RICE UNIVERSITY

Interval Exchange Transformations: Applications of Keane's Construction and Disjointness

by

Jon Chaika

A Thesis Submitted
in Partial Fulfillment of the
Requirements for the Degree

Doctor of Philosophy

Approved, Thesis Committee:

Michael Boshernitzan, Chair
Professor of Mathematics

William Veech,
Edgar Odell Lovett
Professor of Mathematics

Steven Cox,
Professor of
Computational and Applied Mathematics

Alexander Bufetov,
Assistant Professor of Mathematics

Houston, Texas

April, 2010

Abstract

Interval Exchange Transformations: Applications of Keane's Construction and Disjointness

by

Jon Chaika

This thesis is divided into two parts. The first part uses a family of Interval Exchange Transformations constructed by Michael Keane to show that IETs can have some particular behavior including:

1. IETs can be topologically mixing.
2. A minimal IET can have an ergodic measure with Hausdorff dimension α for any $\alpha \in [0, 1]$.
3. The complement of the generic points for Lebesgue measure in a minimal non-uniquely ergodic IET can have Hausdorff dimension 0. Note that this is a dense G_δ set.

The second part shows that almost every pair of IETs are different. In particular, the product of almost every pair of IETs is uniquely ergodic. In proving this we show that any sequence of natural numbers of density 1 contains a rigidity sequence for almost every IET, strengthening a result of Veech.

Acknowledgments

I would like to thank my advisor M. Boshernitzan. I could not have imagined a better advisor. His constant encouragement and joy for mathematics made my graduate experience a pleasure. He carefully chose problems that were original, interesting and continue to open doors for me. Moreover, he has shown me an amazing number of deep insights that inform my understanding of mathematics in general and ergodic theory in particular.

I would like to thank A. Bufetov, D. Damanik and W. Veech for being so wonderful. They were generous with their insights and time. They were always willing to listen to me talk, even when I was not particularly coherent. They helped motivate me and I learned so much from them. I would like to thank R. Hardt and S. Semmes for teaching me a lot and often helping me.

I would like to thank all of my fellow grad students for being such a great community and teaching me so much math. I will avoid naming them for fear of length and omissions.

I would like to thank M. Boshernitzan for posing the questions answered in the second chapter. I would like to thank A. Bufetov for posing the question answered in the third chapter. I would like to thank M. Boshernitzan, A. Bufetov, S. Cox and W. Veech for serving on my thesis committee.

I would like to thank B. Hausman, M. Magee and M. Turner for their frequent assistance during my graduate career.

I would like to thank Evelyn for making every day delightful.

Contents

Chapter 1

Introduction

1.1 Basic dynamics terminology

Let $(X, \mathcal{B}, \mu)$ be a measure space. If $T : X \to X$ is measurable and $\mu(A) = \mu(T^{-1}A)$ for all measurable sets A then T is said to be μ *measure preserving.* If T is μ measure preserving and $\mu(A \Delta T(^{-1}A)) = 0$ only when $\mu(A)$ or $\mu(A^c) = 0$ then T is said to be μ *ergodic.* (Δ denotes symmetric difference.) One of the primary motivations (and tools) for studying ergodic transformations is the Birkhoff Ergodic Theorem.

Theorem 1. *(Birkhoff) Let $(X, \mathcal{B}, \mu)$ be a σ-additive measure space. If T is μ ergodic then for all $f \in L^1(X, \mathcal{B}, \mu)$ we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \int_X f d\mu$ for μ almost every x.*

Informally the Birkhoff Ergodic Theorem says that for ergodic transformations the time average is equal to the space average. It also motivates the following definition:

Definition 1. *Given T: $[0, 1] \to [0, 1]$, a μ ergodic map, we say a point $x_0 \in [0, 1]$ is* generic for μ *if* $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x_0)) = \int_0^1 f d\mu$ *for every* $f \in C([0,1])$.

The definition requires that the limit exists. The generic points can be thought of as an explicit set of μ typical points.

In particular, $\mathcal{B}$ is a Borel σ-algebra and if continuous functions with supremum norm are separable (such as when (X,d) is a compact metric space) then there exist generic points. To see this let $f_1, \ldots$ be a countable dense set in the *sup* norm topology. Let

$$A_i = \{x : \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \int_X f d\mu\}.$$

If $x \in \bigcap_{i=1}^{\infty} A_i$ then for any continuous $f \in L_1(X, \mathcal{B}, \mu)$ we have $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n(x)) = \int_X f d\mu$.

In this construction we use the fact that μ is an ergodic measure for T. If μ and ν are two different ergodic probability measures for T, then $\frac{\mu+\nu}{2}$ is another probability measure, and T is a $\frac{\mu+\nu}{2}$ measure preserving transformation. T is not $\frac{\mu+\nu}{2}$ ergodic. The set of $\frac{\mu+\nu}{2}$ generic points has $\frac{\mu+\nu}{2}$ measure 0. One should observe that the Birkhoff Ergodic Theorem implies that if μ and ν are different ergodic measures of T then they are *singular*, meaning that they have disjoint sets of full measure. If T is a μ ergodic transformation and it has no other ergodic measures it is called *uniquely ergodic.* Uniquely ergodic transformations have only 1 preserved measure.

Another important result in ergodic theory is the Poincaré Recurrence Theorem.

Theorem 2. *(Poincaré) Let $(X, \mathcal{B}, \mu)$ be a finite measure space and T be a μ measure preserving transformation. For any measurable set A, $\hat{A} = \{x \in A : T^n(x) \in A$ for some $n > 0\}$ has $\mu(A \Delta \hat{A}) = 0$.*

This helps justify an important construction in ergodic theory and this thesis, the induced map. If T is μ measure preserving and A is a measurable set, then the Poincaré recurrence theorem tells us that for all but a measure zero set of $x \in A$ there exists $n_x \in \mathbb{N}$ such that $T^{n_x}(x) \in A$. Motivated by this we define the induced map of T on A (or the Poincaré first return map) as $T|_A : A \to A$ by $T|_A(x) = T^{r_x}(x)$ where $r_x = \min\{n \in \mathbb{N} : T^n(x) \in A\}$. $T|_A$ is also μ measure preserving. One can also check that if $B \subset A$ then $T|_B = (T|_A)|_B$.

1.2 What is an IET

Definition 2. *Given* $L = (l_1, l_2, ..., l_d)$ *where* $l_i \geq 0$, $l_1 + ... + l_d = 1$, *we obtain* d *subintervals of* $[0,1)$, $I_1 = [0, l_1), I_2 = [l_1, l_1 + l_2), ..., I_d = [l_1 + ... + l_{d-1}, 1)$. *Given a permutation* π *on* $\{1, 2, ..., d\}$, *we obtain a* d-Interval Exchange Transformation *(d-IET)* T: $[0,1) \to [0,1)$ *which exchanges the intervals* I_i *according to* π. *That is, if* $x \in I_j$ *then*

$$T(x) = x - \sum_{k<j} l_k + \sum_{\pi(k')<\pi(j)} l_{k'}.$$

It follows from the definition that IETs are Lebesgue measure preserving invertible maps of $[0,1)$. They are by construction continuous from the right and have at most $d-1$ discontinuities. The inverse of an IET is also an IET (often with a different permutation.) Rotations can be viewed as 2-IETs with permutation (21).

Interval exchange transformations with a fixed permutation on d-letters are parametrized by the standard simplex in $\mathbb{R}^d$, $\Delta_d = \{(l_1, ..., l_d) : l_i \geq 0, \sum l_i = 1\}$. In this paper, λ denotes Lebesgue measure on the unit interval. The term "almost all" refers to Lebesgue measure on the disjoint union of the simplices corresponding to the permutations that contain some IETs with dense orbits. That is, $\pi(\{1, ..., k\}) \neq \{1, ..., k\}$ for $k < d$ [14, Section 3]. These permutations are called *irreducible.*

The following is one of the main results on IETs and was proven independently by Masur [18] and Veech [23].

Theorem 3. *(Masur, Veech) Let π be an irreducible permutation on d-letters. For almost every $(L_1, L_2, ..., L_d)$ the IET determined by $(L_1, ..., L_d)$ and π is uniquely ergodic with respect to Lebesgue measure.*

1.2.1 The induced map of an IET

Let A be a subinterval of $[0, 1)$. If T is a d-IET then $T|_A$ is at most a $d+2$-IET. If A is bounded by discontinuities then $T|_A$ is at most a d-IET. These observations are classical and follow from the simple fact that the discontinuities of $T|_A$ are pre-images of discontinuities of T or pre-images of endpoints.

Remark 1. If $T|_A$ is a d-IET then one can tabulate the number of hits the j^{th} interval of $T|_A$ makes in the i^{th} interval of T before first return as the ij^{th} entry of a matrix. Notice that the travel of intervals of the induced of an induced map can be kept track of by the product of two of these matrices. This will be used throughout this thesis. We denote this matrix $M(T, A)$.

1.2.2 The Keane condition

An IET, T with discontinuities $\delta_1, ..., \delta_{d-1}$ is said to satisfy the *Keane condition* (also called the infinite distinct orbit condition or idoc) if $\{\delta_1, T\delta_1, ..., T^k\delta_1, ...\}$, $\{\delta_2, T\delta_2, ...\}$,...,$\{\delta_{d-1}, T\delta_{d-1}, ...\}$ are all disjoint infinite sets. The following 2 results motivate introducing this condition [14, Section 3].

Proposition 1. *(Keane) If T satisfies the Keane condition then for any $x \in [0,1)$ the set $\{x, Tx, ...\}$ is dense in $[0,1)$.*

Proposition 2. *(Keane) If π is an irreducible permutation on $\{1, ..., d\}$ and $\{L_1, ..., L_d\}$ are linearly independent over $\mathbb{Q}$ then the IET they define satisfies the Keane condition.*

If π is not irreducible then there are no IETs with permutation π satisfying the Keane condition.

1.3 Rauzy-Veech Induction

Our treatment of Rauzy-Veech induction will be the same as in [23, Section 7]. We recall it here. Let T be a d-IET with permutation π. Let δ_+ be the rightmost discontinuity of T and δ_- be the rightmost discontinuity of T^{-1}. Let $\delta_{max} = \max\{\delta_+, \delta_-\}$. Consider the induced map of T on $[0, \delta_{\max})$ denoted $T|_{[0,\delta_{\max})}$. If $\delta_+ \neq \delta_-$ this is a d-IET on a smaller interval, perhaps with a different permutation.

We can renormalize it so that it is once again a d-IET on $[0,1)$. That is, let $R(T)(x) = T|_{[0,\delta_{\max})}(x\delta_{max})(\delta_{max})^{-1}$. This is the Rauzy-Veech induction of T. To be explicit the Rauzy-Veech induction map is only defined if $\delta_+ \neq \delta_-$. If $\delta_{max} = \delta_+$ we say the first step in Rauzy-Veech induction is a. In this case the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j) & j \leq \pi^{-1}(d) \\ \pi(d) & j = \pi^{-1}(d) + 1 \\ \pi(j-1) & \text{otherwise} \end{cases} . \tag{1.1}$$

We keep track of what has happened under Rauzy-Veech induction by a matrix $M(T,1)$ where

$$M(T,1)[ij] = \begin{cases} \delta_{i,j} & j \leq \pi^{-1}(d) \\ \delta_{i,j-1} & j > \pi^{-1}(d) \text{ and } i \neq d \\ \delta_{\pi^{-1}(d),j} & i = d \end{cases} . \tag{1.2}$$

If $\delta_{max} = \delta_-$ we say the first step in Rauzy-Veech induction is b. In this case the permutation of $R(T)$ is given by

$$\pi'(j) = \begin{cases} \pi(j) & \pi(j) \leq \pi(d) \\ \pi(j)+1 & \pi(d) < \pi(j) < d \\ \pi(d)+1 & \pi(j) = d \end{cases} . \tag{1.3}$$

We keep track of what has happened under Rauzy-Veech induction by a matrix

$$M(T,1)[ij] = \begin{cases} 1 & i = d \text{ and } j = \pi^{-1}(d) \\ \delta_{i,j} & \text{otherwise} \end{cases} . \tag{1.4}$$

The matrices described above depend on whether the step is a or b and the permutation T has. The following well known lemmas which are immediate calculations help motivate the definition of $M(T,1)$.

Lemma 1. *If $R(T) = S_{L,\pi}$ then the length vector of T is comeasurable with $M(T,1)L$.*

Let $M_\Delta = M\mathbb{R}_d^+ \cap \mathring{\Delta}_d$. Recall $\mathring{\Delta}_d$ is the interior of the simplex in $\mathbb{R}^d$.

Lemma 2. *An IET with lengths contained in $M(T,1)_\Delta$ and permutation π has the same first step of Rauzy-Veech induction as T.*

We define the n^{th} matrix of Rauzy-Veech induction by

$$M(T,n) = M(T,n-1)M(R^{n-1}(T),1).$$

All $M(T,n)$ are in $SL_2(\mathbb{Z})$ and have non-negative entries. It follows from Lemma 2 that for an IET with length vector in $M(T,n)_\Delta$ and permutation π the first n steps of Rauzy-Veech induction agree with T. If M is any matrix, $C_i(M)$ denotes the i^{th} column and $C_{max}(M)$ denotes the column with the largest sum of entries. Let $|C_i(M)|$ denote the sum of the entries in the i^{th} column. Versions of the following lemma are well known and we provide a proof for completeness.

Lemma 3. *If $M(R^n(T),k)$ is a positive matrix and $L = \frac{C_i(M(T,n+k))}{|C_i(M(T,n+k))|}$ then $S_{L,\pi}$ agrees with T through the first n steps of Rauzy-Veech induction.*

Proof. By Lemma 1 the length vector for $R^m(S_{L,\pi})$ is $\frac{C_i(M(R^m(T),n+k-m))}{|C_i(M(R^m(T),n+k-m))|}$ for any m where $R^m(S_{L,\pi})$ is defined. By our assumption on the positivity of $M(R^n(T),k)$ the vector $\frac{C_i(M(R^n(T),k))}{|C_i(M(R^n(T),k))|}$ is contained in $\mathring{\Delta}_d$. The lemma follows by Lemma 2 and induction. □

The next definition does not appear in [23] but is important for the last section.

Definition 3. *A matrix M is called ν* balanced *if $\frac{1}{\nu} < \frac{|C_i(M)|}{|C_j(M)|} < \nu$ for all i and j.*

Notice that if M is ν balanced then $|C_i(M)| > \frac{|C_{max}(M)|}{\nu}$.

We remarked earlier that Rauzy-Veech induction may send the IET to an IET with a different permutation. Given a permutation π, its *Rauzy class* is the set of all permutations that can be reached by powers of Rauzy-Veech induction on IETs with permutation π.

Whether the operation of Rauzy-Veech induction is a or b is important. The infinite sequence of a's and b's uniquely determines the IET if it is uniquely ergodic with respect to Lebesgue measure.

1.4 Basic Measure Theory

Theorem 4. *(Borel-Cantelli) Let μ be a measure and $A_1, A_2, \ldots$ be a sequence of μ measurable sets. If $\sum_{n=1}^{\infty} \mu(A_i) < \infty$ then $\mu(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} A_i) = 0$.*

Theorem 5. *(Fubini) Let $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{C}, \nu)$ be two finite measure spaces and $f : X \times Y$ be a measurable function of their product σ-algebra. Then $\int_X (\int_Y f(x,y) d\nu(y)) d\mu(x) = \int_Y (\int_X f(x,y) d\mu(x)) d\nu(y)$.*

The assumption that these are finite measure spaces can be weakened to only assume that they are σ-finite.

