RICE UNIVERSITY

Complex Flow and Transport Phenomena in Porous Media

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A PROPOSAL SUBMITTED
IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE

Doctor of Philosophy

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April, 2010

ABSTRACT

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This thesis analyzes partial differential equations related to the coupled surface and subsurface flows and develops efficient high order discontinuous Galerkin (DG) methods to solve them numerically. Specifically, the coupling of the Navier-Stokes and the Darcy's equations, which is encountered in the environmental problem of groundwater contamination through lakes and rivers, is considered. Predicting accurately the transport of contaminants by this coupled flow is of great importance for the remediation strategies.

The first part of this thesis analyzes a weak formulation of the time-dependent Navier-Stokes equation coupled with the Darcy's equation through the Beavers-Joseph-Saffman condition. The analysis changes depending on whether the inertial forces are included in the interface conditions or not. The inclusion of the inertial forces (Model I) remedies the difficulty in the analysis caused by the nonlinear convection term; however, it does not reflect the physical interactions on the interface correctly. Hence, I also analyze the weak problem by omitting these forces (Model II) which complicates the analysis and necessitates an extra small data condition. For Model I, a fully discrete scheme based on the DG method and the Crank-Nicolson method is introduced. The convergence of the scheme is proven with optimal error estimates.

The second part couples the surface flow and a convection-diffusion type transport with miscible displacement in the subsurface. Initially, I consider the coupled stationary Stokes and Darcy's equations for the flow and establish the existence of a weak solution. Next, imposing additional assumptions on the data, I extend the result to the nonlinear case where the surface flow is given by the Navier-Stokes equation. The analysis also applies to the particular case where the flow is loosely coupled to the transport, that is, the velocity field obtained from the flow is an input for the transport equation. The flow is discretized by combinations of the continuous finite element method and the DG method whereas the discretization of the transport is done by a combined DG and backward Euler methods. The scheme yields optimal error estimates and its robustness for fractured porous media is shown by a numerical example.

Acknowledgments

I owe my deepest gratitude to my Ph.D. advisor Beatrice Rivière for her continuous support and guidance. I greatly admire and respect her for her confidence, kindness, and professional prowess. She is my role model both as a mentor and researcher. Without her enthusiasm, inspiration and sound advice, this thesis wouldn't have been possible.

I am also grateful to the members of my thesis committee, Dan Sorensen, George Hirasaki and Tim Warburton, for their valuable feedback and insightful comments.

I am greatly indebted to Richard Tapia, Chris Lennard, Mike Sussman and my M.Sc. advisor Ali Ülger, who have been great teachers to me in the past, for their support during my job search.

I would like to thank everyone in the CAAM department for their friendship, especially Daria Lawrence for making my transfer to Rice University as smooth as possible.

I wish to thank all my friends who contributed one way or another over these years. Some of them really deserve special thanks. I thank Umut Ermeç for being a good friend for almost 17 years. Meriç Ovacık was always there for me and suffered with me during both my Master's and my Ph.D. studies. I owe Atilla Kılıçarslan, Atila Uçar and Ersin Öksüzoğlu for their continuous morale support. Without them, Houston would be such a boring place. I would like to thank my sister Ayça Çeşmelioğlu for sharing her experience with me and believing in me.

Last but not least, I thank my mother Çiçek Çeşmelioğlu and my father Kadir Çeşmelioğlu for encouraging me to pursue a Ph.D. degree and for their unconditional love. To them I dedicate this thesis.

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Chapter 1

Introduction

The coupling of the Navier-Stokes/Stokes and the Darcy's equations arises in many important engineering problems, an example of which is the contamination of groundwater through lakes and rivers. Everyday more and more contaminants, such as hazardous solids, liquid wastes and toxic wastes, are produced by industries or consumers. These contaminants percolate through lakes, rivers and streams to the groundwater, which is the main source of daily drinking water and irrigation water. It is important to prevent health-threatening situations which may either be caused indirectly by contamination of irrigation water, eventually harming life forms through the food chain, or directly by contamination of drinking water. Development of reliable methods to accurately predict the transport of contaminants for a given time period is extremely important for remediation processes.

The domain of this coupled flow is divided into two subdomains that represent the surface and the subsurface regions. In the surface region, the flow is characterized by the incompressible time-dependent/steady Navier-Stokes/Stokes equations, whereas in the subsurface region, the flow is characterized by the Darcy's equation. For a discussion of the development of these equations, the reader may refer to Darrigol [1]. The coupling of these two different types of flow is accomplished through certain interface conditions. Even though there is no universal agreement on the choice of the right interface conditions, the usual conditions include the Beavers-Joseph-Saffman law [2, 3], the continuity of the normal component of velocity and the balance of forces.

This work also accepts these interface conditions to complete the partial differential equation systems modeling the surface/subsurface flow.

In the coupled surface/subsurface flows, the heterogeneous nature of the reservoir is an important factor determining the properties of the flow. For example, because of the infinite combinations of porous medium structure (arrangement, composition), it is only natural to expect dramatic variations in the permeability (the transmission property of the porous medium) over the region. These variations cause difficulty in the simulations. Discontinuous Galerkin (DG) methods are suitable to overcome this difficulty as the discrete spaces are the discontinuous piecewise polynomial spaces. In spite of being costly for triangular meshes, DG methods are advantageous over other methods in the literature. First, DG methods are ideal for adaptivity since they allow for hanging nodes. This is important to deal with complicated geometries. The continuous finite element methods (FEM) can also handle adaptivity, but they cannot handle meshes with several hanging nodes per edge. Second, with the DG method, it is easy to change the degree of the approximating polynomial to get high order approximations, while this takes much more effort in the case of classical finite element method. Indeed, to change the polynomial degree in a DG code amounts to changing only the routine generating the basis functions, whereas with the finite element code, one has to basically rewrite the code. Another property that the DG methods have, but the FEM methods lack, is the local mass conservation property. In the absence of local mass conservation, the numerical solution of the coupled flow and transport problems in porous media will be unstable [4, p.41]. For these reasons, the discrete schemes I develop are based on the discontinuous Galerkin methods.

The history of the DG methods dates back to 1973 when they were introduced by Reed and Hill for linear hyperbolic type problems to solve transport of neutrons using triangular and quadrilateral elements [5]. These methods were mathematically analyzed later in 1974 by LeSaint and Raviart [6]. The application of these methods to elliptic and parabolic equations was through the introduction of the interior penalty methods of Baker [7]. These methods arose from the observation that inter-element continuity can be imposed weakly instead of being built into the finite element space. Over the years new DG schemes were formulated that use symmetric or non-symmetric bilinear forms, with or without stabilization and penalty terms and written in a mixed or non-mixed form. Cockburn et al. [8] provides a review of the development of the DG methods. The reader can also refer to the recent books by Rivière [4] and Warburton and Hesthaven [9] on the DG methods. Because of the type of equations governing the flow problems, the DG methods considered in this thesis are the interior penalty Galerkin methods that are designed to solve the elliptic and parabolic type of problems. To be specific, symmetric interior penalty Galerkin method (SIPG) introduced by Wheeler [10], Douglas and Dupont [11] and Arnold [12]; non-symmetric interior penalty Galerkin method (NIPG) introduced by Rivière, Wheeler and Girault [13, 14, 15]; incomplete interior penalty Galerkin method (IIPG) introduced by Dawson, Sun and Wheeler [16], Sun and Wheeler [17]; and Oden Babuska Baumann (OBB) method [18] are used. These methods differ either by the sign of the stability term or by the existence of the penalty term. NIPG and SIPG methods have been successfully applied to various flow and transport problems in porous media such as single-phase [13, 19, 20, 21] and two-phase [22, 23, 24, 25, 26, 27, 28] flow problems, linear and reactive transport problems [29, 30] and miscible displacement problems [31, 32, 33, 34].

The first problem of this research, part of which has been studied mathematically in [35, 36], is the coupled time-dependent Navier-Stokes and Darcy's equations. This flow is analyzed in two models depending on the choice of the balance of forces interface condition. In the first model (Model I), the inertial forces are included in the balance of forces, remedying the difficulty in the analysis caused by the nonlinear convection term. However, this condition with the inertial forces does not reflect the physical interactions on the interface correctly [37]. Hence, in the second model (Model II), the balance of forces is considered without the inertial forces, giving a more physical condition. This chapter can be seen as an extension of the steady-state case which has been analyzed by Girault and Rivière [38] and Chidyagwai and Rivière [39]. Girault and Rivière [38] prove the existence of a weak solution under small data condition and its local uniqueness for the steady-state case of Model II. Chidyagwai and Rivière [39] consider non-homogeneous boundary condition for two model problems: one omits the inertial forces as in the paper by Girault and Rivière [38]; the second one includes the inertial forces as in Model I and the existence of a weak solution is proved unconditionally. The weak problem of a similar coupling is analyzed by Badea et al. [40], where an interface problem with Steklov-Poincaré operators is formulated. Removing the nonlinearity from the stationary Navier-Stokes equations leads to the coupling of the Stokes and the Darcy's equations. This problem has been extensively studied in the literature. See, for instance, Layton et al. [41] and Discacciati et al. [42] for the analysis of the weak solution.

Starting from the coupled problem with the mentioned interface conditions, I define a weak solution and prove its existence. The proof is based on a Galerkin technique and uses compactness results in Bochner spaces. In Model II, without the aid of the inertial forces, an extra small data condition is necessary which gives a conditional existence result for the weak solution. Also, under additional assumptions, uniqueness is proved which is only in the local sense for Model II.

For Model I, based on the weak formulation, I propose a fully discrete scheme and prove optimal error estimates in space and a second order error estimate in time. Based on the paper by Çeşmelioğlu and Rivière [36], the discretization is done by discontinuous Galerkin (DG) methods in space and the Crank-Nicolson method in time. The approximation spaces for the fluid velocity and the pressure in the Navier-Stokes region are the discontinuous piecewise polynomials of degree k_1 and $k_1 - 1$. On the other hand, the approximation of the fluid pressure in the Darcy region is done by the discontinuous piecewise polynomials of degree k_2. The reader can refer to [43, 44, 45, 41, 46, 47, 48, 49, 50] for a variety of numerical schemes for the steady-state Stokes/Darcy problem. For the numerical schemes and examples of the steady-state Navier-Stokes/Darcy coupling, one can refer to [40, 38, 39, 51].

To further understand the groundwater contamination by lakes and rivers, I couple the surface/subsurface flow with a convection-diffusion transport equation. The published literature is very sparse on this problem. The mathematical analysis of the miscible displacement problem in subsurface was done in a seminal paper by Alt and Luckhaus [52], and by others such as Marpeau and Saad [53] and Fabrie and Gallouët [54]. My contribution is the analysis of the more general coupling of miscible displacement in porous media with surface flow and transport which to my knowledge is the first analysis of this problem. First, I consider the steady-state case of the Stokes/Darcy flow for the underlying flow problem as presented in [55]. I define the mathematical model and introduce the necessary assumptions on the data. Then I formulate the weak problem. The existence proof is based on a Galerkin approach in time. To define the approximate solution, constant and linear interpolation operators are used as in [52, 53]. Then using compactness results, passing to the limit in the approximate solution gives the existence result for the linear problem. Next, I extend this result to the nonlinear case, that is, I consider the Navier-Stokes/Darcy problem to model the surface/subsurface flow. For this case, only the balance of forces condition excluding the inertial forces is considered. The reason is that the other condition in Model I is relatively easier to prove and gives stronger mathematical results. The proof for this nonlinear problem again is similar to the Stokes case under additional assumptions on the data. This mathematical analysis also applies to the particular case where the flow problem is loosely coupled to the transport problem.

In this loose one-way coupling, the velocity field obtained from the Stokes/Darcy problem becomes an input data for the transport equation. A numerical scheme based on a mixed method for the coupled Stokes/Darcy equations and a local discontinuous Galerkin method [56] for the transport problem has been analyzed for this particular case by Vassilev and Yotov [57]. In this thesis, the numerical analysis and a numerical example from the paper by Çeşmelioğlu et al. [58] are included where the flow problem is approximated by either the DG method or the FEM or by their combination. The transport problem is discretized by a DG method where upwinding, which causes stability without the need for slope limiters, is used for the flux terms in the subsurface [59]. The numerical example aims to show that the methods are robust for fractured porous media.

This thesis is organized as described in the table of contents. Roughly, besides this introduction chapter, there are five more chapters. Chapter 2, titled "Preliminaries", gives necessary notation, definitions and theorems. Chapter 3 studies the time-dependent Navier-Stokes and Darcy coupling problem and the fourth Chapter investigates the Navier-Stokes/Stokes-Darcy-transport problem. Chapter 5 gives conclusions of this thesis and the last chapter discussed possible extensions. The contents of each chapter is described therein.

Chapter 2

Preliminaries

This section provides the well-known definitions, notation, inequalities and theorems as well as the definition of the domain used throughout this thesis. Interested reader should refer to [60, 61, 62, 63, 64, 65, 66, 67] for more details. For any space X, X^2 simply means the product space $X \times X$. The dual space of X is denoted by X' with the duality pairing $\langle \cdot, \cdot \rangle_{X',X}$. The variable $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ denotes the spatial coordinate. We define the gradient of a scalar function $v : \mathbb{R}^2 \to \mathbb{R}$ and the gradient of a vector function $\boldsymbol{v} : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\nabla v = \left(\frac{\partial v}{\partial x_i} \right)_{1 \le i \le 2}, \quad \nabla \boldsymbol{v} = \left(\frac{\partial \boldsymbol{v}}{\partial x_i} \right)_{1 \le i \le 2}.$$

The divergence of a vector function $\boldsymbol{v} = (v_1, v_2)$ is defined by

$$\nabla \cdot \boldsymbol{v} = \sum_{i=1,2} \frac{\partial v_i}{\partial x_i}.$$

Finally, for two vectors $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2), \boldsymbol{u} = (\boldsymbol{u}_1, \boldsymbol{u}_2)$, the dot product is defined to be

$$\boldsymbol{v} \cdot \boldsymbol{u} = \sum_{i=1,2} v_i u_i.$$

Let $\Omega \subset \mathbb{R}^2$. For $1 \le p < \infty$, we define

$$L^p(\Omega) = \{ v : \Omega \to \mathbb{R} : v \text{ is measurable, } \int_\Omega |v|^p d\boldsymbol{x} < \infty \},$$

equiped with the norm

$$\|v\|_{L^p(\Omega)} = \left(\int_\Omega |v|^p d\boldsymbol{x} \right)^{\frac{1}{p}}.$$

The choice $p = \infty$ corresponds to the space of bounded functions defined as

$$L^\infty(\Omega) = \{v : \Omega \to \mathbb{R} : v \text{ is measurable, } ess \sup_{\boldsymbol{x} \in \Omega} |v| < \infty\},$$

equipped with the norm

$$\|v\|_{L^\infty(\Omega)} = ess \sup_{\boldsymbol{x} \in \Omega} |v|.$$

The space $L^p(\Omega)$, $1 \leq p \leq \infty$ is a Banach space with the $\|\cdot\|_{L^p(\Omega)}$ norm. When $p = 2$, the space of square integrable functions $L^2(\Omega)$ is a Hilbert space with the L^2-inner product

$$\begin{aligned}
(v, w)_\Omega &= \int_\Omega vw d\boldsymbol{x}, \text{ for scalar-valued functions } v, w, \\
(\boldsymbol{v}, \boldsymbol{w})_\Omega &= \int_\Omega \boldsymbol{v} \cdot \boldsymbol{w} d\boldsymbol{x}, \text{ for vector-valued functions } \boldsymbol{v}, \boldsymbol{w}, \\
(\boldsymbol{V}, \boldsymbol{W})_\Omega &= \sum_{i,j} \int_\Omega \boldsymbol{V}_{ij} \boldsymbol{W}_{ij} d\boldsymbol{x}, \text{ for matrix-valued functions } \boldsymbol{V}, \boldsymbol{W}.
\end{aligned}$$

Furthermore,

Lemma 1. *Let $1 \leq p \leq \infty$. Then any sequence in $L^p(\Omega)$ that converges with respect to the norm $\|\cdot\|_{L^p(\Omega)}$, has a subsequence that converges pointwise almost everywhere.*

For any continuous function v on $\mathbb{R}^2$, we define its support as

$$supp(v) = \overline{\{\boldsymbol{x} \in \mathbb{R}^2 \,:\, v(\boldsymbol{x}) \neq 0\}}.$$

and denote the space of smooth functions with compact support (or the space of test functions) in Ω by $\mathcal{D}(\Omega)$. The following result is used for the density arguments.

Theorem 2. *For $1 \leq p < \infty$, the space $\mathcal{D}(\Omega)$ is dense in the space $L^p(\Omega)$.*

For a given Banach space B, the Bochner spaces are denoted by $L^k(0, T; B)$, $1 \leq p \leq \infty$, $k \geq 1$. The space $L^k(0, T; B)$ is also a Banach space equipped with the norm $(\int_0^T \|\cdot\|_B^k dt)^{1/k}$ for $1 \le p < \infty$ and $esssup_{t\in(0,T)}\|\cdot\|_B$ for $p = \infty$.

For any integer m, the classical Sobolev space is defined as

$$H^m(\Omega) = \{v \in L^2(\Omega) : \partial^k v \in L^2(\Omega), \quad \forall |k| \le m\},$$

where $\boldsymbol{k} = (k_1, k_2)$, $|\boldsymbol{k}| = k_1 + k_2$, $k_1, k_2 \ge 0$ and $\partial^{\boldsymbol{k}} v = \frac{\partial^{|\boldsymbol{k}|} v}{\partial x_1^{k_1} \partial x_2^{k_2}}$. On the space $H^m(\Omega)$, the seminorm $|\cdot|_{H^m(\Omega)}$ and the norm $\|\cdot\|_{H^m(\Omega)}$ are defined as follows:

$$|v|_{H^m(\Omega)} = \left(\sum_{|\boldsymbol{k}|=m} \int_\Omega |\partial^{\boldsymbol{k}} v|^2 dx\right)^{\frac{1}{2}}, \quad \|v\|_{H^m(\Omega)} = \left(\sum_{0\le j\le m} |v|^2_{H^j(\Omega)} dx\right)^{\frac{1}{2}}.$$

With the inner product $(\cdot,\cdot)_{H^m(\Omega)} = \sum_{0\le|\boldsymbol{k}|\le m}(\partial^{\boldsymbol{k}}\cdot, \partial^{\boldsymbol{k}}\cdot)$, the Sobolev space $H^m(\Omega)$ is a separable Hilbert space . Also note that for $m = 2$, $|v|_{H^1(\Omega)} = \|\nabla v\|_{L^2(\Omega)}$. We also define the Sobolev spaces for fractional indices. The space $H^{m+1/2}(\Omega)$ is the interpolation of the spaces $H^m(\Omega)$ and $H^{m+1}(\Omega)$ which satisfies

$$H^{m+1}(\Omega) \subset H^{m+1/2}(\Omega) \subset H^m(\Omega), \text{ and}$$

$$\forall v \in H^{m+1}(\Omega), \quad \|v\|_{H^{m+1/2}(\Omega)} \le C\|v\|^{\frac{1}{2}}_{H^m(\Omega)}\|v\|^{\frac{1}{2}}_{H^{m+1}(\Omega)},$$

where C is a constant depending on Ω.

To properly define values of Sobolev functions on the boundary, we have the following trace theorem:

Theorem 3. *(Trace theorem) Assume that Ω is bounded with polygonal boundary $\partial\Omega$. Then there exists surjective operators $\gamma_0 : H^r(\Omega) \to H^{r-1/2}(\partial\Omega)$, $r > \frac{1}{2}$ and $\gamma_1 : H^r(\Omega) \to H^{r-3/2}(\partial\Omega)$, $r > \frac{3}{2}$ such that*

$$\forall v \in \mathcal{C}^1(\overline{\Omega}), \quad \gamma_0 v = v|_{\partial\Omega}, \quad \gamma_1 v = \nabla v \cdot \boldsymbol{n}|_{\partial\Omega}$$

where $\boldsymbol{n}$ denotes the outward unit normal of $\partial\Omega$.

For $\Gamma \subset \partial\Omega$, $|\Gamma| \neq 0$, we define

$$H^1_{0,\Gamma}(\Omega) = \{\boldsymbol{v} \in H^1(\Omega) : \boldsymbol{v} = 0 \text{ on } \Gamma\}$$

where $\boldsymbol{v} = 0$ on Γ is interpreted in the sense of the trace of a Sobolev space function and we abuse notation by denoting $\gamma_0 v$ by also v on Γ. When $\Gamma = \partial\Omega$, we denote $H^1_{0,\Gamma}(\Omega) = H^1_0(\Omega)$. The dual space of $L^p(\Omega)$, $1 \leq p < \infty$ is $L^q(\Omega)$ where q is the conjugate of p, that is, $\frac{1}{p} + \frac{1}{q} = 1$. For $1 \leq m < \infty$, we denote by $H^{-m}(\Omega)$, the dual space of $H^m_0(\Omega)$ together with the norm

$$\|f\|_{H^{-m}(\Omega)} = \sup_{v \in H^m_0(\Omega), v \neq 0} \frac{\langle v, f\rangle_{H^{-m}(\Omega), H^m_0(\Omega)}}{\|v\|_{H^m(\Omega)}}.$$

Let X and Y be two Hilbert spaces such that $X \subset Y$ is a continuous embedding. Let $f : [0,T] \to Y$ be an integrable function. We define the extension $\tilde{f}$ of f by

$$\tilde{f}(t) = \begin{cases} f(t), & t \in [0,T] \\ 0, & \text{otherwise} \end{cases}$$

We define the Fourier transform of $\tilde{f}$ by

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} \tilde{f}(t) e^{-2\pi i t\tau} dt, \quad \forall \tau \in \mathbb{R}.$$

Further, for any $\gamma > 0$, we define

$$H^\gamma(0,T;X,Y) = \{f \in L^2(0,T;X) \,:\, |\tau|^\gamma \hat{f} \in L^2(\mathbb{R};Y)\}$$

equipped with the norm

$$\|f\|_{H^\gamma(0,T;X,Y)} = \left(\|f\|^2_{L^2(0,T;X)} + \||\tau|^\gamma \hat{f}\|^2_{L^2(\mathbb{R};Y)}\right)^{\frac{1}{2}}.$$

The space H^γ will be useful in proving strong convergence results via compactness.

We proceed by stating important results of Calculus, Functional Analysis, Real and Complex Analysis, Ordinary Differential Equations and Sobolev Space theory that we frequently use. For the weak formulation of our partial differential equations system, we need formulas to relate the vector identities such as the divergence, the gradient and the Laplacian.

Theorem 4. *(Generalized Green's formula) Assume that Ω is a Lipschitz domain. Let $u \in H^1(\Omega), v \in H^2(\Omega)$. Then*

$$\int_\Omega u\nabla \cdot \boldsymbol{F}\nabla v d\boldsymbol{x} = -\int_\Omega \boldsymbol{F}\nabla v \cdot \nabla u d\boldsymbol{x} + \int_{\partial\Omega} \boldsymbol{F}\nabla v \cdot \boldsymbol{n} u d\sigma$$

where $\boldsymbol{n}$ is the outward unit normal vector of $\partial\Omega$ and $\boldsymbol{F}$ is a matrix-valued function. In particular when $\boldsymbol{F} = \boldsymbol{I}$,

$$\int_\Omega u\Delta v d\boldsymbol{x} = -\int_\Omega \nabla v \cdot \nabla u d\boldsymbol{x} + \int_{\partial\Omega} \nabla v \cdot \boldsymbol{n} u d\sigma$$

To apply the fixed point theorems to a function, we first need to show that this function is really well-defined. Next two theorems are useful to prove the well-definition of these functions. The first theorem supplies us a way to represent uniquely the bounded linear functionals on Hilbert spaces in terms of the inner product.

Theorem 5. *(Riesz representation theorem) Any continuous linear functional L on a Hilbert space $\mathcal{H}$ with the inner product $(\cdot,\cdot)_\mathcal{H}$ has a unique representation, i.e.,*

$$\exists!\, u \in \mathcal{H} : \quad L(v) = (u, v)_\mathcal{H}, \forall v \in \mathcal{H}.$$

Furthermore, the mapping $L \mapsto u$ is an isomorphism of $\mathcal{H}' \to \mathcal{H}$.

The following theorem is an extension of the Riesz representation theorem which is generally used to prove existence and uniqueness.

Theorem 6. *(Lax-Milgram theorem) Let $\mathcal{H}$ be a Hilbert space and $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear mapping such that there exists $\alpha, \beta > 0$ satisfying*

- $|B(u,v)| \leq \alpha \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}, \quad \forall u, v \in \mathcal{H},$ *(continuity)*

- $|B(u,u)| \geq \beta \|u\|^2_{\mathcal{H}}, \quad \forall u \in \mathcal{H}.$ *(coercivity)*

If $L : \mathcal{H} \to \mathbb{R}$ is a bounded linear functional on $\mathcal{H}$, then there exists a unique $u \in \mathcal{H}$ such that

$$B(u,v) = L(v), \quad \forall v \in \mathcal{H}.$$

Moreover, the mapping $L \mapsto u$ is an isomorphism from $\mathcal{H}'$ to $\mathcal{H}$.

The following theorem is a special case of the compactness theorem of Rellich and Kondrachov that is enough for our purposes.

Theorem 7. *(Rellich-Kondrachov theorem) Let $\Omega \subset \mathbb{R}^2$ be an open, bounded Lipschitz domain.*

- *If $0 \leq l < 1$, then $H^1(\Omega)$ is compactly embedded in $H^l(\Omega)$.*

- *$H^1(\Omega)$ is compactly embedded in $L^1(\Omega)$.*

The following theorems are used to prove existence results for nonlinear partial differential equations. The first one is used for the finite dimensional case and the next one is used for the infinite dimensional case.

Theorem 8. *(Corollary to Brouwer's fixed point theorem) Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let $\mathcal{F} : \mathcal{H} \to \mathcal{H}$ be a continuous mapping such that there exists $C > 0$ satisfying*

$$\forall v \in \mathcal{H}, \quad \|v\|_{\mathcal{H}} = C, \quad (\mathcal{F}(v), v)_{\mathcal{H}} \geq 0.$$

Then $\mathcal{F}$ has a zero v_0 in a ball withg radius C of $\mathcal{H}$, i.e.,

$$\exists\, v_0 \in \mathcal{H} : \quad \mathcal{F}(v_0) = 0 \quad \text{and} \quad \|v_0\|_{\mathcal{H}} \leq C.$$

Theorem 9. *(Schauder's fixed point theorem) Let X be a Banach space and $E \subset X$ is nonempty, closed and convex. If $f : E \to X$ is a continuous map such that $f(E) \subset E$ and $f(E)$ is compact, then f has a fixed point in E, i.e., there exists x in E such that $f(x) = x$.*

One of the most important convergence theorems of Lebesgue integration theory is stated next. For this thesis, the a.e. version is chosen.

Theorem 10. *(Lebesgue dominated convergence theorem) Let (X, μ) be a measure space. Suppose that $\{f_n\}$ is a sequence of complex measurable functions defined a.e. in X such that*

$$f = \lim_{n\to\infty} f_n, \quad a.e.\ in\ X$$

If there is $g \in L^1(X)$ such that

$$|f_n| \leq g, \quad n = 1, 2, \ldots, \quad a.e.\ in\ X$$

then $f \in L^1(X)$,

$$\lim_{n\to\infty} \int_X |f_n - f| d\mu = 0$$

and

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_\Omega f d\mu.$$

Weak and weak* topologies possess important compactness properties which allow one to extract weakly and weakly* convergent subsequences from bounded sequences. The following states the compactness property related to the weak topology.

Theorem 11. *In a reflexive Banach space, any bounded set is relatively weakly compact.*

The next theorem is related to the compactness property of the weak* topology.

Theorem 12. *(Banach-Alaoglu theorem)*

Let X be a normed space. Then the unit ball of the dual space X' of X is weakly compact.*

Next two compactness results have been proven by Simon [67]. Let $X \subset B \subset Y$ be Banach spaces with compact embedding $X \hookrightarrow B$.

Theorem 13. *For $1 \leq p \leq \infty$, assume that F is a bounded set in $L^p(0,T;X)$, and*

$$\|f(t+h) - f(t)\|_{L^p(0,T-h;Y)} \to 0 \text{ as } h \to 0, \text{ uniformly for } f \in F.$$

Then F is relatively compact in $L^p(0,T;B)$.

Theorem 14. *Assume that F is a bounded set in $L^\infty(0,T;X)$, and*

$$\{\frac{\partial f}{\partial t} : f \in F\} \text{ is bounded in } L^r(0,T;Y), \quad r > 1.$$

Then F is relatively compact in $\mathcal{C}^0(0,T;B)$.

Theorem 15. *(Schauder's theorem for compact operators) Let X, Y be Banach spaces and $T : X \to Y$ be a bounded linear operator. Then T is a compact operator if and only if $T^\star$ is compact.*

By the Rellich-Kondrachov theorem and Schauder's theorem we deduce the following:

Corollary 16. *$L^\infty(\Omega)$ is compactly embedded in $(H^1(\Omega))'$.*

The following focuses on some well-known inequalities.

Theorem 17. *(Triangle inequality) Let X be a normed space equipped with norm $\|\cdot\|_X$. Then,*

$$\forall x, y \in X, \quad \|x+y\|_X \leq \|x\|_X + \|y\|_X.$$

Theorem 18. *(Young's inequality) Let* $p, q > 1$, $\frac{1}{p}+\frac{1}{q} = 1$. *Then, for any nonnegative* $a, b \in \mathbb{R}$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Using the Young's inequality, one can prove the following theorem:

Theorem 19. *(Hölder's inequality) Let* $1 \leq p_1, \ldots, p_k \leq \infty$ *such that* $\frac{1}{p_1}+\ldots+\frac{1}{p_k} = 1$.

- *Generalized Integral form : If* $f_i \in L^{p_i}(\Omega)$, $i = 1, \ldots, k$, *then*

$$\int_\Omega |f_1 \ldots f_k| d\boldsymbol{x} \leq \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\Omega)}.$$

- *Summation form for finite sums: Let* $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, k$. *Then*

$$\sum_{i=1}^{k} |a_i b_i| \leq (\sum_{i=1}^{k} |a_i|^p)^{\frac{1}{p}} (\sum_{i=1}^{k} |b_i|^q)^{\frac{1}{q}}.$$

Theorem 20. *(Cauchy-Schwarz inequality) Let* X *be an inner product space over the field of real numbers, with inner product* $(\cdot, \cdot)_X$. *Then*

$$|(u, v)| \leq \|u\|_X \|v\|_X, \quad \forall u, v \in X.$$

Remark 21. *Cauchy-Schwarz inequality can be seen as a special case of Hölder's inequality integral form when* $k = 2$, $p_1 = p_2$ *and* $X = L^2(\Omega)$.

The rest of the inequalities and theorems come from the Sobolev space theory.

Theorem 22. *(Sobolev imbedding) Let* $\Omega \subset \mathbb{R}^2$. $H_0^1(\Omega)$ *is compactly imbedded into* $L^p(\Omega)$, *for any* $p < \infty$ *and there exists* $C > 0$ *that depends only on* Ω *such that*

$$\forall v \in H_0^1(\Omega), \quad \|v\|_{L^p(\Omega)} \leq C \|\nabla v\|_{L^2(\Omega)}. \tag{2.1}$$

Remark 23. *When* $r = 2$, *the inequality (2.1) is called Poincaré inequality. By the virtue of this inequality,* $|\cdot|_{H^1(\Omega)}$ *is a norm on* $H_0^1(\Omega)$.

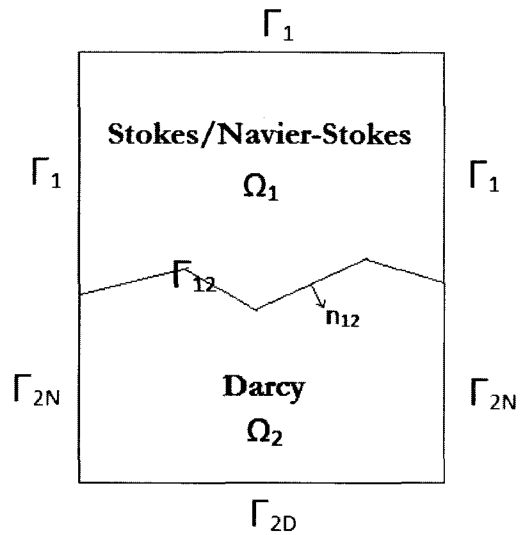


Figure 2.1 : The domain $\Omega = \Omega_1 \cup \Omega_2 \subset \mathbb{R}^2$.

The symmetric deformation tensor $\boldsymbol{D}(\boldsymbol{u}) = \frac{\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T}{2}$ to be used in the Navier-Stokes/Stokes equations satisfies the following inequality:

Theorem 24. *(Korn's inequality) Let $v \in H^1_{0,\Gamma}(\Omega)$ where $|\Gamma| \neq 0$. There exists a constant $C > 0$ such that*

$$\|\boldsymbol{D}(v)\|_{L^2(\Omega)} \leq C\|v\|_{H^1(\Omega)}.$$

Now let us introduce the region of concern $\Omega \subset \mathbb{R}^2$, which is shown in the Figure 2.1, for the flow problems of this thesis. The domain Ω is subdivided into two subregions as $\Omega = \Omega_1 \cup \Omega_2$ where Ω_1 corresponds to the surface region and Ω_2 corresponds to the subsurface region. We assume that Ω is an open, bounded, connected Lipschitz domain with Lipschitz boundary denoted by $\partial\Omega$. The vector $\boldsymbol{n}$ stands for the unit outward normal to $\partial\Omega$.

Let $\partial\Omega_i$, $i = 1, 2$, denote the boundary of Ω_i with exterior unit normal $\boldsymbol{n}_{\Omega_i}$ and define the interface separating Ω_1 and Ω_2 by $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$ with unit normal

$\boldsymbol{n}_{12}$ pointing from Ω_1 to Ω_2. We denote the tangential unit vector of Γ_{12} by $\boldsymbol{\tau}_{12}$. The portion of the boundary $\partial\Omega_i$ different from the interface Γ_{12} is denoted by $\Gamma_i = \partial\Omega_i \backslash \Gamma_{12}$, $i = 1, 2$. The boundary Γ_2 is further decomposed into two disjoint parts to differentiate the Dirichlet and Neumann boundaries; that is, $\Gamma_2 = \Gamma_{2\mathrm{D}} \cup \Gamma_{2\mathrm{N}}$ with $|\Gamma_{2\mathrm{D}}| > 0$.

Finally, some notation related to the DG methods are presented. Let Ω be a polygonal domain subdivided into a regular mesh $\mathcal{E}_h$, which contains triangular or rectangular elements E, with h being the maximum element diameter. We define the discontinuous piecewise polynomial space of degree $r \geq 1$ on mesh $\mathcal{E}_h$ by

$$\mathcal{D}_r(\mathcal{E}_h) = \{v \in L^2(\Omega) \, : \, \forall E \in \mathcal{E}_h, \quad v|_E \in \mathbb{P}_r(E)\}$$

where $\mathbb{P}_r(E)$ is the space of polynomials of degree less than or equal to r, defined on E. For each edge e on the mesh $\mathcal{E}_h$, we pick a unit normal vector $\boldsymbol{n}_e$. If the edge e is shared by two elements, first we order them as E_1 and E_2, then we assume $\boldsymbol{n}_e$ is pointing from E_1 to E_2. If the edge is on the boundary, we pick by convention the outward unit normal. Since continuity across the mesh interior edges is not a requirement for the discrete functions, these functions take different values on different sides of the edge e. This makes new definitions necessary to account for this difference. Define the jump and average values of $v \in \mathcal{D}_r(\mathcal{E}_h)$ on the edge $e \subset \partial E_1 \cap \partial E_2$ by

$$[v] = v|_{E^1} - v|_{E^2}, \quad \{v\} = \frac{v|_{E^1} + v|_{E^2}}{2}.$$

For the case of a boundary edge e which belongs to an element E, by convention we set

$$[v] = \{v\} = v|_E.$$

We denote the length of the edge e by $|e|$.

Chapter 3

Coupling of the Time-Dependent Navier-Stokes and Darcy Equations

Coupling of the incompressible Navier-Stokes and Darcy's equations has been an important multiphysics problem which models the interaction between incompressible free flow and porous media flow. This coupling problem has many applications in natural and industrial settings. It is used, for example, to model groundwater contamination through lakes and rivers, which is an important environmental issue. We depend on groundwater as an important source of daily drinking water and irrigation water. So it is crucial to keep our water free from chemical or organic pollutants if possible. Developing accurate simulation methods to foresee the behavior of contaminants is a necessary component in the remediation of contaminated groundwater.

The first objective of this chapter is to formulate a weak problem to the partial differential equations system governing the coupled flow and then show the existence of a weak solution. The second objective is to introduce and analyze a fully discrete scheme to solve this coupled flow problem. Solving this problem is challenging because of the complicated physical interactions on the interface between the two fluid regions. Appropriate conditions must be chosen to reflect these interactions. Two widely accepted interface conditions, the continuity of the normal component of the velocity and the balance of forces, together with the emprical interface condition of Beavers-Joseph-Saffman are assumed in this work.

This chapter explains the analysis of the coupled time-dependent Navier-Stokes and Darcy's equations with respect to two different models, part of what has been done by Çeşmelioğlu and Rivière [35, 36], in more detail. The difference between these models is the inclusion of the inertial forces in the balance of forces interface condition. It is not clear whether to include the inertial forces is necessary or not. Including them makes sense in mathematical point of view and the mathematical analysis is more easier whereas omitting them is physically more meaningful but much more challenging. Both of these models are governed by the same set of partial differential equations and completing initial and boundary conditions. The first section defines these equations and conditions describing the coupling of the time-dependent Navier-Stokes flow and Darcy flow while pointing out the difference in the balance of forces interface condition to be used in Model I and Model II. The third section provides the derivation of a weak formulation, which is equivalent to the original problem under enough smoothness assumptions. The fourth section establishes the existence and uniqueness of the weak solution to this weak formulation using the Galerkin method. The fifth section introduces a numerical scheme based on DG methods in space and Crank-Nicolson method in time. The sixth section states the necessary properties of the spaces and forms that arise from the numerical scheme. The last two sections focus on the existence and uniqueness results and the error estimates for the discrete solution under appropriate conditions on the data. To my knowledge, this analysis is the first in the literature for the time-dependent coupling problem.

3.1 Model Problem

The governing equations for the coupled surface and subsurface flow depend on the major dynamical laws of continuum mechanics, such as the continuity equation (or conservation of mass) and the momentum equation (or conservation of momentum).

Denote by $\Omega \subset \mathbb{R}^2$ a bounded region decomposed into two disjoint domains; Ω_1 for the Navier-Stokes flow region and Ω_2 for the Darcy flow region. The unknowns are the fluid velocity $\boldsymbol{u}(\boldsymbol{x},t)$ and the fluid pressure $p(\boldsymbol{x},t)$ in the Navier-Stokes region Ω_1 and the fluid pressure $\varphi(\boldsymbol{x},t)$ in the Darcy region Ω_2. The flow in Ω_1 over the time interval $(0,T)$ is characterized by the time-dependent Navier-Stokes equations:

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot (2\nu \boldsymbol{D}(\boldsymbol{u}) - p\mathbf{I}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{\Psi}, \quad \text{in } \Omega_1 \times (0,T), \tag{3.1}$$

where $\nu > 0$ is the kinematic fluid viscosity (measure of the internal resistance of a fluid to flow or to shear) and the vector function $\boldsymbol{\Psi}(\boldsymbol{x},t)$ is a body force, including the gravitational forces, acts on $\Omega_1 \times [0,T]$. The deformation tensor $\boldsymbol{D}(\boldsymbol{u})$ in (3.1) is defined to be the symmetric part of $\nabla \boldsymbol{u}$, that is, $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$. The Navier-Stokes equations defined by (3.1) represents the conservation of momentum.

The flow is incompressible in the Navier-Stokes region Ω_1, which means that the volume of any part of the fluid remains constant during the flow. So, the density remains constant and the mass conservation (or continuity) equation implies

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega_1 \times (0,T). \tag{3.2}$$

The flow in the porous media Ω_2 is characterized by the Darcy's law, which states that the flux is proportional to the pressure gradient:

$$-\nabla \cdot \boldsymbol{K} \nabla \varphi = \Pi, \quad \text{in } \Omega_2 \times (0,T). \tag{3.3}$$

Here, $\boldsymbol{K}(\boldsymbol{x})$ is a positive definite symmetric matrix corresponding to the hydraulic conductivity of Ω_2, that is, the ability of the porous medium to conduct fluids considering the dynamic viscosity [68]. The hydraulic conductivity $\boldsymbol{K}$ depends on space location and may be highly discontinuous. The scalar function $\Pi(\boldsymbol{x},t)$ is an external force, including the gravitational forces, acts on $\Omega_2 \times [0, T]$. Completion of the system (3.1)-(3.3) is through the initial condition

$$\boldsymbol{u} = \boldsymbol{u}_0, \quad \text{in } \Omega_1 \times \{0\}, \tag{3.4}$$

and a set of boundary and interface conditions defined below. On Γ_1, the Dirichlet (or no-slip) boundary condition is assumed,

$$\boldsymbol{u} \;=\; \boldsymbol{0}, \quad \text{on } \Gamma_1 \times (0, T). \tag{3.5}$$

On Γ_2, the Dirichlet and the Neumann (or no-flow) boundary conditions are assumed.

$$\varphi \;=\; 0, \quad \text{on } \Gamma_{2\mathrm{D}} \times (0, T), \tag{3.6}$$

$$\boldsymbol{K}\nabla\varphi \cdot \boldsymbol{n}_{\Omega_2} \;=\; 0, \quad \text{on } \Gamma_{2\mathrm{N}} \times (0, T). \tag{3.7}$$

The flow on different sides of the interface is governed by different types of partial differential equations. Suitable interface conditions are crucial to overcome the incompatibility caused by distinct behaviors of these two flow types. An obvious interface condition is the continuity of the flux (or mass conservation),

$$\boldsymbol{u} \cdot \boldsymbol{n}_{12} = -\boldsymbol{K}\nabla\varphi \cdot \boldsymbol{n}_{12}, \quad \text{on } \Gamma_{12} \times (0, T). \tag{3.8}$$

Correction for the tangential velocity should also be imposed on the interface. The widely accepted Beavers-Joseph-Saffman interface condition [2, 3, 69], based on experimentation and later mathematically justified by Jäger and Mikelić [70], sets the tangential component of the velocity to be proportional to the shear stress.

$$G\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u} \cdot \boldsymbol{\tau}_{12} = -2\nu\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}_{12} \cdot \boldsymbol{\tau}_{12}. \tag{3.9}$$

The positive proportionality constant G in (3.9) is determined experimentally [2, 3, 69]. The last interface condition is the main difference between the two models presented in this thesis. For Model I, the balance of forces includes the inertial forces $\frac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u}$ and given as

$$((-2\nu\boldsymbol{D}(\boldsymbol{u})+p\mathbf{I})\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12}+\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u})=\varphi,\quad \text{on}\Gamma_{12}\times(0,T). \tag{3.10}$$

Chidyagwai and Rivière [39] and Çeşmelioğlu and Rivière [35] consider this condition, which arises naturally from the momentum equation written in divergence form.Also note that (3.10) prevents $p+C$, where C is a constant, to solve the system given a solution p. So there is no need to have an extra condition for uniqueness on the Navier-Stokes pressure p.

For Model II, we omit the inertial forces hence the balance of forces is as follows:

$$((-2\nu\boldsymbol{D}(\boldsymbol{u})+p\mathbf{I})\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12}=\varphi,\quad \text{on } \Gamma_{12}\times(0,T). \tag{3.11}$$

Now that the system describing this surface and subsurface flow is complete, one of the questions that this thesis seeks an answer to is whether there is a solution to (3.1)-(3.9) with the condition (3.10) or (3.11). Rather than looking for a classical solution, a weaker solution $(\boldsymbol{u},p,\varphi)$ in suitable spaces is sought by relaxing the smoothness requirements. We proceed by first showing the existence of a weak solution for Model I and analyzing the proposed numerical method. Then we provide similar results for Model II under additional small data condition.

3.2 Model I with the Inertial Forces on the Interface

As mentioned before, this section considers the time-dependent Navier-Stokes/Darcy coupling where the balance of forces interface condition includes the inertial forces. First, the corresponding weak problem is formulated and the existence of a weak solution is provided. Then a numerical scheme based on the Discontinuous Galerkin methods in space and Crank-Nicolson method in time is derived. Existence of the discrete solution and error analysis are also given in this section.

3.2.1 Weak Formulation

The underlying spaces for a weak solution are defined as follows:

$$\boldsymbol{X} = H^1_{0,\Gamma_1}(\Omega_1)^2, \quad M_1 = L^2(\Omega_1), \quad M_2 = H^1_{0,\Gamma_{2D}}(\Omega_2).$$

For simplicity, I define a form γ which will take into account the interface conditions of the weak formulation as follows:

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}, \quad \forall p, q \in M_2, \quad \gamma(\boldsymbol{u}, p; \boldsymbol{v}, q) = (p - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}), \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u} \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}}.$$

Consequently, observe that

$$\forall \boldsymbol{u} \in \boldsymbol{X}, \quad \forall p \in M_2, \quad \gamma(\boldsymbol{u}, p; \boldsymbol{u}, p) = -\frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{u} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{u} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \geq -\frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}, \boldsymbol{u} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} \quad (3.12)$$

as $\boldsymbol{K}^{-\frac{1}{2}}$ is positive semi-definite. Together with this notation, the following weak formulation is proposed:

Find $(\boldsymbol{u}, p, \varphi) \in (L^2(0,T;\boldsymbol{X}) \cap H^1(0,T;L^2(\Omega_1)^2)) \times L^2(0,T;M_1) \times L^2(0,T;M_2)$ such that

$$(P) \begin{cases} \forall \boldsymbol{v} \in \boldsymbol{X}, \forall q \in M_2, & (\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} \\ & \quad - (p, \nabla \cdot \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2} + \gamma(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) \\ & \quad = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2}, \\ \forall q \in M_1, & (\nabla \cdot \boldsymbol{u}, q)_{\Omega_1} = 0, \\ \forall \boldsymbol{v} \in \boldsymbol{X}, & (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} = (\boldsymbol{u}_0, \boldsymbol{v})_{\Omega_1}. \end{cases}$$

The following lemma shows the equivalence of the original problem and the weak problem under appropriate smoothness assumptions defined in its statement.

Lemma 25. *Assume that*

$$\boldsymbol{\Psi} \in L^2(0,T;L^2(\Omega_1)^2), \quad \Pi \in L^2(0,T;L^2(\Omega_2)) \tag{3.13}$$

and $\boldsymbol{K} \in L^\infty(\Omega_2)^{2\times 2}$ *is uniformly bounded and positive definite in* Ω_2*, i.e., there exists* $\lambda_{min}, \lambda_{max} > 0$ *such that*

$$\lambda_{min}|x|^2 \leq \boldsymbol{K}x \cdot x \leq \lambda_{max}|x|^2, \;\; a.e. \;\; x \in \Omega_2. \tag{3.14}$$

In addition, let $\boldsymbol{u}_0$ *be in* $L^2(\Omega_1)^2$*. Then any solution* $(\boldsymbol{u}, p, \varphi)$ *of* (3.1)-(3.10) *that belongs to* $(L^2(0,T;\boldsymbol{X}) \cap H^1(0,T;L^2(\Omega_1))^2) \times L^2(0,T;M_1) \times L^2(0,T;M_2)$ *is also a solution to* (P)*. Conversely any solution to* (P) *satisfies* (3.1)-(3.10).

Proof. Let $(\boldsymbol{u}, p, \varphi) \in (L^2(0,T;\boldsymbol{X}) \cap H^1(0,T;L^2(\Omega_1)^2)) \times L^2(0,T;M_1) \times L^2(0,T;M_2)$ be a solution to (3.1)-(3.10). Note that because of the assumptions on the data, the following Green's formulas hold [38, p.2056]:

$$\forall \boldsymbol{v} \in H^1(\Omega_1)^2, \quad (\nabla \cdot (2\,\nu \boldsymbol{D}(\boldsymbol{u}) - p\boldsymbol{I}), \boldsymbol{v})_{\Omega_1}$$
$$= -(2\,\nu \boldsymbol{D}(\boldsymbol{u}), \nabla \boldsymbol{v}_{\Omega_1} + (p, \nabla \cdot \boldsymbol{v})_{\Omega_1} + \langle (2\,\nu \boldsymbol{D}(\boldsymbol{u}) - p\boldsymbol{I})\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1}.$$

and

$$\forall q \in H^1(\Omega_2), \quad -(\nabla \cdot \boldsymbol{K}\nabla\varphi, q)_{\Omega_2} = (\boldsymbol{K}\nabla\varphi, q)_{\Omega_2} - \langle \boldsymbol{K}\nabla\varphi \cdot \boldsymbol{n}_{\Omega_2}, q\rangle_{\partial\Omega_2},$$

The first step is to prove that $(\boldsymbol{u}, p, \varphi)$ satisfies the problem (P). For that purpose, let $\boldsymbol{v} \in \boldsymbol{X}$. The scalar product of (3.1) with $\boldsymbol{v} \in \boldsymbol{X}$ over Ω_1 yields

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} - (\nabla \cdot (2\nu \boldsymbol{D}(\boldsymbol{u}) - p\mathrm{I}), \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}.$$

Green's formula applied to the second term gives

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\nu \boldsymbol{D}(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1} - (p, \nabla \cdot \boldsymbol{v})_{\Omega_1} + \langle(-2\nu \boldsymbol{D}(\boldsymbol{u}) + pI)\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}.$$

Observe that by the symmetry property of $\boldsymbol{D}(\boldsymbol{u})$,

$$(\boldsymbol{D}(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1} = \int_{\Omega_1} \sum_{i,j=1}^{2} (\boldsymbol{D}(\boldsymbol{u}))_{ij} (\nabla \boldsymbol{v})_{ij} d\boldsymbol{x} = \int_{\Omega_1} \sum_{i,j=1}^{2} (\boldsymbol{D}(\boldsymbol{u}))_{ji} (\nabla \boldsymbol{v})_{ij} d\boldsymbol{x}$$
$$= \int_{\Omega_1} \sum_{i,j=1}^{2} (\boldsymbol{D}(\boldsymbol{u}))_{ji} ((\nabla \boldsymbol{v})^T)_{ji} d\boldsymbol{x} = (\boldsymbol{D}(\boldsymbol{u}), (\nabla \boldsymbol{v})^T)_{\Omega_1}.$$

Therefore,

$$(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} = (\boldsymbol{D}(\boldsymbol{u}), \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T))_{\Omega_1} = \frac{1}{2}(\boldsymbol{D}(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1} + \frac{1}{2}(\boldsymbol{D}(\boldsymbol{u}), (\nabla \boldsymbol{v})^T)_{\Omega_1} = (\boldsymbol{D}(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1}. \quad (3.15)$$

This and the assumption that $\boldsymbol{v} = 0$ on Γ_1 gives

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\nu \boldsymbol{D}(\boldsymbol{u}), \nabla \boldsymbol{v})_{\Omega_1} - (p, \nabla \cdot \boldsymbol{v})_{\Omega_1} + \langle(-2\nu \boldsymbol{D}(\boldsymbol{u}) + pI)\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}. \quad (3.16)$$

Now let $q \in M_2$. Taking the scalar product of (3.3) with q over Ω_2 yields

$$(-\nabla \cdot \boldsymbol{K}\nabla\varphi, q)_{\Omega_2} = (\Pi, q)_{\Omega_2}.$$

Green's formula, the boundary condition (3.7) and the fact that $\boldsymbol{n}_{\Omega_2} = -\boldsymbol{n}_{12}$ implies

$$(\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2} + \langle(\boldsymbol{K}\nabla\varphi) \cdot \boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} = (\Pi, q)_{\Omega_2}. \quad (3.17)$$

Adding (3.16) and (3.17) yields

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\nu \boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2}$$
$$- (p, \nabla \cdot \boldsymbol{v})_{\Omega_1} + \langle(-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\mathrm{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} + \langle(\boldsymbol{K}\nabla\varphi) \cdot \boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}}$$
$$= (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2}. \quad (3.18)$$

The velocity vector $\boldsymbol{v}$ can be written as the sum of its normal and tangential components, that is,

$$\boldsymbol{v} = (\boldsymbol{v} \cdot \boldsymbol{n}_{12})\boldsymbol{n}_{12} + (\boldsymbol{v} \cdot \boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}.$$

Also, according to Girault and Rivière [38],

$$((2\nu \boldsymbol{D}(\boldsymbol{u}) - p\mathrm{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12} \in L^2(\Gamma_{12}) \tag{3.19}$$

yielding

$$\langle(-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\mathrm{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} = (((-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\mathrm{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12}, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}$$
$$+ (((-2\nu \boldsymbol{D}(\boldsymbol{u}))\boldsymbol{n}_{12}) \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}.$$

Thus, recalling (3.9) and (3.10),

$$\langle(-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\mathrm{I})\boldsymbol{n}_{12}, \boldsymbol{v}\rangle_{\Gamma_{12}} = (\varphi - \frac{1}{2}(\boldsymbol{u} \cdot \boldsymbol{u}), \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}.$$

Further, taking scalar product of (3.8) with $q \in M_2$ on Γ_{12} gives

$$\langle \boldsymbol{K}\nabla\varphi \cdot \boldsymbol{n}_{12}, q\rangle_{\Gamma_{12}} = -(\boldsymbol{u} \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}}.$$

Combining these with (3.18) gives the following equation, which is the exact copy of the first equation in the formulation (Q):

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\nu \boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2} - (p, \nabla \cdot \boldsymbol{v})_{\Omega_1}$$
$$+ \gamma(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2}.$$

Now let $q \in M_1$ and multiply (3.2) by q and integrate over Ω_1 to get $(\nabla \cdot \boldsymbol{u}, q)_{\Omega_1} = 0$. This completes the weak formulation (P).

To show the converse, take a solution $(\boldsymbol{u}, p, \varphi)$ of (P) such that $(\boldsymbol{u}, p, \varphi) \in (L^2(0, T; \boldsymbol{X}) \cap H^1(0, T; L^2(\Omega_1))^2 \times L^2(0, T; M_1) \times L^2(0, T; M_2)$. As $\boldsymbol{u}(t) \in \boldsymbol{X}$ and

$\varphi(t) \in M_2$, by definition of these spaces, equations (3.5) and (3.6) are satisfied immediately. The assumption $(\nabla \cdot \boldsymbol{u}, q)_{\Omega_1} = 0$ for all $q \in M_1$ gives (3.2). To get (3.1), let $\boldsymbol{v} \in \mathcal{D}(\Omega_1)^2$ and $q = 0$. This, using the definition of weak derivatives yields

$$(\frac{\partial \boldsymbol{u}}{\partial t} - 2\nu\nabla \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}.$$

Therefore, in the sense of distributions on Ω_1,

$$\frac{\partial \boldsymbol{u}}{\partial t} - 2\nu\nabla \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{\Psi}. \tag{3.20}$$

which is (3.1). Similarly, letting $\boldsymbol{v} = 0$ and $q \in \mathcal{D}(\Omega_2)$ in the same equation of (P) yields

$$-(\nabla \cdot \boldsymbol{K}\nabla\varphi, q)_{\Omega_2} = (\Pi, q)_{\Omega_2}.$$

Hence, in the distributional sense on Ω_2,

$$-\nabla \cdot \boldsymbol{K}\nabla\varphi = \Pi. \tag{3.21}$$

Hence (3.3) is satisfied. Taking the scalar product of (3.20) with $\boldsymbol{v} \in \boldsymbol{X}$ yields

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} - (2\nu\nabla \cdot \boldsymbol{D}(\boldsymbol{u}), \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\nabla p, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}.$$

By Green's formula, we get

$$\begin{aligned}(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + (2\nu\boldsymbol{D}(\boldsymbol{u}), \nabla\boldsymbol{v})_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} - (p, \nabla \cdot \boldsymbol{v})_{\Omega_1} \\ + \langle(-2\nu\boldsymbol{D}(\boldsymbol{u}) + p\mathbf{I})\boldsymbol{n}_{\Omega_1}, \boldsymbol{v}\rangle_{\partial\Omega_1} = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}.\end{aligned} \tag{3.22}$$

Multiplying (3.21) by $q \in M_2$ and integrating over Ω_2 gives

$$(-\nabla \cdot \boldsymbol{K}\nabla\varphi, q)_{\Omega_2} = (\Pi, q)_{\Omega_2}.$$

As $q \in H^1(\Omega_2)$, applying Green's formula once more gives

$$(\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2} - \langle(\boldsymbol{K}\nabla\varphi) \cdot \boldsymbol{n}_{\Omega_2}, q\rangle_{\Omega_2} = (\Pi, q)_{\Omega_2}. \tag{3.23}$$

Adding (3.22) and (3.23) and using (3.15) gives

$$(\frac{\partial \boldsymbol{u}}{\partial t},\boldsymbol{v})_{\Omega_1} + (2\nu \boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1} - (p,\nabla\cdot\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi,\nabla q)_{\Omega_2}$$

$$+\langle(-2\nu\boldsymbol{D}(\boldsymbol{u})+p\mathrm{I})\boldsymbol{n}_{\Omega_1},\boldsymbol{v}\rangle_{\partial\Omega_1} + \langle-(\boldsymbol{K}\nabla\varphi)\cdot\boldsymbol{n}_{\Omega_2},q\rangle_{\partial\Omega_2} = (\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (\Pi,q)_{\Omega_2}.$$

A comparison of the above equation with (P) yields

$$\forall\boldsymbol{v}\in\boldsymbol{X},\forall q\in M_2,\quad (\varphi-\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}$$

$$-(\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} = \langle(-2\nu\boldsymbol{D}(\boldsymbol{u})+p\mathrm{I})\boldsymbol{n}_{\Omega_1},\boldsymbol{v}\rangle_{\partial\Omega_1} + \langle-(\boldsymbol{K}\nabla\varphi)\cdot\boldsymbol{n}_{\Omega_2},q\rangle_{\partial\Omega_2}. \quad (3.24)$$

Letting $\boldsymbol{v}=0$ in (3.24),

$$(\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} = \langle\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_{\Omega_2},q\rangle_{\partial\Omega_2}. \quad (3.25)$$

Choosing $q=0$ on Γ_{12} and since $q=0$ on Γ_{2D},

$$\langle\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_{\Omega_2},q\rangle_{\Gamma_{2\mathrm{N}}} = 0.$$

which implies (3.7). This, $\boldsymbol{n}_{\Omega_2}=-\boldsymbol{n}_{12}$ on Γ_{12} and $q=0$ on Γ_{2D} reduces (3.25) to

$$\forall q\in M_2,\quad (\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} = -\langle\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_{12},q\rangle_{\Gamma_{12}}$$

which leads to (3.8). Next, taking $q=0$ in (3.24) gives

$$\begin{aligned}\forall\boldsymbol{v}\in\boldsymbol{X},\quad (\varphi-\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} &+ G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12},\boldsymbol{v})_{\Gamma_{12}}\\ &= \langle(-2\nu\boldsymbol{D}(\boldsymbol{u})+p\mathbf{I})\boldsymbol{n}_{12},\boldsymbol{v}\rangle_{\Gamma_{12}}.\end{aligned}$$

Thus,

$$(-2\nu\boldsymbol{D}(\boldsymbol{u})+p\mathbf{I})\boldsymbol{n}_{12} = (\varphi-\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}))\boldsymbol{n}_{12} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}. \quad (3.26)$$

Taking dot product of (3.26) with $\boldsymbol{n}_{12}$ and $\boldsymbol{\tau}_{12}$, respectively, concludes the proof by establishing the conditions (3.9) and (3.10). □

Hence the problem (3.1)-(3.10) is equivalent to the problem (P). In other words, if a weak solution is smooth enough, it is, in fact, a strong solution. So, rather than searching for a strong solution, it suffices to show the existence of a weak solution. We conclude this section by recalling some important inequalities, such as Poincaré, Sobolev's, Korn's and trace inequalities introduced in the preliminary chapter. For any $\boldsymbol{v} \in \boldsymbol{X}$, there exist constants $S_2, S_4, T_2, T_4, C_D > 0$ depending only on Ω_1 such that

$$\|\boldsymbol{v}\|_{L^2(\Omega_1)} \leq S_2 |\boldsymbol{v}|_{H^1(\Omega_1)}, \quad \|\boldsymbol{v}\|_{L^4(\Omega_1)} \leq S_4 |\boldsymbol{v}|_{H^1(\Omega_1)}, \tag{3.27}$$

$$\|\boldsymbol{v}\|_{L^2(\Gamma_{12})} \leq T_2 |\boldsymbol{v}|_{H^1(\Omega_1)}, \quad \|\boldsymbol{v}\|_{L^4(\Gamma_{12})} \leq T_4 |\boldsymbol{v}|_{H^1(\Omega_1)}, \tag{3.28}$$

$$|\boldsymbol{v}|_{H^1(\Omega_1)} \leq C_D \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega_1)}. \tag{3.29}$$

Also, for any $q \in M_2$, there exist $\tilde{S}_2$ depending only on Ω_2 satisfying

$$\|q\|_{L^2(\Omega_2)} \leq \tilde{S}_2 |q|_{H^1(\Omega_2)}, \tag{3.30}$$

3.2.2 Existence of a Weak Solution

The method to prove the existence of a weak solution is the Galerkin method. The idea is to convert the problem to a finite dimensional one by representing the solution in terms of the basis functions of a finite dimensional subspace of the solution space. Then the weak solution is obtained as the limit of the Galerkin approximation. In addition, from the assumption (3.14), we have

$$\frac{1}{\sqrt{\lambda_{max}}} \|\boldsymbol{K}^{\frac{1}{2}} \nabla q\|_{L^2(\Omega_2)} \leq |q|_{H^1(\Omega_2)} \leq \frac{1}{\sqrt{\lambda_{min}}} \|\boldsymbol{K}^{\frac{1}{2}} \nabla q\|_{L^2(\Omega_2)}. \tag{3.31}$$

Now define the product space $\boldsymbol{Y} = \boldsymbol{X} \times M_2$ with the norm

$$\forall (\boldsymbol{v}, q) \in \boldsymbol{Y}, \quad \|(\boldsymbol{v}, q)\|_{\boldsymbol{Y}} = (2\nu \|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}} \nabla q\|^2_{L^2(\Omega_2)})^{\frac{1}{2}}$$

and the associated scalar product

$$\forall(\boldsymbol{v},q),(\mathrm{w},r)\in\boldsymbol{Y},\quad ((\boldsymbol{v},q),(\mathrm{w},r))_{\boldsymbol{Y}}=2\nu(\boldsymbol{D}(\boldsymbol{v}),\boldsymbol{D}(\boldsymbol{w}))_{\Omega_1}+(\boldsymbol{K}\nabla q,\nabla r)_{\Omega_2}$$

derived from the weak formulation. Because of (3.29) and (3.31), the norm $\|(\cdot,\cdot)\|_{\boldsymbol{Y}}$ is equivalent to the following product norm:

$$\forall(\boldsymbol{v},q)\in\boldsymbol{Y},\quad \|(\boldsymbol{v},q)\|=(|\boldsymbol{v}|^2_{H^1(\Omega_1)}+|q|^2_{H^1(\Omega_2)})^{\frac{1}{2}}.$$

So $(\boldsymbol{Y},\|(\cdot,\cdot)\|_{\boldsymbol{Y}})$ is a Hilbert space. Consider now a nicer subspace of $\boldsymbol{Y}$ on which the problem (P) is simplified. This subspace on which the Navier-Stokes pressure p vanishes is the product space of the space of divergence free functions

$$\boldsymbol{V}=\{\boldsymbol{v}\in\boldsymbol{X}:\nabla\cdot\boldsymbol{v}=0\text{ in }\Omega_1\}$$

and M_2, that is, $\boldsymbol{W}=\boldsymbol{V}\times M_2$. The space $\boldsymbol{W}$ is also a Hilbert space with the norm and scalar product of $\boldsymbol{Y}$. Restricting the test functions $\boldsymbol{v}$ to $\boldsymbol{V}$ in (P), we obtain a simpler variational formulation:

Find $(\boldsymbol{u},\varphi)\in(L^2(0,T;\boldsymbol{V})\cap H^1(0,T;L^2(\Omega_1)^2)\times L^2(0,T;M_2)$ such that

$$(P_V)\begin{cases}\forall(\boldsymbol{v},q)\in\boldsymbol{W}, & (\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v})_{\Omega_1}+2\nu(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1}+(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1}\\ & +(\boldsymbol{K}\nabla\varphi,\nabla q)_{\Omega_2}+\gamma(\boldsymbol{u},\varphi;\boldsymbol{v},q)=(\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1}+(\Pi,q)_{\Omega_2},\\ \forall\boldsymbol{v}\in\boldsymbol{V}, & (\boldsymbol{u}(0),\boldsymbol{v})_{\Omega_1}=(\boldsymbol{u}_0,\boldsymbol{v})_{\Omega_1}.\end{cases}$$

Clearly, if $(\boldsymbol{u},p,\varphi)$ is a solution to (P), then $(\boldsymbol{u},\varphi)$ is a solution to (P_V) but not vice versa. So, after showing the existence of a solution $(\boldsymbol{u},\varphi)$ to problem (P_V) using the Galerkin method, a Navier-Stokes pressure p should be constructed such that $(\boldsymbol{u},p,\varphi)$ is a solution to (P).

