This requires that $\operatorname{Re}(\psi \cdot f)$ be (up to an additive constant) the Hilbert transform of $\operatorname{Im}(\psi \cdot f)$ on $\partial\Delta$ for each fixed $t \in I^{k-1}$. By making a linear change of coordinates in $\mathbf{C}^n$, the non-linear equation (H denotes the Hilbert transform on $\partial\Delta$)

$$\operatorname{Re}(\psi \cdot f) = H(\operatorname{Im}(\psi \cdot f)) + \text{const}$$

can be solved by a convergent iterative process (successive approximations) for sufficiently small values of $(\psi \cdot f)$ in some Sobolev norm on the unit circle. Changing the value of t changes the parameters in this integral equation and the constant term. One can show (by using the Sobolev lemma) that the solution $f_t = f \mid \partial\Delta \times t$ is at least C^1 with respect to the parameter t.

Once we have f, we obtain F by using the Cauchy integral formula in terms of $\psi \cdot f$. If we assume that $L_0(M) \neq 0$, then we can compute the Jacobian matrix of the map F and see that, except for a lower dimensional set, this matrix has maximal rank on $\bar{\Delta} \times I^{k-1}$ (see [*3*] and [*13*]). Thus $Q = F(\bar{\Delta} \times I^{k-1})$ is (except for the singular set) a $(k+1)$-dimensional real C^1 manifold immersed in $\mathbf{C}^n$. Moreover, $bQ = F(\partial\Delta \times I^{k-1}) \subset M$ since $F(\partial\Delta \times I^{k-1}) = \psi(f(\partial\Delta \times I^{k-1}))$. Thus we see that if N is a compact neighborhood of 0 in M such that $N \supset F(\partial\Delta \times I^{k-1})$, then the restriction map $\mathcal{O}(N \cup Q) \to \mathcal{O}(N)$ is surjective. This follows from the following commutative diagram (with natural restriction maps)

$$\begin{array}{ccc} \mathcal{O}(N \cup Q) & \longrightarrow & \mathcal{O}(N) \\ \downarrow & & \downarrow \\ \mathcal{O}(Q) & \xrightarrow{\tau} & \mathcal{O}(bQ) \end{array}$$

and the fact that τ is surjective, from Theorem 4.1. This then is a brief outline of the proof of Theorem 3.7.

Theorem 3.4 follows from this result (Theorem 3.7) by applying Bishop's Peak Point Theorem [*1*] to conclude that, on the compact submanifold M, there is at least one point $x \in M$ such that $L_x(M) \neq 0$. Namely, if $L_x(M) \equiv 0$ on M, then through each point $x \in M$, there passes a complex submanifold of $\mathbf{C}^n$ embedded in M. But this contradicts the Peak Point Theorem, since in a neighborhood of a peak point there can be no complex submanifolds, by the maximum principle.

5. *Open Problems*

A. Can the hypothesis of genericity be removed in Theorem 3.7? For instance, are the theorems of Section 3 true for *C-R* submanifolds?

B. Can the compact generic submanifolds of $\mathbf{C}^n$ be classified in some geometric or topological manner? For instance, if M^2 is an orientable compact 2-manifold in "general position" in $\mathbf{C}^2$, then a necessary condition that M^2 be totally real (and hence generic in this case) is that the Euler characteristic $\chi(M^2) = 0$. In other words M^2 must be a torus (see [2]).

C. What can be said about extendibility of submanifolds which are not C-R? For instance, if S^2 (the 2-sphere) is embedded in $\mathbf{C}^2$, is it necessarily extendible? It is known that $S^2 \subset \mathbf{C}^2$ cannot have a well-defined holomorphic tangent bundle, since there is at least one point $x \in S^2$ where $h_x(S^2) = 1$ (see [2]), and we cannot have $h_y(S^2) \equiv 1$ on S^2, since then S^2 would be a complex submanifold of $\mathbf{C}^2$ which is impossible.

D. If K is a holomorphically convex set in a Stein manifold X, then is K a holomorphic set in X^2? Note that the sense in which we use "holomorphically convex" is weaker than that used in Gunning and Rossi [4], where "holomorphic convexity" there means "holomorphically convex with respect to X," i.e., convex with respect to the family of functions $\mathcal{O}(X)$.

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