1.5 The Spectral Theorem

If $f \in L^2(\mu)$ and T is a μ measure preserving transformation then $f \circ T \in L^2(\mu)$, with the same norm. Motivated by this let U_T denote the isometry on $L^2(\mu)$ given by $U_T(f) = f \circ T$. Notice that U_T preserves constant functions. If T has measure preserving inverse then U_T is a unitary operator with $U_T^* = U_{T^{-1}}$.

The spectral theorem for unitary operators states that for each $f \in L^2(\mu)$ there exists a unique measure on the unit circle $\mu_{f,t}$, such that

$$\int_{\mathbb{T}} z^n d\sigma_{f,T} =< f, U_T^n f > \text{ for all } n.$$

In Chapter 3 the spectral theorem will be used to establish that IETs are almost surely different. We will briefly make some remarks that are helpful to motivate this approach. Assume that there exists an increasing sequence of natural numbers $n_1, n_2, \ldots$ such that $\lim\limits_{i \to \infty} \int_0^1 |T^{n_i}x - x| d\lambda = 0$. It follows from either the Lebesgue density theorem or Luzin's theorem that $U_T^{n_i} f$ converges to f in L^2 norm. It follows that $U_T^{n_i}$ converges to the identity in the strong operator topology. It also follows that $\lim\limits_{i \to \infty} \int_{\mathbb{T}} |z^{n_i} - 1|^2 \sigma_{f,T} = 0$. By the fact that convergence in norm implies convergence almost everywhere along a subsequence it follows that there exists $i_1, i_2, \ldots$ such that $\{z : \lim\limits_{j \to \infty} z^{n_{i_j}} = 1\}$ has full $\sigma_{f,T}$ measure. This implies that spectral measures detect information about the measure preserving transformations they are associated with. This argument is developed further in Chapter 3.

Chapter 2

Keane type examples

Michael Keane introduced a construction of a minimal but not uniquely ergodic 4-IET [15]. This construction is based on proving that there are orbits that have asymptotically different distribution. It uses an inductive procedure that provides for a great deal of control. This chapter uses Keane's construction to show that there exists a topologically mixing IET, results on the possible size of ergodic measures in terms of Hausdorff dimension and exotic properties of the distribution of orbits. These results make statements about topology, measure and metric respectively.

2.1 An introduction to Keane type examples

Consider IETs with permutation (4213). Observe that the second interval gets shifted by $l_4 - l_1$. If this difference is small relative to l_2 then much of I_2 gets sent to itself. At the same time, pieces of I_3 do not reach I_2 until they have first reached I_4. This is the heart of the Keane construction. The details of the Keane construction are centered around iterating this procedure by the induced map. Keane considered the induced map on the fourth interval, which we denote $I^{(1)}$. The induced map on this interval is once again a 4-IET. Keane showed that by choosing the lengths appropriately one could ensure that this induced map had the permutation (2431). Name these in reverse order and we once again get a (4213) IET. Motivated by this, we name the 4 exchanged subintervals of $I^{(1)}$ under $T|_{I^{(1)}}$ in reverse order; that is, $I_1^{(1)}$ is the subinterval furthest to the right. Keane also showed that for any choice $m, n \in \mathbb{N}$ one can find an IET whose landing pattern of $I_j^{(1)}$ is given by the columns of following matrix:

$$A_{m,n} = \begin{pmatrix} 0 & 0 & 1 & 1 \\ m-1 & m & 0 & 0 \\ n & n & n-1 & n \\ 1 & 1 & 1 & 1 \end{pmatrix}; \qquad m, n \in \mathbb{N} = \{1, 2, ...\}.$$

In order to see this, pick lengths for $I^{(1)}$ and write it as a column vector. Now assign lengths to the original IET by multiplying this column vector by $A_{m,n}$. The induced map will travel according to this matrix by construction. For instance, if one chooses lengths $[.25, .25, .25, .25]$ for $I^{(1)}$ one gets lengths of

$$[\frac{2}{2+2m-1+4n-1+4}, \frac{2m-1}{2m+4n+4}, \frac{4n-1}{2m+4n+4}, \frac{4}{2m+4n+4}]$$

for the original IET (after renormalizing). For any finite collection of matrices one can iterate this construction. (Assign lengths for $I^{(k)}$ by multiplying the lengths of $I^{(k+1)}$ by $A_{m_{k+1},n_{k+1}}$, multiply the resulting column vector by $A_{m_k,n_k}, \dots$. $I^{(k+1)}$ is defined inductively as the fourth interval of $I^{(k)}$.) Compactness (of $\mathbb{P}^3$, which can be thought of as the parameterizing space of (4213) IETs) ensures that we can pass to an infinite sequence of these matrices.

Since the intervals are named in reverse order, the discontinuity (under the induced map) between $I_2^{(1)}$ and $I_3^{(1)}$ is given by $T^{-1}(\delta_1)$ where δ_1 denotes the discontinuity between I_1 and I_2. As the first row of the matrix suggests $I_1 = T(I_4^{(1)} \cup I_3^{(1)})$. The discontinuity (under the induced map) between $I_1^{(1)}$ and $I_2^{(1)}$ is given by $T^{-m}(\delta_2)$ where δ_2 denotes the discontinuity between I_2 and I_3. As the second row of the matrix suggests

$$I_2 = T(I_2^{(1)} \cup I_1^{(1)}) \cup T^2(I_2^{(1)} \cup I_1^{(1)}) \cup \ldots \cup T^{m-1}(I_2^{(1)} \cup I_1^{(1)}) \cup T^m(I_2).$$

The discontinuity (under the induced map) between $I_3^{(1)}$ and $I_4^{(1)}$ is given by $T^{-n-1}(\delta_3)$ where δ_3 denotes the discontinuity between I_3 and I_4. As the third row of the matrix suggests

$$I_3 = T^m(I_1^{(1)}) \cup T^{m+1}(I_2^{(1)}) \cup T^2(I_4^{(1)} \cup I_3^{(1)}) \cup T^{m+1}(I_1^{(1)}) \cup T^{m+2}(I_2^{(1)}) \cup T^3(I_4^{(1)} \cup I_3^{(1)}) \cup$$

$$\ldots \cup T^{m+n-1}(I_1^{(1)}) \cup T^{m+n}(I_2^{(1)}) \cup T^n(I_4^{(1)} \cup I_3^{(1)}) \cup T^{m+n}(I_1^{(1)}) \cup T^{m+n+1}(I_2^{(1)}) \cup T^{n+1}(I_4^{(1)}).$$

$I_4 = I_4^{(1)} \cup I_3^{(1)} \cup I_2^{(1)} \cup I_1^{(1)}$. As the columns of the matrix suggest, this is also

$$I_4 = T^{n+1}(I_3^{(1)}) \cup T^{m+n+1}(I_2^{(1)}) \cup T^{m+n}(I_1^{(1)}) \cup T^{n+2}(I_4^{(1)}).$$

To recap, the composition of I_j can be given by the j^{th} row of the matrix. The travel before first return of $I_j^{(1)}$ can be given by the j^{th} column. Additionally, because the intervals were named in reverse order, the permutation of the induced map is once again (4213).

It is important for this construction that everything be iterated. The composition of $I_j^{(k)}$ in pieces of $I^{(k+r)}$ is given by $e_j^\tau A_{m_{k+1},n_{k+1}} ... A_{m_{k+r},n_{k+r}}$ (where e_j^τ denotes the transpose pf e_j). Likewise, the travel of $I_j^{(k+r)}$ under $T_{I^{(k)}}$ before first return to $I^{(k+r)}$ is given by $A_{m_{k+1},n_{k+1}} ... A_{m_{k+r},n_{k+r}} e_j$.

Definition 4. *Let $O(I_j^{(k)})$ denote the disjoint images under T of $I_j^{(k)}$ before first return to $I^{(k)}$.*

Now for some explicit statements about the travel of subintervals of $I^{(k)}$ under the induced map $T_{I^{(k)}}$. When $I_3^{(k)}$ returns to $I^{(k)}$ it entirely covers $I_4^{(k)}$. It is a subset of $I_3^{(k)} \cup I_4^{(k)}$. When $I_4^{(k)}$ returns to $I^{(k)}$ it entirely covers $I_1^{(k)}$. It intersects $I_2^{(k)}$. Moreover part of this intersection will stay in $O(I_2^{(k)})$ for the next $m_{k+1} b_{k,2}$ images (the other part $(m_{k+1} - 1) b_{k,2}$.) When $I_2^{(k)}$ returns to $I^{(k)}$ it intersects $I_3^{(k)}$. Moreover this piece of intersection will stay in $O(I_3^{(k)})$ for the next $n_{k+1} b_{k,3}$ images.

Definition 5. *$b_{k,i}$ is the first return time of $I_i^{(k)}$ to $I^{(k)}$.*

Remark 2. $b_{k,i}$ is given by $|A_{m_1,n_1} ... A_{m_k,n_k} e_i|_1$. In particular, $b_{k,2} = m_k b_{k-1,2} + n_k b_{k-1,3} + b_{k-1,4}$ and $b_{k,3} = b_{k-1,1} + (n_k - 1) b_{k-1,3} + b_{k-1,4}$.

Remark 3. $O(I_i^{(k)} = \bigcup_{i=1}^{b_{k,i}-1} T^i(I_j^{(k)})$.

Some facts to keep in mind:

1. The choice of n_k has no effect on $b_{i,2}$ for $i < k$.

2. The choice of n_k has no effect on $b_{i,3}$ for $i < k$.

3. The choice of m_k has no effect on $b_{i,2}$ for $i < k$.

4. The choice of m_k has no effect on $b_{i,3}$ for $i < k + 1$.

2.2 There exists a topologically mixing Keane IET

Definition 6. *Let X be a topological space. A dynamical system $T : X \to X$ is said to be* topologically mixing *if for nonempty open sets U, V there exists $N_{U,V} := N$ such that $T^n(U) \cap V \neq \emptyset$ for all $n \geq N$.*

Theorem 6. *There exists a topologically mixing 4-IET.*

Remark 4. It is classical that aperiodic IETs are measurably conjugate to shift dynamical systems that are continuous. The example presented to prove Theorem 6 has this conjugate system also topologically mixing. The proof is straightforward and will not be presented in this thesis.

Conditions on $b_{k,2}$ and $b_{k,3}$ that ensure topological mixing:

1. $b_{k,2}$ is prime for all k.
2. $b_{i,2} \not| b_{k,3}$ for all $i < k$.
3. The group of multiplicative units mod $\prod_{i=1}^{k} b_{i,2}$ has more than $.5 \prod_{i=1}^{k} b_{i,2}$ elements.
4. $b_{k,2}b_{k+1,3} + b_{k+1,3} + b_{k,4} + b_{k-1,4} < m_{k+1}b_{k,2}$
5. $b_{k,3}b_{k,2} + b_{k,2} < n_{k+1}b_{k,3}$

Theorem 1 will be proven by showing that any Keane IET chosen in this way is Topologically mixing. We first show that the set of such IETs is nonempty.

Lemma 4. *One can choose $b_{k,2}$ and $b_{k,3}$ to fulfill these conditions.*

Proof. By induction. Assume we have chosen $n_1, m_1, n_2, m_2, ..., n_{k-1}, m_{k-1}$; we describe how to choose n_k and then given this n_k how to choose m_k. Consider congruence modulo $g := \prod_{i=1}^{k-1} b_{i,2}$. Choose a congruence class $[f]$ that is in the group of multiplicative units and so that $[f + b_{k-1,3} - b_{k-1,1}]$ is in the multiplicative group of units. This can be done by pigeon hole principle (by condition 3). Pick n_k so that $b_{k,3} \in [f]$ and so that $n_k > \frac{b_{k-1,3}b_{k-1,2}+b_{k-1,2}}{b_{k-1,3}}$. This can be done because $b_{k-1,3}$ is relatively prime to the $b_{i,2}$ for all $i < k$. Next we pick m_k so that $b_{k,2}$ is prime, $m_k > \frac{b_{k-1,2}b_{k,3}+b_{k-1,4}+b_{k,3}}{b_{k-1,2}}$ and condition 3 is satisfied. This is doable because we wish to find a prime in the arithmetic progression $n_k b_{k-1,3} + b_{k-1,4} + b_{k-1,2}\mathbb{N}$ and the starting point and the increment are relatively prime and the other conditions merely require choosing m_k large enough. $\square$

Let $c_k = b_{k+1,2}b_{k+2,3} + b_{k+2,3} + b_{k+2,4} + b_{k+1,4}$; $d_k = b_{k+2,3}b_{k+2,2} + b_{k+2,2}$.

Let J contain at least one level of a tower over $I^{(k)}$. This means that it contains at least 1 level from each of the 4 towers over $I^{(k+2)}$. For all $j > k$, $i > c_j$, $T^i(J)$ intersects every level of every tower over $I^{(j-1)}$. This is proved in the following lemmas. In these arguments it will be important to pick out a level from $O(I_2^{(k+2)})$ and $O(I_3^{(k+2)})$. These will be denoted J'.