Because, the spaces $\boldsymbol{V}$ and M_2 are separable, the product space $\boldsymbol{W}$ is also a separable. Thus, we can find a basis $\{\boldsymbol{w}_i,r_i\}_{i\geq 1}$ of $\boldsymbol{W}$ such that $\boldsymbol{w}_i\in\boldsymbol{V}\cap H^2(\Omega_1)^2$ and $r_i\in$

$M_2 \cap H^2(\Omega_2)$. Fix a positive integer m and let $\boldsymbol{W}_m = \text{span}\{(\boldsymbol{w}_i, r_i), i = 1, \ldots, m\}$. Denote by $\boldsymbol{u}_{0m}$ the orthogonal projection in $L^2(\Omega_1)^2$ of $\boldsymbol{u}_0$ onto $\text{span}\{\boldsymbol{w}_i, i = 1, \ldots, m\}$. Specifically, $\boldsymbol{u}_{0m}$ is chosen to be any element in $\boldsymbol{W}_m$ such that $\boldsymbol{u}_{0m} \to \boldsymbol{u}_0$ strongly in $L^2(\Omega_1)^2$. Then a Galerkin approximation to problem (P_V) is the finite-dimensional problem (P_m) defined as

Find $(\boldsymbol{u}_m, \varphi_m) \in L^2(0, T; \boldsymbol{W}_m)$ with $\boldsymbol{u}_m \in H^1(0, T; L^2(\Omega_1)^2)$ such that

$$(P_m) \begin{cases} \forall (\boldsymbol{v}, q) \in \boldsymbol{W}_m, & (\frac{\partial \boldsymbol{u}_m}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}_m), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{v})_{\Omega_1} \\ & +(\boldsymbol{K}\nabla\varphi_m, \nabla q)_{\Omega_2} + \gamma(\boldsymbol{u}_m, \varphi_m; \boldsymbol{v}, q) = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2}, \\ \forall \boldsymbol{v} \in \boldsymbol{V}_m & (\boldsymbol{u}_m(0), \boldsymbol{v})_{\Omega_1} = (\boldsymbol{u}_{0m}, \boldsymbol{v})_{\Omega_1}. \end{cases}$$

The following shows the existence of a unique solution to (P_m) and also a uniform bound for the solution. If it exists, a solution $(\boldsymbol{u}_m, \varphi_m)$, expanded in terms of the basis functions, is of the form

$$\boldsymbol{u}_m(x,t) = \sum_{j=1}^{m} \alpha_j^m(t)\boldsymbol{w}_j(x), \quad \varphi_m(x,t) = \sum_{j=1}^{m} \beta_j^m(t) r_j(x)$$

where (α_j^m, β_j^m) is selected so that (P_m) is satisfied. Letting $\boldsymbol{v} = \boldsymbol{w}_i$ and $q = r_i$, in (P_m), $i = 1, \ldots, m$, we obtain an equivalent system written in matrix form. For that aim, the following mass and stiffness matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{M}$ are defined:

$$\begin{aligned} \mathbf{A}_{ij} &= (\boldsymbol{w}_j, \boldsymbol{w}_i)_{\Omega_1}, \quad \mathbf{B}_{ij} = 2\nu(\boldsymbol{D}(\boldsymbol{w}_j), \boldsymbol{D}(\boldsymbol{w}_i))_{\Omega_1} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{w}_j \cdot \boldsymbol{\tau}_{12}, \boldsymbol{w}_i \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}, \\ \mathbf{M}_{ij} &= (\boldsymbol{K}\nabla r_j, \nabla r_i)_{\Omega_2}, \quad \mathbf{C}_{ij} = (r_j, \boldsymbol{w}_i \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}, \quad i,j = 1, \ldots, m. \end{aligned}$$

The unknown vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are defined as $\boldsymbol{\alpha}_i = \alpha_i^m$, $\boldsymbol{\beta}_i = \beta_i^m$, $i = 1, \ldots, m$ and we also define the right hand side vectors $\boldsymbol{F}(\boldsymbol{\alpha}), \boldsymbol{b}, \boldsymbol{c}$ and $\mathbf{g}_0$ as follows:

$$(\boldsymbol{F}(\boldsymbol{\alpha}))_i = \boldsymbol{N}_i\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}, \quad \boldsymbol{b}_i = (\boldsymbol{\Psi}, \boldsymbol{w}_i)_{\Omega_1}, \quad \boldsymbol{c}_i = (\Pi, r_i)_{\Omega_2}, \quad (\mathbf{g}_0)_i = (\Pi_m \boldsymbol{u}_0, \boldsymbol{w}_i)_{\Omega_2}$$

where $\boldsymbol{N}_i = \big((\boldsymbol{w}_j \cdot \nabla \boldsymbol{w}_k, \boldsymbol{w}_i)_{\Omega_1} - \frac{1}{2}(\boldsymbol{w}_j \cdot \boldsymbol{w}_k, \boldsymbol{w}_i \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}\big)_{1 \le j,k \le m}$.

With this notation (P_m) is equivalent to the following first order non-homogeneous nonlinear system of ordinary differential equations

$$\begin{cases} \mathbf{A}\boldsymbol{\alpha}' + \mathbf{B}\boldsymbol{\alpha} + \mathbf{F}(\boldsymbol{\alpha}) - \mathbf{C}^T\boldsymbol{\beta} = \boldsymbol{b} \\ \mathbf{M}\boldsymbol{\beta} + \mathbf{C}\boldsymbol{\alpha} = \boldsymbol{c} \\ \mathbf{A}\boldsymbol{\alpha}(0) = \mathbf{g}_0 \end{cases} \tag{3.32}$$

As $\boldsymbol{K}$ is symmetric positive definite and r_i's are linearly independent, $\mathbf{M}$ is also symmetric positive definite. Hence, we can plug $\boldsymbol{\beta} = \mathbf{M}^{-1}\big(\boldsymbol{c} - \mathbf{C}\boldsymbol{\alpha}\big)$ in the first equation. Note also that as $\boldsymbol{w}_i$'s are linearly independent, the Gram matrix $\mathbf{A}$ is invertible and positive definite. Hence (3.32) leads to the following initial value problem:

$$\begin{cases} \boldsymbol{\alpha}'(t) + \mathbf{A}^{-1}(\mathbf{B} + \mathbf{C}^T\mathbf{M}^{-1}\mathbf{C})\boldsymbol{\alpha} = \mathbf{A}^{-1}\big(\boldsymbol{b} - \mathbf{F}(\boldsymbol{\alpha}) + \mathbf{C}^T\mathbf{M}^{-1}\boldsymbol{c}\big) \\ \boldsymbol{\alpha}(0) = \mathbf{A}^{-1}\mathbf{g}_0 \end{cases} \tag{3.33}$$

By Carathéodory's theorem [62, p.43, Thm 1.1], this nonlinear differential system has a maximal solution $\boldsymbol{\alpha}$ defined on some interval $[0, t_m]$. Then, showing a priori bounds on the solution will imply that $t_m = T$. Indeed, I will show later that $\boldsymbol{u}_m$ is bounded in $L^\infty(0, T; L^2(\Omega_1)^2)$ and Carathéodory's theorem will imply that there is a maximal solution $\boldsymbol{\alpha}(\boldsymbol{\alpha}(0); t)$ on some interval $[0, t_m]$ where $0 \le t_m \le T$. Let $[0, t_{max}[$ be the maximal half-open subinterval of $[0, T]$ such that $\boldsymbol{\alpha}(\boldsymbol{\alpha}(0); t)$ exists. Let

$$\mathcal{G}(\boldsymbol{\alpha}(t), t) = \mathbf{A}^{-1}\big(\boldsymbol{b}(t) - \mathbf{F}(\boldsymbol{\alpha}(t)) + \mathbf{C}^T\mathbf{M}^{-1}\boldsymbol{c}(t) - (\mathbf{B} + \mathbf{C}^T\mathbf{M}^{-1}\mathbf{C})\boldsymbol{\alpha}(t)\big).$$

Integrating (3.33), from boundedness of $\mathcal{G}$, there exists $M > 0$ such that

$$\|\boldsymbol{\alpha}(t) - \boldsymbol{\alpha}(s)\| \le \int_s^t \|\mathcal{G}(\boldsymbol{\alpha}(\xi), \xi)\| d\xi \le M(t - s)$$

for any $t, s \in [0, T]$. Hence $\overline{\boldsymbol{\alpha}} = \lim_{t \to t_{max}} \boldsymbol{\alpha}(t)$ exists. We want to show that $t_{max} = T$. Assume otherwise that $t_{max} < T$. Set a new initial value problem as follows:

$$\begin{cases} \boldsymbol{\alpha}'(t) = \mathcal{G}(\boldsymbol{\alpha}(t), t), \\ \boldsymbol{\alpha}(t_{max}) = \overline{\boldsymbol{\alpha}}. \end{cases}$$

Using Carathéodory's theorem once more, we get a solution $\boldsymbol{\alpha}(\boldsymbol{\alpha}(t_{max}), t)$ on $[0, t_m]$. Now consider the function defined by

$$\boldsymbol{\alpha}(t) = \begin{cases} \boldsymbol{\alpha}(\boldsymbol{\alpha}(0); t), & t \in [0, t_{max}); \\ \boldsymbol{\alpha}(\boldsymbol{\alpha}(t_{max}); t - t_{max}), & t \in [t_{max}, t_{max} + t_m]. \end{cases}$$

This is a solution on the interval $[0, t_{max} + t_m]$, a contradiction to the maximality of t_{max}. Thus $t_{max} = T$.

Next, as promised before, a priori estimates for the solution $(\boldsymbol{u}_m, \varphi_m)$ will be derived. Choosing $(\boldsymbol{v}, q) = (\boldsymbol{u}_m, \varphi_m)$ in (P_m) yields

$$\begin{aligned}(\frac{\partial \boldsymbol{u}_m}{\partial t}, \boldsymbol{u}_m)_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}_m), \boldsymbol{D}(\boldsymbol{u}_m))_{\Omega_1} + (\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{u}_m)_{\Omega_1} + (\boldsymbol{K}\nabla\varphi_m, \nabla\varphi_m)_{\Omega_2} \\ + \gamma(\boldsymbol{u}_m, \varphi_m; \boldsymbol{u}_m, \varphi_m) = (\boldsymbol{\Psi}, \boldsymbol{u}_m)_{\Omega_1} + (\Pi, \varphi_m)_{\Omega_2}. \quad (3.34)\end{aligned}$$

To rewrite the third term, observe that, by Green's theorem, for all $\boldsymbol{v} \in \boldsymbol{V}$,

$$\begin{aligned}0 = (\nabla \cdot \boldsymbol{v}, \boldsymbol{v} \cdot \boldsymbol{v})_{\Omega_1} &= -2(\boldsymbol{v}, \boldsymbol{v} \cdot \nabla \boldsymbol{v})_{\Omega_1} + (\boldsymbol{v} \cdot \boldsymbol{n}_{\Omega_1}, \boldsymbol{v} \cdot \boldsymbol{v})_{\partial\Omega_1} \\ &= -2(\boldsymbol{v}, \boldsymbol{v} \cdot \nabla \boldsymbol{v})_{\Omega_1} + (\boldsymbol{v} \cdot \boldsymbol{n}_{12}, \boldsymbol{v} \cdot \boldsymbol{v})_{\Gamma_{12}}.\end{aligned}$$

Hence as $\boldsymbol{u}_m \in \boldsymbol{V}$,

$$(\boldsymbol{u}_m, \boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m)_{\Omega_1} = \frac{1}{2}(\boldsymbol{u}_m \cdot \boldsymbol{n}_{12}, \boldsymbol{u}_m \cdot \boldsymbol{u}_m)_{\Gamma_{12}}.$$

This cancels the same term with the opposite sign in $\gamma(\boldsymbol{u}_m, \varphi_m; \boldsymbol{u}_m, \varphi_m)$. Thus from (3.12), $\gamma(\boldsymbol{u}_m, \varphi_m; \boldsymbol{u}_m, \varphi_m) \geq 0$ which applied to (3.34) yields,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + 2\nu\|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_m\|^2_{L^2(\Omega_2)} \leq (\boldsymbol{\Psi}, \boldsymbol{u}_m)_{\Omega_1} + (\Pi, \varphi_m)_{\Omega_2}.$$

The terms on the right-hand side are bounded using the Cauchy-Schwarz inequality and the inequalities (3.27)-(3.31):

$$(\boldsymbol{\Psi}, \boldsymbol{u}_m)_{\Omega_1} + (\Pi, \varphi_m)_{\Omega_2} \leq \|\boldsymbol{\Psi}\|_{L^2(\Omega_1)} S_2 |\boldsymbol{u}_m|_{H^1(\Omega_1)} + \|\Pi\|_{L^2(\Omega_2)} \tilde{S}_2 |\varphi_m|_{H^1(\Omega_2)}$$
$$\leq \|\boldsymbol{\Psi}\|_{L^2(\Omega_1)} S_2 C_D \|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} + \|\Pi\|_{L^2(\Omega_2)} \tilde{S}_2 \frac{1}{\sqrt{\lambda_{min}}} \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_m\|_{L^2(\Omega_2)}$$
$$\leq \frac{1}{4\nu} S_2^2 C_D^2 \|\boldsymbol{\Psi}\|^2_{L^2(\Omega_1)} + \nu \|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \frac{1}{2} \frac{\tilde{S}_2^{\,2}}{\lambda_{min}} \|\Pi\|^2_{L^2(\Omega_2)} + \frac{1}{2} \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_m\|^2_{L^2(\Omega_2)}.$$

Therefore,

$$\frac{1}{2} \frac{d}{dt} \|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + \nu \|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \frac{1}{2} \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_m\|^2_{L^2(\Omega_2)}$$
$$\leq \frac{1}{4\nu} S_2^2 C_D^2 \|\boldsymbol{\Psi}\|^2_{L^2(\Omega_1)} + \frac{1}{2} \frac{\tilde{S}_2^{\,2}}{\lambda_{min}} \|\Pi\|^2_{L^2(\Omega_2)}.$$

Multiplying this by 2 and integrating from 0 to t concludes

$$\|\boldsymbol{u}_m(t)\|^2_{L^2(\Omega_1)} + 2 \int_0^t \|(\boldsymbol{u}_m, \varphi_m)\|^2_{\boldsymbol{Y}} \leq C_e^2, \tag{3.35}$$

where

$$C_e = \left(\|\boldsymbol{u}_0\|^2_{L^2(\Omega_1)} + \frac{1}{2\nu} S_2^2 C_D^2 \|\boldsymbol{\Psi}\|^2_{L^2(0,T;L^2(\Omega_1))} + \frac{\tilde{S}_2^{\,2}}{\lambda_{min}} \|\Pi\|^2_{L^2(0,T;L^2(\Omega_2))} \right)^{\frac{1}{2}}. \tag{3.36}$$

Therefore, taking supremum over $[0, T]$ yields

$$\sup_{t \in [0,T]} \|\boldsymbol{u}_m(t)\|^2_{L^2(\Omega_1)} + \|(\boldsymbol{u}_m, \varphi_m)\|^2_{L^2(0,T;\boldsymbol{Y})} \leq C_e^2.$$

This a priori bound implies existence of a solution to (3.33) on the interval $[0, T]$. The following theorem summarizes the results so far:

Theorem 26. *Under the assumptions of Lemma 25 there exists a solution $(\boldsymbol{u}_m, \varphi_m) \in \boldsymbol{W}_m$ to the problem (P_m) satisfying*

$$\sup_{t \in [0,T]} \|\boldsymbol{u}_m(t)\|^2_{L^2(\Omega_1)} + \|(\boldsymbol{u}_m, \varphi_m)\|^2_{L^2(0,T;\boldsymbol{Y})} \leq C_e^2, \tag{3.37}$$

where C_e is the constant independent of m defined explicitly by (3.36).

Recall that $(\boldsymbol{u}_m, \varphi_m)$ is an approximation of $(\boldsymbol{u}, \varphi)$. Hence, passing to the limit as $m \to \infty$ will yield the existence of a solution for the problem (P_V). However, certain convergence results for the sequences $\boldsymbol{u}_m$ and φ_m are necessary to validate the passage to the limit. These properties come from the boundedness of $(\boldsymbol{u}_m, \varphi_m)$, some compactness theorems and a Fourier transform in time, as discussed below.

As shown above, the sequence $\{(\boldsymbol{u}_m, \varphi_m)\}_{m\geq 1}$ is bounded in $L^2(0, T, \boldsymbol{W})$. Since $\boldsymbol{W}$ is reflexive, so is $L^2(0, T, \boldsymbol{W})$. Hence, by Theorem 11, there is a subsequence still denoted by $\{(\boldsymbol{u}_m, \varphi_m)\}_{m\geq 1}$ and a pair $(\boldsymbol{u}, \varphi) \in L^2(0, T; \boldsymbol{W})$ such that

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{weakly in} \quad L^2(0, T; \boldsymbol{V}), \quad \text{and} \tag{3.38}$$

$$\varphi_m \to \varphi \quad \text{weakly in} \quad L^2(0, T; M_2). \tag{3.39}$$

Also, since the sequence $\{\boldsymbol{u}_m\}_{m\geq 1}$ is bounded in $L^\infty(0, T; L^2(\Omega_1)^2)$, by the Banach-Alaoglu Theorem 12, there exists a further subsequence, still denoted by $\{\boldsymbol{u}_m\}_{m\geq 1}$ such that for some $\boldsymbol{u}^* \in L^\infty(0, T; L^2(\Omega_1)^2)$,

$$\boldsymbol{u}_m \to \boldsymbol{u}^* \quad \text{weakly * in} \quad L^\infty(0, T; L^2(\Omega_1)^2). \tag{3.40}$$

This implies that

$$\int_0^T (\boldsymbol{u}_m(t) - \boldsymbol{u}^*(t), \boldsymbol{v}(t))_{\Omega_1} dt \to 0, \quad \forall \boldsymbol{v} \in L^2(0, T; L^2(\Omega_1)^2). \tag{3.41}$$

Also, by (3.38),

$$\int_0^T (\boldsymbol{u}_m(t) - \boldsymbol{u}(t), \boldsymbol{v}(t))_{\Omega_1} dt \to 0, \quad \forall \boldsymbol{v} \in L^2(0, T; L^2(\Omega_1)^2). \tag{3.42}$$

Therefore comparing (3.41) and (3.42) gives

$$\forall \boldsymbol{v} \in L^2(0, T; L^2(\Omega_1)^2), \qquad \int_0^T (\boldsymbol{u}(t) - \boldsymbol{u}^*(t), \boldsymbol{v}(t))_{\Omega_1} dt \to 0.$$

Hence,

$$\boldsymbol{u} = \boldsymbol{u}^* \in L^2(0, T; \boldsymbol{V}) \cap L^\infty(0, T; L^2(\Omega_1)^2). \tag{3.43}$$

Next, consider $\psi : [0,T] \to \mathbb{R}$ such that $\psi(T) = 0$ and $\psi \in \mathcal{C}^1([0,T])$. Multiply the first equation in (P_m) by $\psi(t)$ and integrate from 0 to T. Integrating by parts applied to the first term together with the initial condition of (P_m) yields for all $(\boldsymbol{v}, q) \in \boldsymbol{W}$ that

$$\begin{aligned} &-\int_0^T (\boldsymbol{u}_m(t), \psi'(t)\boldsymbol{v})_{\Omega_1} dt - (\boldsymbol{u}_{0m}, \boldsymbol{v})_{\Omega_1}\psi(0) + 2\nu \int_0^T (\boldsymbol{D}(\boldsymbol{u}_m), \psi(t)\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} dt \\ &\quad + \int_0^T (\boldsymbol{u}_m(t) \cdot \nabla \boldsymbol{u}_m(t), \psi(t)\boldsymbol{v})_{\Omega_1} dt + \int_0^T (\boldsymbol{K}\nabla\varphi_m(t), \psi(t)\nabla q)_{\Omega_1} dt \\ &\quad\quad + \int_0^T \gamma(\boldsymbol{u}_m, \varphi_m; \boldsymbol{v}, q) dt = \int_0^T (\boldsymbol{\Psi}(t), \psi(t)\boldsymbol{v})_{\Omega_1} dt + \int_0^T (\Pi(t), \psi(t) q) dt. \end{aligned}$$

By (3.38), (3.39), (3.40), (3.43) and letting $m \to \infty$, in the linear terms, $\boldsymbol{u}_m$ and φ_m can be replaced with $\boldsymbol{u}$ and φ. As $\boldsymbol{u}_m(0) = \boldsymbol{u}_{0m} \to \boldsymbol{u}_0$ strongly in $L^2(\Omega_1)^2$, letting $m \to \infty$, $\boldsymbol{u}_{0m}$ can be replaced with $\boldsymbol{u}_0$. However, passing to the limit in the nonlinear terms and the interface terms is not that easy. For that, the compactness result on $H^\gamma(0, T, \boldsymbol{V}, L^2(\Omega_1)^2)$ [71, p.186] with $0 < \gamma < 1/4$, which requires the boundedness of the sequence $\{\boldsymbol{u}_m\}_{m\geq 1}$ in the space $H^\gamma(0, T, \boldsymbol{V}, L^2(\Omega_1)^2)$, will be used. This boundedness can be shown using a Fourier transform in time. For the details, see A.1. Then, applying the compactness result, another subsequence $\{\boldsymbol{u}_m\}_{m\geq 1}$ can be extracted such that

$$\boldsymbol{u}_m \to \boldsymbol{u} \quad \text{strongly in} \quad L^2(0, T; L^2(\Omega_1)^2). \tag{3.44}$$

Observe also that for any $\boldsymbol{u} \in \boldsymbol{V}$ and any $\boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}$,

$$\begin{aligned} (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w}) &= -(\boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u} \cdot \boldsymbol{n}_{\Omega_1}, \boldsymbol{v} \cdot \boldsymbol{w})_{\partial\Omega_1} \\ &= -(\boldsymbol{u} \cdot \nabla \boldsymbol{w}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u} \cdot \boldsymbol{n}_{12}, \boldsymbol{v} \cdot \boldsymbol{w})_{\Gamma_{12}} \end{aligned} \tag{3.45}$$

Then, using the previous choice of ψ,

$$\int_0^T (\boldsymbol{u}_m(t)\cdot\nabla\boldsymbol{u}_m(t),\psi(t)\boldsymbol{v})_{\Omega_1}dt$$
$$= -\int_0^T (\boldsymbol{u}_m(t)\cdot\psi(t)\nabla\boldsymbol{v},\boldsymbol{u}_m(t))_{\Omega_1}dt + \int_0^T (\boldsymbol{u}_m(t)\cdot\boldsymbol{n}_{12},\boldsymbol{u}_m(t)\cdot\psi(t)\boldsymbol{v})_{\Gamma_{12}}dt.$$

By (3.38) and (3.44),

$$\int_0^T (\boldsymbol{u}_m(t)\cdot\psi(t)\nabla\boldsymbol{v},\boldsymbol{u}_m(t))_{\Omega_1}dt \to \int_0^T (\boldsymbol{u}(t)\cdot\psi(t)\nabla\boldsymbol{v},\boldsymbol{u}(t))_{\Omega_1}dt.$$

Recall that the trace operator from $H^1(\Omega_i)$ to $H^{\frac{1}{2}}(\partial\Omega_i)$ is continuous [60, p.216] for the weak topology. Thus, (3.38) and (3.39) yield

$$\boldsymbol{u}_m|_{\partial\Omega_1} \;\to\; \boldsymbol{u}|_{\partial\Omega_1} \quad \text{weakly in} \quad L^2(0,T;H^{\frac{1}{2}}(\partial\Omega_1)^2), \quad \text{and} \tag{3.46}$$

$$\varphi_m|_{\partial\Omega_2} \;\to\; \varphi|_{\partial\Omega_2} \quad \text{weakly in} \quad L^2(0,T;H^{\frac{1}{2}}(\partial\Omega_2)). \tag{3.47}$$

Also from a Sobolev embedding [60, p.97], after extracting another subsequence,

$$\boldsymbol{u}_m|_{\partial\Omega_1} \to \boldsymbol{u}|_{\partial\Omega_1} \quad \text{strongly in} \quad L^2(0,T;L^4(\partial\Omega_1)^2), \tag{3.48}$$

which allows the passage to the limit in the interface terms.

Finally, for any $(\boldsymbol{v},q)\in\boldsymbol{W}_m$ and $\psi\in\mathcal{C}^1[0,T]$ with $\psi(T)=0$,

$$-\int_0^T (\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\psi'(t)dt + (\boldsymbol{u}_0,\boldsymbol{v})_{\Omega_1}\psi(0) + 2\nu\int_0^T (\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1}\psi(t)dt$$
$$+\int_0^T (\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t),\boldsymbol{v})_{\Omega_1}\psi(t)dt + \int_0^T (\boldsymbol{K}\nabla\varphi(t),\nabla q)_{\Omega_1}\psi(t)dt$$
$$+\int_0^T \gamma(\boldsymbol{u}(t),\varphi(t);\boldsymbol{v},q)dt = \int_0^T (\boldsymbol{\Psi}(t),\boldsymbol{v})_{\Omega_1}\psi(t)dt + \int_0^T (\Pi(t),q)\psi(t)dt. \tag{3.49}$$

The second equation in P_m is true for $\boldsymbol{u}$ and $\boldsymbol{u}_0$ as $\boldsymbol{u}_{0m}\to\boldsymbol{u}_0$ strongly in $L^2(\Omega_1)^2$. Indeed, letting $m\to\infty$ in $(\boldsymbol{u}_m(0),\boldsymbol{v}) = (\boldsymbol{u}_{0m},\boldsymbol{v})$, we obtain

$$(\boldsymbol{u}(0),\boldsymbol{v}) = (\boldsymbol{u}_0,\boldsymbol{v}), \quad \forall\boldsymbol{v}\in\boldsymbol{V}_m. \tag{3.50}$$

Recall that $\{(\boldsymbol{w}_i, r_i)\}_{i\in\mathbb{N}}$ is total in $\boldsymbol{W}$, which means that any $(\boldsymbol{v}, q) \in \boldsymbol{W}$ can be approximated by the elements of $\boldsymbol{W}_m$'s. Therefore, (3.49) holds for any $(\boldsymbol{v}, q) \in \boldsymbol{W}$. As $\mathcal{D}(0,T)$ contains functions which vanish at both 0 and T, the term with $\psi(0)$ can be removed by restricting ψ to $\mathcal{D}(0,T)$. Then (3.49) gives

$$-\int_0^T (\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1}\psi'(t)dt + 2\nu\int_0^T (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1}\psi(t)dt + \int_0^T (\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1}\psi(t)dt$$
$$+\int_0^T (\boldsymbol{K}\nabla\varphi(t), \nabla q)_{\Omega_1}\psi(t)dt + \int_0^T \gamma(\boldsymbol{u}(t), \varphi(t); \boldsymbol{v}, q)\psi(t)dt$$
$$= \int_0^T (\boldsymbol{\Psi}(t), \boldsymbol{v})_{\Omega_1}\psi(t)dt + \int_0^T (\Pi(t), q)_{\Omega_2}\psi(t)dt.$$

By the definition of weak derivatives,

$$-\int_0^T (\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1}\psi'(t)dt = \int_0^T (\boldsymbol{u}'(t), \boldsymbol{v})_{\Omega_1}\psi(t)dt.$$

So, for any $\psi \in \mathcal{D}(0,T)$,

$$\int_0^T \Big(\boldsymbol{u}'(t), \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}(t)), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + \int_0^T (\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi(t), \nabla q)_{\Omega_1}$$
$$+\gamma(\boldsymbol{u}(t), \varphi(t), \boldsymbol{v}, q)\Big)\psi(t)dt = \int_0^T \Big((\boldsymbol{\Psi}(t), \boldsymbol{v})_{\Omega_1} + (\Pi(t), q)_{\Omega_2}\psi(t)\Big)dt.$$

Therefore, for all $(\boldsymbol{v}, q) \in \boldsymbol{W}$,

$$(\boldsymbol{u}', \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_1}$$
$$+\gamma(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} \quad (3.51)$$

in the distributional sense.

To see $\boldsymbol{u}_0 = \boldsymbol{u}(0)$, we multiply (3.51) with $\psi \in \mathcal{C}^1[0,T]$ such that $\psi(T) = 0$. Then integrating from 0 to T and applying integration by parts on the first term yields

$$-\int_0^T (\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1}\psi'(t)dt - (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1}\psi(0) + 2\nu \int_0^T (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1}\psi(t)dt$$
$$+ \int_0^T (\boldsymbol{u}(t)\cdot\nabla\boldsymbol{u}(t), \boldsymbol{v})_{\Omega_1}\psi(t)dt + \int_0^T (\boldsymbol{K}\nabla\varphi(t), \nabla q)_{\Omega_1}\psi(t)dt$$
$$\int_0^T \gamma(\boldsymbol{u}(t), \varphi(t); \boldsymbol{v}, q)dt = \int_0^T (\boldsymbol{\Psi}(t), \boldsymbol{v})_{\Omega_1}\psi(t)dt + \int_0^T (\Pi(t), q)_{\Omega_2}\psi(t)dt.$$

Comparing this with (3.49) yields $(\boldsymbol{u}_0, \boldsymbol{v})_{\Omega_1}\psi(0) = (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1}\psi(0)$. Finally, choosing a nonzero $\psi(0)$ gives $(\boldsymbol{u}_0 - \boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} = 0, \forall \boldsymbol{v} \in \boldsymbol{V}$, completing the existence proof of $(\boldsymbol{u}, \varphi)$ to the problem (P_V) .

In the following, we state the above result in which the a priori estimate is deduced trivially from the approximate case by the weak lower semicontinuity of the norm.

Corollary 27. *Under the same assumptions on the data as in Lemma 25 there exists a solution $(\boldsymbol{u}, \varphi)$ of (P_V). Furthermore, any solution of (P_V) satisfies*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}\|^2_{L^2(\Omega_1)} + \|(\boldsymbol{u}, \varphi)\|^2_{L^2(0,T;\boldsymbol{Y})} \leq C_e^2 \tag{3.52}$$

where C_e is defined by (3.36).

In the following, the uniqueness of the solution $(\boldsymbol{u}, \varphi)$ is provided. The common technique to prove uniqueness is supposing that there are two solutions and showing that they coincide. That being said, assume that $(\boldsymbol{u}, \varphi)$ and $(\tilde{\boldsymbol{u}}, \tilde{\varphi})$ are two solutions of (P_V). Let $\boldsymbol{w} = \boldsymbol{u} - \tilde{\boldsymbol{u}}$ and $r = \varphi - \tilde{\varphi}$. Then, the first equation in (P_V) implies that $(\boldsymbol{w}, r) \in L^2(0, T; \boldsymbol{W})$ satisfies

$$(\frac{\partial \boldsymbol{w}}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{w}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{v})_{\Omega_1}$$
$$+ (\boldsymbol{K}\nabla r, \nabla q)_{\Omega_2} + \gamma(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) - \gamma(\tilde{\boldsymbol{u}}, \tilde{\varphi}; \boldsymbol{v}, q) = 0.$$

Then, choose $\boldsymbol{v} = \boldsymbol{w}$ and $q = r$ in the above equation, use (3.12), add and subtract $\frac{1}{2}(\tilde{\boldsymbol{u}} \cdot \boldsymbol{u}, \boldsymbol{v} \cdot \boldsymbol{n}_{12})$ to get

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + 2\nu\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla r\|^2_{L^2(\Omega_2)} + (\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{w})_{\Omega_1} + (\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1}$$
$$- \frac{1}{2}(\boldsymbol{w}\cdot\boldsymbol{u}, \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\tilde{\boldsymbol{u}}\cdot\boldsymbol{w}, \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \leq 0. \quad (3.53)$$

From (3.45), $(\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} = -(\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} + (\tilde{\boldsymbol{u}}\cdot\boldsymbol{n}_{12}, \boldsymbol{w}\cdot\boldsymbol{w})_{\Gamma_{12}}$, which gives

$$(\tilde{\boldsymbol{u}}\cdot\nabla\boldsymbol{w}, \boldsymbol{w})_{\Omega_1} = \frac{1}{2}(\tilde{\boldsymbol{u}}\cdot\boldsymbol{n}_{12}, \boldsymbol{w}\cdot\boldsymbol{w})_{\Gamma_{12}}.$$

This reduces (3.53) to

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + 2\nu\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla r\|^2_{L^2(\Omega_2)} \leq -(\boldsymbol{w}\cdot\nabla\boldsymbol{u}, \boldsymbol{w})_{\Omega_1}$$
$$- \frac{1}{2}\left((\boldsymbol{w}\cdot\boldsymbol{w}, \tilde{\boldsymbol{u}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{w}\cdot(\boldsymbol{u}+\tilde{\boldsymbol{u}}), \boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\right).$$

The right hand side of the above equation can be bounded, by the virtue of (3.27)-(3.30) and (3.52), with the following expression:

$$\leq \|\boldsymbol{w}\|^2_{L^4(\Omega_1)}\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{w}\|^2_{L^4(\Gamma_{12})}(\|\boldsymbol{u}\|_{L^2(\Gamma_{12})} + 2\|\tilde{\boldsymbol{u}}\|_{L^2(\Gamma_{12})})$$
$$\leq C_D^3\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)}(S_4^2\|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(\Omega_1)} + \frac{1}{2}T_4^2(T_2\|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(\Omega_1)} + 2T_2\|\boldsymbol{D}(\tilde{\boldsymbol{u}})\|_{L^2(\Omega_1)}))$$
$$\leq C_D^3\frac{C_e}{\sqrt{2\nu}}(S_4^2 + \frac{3}{2}T_2T_4^2)\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)}.$$

Thus,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + (2\nu - C_D^3\frac{C_e}{\sqrt{2\nu}}(S_4^2 + \frac{3}{2}T_2T_4^2))\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla r\|^2_{L^2(\Omega_2)} \leq 0.$$

Since $\boldsymbol{w}(0) = 0$, multiplying by 2 and taking the integral from 0 to T yields

$$\|\boldsymbol{w}(T)\|^2_{L^2(\Omega_1)} + 2(2\nu - C_D^3\frac{C_e}{\sqrt{2\nu}}(S_4^2 + \frac{3}{2}T_2T_4^2))\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})}$$
$$+ 2\|\boldsymbol{K}^{\frac{1}{2}}\nabla r\|^2_{L^2(0,T;L^2(\Omega_2)^2)} \leq 0. \quad (3.54)$$

Lastly, imposing the condition

$$(2\nu)^{3/2} > C_D^3 C_e \big(S_4^2 + \frac{3}{2} T_2 T_4^2\big),$$

the inequality (3.54) leads to $(\boldsymbol{w}, r) = (\boldsymbol{0}, 0)$ which gives uniqueness.

So far, under additional assumptions on the data, the uniqueness of the solution $(\boldsymbol{u}, \varphi)$ of the problem (P_V) is proved. The only thing left to show is, given the solution $(\boldsymbol{u}, \varphi)$ of the problem (P_V), the existence of a pressure p, for which $(\boldsymbol{u}, p, \varphi)$ is a solution of the problem (P). I will follow the argument in [71]. Observe first that $(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \mapsto (\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w})_{\Omega_1}$ is a continuous trilinear form on $\boldsymbol{V}$. Therefore, there exists $B(\boldsymbol{u}, \boldsymbol{v}) \in \boldsymbol{V}'$ such that $(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w})_{\Omega_1} = \langle B(\boldsymbol{u}, \boldsymbol{v}), \boldsymbol{w} \rangle_{V',V}$, for all $\boldsymbol{w} \in \boldsymbol{V}$. Let $B\boldsymbol{u} = B(\boldsymbol{u}, \boldsymbol{u})$. Also observe that for any $\boldsymbol{u} \in L^2(0, T; \boldsymbol{V}')$, $B\boldsymbol{u} \in L^1(0, T; \boldsymbol{V}')$. Now, define

$$\boldsymbol{\Upsilon}(t) = \int_0^t \boldsymbol{u}(s) ds, \quad \boldsymbol{\Xi}(t) = \int_0^t \boldsymbol{\Psi}(s) ds, \quad \boldsymbol{\Delta}(t) = \int_0^t B\boldsymbol{u} ds.$$

Then $\boldsymbol{\Upsilon}$, $\boldsymbol{\Psi}$, $\boldsymbol{\Delta} \in \mathcal{C}(0, T; \boldsymbol{V}')$. Integrating (P_V) from 0 to t, choosing $\boldsymbol{v} \in \boldsymbol{V}$ with $\boldsymbol{v} = \boldsymbol{0}$ on Γ_{12} and $q = 0$ yields

$$\forall t \in (0, T), \quad 2\nu(\boldsymbol{D}(\boldsymbol{\Upsilon}(t)), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} = \langle \boldsymbol{u}(0) - \boldsymbol{u}(t) - \boldsymbol{\Delta}(t) + \boldsymbol{\Xi}(t), \boldsymbol{v} \rangle_{V',V}$$

where $\boldsymbol{u}(0) - \boldsymbol{u}(t) - \boldsymbol{\Delta}(t) + \boldsymbol{\Xi}(t) \in \mathcal{C}(0, T; \boldsymbol{V}')$. So, for all $t \in [0, T]$, there exists a $P(t) \in L^2(\Omega_1)$ such that

$$\forall t \in (0, T), \quad \nabla P(t) = \boldsymbol{\Xi}(t) - \boldsymbol{u}(t) + \boldsymbol{u}(0) + 2\nu \nabla \cdot \boldsymbol{D}(\boldsymbol{\Upsilon}(t)) - \boldsymbol{\Delta}(t). \tag{3.55}$$

Because right-hand side of (3.55) belongs to $\mathcal{C}([0, T]; H^{-1}(\Omega_1)^2)$, so does ∇P. The fact that the gradient operator is an isomorphism from $L^2(\Omega_1) \setminus \mathbb{R}$ into $H^{-1}(\Omega_1)^2$ concludes that P belongs to $\mathcal{C}([0, T]; L^2(\Omega_1))$. Differentiating (3.55) with respect to time on $\Omega_1 \times (0, T)$ gives

$$\frac{\partial \boldsymbol{u}}{\partial t} - 2\nu \nabla \cdot \boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + \nabla p = \boldsymbol{\Psi}$$

in the distributional sense where

$$p = \frac{\partial P}{\partial t}.$$

The following theorem concludes this section by stating the existence and uniqueness results obtained above.

Theorem 28. *Let $\boldsymbol{u}_0 \in \boldsymbol{V}$ and suppose that the data assumptions of Lemma 25 holds. If in addition,*

$$(2\nu)^{3/2} > C_D^3 C_e (S_4^2 + \frac{3}{2} T_2 T_4^2),$$

then the problem (P_V) has a unique solution $(\boldsymbol{u}, \varphi) \in (L^2(0,T;\boldsymbol{V}) \cap H^1(0,T;L^2(\Omega_1)^2) \times L^2(0,T;M_2)$ such that

$$\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{L^2(\Omega_1)} + \|(\boldsymbol{u},\varphi)\|^2_{L^2(0,T;\boldsymbol{Y})} \leq C_e^2, \tag{3.56}$$

with the constant defined in Theorem 26. Moreover, there exists $p \in L^2(0,T;M_1)$ such that $(\boldsymbol{u}, p, \varphi)$ is a solution to the problem (P).

Now that the results about existence and uniqueness for the weak solution of (P) are achieved, I will proceed with the formulation of a discrete scheme.

3.2.3 Numerical Scheme

This section contains a more elaborate version of the method given in the paper by Çeşmelioğlu and Rivière [36]. I begin by introducing necessary notation for the space discretization. For $i = 1, 2$, let $\mathcal{E}_h^i$ be a regular mesh of Ω_i consisting of triangles or quadrilaterals. As usual, the size of the mesh is characterized by h, the maximum diameter of the mesh elements. Let Γ_h^1 denote the set of edges that are either in the interior of Ω_1 or on the boundary Γ_1. Let Γ_h^2 denote the set of edges that are either in the interior of Ω_2 or on the Dirichlet boundary Γ_{2D}. The meshes are *not* assumed to match at the interface Γ_{12}. Let k_1 and k_2 be two positive integers. We consider the following finite dimensional spaces for the discretization of the Navier-Stokes velocity, the Navier-Stokes pressure and the Darcy pressure :

$$\boldsymbol{X}_h = (\mathcal{D}_{k_1}(\mathcal{E}_h^1))^2, \quad M_h^1 = \mathcal{D}_{k_1-1}(\mathcal{E}_h^1), \quad M_h^2 = \mathcal{D}_{k_2}(\mathcal{E}_h^2).$$

The discretization of the elliptic operators $-2\nu\nabla \cdot \boldsymbol{D}(\boldsymbol{u})$ and $-\nabla \cdot \boldsymbol{K}\nabla\varphi$ is done by the bilinear forms a_{NS} and a_{D} defined below:

$$\begin{aligned}
\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}_h, \quad & a_{\mathrm{NS}}(\boldsymbol{u}, \boldsymbol{v}) = 2\nu \sum_{E\in\mathcal{E}_h^1} (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_E - 2\nu \sum_{e\in\Gamma_h^1} (\{\boldsymbol{D}(\boldsymbol{u})\boldsymbol{n}_e\}, [\boldsymbol{v}])_e \\
& + 2\epsilon_{\mathrm{NS}}\nu \sum_{e\in\Gamma_h^1} (\{\boldsymbol{D}(\boldsymbol{v})\boldsymbol{n}_e\}, [\boldsymbol{u}])_e + \nu \sum_{e\in\Gamma_h^1} \frac{\sigma_e}{|e|}([\boldsymbol{u}], [\boldsymbol{v}])_e, \\
\forall p, q \in M_h^2, \quad & a_{\mathrm{D}}(p, q) = \sum_{E\in\mathcal{E}_h^2} (\boldsymbol{K}\nabla p, \nabla q)_E - \sum_{e\in\Gamma_h^2} (\{\boldsymbol{K}\nabla p \cdot \boldsymbol{n}_e\}, [q])_e \\
& + \epsilon_{\mathrm{D}} \sum_{e\in\Gamma_h^2} (\{\boldsymbol{K}\nabla q \cdot \boldsymbol{n}_e\}, [p])_e + \sum_{e\in\Gamma_h^2} \frac{\sigma_e}{|e|}([p], [q])_e.
\end{aligned}$$

The symmetrization parameters $\epsilon_{\mathrm{NS}}, \epsilon_{\mathrm{D}}$ take a constant value among $\{-1, 0, 1\}$ that specifies which variation of the primal DG method is being used. For example, the choice $\epsilon_{\mathrm{NS}} = \epsilon_{\mathrm{D}} = 1$ corresponds to the non-symmetric interior penalty Galerkin (NIPG) method, the choice $\epsilon_{\mathrm{NS}} = \epsilon_{\mathrm{D}} = -1$ corresponds to the symmetric interior penalty Galerkin (SIPG) method and the choice $\epsilon_{\mathrm{NS}} = \epsilon_{\mathrm{D}} = 0$ corresponds to the incomplete interior penalty Galerkin (IIPG) method. These interior penalty methods were introduced for the elliptic problem in [10, 13, 16]. The parameters σ_e are positive constants defined for each edge e to be used to penalize the jumps or in other words to control the amount of discontinuity. Denote by $\sigma_{\min}$ the minimum value of σ_e over all edges $e \in \Gamma_h^1 \cup \Gamma_h^2$. From now on, $\sigma_{\min}$ is assumed to be greater than 1, which is necessary for the analysis of the scheme.

The discretization of the pressure term ∇p is done by the bilinear form b_{NS}:

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad \forall p \in M_h^1, \quad b_{\mathrm{NS}}(\boldsymbol{v}, p) = -\sum_{E\in\mathcal{E}_h^1} (p, \nabla\cdot\boldsymbol{v})_E + \sum_{e\in\Gamma_h^1} (\{p\}, [\boldsymbol{v}]\cdot\boldsymbol{n}_e)_e. \quad (3.57)$$

For the discretization of the nonlinear convectionb term $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$, I introduce further notation. For any element E, $\boldsymbol{n}_E$ denotes the outward unit normal to ∂E. The trace of a function $\boldsymbol{v}$ on ∂E coming from the interior of E is denoted by $\boldsymbol{v}^{\mathrm{int}}$, whereas the trace coming from the exterior is denoted by $\boldsymbol{v}^{\mathrm{ext}}$. If the edge belongs to Γ_1, by convention, $\boldsymbol{v}^{\mathrm{int}} = \boldsymbol{v}$ and $\boldsymbol{v}^{\mathrm{ext}} = \boldsymbol{0}$. In a sense, the difference $\boldsymbol{v}^{\mathrm{int}} - \boldsymbol{v}^{\mathrm{ext}}$ is just another way to write the jump of $\boldsymbol{v}$ on the edge. With these notations, the discretization of $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ is through the forms c_{NS} and d_{NS} defined below.

$$\begin{aligned}
\forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{X}_h, \qquad c_{\mathrm{NS}}(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) &= \sum_{E\in\mathcal{E}_h^1} (\boldsymbol{u}\cdot\nabla\boldsymbol{v},\boldsymbol{w})_E + \frac{1}{2}\sum_{E\in\mathcal{E}_h^1} (\nabla\cdot\boldsymbol{u}, \boldsymbol{v}\cdot\boldsymbol{w})_E \\
&\quad -\frac{1}{2}\sum_{e\in\Gamma_h^1} ([\boldsymbol{u}]\cdot\boldsymbol{n}_e, \{\boldsymbol{v}\cdot\boldsymbol{w}\})_e, \\
\forall \boldsymbol{z},\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \boldsymbol{X}_h, \qquad d_{\mathrm{NS}}(\boldsymbol{z},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) &= \sum_{E\in\mathcal{E}_h^1} (|\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E|(\boldsymbol{v}^{\mathrm{int}} - \boldsymbol{v}^{\mathrm{ext}}), \boldsymbol{w}^{\mathrm{int}})_{\partial E_-(\boldsymbol{z})\setminus\Gamma_{12}},
\end{aligned}$$

where $\partial E_-(\boldsymbol{z}) = \{\boldsymbol{x}\in\partial E : \{\boldsymbol{z}(\boldsymbol{x})\}\cdot\boldsymbol{n}_E < 0\}$ is the inflow boundary of ∂E with respect to the vector field $\boldsymbol{z}$. Clearly, the form c_{NS} is linear with respect to all arguments, whereas the form d_{NS} is nonlinear with respect to all of its first argument. The nonlinear d_{NS} uses upwinding along the inflow boundary of ∂E with respect to the vector field $\boldsymbol{z}$.

Group all the linear terms involving $\boldsymbol{u}$ and φ by defining a bilinear form B:

$$\begin{aligned}
B([\boldsymbol{u},\varphi];[\boldsymbol{v},q]) &= a_{\mathrm{NS}}(\boldsymbol{u},\boldsymbol{v}) + a_{\mathrm{D}}(\varphi,q) \\
&+ (\varphi,\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}.
\end{aligned}$$

Note that the semi-column in the definition is just a notation to identify the bilinearity of the form B, that is, the linearity with respect to $(\boldsymbol{u},\varphi)$ and with respect to $(\boldsymbol{v},q)$.

Since the spaces are finite dimensional, the bilinearity of B implies that it is bounded. Also define a form N that combines the discretization of the nonlinear terms:

$$N(\boldsymbol{z},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) = c_{\mathrm{NS}}(\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) + d_{\mathrm{NS}}(\boldsymbol{z},\boldsymbol{u};\boldsymbol{v},\boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v}\cdot\boldsymbol{u},\boldsymbol{w}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}. \tag{3.58}$$

With these notations, the semi-discrete scheme is

Find $\boldsymbol{u}_h \in L^2(0,T;\boldsymbol{X}_h)\cap H^1(0,T;L^2(\Omega_1)^2)$, $p_h \in L^2(0,T;M_h^1)$, $\Phi_h \in L^2(0,T;M_h^2)$ such that for all $t>0$,

$$\forall \boldsymbol{v}\in\boldsymbol{X}_h,\quad \forall q\in M_h^2,\qquad (\frac{\partial \boldsymbol{u}_h}{\partial t},\boldsymbol{v})_{\Omega_1} + B([\boldsymbol{u}_h,\Phi_h];[\boldsymbol{v},q]) + b_{\mathrm{NS}}(\boldsymbol{v},p_h)$$
$$+N(\boldsymbol{u}_h,\boldsymbol{u}_h;\boldsymbol{u}_h,\boldsymbol{v}) = (\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (\Pi,q)_{\Omega_2}, \tag{3.59}$$

$$\forall q\in M_h^1,\qquad b_{\mathrm{NS}}(\boldsymbol{u}_h,q)=0. \tag{3.60}$$

$$\forall \boldsymbol{v}\in\boldsymbol{X}_h,\qquad (\boldsymbol{u}_h(0),\boldsymbol{v})_{\Omega_1} = (\boldsymbol{u}(0),\boldsymbol{v})_{\Omega_1}, \tag{3.61}$$

Lemma 29. *The solution $(\boldsymbol{u},p,\varphi)$ of (3.1)-(3.10) satisfies (3.59)-(3.61) under the additional assumption $\boldsymbol{u}\in L^2(0,T;H^{3/2+\delta}(\Omega_1)^2)$ and $\varphi\in L^2(0,T;H^{3/2+\delta}(\Omega_2))$ for any $\delta>0$.*

Proof. The proof is similar to the continuous case. Let E be any element in $\mathcal{E}_h^1$. Multiply (3.1) by $\boldsymbol{v}\in\boldsymbol{X}_h$ and integrate over E. Using Green's formula and summing over all E's,

$$\sum_{E\in\mathcal{E}_h^1}(\frac{\partial \boldsymbol{u}}{\partial t},\boldsymbol{v})_E + 2\nu\sum_{E\in\mathcal{E}_h^1}(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_E + \sum_{E\in\mathcal{E}_h^1}((-2\nu\boldsymbol{D}(\boldsymbol{u})+p\boldsymbol{I})\boldsymbol{n}_E,\boldsymbol{v})_{\partial E}$$
$$-\sum_{E\in\mathcal{E}_h^1}(p,\nabla\cdot\boldsymbol{v})_E + \sum_{E\in\mathcal{E}_h^1}(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_E = (\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1}.$$

For the boundary integrals switch to edge sums rather than element sums. Consider an interior edge e with neighbors E_1 and E_2. As mentioned in the preliminary chapter, we pick the normal vector of e such that $\boldsymbol{n}_e=\boldsymbol{n}_{E_1}$. Then $\boldsymbol{n}_{E_2}=-\boldsymbol{n}_e$. Summation over all elements have a double effect in terms of edge sums. For example, for any interior edge e, there is a contribution both from E_1 side and E_2 side when we sum over the elements. Together with the regularity of $\boldsymbol{u}$ and p, this means,

$$((-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})|_{E_1}\boldsymbol{n}_{E_1}, \boldsymbol{v}|_{E_1})_e + ((-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})|_{E_2}\boldsymbol{n}_{E_2}, \boldsymbol{v}|_{E_2})_e$$
$$= ((-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})\boldsymbol{n}_e, [\boldsymbol{v}])_e = (\{(-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})\boldsymbol{n}_e\}, [\boldsymbol{v}])_e$$

This implies,

$$\sum_{E\in\mathcal{E}_h^1} (\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_E + 2\nu \sum_{E\in\mathcal{E}_h^1} (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_E - \sum_{E\in\mathcal{E}_h^1} (p, \nabla\cdot \boldsymbol{v})_E + \sum_{E\in\mathcal{E}_h^1} (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_E$$
$$+ \sum_{e\in\Gamma_h^1} (\{(-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})\boldsymbol{n}_e\}, [\boldsymbol{v}])_e + \sum_{e\in\Gamma_{12}} ((-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v})_e = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}.$$

As it is, the method is not stable. Therefore, it is necessary to add the stabilization and penalty terms. The addition of these terms is allowed because they are identically zero by the regularity of the exact solution $\boldsymbol{u}$ and by the boundary condition (3.5).

$$(\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\nu \sum_{E\in\mathcal{E}_h^1} (\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_E - \sum_{E\in\mathcal{E}_h^1} (p, \nabla\cdot \boldsymbol{v})_E + \sum_{E\in\mathcal{E}_h^1} (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_E$$
$$+ \sum_{e\in\Gamma_h^1} (\{(-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})\boldsymbol{n}_e\}, [\boldsymbol{v}])_e + \sum_{e\in\Gamma_{12}} ((-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v})_e$$
$$+ 2\epsilon_{\mathrm{NS}}\nu \sum_{e\in\Gamma_h^1} (\{\boldsymbol{D}(\boldsymbol{v})\}\boldsymbol{n}_e, [\boldsymbol{u}])_e + \nu \sum_{e\in\Gamma_h^1} \frac{\sigma_e}{|e|}([\boldsymbol{u}], [\boldsymbol{v}])_e = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1}. \quad (3.62)$$

Next, multiply (3.3) by $q \in M_h^2$ and integrate over $E \in \mathcal{E}_h^2$. Applying Green's formula and summing over all the elements $E \in \mathcal{E}_h^2$,

$$\sum_{E\in\mathcal{E}_h^2} (\boldsymbol{K}\nabla\varphi, \nabla q)_E - \sum_{E\in\mathcal{E}_h^2} (\boldsymbol{K}\nabla\varphi\cdot \boldsymbol{n}_E, q)_{\partial E} = (\Pi, q)_{\Omega_2}.$$

As in the previous derivation, consider the summation of the boundary terms on edges rather than on elements. By the regularity of φ ,

$$\sum_{E\in\mathcal{E}_h^2}(\boldsymbol{K}\nabla\varphi,\nabla q)_E-\sum_{e\in\Gamma_h^2}(\{\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_e\},[q])_e-\sum_{e\in\Gamma_{2\mathrm{N}}}(\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_e,q)_e$$
$$+\sum_{e\in\Gamma_{12}}(\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_{12},q)_e=(\Pi,q)_{\Omega_2}.$$

Here, the third term can be removed because of (3.7). The stabilization and the penalty terms can be added because of the regularity of φ on Γ_h^2 and the boundary condition (3.6).

$$\sum_{E\in\mathcal{E}_h^2}(\boldsymbol{K}\nabla\varphi,\nabla q)_E-\sum_{e\in\Gamma_h^2}(\{\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_e\},[q])_e+\epsilon_{\mathrm{D}}\sum_{e\in\Gamma_h^2}(\{\boldsymbol{K}\nabla q\cdot\boldsymbol{n}_e\},[\varphi])_e$$
$$+\sum_{e\in\Gamma_h^2}\frac{\sigma_e}{|e|}([\varphi],[q])_e+\sum_{e\in\Gamma_{12}}(\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_{12},q)_e=(\Pi,q)_{\Omega_2}. \quad (3.63)$$

Observe by the regularity of $\boldsymbol{u}$ and as $\nabla\cdot\boldsymbol{u}=0$ on Ω_1,

$$c_{\mathrm{NS}}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})+d_{\mathrm{NS}}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})=\sum_{E\in\mathcal{E}_h^1}(\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_E.$$

Hence, adding (3.62) and (3.63) gives

$$(\frac{\partial\boldsymbol{u}}{\partial t},\boldsymbol{v})_{\Omega_1}+a_{\mathrm{NS}}(\boldsymbol{u},\boldsymbol{v})+b_{\mathrm{NS}}(\boldsymbol{v},p)+a_{\mathrm{D}}(\varphi,q)+c_{\mathrm{NS}}(\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})+d_{\mathrm{NS}}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{u},\boldsymbol{v})$$
$$+\sum_{e\in\Gamma_{12}}((-2\nu\boldsymbol{D}(\boldsymbol{u})+p\boldsymbol{I})\boldsymbol{n}_{12},\boldsymbol{v})_e+\sum_{e\in\Gamma_{12}}(\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_{12},q)_e=(\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1}+(\Pi,q)_{\Omega_2}.$$
$$(3.64)$$

Now decompose $\boldsymbol{v}$ into its normal and tangential components, that is,

$$\boldsymbol{v}=(\boldsymbol{v}\cdot\boldsymbol{n}_{12})\boldsymbol{n}_{12}+(\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})\boldsymbol{\tau}_{12}.$$

For any $e\in\Gamma_{12}$, (3.19), (3.9) and (3.10) implies

$$((-2\nu\boldsymbol{D}(\boldsymbol{u})+p\boldsymbol{I})\boldsymbol{n}_{12},\boldsymbol{v})_e=$$
$$=(((-2\nu\boldsymbol{D}(\boldsymbol{u})+p\boldsymbol{I})\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12},\boldsymbol{v}\cdot\boldsymbol{n}_{12})_e+(((-2\nu\boldsymbol{D}(\boldsymbol{u})+p\boldsymbol{I})\boldsymbol{n}_{12})\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_e$$
$$=(\varphi-\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_e+G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_e.$$

Summing this over all $e \in \Gamma_{12}$ gives

$$\sum_{e\in\Gamma_{12}} ((-2\nu\boldsymbol{D}(\boldsymbol{u}) + p\boldsymbol{I})\boldsymbol{n}_{12}, \boldsymbol{v})_e = (\varphi - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u}), \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}.$$

In a similar fashion, the following holds:

$$\sum_{e\in\Gamma_{12}} (\boldsymbol{K}\nabla\varphi\cdot\boldsymbol{n}_{12}, q)_e = -(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}.$$

With these two equations, (3.64) exactly gives (3.59). To get (3.60), let $E \in \mathcal{E}_h^1$. Multiply (3.2) by $q \in M_h^1$, integrate over $E \in \mathcal{E}_h^1$ and sum over all E to get

$$\sum_{E\in\mathcal{E}_h^1} (\nabla\cdot\boldsymbol{u}, q)_E = 0.$$

Using (3.5) and the regularity of $\boldsymbol{u}$ leads to

$$\sum_{E\in\mathcal{E}_h^1} (\nabla\cdot\boldsymbol{u}, q)_E - \sum_{e\in\Gamma_h^1} (\{q\}, [\boldsymbol{u}]\cdot\boldsymbol{n}_e)_e = 0.$$

Therefore $b_{\mathrm{NS}}(\boldsymbol{u}, q) = 0$ for all $q \in M_h^1$. This completes the derivation of the semi-discrete scheme. □

More notation is necessary to pass from the semi-discrete scheme to the fully-discrete scheme. Let $N_T > 0$ be the number of time steps, t^1 be the first time homstep and define

$$\Delta t = \frac{T - t^1}{N_T - 1}, \quad t^i = t^1 + (i-1)\Delta t, \quad 2 \le i \le N_T.$$

For a sequence $\{\phi^i\}_{i\ge1}$ or for a function $\phi^i = \phi(t^i)$, define

$$\phi^{i+\frac{1}{2}} = \frac{\phi^{i+1} + \phi^i}{2}.$$

The following fully-discrete scheme is obtained from the semi-discrete scheme by applying the Crank-Nicolson method:

Find $\{\boldsymbol{U}_h^i\}_{i\geq 0}$ in $\boldsymbol{X}_h$, $\{P_h^i\}_{i\geq 1} \in M_h^1$ and $\{\Phi_h^i\}_{i\geq 1}$ in M_h^2 such that,

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad (\boldsymbol{U}_h^0, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1}, \tag{3.65}$$

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \forall q \in M_h^2, \quad (\frac{\boldsymbol{U}_h^1 - \boldsymbol{U}_h^0}{t^1}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{U}_h^1, \Phi_h^1]; [\boldsymbol{v}, q]) + b_{\mathrm{NS}}(\boldsymbol{v}, P_h^1)$$
$$+ N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{v}) = (\boldsymbol{\Psi}^1, \boldsymbol{v})_{\Omega_1} + (\Pi^1, q)_{\Omega_2}, \tag{3.66}$$

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \forall q \in M_h^2, \quad (\frac{\boldsymbol{U}_h^{i+1} - \boldsymbol{U}_h^i}{\Delta t}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{U}_h^{i+\frac{1}{2}}, \Phi_h^{i+\frac{1}{2}}]; [\boldsymbol{v}, q])$$
$$\forall i \geq 1, \quad + b_{\mathrm{NS}}(\boldsymbol{v}, P_h^{i+\frac{1}{2}}) + N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{v})$$
$$= (\boldsymbol{\Psi}^{i+\frac{1}{2}}, \boldsymbol{v})_{\Omega_1} + (\Pi^{i+\frac{1}{2}}, q)_{\Omega_2}, \tag{3.67}$$

$$\forall i \geq 0, \forall q \in M_h^1, \quad b_{\mathrm{NS}}(\boldsymbol{U}_h^{i+1}, q)_{\Omega_1} = 0. \tag{3.68}$$

The equation (3.67) corresponds to a Crank-Nicolson discretization, which is chosen to achieve second order error estimates. In order to solve (3.67), the pressure and velocity at time t^1 are needed. I use a lower order and simpler scheme, namely a first order backward Euler scheme (3.66) to compute $\boldsymbol{U}_h^1$, P_h^1 and Φ_h^1. I will show that the resulting scheme is second order in time if the first time step t^1 is chosen appropriately.

