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State Cycles, Quasipositive Modification, and Constructing H-Thick Knots in Khovanov Homology

by

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Abstract

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We study Khovanov homology classes which have state cycle representatives, and examine how they interact with Jacobsson homomorphisms and Lee's map Φ. As an application, we describe a general procedure, quasipositive modification, for constructing H-thick knots in rational Khovanov homology. Moreover, we show that specific families of such knots cannot be detected by Khovanov's thickness criteria. We also exhibit a sequence of prime links related by quasipositive modification whose width is increasing.

Acknowledgments

The author would like to thank John Baldwin, Eric Chesebro, and Matt Hedden for helpful conversations, and Tim Cochran for encouraging him to continue with this problem. The author also thanks Scott Morrison for sharing his recent update to JavaKh which can calculate the Khovanov homology of much larger diagrams, written for calculations in a forthcoming paper [FGMW09].

Contents

Chapter 1

Introduction

1.1 Background

My research lies in knot theory, a branch of low-dimensional topology. A knot is the image of an embedding of the circle, S^1, in the 3 dimensional sphere, S^3; a link is the image of an embedding of several copies of S^1 into S^3, viewed up to ambient isotopy. Link complements are intrinsically related to the study of 3-dimensional manifolds: a theorem of Lickorish and Wallace states that every closed, connected orientable 3-manifold may be obtained by removing a neighborhood of a link and replacing it in a twisted manner.

Instead of working directly from such a 3-dimensional embedding, we will take the common tactic of working from a diagram of a knot. A diagram is just a planar projection, with broken arcs indicating crossings. See Figure 1.1 for an example.

In knot theory, one typically associates algebraic objects, such as numbers, polynomials, and groups, to the knot which remain unchanged by ambient isotopies. One

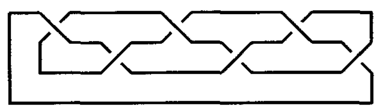

Figure 1.1: A diagram for the knot 6_3.

of the more famous polynomial invariants associated to knots is the Jones polynomial. The Jones polynomial, $V(K)$, is a Laurent polynomial discovered in 1983 by Vaugh Jones, which has been remarkably successful as a knot invariant. For example, it distinguishes a knot from its mirror image, and fully classifies knots of 9 crossings or less. However, it is not a complete knot invariant: for example, $V(5_1) = V(10_{132})$.

My dissertation deals primarily with Khovanov homology, a generalization of the Jones polynomial. Khovanov homology is a collection of abelian groups, indexed by two integer gradings i, j, whose graded Euler characteristic is a normalization of the Jones polynomial. As a knot invariant, it is stronger than the Jones polynomial: for example, the Khovanov homology of 5_1 and 10_{132} differs, even though the two knots have the same Jones polynomial. The underlying chain complex and differential used to construct the homology groups of Khovanov homology are defined in a combinatorial way, based on a diagram of the knot. When working in rational Khovanov homology, one can just view these bigraded groups as rational vector spaces, and represent the homology as a plot of the ranks of each vector space against the bigradings, as shown in Table 1.1.

Khovanov homology has been described as a categorification of the Jones polynomial: a chain complex, where the ranks of vector spaces generalize the polynomial coefficients of the Jones polynomial. This procedure has since been applied to many other polynomials both in topology and in combinatorics, from the Alexander poly-

	-6	-5	-4	-3	-2	-1	0	1	2	3
7										1
5									2	
3								3	1	
1							5	2		
-1						4	4			
-3					4	4				
-5				3	4					
-7			2	4						
-9		1	3							
-11		2								
-13	1									

Table 1.1: Rational Khovanov homology of 9_{41}. The numbers represent the ranks of the appropriate vector spaces. The nontrivial ranks tend to lie on a certain number of diagonals of the form $2i - j = \delta$; here there are two diagonals which have been shaded.

nomial of knot theory to the chromatic polynomial of graph theory.

While Khovanov homology distinguishes many pairs which the Jones polynomial could not, Khovanov homology is also not a complete knot invariant: Liam Watson has shown in [Wat07] that there are infinite families of distinct knots with the same Khovanov homology. It remains an open question for both the Jones polynomial and Khovanov homology as to whether either "distinguishes" the unknot: in other words, is there any nontrivial knot whose respective invariant matches that of the unknot. Partial progress towards this problem on the Khovanov side has been given by [GW08, Hed08, HW08].

1.2 Thickness in Khovanov Homology

After Bar-Natan circulated his initial calculations for Khovanov homology, a striking pattern emerged: when plotting the ranks of the homology according to the bigrading, most small crossing number knots had every nontrivial homology group lying on 2 adjacent "diagonals" of slope 2, with y-intercept of the two diagonals being the signature of the knot ± 1. Knots for which the Khovanov homology lies on only 2 such diagonals are called H-thin, while those with 3 or more diagonals are called H-thick. The number of these diagonals on which the homology is supported is called the homological width. An example of the homology of an H-thick knot with 3 diagonals is given in Table 1.2.

	-4	-3	-2	-1	0	1	2
7							1
5							
3					1	1	
1				1	1		
-1				1	1		
-3		1	1				
-5							
-7	1						

Table 1.2: Rational Khovanov homology of 9_{42}, an H-thick knot. The three diagonals are shaded in different colors.

Recent work of Ozsvath and Manolescu [MO08] has grouped all but one of these small H-thin knots into a single family, called quasi-alternating knots, for which both the Khovanov and Knot Floer homologies are σ thin, in the sense that the knots are H-thin with intercepts related to the signature. For these knots, the reduced Khovanov homology is determined entirely by the Jones polynomial and the signature. From this perspective, the H-thick knots are the knots with "interesting" Khovanov homology.

Most of the general theorems guaranteeing thickness are laid out in [Kho03]. Other authors have come up with upper bounds for the homological width of a knot. But, to actually show a knot is H-thick, one must either do an explicit calculation, or turn to the theorems of Khovanov from [Kho03], summarized below.

Theorem 1.1. *(Khovanov) $K_1 \# K_2$ is H-thick if and only if at least one of K_1, K_2 are H-thick.*

Theorem 1.2. *(Khovanov) The (usual) Jones polynomial of an H-thin knot is alternating.*

Theorem 1.3. *(Khovanov) Adequate non-alternating knots are H-thick.*

Some other knots for which we know the thickness come from explicit calculations of the Khovanov homology for those families. These include some torus knots [Sto09] [Tur08] and pretzel knots [Suz06]. Concurrently with our work, Adam Lowrance [Low09] has produced infinite families of knots of fixed width, by analyzing the long exact sequence for knots gotten by replacing "width-preserving" crossings with rational tangles.

1.3 Results

In this paper, we will demonstrate a new method for constructing H-thick knots, which can produce examples not detected by Khovanov's thickness criterion: in particular, we will construct several infinite families of H-thick hyperbolic knots (hence prime, and not torus knots) that are not adequate and have an alternating Jones polynomial. Instead of fully calculating the Khovanov homology, we will show that certain special homology classes, which have *state cycle* representatives, persist under an operation we call *quasipositive modification.* In brief, a state cycle is when a single generator in the Khovanov chain complex, represented by an enhanced state, is a cycle; quasipositive modification is the process of "gluing" in a quasipositive braid in a way that is compatible with state cycles one wishes to persist. See Figure 1.2 for a schematic of this procedure.

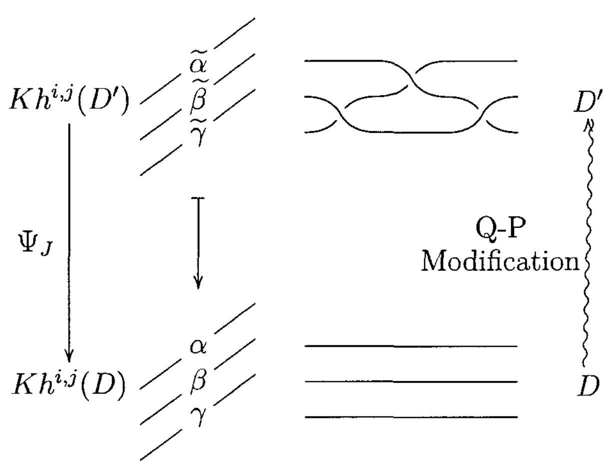

Figure 1.2: A schematic for constructing thick knots via quasipositive modification.