Lemma 5. *At times c_k to d_k J', a level in $O(I_3^{(k+2)})$, intersects every level of $O(I_2^{(k+1)})$.*

Proof. There exists $0 < i \leq b_{k+2,3}$ (it is equal to $b_{k+2,3}$ for $I_3^{(k+2)}$ but for pieces of the orbit it is less than) such that $I_4^{(k+2)} \subset T^i(J')$. So $T^{i+b_{k+2,4}}(J') \cap I_2^{(k+2)} \neq \emptyset$. Also

$$T^{i+b_{k+2,4}+b_{k+1,4}}(J') \cap I_2^{(k+1)} \neq \emptyset.$$

In fact,

$$T^{i+b_{k+2,4}+b_{k+1,4}+jb_{k+2,3}}(J') \cap I_2^{(k+1)} \neq \emptyset$$

for $j < n_{k+3}$ (notice, $T|^j_{I^{(k+2)}}(I_3^{(k+2)}) \cap I_3^{(k+2)} \neq \emptyset$ for $j < n_{k+3}$). Pieces of $I_3^{(k+2)}$ are inserted into $I_2^{(k+1)}$ with a delay of $b_{k+2,3}$ which is coprime to $b_{k+1,2}$. It follows that $T^{c_k}(J')$ intersects every level of $O(I_2^{(k+1)})$. By condition 5 it follows that $T^r(J')$ intersects every level of $O(I_2^{(k+1)})$ for $c_k \leq r \leq d_k$ (because $n_{k+3}b_{k+2,3} > d_k$). Moreover, the pieces inserted take $m_{k+2}b_{k+1,2}$ to leave $O(I_2^{(k+2)})$. Because $m_{k+1}b_{k+1,2} > b_{k+2,3}b_{k+1,2}$ (condition 4) the piece does not leave $O(I_2^{(k+2)})$ before another is inserted into its level. □

Lemma 6. *At times d_k to c_{k+1} J', a level in $O(I_2^{(k+2)})$, intersects every piece of $O(I_3^{(k+2)})$.*

Proof. There exists $0 < i \leq b_{k+2,2}$ (it is equal to $b_{k+2,2}$ for $I_2^{(k+2)}$ but for pieces of the orbit it is less than) such that $I_3^{(k+2)} \cap T^i(J') \neq \emptyset$. Also $I_3^{(k+2)} \cap T^{i+jb_{k+2,2}}(J') \neq \emptyset$ for $j < m_{k+3}$ (notice, $T|^j_{I^{(k)}}(I_2^{(k+2)}) \cap I_2^{(k+2)} \neq \emptyset$ for $j \leq m_{k+3}$). Because $b_{k+2,2}$ is relatively prime to $b_{k+2,3}$ we have $T^{i+jb_{k+2,2}}(J')$ intersects each level of $O(I_3^{(k+2)})$ for $j = b_{k+2,3}$. It follows from condition 4 that $T^r(I_2^{(k+2)})$ intersects each level of $O(I_3^{(k+2)})$ for $d_k \leq r \leq c_{k+1}$ (because $m_{k+3}b_{k+2,2} > b_{k+2,2}b_{k+3,3}$). Moreover, the pieces inserted take $(n_{k+2}-1)b_{k+2,3}$ to leave $O(I_3^{(k+2)})$. Because $(n_{k+3}-1)b_{k+2,3} > b_{k+2,2}b_{k+2,3}$ (condition 5) the piece does not leave $O(I_3^{(k+1)})$ before another is inserted into its level. □

Proof of Theorem 6. For any two intervals J_1, J_2, eventually both contain some level of a tower over $I^{(k_0)}$. This implies that they contain a level from each tower over

$I^{(k)}$ for all $k > k_0 + 1$. This implies that $T^n(J_1) \cap J_2 \neq \emptyset$ for $n \in [c_k, d_k]$ because J_1 contains a level of $I_3^{(k+2)}$ and J_2 contains a level of $I_2^{(k+1)}$. Also $T^n(J_1) \cap J_2 \neq \emptyset$ for $n \in [d_k, c_{k+1}]$ because J_1 contains a level of $I_2^{(k+2)}$ and J_2 contains a level of $I_3^{(k+2)}$. It follows that $T^n(J_1) \cap J_2 \neq \emptyset$ for any $n > c_{k_0+1}$. □

2.2.1 No IET is topologically mixing of all orders

The argument is a straightforward application of [13]. Let T be an d-IET. Observe that a topologically mixing IET must be minimal (otherwise it splits into disjoint invariant components). Let J, J' be any disjoint intervals bounded by discontinuities of T^l for some l, and $n_1, ..., n_{d^2}$ be natural numbers. We will find a violation of topological mixing of order $d^2 + 1$ at bigger times. Pick an interval V such that all of the first returns to V are greater than $\max\{l, n_1, ...n_{d^2}\}$. We may also choose V so that T_V is an s IET for some $s \leq d$. By our assumption that the return times to V are larger than l, each level of a tower over V is either contained in J or disjoint from J. Let $U_1, U_2, ..., U_s$ be its subintervals. T_{U_i} is an s_i-IET for $s_i \leq s$. Call its intervals $U_{i,1}, ..., U_{i,s_i}$ and their return times $r_{i,1}, ...r_{i,s_i}$. If $x \in O(U_i)$ and $x \in O(U_{i,j})$ then $T^{r_{i,j}}(x) \in J$. This is because $x \in T^k(U_i) \subset J$ for some $k < r_i$, in fact $x \in T^k(U_{i,j})$. $T^{r_{i,j}-k}(x) \in U_i$. So $T^k(T^{r_{i,j}-k}(x)) \in T^k(U_i) \subset J$. Therefore $\bigcap\limits_{i,j=1}^{d} T^{r_{i,j}}(J) \cap J' = \emptyset$.

2.3 Measure estimates for Keane's construction

The previous sections discussed the *topological* properties of Keane type IETs. Keane's construction of these IETs was motivated by their measure properties. In

Keane's example we have a non-uniquely ergodic minimal 4-IET T with ergodic measure λ_2 and λ_3. To gain some further intuition consider the product:

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ m-1 & m & 0 & 0 \\ n & n & n-1 & n \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} c+d \\ (m-1)a+mb \\ n(a+b+c+d)-c \\ a+b+c+d \end{pmatrix}$$

Notice that if $a=c=d=0$, $b=1$, m is much bigger than n and large then the resulting column vector has small angle with the original. Likewise, if $a=b=d=0$, $c=1$ and n is large then the resulting column vector has small angle with the original. Motivated by this, we introduce another piece of notation.

Definition 7. *Let* $\bar{A}_{m,n}v = \frac{A_{m,n}v}{|A_{m,n}v|}$, *where* $|w|$ *is the sum of the entries in* w.

Michael Keane showed that if $3n_k \leq m_k \leq \frac{1}{2}n_{k+1}$ and $n_1 \geq 9$ then the IET given by $\lim_{r\to\infty} \bar{A}_{n_1,m_1} \ldots \bar{A}_{n_r,m_r} e_2$ is minimal but not uniquely ergodic. In particular he showed the limit exists. (It is not hard to see that one can remove the assumption on n_1 or any finite number of matrices).

2.3.1 Estimates on the size of intervals with respect to the two ergodic measures

In this section we bound $\lambda_i(I_j^{(k)})$ between two constants. Many of these are needed in the later arguments. We include the rest for completeness.

In these computations, we use jth entry of partial products $\bar{A}_k \ldots \bar{A}_{k+r} e_i$ to estimate $\frac{\lambda_i(I_j^{(k)})}{\lambda_i(I^{(k)})}$. To complete these estimates we remark that $b_{k,2}^{-1} > \lambda_2(I^{(k)}) > \frac{1}{4b_{k,2}}$ (Lemma

28) and $b_{k,3}^{-1} > \lambda_3(I^{(k)}) > \frac{1}{8b_{k,3}}$ (Lemma 27).

Remark 5. The proofs of these lemmas often provide better results than their statements. Additionally, it is often straightforward to provide better estimates, especially under stronger growth conditions on m_i and n_i. Lemma 14, for instance, would be amenable to such an approach.

Proposition 3. $\frac{\lambda_i(I_j^{(k)})}{\lambda_i(I^{(k)})}$ = *the* j*th entry of* $\lim_{r\to\infty} \bar{A}_{m_{k+1},n_{k+1}} \ldots \bar{A}_{m_{k+r},n_{k+r}} e_i$.

Lemma 7. $\frac{\lambda_3(I_2^{(k)})}{\lambda_3(I^{(k)})} \geq \frac{m_{k+1}}{2n_{k+1}n_{k+2}}$.

Proof. It suffices to show that the second entry of $\bar{A}_{m_{k+1},n_{k+1}} \bar{A}_{m_{k+2},n_{k+2}} e_3 > \frac{m_{k+1}}{2n_{k+1}n_{k+2}}$. This is a direct computation. □

Lemma 8. $\frac{\lambda_3(I_2^{(k)})}{\lambda_3(I^{(k)})} \leq \frac{2m_{k+1}}{(n_{k+2}+1)(n_{k+1}+1)}$.

This result is in the proof of Lemma 3 of [15].

Lemma 9. $\frac{\lambda_3(I_3^{(k)})}{\lambda_3(I^{(k)})} \geq 1 - \frac{3}{n_{k+1}}$.

This is Lemma 3 of [15].

Lemma 10. $\frac{\lambda_3(I_4^{(k)})}{\lambda_i(I^{(k)})} \leq \frac{1}{n_{k+1}}$.

Proof. Notice that $I_4^{(k)}$ is the disjoint union of an image of $I_1^{(k+1)}$, an image of $I_2^{(k+1)}$, an image of $I_3^{(k+1)}$ and an image of $I_4^{(k+1)}$ and that $I^{(k)}$ contains at least $n_{k+1} + 1$ disjoint images of $I_j^{(k+1)}$ for each j. □

Lemma 11. $\frac{\lambda_3(I_4^{(k)})}{\lambda_3(I^{(k)})} \geq \frac{1}{2n_{k+1}}$.

Proof. $I_4^{(k)}$ is made up of one disjoint image of each $I_i^{(k+1)}$. $I_3^{(k)}$ is made up of $n_{k+1}-1$ disjoint images of $I_3^{(k+1)}$ and n_{k+1} disjoint images of each of the other $I_i^{(k+1)}$. Therefore, because n_{k+1} disjoint images of $I_4^{(k+1)}$ cover $I_3^{(k+1)}$ and $\frac{\lambda_3(I_4^{(k)})}{\lambda_3(I^{(k)})} > \frac{\lambda_3(I_3^{(k)})}{\lambda_3(I^{(k)})}\frac{1}{n_{k+1}}$. The lemma follows by Lemma 9. ☐

Lemma 12. $\frac{\lambda_3(I_1^{(k)})}{\lambda_3(I^{(k)})} \leq \frac{1}{n_{k+1}}$.

Proof. $I_1^{(k)}$ is made up of a disjoint union of an image of $I_3^{(k+1)}$ and $I_4^{(k+1)}$ each of which has at least $n_{k+1}+1$ disjoint images in $I^{(k)}$. ☐

Lemma 13. $\frac{\lambda_3(I_1^{(k)})}{\lambda_3(I^{(k)})} \geq \frac{1}{3n_{k+1}}$.

Proof. It follows from the composition of $I_i^{(k)}$ by subintervals of $I^{(k+1)}$ that $\lambda_3(I_1^{(k)}) \geq \lambda_3(I_3^{(k+1)})$. The proof follows from Lemmas 11 and 9. ☐

Lemma 14. $\frac{\lambda_2(I_2^{(k)})}{\lambda_2(I^{(k)})} > \frac{0.25m_{k+1}}{n_{k+1}+m_{k+1}+2}$.

Proof. Observe that if $v \in \mathbb{R}_+^4$ is positive, $|v|_1 = 1$ and $v[2] > .25$ then $\bar{A}_{m,n}v[2] > .25$ so long as $m \geq 3n$ and $n > \frac{8}{5}$. By induction, it follows that $\prod_{t=k+1}^{r} \bar{A}_{m_t,n_t}e_2[2] > \frac{0.25m_{k+1}}{n_{k+1}+m_{k+1}+2}$. ☐

Lemma 15. $\frac{\lambda_2(I_3^{(k)})}{\lambda_2(I^{(k)})} \leq \frac{4n_{k+1}}{m_{k+1}}$.

Proof. By the previous proof, $\prod_{t=k+2}^{r} \bar{A}_{m_t,n_t}e_2[2] > \frac{1}{4}$. It follows that $\prod_{t=k+1}^{r} \bar{A}_{m_t,n_t}e_2[3] < \frac{n_{k+1}}{0.25m_{k+1}}$. ☐

Lemma 16. $\frac{\lambda_2(I_3^{(k)})}{\lambda_2(I^{(k)})} \geq \frac{n_{k+1}}{2m_{k+1}}$.

Proof. $\bar{A}_{m_{k+1},n_{k+1}}e_2[3] = \frac{n_{k+1}}{m_{k+1}+n_{k+1}+1} > \frac{n_{k+1}}{2m_{k+1}}$ and $\bar{A}_{m_{k+1},n_{k+1}}e_2[3] < \bar{A}_{m_{k+1},n_{k+1}}e_i[3]$ for $i = 1,3,4$. Thus $\bar{A}_{m_{k+1},n_{k+1}}(\bar{A}_{m_{k+2},n_{k+2}} \cdots \bar{A}_{m_{k+r},n_{k+r}})[3] \geq \frac{n_{k+1}}{2m_{k+1}}$. ☐

This proof is related to Lemma 24

Lemma 17. $\frac{\lambda_2(I_4^{(k)})}{\lambda_2(I^{(k)})} > \frac{1}{2m_{k+1}}$.

Proof. There are at most $m_{k+1} + n_{k+1} + 1$ disjoint images of any $I_i^{(k+1)}$ in $I^{(k)}$. By our standard assumptions $n_{k+1} + 1 < m_{k+1}$. Also $I_4^{(k+1)}$ is made up of one image of each $I_i^{(k+1)}$. ☐

Lemma 18. $\frac{\lambda_2(I_4^{(k)})}{\lambda_2(I^{(k)})} < \frac{4}{m_{k+1}}$.

Proof. By construction the fourth entry of $A_{m_{k+1},n_{k+1}}(\bar{A}_{m_{k+2},n_{k+2}} \dots \bar{A}_{m_{k+r},n_{k+r}})$ is 1. By Lemma 14 the second entry is at least $.25m_{k+1}$. ☐

Lemma 19. $\frac{\lambda_2(I_1^{(k)})}{\lambda_2(I^{(k)})} < \frac{16n_{k+2}+16}{m_{k+1}m_{k+2}}$.

Proof. $I_1^{(k)}$ is made up of one image of $I_3^{(k+1)}$ and one image of $I_4^{(k+1)}$. $\frac{\lambda_2(I_1^{(k)})}{\lambda_2(I^{(k)})} = \frac{\lambda_2(I^{(k+1)})}{\lambda_2(I^{(k)})}\frac{\lambda_2(I_3^{(k+1)} \cup I_4^{(k+1)})}{\lambda_2(I^{(k+1)})}$. By the fact that $I^{(k+1)} = I_4^{(k)}$, Lemmas 15 and 18 this is less than $\frac{4}{m_{k+1}}\frac{4n_{k+2}+4}{m_{k+2}}$. ☐

Lemma 20. $\frac{\lambda_2(I_1^{(k)})}{\lambda_2(I^{(k)})} > \frac{n_{k+2}}{4m_{k+1}m_{k+2}}$.

Proof. $I_1^{(k)}$ contains one image of $I_3^{(k+1)}$. By Lemma 16, $\frac{\lambda_2(I_3^{(k+1)})}{\lambda_2(I^{(k+1)})} > \frac{n_{k+2}}{2m_{k+2}}$ and by Lemma 17, $\frac{\lambda_2(I^{(k+1)})}{\lambda_2(I^{(k)})} > \frac{1}{2m_{k+1}}$. ☐

2.4 Hausdorff dimension for ergodic measures in Keane type examples

In Keane's example we have a non-uniquely ergodic minimal 4-IET T with ergodic measure λ_2 and λ_3. If one assigns lengths to an IET by $l_i = c\lambda_2(I_i) + (1-c)\lambda_3(I_i)$, then the resulting IETs all have the same topological dynamics (see [22, Section 1] for more general discussion). They also all have two ergodic measures that assign the same measure to the 4-subintervals. When $c = 1$ then λ_2 is Lebesgue measure and λ_3 is singular with respect to Lebesgue measure. When $c = 0$ then λ_3 is Lebesgue measure and λ_2 is singular with respect to Lebesgue measure. In the intermediate situation both are absolutely continuous with respect to Lebesgue measure. This is discussed more in Remarks 8 and 10.

Theorem 7. *(a)$H_{dim}(\lambda_2, d_{\lambda_3})$ can take any value in* $[0, 1]$.

(b) $H_{dim}(\lambda_3, d_{\lambda_2})$ can take any value in $[0, 1]$

This result answers a question in [3, Section 6]. If the Hausdorff dimension of an ergodic measure for an IET is zero then the lengths of intervals are not all algebraic [3, Corollary 6.9].