Remark 30. *It is only a technical point to add non-homogeneous boundary conditions for the Darcy problem. For instance, assume that $\varphi = g_{\mathrm{D}}$ on $\Gamma_{2\mathrm{D}}$ with $g_{\mathrm{D}} \in H_{00}^{\frac{1}{2}}(\Gamma_{2\mathrm{D}})$. There exists a function $p_{\mathrm{D}} \in H^1(\Omega_2)$ that vanishes on Γ_{12}, that is equal to g_{D} on $\Gamma_{2\mathrm{D}}$ and such that*

$$\|p_{\mathrm{D}}\|_{H^1(\Omega_2)} \leq C \|g_{\mathrm{D}}\|_{H^{\frac{1}{2}}(\Gamma_{2\mathrm{D}})}.$$

The weak solution becomes $(\boldsymbol{u}, p, \tilde{\varphi})$ where $\tilde{\varphi} = \varphi + p_{\mathrm{D}}$ and with $(\boldsymbol{u}, p, \varphi)$ satisfying problem (P). Next, consider an approximation $p_{h\mathrm{D}} \in M_h^2$ of the lift p_{D}. Then, the numerical solution becomes $(\boldsymbol{U}_h^i, P_h^i, \tilde{\Phi}_h^i)$, where $\tilde{\Phi}_h^i = \Phi_h^i + p_{h\mathrm{D}}$ and $(\boldsymbol{U}_h^i, P_h^i, \Phi_h^i)$ satisfies (3.65)-(3.68) with modified right-hand sides. The analysis given below can *be adapted to the case of non-homogeneous boundary conditions as analyzed for the stationary case by Chidyagwai and Rivière [39].*

In order to prove the existence of a weak solution, it is important to know more about the discrete spaces and the discrete forms we have defined on them.

Properties of Discrete Spaces and Forms

In this section we state important properties of the discrete spaces and the bilinear forms. The propositions presented here are obtained from [72, 73, 74, 38].

The spaces $\boldsymbol{X}_h, M_h^1$ and M_h^2 are equipped with the following norms:

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \qquad \|\boldsymbol{v}\|_{X_h} = \left(2 \sum_{E \in \mathcal{E}_h^1} \|\boldsymbol{D}(\boldsymbol{v})\|^2_{L^2(E)} + \sum_{e \in \Gamma_1^h} \frac{\sigma_e}{|e|} \|[\boldsymbol{v}]\|^2_{L^2(e)} \right)^{\frac{1}{2}},$$

$$\forall q \in M_h^1, \qquad \|q\|_{M_h^1} = \|q\|_{L^2(\Omega_1)},$$

$$\forall q \in M_h^2, \qquad \|q\|_{M_h^2} = \left(\sum_{E \in \mathcal{E}_h^2} \|\boldsymbol{K}^{\frac{1}{2}} \nabla q\|^2_{L^2(E)} + \sum_{e \in \Gamma_2^h} \frac{\sigma_e}{|e|} \|[q]\|^2_{L^2(e)} \right)^{\frac{1}{2}}.$$

Further, we define the discrete divergence-free subspace $\boldsymbol{V}_h$ of $\boldsymbol{X}_h$ as

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{X}_h : \forall q \in M_h^1, \quad b_{\mathrm{NS}}(\boldsymbol{v}, q) = 0\}.$$

The following proposition extends the usual Sobolev imbeddings and trace inequalities to the discontinuous discrete spaces.

Proposition 31. *For any $r \geq 2$, there exist constants C_{1r}, C_2, $\tilde{C}_{1r}$ and $\tilde{C}_{2r}$ independent of h, but dependent on $\sigma_{\min}$ such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \qquad \|\boldsymbol{v}\|_{L^r(\Omega_1)} \leq C_{1r} \|\boldsymbol{v}\|_{X_h}, \tag{3.69}$$

$$\forall q \in M_h^2, \qquad \|q\|_{L^2(\Omega_2)} \leq C_2 \|q\|_{M_h^2}, \tag{3.70}$$

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \qquad \|\boldsymbol{v}\|_{L^r(\Gamma_{12})} \leq \tilde{C}_{1r} \|\boldsymbol{v}\|_{X_h}, \tag{3.71}$$

$$\forall q \in M_h^2, \qquad \|q\|_{L^r(\Gamma_{12})} \leq \tilde{C}_{2r} \|q\|_{M_h^2}. \tag{3.72}$$

The next proposition states the coercivity properties of a_{NS} and a_{D}. These properties are true for the NIPG method, for any $\sigma_{\min}$. However for both SIPG and IIPG methods, coercivity is valid only if $\sigma_{\min}$ is large enough as suggested by Epshteyn and Rivière [75].

Proposition 32. *There exist constants C_3 and C_4, independent of h and ν, such that*

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \qquad C_3\nu\|\boldsymbol{v}\|_{X_h}^2 \leq a_{\mathrm{NS}}(\boldsymbol{v},\boldsymbol{v}), \tag{3.73}$$

$$\forall q \in M_h^2, \qquad C_4\|q\|_{M_h^2}^2 \leq a_{\mathrm{D}}(q,q). \tag{3.74}$$

A straightforward bound for B, which is deduced from (3.73) and (3.74, is given in the following corollary.

Corollary 33. $\forall \boldsymbol{v} \in \boldsymbol{X}_h$, $\forall q \in M_h^2$,

$$B([\boldsymbol{v},q];[\boldsymbol{v},q]) \geq C_3\nu\|\boldsymbol{v}\|_{X_h}^2 + C_4\|q\|_{M_h^2}^2. \tag{3.75}$$

Proof. Since the terms $(q, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$ and $(\boldsymbol{v}\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}$ cancel, we have

$$\begin{aligned} B([\boldsymbol{v},q];[\boldsymbol{v},q]) &= a_{\mathrm{NS}}(\boldsymbol{v},\boldsymbol{v}) + a_{\mathrm{D}}(q,q) + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{v}\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ &\geq C_3\nu\|\boldsymbol{v}\|_{X_h}^2 + C_4\|q\|_{M_h^2}^2. \end{aligned}$$

□

The form $(c_{\mathrm{NS}} + d_{\mathrm{NS}})$ has been extensively studied in the literature. From [72, 76], the following result for N, defined by (3.58), can be deduced.

Proposition 34. *For all $\boldsymbol{u},\boldsymbol{v} \in \boldsymbol{X}_h$,*

$$\begin{aligned} N(\boldsymbol{u},\boldsymbol{u};\boldsymbol{v},\boldsymbol{v}) = \frac{1}{2}\sum_{E\in\mathcal{E}_h^1} \||\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E|^{\frac{1}{2}}[\boldsymbol{v}]\|_{L^2(\partial E_-(\boldsymbol{u})\setminus\partial\Omega_1)}^2 + \||\boldsymbol{u}\cdot\boldsymbol{n}_{\Omega_1}|^{\frac{1}{2}}\boldsymbol{v}\|_{L^2(\Gamma_{1-}(\boldsymbol{u}))}^2 \\ + \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n}_{12}, \boldsymbol{v}\cdot\boldsymbol{v})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{v}, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \end{aligned} \tag{3.76}$$

where the inflow boundary of Γ_1 *is defined by*

$$\Gamma_{1-}(\boldsymbol{u}) = \{\boldsymbol{x} \in \Gamma_1 : \{\boldsymbol{u}(\boldsymbol{x})\} \cdot \boldsymbol{n}_{\Omega_1} < 0\}.$$

The positivity result

$$N(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{u}) \geq 0 \quad \forall \boldsymbol{u} \in \boldsymbol{X}_h \tag{3.77}$$

is a special case of (3.76) obtained by taking $\boldsymbol{u} = \boldsymbol{v}$. The following bounds are important for the uniqueness proof of the numerical solution.

Proposition 35. *There exists a constant* C_5 *independent of* h *and* ν *such that for all* $\boldsymbol{u} \in \boldsymbol{V}_h$, $\boldsymbol{z}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}_h$,

$$|c_{\text{NS}}(\boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w})| + |d_{\text{NS}}(\boldsymbol{z}, \boldsymbol{u}; \boldsymbol{v}, \boldsymbol{w})| \leq C_5 \|\boldsymbol{u}\|_{X_h} \|\boldsymbol{v}\|_{X_h} \|\boldsymbol{w}\|_{X_h}. \tag{3.78}$$

Proof. This result follows from (3.69), (3.70) and Lemma 6.4 of [72]. □

The next proposition is technical and can be found in [38] and included in the Appendix A.2 for completeness.

Proposition 36. *There exists a constant* C_6 *independent of* h *but dependent on* σ_1^{min} *such that for all* $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \boldsymbol{X}_h$,

$$|d_{\text{NS}}(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{u}, \boldsymbol{w}) - d_{\text{NS}}(\boldsymbol{v}, \boldsymbol{v}; \boldsymbol{v}, \boldsymbol{w})| \leq C_6 \|\boldsymbol{u} - \boldsymbol{v}\|_{X_h} \|\boldsymbol{w}\|_{X_h} (\|\boldsymbol{u}\|_{X_h} + \|\boldsymbol{v}\|_{X_h}). \tag{3.79}$$

Now everthing is ready to prove the existence and uniqueness of the discrete solution and derive the error estimates. A version of the Brouwer's fixed point theorem is the key to prove these results.

3.2.4 Existence and Uniqueness of the Numerical Solution

As done in the continuous case, we simplify the problem (3.65)-(3.68) by restricting to the subspace $\boldsymbol{V}_h \subset \boldsymbol{X}_h$ defined in the previous section as

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{X}_h : \forall q \in M_h^1, \quad b_{\rm NS}(\boldsymbol{v}, q) = 0\}.$$

This will remove the $b_{\rm NS}$ terms. Corresponding to $\boldsymbol{Y}$ defined in the continuous case, let $\boldsymbol{Y}_h = \boldsymbol{X}_h \times M_h^2$ equipped with the inner product $((\cdot,\cdot))$ defined by

$$(((\boldsymbol{z}, r), (\boldsymbol{v}, q))) = 2\nu \sum_{E \in \mathcal{E}_h^1} (\boldsymbol{D}(\boldsymbol{z}), \boldsymbol{D}(\boldsymbol{v}))_E + \nu \sum_{e \in \Gamma_h^1} \frac{\sigma_e}{|e|}([\boldsymbol{z}], [\boldsymbol{v}])_e$$
$$+ \sum_{E \in \mathcal{E}_h^2} (\boldsymbol{K} \nabla r, \nabla q)_E + \sum_{e \in \Gamma_h^2} \frac{\sigma_e}{|e|}([r], [q])_e.$$

The norm on $\boldsymbol{Y}_h$ is $\|(\boldsymbol{v}, q)\|_{\boldsymbol{Y}_h} = \left(2\nu \|\boldsymbol{v}\|_{\boldsymbol{X}_h}^2 + \|q\|_{M_h^2}^2\right)^{\frac{1}{2}}$, for all $(\boldsymbol{v}, q) \in \boldsymbol{Y}_h$. Also define the subspace $\boldsymbol{W}_h = \boldsymbol{V}_h \times M_h^2$ of $\boldsymbol{Y}_h$ equipped with the same norm. Clearly from (3.65), the initial velocity $\boldsymbol{U}_h^0$ is uniquely defined. Now the question is if there exists a solution $\{\boldsymbol{U}_h^i, \Phi_h^i\}_{i \geq 1} \in \boldsymbol{W}_h$ satisfying

$$\forall \boldsymbol{v} \in \boldsymbol{V}_h, \forall q \in M_h^2, \quad (\frac{\boldsymbol{U}_h^1 - \boldsymbol{U}_h^0}{t^1}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{U}_h^1, \Phi_h^1]; [\boldsymbol{v}, q])$$
$$+ N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{v}) = (\boldsymbol{\Psi}^1, \boldsymbol{v})_{\Omega_1} + (\Pi^1, q)_{\Omega_2}, \quad (3.80)$$
$$\forall \boldsymbol{v} \in \boldsymbol{V}_h, \forall q \in M_h^2, \quad (\frac{\boldsymbol{U}_h^{i+1} - \boldsymbol{U}_h^i}{\Delta t}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{U}_h^{i+\frac{1}{2}}, \Phi_h^{i+\frac{1}{2}}]; [\boldsymbol{v}, q])$$
$$\forall i \geq 1, \quad + N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{v})$$
$$= (\boldsymbol{\Psi}^{i+\frac{1}{2}}, \boldsymbol{v})_{\Omega_1} + (\Pi^{i+\frac{1}{2}}, q)_{\Omega_2}. \quad (3.81)$$

The following lemma answers the existence question.

Lemma 37. *There exists a solution $\{(\boldsymbol{U}_h^i, \Phi_h^i)\}_{i \geq 1}$ of (3.80)-(3.81) satisfying*

$$\|\boldsymbol{U}_h^1\|_{L^2(\Omega_1)}^2 + t^1 \min(C_3, C_4) \|(\boldsymbol{U}_h^1, \Phi_h^1)\|_{Y_h}^2 \leq \overline{\mathcal{C}}_1^2, \quad (3.82)$$

and for all $2 \le m \le N_T$,

$$\|\boldsymbol{U}_h^m\|_{L^2(\Omega_1)}^2 + \min(C_3, C_4)\Delta t \sum_{i=1}^{m-1} \|(\boldsymbol{U}_h^{i+\frac{1}{2}}, \Phi_h^{i+\frac{1}{2}})\|_{Y_h}^2 \le \overline{\mathcal{C}}^2, \tag{3.83}$$

where the constant $\overline{\mathcal{C}}_1$ *and* $\overline{\mathcal{C}}$ *are defined as follows:*

$$\overline{\mathcal{C}}_1 = (\|\boldsymbol{u}_0\|_{L^2(\Omega_1)}^2 + \frac{C_{12}^2}{C_3\nu} t^1 \|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)}^2 + \frac{C_2^2}{C_4} t^1 \|\Pi^1\|_{L^2(\Omega_2)}^2)^{\frac{1}{2}},$$

$$\overline{\mathcal{C}} = \big(\|\boldsymbol{U}_h^1\|_{L^2(\Omega_1)}^2 + \frac{C_{12}^2}{C_3\nu}\Delta t \sum_{i=1}^{N_T-1} \|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|_{L^2(\Omega_1)}^2 + \frac{C_2^2}{C_4}\Delta t \sum_{i=1}^{N_T-1} \|\Pi^{i+\frac{1}{2}}\|_{L^2(\Omega_2)}^2\big)^{\frac{1}{2}}. \tag{3.84}$$

Proof. The first step is to show that the pair $(\boldsymbol{U}_h^1, \Phi_h^1)$ exists. Define a mapping $\mathcal{F}_1 : \boldsymbol{V}_h \times M_h^2 \to \boldsymbol{V}_h \times M_h^2$ by

$$\forall (\boldsymbol{z}, r),\ (\boldsymbol{v}, q) \in \boldsymbol{V}_h \times M_h^2, \quad ((\mathcal{F}_1(\boldsymbol{z}, r), (\boldsymbol{v}, q))) = (\frac{\boldsymbol{z} - \boldsymbol{U}_h^0}{t^1}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{z}, r]; [\boldsymbol{v}, q])$$
$$+ N(\boldsymbol{z}, \boldsymbol{z}; \boldsymbol{z}, \boldsymbol{v}) - (\boldsymbol{\Psi}^1, \boldsymbol{v})_{\Omega_1} - (\Pi^1, q)_{\Omega_2}.$$

By the Riesz representation theorem and the inequalities (3.27)-(3.30), $\mathcal{F}_1$ is a well-defined mapping from $\boldsymbol{Y}_h$ into itself. From Theorem 8 (Brouwer's fixed point theorem), showing that there is a ball on which $((\mathcal{F}_1(\boldsymbol{z}, r), (\boldsymbol{z}, r))) \ge 0$ implies, that there is a zero $(\boldsymbol{z}^*, r^*)$ of $\mathcal{F}_1$ inside the ball. Clearly, this zero is a solution to (3.80). Taking $(\boldsymbol{v}, q) = (\boldsymbol{z}, r)$ in the definition of $\mathcal{F}_1$ and using (3.75) and (3.76) gives

$$((\mathcal{F}_1(\boldsymbol{z}, r), (\boldsymbol{z}, r))) \ge \frac{1}{2t^1}\|\boldsymbol{z}\|_{L^2(\Omega_1)}^2 - \frac{1}{2t^1}\|\boldsymbol{U}_h^0\|_{L^2(\Omega_1)}^2$$
$$+ C_3\nu\|\boldsymbol{z}\|_{X_h}^2 + C_4\|r\|_{M_h^2}^2 - (\boldsymbol{\Psi}^1, \boldsymbol{z})_{\Omega_1} - (\Pi^1, r)_{\Omega_2}. \tag{3.85}$$

Using the Cauchy-Schwarz inequality, (3.69) and (3.70) yields

$$|(\boldsymbol{\Psi}^1, \boldsymbol{z})_{\Omega_1} + (\Pi^1, r)_{\Omega_2}| \le C_{12}\|\boldsymbol{z}\|_{X_h}\|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)} + C_2\|r\|_{M_h^2}\|\Pi^1\|_{L^2(\Omega_2)}$$
$$\le \frac{C_3\nu}{2}\|\boldsymbol{z}\|_{X_h}^2 + \frac{C_{12}^2}{2C_3\nu}\|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)}^2$$
$$+ \frac{C_4}{2}\|r\|_{M_h^2}^2 + \frac{C_2^2}{2C_4}\|\Pi^1\|_{L^2(\Omega_2)}^2. \tag{3.86}$$

Substituting (3.86) in (3.85) results in

$$\begin{aligned}((\mathcal{F}_1(\boldsymbol{z},r),(\boldsymbol{z},r))) &\geq \frac{1}{2}\min(C_3,C_4)\|(\boldsymbol{z},r)\|_{Y_h}^2 \\ &- \frac{C_{12}^2}{2C_3\nu}\|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)}^2 - \frac{C_2^2}{2C_4}\|\Pi^1\|_{L^2(\Omega_2)}^2 - \frac{1}{2t^1}\|\boldsymbol{U}_h^0\|_{L^2(\Omega_1)}^2.\end{aligned}$$

Therefore, choosing

$$\mathcal{R}_1 = \Big(\frac{1}{\min(C_3,C_4)}\big(\frac{1}{t^1}\|\boldsymbol{U}_h^0\|_{L^2(\Omega_1)}^2 + \frac{C_{12}^2}{C_3\nu}\|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)}^2 + \frac{C_2^2}{C_4}\|\Pi^1\|_{L^2(\Omega_2)}^2\big)\Big)^{\frac{1}{2}} \tag{3.87}$$

concludes that $((\mathcal{F}_1(\boldsymbol{z},r),(\boldsymbol{z},r))) \geq 0$ for $\|(\boldsymbol{z},r)\|_{Y_h} = \mathcal{R}_1$. This yields a solution $(\boldsymbol{U}_h^1,\Phi_h^1)$ in the ball of radius $\mathcal{R}_1$, that is, $(\boldsymbol{U}_h^1,\Phi_h^1)$ satisfies

$$\|(\boldsymbol{U}_h^1,\Phi_h^1)\|_{Y_h} \leq \mathcal{R}_1. \tag{3.88}$$

The next step, which is to show that $(\boldsymbol{U}_h^i,\Phi_h^i)$ satisfying (3.81) exists for all $i \geq 2$, follows a similar argument. So, assume that $\boldsymbol{U}_h^i$ and Φ_h^i are given for some $i \geq 1$. This time we introduce a mapping $\mathcal{F}_i : \boldsymbol{W}_h \to \boldsymbol{W}_h$ defined by

$$\begin{aligned}\forall(\boldsymbol{v},q)\in\boldsymbol{W}_h, ((\mathcal{F}_i(\boldsymbol{z},r),(\boldsymbol{v},q))) &= (\frac{2\boldsymbol{z}-2\boldsymbol{U}_h^i}{\Delta t},\boldsymbol{v})_{\Omega_1} + B([\boldsymbol{z},r];[\boldsymbol{v},q]) \\ &+ N(\boldsymbol{z},\boldsymbol{z};\boldsymbol{z},\boldsymbol{v}) - (\boldsymbol{\Psi}^{i+\frac{1}{2}},\boldsymbol{v})_{\Omega_1} - (\Pi^{i+\frac{1}{2}},q)_{\Omega_2}.\end{aligned}$$

The Riesz representation theorem applied once more shows that $\mathcal{F}_i$ is a well-defined continuous map from $\boldsymbol{W}_h$ into itself. Observe that if $(\boldsymbol{z}^*,r^*)$ is a zero of $\mathcal{F}_i$, then $(2\boldsymbol{z}^*-\boldsymbol{U}_h^i, 2r^*-\Phi_h^i)$ solves (3.81). As before, the definition gives

$$\begin{aligned}((\mathcal{F}_i(\boldsymbol{z},r),(\boldsymbol{z},r))) \geq \frac{1}{\Delta t}\|\boldsymbol{z}\|_{L^2(\Omega_1)}^2 - \frac{1}{\Delta t}\|\boldsymbol{U}_h^i\|_{L^2(\Omega_1)}^2 + C_3\nu\|\boldsymbol{z}\|_{X_h}^2 \\ + C_4\|r\|_{M_h^2}^2 - (\boldsymbol{\Psi}^{i+\frac{1}{2}},\boldsymbol{z})_{\Omega_1} - (\Pi^{i+\frac{1}{2}},r)_{\Omega_2}.\end{aligned}$$

Same inequalities used for (3.86) show that

$$\begin{aligned}|(\boldsymbol{\Psi}^{i+\frac{1}{2}},\boldsymbol{z})_{\Omega_1}| + |(\Pi^{i+\frac{1}{2}},r)_{\Omega_2}| \leq \frac{C_3\nu}{2}\|\boldsymbol{z}\|_{X_h}^2 + \frac{C_{12}^2}{2C_3\nu}\|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|_{L^2(\Omega_1)}^2 \\ + \frac{C_4}{2}\|r\|_{M_h^2}^2 + \frac{C_2^2}{2C_4}\|\Pi^{i+\frac{1}{2}}\|_{L^2(\Omega_2)}^2.\end{aligned}$$

This leads to

$$((\mathcal{F}_i(\boldsymbol{z},r),(\boldsymbol{z},r))) \geq \frac{1}{2}\min(C_3,C_4)\|(\boldsymbol{z},r)\|_{Y_h}^2 - \frac{C_{12}^2}{2C_3\nu}\|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|_{L^2(\Omega_1)}^2 - \frac{C_2^2}{2C_4}\|\Pi^{i+\frac{1}{2}}\|_{L^2(\Omega_2)}^2 - \frac{1}{\Delta t}\|\boldsymbol{U}_h^i\|_{L^2(\Omega_1)}^2.$$

Thus, if

$$\mathcal{R}_i = \Big(\frac{1}{\min(C_3,C_4)}\big(\frac{2}{\Delta t}\|\boldsymbol{U}_h^i\|_{L^2(\Omega_1)}^2 + \frac{C_{12}^2}{C_3\nu}\|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|_{L^2(\Omega_1)}^2 + \frac{C_2^2}{C_4}\|\Pi^{i+\frac{1}{2}}\|_{L^2(\Omega_2)}^2)\Big)^{\frac{1}{2}},$$

then $((\mathcal{F}_i(\boldsymbol{z},r),(\boldsymbol{z},r))) \geq 0$ whenever $\|(\boldsymbol{z},r)\|_{Y_h} = \mathcal{R}_i$. The Brouwer's fixed point theorem now gives a solution $(\boldsymbol{U}_h^{i+1},\Phi_h^{i+1})$ in the ball of radius $\mathcal{R}_i$, i.e.,

$$\|(\boldsymbol{U}_h^{i+1},\Phi_h^{i+1})\|_{Y_h} \leq \mathcal{R}_i.$$

This completes the proof of existence of $\{(\boldsymbol{U}_h^i,\Phi_h^i)\}_{i\geq 1}$ satisfying (3.80)-(3.81). The a priori estimates for $\{(\boldsymbol{U}_h^i,\Phi_h^i)\}_{i\geq 1}$ are hidden in the above proof. Indeed, choose $(\boldsymbol{v},q) = (\boldsymbol{U}_h^1,\Phi_h^1)$ in (3.80) and use (3.76), (3.75), the Cauchy-Schwarz inequality, (3.69) and (3.70) to obtain

$$(\frac{\boldsymbol{U}_h^1-\boldsymbol{U}_h^0}{t^1},\boldsymbol{U}_h^1)_{\Omega_1} + C_3\nu\|\boldsymbol{U}_h^1\|_{\boldsymbol{X}_h}^2 + C_4\|\Phi_h^1\|_{M_h^2}^2 \leq C_{12}\|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)}\|\boldsymbol{U}_h^1\|_{\boldsymbol{X}_h} + C_2\|\Pi^1\|_{L^2(\Omega_2)}\|\Phi_h^1\|_{M_h^2}.$$

Now, the Young's inequality, and the fact that $(a-b)a \geq \frac{1}{2}a^2 - \frac{1}{2}b^2$ for any $a,b\in\mathbb{R}$ leads to

$$\frac{1}{2t^1}\|\boldsymbol{U}_h^1\|_{L^2(\Omega_1)}^2 - \frac{1}{2t^1}\|\boldsymbol{U}_h^0\|_{L^2(\Omega_1)}^2 + C_3\nu\|\boldsymbol{U}_h^1\|_{\boldsymbol{X}_h}^2 + C_4\|\Phi_h^1\|_{M_h^2}^2 \leq \frac{C_{12}^2}{2C_3\nu}\|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)}^2 + \frac{C_3\nu}{2}\|\boldsymbol{U}_h^1\|_{\boldsymbol{X}_h}^2 + \frac{C_2^2}{2C_4}\|\Pi^1\|_{L^2(\Omega_2)}^2 + \frac{C_4}{2}\|\Phi_h\|_{M_h^2}^2.$$

Then, using the definition of $\|(\cdot,\cdot)\|_{\boldsymbol{Y}_h}$ and multiplying by $2t^1$ gives

$$\|\boldsymbol{U}_h^1\|_{L^2(\Omega_1)}^2 + t^1\min(C_3,C_4)\|(\boldsymbol{U}_h^1,\Phi_h^1)\|_{\boldsymbol{Y}_h}^2 \leq \|\boldsymbol{U}_h^0\|_{L^2(\Omega_1)}^2 + t^1\frac{C_{12}^2}{C_3\nu}\|\boldsymbol{\Psi}^1\|_{L^2(\Omega_1)}^2 + t^1\frac{C_2^2}{C_4}\|\Pi^1\|_{L^2(\Omega_2)}^2.$$

For the other estimate, let $(\boldsymbol{v}, q) = (\boldsymbol{U}_h^{i+\frac{1}{2}}, \Phi_h^{i+\frac{1}{2}})$ in (3.81). The results (3.76), (3.75), the Cauchy-Schwarz inequality, (3.69) and (3.70) yields

$$\begin{aligned}(\frac{\boldsymbol{U}_h^{i+1} - \boldsymbol{U}_h^i}{\Delta t}, \boldsymbol{U}_h^{i+\frac{1}{2}})_{\Omega_1} &+ C_3\nu\|\boldsymbol{U}_h^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} + C_4\|\Phi_h^{i+\frac{1}{2}}\|^2_{M_h^2} \\ &\leq C_{12}\|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|_{L^2(\Omega_1)}\|\boldsymbol{U}_h^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h} + C_2\|\Pi^{i+\frac{1}{2}}\|_{L^2(\Omega_2)}\|\Phi_h^{i+\frac{1}{2}}\|_{M_h^2}.\end{aligned}$$

As before, the Young's inequality implies

$$\begin{aligned}\frac{1}{2\Delta t}\|\boldsymbol{U}_h^{i+1}\|^2_{L^2(\Omega_1)} - \frac{1}{2\Delta t}\|\boldsymbol{U}_h^i\|^2_{L^2(\Omega_1)} + \frac{1}{2}\min(C_3, C_4)\|(\boldsymbol{U}_h^{i+\frac{1}{2}}, \Phi_h^{i+\frac{1}{2}})\|^2_{\boldsymbol{Y}_h} \\ \leq \frac{C_{12}^2}{2C_3\nu}\|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|^2_{L^2(\Omega_1)} + \frac{C_2^2}{2C_4}\|\Pi^{i+\frac{1}{2}}\|^2_{L^2(\Omega_2)}.\end{aligned}$$

Multiplying by $2\Delta t$ and summing from 1 to $m-1$ where $2 \leq m \leq N_T$ finally yields

$$\begin{aligned}&\|\boldsymbol{U}_h^m\|^2_{L^2(\Omega_1)} + \Delta t \min(C_3, C_4)\sum_{i=1}^{m-1}\|(\boldsymbol{U}_h^{i+\frac{1}{2}}, \Phi_h^{i+\frac{1}{2}})\|^2_{\boldsymbol{Y}_h} \\ &\leq \|\boldsymbol{U}_h^1\|^2_{L^2(\Omega_1)} + \Delta t\frac{C_{12}^2}{C_3\nu}\sum_{i=1}^{m-1}\|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|^2_{L^2(\Omega_1)} + \Delta t\frac{C_2^2}{C_4}\sum_{i=1}^{m-1}\|\Pi^{i+\frac{1}{2}}\|^2_{L^2(\Omega_2)} \\ &\leq \|\boldsymbol{U}_h^1\|^2_{L^2(\Omega_1)} + \Delta t\frac{C_{12}^2}{C_3\nu}\sum_{i=1}^{N_T-1}\|\boldsymbol{\Psi}^{i+\frac{1}{2}}\|^2_{L^2(\Omega_1)} + \Delta t\frac{C_2^2}{C_4}\sum_{i=1}^{N_T-1}\|\Pi^{i+\frac{1}{2}}\|^2_{L^2(\Omega_2)}.\end{aligned}$$

□

Next lemma gives the uniqueness of the solution under some condition on the data and on the time step.

Lemma 38. *Let $\mathcal{R}_1$ be defined by (3.87) and $\overline{\mathcal{C}}$ defined by (3.84). Under the following condition*

$$\nu^{3/2} > \frac{1}{C_3}(2C_5 + 2C_6 - \tilde{C}_{14}^2\tilde{C}_{12})\max\Big(\mathcal{R}_1, \frac{2^{\frac{1}{2}}\overline{\mathcal{C}}}{(\Delta t\min(C_3, C_4))^{\frac{1}{2}}}\Big),$$

there exists a unique solution $\{(\boldsymbol{U}_h^i, \Phi_h^i)\}_{i\geq 1} \subset \boldsymbol{W}_h$ satisfying (3.80)-(3.81).

Proof. Existence of a solution has already been proven and therefore, enough to show uniqueness of $(\boldsymbol{U}_h^1, \Phi_h^1)$ and $(\boldsymbol{U}_h^{i+\frac{1}{2}}, \Phi_h^{i+\frac{1}{2}})$. Assume that there are two solutions, which are denoted by $(\boldsymbol{U}_h^1, \Phi_h^1)$ and $(\tilde{\boldsymbol{U}}_h^1, \tilde{\Phi}_h^1)$. Let $\boldsymbol{w}^1 = \boldsymbol{U}_h^1 - \tilde{\boldsymbol{U}}_h^1$ and $r^1 = \Phi_h^1 - \tilde{\Phi}_h^1$. It follows from (3.80) that for all $\boldsymbol{v} \in \boldsymbol{X}_h$ and $q \in M_h^2$,

$$(\frac{\boldsymbol{w}^1}{t^1}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{w}^1, r^1]; [\boldsymbol{v}, q]) + N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{v}) - N(\tilde{\boldsymbol{U}}_h^1, \tilde{\boldsymbol{U}}_h^1; \tilde{\boldsymbol{U}}_h^1, \boldsymbol{v}) = 0.$$

Choosing $\boldsymbol{v} = \boldsymbol{w}^1$ and $q = r^1$ and using (3.75) gives

$$\frac{1}{t^1}\|\boldsymbol{w}^1\|_{L^2(\Omega_1)}^2 + C_3\nu\|\boldsymbol{w}^1\|_{X_h}^2 + C_4\|r^1\|_{M_h^2}^2 + N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1) - N(\tilde{\boldsymbol{U}}_h^1, \tilde{\boldsymbol{U}}_h^1; \tilde{\boldsymbol{U}}_h^1, \boldsymbol{w}^1) \leq 0. \quad (3.89)$$

We first consider the forms c_{NS} and d_{NS} that are included in the nonlinear term $N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1) - N(\tilde{\boldsymbol{U}}_h^1, \tilde{\boldsymbol{U}}_h^1; \tilde{\boldsymbol{U}}_h^1, \boldsymbol{w}^1)$. Adding and subtracting $c_{\text{NS}}(\tilde{\boldsymbol{U}}_h^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1)$ results in

$$c_{\text{NS}}(\boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1) - c_{\text{NS}}(\tilde{\boldsymbol{U}}_h^1; \tilde{\boldsymbol{U}}_h^1, \boldsymbol{w}^1) = c_{\text{NS}}(\boldsymbol{w}^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1) + c_{\text{NS}}(\tilde{\boldsymbol{U}}_h^1; \boldsymbol{w}^1, \boldsymbol{w}^1).$$

These terms are bounded by (3.78) and (3.88) as follows,

$$|c_{\text{NS}}(\boldsymbol{w}^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1) + c_{\text{NS}}(\tilde{\boldsymbol{U}}_h^1; \boldsymbol{w}^1, \boldsymbol{w}^1)| \leq C_5\|\boldsymbol{w}^1\|_{X_h}^2(\|\boldsymbol{U}_h^1\|_{X_h} + \|\tilde{\boldsymbol{U}}_h^1\|_{X_h}) \leq \frac{2}{\sqrt{\nu}}C_5\mathcal{R}_1\|\boldsymbol{w}^1\|_{X_h}^2.$$

The terms involving d_{NS} are bounded by Proposition 36 and (3.82),

$$|d_{\text{NS}}(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1) - d_{\text{NS}}(\tilde{\boldsymbol{U}}_h^1, \tilde{\boldsymbol{U}}_h^1; \tilde{\boldsymbol{U}}_h^1, \boldsymbol{w}^1)| \leq C_6\|\boldsymbol{w}^1\|_{X_h}^2(\|\boldsymbol{U}_h^1\|_{X_h} + \|\tilde{\boldsymbol{U}}_h^1\|_{X_h}) \leq \frac{2}{\sqrt{\nu}}C_6\mathcal{R}_1\|\boldsymbol{w}^1\|_{X_h}^2.$$

The remaining nonlinear terms in $N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{w}^1) - N(\tilde{\boldsymbol{U}}_h^1, \tilde{\boldsymbol{U}}_h^1; \tilde{\boldsymbol{U}}_h^1, \boldsymbol{w}^1)$ can be bounded by the Hölder's inequality, (3.71) and (3.88) after adding and subtracting the expression $\frac{1}{2}(\tilde{\boldsymbol{U}}_h^1 \cdot \boldsymbol{U}_h^1, \boldsymbol{w}^1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}$ as follows:

$$\begin{aligned}
&|-\frac{1}{2}(\boldsymbol{U}_h^1 \cdot \boldsymbol{U}_h^1, \boldsymbol{w}^1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\tilde{\boldsymbol{U}}_h^1 \cdot \tilde{\boldsymbol{U}}_h^1, \boldsymbol{w}^1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}| \\
&\quad = |\frac{1}{2}(\boldsymbol{w}^1 \cdot \boldsymbol{U}_h^1, \boldsymbol{w}^1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + \frac{1}{2}(\tilde{\boldsymbol{U}}_h^1 \cdot \boldsymbol{w}^1, \boldsymbol{w}^1 \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}| \\
&\quad \leq \frac{1}{2}\|\boldsymbol{w}^1\|_{L^4(\Gamma_{12})}\|\boldsymbol{w}^1\|_{L^2(\Gamma_{12})}(\|\boldsymbol{U}_h^1\|_{L^4(\Gamma_{12})} + \|\tilde{\boldsymbol{U}}_h^1\|_{L^4(\Gamma_{12})}) \\
&\qquad \leq \frac{1}{2}\tilde{C}_{14}^2\tilde{C}_{12}\|\boldsymbol{w}^1\|_{X_h}^2(\|\boldsymbol{U}_h^1\|_{X_h} + \|\tilde{\boldsymbol{U}}_h^1\|_{X_h}) \leq \frac{\tilde{C}_{14}^2\tilde{C}_{12}\mathcal{R}_1}{\sqrt{\nu}}\|\boldsymbol{w}^1\|_{X_h}^2.
\end{aligned}$$

Combining the bounds above with (3.89) finally gives

$$\frac{1}{t^1}\|\boldsymbol{w}^1\|_{L^2(\Omega_1)}^2 + C_4\|r^1\|_{M_h^2}^2 + (C_3\nu - \frac{2}{\sqrt{\nu}}\mathcal{R}_1(C_5 + C_6) - \frac{\tilde{C}_{14}^2\tilde{C}_{12}}{\sqrt{\nu}}\mathcal{R}_1)\|\boldsymbol{w}^1\|_{X_h}^2 \leq 0.$$

This yields $\boldsymbol{w}^1 = 0$, $r^1 = 0$ and hence $\tilde{\boldsymbol{U}}_h^1 = \boldsymbol{U}_h^1$, $\tilde{\Phi}_h^1 = \Phi_h^1$, if the following condition is satisfied:

$$\nu^{3/2} > \frac{1}{C_3}\mathcal{R}_1(2C_5 + 2C_6 + \tilde{C}_{14}^2\tilde{C}_{12}).$$

Next, fix $i \geq 1$ to show the uniqueness of $(\boldsymbol{U}_h^{i+1}, \Phi_h^{i+1})$. Assume that $(\boldsymbol{U}_h^i, \Phi_h^i)$ exists and is unique. As before, take the differences $\boldsymbol{w}^{i+1} = \boldsymbol{U}_h^{i+1} - \tilde{\boldsymbol{U}}_h^{i+1}$ and $r^{i+1} = \Phi_h^{i+1} - \tilde{\Phi}_h^{i+1}$. Then from (3.81), for any $\boldsymbol{v} \in \boldsymbol{V}_h$ and for any $q \in M_h^2$,

$$\begin{aligned}
(\frac{\boldsymbol{w}^{i+1}}{\Delta t}, \boldsymbol{v}) + B([\boldsymbol{w}^{i+\frac{1}{2}}, r^{i+\frac{1}{2}}]; [\boldsymbol{v}, q]) + N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{v}) \\
- N(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}, \tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}; \tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}, \boldsymbol{v}) = 0.
\end{aligned}$$

Choosing $\boldsymbol{v} = \boldsymbol{w}^{i+\frac{1}{2}}$, $q = r^{i+\frac{1}{2}}$ and using (3.75) gives

$$\begin{aligned}
&\frac{1}{\Delta t}\|\boldsymbol{w}^{i+1}\|_{L^2(\Omega_1)}^2 + C_3\nu\|\boldsymbol{w}^{i+\frac{1}{2}}\|_{X_h}^2 + C_4\|r^{i+\frac{1}{2}}\|_{M_h^2}^2 \\
&\quad + N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{w}^{i+\frac{1}{2}}) - N(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}, \tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}; \tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}, \boldsymbol{w}^{i+\frac{1}{2}}) \leq 0. \quad (3.90)
\end{aligned}$$

As before, we deal with the nonlinear terms by adding and substracting suitable terms. For the terms which involve the form c_{NS}, add and subtract $c_{\mathrm{NS}}(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{w}^{i+\frac{1}{2}})$

to get

$$c_{\mathrm{NS}}(\boldsymbol{U}_h^{i+\frac{1}{2}};\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}) - c_{\mathrm{NS}}(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}};\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}) \\ = c_{\mathrm{NS}}(\boldsymbol{w}^{i+\frac{1}{2}};\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}) + c_{\mathrm{NS}}(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}};\boldsymbol{w}^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}).$$

By (3.78) and the bound (3.83),

$$|c_{\mathrm{NS}}(\boldsymbol{w}^{i+\frac{1}{2}};\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}) + c_{\mathrm{NS}}(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}};\boldsymbol{w}^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}})| \\ \leq C_5\|\boldsymbol{w}^{i+\frac{1}{2}}\|_{X_h}^2(\|\boldsymbol{U}_h^{i+\frac{1}{2}}\|_{X_h} + \|\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}\|_{X_h}) \\ \leq \frac{2^{3/2}C_5\overline{C}}{(\nu\Delta t\min(C_3,C_4))^{\frac{1}{2}}}\|\boldsymbol{w}^{i+\frac{1}{2}}\|_{X_h}^2.$$

The terms involving d_{NS} are bounded by Proposition 36 and the bound (3.83):

$$|d_{\mathrm{NS}}(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{U}_h^{i+\frac{1}{2}};\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}) - d_{\mathrm{NS}}(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}},\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}};\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}})| \\ \leq C_6\|\boldsymbol{w}^{i+\frac{1}{2}}\|_{X_h}^2(\|\boldsymbol{U}_h^{i+\frac{1}{2}}\|_{X_h} + \|\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}\|_{X_h}) \leq \frac{2^{3/2}C_6\overline{C}}{(\nu\Delta t\min(C_3,C_4))^{\frac{1}{2}}}\|\boldsymbol{w}^{i+\frac{1}{2}}\|_{X_h}^2.$$

Lastly, we bound the nonlinear interface terms by adding and subtracting the form $\frac{1}{2}(\tilde{\boldsymbol{U}}_h{}^{i+\frac{1}{2}}\cdot\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$ and using the Hölder's inequality, (3.71) and (3.83):

$$|-\frac{1}{2}(\boldsymbol{w}^{i+\frac{1}{2}}\cdot\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} - \frac{1}{2}(\tilde{\boldsymbol{U}}_h^{i+\frac{1}{2}}\cdot\boldsymbol{w}^{i+\frac{1}{2}},\boldsymbol{w}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}| \\ \leq \frac{2^{\frac{1}{2}}\tilde{C}_{14}^2\tilde{C}_{12}\overline{C}}{(\nu\Delta t\min(C_3,C_4))^{\frac{1}{2}}}\|\boldsymbol{w}^{i+\frac{1}{2}}\|_{X_h}^2.$$

Combining the bounds above with (3.90) leads to

$$\frac{1}{\Delta t}\|\boldsymbol{w}^{i+1}\|_{L^2(\Omega_1)}^2 + C_4\|r^{i+\frac{1}{2}}\|_{M_h^2}^2 + (C_3\nu - \frac{2^{\frac{1}{2}}\overline{C}}{(\nu\Delta t\min(C_3,C_4))^{\frac{1}{2}}}(2C_5+2C_6) \\ - \frac{2^{\frac{1}{2}}\tilde{C}_{14}^2\tilde{C}_{12}\overline{C}}{(\nu\Delta t\min(C_3,C_4))^{\frac{1}{2}}})\|\boldsymbol{w}^{i+\frac{1}{2}}\|_{X_h}^2 \leq 0.$$

Therefore, if

$$\nu^{3/2} > \frac{2^{\frac{1}{2}}\overline{C}}{C_3(\Delta t\min(C_3,C_4))^{\frac{1}{2}}}(2C_5+2C_6-\tilde{C}_{14}^2\tilde{C}_{12}),$$

then the functions $\boldsymbol{w}^{i+\frac{1}{2}}$ and $r^{i+\frac{1}{2}}$ vanish. Observing that $\boldsymbol{w}^{i+\frac{1}{2}} = 0.5\boldsymbol{w}^{i+1}$ and $r^{i+\frac{1}{2}} = 0.5r^{i+1}$ concludes that $\boldsymbol{w}^{i+1} = \boldsymbol{0}$ and $r^{i+1} = 0$ for all $i \geq 1$. □

This completes the proof of existence and uniqueness of the solution $\{(\boldsymbol{U}_h^i, \Phi_h^i)\}_{i\geq 1}$ to the problem restricted to the space $\boldsymbol{V}_h$. Existence and uniqueness of the Navier-Stokes pressure P_h^i, for which $\{(\boldsymbol{U}_h^i, P_h^i, \Phi_h^i)\}_{i\geq 1}$ is a solution of (3.65)-(3.68), is a consequence of the following inf-sup condition: There exists a positive constant β^* independent of h such that

$$\inf_{q\in M_h^1} \sup_{\boldsymbol{v}\in \boldsymbol{X}_h} \frac{b_{\mathrm{NS}}(\boldsymbol{v}, q)}{\|\boldsymbol{v}\|_{\boldsymbol{X}_h}\|q\|_{M_h^1}} \geq \beta^*. \tag{3.91}$$

The proof of this inf-sup condition can be found in [72, 77] and follows a standard argument found, for instance, in [64]. Now that the existence and uniqueness of the numerical solution is established, the next step is to show that if the scheme converges.

3.2.5 Error Analysis

This section derives some error estimates. Decompose the error into an approximation error and a numerical error. For any time $t \geq 0$, let $\tilde{\boldsymbol{u}}(t) \in \boldsymbol{X}_h$ be an approximation of $\boldsymbol{u}(t)$ satisfying

$$b_{\mathrm{NS}}(\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t), q) = 0, \quad \forall q \in M_h^1. \tag{3.92}$$

Existence of such an approximation is given in [72, 77]. Let $\tilde{p}(t) \in M_h^1$ be the L^2-projection of $p(t)$, i.e.,

$$(p(t) - \tilde{p}(t), q)_{\Omega_1} = 0, \quad \forall q \in M_h^1. \tag{3.93}$$

Finally, let $\tilde{\varphi}(t) \in M_h^2$ be an approximation of $\varphi(t)$. In addition, assume that the approximation errors are optimal, that is, for any time $t \geq 0$:

$$\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|_{\boldsymbol{X}_h} \leq Ch^{k_1}|\boldsymbol{u}(t)|_{H^{k_1+1}(\Omega_1)}, \tag{3.94}$$

$$\|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|_{L^2(\Omega_1)} \leq Ch^{k_1+1}|\boldsymbol{u}(t)|_{H^{k_1+1}(\Omega_1)}, \tag{3.95}$$

$$i = 0, 1, \quad \left(\sum_{E \in \mathcal{E}_h^1} \|\nabla^i p(t) - \nabla^i \tilde{p}(t)\|^2_{L^2(E)}\right)^{\frac{1}{2}} \leq Ch^{k_1-i}|p(t)|_{H^{k_1}(\Omega_1)}, \tag{3.96}$$

$$i = 0, 1, \quad \left(\sum_{E \in \mathcal{E}_h^2} \|\nabla^i \varphi(t) - \nabla^i \tilde{\varphi}(t)\|^2_{L^2(E)}\right)^{\frac{1}{2}} \leq Ch^{k_2+1-i}|\varphi(t)|_{H^{k_2+1}(\Omega_2)}. \tag{3.97}$$

Using the triangle inequality and the approximation property (3.94), there is a constant $C_a > 0$ independent of h and ν such that

$$\|\tilde{\boldsymbol{u}}(t)\|_{\boldsymbol{X}_h} \leq \|\boldsymbol{u}(t) - \tilde{\boldsymbol{u}}(t)\|_{\boldsymbol{X}_h} + \|\boldsymbol{u}(t)\|_{\boldsymbol{X}_h} \leq C_a|\boldsymbol{u}(t)|_{H^1(\Omega_1)}. \tag{3.98}$$

In this section, C is a positive generic constant, which may have a different value at different places, independent of h and ν. Denote $\boldsymbol{u}^i = \boldsymbol{u}(t^i)$, $\tilde{\boldsymbol{u}}^i = \tilde{\boldsymbol{u}}(t^i)$, $\varphi^i = \varphi(t^i)$ and $\tilde{\varphi}^i = \tilde{\varphi}(t^i)$ and write for any $i \geq 0$:

$$\boldsymbol{U}_h^i - \boldsymbol{u}^i = \boldsymbol{\chi}^i - \boldsymbol{\eta}^i, \quad \text{where} \quad \boldsymbol{\chi}^i = \boldsymbol{U}_h^i - \tilde{\boldsymbol{u}}^i, \quad \boldsymbol{\eta}^i = \boldsymbol{u}^i - \tilde{\boldsymbol{u}}^i,$$

$$\Phi_h^i - \varphi^i = \xi^i - \zeta^i, \quad \text{where} \quad \xi^i = \Phi_h^i - \tilde{\varphi}^i, \quad \zeta^i = \varphi^i - \tilde{\varphi}^i.$$

Using these decompositions, it is enough to analyze $\boldsymbol{\chi}^i$ and ξ^i as the rest follows from the triangle inequality and the approximation properties. The following theorem states error bounds of the quantities $\boldsymbol{\chi}^i$ and ξ^i.