The first main result is a prescription for what a nontrivial state cycle has to look like. A rough statement is given by:

Theorem 4.8. *For a state cycle to represent a nontrivial homology class, the underlying state must be "almost" adequate, and "most" loops of the 1-block must be* *marked by v_+, while all 0-tracing loops must be marked by v_-.*

A precise statement is given in Chapter 4 where the theorem appears. Our next result is that these state cycles persist under the operation of quasipositive modification:

Theorem 6.4. *Let D be an oriented diagram and α a nontrivial state cycle. Suppose D' is gotten from D by quasipositive modification compatible with α, and that Ψ is the associated Jacobsson homomorphism from $Kh(D')$ to $Kh(D)$. Then there exists a state cycle $\widetilde{\alpha}$ so that $\Psi(\widetilde{\alpha}) = \pm\alpha$. If B is the quasipositive braid associated to this modification, then $\widetilde{\alpha}$ is the enhanced traced state where:*

- *All crossings from D are smoothed as in α.*
- *Negative crossings from B are 1-smoothed, positive crossings from B are 0-smoothed.*
- *Every loop in $\widetilde{\alpha}$ is marked the same as α.*

Furthermore, if multiple such state cycles are compatible with a quasipositive modification, the "lifts" of these state cycles retain the same relative grading difference; this acts as the main workhorse for constructing H-thick knots.

Quasipositive modification is also compatible with Lee's homomorphism, in the following sense:

Theorem 6.6. *Let D be an oriented diagram, and α a nontrivial state cycle of D. Suppose D' is a diagram gotten by quasipositive modification on D compatible with α, and that $\widetilde{\alpha}$ is the lift of α. Then $\Psi_J(\Phi_{Lee}(\widetilde{\alpha})) = \Phi_{Lee}(\Psi_J(\widetilde{\alpha})) = \Phi_{Lee}(\alpha)$.*

This last result is useful in reducing the number of state cycles needed to construct H-thick knots: one can find two nontrivial state cycles in distinct diagonals, and then examine Lee's spectral sequence to see that one of them is paired with a third nonzero diagonal by the Lee homomorphism. This pairing is then preserved under quasipositive modification, so that at least three distinct diagonals will be present in the modified knot.

1.4 Layout

In Chapter 2, we go over the basics of Khovanov homology. Chapter 3 introduces the notion of a state cycle, the special case when a single generator is a cycle in the Khovanov chain complex. Chapter 4 gives the classification of state cycles which represent nontrivial homology classes, and Chapter 5 examines how state cycles interact with various maps and the Lee spectral sequence. Chapter 6 carefully defines quasipositive modification and discusses how it interacts with state cycles and related cycles. Chapter 7 lays out our examples of families of H-thick knots, and details other base knots where this construction works to construct H-thick knots. Included are two infinite families of H-thick knots which cannot be detected by the Khovanov thickness criterion, and a sequence of prime links related by quasipositive modification which have increasing width.

Chapter 2

Short Review of Khovanov Homology

Experts in the field should feel free to skip this section, with the understanding that I follow Bar-Natan's [BN02] convention for labelling generators, and work with $\mathbb{Q}$ coefficients. What follows is a review of the cube of resolutions construction of Khovanov homology.

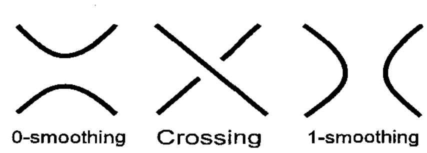

Figure 2.1: Standard smoothing convention for a crossing

Given a diagram D for an oriented link L, one can construct a state for that diagram by replacing every crossing with a choice of smoothing, per the convention in Figure 2.1. The result will be a collection of planar loops, as shown in Figure 2.2.

By choosing an ordering for the crossings, and keeping track of the choice of 0 or

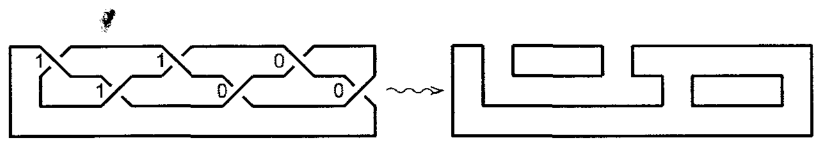

Figure 2.2: On the left, a choice of smoothings has been assigned to every crossing. On the right, each crossing has been replaced by its chosen smoothing, giving a state for the diagram.

1-smoothing for each crossing, one gets a label for the state: a string of 0's and 1's encoding the smoothing choices. This label can also be viewed as a binary coordinate system for a hypercube whose vertices are all the possible states for the diagrams, arranged according to this coordinate system of the labels. This hypercube is what is called the cube of resolutions for the diagram D.

The chain complex for Khovanov homology is constructed by associating a graded vector space to each state, and then organizing them in some fashion that respects the structure of this cube of resolutions. The construction is modelled on a categorification of the Kauffman state-sum formula for the Jones polynomial, with a slightly different normalization.

2.1 Chain Groups

Specifically, let V be the graded vector space over $\mathbb{Q}$ with basis $< v_+, v_- >$ of grading +1, -1 respectively. Given a state σ, let C_σ be the graded vector space $V^{\otimes k}$, where k is the number of loops in σ. Let $h(\sigma)$, the height of the state, be the number of 1-smoothings in the label for σ. Then, the r^{th} unnormalized chain group of the diagram is defined by:

$$C^r(D) = \bigoplus_{h(\sigma)=r} C_\sigma \tag{2.1}$$

This breakdown of generators by state lends itself to viewing generators of the chain groups as enhanced states: states with a basis element of V marked on each loop.

2.2 Differentials

The differential is also defined recursively at the level of these state groups, and further broken down into edge differentials, corresponding to the edges of the cube of resolutions. An edge between two states exists if the labels for the two states differ in only a single place, where a 0 changes to a 1. The associated edge differential is viewed as going from the state with the the 0 label to the state with the 1 label, and corresponds to the cobordism of the change from the 0 smoothing to the 1 smoothing. This cobordism either merges two circles from the initial state, or pinches off a second circle from a circle in the initial state, and the edge differential is defined in accordance with this dichotomy, as shown in Table 2.1. One labels an edge by taking the label of its originating state and substituting a * for the 0 which changes to 1 in the target of the edge.

These edge differentials are then bundled into a state differential, with a sprinkling of negative signs added in such a way that squares in the cube of resolutions anti-commute. Namely, let $|e|$ be the number of 1's which occur in the edge's label before the *, and choose $(-1)^{|e|}$ to be the sign for edge differential d_e. We then combine all

$\bigcirc\ \bigcirc \xrightarrow{\mu} \bigcirc$	$\bigcirc \xrightarrow{\Delta} \bigcirc\ \bigcirc$
$v_+ \otimes v_+ \longmapsto v_+$	$v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+$
$v_+ \otimes v_- \longmapsto v_-$	$v_- \longmapsto v_- \otimes v_-$
$v_- \otimes v_+ \longmapsto v_-$	
$v_- \otimes v_- \longmapsto 0$	

Table 2.1: The edge differential, d_e takes one of the above forms depending on whether the associated cobordism takes two circles to one, or vice versa. Outside of the changed part of the state, the edge differential acts as the identity.

of the state differentials into another direct sum, to obtain the full chain differential:

$$d_\sigma : C_\sigma \longrightarrow C^{r+1} \qquad d^r : C^r \longrightarrow C^{r+1} \tag{2.2}$$

$$d_\sigma = \bigoplus_{\text{edges } e} (-1)^{|e|} d_e \qquad d^r = \bigoplus_{h(\sigma)=r} d_\sigma$$

For our purposes, the most important thing to remember is that edge differentials leave a state's chain group for every 0 in the state's label, and enter the state's chain group for every 1 in the state's label. Much of the analysis of whether things are nontrivial cycles will revolve around careful analysis of the edge differentials from this perspective.

2.3 Gradings

To get the bigrading of $Kh(L)$, there are some index shifts from the writhe and the cube construction that need to be addressed. Since we will be dealing directly with chain generators, it will be sufficient to describe the bigrading of a chain generator.

So, let α_σ be a chain generator based on state σ. Let n_+ and n_- be the number of positive and negative crossings of L respectively, following the usual righthanded sign convention. Let $v_+(\alpha)$ and $v_-(\alpha)$ denote the number of v_+ and v_- elements in the tensor for α. Then, the bigrading (t, q), representing the *homological grading* and *quantum grading* respectively, is given by:

$$t(\alpha) = h(\sigma) - n_- \qquad \text{(homological grading)}$$

$$q(\alpha) = v_+(\alpha) - v_-(\alpha) + h(\sigma) + n_+ - 2n_- \qquad \text{(quantum grading)}$$

All cycles constructed later will be in terms of chain generators, so this gives a concise way to calculate their bigrading.

2.4 Diagonals

Finally, let's review information about the diagonals of Khovanov homology. A common way to present the Khovanov homology of a link is in a table of the following form, where the ranks of the Khovanov homology at bigrading (i, j) are plotted so that homological grading goes horizontally, and quantum grading vertically. Note that the quantum gradings shown are either all even or all odd, because the Khovanov homology of a link of n components is supported only on quantum gradings $j \equiv n \pmod 2$. An example is given in Table 2.2.