Theorem 8. *$(H_{dim}(\lambda_2, d_{\lambda_3}), H_{dim}(\lambda_3, d_{\lambda_2}))$ can take values* $(0, 0), (1, 0), (0, 1)$ *or* $(1, 1)$.

Theorem 9. *If T is a Keane type IET let $G_3(T)$ be the set of λ_3 generic points. There exists a Keane type IET T such that $H_{dim}(G_3(T)^c, d_{\lambda_3}) = 0$.*

This says that all but a set of Hausdorff dimension zero of the points behave λ_3 typically.

2.4.1 Definition of Hausdorff dimension

Let $diam(U) = \sup_{x,y \in U} |x - y|$. Consider a set $S \subset [0, 1)$. We say a collection of open sets $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ is a $\delta > 0$ cover of S if $S \subset \bigcup_{i=1}^{\infty} U_i$ and $diam(U_i) \leq \delta$ $\forall i$. Let

$H^s_\delta(S) = \inf\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\}$ is a δ cover of $S\}$. Let $H^s(S) = \lim_{\delta\to 0^+} H^s_\delta(S)$. Notice that the limit exists. Let $H_{dim}(S) = \inf\{s : H^s(S) = 0\}$. This is equivalent to defining $H_{dim}(S) = \sup\{s : H^s(S) = \infty\}$. We state a few well known properties of Hausdorff dimension.

$H_{dim}(\bigcup_{i=1}^{\infty} S_i) = \sup_i H_{dim}(S_i)$.

$H_{dim} \bigcap_{i=1}^{\infty} S_i \leq \inf_i H_{dim}(S_i)$.

Definition 8. *For a Borel Measure μ we define the* Hausdorff dimension of a probability measure μ *is*

$$H_{dim}(\mu) = \inf\{H_{dim}(M)\colon M \text{ is Borel and } \mu(M) = 1\}.$$

For upper bounds to Hausdorff dimension of a set, explicit constructions are often all that is necessary. For lower bounds Frostman's Lemma is useful.

Lemma 21. *(Frostman) If $B \subset [0, 1)$ be a Borel set. $H^s(B) > 0$ iff there exists a finite radon measure on B, ν, such that $\nu(B(x, r)) \leq r^s$.*

see [19] p.112.

Corollary 1. *If μ is a measure on $[0, 1)$ and $\epsilon_1, \ldots$ is a positive sequence tending to 0 such that $\frac{\epsilon_i}{\epsilon_{i+1}} < C$ for some C and all i then $\mu(B(x, \epsilon_i)) < C(\epsilon_i)^\alpha$ implies $H_{dim}(\mu) \geq \alpha$.*

Lemma 22. *If T is a piecewise isometry then $H_{dim}(T(S)) \leq H_{dim}(S)$.*

This holds for locally Lipshitz maps as well, but this is unnecessary for the present paper.

2.4.2 Estimates towards calculating the Hausdorff dimension for ergodic measures of IETs

For upper bounds to the Hausdorff dimension for an ergodic measure of an IET the following proposition is useful.

Proposition 4. *Let T be a μ ergodic IET and the $H_{dim}(\mu) = t$. If S is a set such that $H_{dim}(S) < t$ then $\mu(S) = 0$.*

Proof. This follows from the countable stability of H_{dim} and ergodicity. If $\mu(S) > 0$ then $\mu(\bigcup_{i=1}^{\infty} T^i(S)) = 1$ by ergodicity. However, by the countable stability of Hausdorff dimension $H_{dim}(\bigcup_{i=1}^{\infty} T^i(S)) = H_{dim}(S)$ because T is a piecewise isometry. $\square$

This proposition says that one needs to only prove upper bounds on part of the measure. If $\mu(S) > 0$ and $H^t(S) = 0$ then $H_{dim}(\mu) \leq t$.

Below is a lemma based adapting Frostman's Lemma to our particular circumstances to provide lower bounds for the Hausdorff dimension of an ergodic measure.

Lemma 23. *If there exists C such that $C\lambda_3(I_i^{(k)})^\alpha > \lambda_2(I_i^{(k)})$ for any k and $i \in \{1, 2, 3, 4\}$ then $H_{dim}(\lambda_2, d_{\lambda_3}) \geq \alpha$. Likewise, if there exists a C such that $C\lambda_2(I_i^{(k)})^\alpha > \lambda_3(I_i^{(k)})$ for any k and $i \in \{1, 2, 3, 4\}$ then $H_{dim}(\lambda_3, d_{\lambda_2}) \geq \alpha$.*

Proof. By Frostman's Lemma it suffices to show that for any interval J $C\lambda_3(J)^\alpha > \lambda_2(J)$. We will show that $\log_{\lambda_3(J)} \lambda_2(J)$ is dominated by something comparable to $\log_{\lambda_3(I_2^{(t)})} \lambda_2(I_2^{(t)})$. This follows form the fact that $I_2^{(k)}$ and $I_3^{(k)}$ are made up of repeating images. To see this assume that we wish to estimate $\log_{\lambda_3(J)} \lambda_2(J)$ mostly covered by images of $I_i^{(k+1)}$ and contained in $I_2^{(k)}$. $I_2^{(k)}$ is made up of repeating unions of images of $I_1^{(k+1)} \cup I_2^{(k+1)}$ so the maximum advantage is either by taking the whole $I_2^{(k)}$ or $I_2^{(k+1)} \cup I_1^{(k+1)} \cup I_2^{(k+1)}$. In either case, $\log_{\lambda_3(J)} \lambda_2(J)$ is dominated by something proportional to $I_2^{(j)}$ for some j. Likewise,if $J \subset I_3^{(k)}$ for pieces in images of $I_3^{(k)}$ one either covers by all of $I_3^{(k)}$ or $I_2^{(k+1)} \cup I_4^{(k+1)} \cup I_3^{(k+1)} \cup I_1^{(k+1)} \cup I_2^{(k+1)}$. $I_1^{(k)}$ and $I_4^{(k)}$ are made up of at most 1 image each of $I_i^{(k+1)}$ for $i \in \{1,2,3,4\}$ and so reduce to these cases. Similar arguments hold for $\log_{\lambda_2(J)} \lambda_3(J)$. $\square$

Lemma 24. *$b_{k,2} > b_{k,i}$ for $i \in \{1,3,4\}$.*

Proof. $b_{k,2} > b_{k,1}$ because the second entry of $A_{m_k,n_k}e_2 = m_k > m_k - 1$ and $m_k - 1$ is the second entry of $A_{m_k,n_k}e_1$. $A_{m_k,n_k}e_2$ agrees with $A_{m_k,n_k}e_1$ in all other entries. $b_{k,2} \geq b_{k,j}$ for $j = 3,4$ because $A_{m_k,n_k}e_2 \geq A_{m_k,n_k}e_j$ in all entries but the first and $m_k A_{m_{k-1},n_{k-1}}e_2 > A_{m_{k-1},n_{k-1}}e_1$ in all entries (the second entry of $A_{m_k,n_k}e_j$ is 0 and the second entry of $A_{m_k,n_k}e_2$ is $m_k e_2$ and also the first entry of $A_{m_k,n_k}e_j = 1$). This argument shows that $A_{m_{k-1},n_{k-1}}A_{m_k,n_k}e_2$ has each entry greater than or equal to the corresponding entries of $A_{m_{k-1},n_{k-1}}A_{m_k,n_k}e_j$ for $j = 3,4$. $\square$

Lemma 25. *$b_{k,2} \leq \prod_{i=1}^{k} 2m_i$.*

Proof. $b_{k,2} = m_k b_{k-1,2} + n_k b_{k-1,3} + b_{k-1,4}$. By Lemma 24 $b_{i,2} \geq b_{i,j}$. By our assumptions $m_i > n_i + 1$. The lemma follows by induction. $\square$

Lemma 26. *$\prod_{i=1}^{k} n_i < b_{k,3}$.*

Proof. $b_{k,3} = b_{k,1} + (n_k - 1)b_{k-1,3} + b_{k-1,4}$. Notice that $b_{i,4} = b_{i-1,1} + n_i b_{i-1,3} + b_{i-1,4} > b_{i,3}$ implying that $b_{k,3} > n_k b_{k-1,3}$. The lemma follows by induction. $\square$

Lemma 27. *$\lambda_3(O(I_3^{(k)})) > \frac{1}{8}$.*

Proof. $n_{k+1}b_{k,3} > \frac{1}{2}b_{k,i}$ for $i \neq 3$. This follows from Lemmas 9, 25 and 26. $\square$

This Lemma establishes that $\lambda_3(I^{(k)})$ is proportional to $b_{k,3}^{-1}$.

Lemma 28. $\lambda_2(O(I_2^{(k)})) > \frac{1}{4}$.

Proof. $b_{k,2} > b_{k,i}$ (Lemma 24) so $\lambda_2(O(I_2^{(k)})) > \frac{\lambda_2(I_2^{(k)})}{\lambda_2(I^{(k)})}$. $\square$

This Lemma establishes the $\lambda_2(I^{(k)})$ is proportional to $b_{k,2}^{-1}$.

Proposition 5. $H_{dim}(\lambda_2, d_{\lambda_3}) \leq H_{dim}(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} O(I_2^{(k)}), d_{\lambda_3})$.

Remark 6. $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} O(I_2^{(k)})$ has positive λ_2 measure and is T invariant except for a set of measure zero (because $\lambda_i(I_2^{(k)}) \to 0$). By ergodicity it has full measure.

Proposition 6. $H_{dim}(\lambda_2, d_{\lambda_3}) \leq \liminf_{k\to\infty} \log_{\lambda_3(I_2^{(k)})} b_{k,2}^{-1}$.

Proof. Assume that $\liminf_{k\to\infty} \log_{\lambda_3(I_2^{(k)})} b_{k,2}^{-1} = s$. It suffices to show that $H_{dim}(\lambda_2, d_{\lambda_3}) < s+\epsilon$ for all $\epsilon > 0$. Let $k_1, k_2, \ldots$ be an increasing sequence of natural numbers such that $\log_{\lambda_3(I_2^{(k_t)})} b_{k_t,2}^{-1} < s+\epsilon$ for all t. Consider $\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} O(I_2^{(k_i)})$. It has positive λ_2 measure by Lemma 28. The naive covering shows that $H^{s+\epsilon}(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} O(I_2^{(k_i)})) = 0$. That is fix $\delta > 0$ and choose n such that $\lambda_3(I_2^{(k_2)}) < \delta$. We bound $H_\delta^{s+\epsilon}(\bigcap_{n=1}^{\infty} \bigcup_{i=n}^{\infty} O(I_2^{(k_i)}))$ by covering each $O(I_2^{(k_i)})$ by $b_{k_i,2}$ images of $I_2^{(k_i)}$. By the fact that $\log_{\lambda_3(I_2^{(k_i)})} b_{k_i,2}^{-1} < s+\epsilon$ for all i it follows that $\sum_{i=1}^{\infty} b_{k_i,2}(\lambda_3(I_2^{(k_i)}))^{s+2\epsilon} < \infty$ and therefore the proposition follows. $\square$

Lemma 29. $H_{dim}(\lambda_2, d_{\lambda_3}) \geq \liminf_{k\to\infty} \log_{\lambda_3(I_2^{(k)})}(\lambda_2(I_2^{(k)}))$.

Proof. By Lemma 23 we have that

$$H_{dim}(\lambda_2, d_{\lambda_3}) \geq \min_{1\leq i\leq 4} \liminf_{k\to\infty} \log_{\lambda_3(I_i^{(k)})}(\lambda_2(I_i^{(k)})).$$

Consider

$$\log_{\frac{\lambda_3(I_i^{(k)})}{\lambda_3(I^{(k)})}\lambda_3(I^{(k)})} \frac{\lambda_2(I_i^{(k)})}{\lambda_2(I^{(k)})}\lambda_2(I^{(k)}).$$

To determine the i that attains the minimum it suffices to consider $\log_{\frac{\lambda_3(I_i^{(k)})}{\lambda_3(I^{(k)})}} \frac{\lambda_2(I_i^{(k)})}{\lambda_2(I^{(k)})}$. For all large k the smallest of these is $\log_{\frac{\lambda_3(I_2^{(k)})}{\lambda_3(I^{(k)})}} \frac{\lambda_2(I_2^{(k)})}{\lambda_2(I^{(k)})} < \log_{\frac{2m_{k+1}}{n_{k+1}n_{k+2}}} \frac{1}{4}$ (see Section 2.3.1). □

I think this is also $\log_{\frac{m_{k+1}}{n_{k+1}n_{k+2}b_{k,2}}} \frac{1}{b_{k,2}}$. I need to have

Proposition 7. $H_{dim}(\lambda_3, d_{\lambda_2}) \leq \liminf_{k\to\infty} \log_{\lambda_2(I_3^{(k)})} b_{k,3}^{-1}$.

The proof is similar to Proposition 6.

Lemma 30. $H_{dim}(\lambda_3, d_{\lambda_2}) \geq \liminf_{k\to\infty} \log_{\lambda_2(I_3^{(k)})}(\lambda_3(I_3^{(k)}))$.

Proof. By Lemma 23 we have that

$$H_{dim}(\lambda_3, d_{\lambda_2}) \geq \min_{1\leq i\leq 4} \liminf_{k\to\infty} \log_{\lambda_2(I_i^{(k)})}(\lambda_3(I_i^{(k)})).$$

Consider

$$\log_{\frac{\lambda_2(I_i^{(k)})}{\lambda_2(I^{(k)})}\lambda_2(I^{(k)})} \frac{\lambda_3(I_i^{(k)})}{\lambda_3(I^{(k)})}\lambda_3(I^{(k)}).$$

To determine the i that attains the minimum it suffices to consider $\log_{\frac{\lambda_2(I_i^{(k)})}{\lambda_2(I^{(k)})}}(\frac{\lambda_3(I_i^{(k)})}{\lambda_3(I^{(k)})})$. The smallest of these is $\log_{\frac{\lambda_2(I_3^{(k)})}{\lambda_2(I^{(k)})}}(\frac{\lambda_3(I_3^{(k)})}{\lambda_3(I^{(k)})}) < \log_{\frac{n_{k+1}}{2m_{k+1}}}(1-\frac{3}{n_{k+1}})$ (see Section 2.3.1). □

2.4.3 Proofs of Theorems

Proof of Theorem 8. Choosing $m_k = n_k^k$ implies that $H_{dim}(\lambda_3, d_{\lambda_2}) = 0$. Likewise, choosing $n_{k+1} = m_k^k$ implies that $H_{dim}(\lambda_2, d_{\lambda_3}) = 0$. Choosing $m_k = 4n_k$ implies that $H_{dim}(\lambda_3, d_{\lambda_2}) = 1$. Lastly, choosing $n_{k+1} = 4m_k$ implies that $H_{dim}(\lambda_2, d_{\lambda_3}) = 1$. By suitable choices of m_k and n_k any of the four possibilities in Theorem 8 can be accomplished. $\square$

Proof of Theorem 7(b). $H_{dim}(\lambda_3, d_{\lambda_2})$ can take any value in $[0,1]$. Pick $\alpha \in [0,1]$. Choose m_k, n_k so that $n_k > b_{k-1,2}^k$ and $m_k = \lfloor n_k^{\frac{1}{\alpha}} \rfloor$. $\frac{1}{\alpha} < \log_{\frac{1}{b_{k,2}}} \frac{1}{b_{k,3}} < \frac{1}{\alpha + \frac{2}{k}}$.