Theorem 39. *Assume that the weak solution of $(\boldsymbol{u}, p, \varphi)$ of problem (P) satisfies $\boldsymbol{u} \in L^2(0,T;H^{k_1+1}(\Omega_1)^2) \cap L^\infty(0,T;H^1(\Omega_1)^2)$, $p \in L^2(0,T;H^{k_1}(\Omega_1))$, and $\varphi \in L^2(0,T;H^{k_2+1}(\Omega_2))$. Further, if $\boldsymbol{u}_0 \in H^{k_1+1}(\Omega_1)^2$, $\boldsymbol{u}_t \in L^\infty(0,T;H^{k_1}(\Omega_1)^2)$,*

$\boldsymbol{u}_{ttt} \in L^\infty(0,T;L^2(\Omega_1)^2)$ and $\nu > \frac{4C_4}{C_3}(C_5 + \frac{3}{2}\tilde{C}_{12}\tilde{C}_{14}^2)\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega_1)^2)}$, then, there exists a constant C independent of h, t^1, Δt and ν such that

$$\|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} + \frac{C_3\nu}{2}t^1\|\boldsymbol{\chi}^1\|^2_{\boldsymbol{X}_h} + C_4 t^1\|\xi^1\|^2_{M_h^2} \leq Ch^{2k_1+2}|\boldsymbol{u}_0|^2_{H^{k_1+1}(\Omega_1)}$$
$$+ C(1+\nu+\nu^{-1})t^1h^{2k_1}|\boldsymbol{u}^1|^2_{H^{k_1+1}(\Omega_1)} + C(1+\nu^{-1})t^1h^{2k_2}|\varphi^1|^2_{H^{k_2+1}(\Omega_2)}$$
$$+ C\nu^{-1}t^1h^{2k_1}|p^1|^2_{H^{k_1}(\Omega_1)} + C\nu^{-1}(t^1)^3\|\boldsymbol{u}_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega_1)^2)}$$
$$+ C\nu^{-1}t^1h^{2k_1}\|\boldsymbol{u}_t\|^2_{L^\infty(0,T;H^{k_1}(\Omega_1)^2)}. \quad (3.99)$$

and for any $m \geq 2$,

$$\|\boldsymbol{\chi}^m\|^2_{L^2(\Omega_1)} + \frac{C_3\nu}{2}\Delta t\sum_{i=1}^{m-1}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} + C_4\Delta t\sum_{i=1}^{m-1}\|\xi^{i+\frac{1}{2}}\|^2_{M_h^2} \leq \|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)}$$
$$+ C(\nu^{-1}+\nu+1)h^{2k_1}|\boldsymbol{u}|^2_{L^2(0,T;H^{k_1+1}(\Omega_1)^2)} + C(\nu^{-1}+1)h^{2k_2}|\varphi|^2_{L^2(0,T;H^{k_2+1}(\Omega_2))}$$
$$+ C\nu^{-1}h^{2k_1}|p|^2_{L^2(0,T;H^{k_1}(\Omega_1))} + C\nu^{-1}h^{2k_1}\|\boldsymbol{u}_t\|^2_{L^\infty(0,T;L^2(\Omega_1)^2)}$$
$$+ C\nu^{-1}\Delta t^4\left(\|\boldsymbol{u}_t\|^2_{L^\infty(0,T;H^1(\Omega_1)^2)} + \|\boldsymbol{u}_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega_1)^2)}\right). \quad (3.100)$$

Proof. From the consistency result of Lemma 29, for any $i \geq 1$, for all $\boldsymbol{v} \in \boldsymbol{X}_h$ and $q \in M_h^2$, the exact solution satisfies

$$(\boldsymbol{u}_t^{i+\frac{1}{2}}, \boldsymbol{v})_{\Omega_1} + B([\boldsymbol{u}^{i+\frac{1}{2}}, \varphi^{i+\frac{1}{2}}];[\boldsymbol{v},q]) + \frac{1}{2}N(\boldsymbol{u}^{i+1},\boldsymbol{u}^{i+1};\boldsymbol{u}^{i+1},\boldsymbol{v})$$
$$+ \frac{1}{2}N(\boldsymbol{u}^i,\boldsymbol{u}^i;\boldsymbol{u}^i,\boldsymbol{v}) + b_{\mathrm{NS}}(\boldsymbol{v},p^{i+\frac{1}{2}}) = (\boldsymbol{\Psi}^{i+\frac{1}{2}},\boldsymbol{v})_{\Omega_1} + (\Pi^{i+\frac{1}{2}},q)_{\Omega_2}, \quad (3.101)$$

Subtract (3.101) from (3.67). Then, add and substract terms with $\tilde{\boldsymbol{u}}$ and $\tilde{\varphi}_m$ to get

$$(\frac{\boldsymbol{\chi}^{i+1}-\boldsymbol{\chi}^i}{\Delta t},\boldsymbol{v})_{\Omega_1} + B([\boldsymbol{\chi}^{i+\frac{1}{2}},\xi^{i+\frac{1}{2}}],[\boldsymbol{v},q]) + b_{\mathrm{NS}}(\boldsymbol{v},P_h^{i+\frac{1}{2}})$$
$$+ N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{U}_h^{i+\frac{1}{2}};\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{v}) = (\boldsymbol{u}_t^{i+\frac{1}{2}},\boldsymbol{v})_{\Omega_1} - (\frac{\tilde{\boldsymbol{u}}^{i+1}-\tilde{\boldsymbol{u}}^i}{\Delta t},\boldsymbol{v})_{\Omega_1} + b_{\mathrm{NS}}(\boldsymbol{v},p^{i+\frac{1}{2}})$$
$$+ B([\boldsymbol{\eta}^{i+\frac{1}{2}},\zeta^{i+\frac{1}{2}}],[\boldsymbol{v},q]) + \frac{1}{2}N(\boldsymbol{u}^{i+1},\boldsymbol{u}^{i+1};\boldsymbol{u}^{i+1},\boldsymbol{v}) + \frac{1}{2}N(\boldsymbol{u}^i,\boldsymbol{u}^i;\boldsymbol{u}^i,\boldsymbol{v}). \quad (3.102)$$

Choosing $\boldsymbol{v} = \boldsymbol{\chi}^{i+\frac{1}{2}}$ and $q = \xi^{i+\frac{1}{2}}$ in (3.102) and using (3.75) yields:

$$\begin{aligned}\frac{1}{2\Delta t}(\|\boldsymbol{\chi}^{i+1}\|^2_{L^2(\Omega_1)} - \|\boldsymbol{\chi}^{i}\|^2_{L^2(\Omega_1)}) + C_3\nu\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} + C_4\|\xi^{i+\frac{1}{2}}\|^2_{M_h^2} \\ \leq -N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) + \frac{1}{2}N(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+\frac{1}{2}}) \\ + \frac{1}{2}N(\boldsymbol{u}^{i}, \boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+\frac{1}{2}}) + (\boldsymbol{u}_t^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}})_{\Omega_1} - (\frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^{i}}{\Delta t}, \boldsymbol{\chi}^{i+\frac{1}{2}})_{\Omega_1} \\ + B([\boldsymbol{\eta}^{i+\frac{1}{2}}, \zeta^{i+\frac{1}{2}}], [\boldsymbol{\chi}^{i+\frac{1}{2}}, \xi^{i+\frac{1}{2}}]) + b_{\mathrm{NS}}(\boldsymbol{\chi}^{i+\frac{1}{2}}, p^{i+\frac{1}{2}} - P_h^{i+\frac{1}{2}}). \quad (3.103)\end{aligned}$$

First, consider the nonlinear terms in (3.103).

$$\begin{aligned}\mathcal{N} = N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) \\ - \frac{1}{2}N(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+\frac{1}{2}}) - \frac{1}{2}N(\boldsymbol{u}^{i}, \boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+\frac{1}{2}}). \quad (3.104)\end{aligned}$$

Because the exact solution is continuous,

$$d_{\mathrm{NS}}(\boldsymbol{u}^{i}, \boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+\frac{1}{2}}) = d_{\mathrm{NS}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+\frac{1}{2}}) = 0$$

So, these can be replaced in (3.104) by the terms $d_{\mathrm{NS}}(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+\frac{1}{2}})$ and $d_{\mathrm{NS}}(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+\frac{1}{2}})$, which are also identically zero. Thus,

$$\begin{aligned}\mathcal{N} = N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) \\ - \frac{1}{2}N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+1}; \boldsymbol{u}^{i+1}, \boldsymbol{\chi}^{i+\frac{1}{2}}) - \frac{1}{2}N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i}; \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+\frac{1}{2}}).\end{aligned}$$

Manipulating these nonlinear terms by adding and subtracting $N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \tilde{\boldsymbol{u}}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}})$, $N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \tilde{\boldsymbol{u}}^{i+\frac{1}{2}}; \tilde{\boldsymbol{u}}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}})$, $N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+\frac{1}{2}}; \tilde{\boldsymbol{u}}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}})$ and $N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+\frac{1}{2}}; \boldsymbol{u}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}})$ leads to the following expression:

$$\begin{aligned}\mathcal{N} = N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{U}_h^{i+\frac{1}{2}}; \boldsymbol{\chi}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) + N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}; \tilde{\boldsymbol{u}}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) \\ - N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{\eta}^{i+\frac{1}{2}}; \tilde{\boldsymbol{u}}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) - N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+\frac{1}{2}}; \boldsymbol{\eta}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) \\ - \frac{1}{4}N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}; \boldsymbol{u}^{i+1} - \boldsymbol{u}^{i}, \boldsymbol{\chi}^{i+\frac{1}{2}}).\end{aligned}$$

Now applying (3.76) gives the following equation:

$$\begin{aligned}\mathcal{N} = \frac{1}{2}\sum_{E\in\mathcal{E}_h^1} \| |\{\boldsymbol{U}_h^{i+\frac{1}{2}}\}\cdot\boldsymbol{n}_E|^{\frac{1}{2}}[\boldsymbol{\chi}^{i+\frac{1}{2}}]\|^2_{L^2(\partial E_-(\boldsymbol{U}_h^{i+\frac{1}{2}})\setminus\partial\Omega_1)} \\ + \||\boldsymbol{U}_h^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{\Omega_1}|^{\frac{1}{2}}\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{L^2(\Gamma_-(\boldsymbol{U}_h^{i+\frac{1}{2}}))} + \frac{1}{2}(\boldsymbol{U}_h^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12},\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{\chi}^{i+\frac{1}{2}})_{\Gamma_{12}} \\ - \frac{1}{2}(\boldsymbol{U}_h^{i+\frac{1}{2}}\cdot\boldsymbol{\chi}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}};\tilde{\boldsymbol{u}}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}}) \\ - N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{\eta}^{i+\frac{1}{2}};\tilde{\boldsymbol{u}}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}}) - N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{u}^{i+\frac{1}{2}};\boldsymbol{\eta}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}}) \\ - \frac{1}{4}N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{u}^{i+1}-\boldsymbol{u}^i;\boldsymbol{u}^{i+1}-\boldsymbol{u}^i,\boldsymbol{\chi}^{i+\frac{1}{2}}). \quad (3.105)\end{aligned}$$

The first two terms of $\mathcal{N}$ are positive. So, it suffices to bound the remaining terms. Rewrite the third and the fourth terms in (3.105) and apply the Hölder's inequality, (3.71), (3.83) and (3.98). This gives

$$\begin{aligned}&|\frac{1}{2}(\boldsymbol{U}_h^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12},\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{\chi}^{i+\frac{1}{2}})_{\Gamma_{12}} - \frac{1}{2}(\boldsymbol{U}_h^{i+\frac{1}{2}}\cdot\boldsymbol{\chi}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}| \\ &\quad = |\frac{1}{2}(\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12},\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{\chi}^{i+\frac{1}{2}})_{\Gamma_{12}} - \frac{1}{2}(\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\cdot\boldsymbol{\chi}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}| \\ &\quad \le \frac{1}{2}\|\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\|_{L^2(\Gamma_{12})}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{L^4(\Gamma_{12})} + \frac{1}{2}\|\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\|_{L^4(\Gamma_{12})}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{L^4(\Gamma_{12})}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{L^2(\Gamma_{12})} \\ &\quad \le \tilde{C}_{12}\tilde{C}_{14}^2\|\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\|_{X_h}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} \\ &\quad \le C_a\tilde{C}_{12}\tilde{C}_{14}^2\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega_1))}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h}.\end{aligned}$$

Applying (3.78), the Hölder's inequality, (3.71) and (3.98), the fifth term in (3.105) can be bounded by

$$\begin{aligned}|N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}};\tilde{\boldsymbol{u}}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})| &\le (C_5+\frac{1}{2}\tilde{C}_{12}\tilde{C}_{14}^2)\|\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} \\ &\le C_a(C_5+\frac{1}{2}\tilde{C}_{12}\tilde{C}_{14}^2)\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega_1)^2)}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h}.\end{aligned}$$

Next, consider the sixth term $N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{\eta}^{i+\frac{1}{2}};\tilde{\boldsymbol{u}}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})$ in (3.105) and analyze the $(c_{\rm NS}+d_{\rm NS})$ term and the interface term, separately. From [72] (see Remark 6.5), Korn's inequality [74] and (3.69),

$$c_{\rm NS}(\boldsymbol{\eta}^{i+\frac{1}{2}};\tilde{\boldsymbol{u}}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})+d_{\rm NS}(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{\eta}^{i+\frac{1}{2}};\tilde{\boldsymbol{u}}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})$$
$$\leq C\|\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}(\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}+\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^4(\Omega_1)}).$$

The interface term is bounded by (3.71) and the Hölder's inequality.

$$\frac{1}{2}(\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\cdot\boldsymbol{\eta}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}\leq\frac{1}{2}\tilde{C}_{14}\tilde{C}_{12}\|\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^4(\Gamma_{12})}.$$

Combining the bounds above and using (3.98), the sixth term in (3.105) gives

$$N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{\eta}^{i+\frac{1}{2}};\tilde{\boldsymbol{u}}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})$$
$$\leq C\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}(\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}+\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^4(\Gamma_{12})}+\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^4(\Omega_1)})$$
$$\leq\nu\delta\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h}+\frac{C}{\nu\delta}(\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h}+\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|^2_{L^4(\Gamma_{12})}+\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|^2_{L^4(\Omega_1)}),$$

where δ is any positive constant (by the Young's inequality) and C is a constant independent of h and ν but dependent on $|\boldsymbol{u}^{i+\frac{1}{2}}|_{H^1(\Omega_1)}$. Similarly the terms $c_{\rm NS}+d_{\rm NS}$ in the expression $N(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{u}^{i+\frac{1}{2}};\boldsymbol{\eta}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})$ are bounded by Remark 6.5 of [72], Korn's inequality [74], (3.69) and from a Sobolev imbedding as follows:

$$c_{\rm NS}(\boldsymbol{u}^{i+\frac{1}{2}};\boldsymbol{\eta}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})+d_{\rm NS}(\boldsymbol{U}_h^{i+\frac{1}{2}},\boldsymbol{u}^{i+\frac{1}{2}};\boldsymbol{\eta}^{i+\frac{1}{2}},\boldsymbol{\chi}^{i+\frac{1}{2}})$$
$$\leq C\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}|\boldsymbol{u}^{i+\frac{1}{2}}|_{H^1(\Omega_1)}\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}.$$

The associated interface term is bounded using (3.71), the Hölder's inequality and a trace inequality,

$$|\frac{1}{2}(\boldsymbol{\eta}^{i+\frac{1}{2}}\cdot\boldsymbol{\chi}^{i+\frac{1}{2}},\boldsymbol{u}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}|\leq C|\boldsymbol{u}^{i+\frac{1}{2}}|_{H^1(\Omega_1)}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^4(\Gamma_{12})}.$$

The bounds above, for some constant C independent of h and ν but dependent on

$|\boldsymbol{u}^{i+\frac{1}{2}}|_{H^1(\Omega_1)}$, yields

$$N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+\frac{1}{2}}; \boldsymbol{\eta}^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}}) \leq C\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}(\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h} + \|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^4(\Gamma_{12})})$$
$$\leq \nu\delta\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} + \frac{C}{\nu\delta}(\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} + \|\boldsymbol{\eta}^{i+\frac{1}{2}}\|^2_{L^4(\Gamma_{12})}).$$

The term $N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i; \boldsymbol{u}^{i+1} - \boldsymbol{u}^i, \boldsymbol{\chi}^{i+\frac{1}{2}})$ simplifies to

$$N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i; \boldsymbol{u}^{i+1} - \boldsymbol{u}^i, \boldsymbol{\chi}^{i+\frac{1}{2}})$$
$$= \sum_{E\in\mathcal{E}_h^1} ((\boldsymbol{u}^{i+1} - \boldsymbol{u}^i)\cdot\nabla(\boldsymbol{u}^{i+1} - \boldsymbol{u}^i), \boldsymbol{\chi}^{i+\frac{1}{2}})_E - \frac{1}{2}((\boldsymbol{u}^{i+1} - \boldsymbol{u}^i)\cdot(\boldsymbol{u}^{i+1} - \boldsymbol{u}^i), \boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$$
$$\leq C\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}\|\nabla(\boldsymbol{u}^{i+1} - \boldsymbol{u}^i)\|^2_{L^2(\Omega_1)}$$

from a Sobolev imbedding, a trace inequality and the bound (3.71). From a Taylor expansion,

$$\boldsymbol{u}^{i+1} - \boldsymbol{u}^i = \Delta t\boldsymbol{u}_t(\tilde{t}^i) \text{ for some } \tilde{t}^i \in (t^i, t^{i+1}).$$

Thus,

$$N(\boldsymbol{U}_h^{i+\frac{1}{2}}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i; \boldsymbol{u}^{i+1} - \boldsymbol{u}^i, \boldsymbol{\chi}^{i+\frac{1}{2}}) \leq C\Delta t^2\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}\|\nabla\boldsymbol{u}_t(\tilde{t}^i)\|^2_{L^2(\Omega_1)}$$
$$\leq \nu\delta\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} + \frac{C}{\nu\delta}\Delta t^4\|\nabla\boldsymbol{u}_t(\tilde{t}^i)\|^4_{L^2(\Omega_1)}.$$

Next, consider the terms

$$\mathcal{D} = (\boldsymbol{u}_t^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}})_{\Omega_1} - (\frac{\tilde{\boldsymbol{u}}^{i+1} - \tilde{\boldsymbol{u}}^i}{\Delta t}, \boldsymbol{\chi}^{i+\frac{1}{2}})_{\Omega_1}$$
$$= (\boldsymbol{u}_t^{i+\frac{1}{2}} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t}, \boldsymbol{\chi}^{i+\frac{1}{2}})_{\Omega_1} + (\frac{\boldsymbol{\eta}^{i+1} - \boldsymbol{\eta}^i}{\Delta t}, \boldsymbol{\chi}^{i+\frac{1}{2}})_{\Omega_1}.$$

Again a Taylor expansion implies the existence of some $t_1^i, t_2^i \in (t^i, t^{i+1})$, such that,

$$\boldsymbol{u}_t^{i+\frac{1}{2}} - \frac{\boldsymbol{u}^{i+1} - \boldsymbol{u}^i}{\Delta t} = \boldsymbol{u}_{ttt}(t_1^i)\frac{\Delta t^2}{8} - \boldsymbol{u}_{ttt}(t_2^i)\frac{\Delta t^2}{24}.$$

Then, the Cauchy-Schwarz and the Young's inequalities and the bound (3.69), for any $\delta_1 > 0$, give

$$|\mathcal{D}| \leq \nu\delta \|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}^2 + \frac{C}{\nu\delta}\big(\Delta t^4 \sum_{\theta=1}^{2} \|\boldsymbol{u}_{ttt}(t_\theta^i)\|_{L^2(\Omega_1)}^2 + \frac{1}{\Delta t^2}\|\boldsymbol{\eta}^{i+1} - \boldsymbol{\eta}^i\|_{L^2(\Omega_1)}^2\big).$$

The interface terms in $B([\boldsymbol{\eta}^{i+\frac{1}{2}}, \zeta^{i+\frac{1}{2}}], [\boldsymbol{\chi}^{i+\frac{1}{2}}, \xi^{i+\frac{1}{2}}])$ are bounded as follows by using (3.71) and (3.72):

$$\begin{aligned}
&|(\zeta^{i+\frac{1}{2}}, \boldsymbol{\chi}^{i+\frac{1}{2}} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} - (\boldsymbol{\eta}^{i+\frac{1}{2}} \cdot \boldsymbol{n}_{12}, \xi^{i+\frac{1}{2}})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{\eta}^{i+\frac{1}{2}} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{\chi}^{i+\frac{1}{2}} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}| \\
&\quad \leq \tilde{C}_{12}\|\zeta^{i+\frac{1}{2}}\|_{L^2(\Gamma_{12})}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h} + \tilde{C}_{22}\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^2(\Gamma_{12})}\|\xi^{i+\frac{1}{2}}\|_{M_h^2} \\
&\qquad + G\tilde{C}_{12}\sqrt{\lambda_{max}}\|\boldsymbol{\eta}^{i+\frac{1}{2}}\|_{L^2(\Gamma_{12})}\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}.
\end{aligned}$$

The remaining terms in $B([\boldsymbol{\eta}^{i+\frac{1}{2}}, \zeta^{i+\frac{1}{2}}], [\boldsymbol{\chi}^{i+\frac{1}{2}}, \xi^{i+\frac{1}{2}}])$ are bounded using standard techniques to discontinuous Galerkin methods. Details can be found in [14, 72]. Therefore, from the approximation results (3.94) and (3.97), Young's inequality implies for any positive constants δ and $\tilde{\delta}$,

$$\begin{aligned}
B([\boldsymbol{\eta}^{i+\frac{1}{2}}, \zeta^{i+\frac{1}{2}}], [\boldsymbol{\chi}^{i+\frac{1}{2}}, \xi^{i+\frac{1}{2}}]) &\leq \nu\delta\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}^2 + \tilde{\delta}\|\xi^{i+\frac{1}{2}}\|_{M_h^2}^2 \\
&\quad + C(\frac{\nu}{\delta} + \frac{1}{\tilde{\delta}} + 1)h^{2k_1}|\boldsymbol{u}^{i+\frac{1}{2}}|_{H^{k_1+1}(\Omega_1)}^2 + C(\frac{1}{\tilde{\delta}} + \frac{1}{\nu\delta})h^{2k_2}|\varphi^{i+\frac{1}{2}}|_{H^{k_2+1}(\Omega_2)}^2.
\end{aligned}$$

Finally it remains to bound $b_{\mathrm{NS}}(\boldsymbol{\chi}^{i+\frac{1}{2}}, p^{i+\frac{1}{2}} - P_h^{i+\frac{1}{2}})$. Start by writing

$$b_{\mathrm{NS}}(\boldsymbol{\chi}^{i+\frac{1}{2}}, p^{i+\frac{1}{2}} - P_h^{i+\frac{1}{2}}) = b_{\mathrm{NS}}(\boldsymbol{\chi}^{i+\frac{1}{2}}, p^{i+\frac{1}{2}} - \tilde{p}_1^{i+\frac{1}{2}}) + b_{\mathrm{NS}}(\boldsymbol{\chi}^{i+\frac{1}{2}}, \tilde{p}^{i+\frac{1}{2}} - P_h^{i+\frac{1}{2}}).$$

The second term vanishes because of (3.68) and since $b(\tilde{\boldsymbol{u}}^{i+\frac{1}{2}}, q) = b(\boldsymbol{u}^{i+\frac{1}{2}}, q) = 0$ for any $q \in M_h^1$. The first term is reduced to

$$b_{\mathrm{NS}}(\boldsymbol{\chi}^{i+\frac{1}{2}}, p^{i+\frac{1}{2}} - \tilde{p}_1^{i+\frac{1}{2}}) = \sum_{e \in \Gamma_h^1}([\boldsymbol{\chi}^{i+\frac{1}{2}} \cdot \boldsymbol{n}_e], \{p^{i+\frac{1}{2}} - \tilde{p}^{i+\frac{1}{2}}\})_e,$$

as $\tilde{p}$ is the L^2-projection of p and as $\nabla\cdot\boldsymbol{\chi}^{i+\frac{1}{2}}\in M_h^1$. The Cauchy-Schwarz inequality, a trace inequality and the approximation result (3.96) give

$$\sum_{e\in\Gamma_h^1}([\boldsymbol{\chi}^{i+\frac{1}{2}}\cdot\boldsymbol{n}_e],\{p^{i+\frac{1}{2}}-\tilde{p}^{i+\frac{1}{2}}\})_e$$

$$\leq \nu\delta\sum_{e\in\Gamma_h^1}\frac{\sigma_e}{|e|}\|[\boldsymbol{\chi}^{i+\frac{1}{2}}]\|_{L^2(e)}^2+\frac{C}{\nu\delta}\sum_{e\in\Gamma_h^1}\frac{|e|}{\sigma_e}\|\{p^{i+\frac{1}{2}}-\tilde{p}^{i+\frac{1}{2}}\}\|_{L^2(e)}^2$$

$$\leq \nu\delta\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}^2+\frac{C}{\nu\delta}h^{2k_1}|p^{i+\frac{1}{2}}|_{H^{k_1}(\Omega_1)}^2.$$

Then, combine the bounds above with (3.103) and choose $\delta=\frac{C_3}{12}$ and $\tilde{\delta}=\frac{C_4}{2}$. The approximation result (3.94) yields

$$\begin{aligned}
&\frac{1}{2\Delta t}(\|\boldsymbol{\chi}^{i+1}\|_{L^2(\Omega_1)}^2-\|\boldsymbol{\chi}^{i}\|_{L^2(\Omega_1)}^2)+\frac{C_4}{2}\|\xi^{i+\frac{1}{2}}\|_{M_h^2}^2\\
&\quad+\Big(\frac{C_3}{2}\nu-C_a(C_5+\frac{3}{2}\tilde{C}_{12}\tilde{C}_{14}^2)\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega_1)^2)}\Big)\|\boldsymbol{\chi}^{i+\frac{1}{2}}\|_{\boldsymbol{X}_h}^2\\
&\leq C\Big((\nu^{-1}+\nu+1)h^{2k_1}|\boldsymbol{u}^{i+\frac{1}{2}}|_{H^{k_1+1}(\Omega_1)}^2+(\nu^{-1}+1)h^{2k_2}|\varphi^{i+\frac{1}{2}}|_{H^{k_2+1}(\Omega_2)}^2\\
&\quad+\nu^{-1}h^{2k_1}|p^{i+\frac{1}{2}}|_{H^{k_1}(\Omega_1)}^2+\nu^{-1}\Delta t^4(\|\boldsymbol{u}_t\|_{L^\infty(0,T;H^1(\Omega_1)^2)}^2+\|\boldsymbol{u}_{ttt}\|_{L^\infty(0,T;L^2(\Omega_1)^2)}^2)\\
&\quad+\nu^{-1}h^{2k_1}\|\boldsymbol{u}_t\|_{L^\infty(0,T;L^2(\Omega_1)^2)}^2\Big).
\end{aligned}\tag{3.106}$$

Multiply the equation (3.106) by $2\Delta t$ and sum from $i=1$ to $i=m-1$, $m\geq 2$. Then under the condition

$$\nu>\frac{4C_a}{C_3}(C_5+\frac{3}{2}\tilde{C}_{12}\tilde{C}_{14}^2)\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega_1)^2)},\tag{3.107}$$

the inequality (3.100) is obtained.

It remains to find a bound for $\|\boldsymbol{\chi}^1\|_{L^2(\Omega_1)}^2$. For this, consider the equation (3.66).

Following a similar derivation as above, the error equation is

$$\begin{aligned}\frac{1}{2t^1}(\|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} - \|\boldsymbol{\chi}^0\|^2_{L^2(\Omega_1)}) + C_3\nu\|\boldsymbol{\chi}^1\|^2_{\boldsymbol{X}_h} + C_4\|\xi^1\|^2_{M_h^2} \\ \leq -N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{\chi}^1) + N(\boldsymbol{u}^1, \boldsymbol{u}^1; \boldsymbol{u}^1, \boldsymbol{\chi}^1) + (\boldsymbol{u}_t^1, \boldsymbol{\chi}^1)_{\Omega_1} - \frac{1}{t^1}(\tilde{\boldsymbol{u}}^1 - \tilde{\boldsymbol{u}}^0, \boldsymbol{\chi}^1)_{\Omega_1} \\ + B([\boldsymbol{\eta}^1, \zeta^1], [\boldsymbol{\chi}^1, \xi^1]) + b_{\mathrm{NS}}(\boldsymbol{\chi}^1, p^1 - P_h^1). \quad (3.108)\end{aligned}$$

The terms in the right-hand side of (3.108) are bounded using a similar argument as above. In fact, the error analysis is simpler. For instance, note that the nonlinear terms are rewritten as

$$\begin{aligned}N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{\chi}^1) - N(\boldsymbol{u}^1, \boldsymbol{u}^1; \boldsymbol{u}^1, \boldsymbol{\chi}^1) &= N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{\chi}^1, \boldsymbol{\chi}^1) + N(\boldsymbol{U}_h^1, \boldsymbol{\chi}^1; \tilde{\boldsymbol{u}}^1, \boldsymbol{\chi}^1) \\ &\quad - N(\boldsymbol{U}_h^1, \boldsymbol{\eta}^1; \tilde{\boldsymbol{u}}^1, \boldsymbol{\chi}^1) - N(\boldsymbol{U}_h^1, \boldsymbol{u}^1; \boldsymbol{\eta}^1, \boldsymbol{\chi}^1).\end{aligned}$$

The resulting inequality similar to (3.106) is

$$\begin{aligned}\frac{1}{2t^1}(\|\boldsymbol{\chi}^1\|^2_{L^2(\Omega_1)} - \|\boldsymbol{\chi}^0\|^2_{L^2(\Omega_1)}) + \big(\frac{C_3}{2}\nu - C_a(C_5 + \frac{3}{2}\tilde{C}_{12}\tilde{C}_{14}^2)\|\boldsymbol{u}\|_{L^\infty(0,T;H^1(\Omega_1)^2)}\big)\|\boldsymbol{\chi}^1\|^2_{\boldsymbol{X}_h} \\ + \frac{C_4}{2}\|\xi^1\|^2_{M_h^2} \leq C(\nu^{-1} + \nu + 1)h^{2k_1}|\boldsymbol{u}^1|^2_{H^{k_1+1}(\Omega_1)} + C(\nu^{-1} + 1)h^{2k_2}|\varphi^1|^2_{H^{k_2+1}(\Omega_2)} \\ + C\nu^{-1}h^{2k_1}|p^1|^2_{H^{k_1}(\Omega_1)} + C\nu^{-1}(t^1)^2\|\boldsymbol{u}_{ttt}\|^2_{L^\infty(0,T;L^2(\Omega_1))} + C\nu^{-1}h^{2k_1}\|\boldsymbol{u}_t\|^2_{L^\infty(0,T;L^2(\Omega_1))}.\end{aligned}$$

Multiplying this by $2t^1$ and using the fact that $\|\boldsymbol{\chi}^0\|_{L^2(\Omega_1)} \leq Ch^{k_1+1}|\boldsymbol{u}_0|_{H^{k_1+1}(\Omega_1)}$ (from the approximation result (3.95)) gives (3.99) under the assumption (3.107). □

Remark 40. *FEM analysis of this problem is simpler and yields the same error estimates [35].*

In order to obtain a scheme that is second order in time, the first time step t^1 has to be chosen small enough, namely $t^1 \leq \Delta t^{4/3}$. The final results are summarized below.

Corollary 41. *Under the assumptions of Theorem 39 and assuming $t^1 \leq \Delta t^2$, there exists a constant C independent of h, t^1 and Δt but dependent on ν and the weak solution, such that*

$$\|\boldsymbol{u}^1 - \boldsymbol{U}_h^1\|^2_{L^2(\Omega_1)} + \nu t^1 \|\boldsymbol{u}^1 - \boldsymbol{U}_h^1\|^2_{\boldsymbol{X}_h} + t^1 \|\varphi^1 - \Phi_h^1\|^2_{M_h^2} \leq C h^{2k_1+2} + C(h^{2k_1} + h^{2k_2} + \Delta t^4),$$

and for any $m \geq 2$,

$$\|\boldsymbol{u}^m - \boldsymbol{U}_h^m\|^2_{L^2(\Omega_1)} + \nu \Delta t \sum_{i=1}^{m-1} \|\boldsymbol{u}^{i+\frac{1}{2}} - \boldsymbol{U}_h^{i+\frac{1}{2}}\|^2_{\boldsymbol{X}_h} + \Delta t \sum_{i=1}^{m-1} \|\varphi^{i+\frac{1}{2}} - \Phi_h^{i+\frac{1}{2}}\|^2_{M_h^2}$$
$$\leq C(h^{2k_1} + h^{2k_2} + \Delta t^4).$$

Remark 42. *The assumption on $\boldsymbol{u}_t$ can be weakened in the following sense. If $\boldsymbol{u}_t$ belongs only to $L^\infty(0, T; L^2(\Omega_1)^2)$, and if the ratio $h/\Delta t$ is bounded above by a constant, then the results of Corollary 41 are valid.*

An error estimate for the Navier-Stokes pressure p is obtained by the inf-sup condition (3.91). The error bounds depend on error estimates of the discrete derivative of the velocity in the L^2-norm, which are not derived.

Theorem 43. *Assume that the weak solution of problem (P) satisfy the regularity assumptions of Theorem 39. In addition, let $\boldsymbol{u} \in L^\infty(0, T; H^{k_1+1}(\Omega_1)^2)$, $p \in L^\infty(0, T; H^{k_1}(\Omega_1)^2)$ and $\varphi \in L^\infty(0, T; H^{k_2+1}(\Omega_2))$. Then there exists a constant C independent of h, t^1 and Δt such that*

$$\|p^1 - P_h^1\|_{L^2(\Omega_1)} \leq \frac{C}{t^1} \|(\boldsymbol{u}^1 - \boldsymbol{U}_h^1) - (\boldsymbol{u}^0 - \boldsymbol{U}_h^0)\|_{L^2(\Omega_1)} + C(h^{k_1} + h^{k_2} + \Delta t^2), \quad (3.109)$$

$$\forall i \geq 1, \quad \|p^{i+\frac{1}{2}} - P_h^{i+\frac{1}{2}}\|_{L^2(\Omega_1)} \leq \frac{C}{\Delta t} \|(\boldsymbol{u}^{i+1} - \boldsymbol{U}_h^{i+1}) - (\boldsymbol{u}^i - \boldsymbol{U}_h^i)\|_{L^2(\Omega_1)}$$
$$+ C(h^{k_1} + h^{k_2} + \Delta t^2). \quad (3.110)$$

Proof. This proof is only a sketch as the argument is standard. From the inf-sup condition, there exists a velocity $\hat{\boldsymbol{v}}^1 \in \boldsymbol{X}_{\mathrm{h}}$ such that

$$\boldsymbol{n}_1(\hat{\boldsymbol{v}}^1, P_h^1 - \tilde{p}^1) = \|P_h^1 - \tilde{p}^1\|^2_{L^2(\Omega_1)}, \quad \|\hat{\boldsymbol{v}}^1\|_{\boldsymbol{X}_h} \leq \frac{1}{\beta^*}\|P_h^1 - \tilde{p}^1\|_{L^2(\Omega_1)}.$$

With the choice $(\boldsymbol{v}, q) = (\hat{\boldsymbol{v}}^1, 0)$, the error equation becomes

$$\begin{aligned}\boldsymbol{n}_1(\hat{\boldsymbol{v}}^1, P_h^1 - \tilde{p}^1) = \frac{1}{t^1}((\boldsymbol{u}^1 - \boldsymbol{U}_h^1) - (\boldsymbol{u}^0 - \boldsymbol{U}_h^0), \hat{\boldsymbol{v}}^1)_{\Omega_1} + B([\boldsymbol{u}^1 - \boldsymbol{U}_h^1, \varphi^1 - \Phi_h^1]; [\hat{\boldsymbol{v}}^1, 0]) \\ + N(\boldsymbol{u}^1, \boldsymbol{u}^1; \boldsymbol{u}^1, \hat{\boldsymbol{v}}^1) - N(\boldsymbol{U}_h^1, \boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \hat{\boldsymbol{v}}^1) + b_{\mathrm{NS}}(\hat{\boldsymbol{v}}^1, p^1 - \tilde{p}^1).\end{aligned}$$

It suffices to bound the terms on the right-hand side. All terms except the first one are bounded using the same techniques as in the proof of Theorem 39. They yield optimal bounds with respect to h and Δt. The first term is simply bounded by using the Cauchy-Schwarz inequality. A similar argument is used to derive (3.110). □

The above error estimate concludes this section on the numerical analysis of the first model of the time-dependent Navier-Stokes equation coupled with the Darcy's equation.

3.3 Model II without the Inertial Forces on the Interface

In the previous section, I analyzed this time-dependent problem with the inertial forces included in the balance of forces. Inclusion of inertial forces in the interface condition makes it easier to analyze the problem which is complicated because of the nonlinear convection term. However, inclusion of inertial forces is not physically meaningful although it is meaningful from the mathematical point of view. So, in this section, the inertial forces are omitted and the more challenging problem is analyzed. Here, we use the same notation as in Section 3.2. There is a minor difference in the boundary conditions. This time rather than the homogeneous Neumann condition on Γ_{2N}, we consider a non-homogeneous condition given as follows:

$$\boldsymbol{K}\nabla\varphi \cdot \boldsymbol{n}_{\Omega_2} = g \text{ on } \Gamma_{2N} \times (0,T).$$

We assume that $|\Gamma_{2D}| \neq 0$. As mentioned above, we no longer have the inertial forces on the interface. Hence the balance of forces is given as

$$((-2\nu \boldsymbol{D}(\boldsymbol{u}) + p\mathbf{I})\boldsymbol{n}_{12}) \cdot \boldsymbol{n}_{12} = \varphi, \quad \text{on } \Gamma_{12} \times (0,T).$$

Finally the initial condition is fixed to be

$$\boldsymbol{u}(0,x) = \boldsymbol{0}, \quad \text{in } \Omega_1. \tag{3.111}$$

The previous assumptions on the data $\boldsymbol{\Psi}, \Pi$ and g are not sufficient for the analysis of the weak problem. The existence of this weak problem will be proven under extra assumptions again using the Galerkin technique. Now, we ask for

$$\boldsymbol{\Psi} \in \mathcal{C}^1(0,T; L^2(\Omega_1)^2), \quad \Pi \in \mathcal{C}^1(0,T; L^2(\Omega_2)), \quad g \in \mathcal{C}^1(0,T; H^{-\frac{1}{2}}(\Gamma_{2N})).$$

3.3.1 Weak Formulation

The Sobolev spaces $\boldsymbol{X}$, M_1 and M_2 are defined the same way as in Model I and the weak formulation corresponding to Model II given as follows :
Find $(\boldsymbol{u}, p, \varphi) \in (L^2(0,T;\boldsymbol{X}) \cap L^\infty(0,T;L^2(\Omega_1)^2)) \times L^2(0,T;H^1(\Omega_1)) \times L^2(0,T;M_2)$ such that $\boldsymbol{u}' \in L^\infty(0,T;L^2(\Omega_1)^2)$ and

$$(\tilde{P}) \begin{cases} \forall \boldsymbol{v} \in \boldsymbol{X}, \forall q \in M_2, & (\frac{\partial \boldsymbol{u}}{\partial t}, \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} \\ & - (p, \nabla\cdot\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2} + \tilde{\gamma}(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) \\ & = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} + (g, q)_{\Gamma_{2N}}, \\ \forall q \in M_1, & (\nabla\cdot\boldsymbol{u}, q)_{\Omega_1} = 0, \\ \forall \boldsymbol{v} \in \boldsymbol{X}, & (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} = 0. \end{cases}$$

Note that there is a slight change in the weak formulation because of the non-homogeneous Neumann condition and the removal of the inertial forces from the interface conditions. Now the solution spaces are different and the form, which includes the interface terms, is defined differently as

$$\forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}, \quad \forall p, q \in M_2,$$
$$\tilde{\gamma}(\boldsymbol{u}, p; \boldsymbol{v}, q) = (p, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u} \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}}. \quad (3.112)$$

With this γ, we have

$$\forall \boldsymbol{v} \in \boldsymbol{X}, \quad \forall q \in M_2, \quad \tilde{\gamma}(\boldsymbol{v}, q; \boldsymbol{v}, q) = G(\boldsymbol{K}^{-\frac{1}{2}}\boldsymbol{v} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \geq 0$$

as $\boldsymbol{K}^{-\frac{1}{2}}$ is positive semi-definite. In addition to the inequalities (3.27)-(3.29) stated previously, we introduce two more inequalities. There exists $T_{12}, T_N > 0$ depending only on Ω_2 satisfying

$$\|q\|_{H^{\frac{1}{2}}(\Gamma_{12})} \leq T_{12}|q|_{H^1(\Omega_2)}, \quad \|q\|_{H^{\frac{1}{2}}(\Gamma_{2N})} \leq T_N|q|_{H^1(\Omega_2)}. \quad (3.113)$$

3.3.2 Existence of a Weak Solution

I will first state the existence theorem and proceed with the proof by pointing out which results still hold and what is different in this case. I will also provide demonstrations of the results when necessary.

Theorem 44. *Suppose that the above assumptions on the data* $\boldsymbol{\Psi}, \Pi, g$ *and* $\boldsymbol{K}$ *hold. Assume also that* $\boldsymbol{u}_0 = \boldsymbol{0}$. *Then under the assumption*

$$\mathcal{A} + \frac{C_D^2 S_2^2}{4\nu}\|\boldsymbol{\Psi}\|^2_{L^\infty(0,T;L^2(\Omega_1)^2)} + \frac{\tilde{S}_2^{\,2}}{\lambda_{min}}\|\Pi\|^2_{L^\infty(0,T;L^2(\Omega_2))} + \frac{T_N^2}{\lambda_{min}}\|g\|^2_{L^\infty(0,T;H^{-\frac{1}{2}}(\Gamma_{2N}))} < \frac{\nu^3}{32 S_4^2 C_D^6} \quad (3.114)$$

the problem (P) has at least one solution $(\boldsymbol{u}, p, \varphi) \in (L^2(0,T;\boldsymbol{V}) \cap H^1(0,T;L^2(\Omega_1)^2) \times L^2(0,T;M_1) \times L^2(0,T;M_2)$ *satisfying*

$$\sup_{t\in[0,T]} \|\boldsymbol{u}(t)\|^2_{L^2(\Omega_1)} + \nu\|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(0,T;L^2(\Omega_2)^2)} \leq \mathcal{M}^2 \quad (3.115)$$

where

$$\mathcal{M} = \left(\frac{C_D^2 S_2^2}{2\nu} \|\boldsymbol{\Psi}\|^2_{L^2(0,T;L^2(\Omega_1)^2)} + \frac{2\tilde{S}_2^{\,2}}{\lambda_{min}} \|\Pi\|^2_{L^2(0,T;L^2(\Omega_2))} + \frac{2T_N^2}{\lambda_{min}} \|g\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_{2N}))} \right)^{\frac{1}{2}}$$

and

$$\mathcal{A} = \mathcal{M} \times \Big(\frac{4C_L^2 T_{12}^2}{\lambda_{min}^2} (\tilde{S}_2^{\,2} \|\Pi(0)\|^2_{L^2(\Omega_2)} + T_N^2 \|g(0)\|^2_{H^{-\frac{1}{2}}(\Gamma_{2N})}) + 2\|\boldsymbol{\Psi}(0)\|^2_{L^2(\Omega_1)}$$
$$+ \frac{C_D^2 S_2^2}{2\nu} \|\boldsymbol{\Psi}'\|^2_{L^2(0,T;L^2(\Omega_1)^2)} + 2\frac{\tilde{S}_2^{\,2}}{\lambda_{min}} \|\Pi'\|^2_{L^2(0,T;L^2(\Omega_2))} + 2\frac{T_N^2}{\lambda_{min}} \|g'\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_{2N}))} \Big)^{\frac{1}{2}}. \quad (3.116)$$

We again use the technique of restricting the problem to the divergence free subspace $\boldsymbol{V}$ of $\boldsymbol{X}$ and consider the weak problem:

Find $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega_1)) \cap L^2(0,T;\boldsymbol{V})$ and $\varphi \in L^2(0,T;M_2)$ such that $\boldsymbol{u}' \in L^\infty(0,T;L^2(\Omega_1))$ and

$$(\tilde{P}_V) \begin{cases} \forall(\boldsymbol{v}, q) \in \boldsymbol{W}, & (\boldsymbol{u}_t, \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi, \nabla q)_{\Omega_2} \\ & +\tilde{\gamma}(\boldsymbol{u}, \varphi, \boldsymbol{v}, q) = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} + (g, q)_{\Gamma_{2N}}, \\ \forall \boldsymbol{v} \in \boldsymbol{V}, & (\boldsymbol{u}(0), \boldsymbol{v})_{\Omega_1} = 0. \end{cases}$$

Theorem 45. *Assume that the assumptions of Theorem 44 hold. Then there exists a solution* $(\boldsymbol{u}, \varphi)$ *to the problem* $(\tilde{P}_V)$ *satisfying (3.115).*

Proof. The proof is the same in essence as the existence proof of the restricted weak problem of Model I and only the differences will be highlighted. We first show existence and uniqueness of a solution $(\boldsymbol{u}_m, \varphi_m)$ of the following finite dimensional problem: Find $(\boldsymbol{u}_m, \varphi_m) \in L^2(0,T;\boldsymbol{W}_m)$ with $\boldsymbol{u}_m \in L^\infty(0,T;L^2(\Omega_1)^2)$ and $\boldsymbol{u}'_m \in L^\infty(0,T;L^2(\Omega_1)^2)$ such that for all $(\boldsymbol{v},q) \in \boldsymbol{W}_m$,

$$(\tilde{P}_m)\begin{cases} (\boldsymbol{u}'_m, \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}_m), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}_m \cdot \nabla \boldsymbol{u}_m, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\varphi_m, \nabla q)_{\Omega_2} \\ \qquad\qquad + \tilde{\gamma}(\boldsymbol{u}_m, \varphi_m; \boldsymbol{v}, q) = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} + (g,q)_{\Gamma_{2N}}, \\ (\boldsymbol{u}_m(0), \boldsymbol{v})_{\Omega_1} = 0. \end{cases}$$

Here the notation $\boldsymbol{u}'_m$ is used for the time derivative of $\boldsymbol{u}_m$. Recall that $\boldsymbol{W}_m$ is the finite dimensional Galerkin space which approximates $\boldsymbol{W}$. Then the problem becomes

$$\begin{cases} \boldsymbol{A}\boldsymbol{\alpha}' + \boldsymbol{B}\boldsymbol{\alpha} + \tilde{\boldsymbol{F}}(\boldsymbol{\alpha}) + \boldsymbol{C}^T\boldsymbol{\beta} = \boldsymbol{b} \\ \boldsymbol{M}\boldsymbol{\beta} + \boldsymbol{C}\boldsymbol{\alpha} = \tilde{\boldsymbol{c}} \\ \boldsymbol{A}\boldsymbol{\alpha}(0) = \boldsymbol{0} \end{cases}$$

with the vector $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ containing the components α_i^m and β_i^m respectively. The matrices are defined exactly the same as in Model I. And the vectors except the following are again defined the same way. Let

$$(\tilde{\boldsymbol{F}}(\boldsymbol{\alpha}))_i = \tilde{\boldsymbol{N}}_i\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}, \quad \tilde{c}_i = (\Pi, r_i)_{\Omega_2} + (g, r_i)_{\Gamma_{2N}}$$

where $\tilde{\boldsymbol{N}}_i = \big((\Phi_j \cdot \nabla\Phi_k, \Phi_i)_{\Omega_1})\big)_{1\le j,k\le m}$ is a matrix for each $i = 1, \dots m$. Thus, solving the problem defined by $(\tilde{P}_m)$ is equivalent to solving

$$\begin{cases} \boldsymbol{\alpha}' + \boldsymbol{A}^{-1}(\boldsymbol{B} - \boldsymbol{D}\boldsymbol{M}^{-1}\boldsymbol{C})\boldsymbol{\alpha} = \boldsymbol{A}^{-1}(\boldsymbol{b} - \tilde{\boldsymbol{F}}(\boldsymbol{\alpha}) - \boldsymbol{D}\boldsymbol{M}^{-1}\tilde{\boldsymbol{c}}) \\ \boldsymbol{\alpha}(0) = \boldsymbol{0}. \end{cases}$$

From the theory of ordinary differential equations [62], there exists a unique maximal solution on the interval $[0, T_m]$ for some T_m such that $0 < T_m \le T$. We need an a priori bound on $(\boldsymbol{u}_m, \varphi_m)$ to conclude that $T_m = T$. Consider equation $(\tilde{P}_m)$ and choose $\boldsymbol{v} = \boldsymbol{u}_m$ and $q = \varphi_m$. After applying Cauchy-Schwarz and Hölder's inequalities and using the nonnegativity of the $\tilde{\gamma}$ term, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + 2\nu\|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_m\|^2_{L^2(\Omega_2)}$$
$$\leq S_4^2C_D^3\|\boldsymbol{D}(\boldsymbol{u}_m)\|^3_{L^2(\Omega_1)} + S_2\|\boldsymbol{\Psi}\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega_1)} + \tilde{S}_2\|\Pi\|_{L^2(\Omega_2)}\|\nabla\varphi_m\|_{L^2(\Omega_2)}$$
$$+ T_N\|g\|_{H^{-\frac{1}{2}}(\Gamma_{2N})}\|\nabla\varphi_m\|_{L^2(\Omega_2)}. \quad (3.117)$$

Thus, by Young's inequality,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + \nu\|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_m\|^2_{L^2(\Omega_2)}$$
$$\leq S_4^2C_D^3\|\boldsymbol{D}(\boldsymbol{u}_m)\|^3_{L^2(\Omega_1)} + \frac{C_D^2S_2^2}{4\nu}\|\boldsymbol{\Psi}\|^2_{L^2(\Omega_1)} + \frac{\tilde{S}_2^{\,2}}{\lambda_{min}}\|\Pi\|^2_{L^2(\Omega_2)} + \frac{T_N^2}{\lambda_{min}}\|g\|^2_{H^{-\frac{1}{2}}(\Gamma_{2N})}. \quad (3.118)$$

The term that gives a problem is the first term on the right hand side of (3.118). We want to hide it in the second term on the left hand side. Observe that under the assumption $\boldsymbol{u}_m(0) = \boldsymbol{0}$, the continuity of the solution implies that there exists $\overline{T}_m > 0$ such that $\overline{T}_m < T_m$ and

$$\forall t \in [0, \overline{T}_m], \quad \|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} < \frac{\nu}{4S_4^2C_D^3}. \quad (3.119)$$

Our aim is to show that (3.119) holds for all $t \in [0, T_m]$. This will give an a priori bound for the Galerkin solution $(\boldsymbol{u}_m, \varphi_m)$ thus enabling us to conclude that $T_m = T$. We will proceed by contradiction and assume that there is a time T^* such that $0 < T^* \leq T_m$ and

$$\|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} < \frac{\nu}{4S_4^2C_D^3},\ 0 \leq t < T^*, \quad \|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} = \frac{\nu}{4S_4^2C_D^3},\ t = T^*. \quad (3.120)$$

Observe that (3.120) suggests $\|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} \leq \frac{\nu}{2S_4^2C_D^3}$ on $[0, T^*]$. Then from (3.118) using Cauchy-Schwarz on the first term, we see that

$$\frac{\nu}{2}\|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_m\|^2_{L^2(\Omega_2)} \leq \|\boldsymbol{u}'_m\|_{L^2(\Omega_1)}\|\boldsymbol{u}_m\|_{L^2(\Omega_1)} + \frac{C_D^2 S_2^2}{4\nu}\|\boldsymbol{\Psi}\|^2_{L^2(\Omega_1)} + \frac{\tilde{S}_2^{\ 2}}{\lambda_{min}}\|\Pi\|^2_{L^2(\Omega_2)} + \frac{T_N^2}{\lambda_{min}}\|g\|^2_{H^{-\frac{1}{2}}(\Gamma_{2N})}. \quad (3.121)$$

Now we need to bound the first term on the right hand side of (3.121). A common approach to find a bound for $\|\boldsymbol{u}'_m\|_{L^2(\Omega_1)}$ is differentiating the first equation $(\tilde{P}_m)$ with respect to t (See [78] for the procedure). As $\boldsymbol{u}'_m(t) \in \boldsymbol{V}_m$ and $\varphi'_m \in M_m$, choosing $\boldsymbol{v} = \boldsymbol{u}'_m$ and $q = \varphi'_m$ yields

$$(\boldsymbol{u}''_m, \boldsymbol{u}'_m)_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{u}'_m), \boldsymbol{D}(\boldsymbol{u}'_m))_{\Omega_1} + (\boldsymbol{u}'_m \cdot \nabla\boldsymbol{u}_m, \boldsymbol{u}'_m)_{\Omega_1} + (\boldsymbol{u}_m \cdot \nabla\boldsymbol{u}'_m, \boldsymbol{u}'_m)_{\Omega_1}$$
$$+(\boldsymbol{K}\nabla\varphi'_m, \nabla\varphi'_m)_{\Omega_2} + \tilde{\gamma}(\boldsymbol{u}'_m, \varphi'_m; \boldsymbol{u}'_m, \varphi'_m) = (\boldsymbol{\Psi}', \boldsymbol{u}'_m)_{\Omega_1} + (\Pi', \varphi'_m)_{\Omega_2} + (g', \varphi'_m)_{\Gamma_{2N}}.$$

Using Hölder's and Cauchy-Schwarz inequalities and nonnegativity of $\tilde{\gamma}$ term, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)} + 2\nu\|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi'_m\|^2_{L^2(\Omega_2)}$$
$$\leq 2S_4^2\|\nabla\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)}\|\nabla\boldsymbol{u}_m\|_{L^2(\Omega_1)} + S_2\|\boldsymbol{\Psi}'_1\|_{L^2(\Omega_1)}\|\nabla\boldsymbol{u}'_m\|_{L^2(\Omega_1)}$$
$$+ \tilde{S}_2\|\Pi'\|_{L^2(\Omega_2)}\|\nabla\varphi'_m\|_{L^2(\Omega_2)} + T_N\|g'\|_{H^{-\frac{1}{2}}(\Gamma_{2N})}\|\nabla\varphi'_m\|_{L^2(\Omega_2)}.$$

Thus, similar to before, we have

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)} + \nu\|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi'_m\|^2_{L^2(\Omega_2)}$$
$$\leq 2S_4^2 C_D^3\|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} + \frac{C_D^2 S_2^2}{4\nu}\|\boldsymbol{\Psi}'\|^2_{L^2(\Omega_1)} + \frac{\tilde{S}_2^{\ 2}}{\lambda_{min}}\|\Pi'\|^2_{L^2(\Omega_2)} + \frac{T_N^2}{\lambda_{min}}\|g'\|^2_{H^{-\frac{1}{2}}(\Gamma_{2N})}. \quad (3.122)$$

Then the assumption (3.120) and the equation (3.122) imply for all $t \in [0, T^\star]$ that,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_m'\|^2_{L^2(\Omega_1)} + \frac{\nu}{2}\|\boldsymbol{D}(\boldsymbol{u}_m')\|^2_{L^2(\Omega_1)} + \frac{1}{2}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_m'\|^2_{L^2(\Omega_2)} \leq \frac{C_D^2 S_2^2}{4\nu}\|\boldsymbol{\Psi}'\|^2_{L^2(\Omega_1)} + \frac{\tilde{S_2}^2}{\lambda_{min}}\|\Pi'\|^2_{L^2(\Omega_2)} + \frac{T_N^2}{\lambda_{min}}\|g'\|^2_{H^{-\frac{1}{2}}(\Gamma_{2N})}. \quad (3.123)$$

Multiply this by two and integrate from 0 to t to obtain

$$\|\boldsymbol{u}_m'(t)\|^2_{L^2(\Omega_1)} - \|\boldsymbol{u}_m'(0)\|^2_{L^2(\Omega_1)} + \nu\int_0^t \|\boldsymbol{D}(\boldsymbol{u}_m')\|^2_{L^2(\Omega_1)}dt + \int_0^t \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_m'\|^2_{L^2(\Omega_2)}dt \leq \frac{C_D^2 S_2^2}{2\nu}\|\boldsymbol{\Psi}'\|^2_{L^2(0,T;L^2(\Omega_1))} + \frac{2\tilde{S_2}^2}{\lambda_{min}}\|\Pi'\|^2_{L^2(0,T;L^2(\Omega_2))} + \frac{2T_N^2}{\lambda_{min}}\|g'\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_{2N}))} \quad (3.124)$$

for all $t \in [0, T^\star]$. To bound the term $\|\boldsymbol{u}_m'(0)\|^2_{L^2(\Omega_1)}$ on the left hand side of (3.124), we use $\boldsymbol{v} = \boldsymbol{u}_m'(0)$ and $q = 0$ in the first equation of $(\tilde{P}_m)$. Since $\boldsymbol{u}_m(0) = \boldsymbol{0}$, this yields the following when evaluated at time $t = 0$:

$$\|\boldsymbol{u}_m'(0)\|^2 + (\varphi_m(0), \boldsymbol{u}_m'(0)\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} = (\boldsymbol{\Psi}(0), \boldsymbol{u}_m'(0)).$$

Therefore we have

$$\|\boldsymbol{u}_m'(0)\|^2_{L^2(\Omega_1)} \leq \|\varphi_m(0)\|_{H^{\frac{1}{2}}(\Gamma_{12})}\|\boldsymbol{u}_m'(0)\cdot\boldsymbol{n}_{12}\|_{(H^{\frac{1}{2}}(\Gamma_{12}))'} + \|\boldsymbol{\Psi}(0)\|_{L^2(\Omega_1)}\|\boldsymbol{u}_m'(0)\|_{L^2(\Omega_1)} \quad (3.125)$$

From Lemma 72 in the Appendix, there exists a constant $C_L > 0$ such that

$$\|\boldsymbol{u}_m'(0)\cdot\boldsymbol{n}_{12}\|_{(H^{\frac{1}{2}}(\Gamma_{12}))'} \leq C_L\|\boldsymbol{u}_m'(0)\|_{L^2(\Omega_1)}.$$

Hence from (3.125),

$$\|\boldsymbol{u}_m'(0)\|_{L^2(\Omega_1)} \leq C_L\|\varphi_m(0)\|_{H^{\frac{1}{2}}(\Gamma_{12})} + \|\boldsymbol{\Psi}(0)\|_{L^2(\Omega_1)}. \quad (3.126)$$

We bound $\|\varphi_m(0)\|_{H^{\frac{1}{2}}(\Gamma_{12})}$ on the right hand side of inequality (3.126) by plugging $\boldsymbol{v} = \mathbf{0}$ and $q = \varphi_m$ in $(\tilde{P_m})$ and evaluating at time $t = 0$. This gives

$$(\boldsymbol{K}\nabla\varphi_m(0), \nabla\varphi_m(0))_{\Omega_2} = (\Pi(0), \varphi_m(0))_{\Omega_2} + (g(0), \varphi_m(0))_{\Gamma_{2N}}.$$

Then (3.31) and the Cauchy-Schwarz inequality imply

$$|\varphi_m(0)|^2_{H^1(\Omega_2)} \leq \frac{1}{\lambda_{min}}\big(\|\Pi(0)\|_{L^2(\Omega_2)}\|\varphi_m(0)\|_{L^2(\Omega_2)} + \|g(0)\|_{H^{-\frac{1}{2}}(\Gamma_{2N})}\|\varphi_m(0)\|_{H^{\frac{1}{2}}(\Gamma_{2N})}\big).$$

Hence by the Poincaré inequality and the trace theorem, we obtain

$$\|\varphi_m(0)\|_{H^{\frac{1}{2}}(\Gamma_{12})} \leq \frac{T_{12}}{\lambda_{min}}\big(\tilde{S}_2\|\Pi(0)\|_{L^2(\Omega_2)} + T_N\|g(0)\|_{H^{-\frac{1}{2}}(\Gamma_{2N})}\big). \tag{3.127}$$

Therefore, (3.124), (3.126) and (3.127) yield

$$\begin{aligned}
&\|\boldsymbol{u}'_m(t)\|^2_{L^2(\Omega_1)} + \nu\int_0^t \|\boldsymbol{D}(\boldsymbol{u}'_m)\|^2_{L^2(\Omega_1)}dt + \int_0^t \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi'_m\|^2_{L^2(\Omega_2)}dt \\
&\quad\leq \frac{4C_L^2T_{12}^2}{\lambda^2_{min}}\Big(\tilde{S_2}^2\|\Pi(0)\|^2_{L^2(\Omega_2)} + T_N^2\|g(0)\|^2_{H^{-\frac{1}{2}}(\Gamma_{2N})}\Big) + 2\|\boldsymbol{\Psi}(0)\|^2_{L^2(\Omega_1)} \\
&\quad+ \frac{C_D^2S_2^2}{2\nu}\|\boldsymbol{\Psi}'\|^2_{L^2(0,T;L^2(\Omega_1))} + \frac{2\tilde{S_2}^2}{\lambda_{min}}\|\Pi'\|^2_{L^2(0,T;L^2(\Omega_2))} + \frac{2T_N^2}{\lambda_{min}}\|g'\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_{2N}))}
\end{aligned} \tag{3.128}$$

for all $t \in [0, T^\star]$. This gives the bound for $\|\boldsymbol{u}'_m\|^2_{L^2(\Omega_1)}$ on the right hand side of the inequality (3.121).

To get a bound for $\|\boldsymbol{u}_m\|_{L^2(\Omega_1)}$ on the right hand side of (3.121), we multiply (3.118) by two, and use the assumption (3.120) which says $S_4^2C_D^3\|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} \leq \nu/2$. This implies

$$\begin{aligned}
&\frac{d}{dt}\|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + \nu\|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi_m\|^2_{L^2(\Omega_2)} \\
&\qquad\qquad\leq \frac{C_D^2S_2^2}{2\nu}\|\boldsymbol{\Psi}\|^2_{L^2(\Omega_1)} + \frac{2\tilde{S_2}^2}{\lambda_{min}}\|\Pi\|^2_{L^2(\Omega_2)} + \frac{2T_N^2}{\lambda_{min}}\|g\|^2_{H^{-\frac{1}{2}}(\Gamma_{2N})}.
\end{aligned}$$

We then integrate both sides from 0 to t for all $0 \leq t \leq T^*$ and use the second condition in $(\tilde{P}_m)$. This yields

$$\begin{aligned}&\|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} + \nu \int_0^t \|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} dt + \int_0^t \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_m\|^2_{L^2(\Omega_2)} dt \\ &\leq \frac{C_D^2 S_2^2}{2\nu} \|\boldsymbol{\Psi}\|^2_{L^2(0,T;L^2(\Omega_1)^2)} + \frac{2\tilde{S}_2^{\,2}}{\lambda_{min}} \|\Pi\|^2_{L^2(0,T;L^2(\Omega_2))} + \frac{2T_N^2}{\lambda_{min}} \|g\|^2_{L^2(0,T;H^{-\frac{1}{2}}(\Gamma_{2N}))} \end{aligned} \tag{3.129}$$

Combining (3.128) and (3.129), we finally have the following bound to be used in (3.121):

$$\|\boldsymbol{u}'_m\|_{L^2(\Omega_1)} \|\boldsymbol{u}_m\|_{L^2(\Omega_1)} \leq \mathcal{A}$$

where $\mathcal{A}$ is defined as in (3.116). Thus we conclude from (3.121) that

$$\begin{aligned}\|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(\Omega_1)} \leq \frac{2}{\nu}\Big(\mathcal{A} &+ \frac{C_D^2 S_2^2}{4\nu} \|\boldsymbol{\Psi}\|^2_{L^\infty(0,T;L^2(\Omega_1)^2)} \\ &+ \frac{\tilde{S}_2^2}{\lambda_{min}} \|\Pi\|^2_{L^\infty(0,T;L^2(\Omega_2))} + \frac{T_N^2}{\lambda_{min}} \|g\|^2_{L^\infty(0,T;H^{-\frac{1}{2}}(\Gamma_{2N}))}\Big).\end{aligned}$$

Since this inequality is valid for $t = T^*$ and because we have made the assumption (3.114) on the data, we conclude that

$$\|\boldsymbol{D}(\boldsymbol{u}_m)\|_{L^2(\Omega_1)} < \frac{\nu}{4 S_4 C_D^3},\ t = T^*$$

which is a contradiction.

To summarize, we showed the existence and uniqueness of the maximal solution $(\boldsymbol{u}_m, \varphi_m)$ on the interval $[0, T_m]$. From the a priori bound (3.129) valid for $[0, T_m]$, we conclude that the solution to the problem defined by $(\tilde{P}_m)$ exists on the whole interval $[0, T]$. Finally, we deduce the $(\boldsymbol{u}_m, \varphi_m)$ version of the bound defined in (3.115). Indeed, taking supremum over $[0, T]$, we obtain for any $m \geq 1$,

$$\sup_{t \in [0,T]} \|\boldsymbol{u}_m(t)\|^2_{L^2(\Omega_1)} + \nu \|\boldsymbol{D}(\boldsymbol{u}_m)\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi_m\|^2_{L^2(0,T;L^2(\Omega_2)^2)} \leq \mathcal{M}^2. \tag{3.130}$$

Note that $\mathcal{M}$ is independent of m. Next step is to pass to the limit in $(\tilde{P}_m)$ to obtain a solution to the problem defined by $(\tilde{P}_V)$.

We start by examining the bound (3.130) on the sequence $\{(\boldsymbol{u}_m, \varphi_m)\}_m$ which will give us the necessary convergence results for $\{\boldsymbol{u}_m\}_m$ and $\{\varphi_m\}_m$. First, (3.130) says that $\{\boldsymbol{u}_m\}_m$ is bounded in $L^2(0,T;\boldsymbol{V})$ and $\{\varphi_m\}_m$ is bounded in $L^2(0,T;M_2)$. As both $\boldsymbol{V}$ and M_2 are reflexive, up to a subsequence, there exists $\boldsymbol{u} \in L^2(0,T;\boldsymbol{V})$ and $\varphi \in L^2(0,T;M)$ such that

$$\boldsymbol{u}_m \to \boldsymbol{u}, \text{ weakly in } L^2(0,T;\boldsymbol{V}), \tag{3.131}$$

$$\varphi_m \to \varphi, \text{ weakly in } L^2(0,T;M). \tag{3.132}$$

The bound (3.130) also says that $\boldsymbol{u}_m$ is bounded in $L^\infty(0,T;\boldsymbol{V})$. This gives a further subsequence, still denoted by $\boldsymbol{u}_m$ such that

$$\boldsymbol{u}_m \to \boldsymbol{u}, \text{ weakly-}\star \text{ in } L^\infty(0,T;L^2(\Omega_1)^2). \tag{3.133}$$

Furthermore, from Lemma A.1 in the Appendix, $\boldsymbol{u}_m$ is bounded in $H^\gamma(0,T;\boldsymbol{V},L^2(\Omega_1)^2)$ for $0<\gamma<\frac{1}{4}$. Hence, from a compactness result [71, p.186], we have

$$\boldsymbol{u}_m \to \boldsymbol{u}, \text{ strongly in } L^2(0,T;L^2(\Omega_1)^2). \tag{3.134}$$

Lastly, we can pass to the limit in the interface terms, as the continuity of the trace operator implies

$$\boldsymbol{u}_m \to \boldsymbol{u}, \text{ weakly in } L^2(0,T;H^{\frac{1}{2}}(\partial\Omega_1)^2), \tag{3.135}$$

$$\varphi_m \to \varphi, \text{ weakly in } L^2(0,T;H^{\frac{1}{2}}(\partial\Omega_2)). \tag{3.136}$$

Using these convergence results, we pass to the limit same way as before which completes the proof of Theorem 45. □

The balance of forces interface condition doesn't really make up for the nonlinear term completely. However, if we assume an additional small data condition, we can obtain local uniqueness of $(\boldsymbol{u}, \varphi)$.