A pattern for the support of the Khovanov homology is that it always lies on a certain number of slope 2 diagonals (this has to do with Khovanov's Krull-Schmidt

	-6	-5	-4	-3	-2	-1	0	1	2	3
7										1
5									2	
3								3	1	
1							5	2		
-1						4	4			
-3					4	4				
-5				3	4					
-7			2	4						
-9		1	3							
-11		2								
-13	1									

Table 2.2: Rational Khovanov homology of 9_{41}, an H-thin knot. Its two diagonals are shaded.

decomposition of the chain complex in [Kho03]). A diagonal of grading δ comprises all $Kh^{i,j}(L)$ of the form $\delta = 2i - j$. When a link's homology is supported on only 2 such diagonals, we say the link is *H-thin*. An example is shown in Table 2.2.

	-4	-3	-2	-1	0	1	2
7							1
5							
3					1	1	
1				1	1		
-1				1	1		
-3		1	1				
-5							
-7	1						

Table 2.3: Rational Khovanov homology of 9_{42}, an H-thick knot. The three diagonals are shaded in different colors.

When a link's Khovanov homology is supported on 3 or more diagonals, we say the link is *H-thick*. An example is seen in Table 2.3. We say that the *width* of a link's Khovanov homology is the number of diagonals on which its Khovanov homology is supported.

Chapter 3

State Cycles

Enhanced states comprise the generators of the Khovanov chain complex. But sometimes, a single enhanced state turns out to be a cycle representative. These special cycles are especially convenient to manipulate, and serve as the foundation for the cycles we will construct explicitly to generate our H-thick families.

Definition 3.1. We say that an enhanced state α is a *state cycle* if the associated element of the chain complex is a cycle; namely, $d(\alpha) = 0$. We say that a state cycle is *nontrivial* if it represents a nontrivial homology class.

By the definition of the Khovanov differential in terms of edge differentials leaving a state's chain group, it is a much simpler task to determine if a single enhanced state α_σ is a cycle than it is to show that it is nontrivial. Indeed, a quick look over the edge differentials in Table 2.1 reveals that for this to happen, every edge exiting σ must fall in the μ case, with the respective loops marked by v_- in α_σ. This forces every edge differential leaving α_σ to vanish; since α_σ lies in $ker(d)$, it is a cycle in $Kh(L)$.

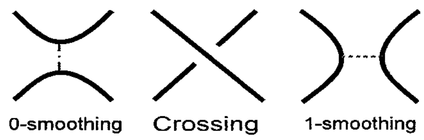

Figure 3.1: Smoothings marked with traces of the crossings. 0-smoothings will be red, dot-and-dashed lines; 1-smoothings will be blue, dotted lines.

To better analyze this situation, it is helpful to record not just the smoothings of σ, but also the traces of the crossings, as shown in Figure 3.1. The *trace* of a crossing is a shadow to show where a crossing has been smoothed to get the state σ, and represents where an edge differential either enters or exits a chain generator based on σ. See Figure 3.2 for a schematic of this relation. A state with all its tracings marked is said to be a *traced state*; an enhanced state with all its tracing marks is said to be an *enhanced trace state*, or *ET state*. See Figure 3.3 for an example of an ET state.

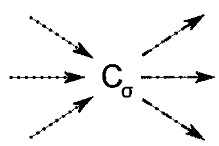

Figure 3.2: Schematic for how traces correspond to edge differentials. Red dot-and-dash edge differentials exit the chain group, and come from the red dot-and-dash 0-traces; blue dotted edge differentials enter the chain group, and come from the blue dotted 1-traces.

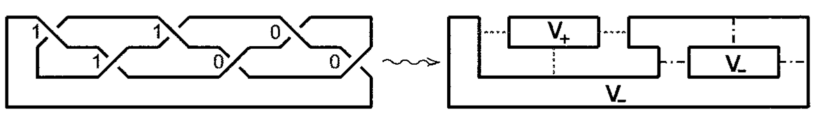

Figure 3.3: On the left, a choice of smoothings has been assigned to every crossing. On the right, an ET state corresponding to $v_+ \otimes v_- \otimes v_-$ is shown.

Let us introduce some further terminology to discuss these traces of crossings. We say that a trace is a *mergetrace* or *pinchtrace* if the trace connects two or one loop in σ, respectively. See Figure 3.4 for some examples. The terminology is meant to suggest that when the crossing associated to a mergetrace is changed to the opposite smoothing, the two loops joined by the mergetrace are merged together; similarly, when the crossing associated to a pinchtrace is changed to the opposite smoothing, the original loop is pinched into a pair of loops.

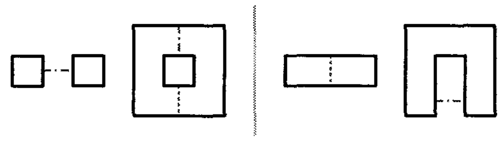

Figure 3.4: The left half shows examples of mergetraces; the right half shows examples of pinchtraces.

We can further differentiate traces by keeping track of which kind of smoothing they are associated to in σ. If a mergetrace is associated to a crossing that has been 0-smoothed in σ, we say it is a *0-mergetrace*, and so on. Returning to the discussion of state cycles, the condition now becomes that σ must be a state such that every 0-trace is a mergetrace, and that every loop touched by a 0-mergetrace must be marked by v_- in α_σ.

Now, let us introduce some terminology describing a state in terms of its traces. A state is said to be *0-merging* if every 0-trace in the state is a mergetrace. Similarly, a state is said to be *1-merging* if every 1-trace is a mergetrace, and so on for *0-pinching* and *1-pinching.* This leads to the definition of an *adequate state* as a state that is both 0-merging and 1-merging (compare to Ozawa's definition [Oza06]).

Unfortunately, the term "adequate" has been overloaded in the literature. Building off the above definition, a diagram is said to be *adequate* if its all-0 state and its all-1 state are both adequate as states, *+ adequate* if its all-0 state is adequate as a state, and *- adequate* if its all-1 state is adequate as a state. Similarly, a link is said to be any of the above if it admits a diagram with that property. This was the context in which Ozawa introduced the notion of an adequate state in [Oza06], where he studied properties of the surfaces associated to such states in the knot complement.

As a final bit of terminology useful in discussing ET states, we say that a loop in σ is *0-tracing* if it is touched by a 0-trace in the traced state σ. The *1-block* of σ is the set of loops which are 1-tracing, but not 0-tracing. It follows from our earlier discussion that for state cycles, the ET state α_σ must be 0-merging and mark every 0-tracing loop with v_-, but there is no condition on loops in the 1-block. In summary, we have:

Proposition 3.2. *Let α_σ be an ET state. Then α_σ is a state cycle if and only if σ is 0-merging and every 0-tracing loop of α_σ is marked by v_-.*

Proof. Suppose α_σ is a state cycle. Since it is only a single generator in the chain complex, the only way it can lie in the kernel of the differential is if every outgoing edge differential is zero. Looking over Table 2.1, one sees that the only way an individual edge differential can be zero is when d_e corresponds to multiplication and acts on two loops marked by v_-. Since we have an outgoing edge differential for every 0-trace in the state, this means that every 0-trace must be a mergetrace and every loop touched by a 0-trace must be marked by v_-, as claimed.

For the opposite direction, the setup guarantees every outgoing edge differential will be zero, by the above discussion. As the total differential is a sum of these edge differentials, α_σ lies in $ker(d)$ and is a state cycle. □

Example 3.1. Suppose D is a + adequate diagram. Let σ_0 denote the all-0 state of D. By definition, σ_0 is an adequate state, and in particular 0-merging. Let α_0 be the ET state of σ_0 where every loop is marked by v_-, as shown in Figure 3.5. Proposition 3.2 tells us that α_0 is a state cycle, but because σ_0 is at the very bottom of the cube of resolutions for D, there are no differentials entering σ_0. Therefore α_0 actually represents a nontrivial homology class, one of minimal homological and quantum grading since it is at the bottom of the cube of resolutions.

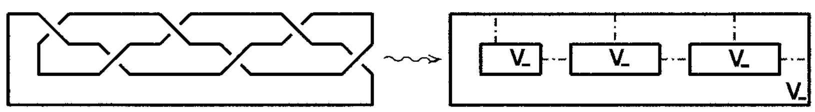

Figure 3.5: On the left is a + adequate diagram of 6_3. On the right, the all-0 ET state α_0 is shown.

Example 3.2. Suppose D is a - adequate diagram. Let σ_1 denote the all-1 state of D. Again, σ_1 is an adequate state, and so is both 0- and 1-merging. Let α_1 be the ET state of σ_1 where every loop is marked by v_+, as shown in Figure 3.6. The set of 0-tracing loops is empty because this is the all-1 state, so Proposition 3.2 trivially holds and α_1 is a state cycle.