$\square$

Proof of Theorem 7(a). $H_{dim}(\lambda_2, d_{\lambda_3})$ can take any value in $[0,1]$. Pick $\alpha \in [0,1]$. Notice that $H_{dim}(\lambda_2, d_{\lambda_3}) = \liminf \frac{\log(b_{k,2})}{\log(\frac{m_{k+1}}{n_{k+1} n_{k+2} b_{k,3}})}$. Choose $m_{k+1} > (n_{k+1} b_{k,3})^k$ and $n_{k+2} = \lfloor m_{k+1}^{\frac{1}{\alpha}} \rfloor$. $\square$

2.4.4 Large sets of generic points

The result of this section is Theorem 9 that the λ_3 generic points can be the complement of a set of Hausdorff dimension 0. Theorem 9 holds in particular when $m_k = 3n_k$ and $n_{k+1} = b_{k,2}^k$.

Definition 9. *Let* $t_k(x) = \min\{n \geq 0 : T^n(x) \in O(I_1^{(k)})\}$.

Proposition 8. *If* $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} O(I_3^{(k)})$ *and* $\lim_{k \to \infty} \frac{b_{k,1}}{t_k(x)} = 0$ *then* x *is* λ_3 *generic.*

Proof. In the proof of Theorem 7 [15], Keane shows that $\prod_{k=1}^{\infty} \bar{A}_{m_k,n_k} e_3$ and $\prod_{k=1}^{\infty} \bar{A}_{m_k,n_k} e_4$ converge to λ_3. Therefore under the conditions of the hypothesis x is generic for λ_3. To see this, consider $b_{k-1,3} < s < b_{k,3}$. x travels through $O(I_3^{(k-1)})$ a times ($a = \frac{t_k}{b_{k-1,3}} - 1$) then through $O(I_4^{(k-1)})$ then through $O(I_1^{(k-1)})$ then it lands back in $O(I_3^{(k-1)})$). By our assumption on t_k eventually the landing in $O(I_3^{(k-1)})$ always dominates, so x is λ_3-generic. $\square$

Proposition 9. *Under appropriate assumptions, the set of points in* $\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} O(I_3^{(k)})$ *not satisfying the hypothesis of Proposition 8 is a set of Hausdorff dimension 0.*

Proof. $I_3^{(k)}$ travel up to n_{k+1} times through $O(I_3^{(k)})$ before traveling through $O(I_4^{(k)})$ and then $O(I_1^{(k)})$. Therefore, the proportion of each level of $O(I_3^{(k)})$ that have $\frac{b_{k,1}}{t_k} < \epsilon$ is $\frac{\epsilon}{b_{k,1}} n_{k+1} b_{k,3}$. There are $b_{k,3}$ such pieces. Therefore if n_{k+1} is chosen so that $(\frac{b_{k,1}}{n_{k+1}})^{\frac{1}{k}} b_{k,3} < \frac{1}{k}$ then the set of $x \in \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} O(I_3^{(k)})$ such that $\limsup_{k \to \infty} \frac{b_{k,1}}{t_k(x)} > 0$ has Hausdorff dimension 0. □

Proof of Theorem 9. By Lemmas 8, 10 and 12 and the independence of the choice of n_{k+1} of the previous n_i and m_i (and therefore $b_{i,j}$ for $i \leq k, j \in \{1,2,3,4\}$) that we may also have $O(I_1^{(k)}) \cup O(I_2^{(k)}) \cup O(I_4^{(k)})$ have Hausdorff dimension 0 by choosing n_{k+1} large enough (or n_{k+2} large enough relative to m_{k+1} for $O(I_2^{(k)})$. The theorem follows with the previous proposition. □

2.5 Exotic shrinking target properties for some Keane type examples

2.5.1 Typical points with respect to one ergodic measure that approximate typical points with respect to the other ergodic measure poorly

First we show that Keane type IETs can have orbits that take a long time to become ϵ dense.

Theorem 10. *There exists T, a minimal 4-IET with 2 ergodic measures, λ_2, λ_3 such that for $\lambda_2 \times \lambda_3$- a.e. pair (x, y)* $\liminf_{n\to\infty} n|T^n(x) - y| = \infty$.

Compare it with Chebyshev's Theorem ([17, Theorem 24]):

Theorem 11. *(Chebyshev) For an arbitrary irrational number α and real number β the inequality $|n\alpha - m - \beta| < \frac{3}{n}$ has an infinite number of integer solutions (n, m).*

In the language of Theorem 10 Chebyshev's Theorem states that if T_α is an irrational rotation, $\liminf_{n\to\infty} n|T_\alpha^n(x) - y| \leq 3$ for any y.

We place the following conditions on m_k and n_k.

1. $(n_k)^3 < m_k$
2. $(b_{k-1,2})^2 < m_k < (b_{k-1,2})^5$
3. $(b_{k,2})^2 2^{2k} m_k < n_{k+1}$.

These conditions provide the following immediate consequences:

1. $b_{k,2} \geq b_{k,j}$ for any j (Lemma 25).
2. $(b_{k-1,2})^3 < b_{k+1,2} < 4(b_{k-1,2})^6$ (direct computation with condition 2).
3. $\lambda_3(O(I_2^k)) < \frac{2}{(b_{k,2})^2}$ (by Lemma 15)
4. $\sum_{k=1}^{\infty} \frac{n_{k+1} b_{k,3}}{b_{k+1,2}}$ converges ($b_{k+1,2} > m_{k+1} b_{k,2}$ and $b_{k,2} > b_{k,3}$).
5. $\sum_{k=1}^{\infty} \frac{n_k}{m_k}$ converges (condition 1).

Picking m_k and n_k as above we choose T to be the IET with permutation (4213) and lengths $(\prod_{k=1}^{\infty} \bar{A}_{m_k,n_k}) e_3$.

Lemma 31. *λ_2 a.e. point is in* $\bigcup_{j=1}^{\infty}\bigcap_{k=j}^{\infty}\bigcap_{r=1}^{b_{k,2}} T^{-r}(O(I_2^{(k)})$

Proof. By condition 1 and Lemmas 15, 18 and 19 it follows that $\sum_{k=1}^{\infty}\frac{\lambda_2(I^{(k)}\backslash I_2^{(k)})}{\lambda_2(I^{(k)})}$ converges. It then follows from the fact that $b_{k,2} \geq b_{k,i}$ (Lemma 24) that $\sum_{k=1}^{\infty}\lambda_2([0,1)\backslash O(I_2^{(k)}))$ converges. The lemma follows from the Borel-Cantelli Theorem with the additional observation that $\frac{1}{m_{k+1}}$ of the measure of $O(I_2^{(k)})$ leaves in the first $b_{k,2}$ steps. □

Definition 10. *Let* $A_{x,r,M,N} = \{y : |T^n(x) - y| < \frac{r}{n}$ *for some* $N < n \leq M\}$

This example relies on showing that $\lambda_3(A_{x,r,M,N})$ is small (see Lemma 33) for a λ_2 large set of x. The following definition provides us with a class of x such that we can control $\lambda_3(A_{x,r,M,N})$ as seen by Lemma 33. This class is also λ_2 large as seen by Lemma 12.

Definition 11. *x is called* k-good *if:*

1. $T^n(x) \in O(I_2^{(k)})$ *for all* $0 \leq n \leq b_{k,2}$.

2. $T^n(x) \in O(I_2^{(k-1)})$ *for all* $0 \leq n \leq (n_k b_{k-1,3})^2$

Lemma 31 shows that λ_2 almost every point satisfies condition (1) for k-good for all $k > N$. The following lemma shows that condition (2) is also satisfied eventually.

Lemma 32. *For λ_2 a.e. x there exists N such that x is k-good for all large enough k.*

Proof. The basic reason λ_2 a.e. x is eventually k-good for all large enough k is that the images of $O(I_2^{(k+1)})$ not in $O(I_2^{(k)})$ are consecutive). This means we need to avoid

$n_{k+1}b_{k,3} + b_{k,4} + (n_k b_{k,3})^2$ image of $O(I_2^{(k+1)})$. Because

$$\frac{n_{k+1}b_{k,3} + b_{k,4} + (n_{k+1}b_{k,3})^2}{b_{k+1,2}} < \frac{(n_{k+1}+1)b_{k,2}}{m_{k+1}b_{k,2}}$$

is a convergent sum, the Borel Cantelli Theorem implies λ_2 almost every x is k-good for all big enough k. (The left hand side is the proportion of the images of $I_2^{(k+1)}$ which are not good.) $\square$

The next lemma shows that if x is $k+1$ good then $A_{x,r,b_{k,2},b_{k+1,2}}$ is small in terms of Lebesgue measure.

Lemma 33. *If x is $k+1$-good for all $k > N$ then $\lambda_3(A_{x,r,b_{k,2},b_{k+1,2}})$ forms a convergent sum.*

Proof. This proof will be carried out by estimating the measure $A_{x,r,b_{k,2},b_{k+1,2}}$ gains when x lands in $O(I_2^{(k)})$ and when it doesn't. Since x is $k+1$-good, the Lebesgue measure $A_{x,r,b_{k,2},b_{k+1,2}}$ gains by not landing in $O(I_2^{(k)})$ is less than

$$\frac{2}{(n_{k+1}b_{k,3})^2}(n_{k+1}b_{k,3} + b_{k,4}) \leq \frac{2(n_{k+1}b_{k,3} + 4b_{k-1,2})}{(n_{k+1}b_{k,3})^2} \leq \frac{4}{n_{k+1}b_{k,3}}.$$

When x lands in $O(I_2^{(k)})$ it either lands in one of the $b_{k-1,2}$ components of which are images of $I_2^{(k-1)}$ (this is $O(I_2^{(k-1)})$) or it doesn't. On each pass through of the orbit, it lands $m_k b_{k-1,2}$ in one of the first $b_{k-1,2}$ images of $I_2^{(k-1)}$, and $n_k b_{k-1,3} + b_{k-1,4}$ times it doesn't. We will estimate $A_{x,r,b_{k,2},b_{k+1,2}}$ by dividing up the orbit into these pieces. When x lands in $O(I_2^{(k-1)})$ the measure of the points its landings place in $A_{x,r,b_{k,2},b_{k+1,2}}$ is at most $\lambda_3(O(I_2^{(k-1)})) + \frac{2r}{b_{k,2}}b_{k-1,2}$. (There are $b_{k-1,2}$ connected components of $O(I_2^{(k-1)})$.)

Otherwise we approximate the measure by :

$$\frac{2}{b_{k,2}} \sum_{i=1}^{b_{k+1,2}/b_{k,2}} \frac{n_k b_{k-1,3} + b_{k-1,4}}{i} \leq \frac{n_k b_{k-1,3} + b_{k-1,4}}{b_{k,2}} \ln(b_{k+1,2}) \leq \frac{n_k b_{k-1,3} + b_{k-1,4}}{b_{k,2}} 7 \ln(b_{k,2}).$$

The left hand side is given by estimating the measure gained by hits in $O(I_2^{(k)}) \backslash O(I_2^{(k-1)}$ $(n_k b_{k-1,3} + b_{k-1,4}$ hits each of which contributes at most $\frac{2}{i b_{k,2}}$ on the i^{th} pass and summing over each pass through $O(I_2^{(k)})$). The first inequality is given by the fact that $b_{k+1,2} > \frac{b_{k+1,2}}{b_{k,2}}$. The final inequality is given by consequence 2. Collecting all of the measure, if x is $k+1$-good then

$$\lambda_3(A_{x,r,b_{k,2},b_{k+1,2}}) \leq \frac{4}{n_{k+1} b_{k,3}} + \frac{2}{(b_{k,2})^2} + \frac{2r}{b_{k,2}} b_{k-1,2} + \frac{n_k b_{k-1,3} + b_{k-1,4}}{b_{k,2}} 7 \ln(b_{k,2})$$

which forms a convergent series due to the at least exponential growth of $b_{k,i}$. □

Proof of Theorem 10. λ_2 a.e. x is eventually $k+1$-good. By Borel-Cantelli for each of these x, Lebesgue a.e. y has $\lim\limits_{n \to \infty} n|T^n(x) - y| = \infty$. The set of all (x, y) such that $\lim\limits_{n \to \infty} n|T^n(x) - y| = \infty$ is measurable, and so has $\lambda_2 \times \lambda_3$ measure 1 (by Fubini's Theorem). □

Remark 7. One can modify conditions 1-3 to achieve $\liminf\limits_{n \to \infty} n^\alpha |T^n x - y| = \infty$ for $0 < \alpha < 1$.

Remark 8. Following [22, Section 1], one can renormalize the IET by choosing the IET S_p, with length vector $\begin{pmatrix} p\lambda_3(I_1) + (1-p)\lambda_2(I_1) \\ p\lambda_3(I_2) + (1-p)\lambda_2(I_2) \\ p\lambda_3(I_3) + (1-p)\lambda_2(I_3) \\ p\lambda_3(I_4) + (1-p)\lambda_2(I_4)) \end{pmatrix}$ and permutation 4213. S_p has the same symbolic dynamics and obeys the same Keane type induction procedure as T (with the same matrices). As a result S has two ergodic measures μ_{S_p}, λ_{S_p} such that $\mu_{S_p}(I_j^{(k)})$ for S_p is the same as $\lambda_2(I_j^{(k)})$ for T and $\lambda_{S_p}(I_j^{(k)})$ for S_p is the same as $\lambda_3(I_j^{(k)})$. Moreover, if $0 < p < 1$ then μ_{S_p} and λ_S are both absolutely continuous and supported on disjoint sets of Lebesgue measure $1-p$ and p respectively. If $p = 1$ the IET is T, if $p = 0$ then μ_{S_0} is Lebesgue measure and λ_{S_0} is singular. As a result the Lebesgue measure of $I_j^{(k)}$ for $S_{.5}$ is at least $.5\max\{\lambda_3(I_j^{(k)}), \lambda_2(I_j^{(k)})\}$ for T. From this it follows that $\liminf_{n\to\infty} n|S_{.5}^n(x) - y| = \infty$ on a set of (x, y) with measure .25 (corresponding to x being chosen from a set of μ_S full measure and y being chosen from a set of λ_S). (See [22, Section 1] for more on renormalizing.)

2.5.2 Two ergodic measures that approximate each other differently

Theorem 12. *There exists a minimal 4-IET with two ergodic measures, λ_2 and λ_3 such that for any $\epsilon > 0$ we have $\liminf n^{1-\epsilon} d(T^n x, y) = 0$ for $\lambda_2 \times \lambda_3$ almost every (x, y) and $\liminf n^{\frac{1}{2}+\epsilon} d(T^n x, y) = \infty$ for $\lambda_3 \times \lambda_2$ almost every (x, y).*

This will be proved in two parts (the $\lambda_3 \times \lambda_2$ statement and the $\lambda_2 \times \lambda_3$ statement) under the assumption that $m_k = k^2 n_k$ and $n_{k+1} = b_{k,2}^2$ and λ_2 is Lebesgue measure (that is $d = d_{\lambda_2}$).

Remark 9. The $\frac{1}{2}$ can be replaced by any $c \in [0, 1)$ with straightforward modification.