Theorem 46. *Under the assumption* $\|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(\Omega_1)} \leq \frac{\nu}{S_4^2 C_D^3}$, *the solution* $(\boldsymbol{u}, \varphi)$ *of* $(\tilde{P}_V)$ *is unique.*

Proof. Let $(\boldsymbol{u}, \varphi)$ and $(\tilde{\boldsymbol{u}}, \tilde{\varphi})$ be two solutions to (P_V). Let $\boldsymbol{w} = \boldsymbol{u} - \tilde{\boldsymbol{u}}$ and $\varphi = \varphi - \tilde{\varphi}$. Then for all $\boldsymbol{v} \in \boldsymbol{V}$ and $q \in M_2$,

$$(\boldsymbol{w}_t, \boldsymbol{v})_{\Omega_1} + 2\nu(\boldsymbol{D}(\boldsymbol{w}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{K} \nabla \varphi, \nabla q)_{\Omega_2}$$
$$\tilde{\gamma}(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) - \tilde{\gamma}(\tilde{\boldsymbol{u}}, \tilde{\varphi}; \boldsymbol{v}, q) = 0$$

and for all $\boldsymbol{v} \in \boldsymbol{V}$,

$$(\boldsymbol{w}(0), \boldsymbol{v})_{\Omega_1} = 0.$$

Letting $\boldsymbol{v} = \boldsymbol{w}$ and $q = \varphi$ yields

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + 2\nu\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2\Omega_1} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u} - \tilde{\boldsymbol{u}} \cdot \nabla \tilde{\boldsymbol{u}}, \boldsymbol{w})_{\Omega_1} + \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi\|^2_{L^2(\Omega_2)} \leq 0$$

We rewrite and bound the third term in the above equation as follows:

$$\begin{aligned} &|(\boldsymbol{w} \cdot \nabla \boldsymbol{u}, \boldsymbol{w})_{\Omega_1} + (\tilde{\boldsymbol{u}} \cdot \nabla \boldsymbol{w}, \boldsymbol{w})_{\Omega_1}| \\ &\qquad \leq \|\boldsymbol{w}\|^2_{L^4(\Omega_1)}\|\nabla \boldsymbol{u}\|_{L^2(\Omega_1)} + \|\tilde{\boldsymbol{u}}\|_{L^4(\Omega_1)}\|\nabla \boldsymbol{w}\|_{L^2(\Omega_1)}\|\boldsymbol{w}\|_{L^4(\Omega_1)} \\ &\qquad \leq 2S_4^2 C_D^3 \|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2(\Omega_1)}\big(\|\boldsymbol{D}(\boldsymbol{u})\|_{L^2(\Omega_1)} + \|\boldsymbol{D}(\tilde{\boldsymbol{u}})\|_{L^2(\Omega_1)}\big). \end{aligned} \quad (3.137)$$

Thus, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + \big(2\nu - 2S_4^2 C_D^3(\|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(\Omega_1)} + \|\boldsymbol{D}(\tilde{\boldsymbol{u}})\|^2_{L^2(\Omega_1)})\big)\|\boldsymbol{D}(\boldsymbol{w})\|^2_{L^2\Omega_1}$$
$$+ \|\boldsymbol{K}^{\frac{1}{2}} \nabla \varphi\|^2_{L^2(\Omega_2)} \leq 0 \quad (3.138)$$

Therefore, given $\|\boldsymbol{D}(\boldsymbol{u})\| \leq \frac{\nu}{S_4^2 C_D^3}$ for any solution of P_V,

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(\Omega_2)} \leq 0. \tag{3.139}$$

Now, integrate this from 0 to t. As, $\boldsymbol{w}(0) = 0$, we get

$$\frac{1}{2}\|\boldsymbol{w}(t)\|^2_{L^2(\Omega_1)} + \int_0^t \|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(\Omega_2)} dt \leq 0. \tag{3.140}$$

for $t \in [0, T]$. This yields $\boldsymbol{w} = 0$, $\varphi = 0$. □

The construction of the Navier-Stokes pressure p from the solution of $(\tilde{P}_V)$ follows the same proof as in the case of Model I.

3.4 Summary

In this chapter, a mathematical model is presented for the coupled surface and subsurface flow. The proposed weak problem is analyzed completely for two different models. For Model I, where we include the inertial forces in the interface conditions, the existence result is obtained unconditionally. However for Model II because of the missing inertial forces, a small data assumption is required to prove existence. Then for Model I, a numerical scheme based on DG methods and Crank-Nicolson method is derived and optimal error estimates in space and second order estimates in time are proved. Similar results have been proved using the FEM method rather than the DG method [35]. The analysis of this method is not included as it is a simplified version of the DG analysis and same error estimates hold.

Chapter 4

Coupling of Surface Flow and Transport with Miscible Displacement

In order to better understand the groundwater contamination problem, we consider the miscible displacement in the subsurface whereas the surface flow is characterized by the steady-state case of the Navier-Stokes/Stokes equations from Chapter 3 where they are coupled with a transport equation. Our motivation is to predict how the coupled surface and subsurface flow carry the pollutants to the groundwater supplies. The first section introduces the model problem for both the Stokes and the Navier-Stokes cases following exactly the notation of Chapter 3 for the flow part. The second section considers the Stokes/Darcy coupling, which can be thought of as the linear case of the Navier-Stokes/Darcy coupling, for the underlying flow problem. This part is a more elaborate version of [55] and proves the existence of a weak solution. In the following section, these results are extended to the full Navier-Stokes/Darcy problem. The numerical analysis and simulations of a special case, which is the one-way coupling of the Navier-Stokes/Darcy flow with the transport equation, are given in the last section. Here the velocity acts like an input to the transport equation. This part comes from [58] where the numerical scheme is based on a combination of FEM and DG method in space and backward Euler method in time. The convergence analysis is provided for this problem and to show the robustness of the derived schemes one numerical example is also presented.

4.1 Model Problem

This section defines the model problem with the assumptions on the data for the coupling of a transport equation with the surface/subsurface flow. The equations are coupled through the velocity field and the concentration. The flow problem of this chapter for the Navier-Stokes problem is the stationary case of Model II of Chapter 3 with minor differences, and for the Stokes problem the nonlinear term is also omitted. For the sake of completeness, we present the problem once more. Let $\boldsymbol{u}$, p and φ denote the fluid velocity in Ω, the Stokes pressure in Ω_1 and the Darcy pressure in Ω_2, respectively. We assume that $|\Gamma_1| > 0$. Let $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. The flow is characterized in the surface Ω_1 by the Stokes equations

$$-\nabla \cdot (2\mu(c)\boldsymbol{D}(\boldsymbol{u}) - p\mathbf{I}) = \boldsymbol{\Psi}, \quad \text{in } \Omega_1 \times (0,T), \tag{4.1}$$

or the Navier-Stokes equations

$$-\nabla \cdot (2\mu(c)\boldsymbol{D}(\boldsymbol{u}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - p\mathbf{I}) = \boldsymbol{\Psi}, \quad \text{in } \Omega_1 \times (0,T), \tag{4.2}$$

and the incompressibility condition

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } \Omega_1 \times (0,T). \tag{4.3}$$

In the subsurface Ω_2, the flow is governed by the Darcy's law

$$\boldsymbol{u} = -\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \qquad \nabla \cdot \boldsymbol{u} = \Pi, \quad \text{in } \Omega_2 \times (0,T). \tag{4.4}$$

Note that we take into account the gravitational pressure drop in the Darcy's equations. The interface conditions are given by the continuity of the flux,

$$\boldsymbol{u}|_{\Omega_1} \cdot \boldsymbol{n}_{12} = \boldsymbol{u}|_{\Omega_2} \cdot \boldsymbol{n}_{12}, \quad \text{on } \Gamma_{12} \times (0,T), \tag{4.5}$$

the Beavers-Joseph-Saffman law [2, 3],

$$GK^{-\frac{1}{2}}\boldsymbol{u}|_{\Omega_1}\cdot\boldsymbol{\tau}_{12} = -2\mu(c)\boldsymbol{D}(\boldsymbol{u}|_{\Omega_1})\boldsymbol{n}_{12}\cdot\boldsymbol{\tau}_{12}, \quad \text{on } \Gamma_{12}\times(0,T), \tag{4.6}$$

and the balance of forces without the inertial forces,

$$((-2\mu(c)\boldsymbol{D}(\boldsymbol{u}|_{\Omega_1}) + p)\boldsymbol{n}_{12})\cdot\boldsymbol{n}_{12} = \varphi, \quad \text{on } \Gamma_{12}\times(0,T). \tag{4.7}$$

The Stokes/Darcy flow is fully coupled to the following diffusion-convection transport equation which defines the concentration (fraction of volume) c of a contaminant transported in the domain Ω over the time interval $(0,T)$.

$$\frac{\partial}{\partial t}(\phi c) - \nabla\cdot(\boldsymbol{F}(\boldsymbol{u})\nabla c - c\boldsymbol{u}) \;=\; \Lambda, \quad \text{in } Q_T. \tag{4.8}$$

This system of equations is subject to the following boundary and initial conditions:

$$\boldsymbol{u} \;=\; \boldsymbol{0}, \quad \text{on } \Gamma_1\times(0,T), \tag{4.9}$$

$$\boldsymbol{u}\cdot\boldsymbol{n} \;=\; \mathcal{U}, \quad \text{on } \Gamma_2\times(0,T), \tag{4.10}$$

$$\boldsymbol{F}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n} \;=\; \begin{cases} (c-\mathcal{C})(\boldsymbol{u}\cdot\boldsymbol{n}), & \text{on } \partial\Omega_{in}\times(0,T) \\ 0, & \text{on } \partial\Omega_{out}\times(0,T) \end{cases}, \tag{4.11}$$

$$c \;=\; c_0, \quad \text{in } \Omega\times\{0\} \tag{4.12}$$

where the inflow boundary and outflow boundaries are defined as

$$\partial\Omega_{in} =: \{x\in\partial\Omega : (\boldsymbol{u}\cdot\boldsymbol{n})(x) < 0\}, \quad \partial\Omega_{out} := \{x\in\partial\Omega : (\boldsymbol{u}\cdot\boldsymbol{n})(x) \geq 0\}.$$

Since $|\Gamma_{2D}| = 0$, the uniqueness of the Darcy pressure is satisfied by the assumption

$$\int_{\Omega_2}\varphi = 0. \tag{4.13}$$

In the following, we define the coefficients of the equations above and set suitable assumptions, which are necessary for the conclusions of this chapter, on these coefficients.

- The fluid viscosity $\mu = \mu(c)$, which measures the resistance of a fluid to flow, belongs to $\mathcal{C}(\mathbb{R}^+;\mathbb{R}^+)$ and there exists μ_L, $\mu_U > 0$ satisfying

$$\mu_L \leq \mu(x) \leq \mu_U \text{ for any } x \in \mathbb{R}^+. \tag{4.14}$$

- The symmetric rate of strain matrix $\boldsymbol{D}(\boldsymbol{u}) = 0.5(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)$ is the same as in Section 3.2 and so satisfies (3.15) and (3.29).

- The vector function $\boldsymbol{\Psi}$ and the scalar functions, Π and Λ are the source/sink terms such that

$$\Pi \geq 0, \quad \Pi \in L^2(0,T;L^2(\Omega_2)), \quad \boldsymbol{\Psi} \in L^2(0,T;L^2(\Omega_1)^2)$$

and

$$\Lambda \geq 0, \quad \Lambda \in L^1(0,T;L^\infty(\Omega)) \cap L^2(0,T;(H^1(\Omega))').$$

- The permeability matrix $\boldsymbol{K} \in L^\infty(\Omega_2)^{2\times 2}$ is a symmetric positive definite matrix bounded from above and below by $k_U > 0$ and $k_L > 0$, that is,

$$\forall \xi \in \mathbb{R}^2, \quad k_L \xi \cdot \xi \leq \boldsymbol{K}\xi \cdot \xi \leq k_U \xi \cdot \xi. \tag{4.15}$$

Remark 47. *In the previous chapter, the matrix $\boldsymbol{K}$ is the hydraulic conductivity which is proportional to the ratio of the permeability to the viscosity. Thus it was a property of both the porous media and the fluid. Here $\boldsymbol{K}$ is only related to the porous material.*

- The fluid density ρ is a positive constant.

- The gravitational acceleration $\boldsymbol{g}$ belongs to $L^\infty(\Omega)^2$.

- The coefficient G that appears in the Beavers-Joseph-Saffman interface condition (4.6) is a positive constant that depends on the properties of the porous medium and is determined experimentally [2, 3].

- The porosity ϕ is defined to be the ratio of the void volume to the total volume. There exists $\phi_L > 0$ such that

$$\phi(x) = 1, \quad \text{a.e. in } \Omega_1, \quad \phi_L \leq \phi(x) \leq 1, \quad \text{a.e. in } \Omega_2. \tag{4.16}$$

- The diffusion/dispersion matrix $\boldsymbol{F}(\boldsymbol{u})$ is equal to $d_m\boldsymbol{I}$ in the surface Ω_1 as in river flow dispersion is not that important because of high velocity. In the subsurface Ω_1, $\boldsymbol{F}(\boldsymbol{u})$ depends on the velocity in the following manner [79]:

$$\boldsymbol{F}(\boldsymbol{u}) = \big(\alpha_t\|\boldsymbol{u}\| + d_m\big)\boldsymbol{I} + \big(\alpha_l - \alpha_t\big)\frac{\boldsymbol{u}\boldsymbol{u}^T}{\|\boldsymbol{u}\|},$$

where $d_m > 0$ is the molecular diffusivity constant, $\alpha_l, \alpha_t \geq 0$ are the longitudinal and transverse dispersivities and $\|\cdot\|$ denotes the Euclidean norm. $\boldsymbol{F}(\boldsymbol{u})$ can be shown to be a continuous and bounded function from $\mathbb{R}^2$ to $\mathbb{R}^{2\times 2}$, that is, there exists $F_C > 0$ and $F_B > 0$ such that

$$\boldsymbol{F}(\boldsymbol{w}) \text{ is measurable } \forall \boldsymbol{w} \in \mathbb{R}^2, \quad \|\boldsymbol{F}(\boldsymbol{w})\| \leq F_C\|\boldsymbol{w}\|, \quad \|\boldsymbol{F}(\boldsymbol{w})\| \leq F_B. \tag{4.17}$$

In addition, $\boldsymbol{F}(\boldsymbol{w})$ is assumed to be uniformly positive definite for all $\boldsymbol{w} \in \mathbb{R}^2$, that is,

$$\exists\, \alpha > 0 : \quad \boldsymbol{F}(\boldsymbol{w})\xi \cdot \xi \geq \alpha\, \xi \cdot \xi, \quad \forall \xi \in \mathbb{R}^2. \tag{4.18}$$

- The boundary flux $\mathcal{U}$ belongs to $L^2(0,T;L^2(\Gamma_2))$. Because of the Neumann boundary condition on the subsurface region, the data Π and $\mathcal{U}$ are assumed to satisfy the compatibility condition

$$\int_{\Gamma_2} \mathcal{U} = \int_{\Omega_2} \Pi. \tag{4.19}$$

We assume that there is a subset of Γ_2 of positive measure, corresponding to an outflow boundary, on which $\mathcal{U}$ is positive. From (4.9), we extend $\mathcal{U}$ to Γ_1 by zero and write:

$$\boldsymbol{u} \cdot \boldsymbol{n} = \mathcal{U}, \quad \text{on} \quad \partial\Omega. \tag{4.20}$$

- The function $\mathcal{C}$ is the prescribed concentration on the inflow boundary such that

$$\mathcal{C} \in L^\infty(\Sigma_T), \quad \mathcal{C} \geq 0, \quad \text{a.e. in } \Sigma_T. \tag{4.21}$$

For any function z, we define the negative part z^- and the positive part z^+ as

$$z^- = \frac{|z| - z}{2}, \quad z^+ = \frac{|z| + z}{2}.$$

Note that $z^+ = \max(0, z)$ and $z^- = \max(0, -z)$. Using these definitions, we rewrite (4.11) as

$$\boldsymbol{F}(\boldsymbol{u})\nabla c \cdot \boldsymbol{n} = (\mathcal{C} - c)\mathcal{U}^-, \quad \text{on } \Sigma_T. \tag{4.22}$$

- The initial concentration $c_0 \in L^\infty(\Omega)$ satisfies

$$c_0 \geq 0, \quad \text{a.e in } \Omega. \tag{4.23}$$

We again recall from the preliminary section two trace inequalities and the Poincaré inequality that we use frequently. Let $\mathcal{D}$ be a bounded domain in $\mathbb{R}^2$. There are constants $M_2, M_4 > 0$ such that for any function $z \in H^1(\mathcal{D})$, we have

$$\|z\|_{L^2(\partial\mathcal{D})} \leq M_2\|z\|_{H^1(\mathcal{D})}, \tag{4.24}$$

$$\|z\|_{L^4(\partial\mathcal{D})} \leq M_4\|z\|_{H^1(\mathcal{D})}. \tag{4.25}$$

In addition, if $z \in H^1(\mathcal{D})$ such that $z = 0$ on a subset of $\mathcal{D}$ or $\int_D z d\boldsymbol{x} = 0$, then there exists $M_S, M_P > 0$ satisfying

$$\|z\|_{L^4(\mathcal{D})} \leq M_S\|\nabla z\|_{L^2(\mathcal{D})}, \tag{4.26}$$

$$\|z\|_{L^2(\mathcal{D})} \leq M_P\|\nabla z\|_{L^2(\mathcal{D})}. \tag{4.27}$$

The next section analyzes the problem when the Stokes case is considered for the surface flow.

4.2 Coupling of the Stokes and Darcy Flow with Transport

The following defines a weak formulation based on the model problem.

4.2.1 Weak Formulation

Let us first define the spaces for the Stokes velocity, the Stokes pressure and the Darcy pressure. The first two spaces are the same as in Chapter 3 but the Darcy pressure space is a little different as (4.13) is assumed for uniqueness.

$$\boldsymbol{X} = H^1_{0,\Gamma_1}(\Omega_1)^2, \quad R_1 = L^2(\Omega_1), \quad R_2 = \{q \in H^1(\Omega_2) : \int_{\Omega_2} q = 0\}. \tag{4.28}$$

Also from Chapter 3, Model II, recall the definition of $\tilde{\gamma}$ for the interface terms. For all $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{X}$, and for all $p, q \in R_2$,

$$\tilde{\gamma}(\boldsymbol{u}, p, \boldsymbol{v}, q) = (p, \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}} \boldsymbol{u} \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u} \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}}$$

which takes a nonnegative value when $\boldsymbol{u} = \boldsymbol{v}$, $p = q$.

Definition 48. *The weak formulation of the coupled flow-transport problem defined by (4.2)-(4.13) is to find* $\boldsymbol{u}|_{\Omega_1} \in L^2(0, T; \boldsymbol{X})$, $p \in L^2(0, T; R_1)$, $\varphi \in L^2(0, T; R_2)$ *and* $c \in L^2(0, T; H^1(\Omega)) \cap L^\infty(Q_T)$ *such that*

$$t \to c(\cdot, t) \in \mathcal{C}([0, T]; (H^1(\Omega))'), \quad t \to \frac{\partial c}{\partial t}(\cdot, t) \in L^2(0, T; (H^1(\Omega))'), \text{ and} \tag{4.29}$$

$$c(\cdot, 0) = c_0(\cdot), \qquad \text{a.e. in } \Omega \tag{4.30}$$

satisfying for all $\boldsymbol{v} \in L^2(0,T;\boldsymbol{X})$, $r \in L^2(0,T;R_1)$ *and* $q \in L^2(0,T;R_2)$,

$$\int_0^T \Big(2(\mu(c)\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}),\nabla q)_{\Omega_2} - (\nabla\cdot\boldsymbol{v},p)_{\Omega_1}$$
$$+ (\nabla\cdot\boldsymbol{u},r)_{\Omega_1} + \gamma(\boldsymbol{u},\varphi;\boldsymbol{v},q)\Big)dt = \int_0^T \Big((\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (\Pi,q)_{\Omega_2} - (\mathcal{U},q)_{\Gamma_2}\Big)dt \quad (4.31)$$

and for all $\psi \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T \langle\phi\frac{\partial c}{\partial t},\psi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt + \int_{Q_T}(\boldsymbol{F}(\boldsymbol{u})\nabla c - c\boldsymbol{u})\cdot\nabla\psi d\boldsymbol{x}dt + \int_{\Sigma_T}(c\mathcal{U}^+ - \mathcal{C}\mathcal{U}^-)\psi d\sigma dt$$
$$= \int_0^T \langle\Lambda,\psi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt. \quad (4.32)$$

The velocity $\boldsymbol{u}|_{\Omega_2} \in L^2(0,T;L^2(\Omega_2)^2)$ in the Darcy region Ω_2 is obtained from the Darcy pressure φ by the equation

$$\boldsymbol{u} = -\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \quad \text{a.e. in } \Omega_2\times(0,T). \quad (4.33)$$

Derivation of the weak formulation :

Let $\psi \in L^2(0,T;H^1(\Omega))$. Multiply (4.8) by ψ, integrate over Q_T and use Green's formula:

$$\int_{Q_T}\frac{\partial}{\partial t}(\phi c)\psi d\boldsymbol{x}dt + \int_{Q_T}(\boldsymbol{F}(\boldsymbol{u})\nabla c - c\boldsymbol{u})\cdot\nabla\psi d\boldsymbol{x}dt - \int_{\Sigma_T}(\boldsymbol{F}(\boldsymbol{u})\nabla c - c\boldsymbol{u})\cdot\boldsymbol{n}\psi d\sigma dt$$
$$= \int_0^T \langle\Lambda,\psi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt.$$

Assuming $\phi\frac{\partial c}{\partial t} \in L^2(0,T;(H^1(\Omega))')$, and observing from (4.22) that

$$(\boldsymbol{F}(\boldsymbol{u})\nabla c - c\boldsymbol{u})\cdot\boldsymbol{n} = \boldsymbol{F}(\boldsymbol{u})\nabla c\cdot\boldsymbol{n} - c(\boldsymbol{u}\cdot\boldsymbol{n})^+ + c(\boldsymbol{u}\cdot\boldsymbol{n})^- = \mathcal{C}\mathcal{U}^- - c\mathcal{U}^+,$$

we obtain

$$\int_0^T \langle\frac{\phi\partial c}{\partial t},\psi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt + \int_{Q_T}(\boldsymbol{F}(\boldsymbol{u})\nabla c - c\boldsymbol{u})\cdot\nabla\psi d\boldsymbol{x}dt + \int_{\Sigma_T}(\mathcal{C}\mathcal{U}^+ - c\mathcal{U}^-)\psi d\sigma dt$$
$$= \int_0^T \langle\Lambda,\psi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt$$

which yields (4.32). The weak formulation for the flow part is gathered similarly as in [38].

4.2.2 Existence of a Weak Solution

The following theorem gives the main result of this section which is the existence of a weak solution.

Theorem 49. *There exists a weak solution $(\boldsymbol{u}, p, \varphi, c)$ to the problem defined in Definition 48. In addition, $(\boldsymbol{u}, \varphi)$ satisfies*

$$2\mu_L \|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(0,T;L^2(\Omega_2)^2)} \leq \frac{M_P^2 C_D^2}{2\mu_L}\|\boldsymbol{\Psi}\|^2_{L^2(0,T;L^2(\Omega_1)^2)} + \frac{3\mu_U}{k_L}\left(M_P^2\|\Pi\|^2_{L^2(0,T;L^2(\Omega_2))} + M_2^2\|\mathcal{U}\|^2_{L^2(0,T;L^2(\Gamma_{12}))} + \frac{\rho^2 T^2}{\mu_L}\right), \quad (4.34)$$

and c satisfies

$$0 \leq c \leq \left\|\frac{\Lambda}{\phi}\right\|_{L^1(0,T;L^\infty(\Omega))} + \max(\|c_0\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}), \quad a.e.\ in\ Q_T. \quad (4.35)$$

The existence result is shown using the method in Chapter 3, which is working on the space of divergence-free functions $\boldsymbol{V}$ defined by

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{X} : \nabla\cdot\boldsymbol{v} = 0 \text{ in } \Omega_1\}.$$

Using this space another variational formulation of (4.31) is defined where the Stokes pressure term p is eliminated, that is,

Find $\boldsymbol{u}|_{\Omega_1} \in L^2(0,T;\boldsymbol{V})$ and $\varphi \in L^2(0,T;R_2)$ such that for all $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V})$ and for all $q \in L^2(0,T;R_2)$,

$$\int_0^T \Big(2(\mu(c)\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \nabla q)_{\Omega_2} + \tilde{\gamma}(\boldsymbol{u},\varphi;\boldsymbol{v},q)\Big)dt = \int_0^T ((\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} - (\mathcal{U}, q)_{\Gamma_2})\,dt. \quad (4.36)$$

The following states the existence theorem for this new problem.

Theorem 50. *There exist* $\boldsymbol{u}|_{\Omega_1} \in L^2(0,T;\boldsymbol{V})$, $\varphi \in L^2(0,T;R_2)$ *and* $c \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega))$ *satisfying the equations (4.29), (4.30), (4.36), (4.32), (4.33) and the stability bounds (4.34) and (4.35).*

The proof follows a similar technique as in [52, 53] and is based on a Galerkin approach in time and consists of several steps. First an intermediate result and related estimates are proved. This solution to the intermediate problem is then used in the definition of the approximate solution. Then passing to the limit in this approximate definition proves existence result for the restricted problem as stated in the Theorem 50. Finally the main result Theorem 64 is deduced by recovering the Stokes pressure p which was lost due to the restriction to $\boldsymbol{V}$.

Approximate solution Extra notation is necessary for both the intermediate and the approximate problems. For a fixed positive integer N, let $\Delta t = \frac{T}{N}$. Let $t_i = i\Delta t$, $i = 0, \ldots, N$. Next, for any Banach space B and for any $z \in L^1(0,T;B)$, define averages at each time step by

$$\overline{z}_0^N = 0, \quad \overline{z}_i^N = \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} z(t)dt, \quad i = 1, \ldots, N. \tag{4.37}$$

This averaging technique is applied to the source terms Λ, $\boldsymbol{\Psi}$, Π, the boundary flux $\mathcal{U}$ and the inflow concentration $\mathcal{C}$ to obtain

$$\overline{\Lambda}^N = (\overline{\Lambda}_0^N, \ldots, \overline{\Lambda}_N^N), \quad \overline{\boldsymbol{\Psi}}^N = (\overline{\boldsymbol{\Psi}}_0^N, \ldots, \overline{\boldsymbol{\Psi}}_N^N), \quad \overline{\Pi}^N = (\overline{\Pi}_0^N, \ldots, \overline{\Pi}_N^N),$$

$$\overline{\mathcal{U}}^N = (\overline{\mathcal{U}}_0^N, \ldots, \overline{\mathcal{U}}_N^N), \quad \overline{\mathcal{C}}^N = (\overline{\mathcal{C}}_0^N, \ldots, \overline{\mathcal{C}}_N^N).$$

Observe that for any $z \in L^\infty(0,T;B)$ and $i = 1, \ldots, N$,

$$\|\overline{z}_i^N\|_B = \Big\|\frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} z(x,t)dt\Big\|_B \leq \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} \|z(x,t)\|_B dt \leq \|z\|_{L^\infty(0,T;B)}.$$

Hence,

$$\|\overline{z}_i^N\|_B \le \|z\|_{L^\infty(0,T;B)}, \quad i = 0, \dots N. \tag{4.38}$$

Also for any $z \in L^p(0,T;B)$, $1 \le p < \infty$, Hölder's inequality imply that for any $i = 1, \dots, N$,

$$\|\overline{z}_i^N\|_B \le \frac{1}{\Delta t}\int_{(i-1)\Delta t}^{i\Delta t} \|z(x,t)\|_B dt \le \frac{1}{(\Delta t)^{\frac{1}{p}}}\left(\int_{(i-1)\Delta t}^{i\Delta t} \|z(x,t)\|_B^p dt\right)^{\frac{1}{p}}.$$

Therefore, $1 \le p < \infty$,

$$\|\overline{z}_i^N\|_B \le \frac{1}{(\Delta t)^{\frac{1}{p}}}\|z\|_{L^p(0,T;B)}, \quad i = 0, \dots N. \tag{4.39}$$

The following proposition introduces the intermediate problem to (4.36) and (4.32).

Proposition 51. *For $n = 0, \dots, N-1$, given $C_n^N \in L^2(\Omega)$, there exists a unique $(\boldsymbol{U}_{n+1}^N, \Phi_{n+1}^N) \in \boldsymbol{V} \times R_2$ satisfying*

$$(P)\begin{cases} \forall(\boldsymbol{v}, q) \in \boldsymbol{V} \times R_2, \quad 2(\mu(C_n^N)\boldsymbol{D}(\boldsymbol{U}_{n+1}^N), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(C_n^N)}(\nabla\Phi_{n+1}^N - \rho\boldsymbol{g}), \nabla q)_{\Omega_2} \\ \qquad +\tilde{\gamma}(\boldsymbol{U}_{n+1}^N, \Phi_{n+1}^N; \boldsymbol{v}, q) = (\overline{\boldsymbol{\Psi}}_{n+1}^N, \boldsymbol{v})_{\Omega_1} + (\overline{\Pi}_{n+1}^N, q)_{\Omega_2} - (\overline{\mathcal{U}}_{n+1}^N, q)_{\Gamma_2}. \end{cases}$$

In Ω_2, if $\boldsymbol{U}_{n+1}^N \in L^2(\Omega_2)^2$ is defined as

$$\boldsymbol{U}_{n+1}^N = -\frac{\boldsymbol{K}}{\mu(C_n^N)}(\nabla\Phi_{n+1}^N - \rho\boldsymbol{g}), \quad in \quad \Omega_2, \tag{4.40}$$

then it satisfies

$$\nabla \cdot \boldsymbol{U}_{n+1}^N = \overline{\Pi}_{n+1}^N, \quad in \quad \Omega_2 \tag{4.41}$$

and

$$\boldsymbol{U}_{n+1}^N \cdot \boldsymbol{n} = \overline{\mathcal{U}}_{n+1}^N, \quad in \quad \Gamma_2. \tag{4.42}$$

Furthermore,

$$2\mu_L\|\boldsymbol{D}(\boldsymbol{U}_{n+1}^N)\|_{L^2(\Omega_1)}^2 + \frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\Phi_{n+1}^N\|_{L^2(\Omega_2)}^2 \le (\mathcal{M}_D^n)^2, \tag{4.43}$$

where $\mathcal{M}_D^n$ *is a constant independent of* $\boldsymbol{U}_{n+1}^N$ *and* Φ_{n+1}^N *and defined by*

$$\mathcal{M}_D^n = \Big(\frac{M_P^2 C_D^2}{2\mu_L} \|\overline{\boldsymbol{\Psi}}_{n+1}^N\|_{L^2(\Omega_1)}^2 + \frac{3\mu_U}{k_L} \big(M_P^2 \|\overline{\Pi}_{n+1}^N\|_{L^2(\Omega_1)}^2 + M_2^2 \|\overline{\mathcal{U}}_{n+1}^N\|_{L^2(\Gamma_1)}^2 + \frac{\rho^2}{\mu_L^2} \|\boldsymbol{K}\boldsymbol{g}\|_{L^2(\Omega_2)}^2 \big) \Big)^{\frac{1}{2}}. \quad (4.44)$$

Proof. The proof of the existence of $(\boldsymbol{U}_{n+1}^N, \Phi_{n+1}^N)$ in a ball of radius $\mathcal{M}_D^n$ with respect to the norm

$$\|(\boldsymbol{v}, q)\| = \Big(2\mu_L \|\boldsymbol{D}(\boldsymbol{U}_{n+1}^N)\|_{L^2(\Omega_1)}^2 + \frac{1}{\mu_U} \|\boldsymbol{K}^{\frac{1}{2}} \nabla \Phi_{n+1}^N\|_{L^2(\Omega_2)}^2 \Big)^{\frac{1}{2}}$$

can be established by a slight modification of the existence proof of [38] which involves a Galerkin approximation and a variant of Brouwer's fixed point theorem. For other proofs refer to [41, 40].

To obtain (4.41), let $\boldsymbol{v} = 0$ and $q \in \mathcal{C}_0^\infty(\Omega_2)$ in (P). Then using (4.40),

$$-(\boldsymbol{U}_{n+1}^N, \nabla q)_{\Omega_2} = (\overline{\Pi}_{n+1}^N, q)_{\Omega_2}.$$

So (4.41) holds in the distributional sense, that is,

$$\nabla \cdot \boldsymbol{U}_{n+1}^N = \overline{\Pi}_{n+1}^N, \quad \text{in } \Omega_2.$$

To show (4.42), let $\boldsymbol{v} \in \mathcal{C}_0^\infty(\Omega_1)^2$ and $q = 0$ in (P). Then

$$2(\mu(C_n^N)\boldsymbol{D}(\boldsymbol{U}_{n+1}^N), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} = (\overline{\boldsymbol{\Psi}}_{n+1}^N, \boldsymbol{v})_{\Omega_1}$$

and together with (3.15) the definition of weak derivatives yields

$$-2\nabla \cdot (\mu(C_n^N)\boldsymbol{D}(\boldsymbol{U}_{n+1}^N)) = \overline{\boldsymbol{\Psi}}_{n+1}^N, \quad \text{in } \Omega_1, \quad (4.45)$$

in the distributional sense. Multiplying this by $\boldsymbol{v} \in \boldsymbol{X}$, integrating over Ω_1 and using Green's formula, we obtain

$$(2(\mu(C_n^N)\boldsymbol{D}(\boldsymbol{U}_{n+1}^N), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} - (2\mu(C_n^N)\boldsymbol{D}(\boldsymbol{U}_{n+1}^N)\boldsymbol{n}, \boldsymbol{v})_{\partial\Omega_1} = (\overline{\boldsymbol{\Psi}}_{n+1}^N, \boldsymbol{v})_{\Omega_1}. \quad (4.46)$$

Next, multiply (4.41) by $q \in R_2$ and use Green's formula to get

$$-(\boldsymbol{U}_{n+1}^N, \nabla q)_{\Omega_2} + (\boldsymbol{U}_{n+1}^N \cdot \boldsymbol{n}, q)_{\partial\Omega_2} = (\overline{\Pi}_{n+1}^N, q)_{\Omega_2}.$$

Adding this to (4.46), comparing the sum with (P) and using (4.40) yields,

$$(2\mu(C_n^N)\boldsymbol{D}(\boldsymbol{U}_{n+1}^N)\boldsymbol{n}_{\Omega_1}, \boldsymbol{v})_{\partial\Omega_1} + \tilde{\gamma}(\boldsymbol{U}_{n+1}^N, \Phi_{n+1}^N; \boldsymbol{v}, q) - (\boldsymbol{U}_{n+1}^N \cdot \boldsymbol{n}, q)_{\partial\Omega_2} = -(\overline{\mathcal{U}}_{n+1}^N, q)_{\Gamma_2}.$$

Letting $\boldsymbol{v} = 0$ in this equation and choosing q such that $q = 0$ on Γ_{12} implies

$$(\boldsymbol{U}_{n+1}^N \cdot \boldsymbol{n}, q)_{\Gamma_2} = (\overline{\mathcal{U}}_{n+1}^N, q)_{\Gamma_2}.$$

Therefore, (4.42) holds. □

Proposition 52. *For $n = 0, 1, \ldots, N-1$, given $C_n^N \in L^2(\Omega)$, there exists $C_{n+1}^N \in H^1(\Omega)$ satisfying*

$$0 \le C_{n+1}^N(x) \le \Delta t \left\| \frac{\overline{\Lambda}_{n+1}^N}{\phi} \right\|_{L^\infty(\Omega)} + \max\left(\|C_n^N\|_{L^\infty(\Omega)}, \|\overline{\mathcal{C}}_{n+1}^N\|_{L^\infty(\partial\Omega)} \right), \quad a.e.\ x \in \Omega \tag{4.47}$$

and for all $\psi \in H^1(\Omega)$,

$$\frac{1}{\Delta t}\int_\Omega \phi(C_{n+1}^N - C_n^N)\psi d\boldsymbol{x} + \int_\Omega (\boldsymbol{F}(\boldsymbol{U}_{n+1}^N)\nabla C_{n+1}^N - C_{n+1}^N \boldsymbol{U}_{n+1}^N) \cdot \nabla\psi d\boldsymbol{x}$$
$$+ \int_{\partial\Omega} (C_{n+1}^N(\overline{\mathcal{U}}_{n+1}^N)^+ - \overline{\mathcal{C}}_{n+1}^N(\overline{\mathcal{U}}_{n+1}^N)^-)\psi d\sigma = \int_\Omega \overline{\Lambda}_{n+1}^N \psi d\boldsymbol{x}. \tag{4.48}$$

where $\boldsymbol{U}_{n+1}^N$ is defined in Proposition 51.

Proof. In the following, the superscript N is dropped for convenience. Let

$$\mathcal{M} = \Delta t \left\| \frac{\overline{\Lambda}_{n+1}}{\phi} \right\|_{L^\infty(\Omega)} + \max\left(\|C_n\|_{L^\infty(\Omega)}, \|\overline{\mathcal{C}}_{n+1}\|_{L^\infty(\partial\Omega)} \right).$$

Define a bounded piecewise function H on $\mathbb{R}$ by

$$H(x) = \begin{cases} 0, & \text{if } x \le 0, \\ x, & \text{if } 0 \le x \le \mathcal{M}, \\ \mathcal{M}, & \text{if } x \ge \mathcal{M}. \end{cases}$$

The existence of $C_{n+1} \in H^1(\Omega)$ will be shown such that for all $\psi \in H^1(\Omega)$,

$$\frac{1}{\Delta t}\int_\Omega \phi(C_{n+1}-C_n)\psi d\boldsymbol{x} + \int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla\psi d\boldsymbol{x} - \int_\Omega H(C_{n+1})\boldsymbol{U}_{n+1}\cdot\nabla\psi d\boldsymbol{x}$$
$$+\int_{\partial\Omega}(C_{n+1}(\overline{\mathcal{U}}_{n+1})^+ - \overline{C}_{n+1}(\overline{\mathcal{U}}_{n+1})^-)\psi d\sigma = \int_\Omega \overline{\Lambda}_{n+1}\psi d\boldsymbol{x}. \quad (4.49)$$

Observe that the solution to (4.49) solves (4.47) and (4.48) if $0 \le C_{n+1} \le \mathcal{M}$, a.e. in Ω. Theorem 9 (Schauder's fixed point theorem) is suitable to show that such a solution exists. Define an operator $\theta : L^2(\Omega) \to L^2(\Omega)$ by $\theta(w) = v$ where v is the unique function of $H^1(\Omega)$ such that for any $\psi \in H^1(\Omega)$,

$$\frac{1}{\Delta t}\int_\Omega \phi\, v\psi d\boldsymbol{x} + \int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla v\cdot\nabla\psi d\boldsymbol{x} + \int_{\partial\Omega} v\,(\overline{\mathcal{U}}_{n+1})^+\psi d\sigma$$
$$= \frac{1}{\Delta t}\int_\Omega \phi\, C_n\psi d\boldsymbol{x} + \int_\Omega H(w)\boldsymbol{U}_{n+1}\cdot\nabla\psi d\boldsymbol{x} + \int_{\partial\Omega}\overline{C}_{n+1}(\overline{\mathcal{U}}_{n+1})^-\psi d\sigma + \int_\Omega \overline{\Lambda}_{n+1}\psi d\boldsymbol{x}. \quad (4.50)$$

Clearly, any fixed point of (4.50) is a solution to (4.49). Well-definition of θ comes from the Lax-Milgram theorem. Indeed, define a bilinear form $\mathcal{B}$ by

$$\mathcal{B}(v,\psi) = \frac{1}{\Delta t}\int_\Omega \phi\, v\psi d\boldsymbol{x} + \int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla v\cdot\nabla\psi d\boldsymbol{x} + \int_{\partial\Omega} v\,(\overline{\mathcal{U}}_{n+1})^+\psi d\sigma,$$

and a linear form $\mathcal{L}$ by

$$\mathcal{L}(\psi) = \frac{1}{\Delta t}\int_\Omega \phi\, C_n\psi d\boldsymbol{x} + \int_\Omega H(w)\boldsymbol{U}_{n+1}\cdot\nabla\psi d\boldsymbol{x} + \int_{\partial\Omega}\overline{C}_{n+1}(\overline{\mathcal{U}}_{n+1})^-\psi d\sigma + \int_\Omega \overline{\Lambda}_{n+1}\psi d\boldsymbol{x}.$$

Then from the Cauchy-Schwarz inequality, (4.16), (4.17), (4.25) and (4.39),

$$|\mathcal{B}(v,\psi)| \le \frac{1}{\Delta t}\|v\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} + \|\boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla v\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}$$
$$+ \|v\|_{L^4(\partial\Omega)}\|(\overline{\mathcal{U}}_{n+1})^+\|_{L^2(\partial\Omega)}\|\psi\|_{L^4(\partial\Omega)}$$
$$\le \frac{1}{\Delta t}\|v\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)} + F_B\|v\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)} + M_4^2\|v\|_{H^1(\Omega)}\|\overline{\mathcal{U}}_{n+1}\|_{L^2(\partial\Omega)}\|\psi\|_{H^1(\Omega)}$$
$$= \left(\frac{1}{\Delta t} + F_B + \frac{M_4^2}{\Delta t^{\frac{1}{2}}}\|\mathcal{U}\|_{L^2(0,T;L^2(\partial\Omega))}\right)\|v\|_{H^1(\Omega)}\|\psi\|_{H^1(\Omega)}.$$

Thus, $\mathcal{B}$ is continuous. Coercivity of $\mathcal{B}$ follows from (4.16) and (4.18).

$$\begin{aligned}\mathcal{B}(v,v) &= \frac{1}{\Delta t}\int_\Omega \phi\, v^2 d\boldsymbol{x} + \int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla v\cdot\nabla v d\boldsymbol{x} + \int_{\partial\Omega}\overline{\mathcal{U}}^+ v^2 d\sigma\\ &\geq \frac{\phi_L}{\Delta t}\|v\|^2_{L^2(\Omega)} + \alpha\|\nabla v\|^2_{L^2(\Omega)} \geq \min\left(\frac{\phi_L}{\Delta t},\alpha\right)\|v\|^2_{H^1(\Omega)}.\end{aligned}$$

Finally, using the bound on the function H, the Cauchy-Schwarz inequality, (4.16), (4.24), (4.38) and (4.39), $\mathcal{L}$ is continuous as shown below:

$$\begin{aligned}|\mathcal{L}(\psi)| &\leq \frac{1}{\Delta t}\|C_n\|_{L^2(\Omega)}\|\psi\|_{L^2(\Omega)} + \mathcal{M}\|\boldsymbol{U}_{n+1}\|_{L^2(\Omega)}\|\nabla\psi\|_{L^2(\Omega)}\\ &\quad + \|\overline{\mathcal{C}}_{n+1}\|_{L^\infty(\partial\Omega)}\|\overline{\mathcal{U}}^-_{n+1}\|_{L^2(\partial\Omega)}\|\psi\|_{L^2(\partial\Omega)} + \|\overline{\Lambda}_{n+1}\|_{(H^1(\Omega))'}\|\psi\|_{H^1(\Omega)}\\ &\leq \Big(\frac{1}{\Delta t}\|C_n\|_{L^2(\Omega)} + \mathcal{M}\|\boldsymbol{U}_{n+1}\|_{L^2(\Omega)}\\ &\quad + \frac{1}{\Delta t^{\frac{1}{2}}}\big(M_2\|\mathcal{C}\|_{L^\infty(\Sigma_T)}\|\mathcal{U}\|_{L^2(0,T;L^2(\partial\Omega))} + \|\Lambda\|_{L^2(0,T;(H^1(\Omega))')}\big)\Big)\|\psi\|_{H^1(\Omega)}.\end{aligned}$$

Hence from Lax-Milgram theorem there exists a unique $v\in H^1(\Omega)$ such that $\mathcal{B}(v,\psi) = \mathcal{L}(\psi)$ for any $\psi\in H^1(\Omega)$.

Schauder's fixed point theorem requires that θ is continuous and $\theta(L^2(\Omega))$ is relatively compact in $L^2(\Omega)$. The relative compactness property will follow from Rellich-Kondrachov theorem [60, see remark 6.3] once $\theta(L^2(\Omega))$ is shown to be bounded in $H^1(\Omega)$. In (4.50), take $\psi = v$,

$$\begin{aligned}&\frac{1}{\Delta t}\int_\Omega \phi\, v^2 d\boldsymbol{x} + \int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla v\cdot\nabla v d\boldsymbol{x} + \int_{\partial\Omega}(\overline{\mathcal{U}}_{n+1})^+ v^2 d\sigma\\ &= \frac{1}{\Delta t}\int_\Omega \phi C_n v d\boldsymbol{x} + \int_\Omega H(w)\boldsymbol{U}_{n+1}\cdot\nabla v d\boldsymbol{x} + \int_{\partial\Omega}\overline{\mathcal{C}}_{n+1}(\overline{\mathcal{U}}_{n+1})^- v d\sigma + \int_\Omega \overline{\Lambda}_{n+1} v d\boldsymbol{x}.\end{aligned}$$

Therefore, by positiveness of the third term, boundedness of H, (4.16), (4.38), (4.39), (4.18) and (4.24),

$$\frac{\phi_L}{\Delta t}\|v\|^2_{L^2(\Omega)} + \alpha\|\nabla v\|^2_{L^2(\Omega)} \leq \frac{1}{\Delta t}\|C_n\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)} + \mathcal{M}\|\boldsymbol{U}_{n+1}\|_{L^2(\Omega)}\|\nabla v\|_{L^2(\Omega)}$$
$$+ \|\overline{\mathcal{C}}_{n+1}\|_{L^\infty(\partial\Omega)}\|\overline{\mathcal{U}}_{n+1}\|_{L^2(\partial\Omega)}\|v\|_{L^2(\partial\Omega)} + \|\overline{\Lambda}_{n+1}\|_{(H^1(\Omega))'}\|v\|_{H^1(\Omega)}$$
$$\leq \mathcal{A}\|\boldsymbol{v}\|_{H^1(\Omega)},$$

where

$$\mathcal{A} = \Big(\frac{1}{\Delta t}\|C_n\|_{L^2(\Omega)} + \mathcal{M}\|\boldsymbol{U}_{n+1}\|_{L^2(\Omega)} + M_2\|\overline{\mathcal{C}}_{n+1}\|_{L^\infty(\partial\Omega)}\|\boldsymbol{U}_{n+1}\|_{L^2(\partial\Omega)} + \|\overline{\Lambda}_{n+1}\|_{(H^1(\Omega))'}\Big).$$

Therefore,

$$\|v\|_{H^1(\Omega)} \leq \frac{\mathcal{A}}{\min(\frac{\phi_L}{2\Delta t}, \frac{\alpha}{2})} \tag{4.51}$$

which means that $\theta(L^2(\Omega))$ is bounded in $H^1(\Omega)$ as and $\mathcal{A}$ is independent of w.

To show the continuity of θ, let $\{w_k\}_k$ be a sequence in $L^2(\Omega)$ such that $w_k \to w$ in $L^2(\Omega)$. Let $v_k = \theta(w_k)$. The convergence $v_k \to \theta(w)$ in $L^2(\Omega)$ will be shown by using the estimate (4.51). First from Lemma 1, convergence of $\{w_k\}_k$ to w in $L^2(\Omega)$ implies that there exists a subsequence w_{k_j}, $w_{k_j} \to w$ a.e. in Ω as $j \to \infty$. As $H(w)$ is bounded and continuous in w, $H(w_{k_j}) \to H(w)$ a.e in Ω as $j \to \infty$. Then by the Lebesgue dominated convergence theorem,

$$H(w_{k_j}) \to H(w) \text{ strongly in } L^2(\Omega). \tag{4.52}$$

By (4.51), $\{v_{k_j}\}_j$ is bounded in $H^1(\Omega)$ so there exists a subsequence still denoted by $\{v_{k_j}\}_j$ such that

$$v_{k_j} \to v \text{ weakly in } H^1(\Omega) \tag{4.53}$$

for some $v \in H^1(\Omega)$. As $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, again, up to a subsequence,

$$v_{k_j} \to v \text{ strongly in } L^2(\Omega). \tag{4.54}$$

Since the trace function is continuous from $L^2(\Omega)$ to $L^2(\partial\Omega)$,

$$v_{k_j} \to v \text{ strongly in } L^2(\partial\Omega). \tag{4.55}$$

Consider (4.50) with v_{k_j} and w_{k_j} in place of v and w. With the above convergence results (4.52), (4.53), (4.54) and (4.55), passing to the limit in (4.50) yields $v = \theta(w)$. Hence $v_{k_j} \to v = \theta(w)$ strongly in $L^2(\Omega)$. Similarly, every subsequence of $\{v_k\}_k$ converging in $L^2(\Omega)$ has limit $\theta(w)$. Therefore $\{v_k\}_k$ has a unique accumulation point. As $\theta(L^2(\Omega))$ is relatively compact in $L^2(\Omega)$, $\theta(w_k) = v_k \to \theta(w)$ in $L^2(\Omega)$. Hence θ is continuous which concludes that there exists a fixed point $C_{n+1} \in H^1(\Omega)$ satisfying (4.50), hence yielding a solution to (4.49).

Next step is to show that $0 \le C_{n+1} \le \mathcal{M}$, a.e. in Ω which proves (4.47) and also implies that $H(C_{n+1}) = C_{n+1}$. This will give (4.48).

Let us first show $C_{n+1} \ge 0$, a.e. in Ω. From Stampacchia [80, p.50], $C_{n+1}^- \in H^1(\Omega)$. In (4.49), let $\psi = -C_{n+1}^-$

$$\begin{aligned} -\frac{1}{\Delta t}\int_\Omega \phi(C_{n+1} - C_n)C_{n+1}^- d\boldsymbol{x} + \int_\Omega H(C_{n+1})\boldsymbol{u}\cdot\nabla C_{n+1}^- d\boldsymbol{x} \\ - \int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla C_{n+1}^- d\boldsymbol{x} - \int_{\partial\Omega}(C_{n+1}\overline{\mathcal{U}}_{n+1}^+ - \overline{\mathcal{C}}_{n+1}\overline{\mathcal{U}}_{n+1}^-)C_{n+1}^- d\sigma \\ + \int_\Omega \overline{\Lambda}_{n+1}C_{n+1}^- d\boldsymbol{x} = 0. \end{aligned}$$

Observe that for any function z,

$$zz^- = \left\{ \begin{array}{cc} -(z^-)^2, & \text{if } z < 0, \\ 0, & \text{otherwise} \end{array} \right\} = -(z^-)^2.$$

Similarly, $\boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla C_{n+1}^- = -\boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}^-\cdot\nabla C_{n+1}^-$. The second term in the equation vanishes since for $C_{n+1} \le 0$, $H(C_{n+1}) = 0$ and for $C_{n+1} \ge 0$, $C_{n+1}^- = 0$.

Therefore,

$$\frac{1}{\Delta t}\int_{\Omega}\phi(C_{n+1}^{-})^{2}d\boldsymbol{x}+\frac{1}{\Delta t}\int_{\Omega}\phi\, C_{n}C_{n+1}^{-}d\boldsymbol{x}+\int_{\Omega}F(\boldsymbol{U}_{n+1})\nabla C_{n+1}^{-}\cdot\nabla C_{n+1}^{-}d\boldsymbol{x}$$
$$+\int_{\partial\Omega}(C_{n+1}^{-})^{2}\overline{\mathcal{U}}_{n+1}^{+}d\sigma+\int_{\partial\Omega}\overline{\mathcal{C}}_{n+1}\overline{\mathcal{U}}_{n+1}^{-}C_{n+1}^{-}d\sigma+\int_{\Omega}\overline{\Lambda}_{n+1}C_{n+1}^{-}d\boldsymbol{x}=0.$$

Observe that $C_0\geq 0$ and $\overline{\mathcal{U}}_{n+1}^{-},\overline{\mathcal{U}}_{n+1}^{+},\overline{\mathcal{C}}_{n+1},\overline{\Lambda}_{n+1}\geq 0$, for all $n\geq 0$. This, together with (4.18) shows that

$$\frac{1}{\Delta t}\int_{\Omega}\phi(C_{1}^{-})^{2}d\boldsymbol{x}+\frac{1}{\Delta t}\int_{\Omega}\phi\, C_{0}C_{1}^{-}d\boldsymbol{x}+\alpha\int_{\Omega}|\nabla C_{1}^{-}|d\boldsymbol{x}+\int_{\partial\Omega}(C_{1}^{-})^{2}\overline{\mathcal{U}}_{1}^{+}d\sigma$$
$$+\int_{\partial\Omega}\overline{\mathcal{C}}_{1}\overline{\mathcal{U}}_{1}^{-}C_{1}^{-}d\sigma+\int_{\Omega}\overline{\Lambda}_{1}C_{1}^{-}d\boldsymbol{x}=0,$$

in which all the terms except the first one are nonnegative. Hence $\frac{1}{\Delta t}\int_{\Omega}\phi(C_1^-)^2d\boldsymbol{x}\leq 0$. This implies $C_1^-=0$, a.e. in Ω as $\phi>0$. In other words, $C_1\geq 0$, a.e. in Ω. Then an induction argument shows that $C_n\geq 0$, a.e. in Ω for all $n\geq 0$.

Now we will show $C_{n+1}\leq\mathcal{M}$, a.e. in Ω by proving that $(C_{n+1}-\mathcal{M})^{+}=0$, a.e. in Ω. As before, from [80], $(C_{n+1}-\mathcal{M})^{+}\in H^{1}(\Omega)$. So let $\psi=(C_{n+1}-\mathcal{M})^{+}$ in (4.49).

$$\frac{1}{\Delta t}\int_{\Omega}\phi(C_{n+1}-C_{n})(C_{n+1}-\mathcal{M})^{+}d\boldsymbol{x}-\int_{\Omega}H(C_{n+1})\boldsymbol{U}_{n+1}\cdot\nabla(C_{n+1}-\mathcal{M})^{+}d\boldsymbol{x}$$
$$+\int_{\Omega}\boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla(C_{n+1}-\mathcal{M})^{+}d\boldsymbol{x}+\int_{\partial\Omega}(C_{n+1}\,\overline{\mathcal{U}}_{n+1}^{+}-\overline{\mathcal{C}}_{n+1}\overline{\mathcal{U}}^{-})(C_{n+1}-\mathcal{M})^{+}d\sigma$$
$$-\int_{\Omega}\overline{\Lambda}_{n+1}(C_{n+1}-\mathcal{M})^{+}d\boldsymbol{x}=0.\quad(4.56)$$

Note that

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla(C_{n+1}-\mathcal{M})^{+}&=\boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla(C_{n+1}-\mathcal{M})\cdot\nabla(C_{n+1}-\mathcal{M})^{+}\\&=\boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla(C_{n+1}-\mathcal{M})^{+}\cdot\nabla(C_{n+1}-\mathcal{M})^{+}.\end{aligned}$$

So, the third term in (4.56) is positive by (4.18). Now let

$$I=-\int_{\Omega}H(C_{n+1})\boldsymbol{U}_{n+1}\cdot\nabla(C_{n+1}-\mathcal{M})^{+}d\boldsymbol{x}+\int_{\partial\Omega}(C_{n+1}\overline{\mathcal{U}}_{n+1}^{+}-\overline{\mathcal{C}}_{n+1}\overline{\mathcal{U}}_{n+1}^{-})(C_{n+1}-\mathcal{M})^{+}d\sigma.$$

From the definition of H, we have

$$H(C_{n+1})\boldsymbol{U}_{n+1}\cdot\nabla(C_{n+1}-\mathcal{M})^+ = \mathcal{M}\boldsymbol{U}_{n+1}\cdot\nabla(C_{n+1}-\mathcal{M})^+, \text{ a.e. in } \Omega.$$

This and the Green's formula gives

$$I = \int_\Omega \mathcal{M}\,\nabla\cdot\boldsymbol{U}_{n+1}(C_{n+1}-\mathcal{M})^+ d\boldsymbol{x} - \int_{\partial\Omega}\mathcal{M}\overline{\mathcal{U}}_{n+1}(C_{n+1}-\mathcal{M})^+ d\sigma$$
$$+ \int_{\partial\Omega}(C_{n+1}\overline{\mathcal{U}}^+_{n+1} - \overline{\mathcal{C}}_{n+1}\overline{\mathcal{U}}^-_{n+1})(C_{n+1}-\mathcal{M})^+ d\sigma.$$

Then by (4.41) and (4.42), we obtain

$$I = \int_{\Omega_2}\mathcal{M}\,\overline{\Pi}_{n+1}(C_{n+1}-\mathcal{M})^+ d\boldsymbol{x} + \int_{\partial\Omega}(C_{n+1}-\mathcal{M})\overline{\mathcal{U}}^+_{n+1}(C_{n+1}-\mathcal{M})^+ d\sigma$$
$$+ \int_{\partial\Omega}(\mathcal{M}-\overline{\mathcal{C}}_{n+1})\overline{\mathcal{U}}^-_{n+1}(C_{n+1}-\mathcal{M})^+ d\sigma.$$

Note that $\mathcal{M}, \overline{\Pi}_{n+1}, (C_{n+1}-\mathcal{M})^+$ and $\overline{\mathcal{U}}^+_{n+1}$ are nonnegative and $\overline{\mathcal{C}}_{n+1} \le \mathcal{M}$. These together with the fact that $(C_{n+1}-\mathcal{M})(C_{n+1}-\mathcal{M})^+ = ((C_{n+1}-\mathcal{M})^+)^2$ yields $I \ge 0$. Then from (4.56) we conclude that

$$\int_\Omega(\phi(C_{n+1}-C_n) - \Delta t\overline{\Lambda}_{n+1})(C_{n+1}-\mathcal{M})^+ d\boldsymbol{x} \le 0.$$

As $C_n + \Delta t\frac{\overline{\Lambda}_{n+1}}{\phi} \le \mathcal{M}$, a.e. in Ω, $C_{n+1} - C_n - \Delta t\frac{\overline{\Lambda}_{n+1}}{\phi} \ge C_{n+1} - \mathcal{M}$, a.e. in Ω. Hence $\int_\Omega \phi((C_{n+1}-\mathcal{M})^+)^2 d\boldsymbol{x} \le 0$ yielding

$$(C_{n+1}-\mathcal{M})^+ = 0, \text{ a.e. in } \Omega.$$

This concludes the proof. □

Let $C_0^N = c_0$, $\Phi_0^N = 0$, $\boldsymbol{U}_0^N = \boldsymbol{0}$ and by Proposition 51 and Proposition 52, define

$$C^N = (C_0^N, \ldots, C_N^N),\ \Phi^N = (\Phi_0^N, \ldots, \Phi_N^N),\ \boldsymbol{U}^N = (\boldsymbol{U}_0^N, \ldots, \boldsymbol{U}_N^N).$$

Now we will define constant and linear interpolation operators for the approximations of $\overline{\Lambda}^N, \overline{\Pi}^N, \overline{\Psi}^N, \overline{\mathcal{C}}^N$ and $C^N, \boldsymbol{U}^N, \Phi^N$.