However, because σ_1 lies at the top of the cube of resolutions, C_{σ_1} is the only chain group of its height h in the Khovanov chain complex for D. This means that every edge differential of height $h-1$ targets C_{σ_1}, allowing us to restrict to a single edge differential for each chain group of height $h-1$. Because every 1-trace of σ_1 is a mergetrace, every incoming edge differential must lie in the Δ half of Table 2.1. By inspection, no edge differential contains a term marking v_+ on every loop of an ET state for σ_1. So, no linear combination of terms can have boundary equal to α_1.

Therefore, α_1 does not lie in $im(d)$ for the Khovanov differential, and hence must represent a nontrivial homology class of $Kh(L)$.

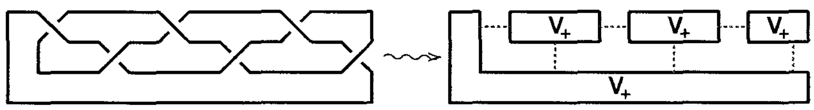

Figure 3.6: On the left is a - adequate diagram of 6_3. On the right, the all-1 ET state α_1 is shown.

Example 3.3. Let D be a braid diagram, and σ be the oriented resolution, the state where every positive crossing has been 0-smoothed, and every negative crossing has been 1-smoothed. This smoothing choice is such that the loops of the resulting state are just the strands of the braid, a concentric set of circles, with each trace going between two strands. So, each trace is a mergetrace, and the state is adequate. Let ψ be the ET state where every loop of σ has been marked by v_-, as shown in Figure 3.7. Proposition 3.2 tells us that ψ is a cycle in Khovanov homology, but more can be said: Plamenevskaya has shown [Pla06] that $[\psi]$ is a transverse knot invariant, the Plamenevskaya class for $Kh(L)$.

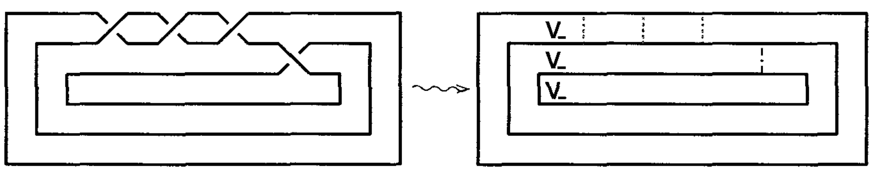

Figure 3.7: On the left is a braid diagram of the negative trefoil. On the right, the ET state for the oriented resolution is shown, a representative for the Plamenevskaya class.

Examples 3.1 and 3.2 are of special interest because the respective state cycles represent nontrivial homology classes. They were a critical part of Khovanov's proof that adequate, nonalternating knots are H-thick [Kho03], and will turn out to be good models for what a general state cycle which represents a nontrivial homology class must look like.

Note that Examples 3.2 and 3.3 choose different values for their respective 1-blocks: v_+ is marked on the 1-block of 3.2 while v_- is marked on the 1-block of the Plamenevskaya class. Furthermore, for all three examples, the underlying state is not simply 0-merging, but actually an adequate state. In fact, every state cycle used in constructing the H-thick families will come from an adequate state.

Chapter 4

Classifying Nontrivial State Cycles

In this section, we will work towards a classification of nontrivial state cycles. The tools are a series of limits on how the 1-block can be assigned, and at the end of the section we will have a strong necessary condition for a state cycle to represent a nontrivial homology class.

The first step towards the classification is to place a restriction on the 1-traces of a nontrivial state cycle:

Proposition 4.1. *Let σ be a state in a diagram D, and α_σ an associated ET state which represents a cycle in Khovanov homology. Suppose σ has a 1-pinchtrace, and let L_σ be the loop this 1-pinchtrace touches. If L_σ lies in the 1-block of σ, then $[\alpha_\sigma]$ is trivial.*

Proof. Let b be the crossing associated to the 1-pinchtrace. Let σ_b be the state obtained by replacing this 1-smoothing by a 0-smoothing, and otherwise matching the smoothings of σ. Let L_1 and L_2 be the two loops L_σ was split into in σ_b. Note that since L_σ lies in the 1-block, it is only touched by 1-traces in σ: this means that splitting it into two loops for σ_b will not affect any of the 0-traces coming from σ, so that they remain mergetraces.

Case 1: Suppose L_σ is marked by v_+. Let β be the ET state associated to σ_b where L_1 and L_2 are marked by v_+, and all other loops match their marking in α_σ. Every 0-trace except that coming from b will be a mergetrace between two loops marked by v_- since α_σ is a state cycle. So, all of the edge differentials applied to β will be 0 except for d_b, the one coming from b. Examining Table 2.1, one sees that d_b takes the $v_+ \otimes v_+$ on L_1 and L_2 to the v_+ on L_σ, and otherwise acts as the identity. In other words, $d_b(\beta) = \alpha_\sigma$, and since all other edge differentials were zero, $d(\beta) = \pm\alpha_\sigma$. It follows that $[\alpha_\sigma]$ is trivial.

Case 2: Suppose L_σ is marked by v_-. Let β be the ET state associated to σ_b where L_1 is marked by v_+, L_2 is marked by v_-, and every other loop matches the marking of α_σ. From the same argument as above, the only nonzero edge differential is d_b; examining Table 2.1, we see that d_b again takes β precisely to α_σ. So, $d(\beta) = \pm\alpha_\sigma$, and hence $[\alpha_\sigma]$ is trivial. $\square$

Remark 4.1.1. In fact, one can loosen this condition to having one of L_1 or L_2 "lie" in the 1-block, in the sense that the only 0-trace touching it is the one associated to b. This only matters for Case 2, where L_σ was marked by v_-: in such a situation, one marks the loop which is only 0-traced by b with v_+, and the other by v_-, and the same proof holds. When both such loops are 0-traced by traces other than b, one has potentially many different nonzero edge differentials, and *a priori* one is not guaranteed a method of cancelling these unwanted factors.

So, every trace touching a loop in the 1-block must be a 1-mergetrace for a state cycle to represent a nontrivial homology class. The next proposition gives some restrictions on how the loops of the 1-block can be marked:

Proposition 4.2. *Let σ be a state in a diagram D, and α_σ an associated ET state which represents a cycle in Khovanov homology. Suppose that there is some pair of loops in σ which are only connected by 1-traces. If those two loops are marked by v_- in α_σ then $[\alpha_\sigma]$ is trivial.*

Proof. Let b denote one of the 1-traces between the two loops in question, and σ_b the state gotten by changing the smoothing for b to a 0-smoothing, and otherwise retaining the smoothing choices of σ. Since b was a 1-mergetrace in σ, b becomes a 0-pinchtrace in σ_b; all other 0-smoothings in σ_b remain 0-mergetraces, since there were no 0-traces between the two loops that have been merged by changing b.

Let β be the ET state for σ_b gotten by marking the newly merged loop by v_-, and all other loops by their markings on α_σ. Since α_σ is a state cycle, every 0-tracing loop is a mergetrace between two loops marked by v_-. This same situation holds for β for every 0-trace except b, since no 0-tracing loop of σ was altered by changing b. Let the edge differential associated to b be denoted d_b; then every edge differential on β except d_b will remain 0, as it each comes from a mergetrace between two loops marked by v_-. In contrast, the edge differential d_b associated to this 0-trace is nonzero: examining Table 2.1 one sees that d_b takes the v_- on the merged loop to $v_- \otimes v_-$. So $d_b(\beta)$ is α_σ, since edge differentials act as the identity outside of the changed part of the state. Because every other edge differential was zero, $d(\beta) = \pm\alpha_\sigma$. Therefore $[\alpha_\sigma]$ is trivial. □

Remark 4.2.1. This is a generalization of Plamenevskaya's triviality condition for her transverse invariant, but the method of proof is different. Compare to Proposition 3 of [Pla06].

Even a single 0-mergetrace between the loops of Proposition 4.2 has the chance to make $[\alpha_\sigma]$ nontrivial, though, depending on how the rest of the cube of resolutions is structured. Each extra 0-mergetrace gives a new 0-pinchtrace, and another nonzero edge differential to complicate the situation.

Adjacency of a pair of v_- loops is not the only problem with v_- loops in the 1-block: in fact, if there is a pair of v_- loops that can be joined by a path of 1-traces in the 1-block, then the ET state is trivial:

Proposition 4.3. *Let σ be a state in a diagram D, and α_σ an associated ET state which represents a cycle in Khovanov homology. Suppose that in α_σ there are two loops L_α and L_γ marked by v_-, and that there exists a path of 1-traces $\{b_1, b_2, \dots, b_n\}$ between these loops, so that every loop in this path (including L_α, L_γ) lies in the 1-block. Then $[\alpha_\sigma]$ is trivial.*

Proof. Essentially, we will construct a telescoping series of boundaries, finishing with the kind of boundary constructed in Proposition 4.2. We will assume that the path chosen is minimal, in the sense that every loop between L_α and L_γ in the path is marked by v_+ in α_σ. Furthermore, Proposition 4.1 lets us assume every 1-trace in this path is a mergetrace, as otherwise we already know that $[\alpha_\sigma]$ is trivial. So, let's establish some notation.