Proposition 10. *For any $\epsilon > 0$ and $\lambda_3 \times \lambda_2$ almost every point (x, y) we have $\liminf_{n\to\infty} n^{\frac{1}{2}+\epsilon} |T^n x - y| = \infty$.*

Lemma 34. *$\lambda_3(\bigcup_{r=1}^{\infty}\bigcap_{k=r}^{\infty}(\bigcap_{t=1}^{\lfloor\frac{1}{k^2}n_{k+1}b_{k,3}\rfloor}T^{-t}(O(I_3^{(k)})))) = 1.$*

Proof. $\lambda_3(\bigcap_{t=1}^{\lfloor\frac{1}{k^2}n_{k+1}b_{k,3}\rfloor}T^{-t}(O(I_3^{(k)}))) \geq (1-\frac{1}{k^2})\lambda_3(O(I_3^{(k)}))$. Also

$$\lambda_3(O(I_2^{(1)}) \cup O(I_2^{(2)}) \cup O(I_2^{(3)})) \leq \frac{1}{k^2})(1 - \frac{1}{b_{k,3}}(\frac{2b_{k,2}m_{k+1}}{n_{k+1}n_{k+2}} + \frac{2b_{k,4}}{n_{k+1}} + \frac{1}{n_{k+1}})).$$

By our assumptions $\frac{b_{k,i}}{b_{k,3}} < 4^k$. Therefore the proposition follows by the Borel-Cantelli Theorem with the observation that $\sum_{k=1}^{\infty}\frac{1}{k^2}+\frac{4^k}{n_{k+1}}$ converge. (The convergence of $\sum_{k=1}^{\infty}\frac{4^k}{n_{k+1}}$ follows by our assumption on the growth of n_k.) ☐

Lemma 35. *If $x \in \bigcap_{t=1}^{\lfloor\frac{1}{k^2}n_{k+1}b_{k,3}\rfloor}T^{-t}(O(I_3^{(k)}))$ then $\lambda_2(\bigcup_{t=\lfloor\frac{1}{(k-1)^2}n_kb_{k-1,3}\rfloor}^{\lfloor\frac{1}{k^2}n_{k+1}b_{k,3}\rfloor}B(T^tx,(\frac{c}{t^\alpha}))) \leq \lambda_2(O(I_3^{(k)})) + (b_{k-1,1}+b_{k-1,4}+b_{k-1,3})2c(\lfloor\frac{1}{k^2}n_kb_{k-1,3}\rfloor)^{-\alpha}$.*

Proof. By our assumption x lies in $O(I_3^{(k)})$ for time described, therefore the measure of the set is at most the measure of a $(\lfloor\frac{1}{k^2}n_{k+1}b_{k,3}\rfloor)^{-0.5}$ neighborhood of $O(I_3^{(k)})$. The lemma follows from observing that $I_3^{(k)}$ travels n_k times through $O(I_3^{(k-1)})$ once through $O(I_1^{(k-1)})$ and once through $O(I_4^{(k-1)})$. One then groups the levels $O(I_3^{(k)})$ by the $O(I_i^{(k-1)})$ that they lie in. ☐

Proof of Proposition 10. By our assumption on n_k, m_k it follows that $\sum_{k=1}^{\infty}\lambda_2(O(I_3^{(k)}))+2c(b_{k-1,1}+b_{k-1,4}+b_{k-1,3})2(\lfloor\frac{1}{k^2}n_kb_{k+1,3}\rfloor)^{\frac{1}{2}+\epsilon}$ converges. By the Borel-Cantelli Theorem it follows that for all such x we have $\lambda_2(\bigcap_{n=1}^{\infty}\bigcup_{i=n}^{\infty}B(T^ix,\frac{c}{i^\alpha}) = 0$. By Fubini's Theorem it follows that for all such x we have $\limsup_{n\to\infty} n^{\frac{1}{2}+\epsilon}|T^nx-y| = \infty$. By Lemma 34 the proposition follows. ☐

Proposition 11. *For any $\epsilon > 0$ and $\lambda_2 \times \lambda_3$ almost every point (x,y) we have $\liminf_{n\to\infty} n^{1-\epsilon}|T^nx-y| = 0$.*

Lemma 36. *If $T^t x \in O(I_2^{(k)})$ for $t < b_{k,3}$ then $\lambda_3(\bigcup_{i=1}^{b_{k,2}} B(T^i x, \frac{1}{b_{k,2}}^{1-\epsilon})) > .5$ for large enough k.*

Proof. By the assumption of the hypothesis $\{x, Tx, ..., T^{b_{k,2}}x\}$ are at least $\lambda_2(I^{(k)})$ dense in $O(I_3^{(k-1)})$. (The condition of the hypothesis ensures that $\{x, Tx, ..., T^{b_{k,2}}x\}$ has n_k hits in each level of $O(I_3^{(k-1)})$ by examining $T|_{I^{(k-1)}}(x)$, $O(I_2^{(k)})$ is $\lambda_2(I^{(k)})$ dense in $O(I_3^{(k-1)})$.) $\lambda_3(I^{(k)}) < \frac{1}{b_{k,4}}$. By our choice of m_k and n_k, $b_{k,4} > b_{k,2}^{1-\epsilon}$ for all large enough k. □

Lemma 37. *The set of points satisfying the hypothesis of the above Lemma has λ_2 measure at least $\frac{1}{8}$.*

Proof. This follows from the fact that $\lambda_2(O(I_2^{(k)})) > \frac{1}{4}$ (Lemma 28) and $1 - \frac{b_{k,3}}{b_{k,2}} > \frac{1}{2}$ of these points satisfy the hypothesis of the lemma. □

Proof of Proposition 11. The proof follows from Fubini and ergodicity. □

Remark 10. One can assign distance by nontrivial linear combinations of λ_2 and λ_3 Theorem 12 holds in these cases as well. (For $d = d_{\lambda_3}$ the argument fails. See Question 2). The reason why is that $\lambda_3(I^{(k)}) < \frac{1}{b_{k,4}}$ and by our choice of m_k and n_k, $b_{k,4} > b_{k,2}^{1-\epsilon}$ for all large enough k. The estimates in Proposition 10 are proportional in this case. Perhaps the most interesting version of Theorem 12 is when $d(x, y) = \frac{1}{2}(\lambda_2([x, y]) + \lambda_3([x, y])$ because the metric gives equal weights to the two measures, but typical points still approach each other differently.

2.6 Conclusion

Michael Keane devised a wonderful class of examples that provide very specified behavior for IETs. The power of this example is its self similar structure and the fact that much of its behavior can be captured by matrix multiplication. This helps to make the behavior easy to understand and demonstrate. Additionally, because the A_{n_k,m_k} arise as matrices of Rauzy-Veech induction these examples can provide intuition to how Rauzy-Veech induction works. As this chapter suggests Keane's class of examples can construct IETs that also have fairly different behavior for their ergodic components.

Question 1. For a given permutation what is the optimal order of topological mixing?

Question 2. Let $S : [0,1) \to [0,1)$ be a minimal IET and have ergodic measures μ_1 and μ_2. Let $d_1(x,y) = \mu_1([\min\{x,y\}, \max\{x,y\}])$. Is it possible for $\liminf n^{.99} d_1(S^n x, y) = \infty$ for $\mu_1 \times \mu_2$ almost every (x,y)? What about with the additional stipulation that $\liminf n^{1-\epsilon} d_1(S^n x, y) = \infty$ for $\mu_2 \times \mu_1$ almost every (x,y) and $\liminf n d_1(S^n x, y) = \infty$ for $\mu_1 \times \mu_1$ almost every (x,y)?

I suspect the answer to these questions is no. I suspect the answer is yes if we remove the requirement that S is an IET.

Question 3. What is the Hausdorff dimension of the set of minimal IETs with an ergodic measure having Hausdorff dimension less than 1? (We consider these IETs as points in $\Delta_3 = \{(l_1, l_2, l_3, l_4) : l_i > 0, \sum l_i = 1\}$.)

For the case of 4-IETs the answer is expected to be 2.5.

Question 4. Can any residual set support an ergodic measure for a minimal IET?

Question 5. Can $H_{dim}(\lambda_2, d_{\lambda_3}), H_{dim}(\lambda_3, d_{\lambda_2})$ take any value in $[0,1]^2$?

I suspect the answer is yes.

Question 6. Can one construct a smooth realization of a Keane type IET?

Chapter 3

IETs are usually different

3.1 Statement of results

Definition 12. *Two measure preserving systems* (T, X, μ) *and* (S, Y, ν) *are called disjoint (or have trivial joinings) if* $\mu \times \nu$ *is the only invariant measure of* $T \times S \colon X \times Y \to X \times Y$ *by* $(T \times S)(x, y) = (Tx, Sy)$ *with projections* μ *and* ν*.*

The main result of this chapter is:

Theorem 13. *Let* $T \colon X \to X$ *be* μ *ergodic.* (T, X, μ) *is disjoint from almost every IET.*

We remark that this is a strong way of saying that 2 IETs are different. T and STS^{-1} have many nontrivial joinings, for instance (x, Sx) supports an invariant measure with both projections λ (as does (x, ST^7x)). This implies that if the transformations are conjugate then they won't be disjoint. Similarly if $\sigma T = f\sigma$ and $\rho S = f\rho$ where $f : X \to X$ preserves μ which satisfies $\mu(A) = \lambda(\sigma^{-1}(A)) = \lambda(\rho^{-1}(A))$ for all measurable A, then $\underset{x \in X}{\cup} \sigma^{-1}(x) \times \rho^{-1}(x)$ supports an invariant measure of $T \times S$ with projections λ. This is said disjointness implies no common factors (a stronger condition than not being conjugate). Note that if S and T are IETs STS^{-1} is an IET so every IET shares non-trivial joinings with uncountably many other IETs. We prove Theorem 13 by the following criterion [12, Theorem 2.1], see also [21, Lemma 1] and [11, Theorem 6.28].

Theorem 14. *(Hahn and Parry) If T_1 and T_2 are ergodic transformations of (X_1, B_1, m_1) and (X_2, B_2, m_2) respectively, and if U_{T_1} and U_{T_2} are spectrally singular modulo constants then T_1 and T_2 are disjoint.*

A key result was established in proving Theorem 13 which requires a definition.

Definition 13. *Let $T : [0,1) \to [0,1)$ be a μ measure preserving transformation. $n_1, n_2, \ldots$ is called a* rigidity sequence *for T if $\lim_{i \to \infty} \int_{[0,1)} |T^{n_i}x - x| d\mu = 0$.*

Theorem 15. *Let A be a sequence of natural numbers with density 1. Almost every IET has a rigidity sequence contained in A.*

This a strengthening of an earlier Theorem of Veech ([24, Part I, Theorem 1.4]) which proved that almost every IET has a rigidity sequence (choose N_i corresponding to ϵ_i where $\lim_{i \to \infty} \epsilon_i = 0$).

Theorem 16. *(Veech) For almost every interval exchange transformation T, with irreducible permutation, and given $\epsilon > 0$ there are $N \in \mathbb{N}$, and an interval $J \subset [0,1)$ such that:*

1. *$J \cap T^n(J) = \emptyset$ for $0 < n < N$.*

2. T is continuous on $T^n(J)$ for $0 \leq n < N$.

3. $\lambda(\bigcup_{n=1}^{N} T^n(J)) > 1 - \epsilon$.

4. $\lambda(T^N(J) \cap J) > (1 - \epsilon)\lambda(J)$.

3.2 Proof of Theorem 15

Theorem 15 follows from the following proposition.

Proposition 12. *Let $A \subset \mathbb{N}$ be a sequence of density 1. For every $\epsilon > 0$ and almost every IET S, there exists $n_\epsilon \in A$ such that $\int_0^1 |S^{n_\epsilon}(x) - x| d\lambda < \epsilon$.*

This proposition implies Theorem 15 because the countable intersection of sets of full measure has full measure.

Motivated by this proposition if $\int_0^1 |T^n(x) - x| d\lambda < \epsilon$ we say *n is an ϵ rigidity time for T.*

Throughout this section we will assume that the IETs are in a fixed Rauzy class $\mathfrak{R}$, which contains d-IETs with some irreducible permutations. Let r denote the number of different permutations IETs in $\mathfrak{R}$ may have. Let $\mathbf{m}_{\mathfrak{R}}$ denote Lebesgue measure on $\mathfrak{R}$ (the disjoint union of r simplices in $\mathbb{R}^d$).

Proposition 12 will be proved by showing that there is a particular reason for ϵ rigidity (called acceptable ϵ rigidity) that occurs often in many $P_i := [2^i, 2^{i+1}]$ (Proposition 15) but rarely occurs for any fixed n (Lemma 44). For every IET S satisfying the Keane condition, and every i there exists some n such that $|C_{max}(M(S,n))| \in P_i$. In general there can be more than one such n.

For each of the permutations $\pi_1, ..., \pi_r$ that an IET in $\mathfrak{R}$ may have, fix a finite sequence of Rauzy-Veech induction steps ω_i, which gives a positive matrix. That is each letter of ω_i will be one of the two types of Rauzy-Veech steps (a or b) and the product of the sequence of the associated matrices starting from permutation π_i provides a positive Rauzy-Veech matrix. Let $M(\omega_i)$ denote this matrix. Let $|\omega_i|$ denote the number of steps in ω_i. Let $p_i = \mathbf{m}_{\mathfrak{R}}(M(\omega_i)_\Delta)$.

Definition 14. *We say a pair $(M, C_{max}(M))$ is* acceptable *if $M = M(T,n)$, $R^{n-|\omega_i|}(T)$ has permutation π_i and $M(R^{n-|\omega_i|}(T), |\omega_i|) = M(\omega_i)$.*

If $(M, C_{max}(M))$ is an acceptable pair then M is called an acceptable matrix.

Informally, if $M = M(T,n)$ then the pair $(M, C_{max}(M))$ is acceptable if the last steps in Rauzy-Veech induction for an IET with length vector in M_Δ agrees with some ω_i and the permutation of $R^{n-|\omega_i|}(T)$ is π_i.

Remark 11. In the remainder of this section we will use the fact that if $R^n(T_{L,\pi})$ has permutation π_i then for any IET S with length vector in $(M(T_{L,\pi}, n)M(\omega_i))_\Delta$ and permutation π the pair $(M(S, n+|\omega_i|), C_{max}(M(S, n+|\omega_i|)))$ is acceptable.

Lemma 38. *There exists ν such that any acceptable matrix is ν balanced.*

Proof. Let M_1 be a positive matrix. Observe that if M_2 is a matrix with nonnegative entries then M_2M_1 is at worst $\max\limits_{i,j,k} \frac{M_1[i,j]}{M_1[i,k]}$ balanced. Since there are only finitely many $M(\omega_i)$ and they are all positive the lemma follows. In particular, we can chose $\nu = \max\limits_{t} \max\limits_{i,j,k} \frac{M(\omega_t)[i,j]}{M(\omega_t)[i,k]}$. □

Lemma 39. *For any d-column C, $|\{M\colon (M,C) \text{ is an acceptable pair }\}| \leq r^2$.*

That is, any d-column can appear in at most r^2 different acceptable pairs.