Definition 53. *Let B be a Banach space. For $\boldsymbol{\xi} = (\xi_0, \ldots, \xi_N) \in B^{N+1}$, define $I_0\boldsymbol{\xi}, I_1\boldsymbol{\xi} : [0,T] \to B$ by*

$$I_0\boldsymbol{\xi}(t) = \begin{cases} \xi_0, & t = 0 \\ \xi_{n+1}, & \text{if } n\Delta t < t \le (n+1)\Delta t, \quad n = 0, \ldots, N-1 \end{cases}$$

and

$$I_1\boldsymbol{\xi}(t) = \left(1 + n - \frac{t}{\Delta t}\right)\xi_n + \left(\frac{t}{\Delta t} - n\right)\xi_{n+1}\,,\ \text{if } n\Delta t \le t \le (n+1)\Delta t,\ n = 0, \ldots, N-1.$$

Also define $\widetilde{I_0}$ to be the extension of the constant interpolation operator such that

$$\widetilde{I_0\boldsymbol{\xi}}(t) = \begin{cases} \xi_0 & t \in [-\Delta t, 0], \\ \xi_{n+1} & t \in (n\Delta t, (n+1)\Delta t], \quad n = 0, \ldots N-1. \end{cases}$$

Observe that $I_1\boldsymbol{\xi}$ is continuous and,

$$\frac{\partial}{\partial t} I_1\boldsymbol{\xi}(t) = \frac{1}{\Delta t}(\xi_{n+1} - \xi_n)\,, \text{ if } n\Delta t < t < (n+1)\Delta t, \quad n = 0, \ldots, N-1. \tag{4.57}$$

Also for all $1 \le p < \infty$,

$$\|I_0\boldsymbol{\xi}\|_{L^p(0,T;B)} = (\Delta t \sum_{n=1}^{N} \|\xi_n\|_B^p)^{\frac{1}{p}}, \tag{4.58}$$

the proof of which is included in the Appendix A.4. For $p = \infty$,

$$\|I_0\boldsymbol{\xi}\|_{L^\infty(0,T;B)} = ess \sup_{t\in[0,T]} \|I_0\boldsymbol{\xi}(t)\|_B = \max_{n=1,\ldots,N} \|\xi_n\|_B. \tag{4.59}$$

Furthermore, from Appendix A.4, for $z \in L^p(0,T;B)$, if $\overline{z}^N = (\overline{z}_0^N, \ldots, \overline{z}_N^N)$ is defined as in (4.37), then for all $1 \le p < \infty$,

$$I_0\overline{z}^N \to z \text{ strongly in } L^p(0,T;B) \text{ as } N \to \infty. \tag{4.60}$$

With these properties of the constant and linear interpolation operators, integrating (4.48) and (P) from $n\Delta t$ to $(n+1)\Delta t$, summing from $n = 0$ to $n = N-1$ and using (4.40) yields the following definition of the approximate solution to the Stokes-Darcy-transport problem.

Definition 54. *(Definition of the approximate solution) For all* $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V})$ *and for all* $q \in L^2(0,T;R_2)$,

$$\int_0^T \Big(2(\mu(\widetilde{I_0C}^N_{\Delta t})\boldsymbol{D}(I_0\boldsymbol{U}^N), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + \big(\frac{\boldsymbol{K}}{\mu(\widetilde{I_0C}^N_{\Delta t})}(\nabla I_0\Phi^N - \rho\boldsymbol{g}), \nabla q\big)_{\Omega_2}$$
$$+ (I_0\Phi^N, \boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-\frac{1}{2}}I_0\boldsymbol{U}^N\cdot\boldsymbol{\tau}_{12}, \boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - (I_0\boldsymbol{U}^N\cdot\boldsymbol{n}_{12}, q)_{\Gamma_{12}}\Big)dt$$
$$= \int_0^T \Big((I_0\overline{\boldsymbol{\Psi}}^N, \boldsymbol{v})_{\Omega_1} + (I_0\overline{\Pi}^N, q)_{\Omega_2} - (I_0\overline{\mathcal{U}}^N, q)_{\Gamma_2}\Big)\,dt \quad (4.61)$$

where

$$I_0\boldsymbol{U}^N = -\frac{\boldsymbol{K}}{\mu(\widetilde{I_0C}^N_{\Delta t})}(\nabla I_0\Phi^N - \rho\boldsymbol{g}), \quad in\ \Omega_2\times(0,T). \quad (4.62)$$

and the concentration equation is defined as

$$\int_0^T \langle\frac{\partial}{\partial t}I_1C^N, \psi\rangle_{(H^1(\Omega))', H^1(\Omega)}dt - \int_{Q_T} I_0C^N I_0\boldsymbol{U}^N\cdot\nabla\psi d\boldsymbol{x}dt$$
$$+ \int_{Q_T}\boldsymbol{F}(I_0\boldsymbol{U}^N)\nabla I_0C^N\cdot\nabla\psi d\boldsymbol{x}dt + \int_{\Sigma_T}(I_0C^N(I_0\overline{\mathcal{U}}^N)^+ - I_0\overline{\mathcal{C}}^N(I_0\overline{\mathcal{U}}^N)^-)\psi d\sigma dt$$
$$- \int_0^T \langle I_0\overline{\Lambda}^N, \psi\rangle_{(H^1(\Omega))', H^1(\Omega)}dt = 0, \quad (4.63)$$

for all $\psi \in L^2(0,T;H^1(\Omega))$*. The function* $\widetilde{I_0C}^N_{\Delta t}$ *denotes the translated function:* $\widetilde{I_0C}^N_{\Delta t}(x,t) = \widetilde{I_0C}^N(x, t-\Delta t)$*. Furthermore, multiplying by* Δt *and summing from* $n = 0$ *to* $N-1$ *both sides of the bound (4.43), we obtain*

$$2\mu_L\|\boldsymbol{D}(I_0\boldsymbol{U}^N)\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla(I_0\Phi^N)\|^2_{L^2(0,T;L^2(\Omega_2)^2)}$$
$$\leq \frac{M_P^2C_D^2}{2\mu_L}\|\boldsymbol{\Psi}\|^2_{L^2(0,T;L^2(\Omega_1)^2)} + \frac{3\mu_U}{k_L}\Big(M_P^2\|\Pi\|^2_{L^2(0,T;L^2(\Omega_2))}$$
$$+ M_2^2\|\mathcal{U}\|^2_{L^2(0,T;L^2(\Gamma_{12}))} + \frac{\rho^2T^2}{\mu_L}\|\boldsymbol{K}\boldsymbol{g}\|^2_{L^2(\Omega)}\Big). \quad (4.64)$$

We will pass to the limit in this definition. First we need some bounds for the approximate solution, which are derived in the next section.

Stability bounds The first proposition of this section gives a uniform L^∞-bound for $I_0 C^N$ which will be used when passing to the limit. A slightly more general version of this result can be found in [53].

Proposition 55. *For $n = 0, \dots, N$*

$$0 \leq C_n^N(x) \leq \mathcal{N}, \;\; a.e. \;\; x \in \Omega, \tag{4.65}$$

where $\mathcal{N}$ is the right-hand side of (4.35), i.e.,

$$\mathcal{N} = \left\| \frac{\Lambda}{\phi} \right\|_{L^1(0,T;L^\infty(\Omega))} + \max(\|c_0\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}).$$

Proof. For readibility again, we drop the superscript N. Using (4.47) and (4.38) recursively, for all $n = 1, \dots, N$, we obtain for a.e. $x \in \Omega$ that

$$\begin{aligned} 0 \leq C_n(x) &\leq \Delta t \left\| \frac{\overline{\Lambda}_n}{\phi} \right\|_{L^\infty(\Omega)} + \max\left(\|C_{n-1}\|_{L^\infty(\Omega)}, \|\overline{\mathcal{C}}_n\|_{L^\infty(\partial\Omega)} \right) \\ &\leq \Delta t \left\| \frac{\overline{\Lambda}_n}{\phi} \right\|_{L^\infty(\Omega)} + \max\left(\left(\Delta t \left\| \frac{\overline{\Lambda}_{n-1}}{\phi} \right\|_{L^\infty(\Omega)} + \max\left(\|C_{n-2}\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)} \right) \right), \|\mathcal{C}\|_{L^\infty(\Sigma_T)} \right) \\ &\leq \Delta t \left\| \frac{\overline{\Lambda}_{n-1}}{\phi} \right\|_{L^\infty(\Omega)} + \Delta t \left\| \frac{\overline{\Lambda}_n}{\phi} \right\|_{L^\infty(\Omega)} + \max\left(\|C_{n-2}\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)} \right) \leq \dots \\ &\dots \leq \Delta t \sum_{i=1}^n \left\| \frac{\overline{\Lambda}_i}{\phi} \right\|_{L^\infty(\Omega)} + \max(\|C_0\|_{L^\infty(\Omega)}, \|\mathcal{C}\|_{L^\infty(\Sigma_T)}). \end{aligned}$$

Observe from the proof of (4.38) that we have

$$\begin{aligned} \Delta t \sum_{i=1}^n \left\| \frac{\overline{\Lambda}_i}{\phi} \right\|_{L^\infty(\Omega)} &\leq \sum_{i=1}^n \int_{(i-1)\Delta t}^{i\Delta t} \left\| \frac{\Lambda(t)}{\phi} \right\|_{L^\infty(\Omega)} dt \\ &\leq \int_0^T \left\| \frac{\Lambda(t)}{\phi} \right\|_{L^\infty(\Omega)} dt = \left\| \frac{\Lambda}{\phi} \right\|_{L^1(0,T;L^\infty(\Omega))}. \end{aligned}$$

Then the result follows from this and the assumption that $C_0 = c_0$. □

Remark 56. *It is trivial to deduce the following uniform bounds for $I_0 C^N$ and $I_1 C^N$:*

$$0 \leq I_0 C^N(x,t) \leq \mathcal{N}, \quad 0 \leq I_1 C^N(x,t) \leq \mathcal{N}, \;\; a.e. \;\; x \in \Omega, \forall\, t \in (0,T). \tag{4.66}$$

The next proposition gives uniform bounds for the terms related to the Stokes-Darcy flow.

Proposition 57. *There exists a constant $\mathcal{M}$ independent of N such that*

$$\|I_0\boldsymbol{U}^N\|_{L^2(0,T;L^2(\Omega)^2)} \leq \mathcal{M}. \tag{4.67}$$

Furthermore,

$$\begin{aligned}
\|I_0\overline{\Lambda}^N\|_{L^1(Q_T)} &\leq \|\Lambda\|_{L^1(Q_T)}, &(4.68)\\
\|I_0\overline{\Lambda}^N\|_{L^2(0,T;(H^1(\Omega))')} &\leq \|\Lambda\|_{L^2(0,T;(H^1(\Omega))')}, &(4.69)\\
\|I_0\overline{\mathcal{U}}^N\|_{L^1(\Sigma_T)} &\leq \|\mathcal{U}\|_{L^1(\Sigma_T)}, &(4.70)\\
\|I_0\overline{\mathcal{U}}^N\|_{L^2(\Sigma_T)} &\leq \|\mathcal{U}\|_{L^2(\Sigma_T)}. &(4.71)
\end{aligned}$$

Proof. The estimates (4.68), (4.69), (4.70) and (4.71) are easy consequences of (A.13). To obtain (4.67), note from (4.64) that we have a uniform $L^2(0,T;L^2(\Omega_2)^2)$-estimate for $\nabla I_0\Phi^N$ with respect to N. This gives a bound for $I_0\boldsymbol{U}^N$ on Ω_2 as a result of (4.62) and (4.14). Similarly, we have a uniform $L^2(0,T;L^2(\Omega_1)^{2\times 2})$ bound for $\boldsymbol{D}(I_0\boldsymbol{U}^N)$. This implies a uniform $L^2(0,T;L^2(\Omega_1)^2)$ bound for $I_0\boldsymbol{U}^N$ in Ω_1 from Poincaré inequality (4.27). Therefore (4.67) holds. □

The following result gives various bounds for the interpolation of the concentration.

Proposition 58. *There exists a constant $\mathfrak{M}$ independent of N such that*

$$\begin{aligned}
\|I_0C^N\|_{L^2(0,T;H^1(\Omega))} &\leq \mathfrak{M}, &(4.72)\\
\forall t' > 0, \quad \|I_0C^N_{-t'} - I_0C^N\|^2_{L^2((0,T-t');L^2(\Omega))} &\leq \mathfrak{M}\, t', &(4.73)\\
\|\frac{\partial}{\partial t}I_1C^N\|_{L^2(0,T;(H^1(\Omega))')} &\leq \mathfrak{M}, &(4.74)\\
\|I_1C^N - I_0C^N\|^2_{L^2(0,T;(H^1(\Omega))')} &\leq \mathfrak{M}\, \Delta t, &(4.75)
\end{aligned}$$

$$\|\overline{\mathcal{C}}_n^N\|_{L^\infty(\partial\Omega)} \le \|\mathcal{C}\|_{L^\infty(\Sigma_T)}, \tag{4.76}$$

$$\|I_0\overline{\mathcal{C}}^N\|_{L^\infty(\partial\Omega)} \le \|\mathcal{C}\|_{L^\infty(\Sigma_T)}, \tag{4.77}$$

where $C_{t'}(x,t) = C(x,t-t')$ is the translation of C to $(0,T-t')$.

Proof. The inequality (4.76) follows from (4.38) and the last estimate (4.77) is a direct consequence of (A.13). We will prove the first four bounds. In (4.48), omitting the superscript N and letting $\psi = C_{n+1}$, we have

$$\frac{1}{\Delta t}\int_\Omega \phi(C_{n+1}-C_n)C_{n+1}d\boldsymbol{x}+\int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla C_{n+1}d\boldsymbol{x}-\int_\Omega C_{n+1}\boldsymbol{U}_{n+1}\cdot\nabla C_{n+1}d\boldsymbol{x}$$
$$+\int_{\partial\Omega}(C_{n+1}(\overline{\mathcal{U}}_{n+1})^+ - \overline{\mathcal{C}}_{n+1}(\overline{\mathcal{U}}_{n+1})^-)C_{n+1}d\sigma = \int_\Omega \overline{\Lambda}_{n+1}C_{n+1}d\boldsymbol{x}.$$

By Green's formula and (4.42) we rewrite the third terms as

$$\int_\Omega C_{n+1}\boldsymbol{U}_{n+1}\cdot\nabla C_{n+1}d\boldsymbol{x} = -\int_\Omega \nabla(C_{n+1}\boldsymbol{U}_{n+1})C_{n+1}d\boldsymbol{x} + \int_{\partial\Omega}\overline{\mathcal{U}}_{n+1}C_{n+1}^2 d\sigma$$
$$= -\int_\Omega (C_{n+1}\nabla\cdot\boldsymbol{U}_{n+1} + \nabla C_{n+1}\cdot\boldsymbol{U}_{n+1})C_{n+1}d\boldsymbol{x} + \int_{\partial\Omega}\overline{\mathcal{U}}_{n+1}C_{n+1}^2 d\sigma.$$

Since we have (4.41), this implies

$$2\int_\Omega C_{n+1}\boldsymbol{U}_{n+1}\cdot\nabla C_{n+1}d\boldsymbol{x} = -\int_\Omega \nabla\cdot\boldsymbol{U}_{n+1}C_{n+1}^2 d\boldsymbol{x} + \int_{\partial\Omega}\overline{\mathcal{U}}_{n+1}C_{n+1}^2 d\sigma$$
$$= -\int_{\Omega_2}\overline{\Pi}_{n+1}C_{n+1}^2 d\boldsymbol{x} + \int_{\partial\Omega}\overline{\mathcal{U}}_{n+1}C_{n+1}^2 d\sigma.$$

Then,

$$\frac{1}{\Delta t}\int_\Omega \phi(C_{n+1}-C_n)C_{n+1}d\boldsymbol{x}+\int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla C_{n+1}d\boldsymbol{x}+\frac{1}{2}\int_{\Omega_2}\overline{\Pi}_{n+1}C_{n+1}^2 d\boldsymbol{x}$$
$$-\frac{1}{2}\int_{\partial\Omega}\overline{\mathcal{U}}_{n+1}C_{n+1}^2 d\sigma+\int_{\partial\Omega}(C_{n+1}(\overline{\mathcal{U}}_{n+1})^+-\overline{\mathcal{C}}_{n+1}(\overline{\mathcal{U}}_{n+1})^-)C_{n+1}d\sigma = \int_\Omega \overline{\Lambda}_{n+1}C_{n+1}d\boldsymbol{x}.$$

Note that $(\overline{\mathcal{U}}_{n+1})^+ - \frac{1}{2}\overline{\mathcal{U}}_{n+1} = \frac{1}{2}|\overline{\mathcal{U}}_{n+1}|$. So,

$$\frac{1}{\Delta t}\int_\Omega \phi(C_{n+1}-C_n)C_{n+1}d\boldsymbol{x}+\int_\Omega \boldsymbol{F}(\boldsymbol{U}_{n+1})\nabla C_{n+1}\cdot\nabla C_{n+1}d\boldsymbol{x}+\frac{1}{2}\int_{\Omega_2}\overline{\Pi}_{n+1}C_{n+1}^2 d\boldsymbol{x}$$
$$+\frac{1}{2}\int_{\partial\Omega}|\overline{\mathcal{U}}_{n+1}|C_{n+1}^2 d\sigma = \int_{\partial\Omega}\overline{\mathcal{C}}_{n+1}(\overline{\mathcal{U}}_{n+1})^- C_{n+1}d\sigma + \int_\Omega \overline{\Lambda}_{n+1}C_{n+1}d\boldsymbol{x}.$$

Using the assumption $\Pi \geq 0$ and (4.18),

$$\frac{1}{\Delta t}\int_\Omega \phi(C_{n+1} - C_n)C_{n+1}d\boldsymbol{x} + \alpha\int_\Omega |\nabla C_{n+1}|^2 d\boldsymbol{x} \\ \leq \int_{\partial\Omega} \overline{\mathcal{C}}_{n+1}(\overline{\mathcal{U}}_{n+1})^- C_{n+1}d\sigma + \int_\Omega \overline{\Lambda}_{n+1}C_{n+1}d\boldsymbol{x}.$$

Finally noting that $\frac{1}{2}(C_{n+1}^2 - C_n^2) \leq (C_{n+1} - C_n)C_{n+1}$, we further obtain

$$\frac{1}{2\Delta t}\int_\Omega \phi(C_{n+1}^2 - C_n^2)d\boldsymbol{x} + \alpha\|\nabla C_{n+1}\|_{L^2(\Omega)}^2 \leq \int_{\partial\Omega} \overline{\mathcal{C}}_{n+1}(\overline{\mathcal{U}}_{n+1})^- C_{n+1}d\sigma + \int_\Omega \overline{\Lambda}_{n+1}C_{n+1}d\boldsymbol{x}.$$

This, (4.65) and (4.38) implies

$$\frac{1}{2\Delta t}\int_\Omega \phi(C_{n+1}^2 - C_n^2)d\boldsymbol{x} + \alpha\|\nabla C_{n+1}\|_{L^2(\Omega)}^2 \\ \leq \mathcal{N}\|\overline{\Lambda}_{n+1}\|_{L^1(\Omega)} + \mathcal{N}\|\mathcal{C}\|_{L^\infty(\Sigma_T)}\|\overline{\mathcal{U}}_{n+1}\|_{L^1(\partial\Omega)}.$$

Multiplying by $2\Delta t$, summing from 0 to $m-1$, for any $1 \leq m \leq N$, and using (4.58), (A.13) and (4.70) we get

$$\int_\Omega \phi\, C_m^2 d\boldsymbol{x} + 2\alpha\sum_{n=0}^{m-1}\Delta t\|\nabla C_{n+1}\|_{L^2(\Omega)}^2 \leq \int_\Omega \phi\, C_0^2 d\boldsymbol{x} + 2\mathcal{N}\sum_{n=0}^{m-1}\Delta t\|\overline{\Lambda}_{n+1}\|_{L^1(\Omega)} \\ + 2\mathcal{N}^2\sum_{n=0}^{m-1}\Delta t\|\overline{\mathcal{U}}_{n+1}\|_{L^1(\partial\Omega)} = \int_\Omega \phi C_0^2 d\boldsymbol{x} + 2\mathcal{N}\|I_0\overline{\Lambda}\|_{L^1(Q_T)} + 2\mathcal{N}^2\|I_0\overline{\mathcal{U}}\|_{L^1(\Sigma_T)} \\ \leq \int_\Omega \phi C_0^2 d\boldsymbol{x} + 2\mathcal{N}\|\Lambda\|_{L^1(Q_T)} + 2\mathcal{N}^2\|\mathcal{U}\|_{L^1(\Sigma_T)}.$$

Therefore from (4.16), for all $1 \leq m \leq N$,

$$\phi_L\|C_m\|_{L^2(\Omega)}^2 + 2\alpha\sum_{n=0}^{m-1}\Delta t\|\nabla C_{n+1}\|_{L^2(\Omega)}^2 \leq A, \tag{4.78}$$

where $A = \int_\Omega \phi C_0^2 d\boldsymbol{x} + 2\mathcal{N}\|\Lambda\|_{L^1(Q_T)} + 2\mathcal{N}^2\|\mathcal{U}\|_{L^1(\Sigma_T)}$.This implies (4.72) as

$$\|I_0C\|_{L^2(0,T;H^1(\Omega))} = \left(\sum_{n=0}^{N-1} \Delta t \|C_{n+1}\|^2_{H^1(\Omega)}\right)^{\frac{1}{2}}$$
$$= \left(\sum_{n=0}^{N-1} \Delta t (\|C_{n+1}\|^2_{L^2(\Omega)} + \|\nabla C_{n+1}\|^2_{L^2(\Omega)})\right)^{\frac{1}{2}}$$
$$\leq \left(\left(\sum_{n=0}^{N-1} \Delta t \frac{A}{\phi_L}\right) + \frac{A}{2\alpha}\right)^{\frac{1}{2}} = \left(\frac{AT}{\phi_L} + \frac{A}{2\alpha}\right)^{\frac{1}{2}}.$$

What comes next is the proof of (4.73), which comes from [81, 53]. Fix $t' > 0$. As before define $\lceil t \rceil = \min\{n \in \mathbb{Z} : t \leq n\}$. Then

$$\int_\Omega \phi (I_0C_{-t'} - I_0C)^2 d\boldsymbol{x} = \int_\Omega \phi \left(C_{\lceil \frac{t+t'}{\Delta t} \rceil} - C_{\lceil \frac{t}{\Delta t} \rceil}\right)^2 d\boldsymbol{x}$$
$$= \int_\Omega \phi \left(C_{\lceil \frac{t+t'}{\Delta t} \rceil} - C_{\lceil \frac{t}{\Delta t} \rceil}\right)\left(C_{\lceil \frac{t+t'}{\Delta t} \rceil} - C_{\lceil \frac{t}{\Delta t} \rceil}\right) d\boldsymbol{x}$$
$$= \int_\Omega \phi \sum_{n=n_0(t)}^{n_1(t)-1} (C_{n+1} - C_n)(C_{n_1(t)} - C_{n_0(t)}) d\boldsymbol{x},$$

where $n_0(t) = \lceil \frac{t}{\Delta t} \rceil$ and $n_1(t) = \lceil \frac{t+t'}{\Delta t} \rceil$. Multiplying (4.48) by Δt, summing from $n_0(t)$ to $n_1(t) - 1$ and choosing $\psi = C_{n_1(t)} - C_{n_0(t)}$ we have

$$J := \int_\Omega \sum_{n=n_0(t)}^{n_1(t)-1} \phi (C_{n+1} - C_n)(C_{n_1(t)} - C_{n_0(t)}) d\boldsymbol{x}$$
$$= -\Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_\Omega \left(\boldsymbol{F}(\boldsymbol{U}_{n+1}) \nabla C_{n+1} - C_{n+1} \boldsymbol{U}_{n+1}\right) \cdot \nabla (C_{n_1(t)} - C_{n_0(t)}) d\boldsymbol{x}$$

$$- \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\partial\Omega} (C_{n+1}(\overline{\mathcal{U}}_{n+1})^+ - \overline{\mathcal{C}}_{n+1}(\overline{\mathcal{U}}_{n+1})^-)(C_{n_1(t)} - C_{n_0(t)}) d\sigma$$
$$+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_\Omega \overline{\Lambda}_{n+1}(C_{n_1(t)} - C_{n_0(t)}) d\boldsymbol{x}.$$

Then from (4.17) and (4.65), we obtain

$$J \le \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_\Omega \big(F_B|\nabla C_{n+1}| + \mathcal{N}|\boldsymbol{U}_{n+1}|\big)|(|\nabla C_{n_1(t)}| + |\nabla C_{n_0(t)}|)d\boldsymbol{x}$$
$$+ \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_{\partial\Omega} 2\mathcal{N}(\mathcal{N} + |\overline{\mathcal{C}}_{n+1}|)|\overline{\mathcal{U}}_{n+1}|d\sigma + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_\Omega 2\mathcal{N}|\overline{\Lambda}_{n+1}|d\boldsymbol{x}$$

Applying Young's inequality to the first term of the right hand side of the above inequality implies

$$J \le \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \Big(\int_\Omega (F_B^2|\nabla C_{n+1}|^2 + \mathcal{N}^2|\boldsymbol{U}_{n+1}|^2 + 2\mathcal{N}|\overline{\Lambda}_{n+1}|)d\boldsymbol{x}$$
$$+ \int_{\partial\Omega} 2\mathcal{N}(\mathcal{N} + |\overline{\mathcal{C}}_{n+1}|)|\overline{\mathcal{U}}_{n+1}|d\sigma\Big) + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_\Omega |\nabla C_{n_1(t)}|^2 + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} \int_\Omega |\nabla C_{n_0(t)}|^2.$$

Now, define

$$p_n := \int_\Omega (F_B^2|\nabla C_n|^2 + \mathcal{N}^2|\boldsymbol{U}_n|^2 + 2\mathcal{N}|\overline{\Lambda}_n|)d\boldsymbol{x} + \int_{\partial\Omega} 2\mathcal{N}(\mathcal{N} + |\overline{\mathcal{C}}_n|)|\overline{\mathcal{U}}_n|d\sigma$$

and

$$q_n := \int_\Omega |\nabla C_n|^2.$$

Therefore, we can rewrite

$$\int_\Omega \phi(I_0 C_{-t'} - I_0 C)^2 d\boldsymbol{x} \le \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} p_{n+1} + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} q_{n_1(t)} + \Delta t \sum_{n=n_0(t)}^{n_1(t)-1} q_{n_0(t)}.$$

Now let

$$\chi_n(t, t+t') = \begin{cases} 1, & \text{if } n\Delta t \in [t, t+t') \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\int_0^{T-t'} \sum_{n=n_0(t)}^{n_1(t)-1} p_{n+1} dt = \int_0^{T-t'} \sum_{n=0}^{N-1} p_{n+1}\chi_n(t, t+t')dt = \sum_{n=0}^{N-1} p_{n+1} \int_0^{T-t'} \chi_n(t, t+t')dt$$
$$\le \sum_{n=0}^{N-1} p_{n+1} \int_{\mathbb{R}} \chi_n(t, t+t')dt = \sum_{n=0}^{N-1} p_{n+1} \int_{n\Delta t - t'}^{n\Delta t} dt = t' \sum_{n=0}^{N-1} p_{n+1}.$$

Observe that $n_0(t) = m$ for some $m \in \mathbb{N}$ if and only if $t \in ((m-1)\Delta t, m\Delta t]$. Then

$$\int_0^{T-t'} \sum_{n=n_0(t)}^{n_1(t)-1} q_{n_0(t)} dt = \int_0^{T-t'} \sum_{n=0}^{N-1} q_{n_0(t)} \chi_n(t, t+t') dt = \int_0^{T-t'} q_{n_0(t)} \sum_{n=0}^{N-1} \chi_n(t, t+t') dt$$
$$\leq \sum_{m=1}^{N} \int_{(m-1)\Delta t}^{m\Delta t} q_m \sum_{n=0}^{N-1} \chi_n(t, t+t') dt \leq \sum_{m=1}^{N} q_m \sum_{n=0}^{N-1} \int_{(m-1)\Delta t}^{m\Delta t} \chi_n(t, t+t') dt$$
$$= \sum_{m=1}^{N} q_m \sum_{n=0}^{N-1} \int_{(2m-n-1)\Delta t}^{(2m-n)\Delta t} \chi_n(s + (n-m)\Delta t, s + (n-m)\Delta t + t') ds$$
$$= \sum_{m=1}^{N} q_m \sum_{n=0}^{N-1} \int_{(2m-n-1)\Delta t}^{(2m-n)\Delta t} \chi_m(s, s+t') ds$$
$$\leq \sum_{m=1}^{N} q_m \int_{\mathbb{R}} \chi_m(s, s+t') ds = \sum_{m=1}^{N} q_m \int_{m\Delta t - t'}^{m\Delta t} ds = t' \sum_{m=1}^{N} q_m.$$

Similarly,

$$\int_0^{T-t'} \sum_{n=n_0(t)}^{n_1(t)+1} q_{n_1(t)} dt = t' \sum_{m=1}^{N} q_m.$$

Therefore, from (4.16),

$$\|I_0 C_{-t'} - I_0 C\|^2_{L^2((0,T-t');L^2(\Omega))} = \int_0^{T-t'} \int_\Omega (I_0 C_{-t'} - I_0 C)^2 d\boldsymbol{x} \leq t' \frac{\Delta t}{\phi_L} \sum_{n=1}^{N} (p_n + 2q_n).$$

Let us see that $\Delta t \sum_{n=1}^{N} p_n$ and $\Delta t \sum_{n=1}^{N} q_n$ are bounded uniformly in N. From Cauchy-Schwarz inequality,

$$\Delta t \sum_{n=1}^{N} p_n \leq F_B^2 \Delta t \sum_{n=1}^{N} \|\nabla C_n\|^2_{L^2(\Omega)} + \Delta t \sum_{n=1}^{N} \mathcal{N}^2 \|\boldsymbol{U}_n\|^2_{L^2(\Omega)} + 2\mathcal{N} \Delta t \sum_{n=1}^{N} \|\overline{\Lambda}_n\|_{L^1(\Omega)}$$
$$+ 2\mathcal{N}^2 \Delta t \sum_{n=1}^{N} \|\overline{\mathcal{U}}_n\|_{L^1(\partial\Omega)} + 2\mathcal{N} \Delta t \sum_{n=1}^{N} \|\overline{\mathcal{C}}_n\|_{L^2(\partial\Omega)} \|\overline{\mathcal{U}}_n\|_{L^2(\partial\Omega)}.$$

Then, (4.58), (4.67), (4.68), (4.70), (4.71), (4.77) and (4.78) imply

$$\Delta t \sum_{n=1}^{N} p_n \leq F_B^2 \frac{A}{2\alpha} + \mathcal{N}^2 \mathcal{M}^2 + 2\mathcal{N} \|\Lambda\|_{L^1(Q_T)} + 2\mathcal{N}^2 \|\mathcal{U}\|_{L^1(\Sigma_T)} + 2\mathcal{N} \|\mathcal{C}\|_{L^2(\Sigma_T)} \|\mathcal{U}\|_{L^2(\Sigma_T)}.$$

Again from (4.78),

$$\Delta t \sum_{n=1}^{N} q_n = \Delta t \sum_{n=1}^{N} \|\nabla C_n\|^2_{L^2(\Omega)} \leq \frac{A}{2\alpha}.$$

Therefore, $\Delta t \sum_{n=1}^{N} p_n$ and $\Delta t \sum_{n=1}^{N} q_n$ are bounded uniformly in N implying

$$\|I_0 C_{-t'} - I_0 C\|^2_{L^2(0,T-t';L^2(\Omega))} \leq \mathfrak{M}\, t',$$

where $\mathfrak{M}$ is a constant independent of N. Let us prove (4.74). From (4.57),

$$\begin{aligned}\left\|\frac{\partial}{\partial t} I_1 C\right\|^2_{L^2(0,T;(H^1(\Omega))')} &= \int_0^T \left\|\frac{\partial}{\partial t} I_1 C\right\|^2_{(H^1(\Omega))'} dt \\ &= \sum_{m=0}^{N-1} \int_{m\Delta t}^{(m+1)\Delta t} \frac{1}{(\Delta t)^2} \|C_{m+1} - C_m\|^2_{(H^1(\Omega))'} dt = \frac{1}{\Delta t} \sum_{m=0}^{N-1} \|C_{m+1} - C_m\|^2_{(H^1(\Omega))'}.\end{aligned}$$

To bound this, Cauchy-Schwarz inequality, Hölder's inequality and (4.17) are applied to (4.48). This yields for all $\psi \in H^1(\Omega)$,

$$\begin{aligned}&\frac{1}{\Delta t} |\langle \phi(C_{n+1} - C_n), \psi \rangle_{(H^1(\Omega))', H^1(\Omega)}| \\ &\qquad \leq \left(F_B \|\nabla C_{n+1}\|_{L^2(\Omega)} + \|C_{n+1}\|_{L^\infty(\Omega)} \|\boldsymbol{U}_{n+1}\|_{L^2(\Omega)}\right) \|\nabla \psi\|_{L^2(\Omega)} \\ &\quad + \left(\|C_{n+1}\|_{L^4(\partial\Omega)} \|\psi\|_{L^4(\partial\Omega)} + \|\overline{\mathcal{C}}_{n+1}\|_{L^\infty(\partial\Omega)} \|\psi\|_{L^2(\partial\Omega)}\right) \|\overline{\mathcal{U}}_{n+1}\|_{L^2(\partial\Omega)} \\ &\qquad\qquad + \|\overline{\Lambda}_{n+1}\|_{(H^1(\Omega))'} \|\psi\|_{H^1(\Omega)}.\end{aligned}$$

Then by (4.24), (4.25), (4.65) and (4.76) we have

$$\begin{aligned}\frac{1}{\Delta t} |\langle \phi(C_{n+1} - C_n), \psi \rangle_{(H^1(\Omega))', H^1(\Omega)}| &\leq \left(F_B \|\nabla C_{n+1}\|_{L^2(\Omega)} + \mathcal{N} \|\boldsymbol{U}_{n+1}\|_{L^2(\Omega)}\right. \\ &\quad \left. + (M_4^2 \|C_{n+1}\|_{H^1(\Omega)} + M_2 \|\mathcal{C}\|_{L^\infty(\Sigma_T)}) \|\overline{\mathcal{U}}_{n+1}\|_{L^2(\partial\Omega)} + \|\overline{\Lambda}_{n+1}\|_{(H^1(\Omega))'}\right) \|\psi\|_{H^1(\Omega)}.\end{aligned}$$

Taking supremum over all $\psi \in H^1(\Omega)$ such that $\|\psi\|_{H^1(\Omega)} = 1$, using (4.16), (4.39) and (4.78), we see that there exists a constant M independent of N such that

$$\begin{aligned}\frac{1}{\Delta t^2} \|C_{n+1} - C_n\|^2_{(H^1(\Omega))'} &\leq M(\|\nabla C_{n+1}\|^2_{L^2(\Omega)} + \|\boldsymbol{U}_{n+1}\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{\Delta t^2} \|\mathcal{U}\|^2_{L^2(\Sigma_T)} + \|\overline{\mathcal{U}}_{n+1}\|^2_{L^2(\partial\Omega)} + \|\overline{\Lambda}_{n+1}\|^2_{(H^1(\Omega))'}).\end{aligned}$$

Multiplying by Δt, summing from 0 to $N-1$ and using (4.58), (4.72) and (A.13) we obtain (4.74). (4.75) follows from (4.74) as

$$\|I_1 C - I_0 C\|^2_{L^2(0,T;(H^1(\Omega))')} = \sum_{m=0}^{N-1} \int_{m\Delta t}^{(m+1)\Delta t} \|(1+m-\frac{t}{\Delta t})(C_m - C_{m+1})\|^2_{(H^1(\Omega))'} dt$$

$$= \sum_{m=0}^{N-1} \|C_m - C_{m+1}\|^2_{(H^1(\Omega))'} \int_{m\Delta t}^{(m+1)\Delta t} (1+m-\frac{t}{\Delta t})^2 dt = \frac{\Delta t}{3} \sum_{m=0}^{N-1} \|C_m - C_{m+1}\|^2_{(H^1(\Omega))'}.$$

□

Passing to the limit Passing to the limit in (4.61)-(4.63) requires certain convergence properties that we now state and prove.

Proposition 59. *There exists a subsequence of $\{C^N\}_{N\geq 1}$ still denoted by $\{C^N\}_{N\geq 1}$ and a function $c \in L^\infty(Q_T) \cap L^2(0,T;H^1(\Omega))$ such that $t \to c(\cdot,t) \in \mathcal{C}([0,T];(H^1(\Omega))')$ satisfying*

$$I_0 C^N \rightarrow c \text{ weakly -}\star \text{ in } L^\infty(Q_T), \tag{4.79}$$

$$I_0 C^N \rightarrow c \text{ weakly in } L^2(0,T;H^1(\Omega)), \tag{4.80}$$

$$I_0 C^N \rightarrow c \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \tag{4.81}$$

$$I_0 C^N \rightarrow c \text{ strongly in } L^2(\Sigma_T), \tag{4.82}$$

$$\frac{\partial}{\partial t} I_1 C^N \rightarrow \frac{\partial}{\partial t} c \text{ weakly in } L^2(0,T;(H^1(\Omega))'), \tag{4.83}$$

$$I_1 C^N \rightarrow c \text{ strongly in } \mathcal{C}([0,T];(H^1(\Omega))'), \tag{4.84}$$

$$I_0 \overline{\Lambda}^N \rightarrow \Lambda \text{ strongly in } L^2(0,T;(H^1(\Omega))'), \tag{4.85}$$

$$I_0 \overline{\mathcal{C}}^N \rightarrow \mathcal{C} \text{ strongly in } L^2(\Sigma_T), \tag{4.86}$$

as $N \to \infty$.

Proof. The last two convergence results follow trivially from (4.60). To prove the rest we will use the estimates from the previous section. From (4.66) and (4.72), we know that $\{I_0 C^N\}_N$ is bounded both in $L^\infty(Q_T)$ and in $L^2(0,T;H^1(\Omega))$. Because $L^\infty(Q_T) = (L^1(Q_T))'$, by Theorem 12 (Banach-Alaoglu theorem), we can extract a subsequence still denoted by $\{C^N\}_{N\geq 1}$ (from now on we will denote each extracted subsequence by $\{C^N\}_{N\geq 1}$) and find a function $c \in L^\infty(Q_T)$ such that (4.79) holds. Next the reflexivity of the space $L^2(0,T;H^1(\Omega))$ implies that there exists a subsequence $\{C^N\}_{N\geq 1}$ and a function $c_1 \in L^2(0,T;H^1(\Omega))$ such that $I_0 C^N \to c_1$ weakly in $L^2(0,T;H^1(\Omega))$. This also implies that $I_0 C^N \to c_1$ weakly-$\star$ in $L^\infty(Q_T)$. Therefore, $c_1 = c$ by uniqueness of the weak-$\star$ limits. Hence (4.80) holds. From (4.73), $\|I_0 C^N_{-t'} - I_0 C^N\|_{L^2((0,T-t');L^2(\Omega))} \to 0$ as $t' \to 0$ uniformly for all N. Theorem 7 states that $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$. So applying Theorem 13 we can find a subsequence $\{C^N\}_N$ and a function $c_2 \in L^2(Q_T)$ such that $I_0 C^N \to c_2$ strongly in $L^2(Q_T)$. This further implies the weak convergence in $L^2(Q_T)$. But (4.80) gives weak convergence in $L^2(Q_T)$ as well. Therefore, $c_2 = c$ by the uniqueness of the weak limits and hence (4.81) holds. Similarly, by Theorem 7, as $H^1(\Omega)$ is compactly embedded in $H^{\frac{1}{2}}(\Omega)$, so we can find a subsequence $\{C^N\}_{N\geq 1}$ such that $I_0 C^N \to c$ strongly in $L^2(0,T;H^{\frac{1}{2}}(\Omega))$. Then the continuity of the trace operator gives (4.82).

Recall from (4.66) that $I_1 C^N$ is uniformly bounded. So again by the Banach-Alaoglu theorem, up to a subsequence, there exists $c_3 \in L^\infty(Q_T)$ such that

$$I_1 C^N \to c_3 \text{ weakly-}\star \text{ in } L^\infty(Q_T).$$

The bound (4.74) and the reflexivity of $L^2(0,T;(H^1(\Omega))')$ gives a subsequence for which we have (again by uniqueness of weak-$\star$ limits $c_3 = c$)

$$\frac{\partial}{\partial t} I_1 C^N \to \frac{\partial}{\partial t} c \text{ weakly in } L^2(0,T;(H^1(\Omega))').$$

We know that $\{I_1 C^N\}_{N\geq 1}$ is bounded in $L^\infty(Q_T)$ by (4.66) and $\{\frac{\partial}{\partial t} I_1 C^N\}_{N\geq 1}$ is bounded in $L^2(0,T;(H^1(\Omega))')$ by (4.74). Also by Corollary 16, $L^\infty(\Omega)$ is compactly embedded in $(H^1(\Omega))'$. Then (4.84) is a consequence of Theorem 14 which implies that there exists a subsequence $\{C^N\}_N$ and a function $c_4 \in \mathcal{C}([0,T];(H^1(\Omega))')$ such that

$$I_1 C^N \to c_4 \text{ strongly in } \mathcal{C}([0,T];(H^1(\Omega))').$$

The bound (4.75) implies

$$I_1 C^N - I_0 C^N \to 0 \text{ strongly in } L^2(0,T;(H^1(\Omega))').$$

This together with (4.81) yields

$$I_1 C^N \to c \text{ strongly in } L^2(0,T;(H^1(\Omega))')$$

and thus $c_4 = c$ yielding (4.84). □

Proposition 60. *The following convergence results hold.*

$$I_0 \overline{\Pi}^N \to \Pi \textit{ strongly in } L^2(0,T;L^2(\Omega_2)), \tag{4.87}$$

$$I_0 \overline{\boldsymbol{\Psi}}^N \to \boldsymbol{\Psi} \textit{ strongly in } L^2(0,T;L^2(\Omega_1)^2), \tag{4.88}$$

$$I_0 \overline{\mathcal{U}}^N \to \mathcal{U} \textit{ strongly in } L^2(\Sigma_T), \tag{4.89}$$

and there exists $\boldsymbol{u} \in L^2(Q_T)^2$ such that

$$I_0 \boldsymbol{U}^N \to \boldsymbol{u} \textit{ strongly in } L^2(Q_T)^2. \tag{4.90}$$

Proof. The results (4.87), (4.88) and (4.89) are direct consequences of (4.60). For (4.90), consider the following problem where c is the limit found in Proposition 59. Find $(\boldsymbol{u}|_{\Omega_1}, \varphi) \in L^2(0,T;\boldsymbol{V}) \times L^\infty(0,T;R_2)$ satisfying

$$\int_0^T \Big(2(\mu(c)\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}),\nabla q)_{\Omega_2} + \tilde{\gamma}(\boldsymbol{u},\varphi;\boldsymbol{v},q)$$

$$= \int_0^T ((\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (\Pi,q)_{\Omega_2} - (\mathcal{U},q)_{\Gamma_2})\,dt, \tag{4.91}$$

for all $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V})$ and for all $q \in L^2(0,T;R_2)$. It is known that there exists a unique solution $(\boldsymbol{u},\varphi)$ to this problem [40, 38]. Next define $\boldsymbol{u}|_{\Omega_2} \in L^2(0,T;L^2(\Omega_2)^2)$ as

$$\boldsymbol{u} = -\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \quad \text{a.e. in } \Omega_2 \times (0,T).$$

The difference of (4.61) and (4.114) yields

$$\begin{aligned}
&\int_0^T \Big(2(\mu(\widetilde{I_0C}_{\Delta t}^N)\boldsymbol{D}(I_0\boldsymbol{U}^N) - \mu(c)\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} \\
&+ (\frac{\boldsymbol{K}}{\mu(\widetilde{I_0C}_{\Delta t}^N)}(\nabla I_0\Phi^N - \rho\boldsymbol{g}) - \frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}),\nabla q)_{\Omega_2} + \tilde{\gamma}(I_0\boldsymbol{U}^N - \boldsymbol{u}, I_0\Phi^N - \varphi; \boldsymbol{v}, q) \\
&\qquad = \int_0^T \Big((I_0\overline{\boldsymbol{\Psi}}^N - \boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (I_0\overline{\Pi}^N - \Pi, q)_{\Omega_2} - (I_0\overline{\mathcal{U}}^N - \mathcal{U}, q)_{\Gamma_2}\Big)\,dt. \quad (4.92)
\end{aligned}$$

Observe that the first and the second terms can be written as

$$\begin{aligned}
&\int_0^T 2(\mu(\widetilde{I_0C}_{\Delta t}^N)\boldsymbol{D}(I_0\boldsymbol{U}^N) - \mu(c)\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1}dt \\
&\qquad = \int_0^T 2\left(\mu(\widetilde{I_0C}_{\Delta t}^N)(\boldsymbol{D}(I_0\boldsymbol{U}^N) - \boldsymbol{D}(\boldsymbol{u})),\boldsymbol{D}(\boldsymbol{v})\right)_{\Omega_1} dt \\
&\qquad\qquad + \int_0^T 2\left((\mu(\widetilde{I_0C}_{\Delta t}^N) - \mu(c))\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v})\right)_{\Omega_1} dt
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T (\frac{\boldsymbol{K}}{\mu(\widetilde{I_0C}_{\Delta t}^N)}(\nabla I_0\Phi^N - \rho\boldsymbol{g}) - \frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}),\nabla q)_{\Omega_2}dt \\
&= \int_0^T (\frac{\boldsymbol{K}}{\mu(\widetilde{I_0C}_{\Delta t}^N)}(\nabla I_0\Phi^N - \nabla\varphi),\nabla q)_{\Omega_2}dt + \int_0^T ((\frac{1}{\mu(\widetilde{I_0C}_{\Delta t}^N)} - \frac{1}{\mu(c)})\boldsymbol{K}\nabla\varphi,\nabla q)_{\Omega_2}dt \\
&\qquad - \int_0^T ((\frac{\boldsymbol{K}}{\mu(\widetilde{I_0C}_{\Delta t}^N)} - \frac{\boldsymbol{K}}{\mu(c)})\rho\boldsymbol{g},\nabla q)_{\Omega_2}dt.
\end{aligned}$$

With these, letting $\boldsymbol{v} = I_0\boldsymbol{U}^N - \boldsymbol{u}$, $q = I_0\Phi^N - \varphi$ in (4.92), using the nonnegativity of the form $\tilde{\gamma}$, and the bounds (4.14) and (4.15) give

$$\int_0^T \Big(2\mu_L\|\boldsymbol{D}(I_0\boldsymbol{U}^N-\boldsymbol{u})\|^2_{L^2(\Omega_1)}+\frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla(I_0\Phi^N-\varphi)\|^2_{L^2(\Omega_2)}\Big)dt$$
$$\leq \int_0^T \Big((I_0\overline{\boldsymbol{\Psi}}^N-\boldsymbol{\Psi},I_0\boldsymbol{U}^N-\boldsymbol{u})_{\Omega_1}+(I_0\overline{\Pi}^N-\Pi,I_0\Phi^N-\varphi)_{\Omega_2}$$
$$-(I_0\overline{\mathcal{U}}^N-\mathcal{U},I_0\Phi^N-\varphi)_{\Gamma_2}-2((\mu(\widetilde{I_0C}^N_{\Delta t})-\mu(c))\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(I_0\boldsymbol{U}^N-\boldsymbol{u}))_{\Omega_1}$$
$$-((\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})}-\frac{1}{\mu(c)})\boldsymbol{K}\nabla\varphi,\nabla(I_0\Phi^N-\varphi))_{\Omega_2}\Big)dt$$
$$+\int_0^T(\rho(\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})}-\frac{1}{\mu(c)})\boldsymbol{K}\boldsymbol{g},\nabla(I_0\Phi^N-\varphi)_{\Omega_2})dt.$$

Using Cauchy-Schwarz, Poincaré (4.27) and Young's inequalities together with (3.29) and (4.24) yield

$$\frac{\mu_L}{C_D^2}\|\nabla(I_0\boldsymbol{U}^N-\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times2})}+\frac{k_L}{2\mu_U}\|\nabla(I_0\Phi^N-\varphi)\|^2_{L^2(0,T;L^2(\Omega_2)^2)}$$
$$\leq M\Big(\|I_0\overline{\boldsymbol{\Psi}}^N-\boldsymbol{\Psi}\|^2_{L^2(0,T;L^2(\Omega_1)^2)}+\|I_0\overline{\Pi}^N-\Pi\|^2_{L^2(0,T;L^2(\Omega_2))}+\|I_0\overline{\mathcal{U}}^N-\mathcal{U}\|^2_{L^2(0,T;L^2(\Gamma_2))}$$
$$+\|(\mu(\widetilde{I_0C}^N_{\Delta t})-\mu(c))\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1))}+\|(\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})}-\frac{1}{\mu(c)})\boldsymbol{K}\nabla\varphi\|^2_{L^2(0,T;L^2(\Omega_2)^{2\times2})}$$
$$+\|\rho(\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})}-\frac{1}{\mu(c)})\boldsymbol{K}\boldsymbol{g}\|^2_{L^2(0,T;L^2(\Omega_2))}\Big),$$

where $M>0$ is a generic constant independent of N. Then by boundedness and continuity of μ, (4.74), (4.87), (4.88) and (4.89) together with the Lebesgue dominated convergence theorem imply as $N\to\infty$ that

$$\mu_L\|\boldsymbol{D}(I_0\boldsymbol{U}^N-\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times2})}+\frac{1}{2\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla(I_0\Phi^N-\varphi)\|^2_{L^2(0,T;L^2(\Omega_2)^2)}\to 0.$$

Thus, as $N\to\infty$,

$$\nabla I_0\boldsymbol{U}^N \;\to\; \nabla\boldsymbol{u} \text{ strongly in } L^2(0,T;L^2(\Omega_1)^{2\times2}), \tag{4.93}$$
$$\nabla I_0\Phi^N \;\to\; \nabla\varphi \text{ strongly in } L^2(0,T;L^2(\Omega_2)^2). \tag{4.94}$$

The result (4.90) follows from (4.62), the continuity of μ, (4.27), Proposition 59, (4.93) and (4.94). □

Proof of Theorem 50 We are now ready to prove the existence result for the weak solution of the restricted problem. Recall that in order to obtain a weak solution we need to pass to the limit in the approximate solution equations (4.61)-(4.64). Passing to the limit in the flow equations (4.61) and (4.62) and the bound (4.64) is easy due to the continuity and the bound (4.14) of μ, (4.93) and (4.94). The convergence result (4.83) implies that

$$\frac{\partial}{\partial t} I_1 C^N \to \frac{\partial}{\partial t} c \text{ weakly-}\star \text{ in } L^2(0,T;(H^1(\Omega))').$$

Thus,

$$\lim_{N\to\infty} \int_0^T \langle \frac{\partial}{\partial t} I_1 C^N, \psi \rangle dt = \int_0^T \langle \frac{\partial}{\partial t} c, \psi \rangle dt, \quad \forall \psi \in L^2(0,T;H^1(\Omega)). \tag{4.95}$$

Note that

$$\begin{aligned}(I_0C^N I_0\boldsymbol{U}^N, \nabla\psi)_{Q_T} - (c\boldsymbol{u}, \nabla\psi)_{Q_T} &= ((I_0C^N - c)I_0\boldsymbol{U}^N, \nabla\psi)_{Q_T} + (c(I_0\boldsymbol{U}^N - \boldsymbol{u}), \nabla\psi)_{Q_T} \\ &:= I_1 + I_2.\end{aligned}$$

The first part I_1 of the above equation converges to zero by (4.79) and (4.67) and the second part I_2 converges to zero by (4.90) and the result from Proposition 59 saying that $c \in L^\infty(Q_T)$. Thus

$$\lim_{N\to\infty} (I_0C^N I_0\boldsymbol{U}^N, \nabla\psi)_{Q_T} = (c\boldsymbol{u}, \nabla\psi)_{Q_T}. \tag{4.96}$$

To pass to the limit in the third term in (4.63), we write:

$$\begin{aligned}&\int_{Q_T} \boldsymbol{F}(I_0\boldsymbol{U}^N)\nabla I_0C^N \cdot \nabla\psi d\boldsymbol{x}dt - \int_{Q_T} \boldsymbol{F}(\boldsymbol{u})\nabla c \cdot \nabla\psi d\boldsymbol{x}dt \\ &= \int_{Q_T} (\boldsymbol{F}(I_0\boldsymbol{U}^N) - \boldsymbol{F}(\boldsymbol{u}))\nabla I_0C^N \cdot \nabla\psi d\boldsymbol{x}dt + \int_{Q_T} \boldsymbol{F}(I_0\boldsymbol{U}^N)(\nabla I_0C^N - \nabla c) \cdot \nabla\psi d\boldsymbol{x}dt \\ &:= J_1 + J_2.\end{aligned}$$

The strong convergence $I_0\boldsymbol{U}^N - \boldsymbol{u} \to 0$ in $L^2(Q_T)^2$ implies $I_0\boldsymbol{U}^N - \boldsymbol{u} \to 0$ a.e. in Q_T up to a subsequence. So as $\boldsymbol{F}$ is continuous,

$$\boldsymbol{F}(I_0\boldsymbol{U}^N) - \boldsymbol{F}(\boldsymbol{u}) \to 0 \text{ a.e. in } Q_T.$$

Therefore J_1 converges to zero. By (4.80) and the bound (4.17) on $\boldsymbol{F}$, J_2 converges to zero as well. Hence

$$\lim_{N\to\infty}\int_{Q_T} \boldsymbol{F}(I_0\boldsymbol{U}^N)\nabla I_0 C^N \cdot \nabla\psi d\boldsymbol{x}dt = \int_{Q_T} \boldsymbol{F}(\boldsymbol{u})\nabla c \cdot \nabla\psi d\boldsymbol{x}dt. \tag{4.97}$$

The boundary terms in (4.63) are handled as follows

$$\begin{aligned}\int_{\Sigma_T} &(I_0C^N(I_0\overline{\mathcal{U}}^N)^+ - I_0\overline{\mathcal{C}}^N(I_0\overline{\mathcal{U}}^N)^-)\psi d\sigma dt - \int_{\Sigma_T}(c\mathcal{U}^+ - \mathcal{C}\mathcal{U}^-)\psi d\sigma dt \\ &= \int_{\Sigma_T}(I_0C^N - c)(I_0\overline{\mathcal{U}}^N)^+\psi d\sigma dt + \int_{\Sigma_T} c((I_0\overline{\mathcal{U}}^N)^+ - \mathcal{U})^+)\psi d\sigma dt \\ &\quad - \int_{\Sigma_T}(I_0\overline{\mathcal{C}}^N - \mathcal{C})(I_0\overline{\mathcal{U}}^N)^-\psi d\sigma dt - \int_{\Sigma_T}\mathcal{C}((I_0\overline{\mathcal{U}}^N)^- - \mathcal{U}^-)\psi d\sigma dt. \end{aligned} \tag{4.98}$$

By (4.82), (4.86) and (4.89), up to a subsequence, the terms on the right hand side of (4.98) converge to zero. Hence,

$$\begin{aligned}\lim_{N\to\infty}\int_{\Sigma_T} I_0C^N(I_0\overline{\mathcal{U}}^N)^+\psi d\sigma dt - \int_{\Sigma_T} I_0\overline{\mathcal{C}}^N(I_0\overline{\mathcal{U}}^N)^-\psi d\sigma dt \\ = \int_{\Sigma_T} c\mathcal{U}^+\psi d\sigma dt - \int_{\Sigma_T}\mathcal{C}\mathcal{U}^-\psi d\sigma dt. \end{aligned} \tag{4.99}$$

Finally from (4.85),

$$\lim_{N\to\infty}\int_0^T \langle I_0\overline{\Lambda}^N, \phi\rangle_{(H^1(\Omega))', H^1(\Omega)} dt = \int_0^T \langle \Lambda, \phi\rangle_{(H^1(\Omega))', H^1(\Omega)} dt. \tag{4.100}$$

Combining (4.95), (4.96), (4.97), (4.99) and (4.100), we obtain (4.63). We also need to prove the following to complete the proof of Theorem 64:

$$c(x,0) = c_0(x), \tag{4.101}$$

$$0 \le c(x,t) \le \mathcal{N}, \quad \text{a.e. } (x,t) \in Q_T, \tag{4.102}$$

where $\mathcal{N}$ is defined in Proposition 55. To prove (4.101), we observe from (4.84) that $I_1C^N(0,\cdot) \rightarrow c(0,\cdot)$ strongly in $(H^1(\Omega))'$. But $I_1C^N(0) = c_0$ for all N. So $c(\cdot,0) = c_0(\cdot)$, a.e. in Ω. For (4.102), recall that we have a uniform bound (4.66) on I_0C^N. Then letting $N \rightarrow \infty$ and using the Lebesgue dominated convergence theorem, we finally get

$$0 \leq c(x,t) \leq \mathcal{N} \text{ a.e in } Q_T.$$

Proof of Theorem 64 The following completes the proof of the main result of this section. The existence of a weak solution $(\boldsymbol{u},\varphi) \in L^2((0,T);\boldsymbol{V}) \times L^2((0,T);R_2)$ is established above. Hence as the final step, we recover the Stokes pressure p using an inf-sup condition.

Lemma 61. *For any $q \in L^2(0,T;R_1)$, there exists $\boldsymbol{v} \in L^2(0,T;\boldsymbol{X})$ such that $\nabla\cdot\boldsymbol{v} = q$ in $(0,T)\times\Omega_1$ and*

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{X})} \leq \beta\|q\|_{L^2(0,T;R_1)},$$

for some positive constant $\beta > 0$ independent of $\boldsymbol{v}$ and q.

Proof. Let $q \in L^2(0,T;R_1)$. For a.e $t \in [0,T]$, define $q^t(x) = q(x,t)$, for a.e $x \in \Omega_1$. Then $q^t \in R_1$. From the inf-sup condition [38, Lemma 1.2], there exists $\boldsymbol{v}^t \in \boldsymbol{X}$ and $\beta > 0$ independent of q^t, such that

$$\nabla\cdot\boldsymbol{v}^t = q^t \text{ in } \Omega_1, \quad \|\nabla\boldsymbol{v}^t\|_{L^2(\Omega_1)} \leq \beta\|q^t\|_{L^2(\Omega_1)}.$$

Now, set $\boldsymbol{v}(x,t) = \boldsymbol{v}^t(x)$, for a.e $(x,t) \in \Omega_1\times[0,T]$. Then $\nabla\cdot\boldsymbol{v} = q$ and as $q \in L^2(0,T;R_1)$, $\nabla\cdot\boldsymbol{v} \in L^2(0,T;\boldsymbol{X})$. Integrating the square of the above inequality from 0 to T in time, we also have

$$\|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{X})} \leq \beta\|q\|_{L^2(0,T;R_1)}.$$

□

Equivalently, we have the following inf-sup condition: there exists a constant $\beta > 0$ such that

$$\inf_{q \in L^2(0,T;R_1)} \sup_{\boldsymbol{v} \in L^2(0,T;\boldsymbol{X})} \frac{\int_0^T (q, \nabla \cdot \boldsymbol{v})_{\Omega_1}}{\|q\|_{L^2(0,T;R_1)} \|\boldsymbol{v}\|_{L^2(0,T;\boldsymbol{X})}} \geq \beta.$$

This trivially implies that

$$\inf_{q \in L^2(0,T;R_1)} \sup_{(\boldsymbol{v},r) \in L^2(0,T;\boldsymbol{X} \times R_2)} \frac{\int_0^T (q, \nabla \cdot \boldsymbol{v})_{\Omega_1}}{\|q\|_{L^2(0,T;R_1)} \|(\boldsymbol{v}, r)\|_{L^2(0,T;\boldsymbol{X} \times R_2)}} \geq \beta.$$

From (4.31), we have for any $\boldsymbol{v} \in L^2(0,T;\boldsymbol{X})$ and $q \in L^2(0,T;R_2)$:

$$\int_0^T (\nabla \cdot \boldsymbol{v}, p)_{\Omega_1} dt = L(\boldsymbol{v}, q), \tag{4.103}$$

where L is a continuous linear functional on $L^2(0,T;\boldsymbol{X}) \times L^2(0,T;R_2)$:

$$L(\boldsymbol{v}, q) = \int_0^T \Big(2(\mu(c)\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \nabla q)_{\Omega_2} + \tilde{\gamma}(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) - (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} - (\Pi, q)_{\Omega_2} + (\mathcal{U}, q)_{\Gamma_2} \Big) dt. \tag{4.104}$$

As $(\boldsymbol{u}, \varphi)$ solves (4.36), L vanishes on the space $L^2(0,T;\boldsymbol{V}) \times L^2(0,T;R_2)$. Thus, from [64, Lemma 4.1], there exists a unique $p \in L^2(0,T;R_1)$ such that for all $(\boldsymbol{v}, q) \in L^2(0,T;\boldsymbol{X}) \times L^2(0,T;R_2)$, (4.103) holds. This completes the proof of Theorem 64.