Let σ_i be the state where crossing b_i has been 0-smoothed, and all other smoothings match those of σ. Let d_i be the edge differential from σ_i associated to the trace b_i. Let L_1 be L_α, and L_i the loop besides L_{i-1} which trace b_{i-1} touches: L_γ then becomes L_{n+1}. Furthermore, note that in σ_i, the loops L_i and L_{i+1} are merged, while every other loop remains the same. Denote this merged loop in σ_i by M_i.

Let β_i be the ET state associated to σ_i where the loops L_1 and M_i are marked by v_+ and all other loops match the marking of α_σ. Let δ_i be the ET state associated to σ where L_1 and L_i are marked by v_+, L_{i+1} is marked by v_-, and all other loops are marked as in α_σ. See Figure 4.1 for an illustration.

Because the only traces between each pair L_i and L_{i+1} are 1-traces by assumption, the only new edge differential out of β_i will be d_i. Furthermore, since L_i, L_{i+1} lie in the 1-block, d_i is the only edge differential out of β_i which involves the merged loop M_i. This means that all the other edge differentials remain zero: none of the 0-tracing loops from α_σ are marked differently, and none of the 0-traces from α_σ have been changed in β_i, so that each edge differential other than d_i comes from a mergetrace between two loops marked by v_-.

Consulting Table 2.1, we see that d_i acting on β_i will take the v_+ on loop M_i to a pair of ET states, where $v_- \otimes v_+ + v_+ \otimes v_-$ suggests the marking on loops L_i and L_{i+1}, and otherwise the markings match α_σ. In particular, writing only the tensor

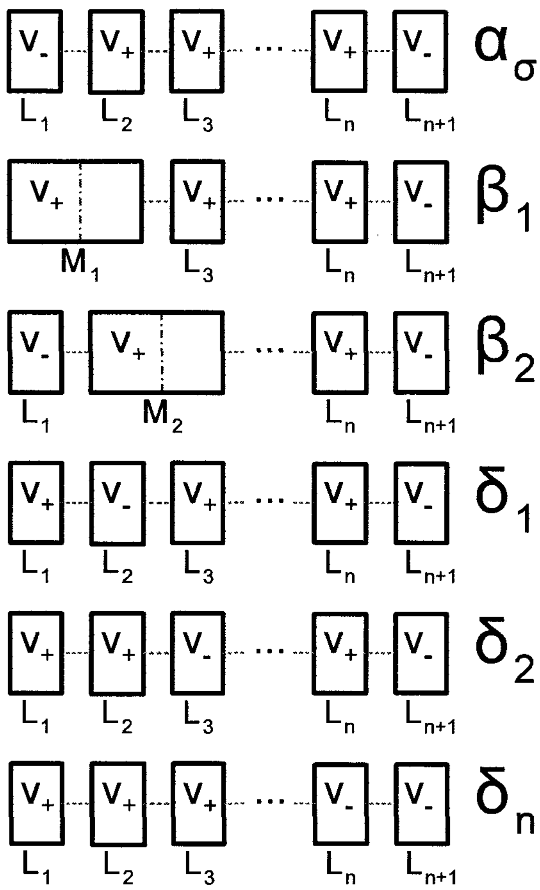

Figure 4.1: ET states showing relevant loops for α_σ, and the β_i and δ_i.

elements for the loops L_i, and marking the cobordant loops in parentheses:

$$d_1(\beta_1) = (v_- \otimes v_+) \otimes \cdots \otimes v_- + (v_+ \otimes v_-) \otimes \cdots \otimes v_- \tag{4.1}$$

$$= \alpha_\sigma + \delta_1 \tag{4.2}$$

$$d_2(\beta_2) = v_+ \otimes (v_- \otimes v_+) \otimes \cdots \otimes v_- + v_+ \otimes (v_+ \otimes v_-) \otimes \cdots \otimes v_- \tag{4.3}$$

$$= \delta_1 + \delta_2 \tag{4.4}$$

$$d_i(\beta_i) = \delta_{i-1} + \delta_i \tag{4.5}$$

$$\sum_{i=1}^{n}(-1)^{i+1} d_i(\beta_i) = \alpha_\sigma + (-1)^{n+1}\delta_n \tag{4.6}$$

Now, note that in δ_n we have an ET state where there are loops L_n, L_{n+1} marked with v_- and joined only with 1-traces. The proof of Proposition 4.2 gives us an explicit boundary for δ_n: let γ be the ET state associated to σ_n where M_n is marked by v_-, L_1 is marked by v_+, and other loops are marked as in α_σ. Then $d(\gamma) = \pm\delta_n$. Adding this to our sum above gives an explicit boundary for α_σ, modulo some sign changes which come from the signs of the edge differentials when passing from d_i to d. Therefore, $[\alpha_\sigma]$ is trivial. □

Remark 4.3.1. The restriction that each loop in this path must lie in the 1-block is needed to guarantee triviality. Even a single 0-trace touching one of the loops will give a second potential nonzero edge differential for this construction that can obstruct triviality, as the following example shows:

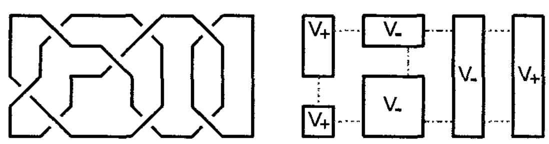

Figure 4.2: On the left is a diagram for a link, and on the right is a particular enhanced state giving a state cycle. Direct calculation shows that this state cycle represents a nontrivial homology class.

Example 4.1. Consider the link diagram and enhanced state for that diagram shown in Figure 4.2. The two leftmost loops marked by v_- can be joined by a path of loops joined by 1-traces, but direct calculation shows that the associated homology class is nontrivial. So, the extra 0-traces coming off the starting and ending loops can sometimes obstruct triviality.

Using these propositions, we can give a fairly restricted picture of what enhanced states can represent nontrivial homology classes. Roughly speaking, the underlying state must be "almost" adequate, and "most" loops of the 1-block must be marked by v_+, while all 0-tracing loops must be marked by v_-. To be precise about what "most" loops means, we will need to introduce a number of definitions related to a graph theoretic interpretation of the traced states.

Definition 4.4. Given a state σ, the associated *state graph* Γ_σ is constructed by taking the loops of σ to be vertices, and the traces to be edges.

Remark 4.4.1. In general this is really a pseudograph, because an edge may start and end at the same vertex if the state is not adequate, and two vertices may be joined by multiple edges. When the state is adequate, the state graph is an honest multigraph.

The state graph, and subgraphs of the state graph, turns out to be a useful tool for describing conditions for triviality and nontriviality of a state cycle. For example, consider the following definition:

Definition 4.5. We say that a state is *even* if every circuit in its state graph Γ_σ has even length. Otherwise, we say that the state is *odd.*

Remark 4.5.1. A pinchtrace gives a closed path of length 1 from the loop it joins to itself. So, an even state is also an adequate state.

Remark 4.5.2. A graph theory consequence is that if σ is even, Γ_σ is 2-colorable. Namely, there is an assignment of a color to every vertex, using only 2 colors, so that two vertices joined by an edge have distinct colors. One can view a 2-coloring as a sign choice for something associated to each vertex, so that one color represents "+" and the other represents "-".

Example 4.2. The primary example of an even state is the Seifert state σ_s, the state gotten by choosing the smoothing for each crossing corresponding to that used in Seifert's algorithm. σ_s is also called the oriented resolution, because the smoothing choice is the one consistent with the orientation of the link: positive crossings are 0-smoothed, and negative crossings are 1-smoothed, as shown in Figure 4.3. For an example of a Seifert state and graph, see Figure 4.4.

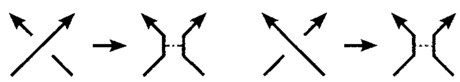

Figure 4.3: The smoothing choice for Seifert's algorithm is the smoothing consistent with orientation.

The reason the Seifert state is even has to do with the surface Σ associated to this state by Seifert's algorithm, formed by gluing disks to each loop of the state, and twisted bands to every trace which match the original crossing. An odd path of traces means there is an annulus with an odd number of twists contained in the surface (gotten by following the twisted bands associated to the traces of the path), contradicting orientability of the Seifert's algorithm surface.

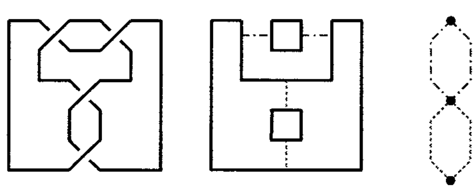

Figure 4.4: On the left is a diagram of the figure 8. In the middle is its Seifert state. On the right is the state graph for this state. By inspection this is an even state.