Proof. Assume C belongs to two different acceptable pairs $(M(T,n),C)$, and $(M(S,n'),C)$ where both T and S have permutation π_i. The acceptable sequence of steps ω_j for T and $\omega_{j'}$ for S are different. This is because if $\omega_j = \omega_{j'}$ then the last $|\omega_j|$ steps of Rauzy-Veech induction are the same. However, since $C = C_{max}(M(T,n)) = C_{max}(M(S,n'))$ and S and T have the same starting permutation, Lemma 3 implies that all but the last $|\omega_j|$ steps of Rauzy-Veech induction are the same and therefore $M(T,n) = M(S,n')$. Lemma 39 follows because there are r choices of ω_j and r choices of starting permutation. □

Proposition 13. *For $\mathbf{m}_{\mathfrak{R}}$-almost every IET S, the set of natural numbers*

$$\{i : \textit{ for some } n,\ |C_{max}(M(S,n))| \in P_i \textit{ and } (M(S,n), C_{max}(M(S,n))) \textit{ is an acceptable pair } \} \tag{3.1}$$

has positive lower density.

The following two lemmas are used in the proof of Proposition 13.

Lemma 40. *For $\mathbf{m}_{\mathfrak{R}}$-almost every IET S, and all sufficiently large ν_0, the set of natural numbers*

$$G(S) := \{i : \textit{for some } n,\ |C_{max}(M(S,n))| \in P_i \textit{ and } M(S,n) \textit{ is } \nu_0 \textit{ balanced}\}$$

has positive lower density.

Remark 12. It is not claimed that a positive lower density of the Rauzy-Veech induction matrices are balanced.

To prove this we use [16, Corollary 1.7].

Proposition 14. *(Kerckhoff) At any stage of the [Rauzy-Veech] expansion of S the columns of M(S) will become ν_0 distributed [i.e. ν_0 balanced] with probability ρ before the maximum norm of the columns increases by a factor of K^d. ν_0 and ρ are constants depending only on K and d.*

Remark 13. In [16] the term "ν_0 distributed" has the same meaning in as "ν_0 balanced" has here.

Proof of Lemma 40. Consider the independent μ distributed random variables $F_1, F_2, \ldots$ where μ takes value 1 with probability ρ and 0 with probability $1-\rho$. By the strong law of large numbers, for $\mu^{\mathbb{N}}$-almost every t we have $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} F_i(t)}{n} = \rho$. By the previous proposition, given $G(S) \cap [0, N]$ the conditional probability that $N + i \in G(S)$ for some $0 < i \leq \lceil d \log_2(K) \rceil$ is at least ρ. Thus for any natural numbers $n_1, n_2, \ldots, n_k$

$$\mathbf{m}_{\mathfrak{R}}(\{S : [n_i \lceil d \log_2(K) \rceil, (n_i + 1)] \lceil d \log_2(K) \rceil] \cap G(S) \neq \emptyset \, \forall i \leq k\})$$
$$\geq \mu^{\mathbb{N}}(\{t : F_{n_i}(t) = 1 \, \forall i \leq k\}). \quad (3.2)$$

This implies that from $\mathbf{m}_{\mathfrak{R}}$-almost every S, $G(S)$ has lower density at least $\frac{\rho}{\lceil d \log_2(K) \rceil}$.

□

Lemma 41. *(Kerckhoff) If M is ν_0 balanced and $W \subset \Delta_d$ is a measurable set, then*

$$\frac{\mathbf{m}_{\mathfrak{R}}(W)}{\mathbf{m}_{\mathfrak{R}}(\Delta_d)} < \frac{\mathbf{m}_{\mathfrak{R}}(MW)}{\mathbf{m}_{\mathfrak{R}}(M\Delta_d)} (\nu_0)^{-d}.$$

This is [16, Corollary 1.2]. See [22, Section 5] for details.

Proof of Proposition 13. By Lemma 41 if $M(T,n)$ is ν_0 balanced and $R^n(T)$ has permutation π_i then the $\frac{\mathbf{m}_{\mathfrak{R}}(M(T,n)M(\omega_i)\Delta_d)}{\mathbf{m}_{\mathfrak{R}}(M(T,n)\Delta_d)} \geq \nu_0^{-d}p_i$. In words: given that $M(T,n)$ is ν balanced and that $R^n(T)$ has permutation π_i, the conditional probability that $(M(T,n+|\omega_i|), C_{max}(M(T,n+|\omega_i|)))$ is an acceptable pair is at least $\nu_0^{-d}p_i$. Considering each π_i, the proposition follows analogously to Lemma 40. $\square$

Definition 15. *Let S be an IET. If $(M(S,n), C_{max}(M(S,n)))$ is acceptable and $m = |C_{max}(M(S,n))|$ is an ϵ rigidity time for S then m is called an* acceptable ϵ rigidity time for S.

Proposition 15. *For every $\epsilon > 0$, $\mathbf{m}_{\mathfrak{R}}$-almost every IET S, the set of natural numbers*

$$G_\epsilon(S) := \{i : P_i \text{ contains an acceptable } \epsilon \text{ rigidity time for } S\}$$

has positive lower density.

Proof. Consider an IET $S_{L,\pi} = S$ such that $(M(S,n), C_k(M(S,n)))$ is an acceptable pair (in particular, $C_k(M(S,n)) = C_{max}(M(S,n)))$. For ease of notation let $M' = M(S,n)$. Let $W_{k,\epsilon} = \{(l_1, l_2, ..., l_d) : l_i > 0\,\forall i,\, l_k > 1 - \frac{\epsilon}{3}, \sum l_i = 1\}$. If $L \in W_{k,\epsilon}$ then $T_{\frac{M'L}{|M'L|},\pi}$ has an ϵ rigidity time of $|C_k(M')|$. This is the reason for rigidity used to prove Theorem 1.3 and 1.4 [24, pages 1337-1338]. If M' is acceptable then Lemma 38 states that M' is ν balanced. It then follows by Lemma 41 that the proportion of M'_Δ which has $|C_k(M')|$ as an ϵ rigidity time is at least $\nu^{-d}\mathbf{m}_{\mathfrak{R}}(W_{k,\epsilon})$. Thus if $i_1 < i_2 < ... \in G(S)$ then the probability that $i_f \in G_\epsilon(S)$ is at least $\nu^{-d}\mathbf{m}_{\mathfrak{R}}(W_{k,\epsilon})$ regardless of which $i_k \in G_\epsilon(S)$ for $k < f$. The proposition follows analogously to Lemma 40. $\square$

Before proving Proposition 12 we provide the following lemmas.

Lemma 42. *There exists $b \in \mathbb{R}$ such that for any $n \in \mathbb{N}$,*

$$|\{M\colon M \text{ is acceptable and } |C_{max}(M)| = n\}| \leq bn^{d-1}.$$

Remark 14. The constant b depends only on our Rauzy class $\mathfrak{R}$. It is not claimed that for every $n \in \mathbb{N}$ there exists an acceptable matrix M with $|C_{max}(M)| = n$.

Proof. By Lemma 39 each column C can be $|C_{max}(M)|$ for at most r^2 different acceptable matrices M. By induction on d, $O(n^{d-1})$ different d-columns with non-negative integer entries have the sum of their entries equal to n. $\square$

Lemma 43. *(Veech) If M is a matrix given by Rauzy-Veech induction, then*

$$\mathbf{m}_{\mathfrak{R}}(M_\Delta) = c_{\mathfrak{R}} \prod_{i=1}^{d} |C_i(M)|^{-1}.$$

This is [22, equation 5.5]. An immediate consequence of it is that any ν balanced Rauzy-Veech matrix M has $\mathbf{m}_{\mathfrak{R}}(M_\Delta) \leq c_{\mathfrak{R}} \nu^{d-1} |C_{max}(M)|^{-d}$. The previous two lemmas give the following result.

Lemma 44. *The $\mathbf{m}_{\mathfrak{R}}$-measure of IETs that have acceptable pairs with the same $|C_{max}|$ is at most $O(|C_{max}|^{-1})$.*

Proof of Proposition 12. By Lemma 44 and the fact that A has density 1,

$$\lim_{i \to \infty} \mathbf{m}_{\mathfrak{R}}(\{T\colon \exists n \text{ with } M(T,n) \text{ acceptable and } |C_{max}(M(T,n))| \in P_i \backslash A\}) = 0.$$

Therefore, Proposition 15 implies that for any $\epsilon > 0$, almost every IET has an acceptable ϵ rigidity time in A. In fact, almost every IET has an ϵ rigidity time in $P_i \cap A$ for a positive upper density set of i. $\square$

Remark 15. To be explicit, Proposition 15 shows that for any sequence A with density 1, and any $\epsilon > 0$, for almost every IET the integer N in Veech's Theorem 16 can be chosen from A.

3.3 Consequences of Section 2

In this section we glean some consequences of the proofs in the previous section. One of these (Corollary 5) follows from [1, Theorem A] and is used in the proof of Theorem 13. It is proven independently of [1, Theorem A] in this section.

Corollary 2. *Let A be a sequence of natural numbers with density 1. A residual set of IETs has a rigidity sequence contained in A.*

Proof. Take the interior of the set $W_{k,\epsilon}$ considered in the proof of Proposition 15. In this way one obtains that the set of IETs with an ϵ rigidity time in A contains an open set of full measure (therefore dense). Intersecting over ϵ shows that a residual set of IETs has a rigidity sequence in any sequence of density 1. $\square$

The number of columns that can appear in Rauzy-Veech matrices grows at least like $u_{\mathfrak{R}}R^d$ (where the constant $u_{\mathfrak{R}}$ depends on $\mathfrak{R}$). Briefly, in order to collect a positive measure of IETs having admissible matrices M, with $|C_{max}(M)| \in P_k$, Lemma 43 implies that there needs of be more than $u_{\mathfrak{R}}(2^k)^d$ admissible matrices with $|C_{max}| \in P_k$. This provides a partial answer to the first question in [24, Part II, Questions 10.7] which considers asymptotics for the growth of so called primitive IETs.

The next result provides a slight improvement of Theorem 15 and uses the following definition.

Definition 16. *Let S be an IET. We say m is an* expected ϵ rigidity time *for S if there exists an n such that that the following two conditions are met.*

1. *$(M(S,n), C_{max}(M(S,n)))$ is acceptable and $m = |C_{max}(M(S,n))|$.*

2. *$C_{max}(M(S,n)) = C_k(M(S,n))$ and $R^n(S)$ lies in the set $W_{k,\epsilon}$ defined in the proof of Proposition* 15.

Every expected ϵ rigidity time is an acceptable ϵ rigidity time.

Corollary 3. *For every $\epsilon > 0$ and Rauzy class $\mathfrak{R}$ there is a constant $a_{\mathfrak{R}}(\epsilon) < 1$ such that any sequence of natural numbers A with density $a_{\mathfrak{R}}(\epsilon)$ has a rigidity sequence for all but a $\mathbf{m}_{\mathfrak{R}}$-measure ϵ set of IETs.*

Proof. First note that the set of IETs having a rigidity sequence contained in A is measurable. Let $e_{\mathfrak{R}}(\epsilon)$ denote $\mathbf{m}_{\mathfrak{R}}(W_{k,\epsilon})$. Let $M = M(T_{L,\pi}, n)$ be an acceptable matrix. By the bound on distortion in Lemma 41,

the conditional probability of an IET in M_Δ and permutation π having an expected ϵ rigidity time $|C_{max}(M)|$ is proportional to $e_{\mathfrak{R}}(\epsilon)$. This uses Lemma 38 which states that if M is an acceptable matrix then M is ν balanced. An analogous argument to Lemma 40 shows that there exists $c_1 > 0$ such that the set

$$\{i \colon \exists\, m \in P_i \text{ which is an expected } \epsilon \text{ rigidity time for } T\}$$

has lower density at least $c_1 e_{\mathfrak{R}}(\epsilon)$ for almost every T.

Lemma 44 establishes that there exists $c_2 > 0$ such that

$$\mathbf{m}_{\mathfrak{R}}(\{T\colon n \text{ is an expected } \epsilon \text{ rigidity time for } T\}) < c_2 e_{\mathfrak{R}}(\epsilon) n^{-1}$$

for all n. Thus, for any $\epsilon > 0$ a set of natural numbers with density $1 - \delta$ contains an ϵ expected rigidity time for all but a set of IETs of measure $2\delta\frac{c_2}{c_1}$ and the corollary follows. □

Remark 16. Recall that ν depends on the choices of ω_i that define acceptable pairs. The constants c_1 and c_2 depend on ν.

Corollary 3 gives two further corollaries.

Corollary 4. *Almost every IET has a rigidity sequence which is shared by a $\mathbf{m}_{\mathfrak{R}'}$-measure zero set of IETs for all $\mathfrak{R}'$ simultaneously.*

Proof. It suffices to show that for any $\delta > 0$ and Rauzy class $\mathfrak{R}'$ all but a set of $\mathbf{m}_{\mathfrak{R}}$-measure δ IETs have a rigidity sequence that is not a rigidity sequence for $\mathbf{m}_{\mathfrak{R}'}$-almost every IET. Given $\epsilon_1, \epsilon_2 > 0$ and a Rauzy class, $\mathfrak{R}'$ consider the set

$$A_{\mathfrak{R}'}(\epsilon_1, \epsilon_2) = \{n\colon n \text{ is an } \epsilon_1 \text{ rigidity time for a set of IETs of } \mathbf{m}_{\mathfrak{R}'}\text{-measure at least } \epsilon_2\}. \tag{3.3}$$

If $\epsilon_2 > 0$ and $\mathfrak{R}'$ are fixed then the density of this set goes to zero with ϵ_1. To see this, observe that if n_1 and n_2 are ϵ rigidity times for T then $n_1 - n_2$ is a 2ϵ rigidity time for T. It follows that if $\epsilon < \frac{1}{2}\min\limits_{0<n\leq M} \int |T^n x - x| d\lambda$ then $\{r+1, r+2, ..., r+M\}$ can contain at most one ϵ rigidity time for T. Choose $\epsilon_1(k)$ so that the (upper) density of $A_{\mathfrak{R}'}(\epsilon_1(k), \frac{1}{k})$ is less than $1 - a_{\mathfrak{R}}(\delta)$. By Corollary 3, all but a $\mathbf{m}_{\mathfrak{R}}$-measure δ set of IETs

have a rigidity sequence in the complement of $A_{\mathfrak{R}'}(\epsilon_1(k), \frac{1}{k})$ (which can be shared by a set of IETs with $\mathbf{m}_{\mathfrak{R}'}$-measure at most $\frac{1}{k}$). Consider the countable intersection over k of these nested sets of $\mathbf{m}_{\mathfrak{R}}$-measure $1-\delta$. For each IET T in this set let n_i be a $\frac{1}{i}$ rigidity time for T lying in the complement of $A_{\mathfrak{R}'}(\epsilon_1(i), \frac{1}{i})$. Therefore, $n_1, n_2, \ldots$ is a rigidity sequence for T that is not a rigidity sequence for $\mathbf{m}_{\mathfrak{R}'}$-almost every IET. □

Corollary 5. *For every $\alpha \notin \mathbb{Z}$, almost every IET does not have $e^{2\pi i\alpha}$ as an eigenvalue.*

We will prove this corollary independently of [1, Theorem A], from which it immediately follows.