Remark 62. *This inf-sup condition also shows that the weak problems (4.31) and (4.36) are equivalent.*

This concludes the analysis of the weak formulation of the Stokes-Darcy-transport problem. Next section proves existence result for the Navier Stokes-Darcy-transport problem where we added the nonlinearity to the system.

4.3 Coupling of the Navier-Stokes and Darcy Flow with Transport

We accept the same problem as in Section 4.1 but with the Navier-Stokes equations for the surface flow rather than the Stokes equations. The existence proof for the Stokes problem hold for the most part in this case, so this section will only be pointing out the differences and modifications. First we recall the Navier-Stokes equations where this time on Ω_1, $\boldsymbol{u}$ denotes the Navier-Stokes velocity and p denotes the Navier-Stokes pressure.

$$\frac{\partial \boldsymbol{u}}{\partial t} - \nabla \cdot (2\mu(c)\boldsymbol{D}(\boldsymbol{u}) - p\mathrm{I}) + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = \boldsymbol{\Psi}, \quad \text{in} \quad \Omega_1 \times (0,T) \tag{4.105}$$

Also, the balance of forces interface condition will be the same as in Model II of Chapter 3 as the other case is simpler.

4.3.1 Weak Formulation

The underlying spaces are defined exactly the same as in (4.28) from the Stokes case. Although the weak formulation differs only in the flow equation by the addition of the nonlinear term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$, for integrity the weak problem definition is presented below:

Definition 63. *The weak formulation of the coupled flow-transport problem defined by (4.2)-(4.13) is to find* $\boldsymbol{u}|_{\Omega_1} \in L^2(0,T;\boldsymbol{X})$, $p \in L^2(0,T;R_1)$, $\varphi \in L^2(0,T;R_2)$ *and* $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ *such that*

$$t \to c(\cdot,t) \in \mathcal{C}([0,T];(H^1(\Omega))'), \quad t \to \frac{\partial c}{\partial t}(\cdot,t) \in L^2(0,T;(H^1(\Omega))') \tag{4.106}$$

$$\textit{and} \quad c(\cdot,0) = c_0(\cdot) \ \textit{a.e. in}\ \Omega \tag{4.107}$$

satisfying for all $\boldsymbol{v} \in L^2(0,T;\boldsymbol{X})$, $r \in L^2(0,T;R_1)$ *and* $q \in L^2(0,T;R_2)$,

$$\int_0^T \Big(2(\mu(c)\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}),\nabla q)_{\Omega_2} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1} - (\nabla\cdot\boldsymbol{v},p)_{\Omega_1}$$
$$+\tilde{\gamma}(\boldsymbol{u},\varphi;\boldsymbol{v},q)\Big)dt = \int_0^T \Big((\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (\Pi,q)_{\Omega_2} - (\mathcal{U},q)_{\Gamma_2}\Big)dt, \quad (4.108)$$

and for all $\psi \in L^2(0,T;H^1(\Omega))$,

$$\int_0^T \langle\phi\frac{\partial c}{\partial t},\psi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt + \int_{Q_T}(\boldsymbol{F}(\boldsymbol{u})\nabla c - c\boldsymbol{u})\cdot\nabla\psi d\boldsymbol{x}dt + \int_{\Sigma_T}(c\mathcal{U}^+ - \mathcal{C}\mathcal{U}^-)\psi d\sigma dt$$
$$= \int_0^T \langle\Lambda,\psi\rangle_{(H^1(\Omega))',H^1(\Omega)}dt. \quad (4.109)$$

The velocity $\boldsymbol{u}|_{\Omega_2} \in L^2(0,T;L^2(\Omega_2)^2)$ in the Darcy region Ω_2 is obtained from the Darcy pressure φ by the equation

$$\boldsymbol{u} = -\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \quad \text{a.e. in } \Omega_2\times(0,T). \quad (4.110)$$

4.3.2 Existence of a Weak Solution

The following theorem gives the existence result for this formulation. There is a difference in the statement of the theorem compared to the Stokes case. Here we need an additional smallness assumption for the data or in other words, we need the viscosity to be big enough.

Theorem 64. *Assume that*

$$\mu_L^{3/2} > C_D^3 M_S^2 \|\mathcal{M}_D\|_{L^\infty(0,T)} \quad (4.111)$$

where

$$\mathcal{M}_D(t) = \Big(\frac{C_D^2 M_P^2}{\mu_L}\|\boldsymbol{\Psi}(t)\|^2_{L^2(\Omega_1)}$$
$$+\frac{3\mu_U}{k_L}\big(\frac{\rho^2}{\mu_L^2}\|\boldsymbol{K}\boldsymbol{g}\|^2_{L^2(\Omega_2)} + M_P^2\|\Pi(t)\|^2_{L^2(\Omega_2)} + M_2^2\|\mathcal{U}(t)\|^2_{L^2(\Gamma_2)}\big)\Big)^{\frac{1}{2}},$$

a.e. in $(0,T)$. *Then there exists a weak solution* $(\boldsymbol{u},p,\varphi,c)$ *to the problem defined in Definition 63. In addition,* $(\boldsymbol{u},\varphi)$ *satisfies*

$$\mu_L\|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})}+\frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(0,T;L^2(\Omega_2)^2)}\leq\|\mathcal{M}_D\|^2_{L^2(0,T)},$$

and c *satisfies*

$$0\leq c(x,t)\leq\left\|\frac{\Lambda}{\phi}\right\|_{L^1(0,T;L^\infty(\Omega))}+\max(\|c_0\|_{L^\infty(\Omega)},\|\mathcal{C}\|_{L^\infty(\Sigma_T)}),\quad a.e.\ (x,t)\in Q_T. \tag{4.112}$$

Remark 65. *We can obtain stronger mathematical results if we add inertial forces to the balance of forces as in Model I of Chapter 3. The resulting weak problem contains an additional term, namely* $-\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{u},\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}$ *in the left-hand side of (4.108).*

As before we consider the problem restricted to the divergence free subspace and drop the term with the Navier-Stokes pressure. We again use the Galerkin approach. This time the definition of the intermediate problem given in Proposition 51 includes the term $(\boldsymbol{U}^N_{n+1}\cdot\nabla\boldsymbol{U}^N_{n+1},\boldsymbol{v})_{\Omega_1}$ corresponding to the nonlinearity of the Navier-Stokes equations. Because of this addition, to prove the intermediate result, we ask for the extra assumption namely

$$\mu_L^{3/2}>C_D^3M_S^2\mathcal{M}_D^n$$

where

$$\mathcal{M}_D^n=\Big(\frac{M_1^2M_P^2}{\mu_L}\|\overline{\Psi}^N_{n+1}\|^2_{L^2(\Omega_1)}+\frac{3\mu_U}{k_L}\big(\frac{\rho^2}{\mu_L^2}\|\boldsymbol{Kg}\|^2_{L^2(\Omega_2)}+M_P^2\|\overline{\Pi}^N_{n+1}\|^2_{L^2(\Omega_2)}+M_2^2\|\overline{\mathcal{U}}^N_{n+1}\|^2_{L^2(\Gamma_2)}\big)\Big)^{\frac{1}{2}}.$$

Indeed, this is a consequence of (4.111) using (4.38). This assumption then implies that on the sphere of radius $\mathcal{M}_D^n$, we have

$$\|\nabla\boldsymbol{U}^N_{n+1}\|_{L^2(\Omega_1)}\leq\frac{\mathcal{M}_D^nC_D}{\sqrt{\mu_L}}<\frac{\mu_L}{C_D^2M_S^2}.$$

Then we can hide the nonlinear terms in $\|\boldsymbol{D}(\boldsymbol{U}_{n+1}^N)\|_{L^2(\Omega_1)}$ as follows by Hölder's inequality and Sobolev's inequality:

$$
\begin{aligned}
|(\boldsymbol{U}_{n+1}^N \cdot \nabla \boldsymbol{U}_{n+1}^N, \boldsymbol{U}_{n+1}^N)_{\Omega_1}| \leq \|\boldsymbol{U}_{n+1}^N\|_{L^4(\Omega_1)}^2 \|\nabla \boldsymbol{U}_{n+1}^N\|_{L^2(\Omega_1)} \leq M_S^2 \|\nabla \boldsymbol{U}_{n+1}^N\|_{L^2(\Omega_1)}^3 \\
< \frac{\mu_L}{C_D^2} \|\nabla \boldsymbol{U}_{n+1}^N\|_{L^2(\Omega_1)}^2 < \mu_L \|\boldsymbol{D}(\boldsymbol{U}_{n+1}^N)\|_{L^2(\Omega_1)}.
\end{aligned}
$$

The bound obtained in this case is the same as (4.43) written in a more compact form for simplification. Other than this there is no difference to the proof of Proposition 51. Proposition 52 about the existence of concentration C_n^N of the intermediate problem still holds. As before, we obtain the approximate solution after integrating the intermediate equations from $n\Delta t$ to $(n+1)\Delta t$ and summing from $n = 0$ to $n = N-1$. This results in an approximate solution definition which is only different in the inclusion of the term $\int_0^T (I_0 \boldsymbol{U}^N \cdot \nabla I_0 \boldsymbol{U}^N, \boldsymbol{v})_{\Omega_1} dt$ and the bound of the approximate solution $(I_0 \boldsymbol{U}^N, I_0 \Phi^N)$. The bound we have in this case is

$$
\begin{aligned}
\mu_L \|\boldsymbol{D}(I_0 \boldsymbol{U}^N)\|_{L^2(0,T;L^2(\Omega_1)^{2\times 2})}^2 + \frac{1}{\mu_U} \|\boldsymbol{K}^{\frac{1}{2}} \nabla I_0 \Phi^N\|_{L^2(0,T;L^2(\Omega_2)^2)}^2 \\
\leq \Big(\frac{C_D^2 M_P^2}{\mu_L} \|\boldsymbol{\Psi}\|_{L^2(0,T;L^2(\Omega_1)^2)}^2 + \frac{3\mu_U}{k_L} \big(\frac{\rho^2 T}{\mu_L^2} \|\boldsymbol{K}\boldsymbol{g}\|_{L^2(\Omega_2)}^2 \\
+ M_P^2 \|\Pi\|_{L^2(0,T;L^2(\Omega_2))}^2 + M_2^2 \|\mathcal{U}\|_{L^2(0,T;L^2(\Gamma_2))}^2 \big) \Big). \quad (4.113)
\end{aligned}
$$

Again every result holds as they are except the strong convergence (4.90) of $I_0 \boldsymbol{U}^N$ presented in Proposition 60. For that, we need some modification in the proof. Recall that in the proof we considered a problem where c is the limit found in Proposition 59. This time the problem we define with the limit c is the following: Find $(\boldsymbol{u}|_{\Omega_1}, \varphi) \in L^2(0,T;\boldsymbol{V}_1) \times L^2(0,T;R_2)$ satisfying

$$
\begin{aligned}
\int_0^T \Big(2(\mu(c)\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \nabla q)_{\Omega_2} + (\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v})_{\Omega_1} \\
+ \tilde{\gamma}(\boldsymbol{u}, \varphi; \boldsymbol{v}, q) \Big) dt = \int_0^T ((\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi, q)_{\Omega_2} - (\mathcal{U}, q)_{\Gamma_2}) \, dt, \quad (4.114)
\end{aligned}
$$

for all $\boldsymbol{v} \in L^2(0,T;\boldsymbol{V}_1)$ and for all $q \in L^2(0,T;R_2)$. The existence result of $(\boldsymbol{u},\varphi)$ to this problem is an easy modification of [38] under the condition (4.111). For this, consider finding $(\boldsymbol{u}^t|_{\Omega_1},\varphi^t) \in \boldsymbol{V} \times R_2$ satisfying

$$2(\mu(c(t))\boldsymbol{D}(\boldsymbol{u}^t),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu(c(t))}(\nabla\varphi^t - \rho\boldsymbol{g}),\nabla q)_{\Omega_2} + \tilde{\gamma}(\boldsymbol{u}^t,\varphi^t;\boldsymbol{v},q)$$
$$- (\boldsymbol{u}^t\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} = (\boldsymbol{\Psi}(t),\boldsymbol{v})_{\Omega_1} + (\Pi(t),q)_{\Omega_2} - (\mathcal{U}(t),q)_{\Gamma_2}, \quad (4.115)$$

for a.e. $t \in (0,T)$. Then assuming (4.111) we also have

$$\mu(c(t))^{3/2} > C_D^3 M_S^2 \mathcal{M}_D(t) \text{ for a.e. } t \in (0,T).$$

Then there exists $(\boldsymbol{u}^t,\varphi^t) \in \boldsymbol{V} \times R_1$ satisfying (4.115), such that

$$\mu_L\|\boldsymbol{D}(\boldsymbol{u}^t)\|^2_{L^2(\Omega_1)} + \frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi^t\|^2_{L^2(\Omega_2)} \le \mathcal{M}_D(t)^2. \quad (4.116)$$

Let $\boldsymbol{u}(\boldsymbol{x},t) = \boldsymbol{u}^t(\boldsymbol{x})$. Integrating (4.115) and (4.116) from 0 to T, we get (4.114) and

$$\mu_L\|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla\varphi\|^2_{L^2(0,T;L^2(\Omega_2)^2)} \le \|\mathcal{M}_D\|^2_{L^2(0,T)}. \quad (4.117)$$

Define $\boldsymbol{u}|_{\Omega_2} \in L^2(0,T;L^2(\Omega_2)^2))$ as

$$\boldsymbol{u} = -\frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}), \quad \text{a.e. in } \Omega_2 \times (0,T).$$

As before we look at the difference between the equations (4.61) and (4.114). This yields

$$\int_0^T \Big(2(\mu(\widetilde{I_0C}^N_{\Delta t})\boldsymbol{D}(I_0\boldsymbol{U}^N) - \mu(c)\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1}$$
$$+ (\frac{\boldsymbol{K}}{\mu(\widetilde{I_0C}^N_{\Delta t})}(\nabla I_0\Phi^N - \rho\boldsymbol{g}) - \frac{\boldsymbol{K}}{\mu(c)}(\nabla\varphi - \rho\boldsymbol{g}),\nabla q)_{\Omega_2}$$
$$+ (I_0\boldsymbol{U}^N\cdot\nabla I_0\boldsymbol{U}^N - \boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1} + \tilde{\gamma}(I_0\boldsymbol{U}^N - \boldsymbol{u}, I_0\Phi^N - \varphi;\boldsymbol{v},q)\Big)dt$$
$$= \int_0^T \Big((I_0\overline{\boldsymbol{\Psi}}^N - \boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (I_0\overline{\Pi}^N - \Pi,q)_{\Omega_2} - (I_0\overline{\mathcal{U}}^N - \mathcal{U},q)_{\Gamma_2}\Big)\,dt. \quad (4.118)$$

We deal with all the terms except the nonlinear ones exactly the same way as before. To handle the nonlinear terms we write

$$\int_0^T (I_0\boldsymbol{U}^N\cdot\nabla I_0\boldsymbol{U}^N-\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1}dt$$
$$=\int_0^T\Big(\big((I_0\boldsymbol{U}^N-\boldsymbol{u})\cdot\nabla I_0\boldsymbol{U}^N,\boldsymbol{v}\big)_{\Omega_1}+\big(\boldsymbol{u}\cdot\nabla(I_0\boldsymbol{U}^N-\boldsymbol{u}),\boldsymbol{v}\big)_{\Omega_1}\Big)dt.$$

We bound the integrand of the above equation by using Hölder's inequality, (3.29) and (4.26) as follows

$$\begin{aligned}|\big((I_0\boldsymbol{U}^N-\boldsymbol{u})\cdot\nabla I_0\boldsymbol{U}^N,\boldsymbol{v}\big)_{\Omega_1}+\big(\boldsymbol{u}\cdot\nabla(I_0\boldsymbol{U}^N-\boldsymbol{u}),\boldsymbol{v}\big)_{\Omega_1}|\\ \leq C_D^2M_S^2\Big(\|\nabla I_0\boldsymbol{U}^N\|_{L^2(\Omega_1)}+\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)}\Big)\|\boldsymbol{D}(I_0\boldsymbol{U}^N-\boldsymbol{u})\|_{L^2(\Omega_1)}\|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega_1)}\end{aligned} \tag{4.119}$$

Then letting $\boldsymbol{v}=I_0\boldsymbol{U}^N-\boldsymbol{u}$, $q=I_0\Phi^N-\varphi$ in (4.118), using (4.15), (4.14), the nonnegativity of the $\tilde{\gamma}$ term and (4.119), we have

$$\begin{aligned}&\int_0^T\Big(\big(2\mu_L-C_D^2M_S^2(\|\nabla I_0\boldsymbol{U}^N\|_{L^2(\Omega_1)}+\|\nabla\boldsymbol{u}\|_{L^2(\Omega_1)})\big)\|\boldsymbol{D}(I_0\boldsymbol{U}^N-\boldsymbol{u})\|^2_{L^2(\Omega_1)}\\
&\quad+\frac{1}{\mu_U}\|\boldsymbol{K}^{\frac12}\nabla(I_0\Phi^N-\varphi)\|^2_{L^2(\Omega_2)}\Big)dt\leq\int_0^T\Big((I_0\overline{\boldsymbol{\Psi}}^N-\boldsymbol{\Psi},I_0\boldsymbol{U}^N-\boldsymbol{u})_{\Omega_1}\\
&\qquad+(I_0\overline{\Pi}^N-\Pi,I_0\Phi^N-\varphi)_{\Omega_2}-(I_0\overline{\mathcal{U}}^N-\mathcal{U},I_0\Phi^N-\varphi)_{\Gamma_2}\\
&\qquad-2\big((\mu(\widetilde{I_0C}^N_{\Delta t})-\mu(c))\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(I_0\boldsymbol{U}^N-\boldsymbol{u})\big)_{\Omega_1}\\
&\qquad-\big((\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})}-\frac{1}{\mu(c)})(\boldsymbol{K}(\nabla\varphi-\rho\boldsymbol{g}),\nabla(I_0\Phi^N-\varphi)\big)_{\Omega_2}\Big)dt.\end{aligned}$$

Observe from (4.43), which still holds for the Navier-Stokes case with the modification of the first coefficient, that if we take the maximum over $n=1,\ldots,N$ and recalling (4.38) and (4.59),

$$\mu_L\|\boldsymbol{D}(I_0\boldsymbol{U}^N)\|^2_{L^\infty(0,T;L^2(\Omega_1)^{2\times2})}\leq\max_{n=1,\ldots,N}(\mathcal{M}_D^n)^2\leq\|\mathcal{M}_D\|^2_{L^\infty(0,T)}.$$

Also, from (4.116),

$$\mu_L \|\boldsymbol{D}(\boldsymbol{u})\|^2_{L^\infty(0,T;L^2(\Omega_1)^{2\times 2})} \le \|\mathcal{M}_D\|^2_{L^\infty(0,T)}.$$

Therefore,

$$\int_0^T 2\Big(\big(\frac{\mu_L}{C_D^2} - \frac{C_D^3 M_S^2}{\mu_L^{\frac{1}{2}}}\|\mathcal{M}_D\|_{L^\infty(0,T)}\big)\|\boldsymbol{D}(I_0\boldsymbol{U}^N - \boldsymbol{u})\|^2_{L^2(\Omega_1)} + \frac{1}{\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla(I_0\Phi^N - \varphi)\|^2_{L^2(\Omega_2)}\Big)dt \le \int_0^T \Big((I_0\overline{\boldsymbol{\Psi}}^N - \boldsymbol{\Psi}, I_0\boldsymbol{U}^N - \boldsymbol{u})_{\Omega_1} + (I_0\overline{\Pi}^N - \Pi, I_0\Phi^N - \varphi)_{\Omega_2} - 2\big((\mu(\widetilde{I_0C}^N_{\Delta t}) - \mu(c))\boldsymbol{D}(\boldsymbol{u}), \boldsymbol{D}(I_0\boldsymbol{U}^N - \boldsymbol{u})\big)_{\Omega_1} - (I_0\overline{\mathcal{U}}^N - \mathcal{U}, I_0\Phi^N - \varphi)_{\Gamma_2} - \big((\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})} - \frac{1}{\mu(c)})(\boldsymbol{K}(\nabla\varphi - \rho\boldsymbol{g}), \nabla(I_0\Phi^N - \varphi))_{\Omega_2}\Big)dt.$$

Using Cauchy-Schwarz inequality, Poincaré inequality (4.27) and Young's inequality together with (4.24) gives

$$\big(\mu_L - \frac{C_D^3 M_S^2}{\mu_L^{\frac{1}{2}}}\|\mathcal{M}_D\|_{L^\infty(0,T)}\big)\|\boldsymbol{D}(I_0\boldsymbol{U}^N - \boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \frac{1}{2\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla(I_0\Phi^N - \varphi)\|^2_{L^2(0,T;L^2(\Omega_2)^2)} \le M\Big(\|I_0\overline{\boldsymbol{\Psi}}^N - \boldsymbol{\Psi}\|^2_{L^2(0,T;L^2(\Omega_1)^2)} + \|I_0\overline{\Pi}^N - \Pi\|^2_{L^2(0,T;L^2(\Omega_2))} + \|I_0\overline{\mathcal{U}}^N - \mathcal{U}\|^2_{L^2(0,T;L^2(\Gamma_2))} + \|(\mu(\widetilde{I_0C}^N_{\Delta t}) - \mu(c))\boldsymbol{D}(\boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \|(\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})} - \frac{1}{\mu(c)})\boldsymbol{K}\nabla\varphi\|^2_{L^2(0,T;L^2(\Omega_2)^2)} + \|\rho(\frac{1}{\mu(\widetilde{I_0C}^N_{\Delta t})} - \frac{1}{\mu(c)})\boldsymbol{K}\boldsymbol{g}\|^2_{L^2(0,T;L^2(\Omega_2))}\Big),$$

where M is a generic constant independent of N. Then by uniform boundedness (4.14) and continuity of μ, (4.74), (4.87), (4.88) and (4.89) together with the Lebesgue dominated convergence theorem imply

$$\big(\mu_L - \frac{C_D^3 M_S^2}{\mu_L^{\frac{1}{2}}}\|\mathcal{M}_D\|_{L^\infty(0,T)}\big)\|\boldsymbol{D}(I_0\boldsymbol{U}^N - \boldsymbol{u})\|^2_{L^2(0,T;L^2(\Omega_1)^{2\times 2})} + \frac{1}{2\mu_U}\|\boldsymbol{K}^{\frac{1}{2}}\nabla(I_0\Phi^N - \varphi)\|^2_{L^2(0,T;L^2(\Omega_2)^2)} \to 0, \text{ as } N \to \infty.$$

Thus, because of the small data condition (4.111) and (3.29), letting $N \to \infty$, we again obtain

$$\nabla I_0 \boldsymbol{U}^N \quad \to \quad \nabla \boldsymbol{u} \text{ strongly in } L^2(0,T;L^2(\Omega_1)^{2\times 2}), \tag{4.120}$$

$$\nabla I_0 \Phi^N \quad \to \quad \nabla \varphi \text{ strongly in } L^2(0,T;L^2(\Omega_2)^2). \tag{4.121}$$

Then (4.90) follows from (4.62), the continuity of μ, (4.27), proposition 59, (4.120) and (4.121).

The rest of the proof works the same way except that in the last step in recovering the Navier-Stokes pressure p, the linear function L now includes the term $(\boldsymbol{u}\cdot\nabla\boldsymbol{u}, \boldsymbol{v})_{\Omega_1}$ in the integrand of the right hand side.

Next section deals with the numerical approximation of the special case of the problems described in the previous two sections.

4.4 One-Way Coupling of the Navier-Stokes/Stokes and Darcy Flow with Transport

The contents of this section comes from a joint work with P. Chidyagwai and B. Rivière [58]. This section drops the assumption that μ is a function of the concentration c and simply sets it equal to a positive constant. Hence, the coupling is a one-way coupling, in the sense that the velocity field obtained from solving the surface/subsurface flow problem, becomes an input data for the transport problem. We also assume that the Dirichlet boundary $|\Gamma_{2D}| \neq 0$ and that it is contained in the outflow boundary, that is,

$$\Gamma_{2D} \subset \{\boldsymbol{x} \in \partial\Omega : \mathcal{U}(\boldsymbol{x}) \geq 0\}$$

Hence, the analysis of the previous section is still valid in this case and the weak formulation and the deduced existence result is stated below. The weak formulation of the coupled flow problem is to find $\boldsymbol{u} \in \boldsymbol{X}, p \in R^1, \varphi \in R^2$ and $c \in L^2(0,T;H^1(\Omega)) \cap L^\infty(Q_T)$ such that

$$t \to c(t,\cdot) \in \mathcal{C}([0,T];(H^1(\Omega))'), \quad t \to \frac{\partial c}{\partial t}(t,\cdot) \in L^2((0,T);(H^1(\Omega))'), \tag{4.122}$$

$$c(0,x) = c_0(x), \text{ a.e. } x \in \Omega. \tag{4.123}$$

and satisfying for all $\boldsymbol{v} \in \boldsymbol{X}, \forall r \in R^1, \forall q \in R^2$,

$$\begin{aligned} 2\mu(\boldsymbol{D}(\boldsymbol{u}),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1} + (\boldsymbol{u}\cdot\nabla\boldsymbol{u},\boldsymbol{v})_{\Omega_1} + (\frac{\boldsymbol{K}}{\mu}\nabla\varphi,\nabla q)_{\Omega_2} - (\nabla\cdot\boldsymbol{v},p)_{\Omega_1} + (\varphi,\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}} \\ + G(\boldsymbol{K}^{\frac{1}{2}}\boldsymbol{u}\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}} - (\boldsymbol{u}\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}} + (\nabla\cdot\boldsymbol{u}_1,r)_{\Omega_1} \\ = (\boldsymbol{\Psi},\boldsymbol{v})_{\Omega_1} + (\Pi + \frac{\boldsymbol{K}}{\mu}\rho\boldsymbol{g},q)_{\Omega_2} + (\mathcal{U},q)_{\Gamma_2} \end{aligned} \tag{4.124}$$

and for all $z \in L^2(0,T);H^1(\Omega)$,

$$\begin{aligned} \int_0^T \langle\varphi\frac{\partial c}{\partial t},z\rangle_{(H^1(\Omega))',(H^1(\Omega))}dt - \int_{Q_T} c\boldsymbol{u}\cdot\nabla z d\boldsymbol{x}dt + \int_{Q_T} F(\boldsymbol{u})\nabla c\cdot\nabla z d\boldsymbol{x}dt \\ + \int_{\Sigma_T}(c\mathcal{U}^+ - \mathcal{C}\mathcal{U}^-)zd\sigma dt = \int_{Q_T}\Lambda z d\boldsymbol{x}dt. \end{aligned} \tag{4.125}$$

From the results of the previous sections, we obtain the following existence result:

Theorem 66. *Assume that* $\boldsymbol{\Psi} \in L^2(\Omega_1)^2$, $\Pi \geq 0$, $\Pi \in L^2(\Omega_2)$ *and* $\Lambda \geq 0$, $\Lambda \in L^1(0,T;L^\infty(\Omega)) \cap L^2(0,T;L^2(\Omega))$. *There exists a constant* $\tilde{M} > 0$ *such that if*

$$\mu^2 > \tilde{M}\Big(\|\boldsymbol{\Psi}\|^2_{L^2(\Omega_1)} + \|\boldsymbol{K}\boldsymbol{g}\|^2_{L^2(\Omega_2)} + \mu^2\big(\|\Pi\|^2_{L^2(\Omega_2)} + \|\mathcal{U}\|^2_{L^2(\Gamma_2)}\big)\Big)^{\frac{1}{2}},$$

then there exists a weak solution $(\boldsymbol{u},p,\varphi,c)$ *to the weak problem defined in (4.122)-(4.125).*

Remark 67. *Similar results hold if the interface condition with the inertial forces defined in Model II is used. The coupled flow problem with this interface condition has been studied numerically by Chidyagwai and Rivière [39]. If the Stokes equations* *are used rather than the Navier-Stokes equations, then there is no need for a small data condition like the one given in Theorem 66 above.*

We direcly move to the numerical analysis of the problem. The flow problem is approximated by a combination of the FEM and DG method. The transport problem is solved by a DG method that upwinds the numerical fluxes in the subsurface region [59]. In this case, one does not need to use slope limiters. In the following the numerical schemes are defined and error estimates are obtained and the schemes are tried on a numerical example to show the robustness of the methods for fractured porous media. The chapter proceeds by assuming that the free flow is governed by the Navier-Stokes equation and the simplifications are mentioned if the Stokes equation is used instead of the Navier-Stokes equation.

4.4.1 Numerical Scheme

Let $\mathcal{E}_h$ be a regular family of triangulations of $\overline{\Omega}$ (see [82]) and let h denote the maximum diameter of the triangles. We assume that the interface Γ_{12} is a finite union of triangle edges. The restriction of $\mathcal{E}_h$ to Ω_i is also a regular family of triangulations of $\overline{\Omega}_i$; we denote it by $\mathcal{E}_h^i$ and impose that the two meshes $\mathcal{E}_h^i$ coincide at the interface Γ_{12}. This restriction simplifies the discussion, but it can be relaxed. We accept the rest of the notation about the mesh as it is.

Numerical Approximation of Flow Problem

The approximation of the flow problem is done using three different schemes based on combinations of the FEM and the DG method. For now the discretization of the flow problem is introduced in a general form. Formally, the discrete weak formulation of (3.38)-(3.44) can be written as:

Find $\boldsymbol{U}_h^1 \in \boldsymbol{X}_h, P_h \in R_h^1, \Phi_h \in R_h^2$ such that

$$\begin{aligned} \forall \boldsymbol{v} \in \boldsymbol{X}_h, \forall q \in R_h^2, \qquad & a_{\rm NS}(\boldsymbol{U}_h^1, \boldsymbol{v}) + b_{\rm NS}(\boldsymbol{v}, P_h) + k_{\rm NS}(\boldsymbol{U}_h^1; \boldsymbol{U}_h^1, \boldsymbol{v}) + a_{\rm D}(\Phi_h, q) \\ & + \tilde{\gamma}(\boldsymbol{U}_h^1, \Phi_h; \boldsymbol{v}, q) = L(\boldsymbol{v}, q), \\ \forall r \in R_h^1, \qquad & b_{\rm NS}(\boldsymbol{U}_h^1, r) = 0, \\ & \int_{\Omega_1} P_h + \int_{\Omega_2} \Phi_h = 0. \end{aligned}$$

Denote by $\boldsymbol{U}_h^1$ the resulting velocity field of the coupled Navier-Stokes and Darcy equations. The velocity $\boldsymbol{U}_h^1$ is defined in Ω by:

$$\boldsymbol{U}_h^1 = \begin{cases} \boldsymbol{U}_h^1, & \text{in } \Omega_1 \\ -\frac{\boldsymbol{K}}{\mu}(\nabla \Phi_h - \rho \boldsymbol{g}), & \text{in } \Omega_2 \end{cases} \tag{4.126}$$

The form L is defined as:

$$L(\boldsymbol{v}, q) = (\boldsymbol{\Psi}, \boldsymbol{v})_{\Omega_1} + (\Pi + \frac{\boldsymbol{K}}{\mu}\rho \boldsymbol{g}, q)_{\Omega_2} + (\mathcal{U}, q)_{\Gamma_2}$$

and the form $\tilde{\gamma}$ is given in (3.112). The following sections describe the forms $a_{\rm NS}$, $a_{\rm D}$, $b_{\rm NS}$ and $k_{\rm NS}$ corresponding to different schemes which were studied in [38, 39, 51]. For completeness all methods are defined below and the results are stated together with the results for concentration.

DG Method The primal DG method is applied to both the Navier-Stokes equations and the Darcy equations. The notation will be the same as in Section 3.2.3. To simplify the text, we assume that $\sigma_e = \sigma$ and $\epsilon_{\rm NS} = \epsilon_{\rm D} = \epsilon$ are fixed constants for both forms $a_{\rm NS}$ and $a_{\rm D}$. Let $k_1, k_2 \geq 1$ be integers and set the discrete spaces as

$$\boldsymbol{X}_h^1 = \mathcal{D}_{k_1}(\mathcal{E}_h^1), \quad R_h^1 = \mathcal{D}_{k_1-1}(\mathcal{E}_h^1), \quad R_h^2 = \mathcal{D}_{k_2}(\mathcal{E}_h^2).$$

The forms $a_{\rm NS}$ and $d_{\rm NS}$ are exactly the same as in Section 3.2.3. However, the form $a_{\rm D}$, presented below, has an extra μ^{-1} coefficient since compared to (3.3), (4.4) has an extra $1/\mu$.

$$\forall z_h^2, q_h^2 \in M_2^h, \quad a_{\mathrm{D}}(z_h, q_h) = \mu^{-1} \sum_{E \in \mathcal{E}_h^2} (\boldsymbol{K} \nabla z_h, \nabla q_h)_E - \mu^{-1} \sum_{e \in \Gamma_h^2} (\{\boldsymbol{K} \nabla z_h \cdot \boldsymbol{n}_e\}, [q_h])_e$$

$$+\mu^{-1} \epsilon_2 \sum_{e \in \Gamma_h^2} (\{\boldsymbol{K} \nabla q_h \cdot \boldsymbol{n}_e\}, [z_h])_e + \sum_{e \in \Gamma_h^2} \frac{\sigma}{|e|} ([z_h], [q_h])_e, \tag{4.127}$$

We define the nonlinear form using the definition (3.58) as follows:

$$\forall \boldsymbol{u}_h, \boldsymbol{v}_h, \boldsymbol{w}_h \in \boldsymbol{X}_h^1, \quad k_{\mathrm{NS}}(\boldsymbol{u}_h; \boldsymbol{v}_h, \boldsymbol{w}_h) = N(\boldsymbol{u}_h, \boldsymbol{u}_h; \boldsymbol{v}_h, \boldsymbol{w}_h).$$

In this case, the norms associated with the discrete spaces are:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}_h^1} = \left(\sum_{E \in \mathcal{E}_h^1} \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h^1 \cup \Gamma_1} |e|^{-1} \|[\boldsymbol{v}]\|_{L^2(e)}^2 \right)^{\frac{1}{2}}$$

$$\|q\|_{R_h^1} = \|q\|_{L^2(\Omega_1)}$$

$$\|q\|_{R_h^2} = \left(\sum_{E \in \mathcal{E}_h^2} \|\boldsymbol{K}^{\frac{1}{2}} \nabla q\|_{L^2(E)}^2 + \sum_{e \in \Gamma_h^2} |e|^{-1} \|[q]\|_{L^2(e)}^2 \right)^{\frac{1}{2}}.$$

FEM Method In this second approach, the discrete spaces are conforming spaces of order k_1 for Ω_1 and k_2 for Ω_2. For instance, to approximate the Navier-Stokes velocity and pressure, one can use the MINI elements [83] of order one and the Taylor-Hood elements [84] of order two. These spaces satisfy an inf-sup condition, with an inf-sup constant independent of h. The Darcy pressure space is

$$R_h^2 = \{q_h \in \mathcal{C}(\overline{\Omega}_2) : \, q_h|_E \in \mathbb{P}_{k_2}(E), \forall E \in \mathcal{E}_h^2\}.$$

The FEM spaces are equipped with the following norms:

$$\|\boldsymbol{v}\|_{\boldsymbol{X}_h^1} = \|\boldsymbol{D}(\boldsymbol{v})\|_{L^2(\Omega_1)}, \quad \|q\|_{R_h^1} = \|q\|_{L^2(\Omega_1)}, \quad \|q\|_{R_h^2} = \|\boldsymbol{K}^{\frac{1}{2}} \nabla q\|_{L^2(\Omega_2)}$$

The bilinear forms are

$$a_{\rm NS}(\boldsymbol{v}_h, \boldsymbol{w}_h) = 2\mu(\boldsymbol{D}(\boldsymbol{v}_h), \boldsymbol{D}(\boldsymbol{w}_h))_{\Omega_1}, \tag{4.128}$$

$$b_{\rm NS}(\boldsymbol{v}_h, r_h) = -(r_h, \nabla \cdot \boldsymbol{v}_h)_{\Omega_1}, \tag{4.129}$$

$$a_{\rm D}(z_h, q_h) = (\boldsymbol{K}\nabla z_h, \nabla q_h)_{\Omega_2} \tag{4.130}$$

$$k_{\rm NS}(\boldsymbol{z}_h; \boldsymbol{v}_h, \boldsymbol{w}_h) = \frac{1}{2}(\boldsymbol{z}_h \cdot \nabla \boldsymbol{v}_h, \boldsymbol{w}_h)_{\Omega_1} - \frac{1}{2}(\boldsymbol{z}_h \cdot \nabla \boldsymbol{w}_h, \boldsymbol{v}_h)_{\Omega_1} + \frac{1}{2}(\boldsymbol{z}_h \cdot \boldsymbol{n}_{12}, \boldsymbol{v}_h \cdot \boldsymbol{w}_h)_{\Gamma_{12}}, \tag{4.131}$$

FEM/DG Method In this third approach, we propose to employ the FEM to solve the Navier-Stokes equations in Ω_1 and to employ the DG method to solve the Darcy equations in Ω_2. Conforming element spaces of order k_1 are used for the spaces $\boldsymbol{X}_h^1$ and R_h^1, and discontinuous piecewise polynomials of degree k_2 are used for the space R_h^2. The bilinear forms are the forms defined by (4.128), (4.129), (4.131) and (4.127).

Numerical Approximation of the Transport Problem

The transport equation (4.8) is discretized by a combined backward Euler and DG method. Let Δt be a positive time step and let $t^j = j\Delta t$ denote the time at the j^{th} step. Let

$$Q^h = \mathcal{D}_r(\mathcal{E}_h^2).$$

The approximation of the initial concentration is obtained by an L^2 projection:

$$\forall q_h \in Q_h, \quad (C_h^0, q_h)_\Omega = (c_0, q_h)_\Omega$$

For any $j \geq 0$, the approximation C_h^{j+1} of the concentration c at time t^{j+1} is defined by the following discrete variational problem.

$$\forall q_h \in Q_h, \quad \varphi(\frac{C_h^{j+1} - C_h^j}{\Delta t}, q_h)_\Omega + a_T(\boldsymbol{U}_h^1; C_h^{j+1}, q_h) + d_T(\boldsymbol{U}_h^1; C_h^{j+1}, q_h) = L_T^{j+1}(q_h) \tag{4.132}$$

where the bilinear form a_T is a DG discretization of the operator $-\nabla \cdot (F(\boldsymbol{u})\nabla c)$ and the bilinear form d_T is a DG discretization of the operator $\nabla \cdot (\boldsymbol{u}c)$. Before defining these forms, we introduce the upwind value $q_h^\uparrow$ of a function q_h in Q_h with respect to the velocity field $\boldsymbol{U}_h^1$, defined by (4.126). Let e be an edge shared by the elements E_1 and E_2 and let the unit normal vector $\boldsymbol{n}_e$ point outward of E_1.

$$q_h^\uparrow = \begin{cases} q_h|_{E_1} & \text{if } \{\boldsymbol{U}_h^1\} \cdot \boldsymbol{n}_e > 0, \\ q_h|_{E_2} & \text{if } \{\boldsymbol{U}_h^1\} \cdot \boldsymbol{n}_e \leq 0. \end{cases}$$

The penalty parameter is denoted by σ_e as it varies from edge to edge. The symmetrization parameter is denoted by $\epsilon_T \in \{-1, 1\}$. The forms a_T, d_T, L_T are given below for any θ_h, q_h in Q_h:

$$\begin{aligned} a_T(\boldsymbol{U}_h^1; \theta_h, q_h) &= \sum_{E \in \mathcal{E}_h^1} (F(\boldsymbol{U}_h^1)\nabla\theta_h, \nabla q_h)_E + \sum_{e \in \Gamma_h} |e|^{-1}(\sigma_e[\theta_h], [q_h])_e \\ &- \sum_{e \in \Gamma_h} ((F(\boldsymbol{U}_h^1)\nabla\theta_h \cdot \boldsymbol{n}_e)^\uparrow, [q_h])_e + \epsilon_T \sum_{e \in \Gamma_h} ((F(\boldsymbol{U}_h^1)\nabla q_h \cdot \boldsymbol{n}_e)^\uparrow, [\theta_h])_e \\ &+ \sum_{e \in \partial\Omega} (\theta_h, \mathcal{U}^+ q_h)_e, \\ d_T(\boldsymbol{U}_h^1; \theta_h, q_h) &= -\sum_{E \in \mathcal{E}_h^1} (\theta_h \boldsymbol{U}_h^1, \nabla q_h)_E + \sum_{e \in \Gamma_h} (\theta_h^\uparrow \{\boldsymbol{U}_h^1 \cdot \boldsymbol{n}_e\}, [q_h])_e, \\ L_T^{j+1}(q_h) &= \int_\Omega \Lambda(t^{j+1}) q_h + \int_{\partial\Omega} \mathcal{C}(t^{j+1}) \mathcal{U}^- q_h. \end{aligned}$$

This scheme uses an improved DG method in which the diffusive fluxes are upwinded whereas in the standard DG method the diffusive fluxes are averaged. The improved method is more stable and does not require the use of slope limiters [59]. The space Q_h is equipped with the following semi-norm:

$$|q_h|_{Q_h} = \Big(\sum_{E\in\mathcal{E}_h^1} \|\nabla q_h\|_{L^2(E)}^2 + \sum_{e\in\Gamma_h} |e|^{-1} \|\sigma_e^{\frac{1}{2}}[q_h]\|_{L^2(e)}^2 \Big).$$

We now recall the coercivity property of the form a_T: there is a constant $\kappa > 0$ such that

$$\forall q_h \in Q_h, \quad a_T(\boldsymbol{U}_h^1; q_h, q_h) \geq \kappa |q_h|_{Q_h}^2 + \|(\mathcal{U}^+)^{\frac{1}{2}} q_h\|_{L^2(\partial\Omega)}^2. \tag{4.133}$$

This is straightforward for the NIPG method ($\epsilon_T = 1$) and in that case the constant $\kappa = \min(1, \alpha)$ where α is the lower bound for $F(\boldsymbol{u})$. For the SIPG method ($\epsilon_T = -1$), we use the fact that the matrix $F(\boldsymbol{U}_h^1)$ is bounded above and the coercivity is obtained if the penalty parameter is large enough.

We will use the following inverse inequality for the existence and uniqueness result corresponding to the concentration. There is a constant $M > 0$ independent of h such that

$$\forall q_h \in Q_h, \forall E \in \mathcal{E}_h^1, \quad \|q_h\|_{L^\infty(E)} \leq M h^{-1} \|q_h\|_{L^2(E)}. \tag{4.134}$$

4.4.2 Existence and Uniqueness of the Numerical Solution

Flow Problem

The discretization with DG method and the FEM/DG method of the flow problem were analyzed in [38, 39, 51] for different boundary conditions for the Darcy pressure. It is a technicality to redo the analysis for the case of Neumann boundary condition. A similar analysis can be done for the FEM method. Existence and uniqueness of the numerical solution $(\boldsymbol{U}_h^1, P_h, \Phi_h)$ are obtained under small data condition.

Concentration Problem

To prove existence and uniqueness of the discrete solution of the concentration problem, it suffices to show uniqueness since the system is linear. Clearly the initial concentration is uniquely defined. Fix $j \geq 0$. Let $\theta_h = C_h^{j+1} - \tilde{C}_h^{j+1}$ be the difference of two solutions of (4.132). The function θ_h satisfies

$$\frac{\varphi}{\Delta t}\|\theta_h\|_{L^2(\Omega)}^2 + a_T(\boldsymbol{U}_h^1;\theta_h,\theta_h) + d_T(\boldsymbol{U}_h^1;\theta_h,\theta_h) = 0.$$

Next, we use coercivity of a_T (4.133):

$$\frac{\varphi}{\Delta t}\|\theta_h\|_{L^2(\Omega)}^2 + \kappa|\theta_h|_{Q_h}^2 \leq |d_T(\boldsymbol{U}_h^1;\theta_h,\theta_h)|.$$

The first term in $d_T(\boldsymbol{U}_h^1;\theta_h,\theta_h)$ is bounded using Cauchy-Schwarz inequality, the inverse inequality (4.134) and the bound (4.138).

$$\begin{aligned}
|\sum_{E\in\mathcal{E}_h^1}(\theta_h\boldsymbol{U}_h^1,\nabla\theta_h)_E| &\leq \sum_{E\in\mathcal{E}_h^1}\|\theta_h\|_{L^\infty(E)}\|\boldsymbol{U}_h^1\|_{L^2(E)}\|\nabla\theta_h\|_{L^2(E)}\\
&\leq Mh^{-1}\sum_{E\in\mathcal{E}_h^1}\|\theta_h\|_{L^2(E)}\|\boldsymbol{U}_h^1\|_{L^2(E)}\|\nabla\theta_h\|_{L^2(E)} \leq M\overline{M}h^{-1}\sum_{E\in\mathcal{E}_h^1}\|\theta_h\|_{L^2(E)}\|\nabla\theta_h\|_{L^2(E)}\\
&\leq \frac{M^2\overline{M}^2}{\kappa h^2}\|\theta_h\|_{L^2(\Omega)}^2 + \frac{\kappa}{4}\sum_{E\in\mathcal{E}_h^1}\|\nabla\theta_h\|_{L^2(E)}^2.
\end{aligned}$$

The second term in $d_T(\boldsymbol{U}_h^1;\theta_h,\theta_h)$ is bounded similarly, but here we take advantage of the penalty term:

$$\begin{aligned}
|\sum_{e\in\Gamma_h}(\theta_h^\uparrow\{\boldsymbol{U}_h^1\cdot\boldsymbol{n}_e\},[\theta_h])_e| &\leq M\sum_{e\in\Gamma_h}|e|^{-\frac{1}{2}}\|\sigma_e^{\frac{1}{2}}[\theta_h]\|_{L^2(e)}h^{\frac{1}{2}}\|\theta_h^\uparrow\|_{L^\infty(e)}\|\{\boldsymbol{U}_h^1\cdot\boldsymbol{n}_e\}\|_{L^2(e)}\\
&\leq M\sum_{e\in\Gamma_h}|e|^{-\frac{1}{2}}\|\sigma_e^{\frac{1}{2}}[\theta_h]\|_{L^2(e)}h^{\frac{1}{2}}h^{-1}\|\theta_h\|_{L^2(E_e^{12})}\|\{\boldsymbol{U}_h^1\cdot\boldsymbol{n}_e\}\|_{L^2(e)}\\
&\leq M\sum_{e\in\Gamma_h}|e|^{-\frac{1}{2}}\|\sigma_e^{\frac{1}{2}}[\theta_h]\|_{L^2(e)}h^{\frac{1}{2}}h^{-1}\|\theta_h\|_{L^2(E_e^{12})}h^{-\frac{1}{2}}\|\boldsymbol{U}_h^1\|_{L^2(E_e^{12})}.
\end{aligned}$$

In the bound above we have used the inverse inequality $\|\boldsymbol{U}_h^1\|_{L^2(e)} \leq M h^{-\frac{1}{2}} \|\boldsymbol{U}_h^1\|_{L^2(E)}$. We also defined the union of the elements who share the edge e by E_e^{12}. Next, we use the bound on the discrete velocity (4.138) and we obtain:

$$|\sum_{e\in\Gamma_h}(\theta_h^{\uparrow}\{\boldsymbol{U}_h^1\cdot\boldsymbol{n}_e\},[\theta_h])_e| \leq \frac{M^2\overline{M}^2}{h^2\kappa}\|\theta_h\|_{L^2(\Omega)}^2 + \frac{\kappa}{4}\sum_{e\in\Gamma_h}|e|^{-1}\|\sigma_e^{\frac{1}{2}}[\theta_h]\|_{L^2(e)}^2.$$

Therefore we have

$$(\frac{1}{\Delta t} - \frac{2M^2\overline{M}^2}{\kappa h^2})\|\theta_h\|_{L^2(\Omega)}^2 + \frac{3\kappa}{4}|\theta_h|_{Q_h}^2 \leq 0.$$

We conclude that $\theta_h = 0$ if the time step satisfies the following condition:

$$\Delta t < \frac{\kappa h^2}{2M^2\overline{M}^2}.$$

We summarize our result below.

Lemma 68. *There is a constant $M_0 > 0$ such that if $\Delta t < M_0 h^2$, there is a unique solution to the scheme (4.132).*

4.4.3 Error Analysis

Flow Problem

Convergence rates are optimal [38, 39, 51]. More precisely, there is a constant M independent of h such that

$$\|\boldsymbol{u}_1 - \boldsymbol{U}_h^1\|_{\boldsymbol{X}_h^1} + \|p_1 - P_h\|_{R_h^1} + \|p_2 - \Phi_h\|_{R_h^2} \leq M(h^{k_1} + h^{k_2}). \tag{4.135}$$

Using the fact that $\|\cdot\|_{L^2(\Omega_1)} \leq M\|\cdot\|_{\boldsymbol{X}_h^1}$, we obtain an error bound of the velocity field in the L^2-norm.

$$\|\boldsymbol{u} - \boldsymbol{U}_h^1\|_{L^2(\Omega)} \leq M(h^{k_1} + h^{k_2}). \tag{4.136}$$

As a consequence, using a trace theorem, an inverse inequality, and the Lagrange interpolant of $\boldsymbol{u}$, we have

$$\forall e \in \Gamma_h, \quad \|\boldsymbol{u} - \boldsymbol{U}_h^1\|_{L^2(e)} \leq M(h^{k_1-\frac{1}{2}} + h^{k_2-\frac{1}{2}}). \tag{4.137}$$

One can also show that the velocity $\boldsymbol{U}_h^1$ is bounded in the L^2 norm by the data: there is a constant $\overline{M} > 0$ independent of h, but dependent on the data μ, $\|\boldsymbol{\Psi}\|_{L^2(\Omega_1)}$, $\|\Pi\|_{L^2(\Omega_2)}$ and $\|\mathcal{U}\|_{L^2(\partial\Omega)}$, such that

$$\|\boldsymbol{U}_h^1\|_{L^2(\Omega)} \leq \overline{M}. \tag{4.138}$$

Concentration problem

We decompose the error at time t^j into an approximation error η and a numerical error ξ. Let $\tilde{c} \in Q_h \cap \mathcal{C}(\overline{\Omega})$ be an approximation of c in the sense that the following approximation bounds [61, p.111] hold:

$$\|c(t^j) - \tilde{c}(t^j)\|_{L^2(\Omega)} \leq Mh^{r+1}\|c(t^j)\|_{H^{r+1}(\Omega)}, \; \|\nabla(c(t^j) - \tilde{c}(t^j))\|_{L^2(\Omega)} \leq Mh^{r}\|c(t^j)\|_{H^{r+1}(\Omega)},$$

$$\|c(t^j) - \tilde{c}(t^j)\|_{L^\infty(\Omega)} \leq Mh^{r+1}\|c(t^j)\|_{H^{r+1}(\Omega)}, \; \|\nabla(c(t^j) - \tilde{c}(t^j))\|_{L^\infty(\Omega)} \leq Mh^{r}\|c(t^j)\|_{H^{r+1}(\Omega)}.$$

We write

$$C_h^j - c(t^j) = \eta^j - \xi^j, \quad \eta^j = C_h^j - \tilde{c}(t^j), \quad \xi^j = c(t^j) - \tilde{c}(t^j).$$

Theorem 69. *Under the assumption of Lemma 68 and the additional regularity assumption $c \in L^2(0,T; H^{r+1}(\Omega)) \cap W^{1,\infty}(\Omega)$, $c_t \in L^2(0,T; H^r(\Omega))$, $c_0 \in H^r(\Omega)$, there is a constant M independent of h and Δt such that for all $m \geq 1$, such that for Δt small enough, we have the error bound*

$$\|\eta^m\|_{L^2(\Omega)}^2 + \kappa\Delta t \sum_{j=1}^{m} |\eta^j|_{Q_h}^2 + \Delta t \sum_{j=1}^{m} \||\mathcal{U}|^{\frac{1}{2}}\eta^j\|_{\partial\Omega}^2 \leq M(h^{2r} + h^{2k_1} + h^{2k_2} + \Delta t^2)$$

Proof. The error equation becomes

$$(\varphi\frac{\eta^{j+1}-\eta^j}{\Delta t}, q_h)_\Omega + a_T(\boldsymbol{U}_h^1;\eta^{j+1}, q_h) + d_T(\boldsymbol{u};\eta^{j+1}, q_h) = (\varphi\frac{\partial\xi}{\partial t}(t^{j+1}), q_h)_\Omega$$
$$+ (\varphi\frac{\partial\tilde{c}}{\partial t}(t^{j+1}) - \varphi\frac{\tilde{c}^{j+1}-\tilde{c}^j}{\Delta t}, q_h)_\Omega + d_T(\boldsymbol{u}-\boldsymbol{U}_h^1;\eta^{j+1}, q_h) + a_T(\boldsymbol{U}_h^1;\xi^{j+1}, q_h)$$
$$+ d_T(\boldsymbol{U}_h^1;\xi^{j+1}, q_h) + d_T(\boldsymbol{u}-\boldsymbol{U}_h^1; c(t^{j+1}), q_h) + a_T(\boldsymbol{u}; c(t^{j+1}), q_h) - a_T(\boldsymbol{U}_h^1; c(t^{j+1}), q_h).$$

for all $q_h \in Q_h$. We take $q_h = \eta^{j+1}$ and we use coercivity of a_T:

$$\frac{\varphi}{2\Delta t}(\|\eta^{j+1}\|^2_{L^2(\Omega)} - \|\eta^j\|^2_{L^2(\Omega)}) + \kappa|\eta^{j+1}|^2_{Q_h} + d_T(\boldsymbol{u};\eta^{j+1},\eta^{j+1}) + \|(\mathcal{U}^+)^{\frac{1}{2}}\eta^{j+1}\|^2_{L^2(\partial\Omega)}$$
$$\leq |(\frac{\partial\xi}{\partial t}(t^{j+1}), \eta^{j+1})_\Omega| + |(\frac{\partial\tilde{c}}{\partial t}(t^{j+1}) - \frac{\tilde{c}^{j+1}-\tilde{c}^j}{\Delta t}, \eta^{j+1})_\Omega| + |d_T(\boldsymbol{u}-\boldsymbol{U}_h^1;\eta^{j+1},\eta^{j+1})|$$
$$+ |a_T(\boldsymbol{U}_h^1;\xi^{j+1},\eta^{j+1})| + |d_T(\boldsymbol{U}_h^1;\xi^{j+1},\eta^{j+1})| + |d_T(\boldsymbol{u}-\boldsymbol{U}_h^1; c(t^{j+1}),\eta^{j+1})|$$
$$+ |a_T(\boldsymbol{u}; c(t^{j+1}),\eta^{j+1}) - a_T(\boldsymbol{U}_h^1; c(t^{j+1}),\eta^{j+1})| \quad (4.139)$$

Since the weak solution satisfies $\nabla\cdot\boldsymbol{u}|_{\Omega_1} = \boldsymbol{0}$ and $\nabla\cdot\boldsymbol{u}|_{\Omega_2} = \Pi \geq 0$, we use integration by parts and obtain:

$$d_T(\boldsymbol{u};\eta^{j+1},\eta^{j+1}) + \|(\mathcal{U}^+)^{\frac{1}{2}}\eta^{j+1}\|^2_{L^2(\partial\Omega)} = \frac{1}{2}(\mathcal{U}^+, (\eta^{j+1})^2)_{\partial\Omega} + \frac{1}{2}(\mathcal{U}^-, (\eta^{j+1})^2)_{\partial\Omega} \geq 0$$

We now bound the first and second terms in the right-hand side of (4.139), under the regularity assumption for the exact solution c.

$$|(\frac{\partial\xi}{\partial t}(t^{j+1}), \eta^{j+1})_\Omega| \leq \|\eta^{j+1}\|^2_{L^2(\Omega)} + Mh^{2r}\|\frac{\partial c}{\partial t}(t^{j+1})\|^2_{H^r(\Omega)}.$$
$$|(\frac{\partial\tilde{c}}{\partial t}(t^{j+1}) - \frac{\tilde{c}^{j+1}-\tilde{c}^j}{\Delta t}, \eta^{j+1})_\Omega| \leq \|\eta^{j+1}\|^2_{L^2(\Omega)} + \frac{\Delta t}{12}\int_{t^j}^{t^{j+1}} \|\frac{\partial^2\tilde{c}}{\partial t^2}\|^2_{L^2(\Omega)}.$$

We now bound the d_T terms. Using standard techniques, inequality (4.134), we obtain

$$d_T(\boldsymbol{u}-\boldsymbol{U}_h^1;\eta^{j+1},\eta^{j+1}) \leq Mh^{-1}\|\eta^{j+1}\|_{L^2(\Omega)}\|\boldsymbol{u}-\boldsymbol{U}_h^1\|_{L^2(\Omega)}|\eta^{j+1}|_{Q_h}.$$

Using the velocity bound (4.136) and the fact that $k_1 \geq 1, k_2 \geq 1$, we have

$$d_T(\boldsymbol{u}-\boldsymbol{U}_h^1;\eta^{j+1},\eta^{j+1}) \leq \frac{\kappa}{8}|\eta^{j+1}|^2_{Q_h} + M\|\eta^{j+1}\|^2_{L^2(\Omega)}.$$

Similarly, using (4.138), we have

$$d_T(\boldsymbol{U}_h^1; \xi^{j+1}, \eta^{j+1}) \leq M\|\xi^{j+1}\|_{L^\infty(\Omega)}\|\boldsymbol{U}_h^1\|_{L^2(\Omega)}|\eta^{j+1}|_{Q_h} \leq M\|\xi^{j+1}\|_{L^\infty(\Omega)}|\eta^{j+1}|_{Q_h}.$$

and using (4.136), (4.137) and the boundedness of the weak solution, we have

$$d_T(\boldsymbol{u} - \boldsymbol{U}_h^1; c(t^{j+1}), \eta^{j+1}) \leq M\|c(t^{j+1})\|_{L^\infty(\Omega)}|\eta^{j+1}|_{Q_h}\big(\|\boldsymbol{u} - \boldsymbol{U}_h^1\|_{L^2(\Omega)}$$
$$+ \big(\sum_{e\in\Gamma_h} |e|\|\boldsymbol{u} - \boldsymbol{U}_h^1\|^2_{L^2(e)}\big)^{\frac{1}{2}}\big) \leq \frac{\kappa}{8}|\eta^{j+1}|^2_{Q_h} + M(h^{2k_1} + h^{2k_2}).$$

The diffusive term $a_T(\boldsymbol{U}_h^1; \xi^{j+1}, \eta^{j+1})$ is bounded using standard techniques.

$$a_T(\boldsymbol{U}_h^1; \xi^{j+1}, \eta^{j+1}) \leq \frac{\kappa}{8}|\eta^{j+1}|^2_{Q_h} + \frac{1}{8}\|(\mathcal{U}^+)^{\frac{1}{2}}\eta^{j+1}\|^2_{L^2(\partial\Omega)} + Mh^{2r}\|c(t^{j+1})\|^2_{H^{r+1}(\Omega)}.$$

To bound the remaining diffusive terms, we use the boundedness of c, the Lipschitz continuity of F and the bounds (4.136), (4.137) [Note: here we need $\|\nabla c\|_{L^\infty(E,e)} < M$]

$$a_T(\boldsymbol{u}; c(t^{j+1}), \eta^{j+1}) - a_T(\boldsymbol{U}_h^1; c(t^{j+1}), \eta^{j+1}) \leq \frac{\kappa}{8}|\eta^{j+1}|^2_{Q_h} + M(h^{2k_1} + h^{2k_2}).$$

We can now conclude by combining all bounds, summing over the time steps, and using Gronwall's inequality. □

4.4.4 Numerical Example

In this section, we show that our schemes are robust for fractured porous medium. For more numerical examples of heterogeneous porous media see [58]. We also investigate the effect of different approximations of velocity on the concentration solution. In the following, the fluid viscosity is equal to 1, and the Beavers-Joseph-Saffman constant is equal to to 0.1. Meshes are generated using Gmsh [85], visualization is done using Tecplot [86] and the simulations are done using software developed by Rivière. Uniqueness of the pressure is obtained by imposing a Dirichlet boundary condition on part of the subsurface boundary.