The fact that even states can be 2-colored turns out to be quite useful: using this 2-coloring turns out to be the key step in showing there are two nontrivial state cycles based on the all-1 state when it is even (Theorem 4.10). But, for general state cycles, we will want to consider similar notions of even and oddness, restricted to the 1-block. To describe these notions, we will examine a particular subgraph of the state graph:

Definition 4.6. Given a state σ, the associated *1-block graph* Γ_1 is constructed by taking the loops of the 1-block of σ to be vertices, and the 1-traces to be edges.

In general, Γ_1 will not be connected, and its connected components turn out to be the natural setting for describing restrictions on the 1-block.

Definition 4.7. A *connected component of the 1-block* refers to the set of loops from a connected component of Γ_1. We say that a connected component of the 1-block is *even* if every circuit in that component of Γ_1 has even length, and *odd* otherwise.

Now we are ready to state the classification theorem precisely. Restrictions (S1) and (S2) give the precise definition of an "almost" adequate state, and restrictions (L2)-(L4) spell out the full obstructions to having loops in the 1-block marked by v_-:

Theorem 4.8 (State Cycle Classification). *For a state cycle α based on state σ to represent a nontrivial homology class in Khovanov homology, it must satisfy the following restrictions:*

(S1) σ must be 0-merging.

(S2) σ can have no 1-pinchtraces touching loops in its 1-block.

(L1) 0-tracing loops of α must be marked by v_-.

(L2) No pair of loops both marked by v_- can be joined by only 1-traces.

(L3) Every loop in an odd connected component of the 1-block must be marked by v_+.

(L4) At most one loop in an even connected component of the 1-block may be marked by v_-; all other loops in that component must be marked by v_+.

Furthermore, other than condition (L2), changing which loop is marked by v_- for a fixed even component from (L4) only changes the sign of the resulting homology class.

Proof. Most of the tools have been assembled already. First we will deal with the restrictions on the underlying state:

(L1) : Proposition 3.2 tells us that the underlying state for any state cycle must be 0-merging.

(L2) : Proposition 4.1 tells us that every 1-trace between loops in the 1-block must be a mergetrace.

Now, we will address the restrictions on the loop markings.

(S1) : Proposition 3.2 tells us that the 0-tracing loops of any state cycle must be marked by v_-.

(S2) : Proposition 4.2 obstructs such a pair of loops marked by v_- which are joined by only 1-traces.

(S3) : If there are two or more loops in this odd component marked by v_-, then there is a path of 1-traces from one to the other involving only loops in the 1-block, so Proposition 4.3 tells us $[\alpha] = 0$.

Suppose only one loop in the odd connected component is marked by v_-. Denote this loop by L_a. Since the component is odd, there is some closed path of 1-traces in that component with odd length. Let this path be P. We can construct a path P_a from L_a to itself of odd length by taking a path P' from L_a to any loop of this odd path, follow the odd path P, and then follow the reverse of path P' back to L_a. The length of P_a is then twice the length of P' plus the length of P, and so must be odd.

Let's view P_a as a sequence T_i of traces going from loop L_i to L_{i+1}, of length n. To each trace T_i, we can associate an enhanced state β_i, in the following way. The underlying state for β_i is σ, modified by changing the trace T_i from 1 to a 0. The loops of β_i will have a merged loop M_i corresponding to the change of the mergetrace T_i to a pinchtrace, and will otherwise match the loops of σ. So, we will label the loops of β_i by the matching label of α for every loop that was unchanged, and by v_+ for the merged loop. See Figure 4.5.

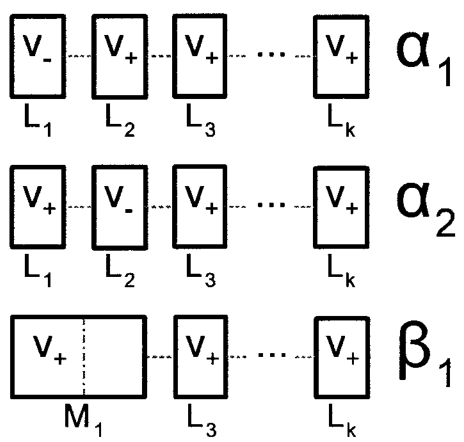

Figure 4.5: Schematic showing the local picture for the α_i and β_i in Case 1b.

Following the loop numbering of trace T_i joining loop L_i to loop L_{i+1}, let α_i denote the enhanced state where σ is marked so that a v_- is on loop L_i, and all other loops are marked by v_+, as shown in Figure 4.5. By this convention, $\alpha_1 = \alpha_{n+1} = \alpha$. By construction, $d(\beta_i) = \pm(\alpha_i + \alpha_{i+1})$: there is only one edge differential from each state to the all-1 state, and we are in the case $\Delta(v_+) = v_- \otimes v_+ + v_+ \otimes v_-$ of Table 2.1. For our purposes, we can ignore this sign associated to the edge differential during calculations and correct by the correct sign at the end.

With this setup, the claim is that $2\alpha = \sum_{i=1}^{n}(-1)^{i+1}d(\beta_i)$. The reason is that this is a telescoping series - observe that $d(\beta_i)-d(\beta_{i+1}) = (\alpha_i+\alpha_{i+1})-(\alpha_{i+1}+\alpha_{i+2}) = \alpha_i-\alpha_{i+2}$. So, since the path is of odd length, the alternating sum collapses to $\alpha_1 + \alpha_{n+1} = 2\alpha$, as claimed. It follows that $[\alpha]$ is trivial.

(S4) : If there are two or more loops in this component marked by v_-, then there is a path of 1-traces from one to the other involving only loops in the 1-block, so Proposition 4.3 tells us $[\alpha] = 0$.

The last claim to check is that if some loop in such an even component of the 1-block is marked by v_-, then the choice of which loop in that component to mark v_- only changes the sign of the resulting homology class.

Label the loops of this component of the 1-block by L_1 to L_n, and the traces by T_1 to T_m. Let α_i be the enhanced state where loop L_i is marked by v_-, the other L_k are marked by v_+, and loops outside this component are marked as in α. Let σ_j be the state where σ is modified by changing trace T_j from 1 to 0. Because of restriction (L2), we know that T_j is a mergetrace in σ, so σ_j will have one loop M_j which was merged by the change of smoothings, and otherwise will match the loop structure of σ. Let β_j be the enhanced state where σ_j is the underlying state, M_j and the other L_k are all marked by v_+, and the other loops retain their marking from α.

Given i and j, there is some path of traces from L_i to L_j in this connected component of the 1-block. Let this path of traces be $T'_1, \ldots, T'_p$, and trace T'_h join loop L'_h to loop L'_{h+1}. Similarly, let α'_h and β'_h be the obvious analogues in this renumbering of loops and traces. By this setup, $d(\beta'_h) = \alpha'_h + \alpha'_{h+1}$, and we have the familiar telescoping series $\sum(-1)^{h+1}d(\beta'_h) = \alpha'_1 \pm \alpha'_p = \alpha_i \pm \alpha_j$. So, up to sign any choice of α_i represents the same homology class, as claimed. □

Remark 4.8.1. While this classification theorem narrows down the candidates for nontrivial state cycles, it is a necessary condition, not a sufficient condition. So one still must check that a particular candidate represents a nontrivial homology class in practice.

In general, there may be several nontrivial state cycles associated to a given state. However, there is a unique state cycle for a given state which has maximal quantum grading:

Definition 4.9. Given an almost adequate state σ, we say that the *maximal state cycle* for σ is enhanced state where the 0-tracing loops of σ are marked by v_-, and the loops of the 1-block are marked by v_+.

Note that this paradigm of v_- for 0-tracing loops, v_+ for the 1-block, matches up nicely with our extreme state cycles for adequate knots in Examples 3.1 and 3.2: α_0 has only 0-tracing loops, each of which is marked by v_-; α_1 has only a 1-block, each loop of which is marked by v_+.

For the families of H-thick knots and links in Chapter 7, we will only need such maximal state cycles. However, nonmaximal state cycles can still represent nontrivial homology classes:

Theorem 4.10. *Suppose that the all-1 state σ for a nonsplit link diagram is even. Then there are always two nontrivial homology classes represented by state cycles based on σ: one is uniquely represented by marking a v_+ on every loop, and the other is represented up to sign by marking a single loop of choice by v_- and all other loops by v_+.*

Before going through the full proof, let's go through a small example.

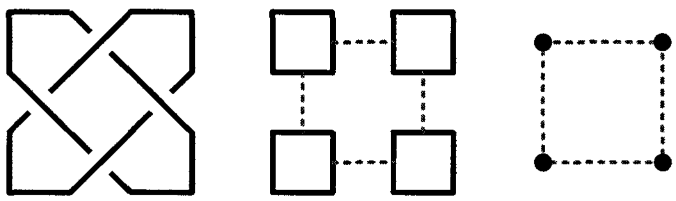

Figure 4.6: On the left side is a diagram for the mirror of Solomon's knot. In the middle is the all-1 state, and on the right is the all-1 state's state graph.