Theorem 17. *(Avila and Forni) If π is an irreducible permutation that is not a rotation, then almost every IET with permutation π is weak mixing.*

The proof is split into the case of rational α and the case of irrational α. If T has $e^{2\pi i\alpha}$ as an eigenvalue for some rational $\alpha \notin \mathbb{Z}$ then it is not totally ergodic. This is not the case for almost every IET [24, Part I, Theorem 1.7].

Theorem 18. *(Veech) Almost every IET is totally ergodic.*

It suffices to consider irrational α and show that for any $\delta > 0$ and $\mathfrak{R}$, the set of IETs having $e^{2\pi i\alpha}$ as an eigenvalue has $\mathbf{m}_{\mathfrak{R}}$-outer measure less than δ. If $e^{2\pi i\alpha}$ is an eigenvalue for T then rotation by α is a factor of T. However, rigidity sequences of a transformation are also rigidity sequences for the factor. For every irrational α and $e > 0$ there is a sequence of density $1-e$ that contains no rigidity sequence for rotation by α. To see this, observe that if n_1 and n_2 are ϵ rigidity times for T then $n_1 - n_2$ is a 2ϵ rigidity time for T. It follows that if $\epsilon < \frac{1}{2} \min_{0<n\leq M} \int |T^n x - x| d\lambda$

then $\{k+1, k+2, ..., k+M\}$ can contain at most one ϵ rigidity time for T. Choose $e < 1 - a_{\mathfrak{R}}(\delta)$ and pick a sequence of density $1-e$ containing no rigidity sequence for rotation by α. The IETs having a rigidity sequence in this sequence have $\mathbf{m}_{\mathfrak{R}}$-measure at least $1-\delta$ and Corollary 5 follows.

Remark 17. Every sequence of density 1 contains a rigidity sequence for rotation by α.

3.4 The spectral argument

Given a μ measure preserving dynamical system T, let U_T be the unitary operator on $L^2(\mu)$ given by $U_T(f) = f \circ T$. Let $L^2_0(\mu)$ denote the set of L^2 functions with integral zero. If $f \in L^2$ let $\sigma_{f,T}$ be the spectral measure for f and U_T, that is the unique measure on $\mathbb{T}$ such that

$$\int_{\mathbb{T}} z^n d\sigma_{f,T} = < f, U_T^n f > \text{ for all } n.$$

Fix $T\colon [0,1) \to [0,1)$, a μ ergodic transformation. We will show that for any S in a full measure set of IETs $\sigma_{f,T}$ is singular with respect to $\sigma_{g,S}$ for any $f, g \in L^2_0$. By Theorem 14, this establishes Theorem 13. Let H_{pp} be the closure of the subspace of L^2_0 spanned by non-constant eigenfunctions of U_T (where the spectral measures are atomic) and H_c be its orthogonal complement (where the spectral measures are continuous).

Lemma 45. *If $f \in H_{pp}$ then for almost every IET S, $\sigma_{f,T}$ is singular with respect to $\sigma_{g,S}$ for any $g \in L^2_0$.*

Proof. Let $f \in H_{pp}$. $\sigma_{f,T}$ is an atomic measure supported on the $e^{2\pi i \alpha}$ that are eigenvalues of U_T. If $\sigma_{f,T}$ is nonsingular with respect to $\sigma_{g,S}$ then U_T and U_S share an eigenvalue (other than the simple eigenvalue 1 corresponding to constant functions). The set of eigenvalues of U_T is countable because H_{pp} has a countable basis of eigenfunctions. The lemma follows from the fact that the set of IETs having a particular eigenvalue has measure zero (Corollary 5) and the countable union of measure zero sets has measure zero. □

Lemma 46. *If $f \in H_c$ then for almost every IET S, $\sigma_{f,T}$ is singular with respect to $\sigma_{g,S}$ for any $g \in L_0^2$.*

To prove this lemma we use Wiener's Lemma (see e.g. [5, Lemma 4.10.2]) and its immediate corollary.

Lemma 47. *For a finite measure μ on $\mathbb{T}$ set $\hat{\mu}(k) = \int_{\mathbb{T}} z^k d\mu(z)$. $\lim_{n\to\infty} n^{-1} \sum_{k=0}^{n-1} |\hat{\mu}(k)|^2 = 0$ iff μ is continuous.*

Corollary 6. *For a finite continuous measure μ on $\mathbb{T}$ there exists a density 1 sequence A, such that $\lim_{k\in A} \hat{\mu}(k) = 0$.*

Proof of Lemma 46. Decompose H_c into the direct sum of mutually orthogonal H_{f_i}, where each H_{f_i} is the cyclic subspace generated by f_i under U_T (and $U_T^{-1} = U_T^*$). By Corollary 6, for each i there exists a density 1 set of natural numbers B_i such that $\lim_{n\in B_i} \int_{\mathbb{T}} z^n d\sigma_{f_i,T} = 0$. Choose N_j increasing such that for each j we have $\inf_{n>N_j} \frac{|B_i \cap [0,n]|}{n} > 1 - 2^{-j}$. Let $A_i := \bigcup_{j=1}^{\infty} ([N_j, N_{j+1}] \bigcap_{k=-j}^{j} B_i + k)$. By construction, $(A_i - k)\backslash B_i$ is a finite set for any $k \in \mathbb{Z}$. Therefore, $\lim_{n\in A_i} \int_{\mathbb{T}} z^{n+k} d\sigma_{f_i,T} = 0$ for any $k \in \mathbb{Z}$. Thus, for any

$h \in H_{f_i}$ it follows that $\lim\limits_{n \in A_i} \int_{\mathbb{T}} z^{k+n} d\sigma_{h,T} = 0$ for any k. This follows from the fact that $\sigma_{h,T} \ll \sigma_{f_i,T}$, the span of z^k is dense in L_2 and $|\int_{\mathbb{T}} z^r d\mu| \leq \mu(\mathbb{T})$. Since there are only a countable number of H_{f_i}, there exists a density 1 sequence A such that for any i and $h \in H_{f_i}$ we have that $\lim\limits_{n \in A_i} \int_{\mathbb{T}} z^{k+n} d\sigma_{h,T} = 0$ for any k. (To see this pick M_j such that $\inf\limits_{M_j < n} \frac{|A_i \cap [0,n]|}{n} > 1 - 2^{-j}$ for any $i \leq j$. Let $A = \bigcup\limits_{j=1}^{\infty} [M_j, M_{j+1}] \cap A_1 \cap \ldots \cap A_j$.) It follows that for any $h \in H_c$, $\lim\limits_{n \in A_i} \int_{\mathbb{T}} z^{k+n} d\sigma_{h,T} = 0$ for any k. This uses the fact that if g_1 and g_2 lie in orthogonal cyclic subspaces then $\sigma_{g_1+g_2,T}$ is $\sigma_{g_1,T} + \sigma_{g_2,T}$.

Let S be any IET with a rigidity sequence contained in A, which almost every IET has by Theorem 15. We now show that $\sigma_{g,S}$ is singular with respect to $\sigma_{f,T}$ for any $f \in H_c$ and $g \in L_0^2(\lambda)$. Notice that since $n_1, n_2, \ldots$ is a rigidity sequence for S, $\lim\limits_{i \to \infty} \int_{\mathbb{T}} |z^{n_i} - 1|^2 d\sigma_{g,S} = 0$. Because L^2 convergence implies convergence almost everywhere along a subsequence, it follows that there exists $i_1, i_2, \ldots$ such that $\sigma_{g,S}(\{z : \lim\limits_{j \to \infty} z^{n_{i_j}} \to 1\}) = \sigma_{g,S}(\mathbb{T})$. However, $\lim\limits_{i \to \infty} \int_C z^{n_i} \sigma_{f,T} \to 0$ for any measurable $C \subset \mathbb{T}$. This is because $\int_C z^{n_i} \sigma_{f,T} = \int_{\mathbb{T}} z^{n_i} \chi_C(z) \sigma_{f,T}$ and χ_C can be approximated in $L_2(\sigma_{f,T})$ by polynomials. The construction of A in the previous paragraph shows that $\lim\limits_{n \in A} \int_{\mathbb{T}} p(z) z^n d\sigma_{f,T} = 0$ for any polynomial p. It follows that $\sigma_{g,S}$ is singular with respect to $\sigma_{f,T}$ for any $f \in H_c$ and $g \in L_0^2(\lambda)$. $\square$

Theorem 13 follows by considering the intersection of the two full measure sets of IETs and the fact that if $g_1 \in H_{pp}$ and $g_2 \in H_c$ then $\sigma_{g_1+g_2,T}$ is $\sigma_{g_1,T} + \sigma_{g_2,T}$.

Remark 18. Motivating the proof is: If μ and ν are probability measures on S^1 such that $z^{n_i} \to f$ weakly in $L_2(\mu)$ and $z^{n_i} \to g$ weakly in $L_2(\nu)$ and $f(z) \neq g(z)$ for all z then ν and μ are singular.

Remark 19. A possibly more checkable result follows from the above proof. Assume A is a mixing sequence for T (that is, $\lim_{n \in A} \mu(B \cap T^n(B')) = \mu(B)\mu(B')$ for all measurable B and B') then any S having a rigidity sequence in A is disjoint from T. Note that weak mixing transformations have mixing sequences of density 1.

Remark 20. Given a family of transformations $\mathcal{F}$ with a measure η on $\mathcal{F}$ any μ ergodic $T\colon X \to X$ will be disjoint for η-almost every $S \in \mathcal{F}$ if:

1. Any sequence of density 1 is a rigidity sequence for η-almost every $S \in \mathcal{F}$.

2. $\eta(\{S \in \mathcal{F} : \alpha \text{ is an eigenvalue for } S\}) = 0$ for any $\alpha \neq 1$.

Additionally, the previous section shows that a slightly stronger version of condition 1 and η-almost sure total ergodicity implies condition 2. Condition 1 on its own does not imply condition 2 (let $\mathcal{F}$ be the set of 1 element, rotation by α_0).

3.5 Concluding remarks

First, a consequence of Theorem 13 that is interesting in its own right.

Corollary 7. *For almost every pair of IETs T, S the transformation $T \times S$ is uniquely ergodic with respect to Lebesgue measure on $[0,1)^2$.*

Proof. This follows from the fact that almost every IET is uniquely ergodic ([18] and [23]) and the following Lemma. ☐

Lemma 48. *If T and S are uniquely ergodic with respect to μ and ν respectively then any preserved measure of $T \times S$ has projections μ and ν.*

Proof. Consider η, a preserved measure of $T \times S$.

$$\eta(A \times Y) = \eta(T^{-n} \times S^{-n}(A \times Y)) = \eta(T^{-n}(A) \times Y).$$

Therefore, $\mu_1(A) := \eta(A \times Y)$ is preserved by T and so it is μ. For the other projection the proof is similar. $\square$

More is true in fact, for Leb $\times ... \times$ Leb almost every n-tuple of IETs $(S_1, ..., S_n)$, $S_1 \times ... \times S_n$ is uniquely ergodic and S_1 is disjoint from $S_2 \times S_3 \times ... \times S_n$.

Corollary 7 has an application. Consider $T \times S$. In our context, unique ergodicity implies minimality, which implies uniformly bounded return time to a fixed rectangle. Therefore, if we choose a rectangle $V \subset [0,1) \times [0,1)$ then the induced map of $T \times S$ on V is almost surely (in (T, S) or even S if T is uniquely ergodic) an exchange of a finite number of rectangles.

Theorem 13 also strengthens Corollary 5 because transformations are not disjoint from their factors.

Corollary 8. *No transformation is a factor of a positive measure set of IETs.*

Bibliography

[1] Avila, A; Forni, G: Weak mixing for interval exchange transformations and translation flows, *Ann. of Math. (2)* **165** (2007), 637–664.

[2] Berthe, V; Chekhova, N; Ferenczi, S: Covering numbers: arithmetics and dynamics for rotations and interval exchanges. *J. D'Analyse Math.* **79** 1 (1999) 1-31

[3] Boshernitzan, M: Quantitative recurrence results. *Invent. Math.* **113** (1993), no. 3, 617–631.

[4] Boshernitzan, M; Chaika, J: Diophantine properties of IETs and general systems: Quantitative proximality and connectivity. Preprint.

[5] Brin, M; Stuck, G: Introduction to dynamical systems. Cambridge University Press, Cambridge, 2002.

[6] Bufetov, A: Decay of correlations for the Rauzy-Veech-Zorich induction map on the space of interval exchange transformations and the central limit theorem for the Teichmller flow on the moduli space of abelian differentials. *J. Amer. Math. Soc.* **19** (2006), no. 3, 579–623

[7] Cornfeld I. P; Fomin S. V; Sinai, Ya. G: Ergodic theory. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 245. Springer-Verlag, New York, 1982.

[8] del Junco, Andrs: Disjointness of measure-preserving transformations, minimal self-joinings and category. I (College Park, Md., 1979–80), pp. 81–89, Progr. Math., 10, Birkhuser, Boston, Mass., 1981.

[9] El Abdalaoui, El Houcein: La singularité mutuelle presque sûre du spectre des transformations d'Ornstein. *Israel J. Math.* **112** (1999), 135–155

[10] Fraczek, Krzysztof; Lemanczyk, Mariusz: On disjointness properties of some smooth flows. *Fund. Math.* **185** (2005), no. 2, 117–142

[11] Glasner, E: Ergodic theory via joinings. Mathematical Surveys and Monographs, 101. American Mathematical Society, Providence, RI, 2003.

[12] Hahn, F; Parry, W: Some Characteristic Properties of dynamical systems with quasi discrete spectrum. *Math. Systems Theory* **2** (1968) 179-190.

[13] Katok, A: Interval exchange transformations and some special flows are not mixing, *Israel J. Math* **35** (4) (1980) 301-310

[14] Keane, M: Interval exchange transformations, *Math. Z.* **141**, 25-31 (1975)

[15] Keane, M: Non-ergodic interval exchange transformations, *Israel J. Math.* **26** (2) (1977) 188-196.

[16] Kerckhoff, S. P: Simplicial systems for interval exchange maps and measured foliation. *Ergod. Th. & Dynam. Sys.* **5** (1985), 257-271

[17] Khinchin, A: Continued Fractions, Dover.

[18] Masur, H: Interval exchange transformations and measured foliations. *Ann. of Math.* (2) **115** (1982) 168-200.

[19] Mattila, P: Geometry of sets and measures in Euclidean spaces. Cambridge University Press 1995.

[20] Sinai, Y; Ulcigrai, C: Renewal-type limit theorem for the Gauss map and continued fractions. *Ergod Th. & Dynam. Sys.* **28** (2008), no. 2, 643–655.

[21] Thouvenot, J.-P: Some properties and applications of joinings in ergodic theory. Ergodic theory and its connections with harmonic analysis (Alexandria, 1993), 207–235, London Math. Soc. Lecture Note Ser., 205, Cambridge Univ. Press, Cambridge, 1995.

[22] Veech, W: Interval exchange transformations. *J. D'Analyse Math.* **33** (1978) 222-272.

[23] Veech, W: Gauss measures for transformations on the space of interval exchange maps. *Ann. of Math. (2)* 115 (1982) 201-242.

[24] Veech W: The Metric Theory of interval exchange transformations, *American Journal of Mathematics* **106** (6) (1984) 1331-1422.

[25] Viana, M: Ergodic theory of interval exchange maps. *Rev. Mat. Complut.* **19** (2006), no. 1, 7100.