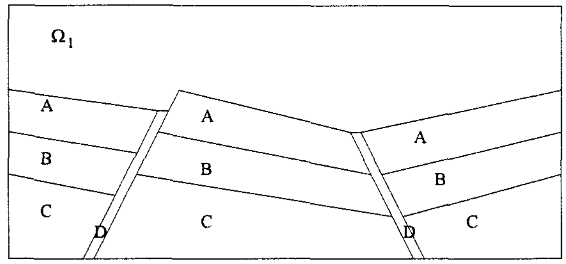


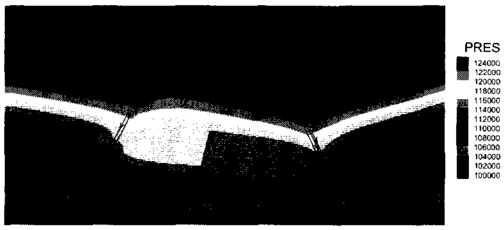
Figure 4.1 : Domain for surface coupled with fractured subsurface. Permeability value is 10^{-9} in A region, 10^{-5} in B region, 10^{-7} in C region and 10^{-4} in D region (slanted fractures).

Fractured Subsurface

In this example, the porous medium $\Omega = (0, 12) \times (0, 6)$ contains three horizontal layers of varying permeability that are intersected by two slanted faults. The permeability matrix is equal to $10^{-4}\mathbf{I}$, $10^{-9}\mathbf{I}$, $10^{-5}\mathbf{I}$, $10^{-7}\mathbf{I}$ in the faults, the top layer, the middle layer and the bottom layer respectively (see Fig. 4.1). First for the flow problem, we impose a parabolic velocity profile on the left vertical boundary of Ω_1 and a similar profile on the right vertical boundary of Ω_1 but with a smaller magnitude. Zero Neumann boundary conditions are imposed on the Darcy pressure for the vertical boundaries of Ω_2 and Dirichlet pressure is prescribed on bottom horizontal boundary. The Dirichlet values are given below:

$$\forall y \geq 4, \quad \boldsymbol{u}_1(0, y) = (0.25(y-4)(8-y), 0), \quad \boldsymbol{u}_1(12, y) = ((3/16)(y-4)(8-y), 0),$$
$$\forall 0 \leq x \leq 12, \quad \boldsymbol{u}_1(x, 6) = (1, 0), \quad p_2(x, 0) = 10^5.$$

Fig. 4.2 shows the pressure contours and the velocity field obtained with the DG method of first and second order, which yields 8707 and 17679 degrees of freedom respectively. The pressure follows a vertical gradient, and thus the velocity in the



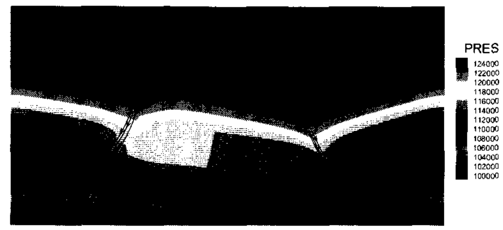


Figure 4.2 : Pressure and velocity field obtained with the DG method of order one (left figure) and order two (right figure).

middle layer (denoted by B on Fig. 4.1) remains small. For this example, we also solve the flow problem using the FEM/DG method of order one. The MINI elements are used for the Navier-Stokes region. Discontinuous piecewise linear or quadratic approximations are used in the Darcy region. Fig. 4.3 shows the pressure contours and streamlines obtained on the same mesh as the solutions in Fig. 4.2. Using FEM/DG is computationally cheaper than DG alone, as the number of degrees of freedom is 7899 and 14766 for piecewise linears and quadratics respectively. However we observe that even though the streamlines are similar, the values for the pressure differ. If we solve the problem on a finer mesh, the pressure values match those obtained by the DG scheme (see Fig. 4.4). The number of degrees of freedom is 125043 and 234915 for piecewise linears and quadratics respectively. Similar conclusions can be made if the FEM scheme is used in the whole domain. The method of order one yields the smallest number of degrees of freedom (2196), however the solution is not accurate enough and the mesh needs to be finer.

Next we describe the parameters chosen for the transport problem. The coefficients are: $\varphi = 0.2$, $\alpha_l = 0.1$, $\alpha_t = 0.01$, $\mathcal{C} = 0$, $d_m = 10^{-4}$ in Ω_2, $d_m = 10^{-2}$ in Ω_1.

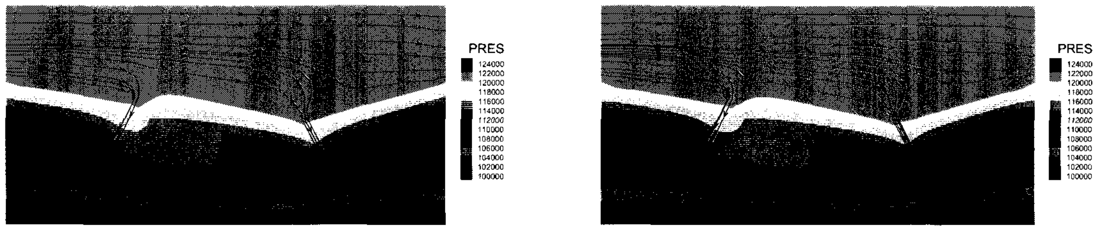


Figure 4.3 : Pressure and velocity field obtained with the FEM/DG method of order one (left figure) and order two (right figure).

We simulate the leakage of a contaminant in the surface. The initial concentration is equal to one in a localized region in the surface, and zero elsewhere. In addition, there is a temporary source of contaminant (for $t \le t^*$, with $t^* = 3$) defined by:

$$f(t,x,y) = \begin{cases} 0.5, & t < 3, \text{ and } ((x-2.0)^2 + (y-5.1)^2)^{\frac{1}{2}} \le 0.5 \\ 0, & \text{otherwise} \end{cases}$$

We obtain the numerical approximation of the concentration by the DG method with parameters $r = \epsilon = \sigma = 1$. In Fig. 4.5, 4.6, 4.7, we show the concentration contours at different times in the case where the numerical approximation of the velocity is obtained by DG (with $k_1 = k_2 = 2$), FEM/DG (with $k_1 = 1$ and $k_2 = 2$) and FEM (with $k_1 = k_2 = 1$) schemes. We note that the mesh used for the transport problem is the same as the one used in Fig. 4.2 and Fig. 4.3. The overall behavior of the solution is as expected: the contaminant is transported faster in the surface region, and some of it penetrates the subsurface via the slanted fractures. Because of the intermediate value of the permeability in the middle layer, some of the contaminant appears in part of region B neighboring the fractures. The interest of this example is

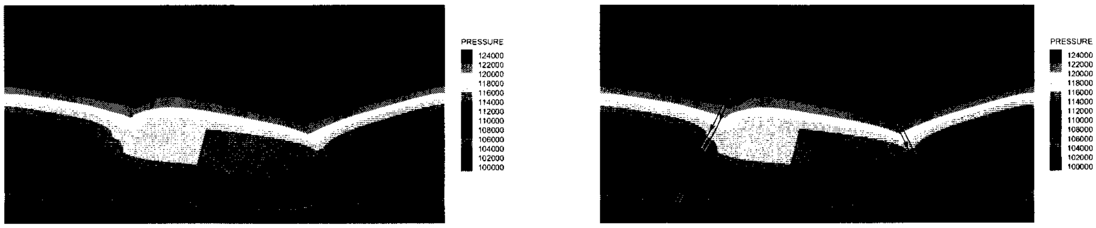


Figure 4.4 : Pressure and velocity field obtained with the FEM/DG method of order one (left figure) and order two (right figure) on a very fine mesh.

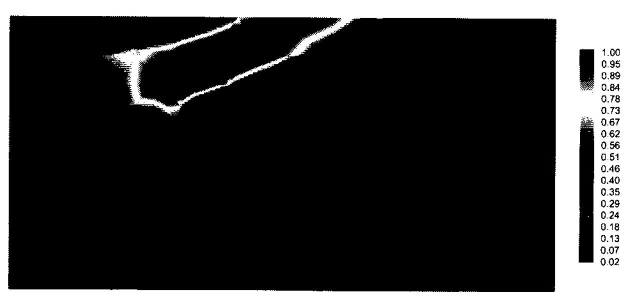
to see that the poor/good accuracy of the input velocity has an important effect on the concentration solution. At the times t_1 and t_2, solutions obtained with FEM/DG or FEM input velocities are similar. At the time t_3 (which is greater than t^*, the time when the source disappears), we observe an unphysical accumulation of mass at the outflow boundary of the left fracture if the FEM velocity is used. The use of DG in the subsurface region for the flow problem removes this numerical problem. We also note that the solution obtained with DG input velocity differs from the other two solutions. The contaminant plume appears to be less diffusive, and further along the x-axis. This is particularly clear in Fig. 4.7, where we see that the left fracture contains very little contaminant if the input velocity is obtained with DG. In addition, a larger amount of contaminant has reached the second fracture.

4.5 Summary

The coupling of surface/subsurface flow and transport is studied theoretically and numerically by the use of finite element methods and discontinuous Galerkin methods. It is shown that the DG scheme is robust and yields accurate solutions for fractured

(a) Input DG

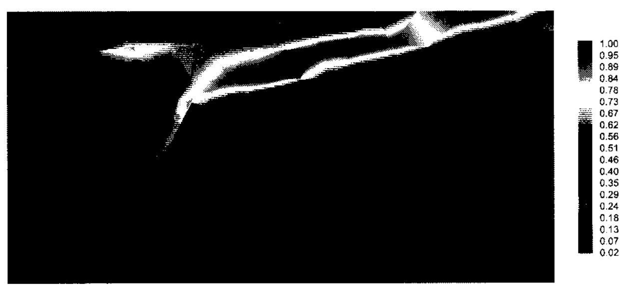
(b) Input FEM/DG



(c) Input FEM

Figure 4.5 : Concentration contours at time t_1 with input velocity obtained from DG (a), FEM/DG (b) and FEM (c).

(a) Input DG

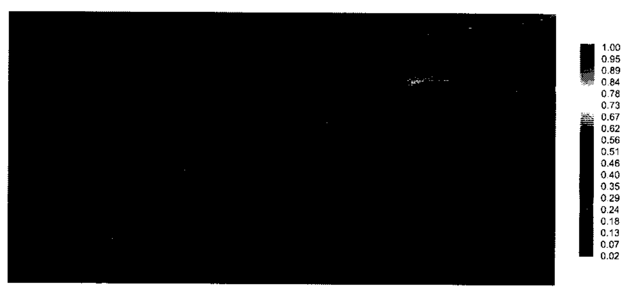


(b) Input FEM/DG

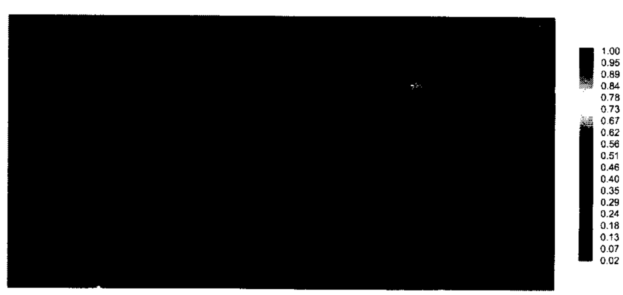
(c) Input FEM

Figure 4.6 : Concentration contours at time $t_2 = 2t_1$ with input velocity obtained from DG (a), FEM/DG (b) and FEM (c).

(a) Input DG



(b) Input FEM/DG



(c) Input FEM

Figure 4.7 : Concentration contours at time $t_3 = 5t_1$ with input velocity obtained from DG (a), FEM/DG (b), and FEM (c).

subsurface. A finer mesh is needed to obtain an accurate FEM/DG or FEM velocity. If one is constrained to use the same computational mesh for both flow and transport, then the most economical solution is still given by the DG method. It would be of interest to study the effects of projection of the velocity field, if independent meshes are used for the flow and transport problems.

Chapter 5

Conclusions

The first chapter of this thesis gives the first mathematical analysis of the coupled time-dependent Navier-Stokes and Darcy equations. The standard transmissibility conditions, namely the continuity of the flux, the Beavers-Joseph-Saffman condition and the balance of forces, are assumed on the interface separating the surface and subsurface. The last of these conditions is considered in two versions. First version adds the inertial forces $\frac{1}{2}\boldsymbol{u} \cdot \boldsymbol{u}$ to the condition hence canceling the problematic term. This violates the physical laws but is mathematically more convenient. I presented a weak formulation of this version and proved the existence of a weak solution. The second version without the additional inertial forces however is mathematically challenging and asks for an additional requirement on the data. Hence the existence of the weak solution in this second case is proved conditionally. This chapter also numerically analyzes the first version discretized with the DG methods and the Crank-Nicolson scheme. I showed that the error is optimal in space and second order in time. I have also discretized the same problem by the continuous FEM rather than the DG method and the results are similar [35]. Thus they are not included in here. This part of my thesis can also be seen as completing the series of papers on the Navier-Stokes/Darcy coupling [39, 51, 38] by extending the results to the time-dependent case.

The second problem of this thesis is again based on the coupling of Navier-Stokes and Darcy's equations. This flow is coupled to a convection-diffusion transport equation to account for the contaminant concentration in the problem of groundwater contamination through rivers. The published literature is very sparse on the coupling of Navier-Stokes/Darcy-transport problem. In this chapter, I first proved existence result for a weak solution for the linear case (Stokes and Darcy's equations) where the nonlinearity is neglected. I built the Navier-Stokes analysis on the analysis of this simpler case while pointing out the differences in between. I determined the additional small data assumption in order to have the existence of a weak solution. Furthermore, I provided numerical analysis of the scheme derived by using FEM and DG methods and presented a numerical example that shows that the DG scheme is robust and yields accurate solutions for fractured subsurfaces. The conclusion from the results is that a finer mesh is needed to obtain an accurate FEM/DG or FEM velocity. If one is constrained to use the same computational mesh for both flow and transport, then the most economical solution is still given by the DG method.

Chapter 6

Current and Future Work

Similar to the numerical analysis of Model I of Chapter 3, we can obtain convergence results under additional assumptions. The backward Euler method, which is chosen for simplicity, is applied to the fully coupled linearized problem. The next section describes the numerical scheme based on DG methods derived for Model II of Chapter 3.

6.1 Numerical Scheme For Model II

Let $\boldsymbol{X}_h, M_h^1, M_h^2$ denote the finite element spaces for the discretization of the Navier-Stokes velocity, Navier-Stokes pressure and Darcy pressure. The terms a_{NS} and a_{D} stand for the discretization of the elliptic operators $-2\mu\nabla \cdot D(\boldsymbol{u})$ and $-\nabla \cdot \boldsymbol{K}\nabla\varphi$. The discretization of the pressure term ∇p is denoted by the bilinear form b_{NS} and the discretization of the nonlinear terms is taken care of by the term c_{NS}. Lastly, the interface terms are combined in $\tilde{\gamma}$. Let $N \geq 1$ and $\Delta t = \frac{T}{N}$. Define $t^i = i\Delta t$. Then the fully discrete scheme is given by:

Find $\{(\boldsymbol{U}_h^i, P_h^i, \Phi_h^i\}_i \in \boldsymbol{X}_h \times M_h^1 \times M_h^2$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, q \in M_h^2, \quad \left(\frac{\boldsymbol{U}_h^{i+1} - \boldsymbol{U}_h^i}{\Delta t}, \boldsymbol{v}\right)_{\Omega_1} + a_{\text{NS}}(\boldsymbol{U}_h^{i+1}, \boldsymbol{v}) + N(\boldsymbol{U}_h^i, \boldsymbol{U}_h^i; \boldsymbol{U}_h^{i+1}, v)$$

$$+b_{\text{NS}}(p^{i+1}, \boldsymbol{v}) + a_{\text{D}}(\varphi^{i+1}, q) + \gamma(\boldsymbol{U}_h^{i+1}, \varphi^{i+1}; \boldsymbol{v}, q) = L(\boldsymbol{v}, q)$$

$$\forall q \in M_h^1, \quad b_{\text{NS}}(q, \boldsymbol{U}_h^{i+1}) = 0$$

$$\forall \boldsymbol{v} \in \boldsymbol{X}_h, \quad (\boldsymbol{U}_h^0, \boldsymbol{v})_{\Omega_1} = (\boldsymbol{u}_0, \boldsymbol{v})_{\Omega_1}$$

The forms a_{NS}, b_{NS} and a_{D} are defined similarly as in Section 3.2. The form N in this case is free of the discrete form of the inertial forces. Under additional assumptions and using standard techniques, one can show that there exists a unique discrete solution and that the error is optimal.

The numerical results obtained for one-way coupling problem can be extended to the full-coupling problem. The next section describes the numerical scheme.

6.2 Numerical Scheme for the Fully Coupled Flow and Transport

Denote by $\boldsymbol{X}_h$, R_h^1, R_h^2 and Q_h the discrete spaces. The choice for the time-discretization is the backward Euler method. Let Δt be a positive time step and let $t_j = j\Delta t$ denote the time at the j^{th} step. The fully-discrete problem is as follows:

Find $(\boldsymbol{U}_h^i)_{0\le i\le N} \in (\boldsymbol{X}_h)^{N+1}$, $(P_h^i)_{0\le i\le N} \in (R_h^1)^{N+1}$, $(\Phi_h^i)_{0\le i\le N} \in (R_h^2)^{N+1}$, $(C_h^i)_{0\le i\le N} \in (Q_h)^{N+1}$ such that

$$\forall z_h \in Q_h, \quad (C_h^0, z_h)_\Omega = (c_0, z_h)_\Omega,$$

for all $0 \le i \le N-1$, $\int_{\Omega_1} P_h^{i+1} = 0$ and

$$\begin{aligned}\forall \boldsymbol{v}^h \in \boldsymbol{X}_h, \forall r_h \in R_h^1, \quad & \tilde{a}_{\mathrm{NS}}(C_h^{i+1}; \boldsymbol{U}_h^{i+1}, \boldsymbol{v}_h) + \tilde{b}_{\mathrm{NS}}(\boldsymbol{v}_h, P_h^{i+1}) + \tilde{a}_{\mathrm{D}}(C_h^{i+1}; \Phi_h^{i+1}, z_h) \\ & + \tilde{k}_{\mathrm{NS}}(\boldsymbol{U}_h^{i+1}; \boldsymbol{U}_h^{i+1}, \boldsymbol{v}) + \tilde{\gamma}(\boldsymbol{U}_h^{i+1}, \Phi_h^{i+1}; \boldsymbol{v}_h, z_h) = \tilde{L}(C_h^{i+1}; \boldsymbol{v}_h, z_h),\end{aligned}$$

$$\forall z_h \in R_h^2, \quad b_{\mathrm{NS}}(\boldsymbol{U}_h^{i+1}, r_h) = 0, \text{ and}$$

$$\forall q_h \in Q_h, \quad \phi\Big(\frac{C_h^{i+1} - C_h^i}{\Delta t}, q_h\Big)_\Omega + a_{\mathrm{T}}(\boldsymbol{U}_h^{i+1}; C_h^{i+1}, q_h) + d_{\mathrm{T}}(\boldsymbol{U}_h^{i+1}; C_h^{i+1}, q_h) = L_{\mathrm{T}}^{j+1}(q_h)$$

and the discrete velocity on Ω_2 is given as

$$\boldsymbol{U}_h^{i+1} = -\frac{\boldsymbol{K}}{\mu(C_h^{i+1})}(\nabla \Phi_{i+1}^h - \rho \boldsymbol{g}), \quad \text{in } \Omega_2.$$

The forms $\tilde{a}_{\mathrm{NS}}$, $\tilde{b}_{\mathrm{NS}}$, $\tilde{a}_{\mathrm{D}}$ and $\tilde{k}_{\mathrm{NS}}$ this time incorporates the concentration into the scheme. So they will be defined in a slightly different way. This gives us a nonlinear system of equations. In addition to the existence and uniqueness of this fully discrete system, stability of the scheme can be proved.

6.3 Coupling of the Navier-Stokes/Stokes Flow with Two Phase Flow

Rather than coupling the surface flow with the single phase (Darcy) flow, I plan to consider the coupling with the two phase flow in the subsurface which is the simultaneous flow within a porous medium of two immiscible fluids such as oil and water. Immiscibility of fluids means that there is no mass transfer between the fluids. Two-phase flow occurs in a variety of flow phenomena in the subsurface. One example is the oil flow in reservoirs. See, for instance, Aziz and Settari [87], Parker [88] and Wheeler [89] for oil-reservoir modeling. NIPG and SIPG methods have been successfully applied to the two-phase flow problem [22, 23, 24, 25, 26, 27, 28]. My plan in the general sense is to combine the results for the surface/subsurface flow with the results for the two-phase flow.

Appendix A

Results for Chapter 3 and Chapter 4

A.1 Boundedness of $\{\boldsymbol{u}_m\}_{m\geq 1}$ in $H^\gamma(0,T;\boldsymbol{V},L^2(\Omega_1)^2)$ for $0<\gamma<\frac{1}{4}$.

In the following, C will be a generic constant independent of m. The proof is modified from [71, p.193] and [90, p.163] and uses Fourier transforms. Recall from Chapter 2 that the extension to $\mathbb{R}$ of a function f is denoted by $\tilde{f}$ and the Fourier transform of $\tilde{f}$ is denoted by $\hat{f}$. By Theorem 26, $\boldsymbol{u}_m$ is bounded in $L^2(0,T;\boldsymbol{V})$. Hence, it is enough to bound $\||\tau|^\gamma \hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(0,T;L^2(\Omega_1)^2)}$ to obtain boundedness in $H^\gamma(0,T;\boldsymbol{V},L^2(\Omega_1)^2)$. Extending the functions, the first equation in (P_m) is equivalent to

$$\frac{\partial}{\partial t}(\tilde{\boldsymbol{u}}_m(t),\boldsymbol{v})_{\Omega_1}+2\nu(\boldsymbol{D}(\tilde{\boldsymbol{u}}_m(t)),\boldsymbol{D}(\boldsymbol{v}))_{\Omega_1}+(\tilde{\boldsymbol{u}}_m(t)\cdot\nabla\tilde{\boldsymbol{u}}_m(t),\boldsymbol{v})_{\Omega_1}+(\boldsymbol{K}\nabla\tilde{p}_m(t),\nabla q)_{\Omega_2}$$
$$+(\tilde{p}_m(t)-\frac{1}{2}(\tilde{\boldsymbol{u}}_m(t)\cdot\tilde{\boldsymbol{u}}_m)(t),\boldsymbol{v}\cdot\boldsymbol{n}_{12})_{\Gamma_{12}}+G(\boldsymbol{K}^{-1/2}\tilde{\boldsymbol{u}}_m(t)\cdot\boldsymbol{\tau}_{12},\boldsymbol{v}\cdot\boldsymbol{\tau}_{12})_{\Gamma_{12}}$$
$$-(\tilde{\boldsymbol{u}}_m(t)\cdot\boldsymbol{n}_{12},q)_{\Gamma_{12}}=(\tilde{\boldsymbol{\Psi}}(t),\boldsymbol{v})_{\Omega_1}+(\tilde{\Pi}(t),q)_{\Omega_2}\quad\text{(A.1)}$$

for all $t\in\mathbb{R}$, for all $\boldsymbol{v}\in\boldsymbol{V}_m$ and for all $q\in M_m$. Let us now find a more suitable expression for the first term of this equation. By the definition of weak derivative and regularity of $\boldsymbol{u}_m$, for any $\phi\in\mathcal{D}(\mathbb{R})$,

$$\int_0^T\frac{\partial}{\partial t}(\tilde{\boldsymbol{u}}_m(t),\boldsymbol{v})_{\Omega_1}\phi(t)dt=-\int_{\mathbb{R}}(\tilde{\boldsymbol{u}}_m(t),\boldsymbol{v})_{\Omega_1}\phi' dt=-\int_0^T(\boldsymbol{u}_m(t),\boldsymbol{v})_{\Omega_1}\phi' dt$$
$$=\int_0^T\frac{\partial}{\partial t}(\boldsymbol{u}_m(t),\boldsymbol{v})_{\Omega_1}\phi(t)dt+(\boldsymbol{u}_m(0),\boldsymbol{v})_{\Omega_1}\phi(0)-(\boldsymbol{u}_m(T),\boldsymbol{v})_{\Omega_1}\phi(T)$$
$$=\int_{\mathbb{R}}\Big((\widetilde{\frac{\partial}{\partial t}\boldsymbol{u}_m}(t),\boldsymbol{v})_{\Omega_1}+(\boldsymbol{u}_m(t),\boldsymbol{v})_{\Omega_1}\delta_0-(\boldsymbol{u}_m(t),\boldsymbol{v})_{\Omega_1}\delta_T\Big)\phi(t)dt.$$

where δ_0 and δ_T are Dirac delta functions centered at 0 and T, respectively. So, in the sense of distributions in $\mathbb{R}$,

$$\frac{\partial}{\partial t}(\tilde{\boldsymbol{u}}_m, \boldsymbol{v})_{\Omega_1} = (\widetilde{\frac{\partial}{\partial t}\boldsymbol{u}_m}, \boldsymbol{v})_{\Omega_1} + (\boldsymbol{u}_m(t), \boldsymbol{v})_{\Omega_1}\delta_0 - (\boldsymbol{u}_m(t), \boldsymbol{v})_{\Omega_1}\delta_T. \tag{A.2}$$

For the third term in (A.1), we use

$$(\tilde{\boldsymbol{u}}_m \cdot \nabla\tilde{\boldsymbol{u}}_m, \boldsymbol{v})_{\Omega_1} = -(\tilde{\boldsymbol{u}}_m \cdot \nabla\boldsymbol{v}, \tilde{\boldsymbol{u}}_m)_{\Omega_1} + (\tilde{\boldsymbol{u}}_m \cdot \boldsymbol{v}, \tilde{\boldsymbol{u}}_m \cdot \boldsymbol{n}_{12})_{\Gamma_{12}}.$$

Note that,

$$(\tilde{\boldsymbol{u}}_m \cdot \nabla\boldsymbol{v}, \tilde{\boldsymbol{u}}_m)_{\Omega_1} = (\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m, \nabla\boldsymbol{v})_{\Omega_1}, \quad (\tilde{\boldsymbol{u}}_m \cdot \boldsymbol{v}, \tilde{\boldsymbol{u}}_m \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} = (\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m, \boldsymbol{v} \otimes \boldsymbol{n}_{12})_{\Gamma_{12}},$$

where $\otimes$ denote the outer product of two vectors. Using (A.1), (A.2) and the above observations, for all $1 \leq i \leq m$ and for all $t \in \mathbb{R}$,

$$\begin{aligned} &\frac{\partial}{\partial t}(\tilde{\boldsymbol{u}}_m, \boldsymbol{v})_{\Omega_1} + 2\nu(D(\tilde{\boldsymbol{u}}_m), D(\boldsymbol{v}))_{\Omega_1} - (\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m, \nabla\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\tilde{p}_m, \nabla q)_{\Omega_2} \\ &+ (\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m, \boldsymbol{v} \otimes \boldsymbol{n}_{12})_{\Gamma_{12}} + (\tilde{p}_m - \frac{1}{2}(\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m), \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} + G(\boldsymbol{K}^{-1/2}\tilde{\boldsymbol{u}}_m \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} \\ &- (\tilde{\boldsymbol{u}}_m \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}} = (\tilde{\boldsymbol{\Psi}}, \boldsymbol{v})_{\Omega_1} + (\tilde{\Pi}, q)_{\Omega_2} + (\boldsymbol{u}_m(t), \boldsymbol{v})_{\Omega_1}\delta_0 - (\boldsymbol{u}_m(t), \boldsymbol{v})_{\Omega_1}\delta_T. \end{aligned} \tag{A.3}$$

Taking Fourier transform of the above equation,

$$\begin{aligned} &2\pi i\tau(\hat{\boldsymbol{u}}_m(\tau), \boldsymbol{v})_{\Omega_1} + 2\nu(D(\hat{\boldsymbol{u}}_m(\tau)), D(\boldsymbol{v}))_{\Omega_1} - (\widehat{\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m}(\tau), \nabla\boldsymbol{v})_{\Omega_1} + (\boldsymbol{K}\nabla\hat{p}_m(\tau), \nabla q)_{\Omega_2} \\ &+ (\widehat{\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m}, \boldsymbol{v} \otimes \boldsymbol{n}_{12})_{\Gamma_{12}} - (\hat{p}_m(\tau) - \frac{1}{2}(\widehat{\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m})(\tau), \boldsymbol{v} \cdot \boldsymbol{n}_{12})_{\Gamma_{12}} \\ &- (\hat{\boldsymbol{u}}_m(\tau) \cdot \boldsymbol{n}_{12}, q)_{\Gamma_{12}} + G(\boldsymbol{K}^{-1/2}\hat{\boldsymbol{u}}_m(\tau) \cdot \boldsymbol{\tau}_{12}, \boldsymbol{v} \cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}} = \langle \hat{\boldsymbol{\Psi}}(\tau), \boldsymbol{v} \rangle_{\boldsymbol{V}', \boldsymbol{V}} \\ &+ \langle \hat{\Pi}(\tau), q \rangle_{M', M} + (\boldsymbol{u}_{0m}, \boldsymbol{v})_{\Omega_1} - (\boldsymbol{u}_m(T), \boldsymbol{v})_{\Omega_1} e^{-2\pi i T\tau}. \end{aligned}$$

Letting $\boldsymbol{v} = \hat{\boldsymbol{u}}_m(\tau)$ and cancelling $(\hat{\boldsymbol{u}}_m(\tau)\cdot \boldsymbol{n}_{12}, \hat{p}_m(\tau))_{\Gamma_{12}}$ terms we obtain

$$
\begin{aligned}
2\pi i\tau \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} &+ 2\nu\|D(\hat{\boldsymbol{u}}_m(\tau))\|^2_{L^2(\Omega_1)} - (\widehat{\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m}(\tau), \nabla \hat{\boldsymbol{u}}_m(\tau))_{\Omega_1} \\
&+ (\boldsymbol{K}\nabla \hat{p}_m(\tau), \nabla \hat{p}_m(\tau))_{\Omega_2} + (\widehat{\tilde{\boldsymbol{u}}_m(t) \otimes \tilde{\boldsymbol{u}}_m(t)}(\tau), \hat{\boldsymbol{u}}_m(\tau)\otimes \boldsymbol{n}_{12})_{\Gamma_{12}} \\
&+ G(\boldsymbol{K}^{-1/2}\hat{\boldsymbol{u}}_m(\tau)\cdot \boldsymbol{\tau}_{12}, \hat{\boldsymbol{u}}_m(\tau)\cdot \boldsymbol{\tau}_{12})_{\Gamma_{12}}) - (\frac{1}{2}(\widehat{\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m}(\tau), \hat{\boldsymbol{u}}_m(\tau)\cdot \boldsymbol{n}_{12})_{\Gamma_{12}} \\
&= \langle \hat{\boldsymbol{\Psi}}(\tau), \hat{\boldsymbol{u}}_m(\tau)\rangle_{\boldsymbol{V}', \boldsymbol{V}} + \langle \hat{\Pi}(\tau), \hat{p}_m(\tau)\rangle_{M', M} \\
&\quad + (\boldsymbol{u}_{0m}, \hat{\boldsymbol{u}}_m(\tau))_{\Omega_1} - (\boldsymbol{u}_m(T), \hat{\boldsymbol{u}}_m(\tau))_{\Omega_1} e^{-2\pi i T\tau}. \qquad (A.4)
\end{aligned}
$$

Observe that, second, fourth and sixth terms are real. Then taking the imaginary part of (A.4) yields

$$
\begin{aligned}
2\pi\tau\|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} = \operatorname{Im}\Big(&(\widehat{\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m}(\tau), \nabla \hat{\boldsymbol{u}}_m(\tau))_{\Omega_1} + \frac{1}{2}(\widehat{\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m}(\tau), \hat{\boldsymbol{u}}_m(\tau)\cdot \boldsymbol{n}_{12})_{\Gamma_{12}} \\
&- (\widehat{\tilde{\boldsymbol{u}}_m(t) \otimes \tilde{\boldsymbol{u}}_m(t)}, \hat{\boldsymbol{u}}_m(\tau)\otimes \boldsymbol{n}_{12})_{\Gamma_{12}} + \langle \hat{\boldsymbol{\Psi}}(\tau), \hat{\boldsymbol{u}}_m(\tau)\rangle_{\boldsymbol{V}', \boldsymbol{V}} \\
&+ \langle \hat{\Pi}(\tau), \hat{p}_m(\tau)\rangle_{M', M} + (\boldsymbol{u}_{0m}, \hat{\boldsymbol{u}}_m(\tau)) - (\boldsymbol{u}_m(T), \hat{\boldsymbol{u}}_m(\tau))e^{-2\pi i T\tau}\Big).
\end{aligned}
$$

Using the Cauchy-Schwarz inequality,

$$
\begin{aligned}
2\pi|\tau|\|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} &\le \|\widehat{\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m}(\tau)\|_{L^2(\Omega_1)}\big(\|\hat{\boldsymbol{u}}_m(\tau)\|_{H^1(\Omega_1)} + \|\hat{\boldsymbol{u}}_m(\tau)\otimes \boldsymbol{n}_{12}\|_{L^2(\Gamma_{12})}\big) \\
&+ \frac{1}{2}\|\widehat{\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m}(\tau)\|_{L^2(\Gamma_{12})}\|\hat{\boldsymbol{u}}_m(\tau)\|_{L^2(\Gamma_{12})} + \|\hat{\boldsymbol{\Psi}}(\tau)\|_{\boldsymbol{V}'}\|\hat{\boldsymbol{u}}_m(\tau)\|_{\boldsymbol{V}} + \|\hat{\Pi}(\tau)\|_{M'}\|\hat{p}_m(\tau)\|_{M} \\
&+ \big(\|\boldsymbol{u}_{0m}\|_{L^2(\Omega_1)} + \|\boldsymbol{u}_m(T)\|_{L^2(\Omega_1)}\big)\|\hat{\boldsymbol{u}}_m(\tau)\|_{L^2(\Omega_1)}.
\end{aligned}
$$

To bound the right hand side a series of estimates are needed. Applying the Hölder's inequality and using (3.37),

$$
\begin{aligned}
\|\widehat{\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m}(\tau)\|_{L^2(\Omega_1)} &= \Big\|\int_{\mathbb{R}} \tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m(t) e^{-2\pi i t\tau} dt\Big\|_{L^2(\Omega_1)} \le \int_{\mathbb{R}} \|\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m(t)\|_{L^2(\Omega_1)} dt \\
&= \int_0^T \|\boldsymbol{u}_m(t)\otimes \boldsymbol{u}_m(t)\|_{L^2(\Omega_1)} dt = \int_0^T \left(\sum_{i=1}^n \sum_{j=1}^n \|u^i_m(t) u^j_m(t)\|^2_{L^2(\Omega_1)}\right)^{1/2} dt \\
&\le \int_0^T \sum_{i=1}^n \|u^i_m(t)\|^2_{L^2(\Omega_1)} dt = \int_0^T \|\boldsymbol{u}_m\|^2_{L^2(\Omega_1)} dt = \|\boldsymbol{u}_m\|^2_{L^2(0,T;L^2(\Omega_1)^2)}.
\end{aligned}
$$

Then $\|\widehat{\tilde{\boldsymbol{u}}_m \otimes \tilde{\boldsymbol{u}}_m}(\tau)\|_{L^2(\Omega_1)}$ is bounded by (3.37) which also says that $\|\boldsymbol{u}_{0m}\|_{L^2(\Omega_1)} = \|\boldsymbol{u}_m(0)\|_{L^2(\Omega_1)}$ and $\|\boldsymbol{u}_m(T)\|_{L^2(\Omega_1)}$ are bounded. Further,

$$\begin{aligned} \|\hat{\Pi}(\tau)\|_{M'} &= \|\int_{\mathbb{R}} \tilde{\Pi}(t)e^{-2\pi i t\tau}dt\|_{M'} \leq \int_{\mathbb{R}} \|\tilde{\Pi}(t)\|_{M'}dt = \int_0^T \|\Pi(t)\|_{M'}dt, \\ \|\hat{\boldsymbol{\Psi}}(\tau)\|_{\boldsymbol{V}'} &= \|\int_{\mathbb{R}} \tilde{\boldsymbol{\Psi}}(t)e^{-2\pi i t\tau}dt\|_{\boldsymbol{V}'} \leq \int_{\mathbb{R}} \|\tilde{\boldsymbol{\Psi}}(t)\|_{\boldsymbol{V}'}dt = \int_0^T \|\boldsymbol{\Psi}(t)\|_{\boldsymbol{V}'}dt. \end{aligned}$$

Observe that,

$$\begin{aligned} \|\hat{\boldsymbol{u}}_m(\tau) \otimes \boldsymbol{n}_{12}\|_{L^2(\Gamma_{12})} &= \left(\sum_i \sum_j \|\hat{u}^i_m(\tau)n^j_{12}\|^2_{L^2(\Gamma_{12})}\right)^{\frac{1}{2}} \\ &\leq \|\hat{\boldsymbol{u}}_m(\tau)\|_{L^2(\Gamma_{12})}\|\boldsymbol{n}_{12}\|_{L^2(\Gamma_{12})} \leq C\|\hat{\boldsymbol{u}}_m(\tau)\|_{H^1(\Omega)}. \end{aligned}$$

The last bound needed is for $\|\widehat{\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m}(\tau)\|_{L^2(\Gamma_{12})}$.

$$\begin{aligned} \|\widehat{\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m}(\tau)\|_{L^2(\Gamma_{12})} &\leq \|\int_{\mathbb{R}} \tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m(\tau)e^{-2\pi i t\tau}dt\|_{L^2(\Gamma_{12})} \leq \int_{\mathbb{R}} \|\tilde{\boldsymbol{u}}_m \cdot \tilde{\boldsymbol{u}}_m(t)\|_{L^2(\Gamma_{12})}dt \\ &= \int_0^T \|\boldsymbol{u}_m \cdot \boldsymbol{u}_m(t)\|_{L^2(\Gamma_{12})}dt \leq C\int_0^T \|\boldsymbol{u}_m\|^2_{L^4(\Gamma_{12})} \leq C\|\boldsymbol{u}_m\|^2_{L^2(0,T;H^1(\Omega_1)^2)}. \end{aligned}$$

Combining all of these,

$$2\pi|\tau|\|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} \leq C\big(\|\hat{\boldsymbol{u}}_m(\tau)\|_{L^2(\Omega_1)}+\|\hat{\boldsymbol{u}}_m(\tau)\|_{H^1(\Omega_1)}+\|\hat{\boldsymbol{u}}_m(\tau)\|_{\boldsymbol{V}}+\|\hat{p}_m(\tau)\|_M\big). \tag{A.5}$$

Now fix $\sigma \in]\frac{1}{2}, 1[$. Note that Fourier transformation preserves the norm in $L^2(\mathbb{R})$ by Parseval's equality. Then,

$$\int_{\mathbb{R}} \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{H^1(\Omega)}d\tau = \int_{\mathbb{R}} \|\tilde{\boldsymbol{u}}_m(t)\|^2_{H^1(\Omega)}dt = \int_0^T \|\boldsymbol{u}_m(t)\|^2_{H^1(\Omega)}dt = \|\boldsymbol{u}_m\|^2_{L^2(0,T;H^1(\Omega_1)^2)}.$$

Hence,

$$\int_{\mathbb{R}} \frac{\|\hat{\boldsymbol{u}}_m(\tau)\|_{H^1(\Omega_1)}}{1+|\tau|^\sigma}d\tau \leq \|\boldsymbol{u}_m\|^2_{L^2(0,T;H^1(\Omega_1)^2)} \left(\int_{\mathbb{R}} \frac{1}{(1+|\tau|^\sigma)^2}d\tau\right)^{\frac{1}{2}}.$$

Let $M := \left(\int_{\mathbb{R}} \frac{1}{(1+|\tau|^\sigma)^2} d\tau\right)^{\frac{1}{2}} < \infty$. Then,

$$\int_{\mathbb{R}} \frac{\|\hat{\boldsymbol{u}}_m(\tau)\|_{H^1(\Omega_1)}}{1+|\tau|^\sigma} d\tau \leq M \|\boldsymbol{u}_m\|_{L^2(0,T;H^1(\Omega_1)^2)},$$

and similarly,

$$\int_{\mathbb{R}} \frac{\|\hat{\boldsymbol{u}}_m(\tau)\|_{L^2(\Omega_1)}}{1+|\tau|^\sigma} d\tau \leq M \|\boldsymbol{u}_m\|_{L^2(0,T;L^2(\Omega_1)^2)}, \quad \int_{\mathbb{R}} \frac{\|\hat{\boldsymbol{u}}_m(\tau)\|_{\boldsymbol{V}}}{1+|\tau|^\sigma} d\tau \leq M \|\boldsymbol{u}_m\|_{L^2(0,T;\boldsymbol{V})},$$

and

$$\int_{\mathbb{R}} \frac{\|\hat{p}_m(\tau)\|_M}{1+|\tau|^\sigma} d\tau \quad \leq \quad M \|\varphi_m\|_{L^2(0,T;M)},$$

which are bounded by (3.37). Therefore (A.5) gives

$$\int_{\mathbb{R}} \frac{|\tau| \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)}}{1+|\tau|^\sigma} d\tau \leq C,$$

where $C > 0$ is a generic constant independent of m. Observe that for $|\tau| < 1$, $\frac{1}{1+|\tau|^\sigma} > \frac{1}{2}$ and for $|\tau| > 1$, $\frac{|\tau|^{1-\sigma}}{2} \leq \frac{|\tau|}{1+|\tau|^\sigma}$. Then,

$$\begin{aligned}
\int_{\mathbb{R}} \frac{|\tau| \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)}}{1+|\tau|^\sigma} d\tau &= \int_{-1}^{1} \frac{|\tau| \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)}}{1+|\tau|^\sigma} d\tau + \int_{|\tau|>1} \frac{|\tau| \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)}}{1+|\tau|^\sigma} d\tau \\
&\geq \frac{1}{2} \int_{-1}^{1} |\tau| \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} d\tau + \frac{1}{2} \int_{|\tau|>1} |\tau|^{1-\sigma} \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} d\tau.
\end{aligned}$$

Thus,

$$\int_{|\tau|>1} |\tau|^{1-\sigma} \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} d\tau \leq C$$

Consequently,

$$\int_{\mathbb{R}} |\tau|^{1-\sigma} \|\hat{\boldsymbol{u}}_m(\tau)\|^2_{L^2(\Omega_1)} d\tau \leq \|\boldsymbol{u}_m\|^2_{L^2(0,T;L^2(\Omega_1)^2)} + C \leq C.$$

Therefore, $\boldsymbol{u}_m$ is bounded in $H^\gamma(0,T;\boldsymbol{V}, L^2(\Omega_1)^2)$ where $0 < \gamma = \frac{1-\sigma}{2} < \frac{1}{4}$.

Remark 70. *This proof works also for Model II with minor modifications because of the extra right hand side term coming from the nonhomogeneous Neumann boundary condition and as the inertial forces are omitted.*

A.2 Bounds for Discrete Forms

This section contains results from [38]. For the proof of Proposition 36, the following result is necessary:

Lemma 71. *There exists a constant C, independent of h, but dependent on $\sigma_1^{\min}$, such that for all edges $e \in \Gamma_h^1$,*

$$\Big| \sum_{E\in\mathcal{E}_h^1} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{ext}}-\boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{u},\boldsymbol{v})\cap e} - \sum_{E\in\mathcal{E}_h^1} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{ext}}-\boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{v},\boldsymbol{u})\cap e}\Big|$$
$$\leq C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^2(\Delta_e)}\frac{\sigma_e}{|e|}\|[\boldsymbol{u}]\|_{L^2(e)}\|[\boldsymbol{w}]\|_{L^2(e)}, \quad \text{(A.6)}$$

where

$$\partial E_-^*(\boldsymbol{u},\boldsymbol{v}) = \{\boldsymbol{x}\in\partial E \,:\, \{\boldsymbol{u}(\boldsymbol{x})\}\cdot\boldsymbol{n}_E < 0 \text{ and } \{\boldsymbol{v}(\boldsymbol{x})\}\cdot\boldsymbol{n}_E \neq 0\},$$

and Δ_e is the union of elements of $\mathcal{E}_h^1$ adjacent to e.

Proof. Let $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}\in\boldsymbol{X}_h$ and define the set

$$\partial E_-(\boldsymbol{u},-\boldsymbol{v}) = \{\boldsymbol{x}\in\partial E \,:\, \{\boldsymbol{u}(\boldsymbol{x})\}\cdot\boldsymbol{n}_E < 0 \text{ and } \{\boldsymbol{v}(\boldsymbol{x})\}\cdot\boldsymbol{n}_E > 0\}.$$

Consider first an interior edge e in Γ_h^1. The proof is based on the identity (see formula (5.32), Chapter IV, [64]):

$$\sum_{E\in\mathcal{E}_h^1} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{ext}}-\boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{u},\boldsymbol{v})\cap e} - \sum_{E\in\mathcal{E}_h^1} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{ext}}-\boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{v},\boldsymbol{u})\cap e}$$
$$= -\sum_{E\in\mathcal{E}_h^1} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{ext}}-\boldsymbol{u}^{\text{int}}), \boldsymbol{w}^{\text{ext}}-\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u},-\boldsymbol{v})\cap e} =: A,$$

and on the remark that on $\partial E_-(\boldsymbol{u},-\boldsymbol{v})$, we have

$$|\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E| < |\{\boldsymbol{u}-\boldsymbol{v}\}\cdot\boldsymbol{n}_E|\,. \quad \text{(A.7)}$$

Therefore

$$|A| \leq \|\{\boldsymbol{u}-\boldsymbol{v}\} \cdot \boldsymbol{n}_E\|_{L^\infty(e)} \|[\boldsymbol{u}]\|_{L^2(e)} \|[\boldsymbol{w}]\|_{L^2(e)}. \tag{A.8}$$

As $\boldsymbol{u}$ and $\boldsymbol{v}$ belong to a finite-dimensional space in each element E, we easily deduce that

$$\frac{|e|}{\sigma_e} \|\{\boldsymbol{u}-\boldsymbol{v}\}\|_{L^\infty(e)} \leq C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^2(\Delta_e)}\,, \tag{A.9}$$

where Δ_e is the union of all elements of $\mathcal{E}_h^1$ adjacent to e, and C is a constant that depends on $\sigma_1^{\min}$, but not on h. Then (A.6) follows easily from (A.8) and (A.9).

Next, we prove the result for a boundary edge e. In this case, we easily obtain that

$$A = -\sum_{E\in\mathcal{E}_h^1} (\boldsymbol{u}\cdot\boldsymbol{n}_E\boldsymbol{u}, \boldsymbol{w})_{\partial E_-(\boldsymbol{u},-\boldsymbol{v})\cap e} + \sum_{E\in\mathcal{E}_h^1} (\boldsymbol{u}\cdot\boldsymbol{n}_E\boldsymbol{u}, \boldsymbol{w})_{\partial E_-(\boldsymbol{v},-\boldsymbol{u})\cap e}.$$

The proof is concluded as above by noting that (A.7) holds also on $\partial E_-(\boldsymbol{v}, -\boldsymbol{u})$. □

The following proves Proposition 36.

Proof. We first note that for any $\boldsymbol{u} \in \boldsymbol{X}_h$, on any fixed edge e, we have either $\{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \equiv 0$ or $\{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \neq 0$ except possibly on a finite number of points, in which case $\{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \neq 0$ a.e.. Therefore, Γ_h^1 can be partitioned into $\Gamma_h^1 = \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$, with

$$\mathcal{F}_1 = \{e : \{\boldsymbol{u}\}\cdot\boldsymbol{n}_e = 0 \text{ on } e \quad \text{and } \{\boldsymbol{v}\}\cdot\boldsymbol{n}_e \neq 0 \text{ on } e \quad \text{a.e.}\},$$

$$\mathcal{F}_2 = \{e : \{\boldsymbol{v}\}\cdot\boldsymbol{n}_e = 0 \text{ on } e \quad \text{and } \{\boldsymbol{u}\}\cdot\boldsymbol{n}_e \neq 0 \text{ on } e \quad \text{a.e.}\},$$

$$\mathcal{F}_3 = \Gamma_h^1 \setminus (\mathcal{F}_1 \cup \mathcal{F}_2).$$

We then have

$$d_{NS}(\boldsymbol{u},\boldsymbol{u};\boldsymbol{u},\boldsymbol{w}) - d_{NS}(\boldsymbol{v},\boldsymbol{v};\boldsymbol{v},\boldsymbol{w}) = \sum_{i=1}^{3}\sum_{e\in\mathcal{F}_i}\sum_{E\in\mathcal{E}_h^1} (\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}}-\boldsymbol{u}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u})\cap e}$$

$$-\sum_{i=1}^{3}\sum_{e\in\mathcal{F}_i}\sum_{E\in\mathcal{E}_h^1} (\{\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}}-\boldsymbol{v}^{\text{ext}}), \boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e} = \sum_{i=1}^{3} Q_i.$$

We now consider each subset $\mathcal{F}_i$ separately:

$$Q_1 = \sum_{e\in\mathcal{F}_1}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{u}-\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}}-\boldsymbol{v}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}$$
$$\leq C(\|\boldsymbol{v}\|_{\boldsymbol{X}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{L^4(\Omega_1)}\|\boldsymbol{w}\|_{L^4(\Omega_1)},$$

similarly,

$$Q_2 = \sum_{e\in\mathcal{F}_2}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{u}-\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{v}^{\text{int}}-\boldsymbol{v}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}$$
$$\leq C(\|\boldsymbol{u}\|_{\boldsymbol{X}_h}\|\boldsymbol{u}-\boldsymbol{v}\|_{L^4(\Omega_1)}\|\boldsymbol{w}\|_{L^4(\Omega_1)};$$

finally,

$$\begin{aligned}Q_3 = &\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}}-\boldsymbol{u}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{u})\cap e}\\
&-\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}}-\boldsymbol{u}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\\
&+\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{u}-\boldsymbol{v}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}}-\boldsymbol{u}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}\\
&+\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{v}\}\cdot\boldsymbol{n}_E((\boldsymbol{u}^{\text{int}}-\boldsymbol{v}^{\text{int}})-(\boldsymbol{u}^{\text{ext}}-\boldsymbol{v}^{\text{int}})),\boldsymbol{w}^{\text{int}})_{\partial E_-(\boldsymbol{v})\cap e}.\end{aligned} \quad (A.10)$$

The first two terms in the right-hand side of (A.10) are equivalently rewritten as

$$\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}}-\boldsymbol{u}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{u},\boldsymbol{v})\cap e}$$
$$-\sum_{e\in\mathcal{F}_3}\sum_{E\in\mathcal{E}_h^1}(\{\boldsymbol{u}\}\cdot\boldsymbol{n}_E(\boldsymbol{u}^{\text{int}}-\boldsymbol{u}^{\text{ext}}),\boldsymbol{w}^{\text{int}})_{\partial E_-^*(\boldsymbol{v},\boldsymbol{u})\cap e},$$

and in view of Lemma 71, are bounded by:

$$C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^2(\Omega_1)}\sum_{e\in\mathcal{F}_3}\frac{\sigma_e}{|e|}\|[\boldsymbol{u}]\|_{L^2(e)}\|[\boldsymbol{w}]\|_{L^2(e)} \leq C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^2(\Omega_1)}\|\boldsymbol{u}\|_{\boldsymbol{X}_h}\|\boldsymbol{w}\|_{\boldsymbol{X}_h}.$$

The second and third lines in the right-hand side of (A.10) are easily bounded respectively by

$$C\|\boldsymbol{u}-\boldsymbol{v}\|_{L^4(\Omega_1)}\|\boldsymbol{u}\|_{\boldsymbol{X}_h}\|\boldsymbol{w}\|_{L^4(\Omega_1)} \quad \text{and} \quad C\|\boldsymbol{v}\|_{L^4(\Omega_1)}\|\boldsymbol{u}-\boldsymbol{v}\|_{\boldsymbol{X}_h}\|\boldsymbol{w}\|_{L^4(\Omega_1)}.$$

Then (3.79) follows from the above bounds, (3.69) and a Korn's inequality [74]. □

A.3 Bound for $\boldsymbol{u}_m'(0)\cdot\boldsymbol{n}_{12}$ in $(H^{\frac{1}{2}}(\Gamma_{12}))'$

The following is used for the existence result of Model II.

Lemma 72. *For any $\boldsymbol{v}\in\boldsymbol{V}$, there exists $C_L>0$ such that*

$$\|\boldsymbol{v}\cdot\boldsymbol{n}_{12}\|_{(H^{\frac{1}{2}}(\Gamma_{12}))'} \le C_L\|\boldsymbol{v}\|_{L^2(\Omega_1)}.$$

Proof. We use the definition of the dual space norm. Recall that

$$\|\boldsymbol{v}\cdot\boldsymbol{n}_{12}\|_{(H^{\frac{1}{2}}(\Gamma_{12}))'} = \sup_{\varphi\in H^{\frac{1}{2}}(\Gamma_{12})} \frac{\langle \boldsymbol{v}\cdot\boldsymbol{n}_{12},\varphi\rangle_{\Gamma_{12}}}{\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{12})}}.$$

Then let $\mathcal{R}: H_0^{\frac{1}{2}}(\Gamma_{12}) \to H^1(\Omega_1)$ be a continuous extension (lifting) operator such that there exists $C_L>0$ satisfying

$$\|\mathcal{R}(\varphi)\|_{H^1(\Omega_1)} \le C_L\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{12})}.$$

Observe that $\boldsymbol{v}\cdot\boldsymbol{n}_{\partial\Omega_1}=0$ on Γ_1 as $\boldsymbol{v}=0$ on Γ_1. This implies

$$\langle \boldsymbol{v}\cdot\boldsymbol{n}_{12},\varphi\rangle_{\Gamma_{12}} = \langle \boldsymbol{v}\cdot\boldsymbol{n}_{12},\mathcal{R}(\varphi)\rangle_{\Gamma_{12}} = \langle \boldsymbol{v}\cdot\boldsymbol{n}_{\partial\Omega_1},\mathcal{R}(\varphi)\rangle_{\partial\Omega_1}.$$

As $\nabla\cdot\boldsymbol{v}=0$ on Ω_1, using Green's formula,

$$\langle \boldsymbol{v}\cdot\boldsymbol{n}_{\partial\Omega_1},\mathcal{R}(\varphi)\rangle_{\partial\Omega_1} = \int_{\Omega_1}\mathcal{R}(\varphi)\nabla\cdot\boldsymbol{v} + \int_{\Omega_1}\boldsymbol{v}\cdot\nabla\mathcal{R}(\varphi) = \int_{\Omega_1}\boldsymbol{v}\cdot\nabla\mathcal{R}(\varphi)$$

These imply that

$$|\langle \boldsymbol{v} \cdot \boldsymbol{n}_{12}, \varphi \rangle_{\Gamma_{12}}| \leq \|\boldsymbol{v}\|_{L^2(\Omega_1)} |\mathcal{R}(\varphi)|_{H^1(\Omega_1)} \leq C_L \|\boldsymbol{v}\|_{L^2(\Omega_1)} \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_{12})}$$

Therefore,

$$\|\boldsymbol{v} \cdot \boldsymbol{n}_{12}\|_{(H^{1/2}(\Gamma_{12}))'} \leq C_L \|\boldsymbol{v}\|_{L^2(\Omega_1)}$$

□

A.4 Properties of the Linear Interpolation Operator I_0

This section contains properties of the interpolation operators that are used to construct the approximate solution for the weak problems of Chapter 4. The following result can be found in [53]. For completeness, the sketch of the proof is given here.

Lemma 73. *For $z \in L^p(0, T; B)$, let $\overline{z}^N = (\overline{z}_0^N, \ldots, \overline{z}_N^N)$ where $\overline{z}_i^N$ is the average on the interval $[(i-1)\Delta t, i\Delta t]$ defined as in (4.37). Then for all $1 \leq p < \infty$,*

$$\|I_0 \overline{z}^N\|_{L^p(0,T;B)} = \left(\Delta t \sum_{n=1}^{N} \|\overline{z}_n^N\|_B^p \right)^{\frac{1}{2}}, \tag{A.11}$$

and

$$I_0 \overline{z}^N \to z \text{ strongly in } L^p(0, T; B) \text{ as } N \to \infty. \tag{A.12}$$

Furthermore, for all $1 \leq p \leq \infty$,

$$\|I_0 \overline{z}^N\|_{L^p(0,T;B)} \leq \|z\|_{L^p(0,T;B)}. \tag{A.13}$$

Proof. For $1 \leq p < \infty$,

$$\|I_0 \overline{z}^N\|_{L^p(0,T;B)} == \left(\sum_{n=1}^{N} \int_{(n-1)\Delta t}^{n\Delta t} \|\overline{z}_n^N\|_B^p dt \right)^{\frac{1}{p}} = \left(\Delta t \sum_{n=1}^{N} \|\overline{z}_n^N\|_B^p \right)^{\frac{1}{p}}.$$

Then Hölder's inequality and (4.58) imply

$$\|I_0\overline{z}^N\|_{L^p(0,T;B)} = \left(\Delta t \sum_{n=1}^{N} \|\frac{1}{\Delta t}\int_{(n-1)\Delta t}^{n\Delta t} z(t)dt\|_B^p\right)^{\frac{1}{p}}$$
$$\leq \left(\sum_{n=1}^{N}\int_{(n-1)\Delta t}^{n\Delta t} \|z(t)\|_B^p dt\right)^{\frac{1}{p}} = \left(\int_0^T \|z(t)\|_B^p dt\right)^{\frac{1}{p}} = \|z\|_{L^p(0,T;B)}.$$

Also by (4.59) and (4.38),

$$\|I_0\overline{z}^N\|_{L^\infty(0,T;B)} = \max_{n=1,\dots,N} \|\overline{z}_n^N\|_B \leq \frac{1}{\Delta t}\max_{n=1,\dots,N}\int_{(n-1)\Delta t}^{n\Delta t} \|z(t)\|_B dt = \|z\|_{L^\infty(0,T;B)}.$$

Therefore the result holds for $1 \leq p \leq \infty$. We will first show the last result for $z \in \mathcal{C}([0,T];B)$. Then we will conclude by a density argument as $\mathcal{C}(0,T;B)$ is dense in $L^p(0,T;B)$.

Let $\epsilon_0 > 0$. Let χ_n denote the characteristic function on the interval $(n\Delta t,(n+1)\Delta t)$. Then,

$$\|I_0\overline{z}^N - z\|_{L^p(0,T;B)}^p = \int_0^T \|I_0\overline{z}^N(t) - z(t)\|_B^p dt = \int_0^T \chi_n(t)\|z_{n+1} - z(t)\|_B^p dt$$
$$= \int_0^T \chi_n(t)\left\|\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} z(s)ds - z(t)\right\|_B^p dt$$
$$\leq \int_0^T \chi_n(t)\left(\frac{1}{\Delta t}\int_{n\Delta t}^{(n+1)\Delta t} \|z(s)ds - z(t)\|_B\right)^p dt$$

As any continuous function on a compact set is uniformly continuous, z is uniformly continuous on $[0,1]$. So for any $\epsilon_1 > 0$, we can find $\delta > 0$ such that for any $t,s \in [0,1]$, $|t-s| < \delta$ implies $\|z(t) - z(s)\|_B < \epsilon_1$. Let N_0 be such that $\frac{T}{N_0} < \delta$. Then for all $N \geq N_0$, $\Delta t < \delta$. Thus for $0 < \epsilon_1 < T^{-\frac{1}{p}}\epsilon_0$,

$$\|I_0\overline{z}^N - z\|_{L^p(0,T;B)}^p \leq \int_0^T \chi_n(t)\epsilon_1^p dt = \epsilon'^p\Delta t \leq T\epsilon_1^p < \epsilon_0^p \tag{A.14}$$

yielding $I_0\overline{z}^N \to z$ in $L^p(0,T;B)$ for any $z \in \mathcal{C}([0,T];B)$. The result then follows by the density. □

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