Example 4.3. Consider the mirror of Solomon's knot (which is really a link). As shown in Figure 4.6, its all-1 state σ is even, with a square state graph. We already know by Example 3.2 that marking every loop of this state by v_+ gives a state cycle representing a nontrivial homology class. So, let's number the four loops of the state and let α_i be the enhanced state where loop L_i is marked by v_- and all other loops are marked by v_+. Similarly, number the traces so that trace T_i goes from L_i to L_{i+1}, with T_4 going between L_4 and L_1. Finally, let β_i be the enhanced state where trace T_i has been changed from 1 to 0, and all loops are marked by v_+. See Figure 4.7.

This setup should be familiar by now, and it should be no surprise that $d(\beta_i) = \alpha_i + \alpha_{i+1}$, up to sign. Setting up the usual telescoping series, this means that $\alpha_i \pm \alpha_j = \sum(-1)^{k+1} d(\beta_k)$, so that up to sign the α_i represent the same homology class. Because

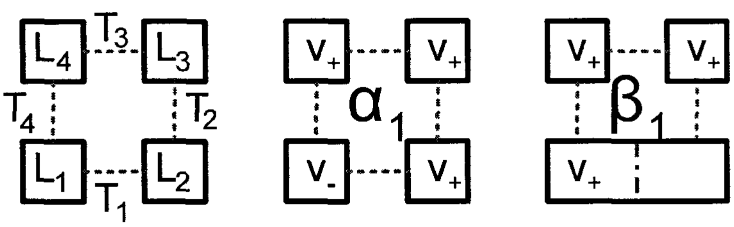

Figure 4.7: On the left side is the numbering convention for loops and traces of the all-1 state. In the middle is an enhanced state for α_1, and on the right is an enhanced state for β_1.

there are only even closed paths in this state graph, however, we cannot use the same trick we used in Case 1b of Theorem 4.8 to conclude that each $[\alpha_i]$ is trivial.

So, let's try to show that α_1 does not lie in the image of d. Every edge differential entering C_σ takes the Δ form of Table 2.1, because every trace in σ is a mergetrace. Since there is only a single v_- in α_1, there is no chance for a $v_- \otimes v_-$ term to be nontrivially involved in any linear combination resulting in α_1. So, we may restrict our attention to the other kind of Δ output, that of the form $v_+ \otimes v_- + v_- \otimes v_+$, which comes from the image of the β_i under the differential. In particular, $[\alpha_1]$ is trivial if and only if α_1 is some nontrivial linear combination of the $d(\beta_i)$.

Suppose $\alpha_1 = \sum a_i d(\beta_i)$. Approaching this via linear algebra, we get a simple system of equations, knowing that the coefficient of all α_i other than $i = 1$ must be zero:

$$a_1 + a_2 = 0$$

$$a_2 + a_3 = 0$$

$$a_3 + a_4 = 0$$

Consequently, $a_1 + a_4 = 0$. But, that sum was the coefficient of α_1 from the sum, so α_1 cannot be realized as such a linear combination.

This works for small cases, but seems to hide the combinatorics of the state: ideally a proof of nontriviality would exploit this added structure to take advantage of the hypothesis. One way to take this into account is to associate such a linear combination to a weighted state graph, where the edges of the state graph have weight a_i.

Note that $d(\beta_i)$ contributes a_i copies of α_i and α_{i+1}. By interpreting the edge T_i as the $d(\beta_i)$, and the vertices bounding that edge as the associated state cycles α_i and α_{i+1}, we can interpret the linear algebra constraint on the coefficients as the condition that the sum of the weights around every vertex other than L_1 must be 0.

In our case, every vertex is adjacent to exactly two edges. So, the weight condition means that the two weights must have the same magnitude and opposite sign. As we travel around the square, the weights follow this pattern: alternating in sign, same magnitude. So, let's pick an edge adjacent to L_1 to start a traversal, and propagate this condition on the edge weights until we return to L_1. Starting with a_1, the other weights must then be $-a_1, a_1, -a_1$, and the sum of the two weights of the edges around

L_1 are then $a_1 - a_1 = 0$. This same pattern holds any time we have such an even closed paths based at L_1. See Figure 4.8.

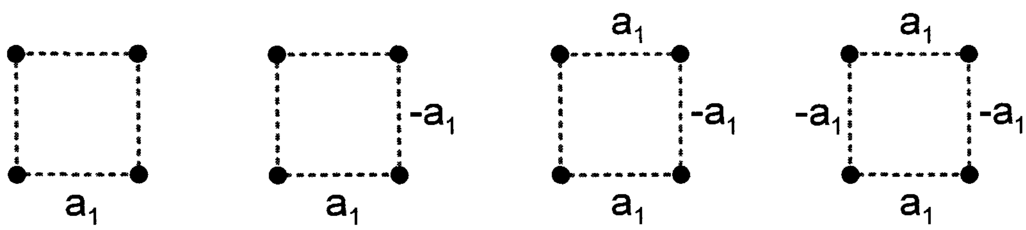

Figure 4.8: From left to right, a propagation of the weight condition over a traversal of a closed path based at L_1.

The weighted graph interpretation may be overkill for an easy case like this, but the idea is to use this interpretation to take advantage of geometric information in solving the associated linear algebra problem. The general case involves a more complicated graph, but it turns out the fact that the graph is 2-colorable is enough to force the linear combination to be trivial, with this setup.

So, let's generalize the weighted graph approach of Example 4.3 to the general case of Theorem 4.10.

Proof of Theorem 4.10. The first nontrivial homology class associated to this even all-1 state comes from marking every loop by v_+. By the discussion of Example 3.2, we know this enhanced state represents a nontrivial homology class.

The new content of this theorem comes in dealing with the enhanced states which mark a single loop by v_- and all other loops by v_+. The first claim is that our choice of loop to mark only changes the sign of the associated homology class.

Label the loops of the traced all-1 state σ by L_1 to L_n, and the traces by T_1 to T_m. Let α_i be the enhanced state where loop L_i is marked by v_- and all other loops are marked by v_+. Let σ_j be the state where σ is modified by changing trace T_j from 1 to 0. Since σ is adequate, σ_j will have one loop M_j which was merged by the change of smoothings, and otherwise will match the loop structure of σ. Let β_j be the enhanced state where σ_j is the underlying state and every loop is marked by v_+.

Given i and j, there is some path of traces from L_i to L_j in σ because σ is a state of a nonsplit link. Let this path of traces be $T'_1, \ldots, T'_p$, and trace T'_h join loop L'_h to loop L'_{h+1}. Similarly, let α'_h and β'_h be the obvious analogues in this renumbering of loops and traces. By this setup, $d(\beta'_h) = \alpha'_h + \alpha'_{h+1}$, and we have the familiar telescoping series $\sum(-1)^{h+1}d(\beta'_h) = \alpha'_1 \pm \alpha'_p = \alpha_i \pm \alpha_j$. So, up to sign any choice of α_i represents the same homology class.

The meat of the proof is showing that this homology class represented by α_1 is nontrivial. As was the case in Example 4.3, this boils down to showing that no nontrivial linear combination of the $d(\beta_i)$ can equal α_1. To show this, we will reinterpret such a linear combination as a weighted state graph Γ_σ. If $\alpha_1 = \sum a_i d(\beta_i)$, we will place weight a_i on the edge associated to trace T_i. The sum of weights about the vertex for L_i corresponds to the coefficient of α_i in this linear combination, so the sum of weights about every vertex except L_1 must be zero.

Because σ is even, the state graph is 2-colorable. Since the state graph is path-connected, a choice of color on one vertex determines the 2-coloring, so let's use the colors "+" and "-", and assume that vertex L_1 is colored "-". Let $\mathrm{sign}(L_i)$ denote this color, overloaded to be an actual sign choice.

Now, let's use this sign choice to take an alternating sum of the weights around every vertex. Let $\mathrm{weight}(L_i)$ denote the sum of the weights around vertex L_i. We claim that $\sum_{i=1}^{n} \operatorname{sign}(L_i) \operatorname{weight}(L_i) = 0$. To see this, note that if we ignore the sign choice, every edge's weight is counted exactly twice, once for each vertex at the endpoints of the edge. But, the 2-coloring guarantees that in this weighted sum, the two occurences of weight show up with opposite sign, since if two vertices are joined by an edge in the 2-coloring, they have opposite colors, or opposite signs in this color scheme.

We chose the sign of the L_i vertex to be negative, which means that we can bring it to the other side of this equality to conclude that $\operatorname{weight}(L_1) = \sum_{i=2}^{n} \operatorname{sign}(L_i) \operatorname{weight}(L_i)$. But, $\operatorname{weight}(L_i) = 0$ for every i other than 1 by construction, so $\operatorname{weight}(L_1) = 0$. This means that the coefficient of α_1 in the original linear combination $\sum a_i d(\beta_i)$ must be zero. In other words, α_1 does not lie in the image of the differential, and represents a nontrivial homology class. □

In fact, the proof of Theorem 4.10 can be generalized to a larger class of state cycles. Towards this end, we need to consider yet another subgraph of the state graph.

Definition 4.11. Let σ be a state in a diagram. Let Λ_1 denote the graph whose vertices correspond to the 1-tracing loops of σ (i.e. loops touched by at least one 1-trace). We say that σ is *1-even* if Λ_1 has only even circuits. If α is any state cycle based on σ, we say that α is *1-even* if σ is 1-even.

Definition 4.12. Let α be a state cycle based on state σ. We say that α is *1-isolated* if every connected component of Λ_1 has at most one loop marked by v_-.

Theorem 4.13. *Let α be a state cycle which is 1-even and 1-isolated. Then α represents a nontrivial homology class.*

Proof. The idea here is that with this setup, each connected component of Λ_1 looks exactly like the state graph we considered in Theorem 4.10. Let the underlying state of α be σ. The 1-traces of σ are the source of edge differentials that target C_σ, the chain group in the Khovanov chain complex in which α lies. The difference here is that since σ is no longer the all-1 state, there will be other chain groups of the same height as C_σ. This means that, for some 1-trace T of σ, if we have $d_T(\gamma) = \alpha$, $d(\gamma)$ may also have nonzero image in other chain groups besides C_σ.

However, our strategy here is to simply restrict our attention to the component of the differential which targets C_σ, which we will denote d_σ. What we will show is that there is no γ so that $d_\sigma(\gamma) = \alpha$. Consequently, there can be no γ so that $d(\gamma) = \alpha$, and $[\alpha] \neq 0$.

Given a 1-trace T of σ, let σ_T be the state where the T is changed from a 1-trace to a 0-trace, and all other smoothings remain the same, and let d_T denote the edge differential $d_T : C_{\sigma_T} \longrightarrow C_\sigma$. Thinking through the definitions, it is easy to see that $d_\sigma = \sum_{T \text{ 1-trace}} (-1)^{|T|} d_T$. Since we are only considering this restricted differential, we can safely ignore the signs in front of these edge differentials: we can pick a negative basis for C_{σ_T} any time the sign for d_T should be negative, and end up with the same result as if we had ignored the sign and chosen the positive basis. So, we will view d_σ as simply $\sum_{T \text{ 1-trace}} d_T$.

Because α is 1-even, we know that every 1-trace is a mergetrace (a 1-pinchtrace would give an odd closed path in Λ_1). So, we know that every edge differential d_T will take the Δ form. In general, Δ can locally return either $v_- \otimes v_-$ or $v_+ \otimes v_- + v_- \otimes v_+$, where the tensor here corresponds to a 1-trace joining the two loops in σ. However, because α is 1-isolated, each connected component of Λ_1 has at most one loop marked by v_-, so that the $v_- \otimes v_-$ form of Δ cannot be involved in any potential boundary for α. So, the only terms of interest will have the local form $v_+ \otimes v_- + v_- \otimes v_+$.

Label the 1-traces of σ by T_1 through T_n, and let β_i denote the enhanced state based on σ_{T_i} where the merged loop, and every loop in the connected component of Λ_1 in which this loop lies, are marked by v_+, and every other loop is marked as in α. We have argued that the only possible linear combination whose d_σ image can be α must take the form $\sum_{i=1}^{n} a_i\beta_i$.

Now, we may proceed exactly as in Theorem 4.10, building a weighted graph version of Λ_1, where the weights of the edges are the a_i, and the sum of weights around all but one vertex of every connected component of Λ_1 must be 0. Since Λ_1 is even, it is 2-colorable, and the alternating sum of weights arguments lets us conclude again that the sum of weights around *every* vertex is 0. Consequently, $\sum_{i=1}^{n} d_\sigma(a_i\beta_i) = 0$, so that α cannot lie in the image of d_σ. It then follows that $[\alpha] \neq 0$, as desired. $\square$

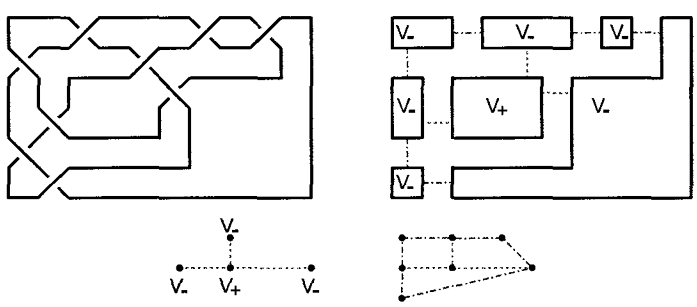

Figure 4.9: Example of a 1-even state cycle which is not 1-isolated, and represents a trivial homology class. On the top left is the diagram of the knot; on the top right is the state cycle in question. The bottom right shows the state graph for the state cycle, and the bottom left shows Λ_1, with the vertices marked as in the state cycle. Clearly Λ_1 is 1-even but not 1-isolated.

Example 4.4. As an example of why the 1-isolated condition is needed, Figure 4.9 shows a state cycle which is 1-even, but not 1-isolated. Calculation shows that the associated homology class is trivial.

However, there is still a lot of room for improvement in terms of detecting non-triviality of a state cycle, as the following examples show:

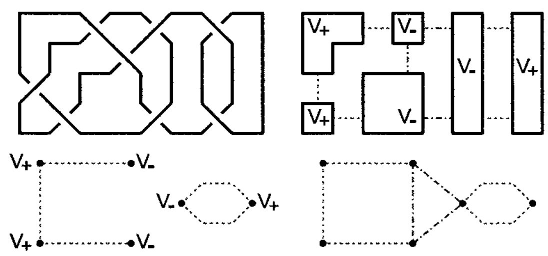

Figure 4.10: Example of a 1-even state cycle which is not 1-isolated, and represents a nontrivial homology class. On the top left is the diagram of the link; on the top right is the state cycle in question. The bottom right shows the state graph for the state cycle, and the bottom left shows Λ_1, with the vertices marked as in the state cycle. Clearly Λ_1 is 1-even, but since the lefthand component of Λ_1 has two loops marked by v_-, it is not 1-isolated.

Example 4.5. Often 1-even state cycles which are not 1-isolated still represent non-trivial homology classes. An example is illustrated in Figure 4.10, which can be shown by direct calculation to represent a nontrivial homology class. Presumably there is some condition on the 0-tracing loops that can extend this result to a larger class of 1-even state cycles.

Example 4.6. Similarly, it is possible for a 1-isolated state cycle which is not 1-even to represent a nontrivial homology class. An example is illustrated in Figure 4.11, which can be shown by direct calculation to represent a nontrivial homology class.

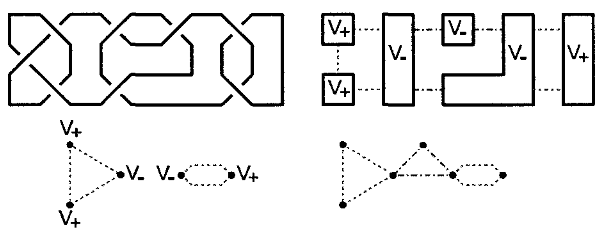

Figure 4.11: Example of a 1-isolated state cycle which is not 1-even, and represents a nontrivial homology class. On the top left is the diagram of the link; on the top right is the state cycle in question. The bottom right shows the state graph for the state cycle, and the bottom left shows Λ_1, with the vertices marked as in the state cycle. Clearly Λ_1 is 1-isolated, but since the lefthand component of Λ_1 has an odd length closed path, it is not 1-even.

Chapter 5

Maps acting on State Cycles

This section examines how state cycles interact with various homomorphisms and spectral sequences. Here we follow the paradigm that sometimes one can understand objects better by studying the maps which act on them.

The first maps we will examine are the Jacobsson homomorphisms, which give a way to relate Khovanov homology classes of two different links. It turns out that in certain situations, Jacobsson homomorphisms can take state cycles to state cycles. Suppose α, β are state cycles, $[\alpha] \neq 0$, and $\Psi(\beta) = \alpha$, for some Jacobsson homomorphism Ψ. It follows that $[\beta] \neq 0$, since homomorphisms take 0 to 0. In this way, Jacobsson homomorphisms can let us "lift" nontrivial state cycles to nontrivial state cycles, under the right setup.

The second kind of map we will examine is the Lee homomorphism Φ_{Lee}. Because of the convergence of an associated spectral sequence, in many cases Φ gives a way to organize the homology classes of a fixed diagram D into pairs of the form $(\alpha, \Phi(\alpha))$, of relative bidegree (1,4). We will see later that when the α of such a Lee pair is a nontrivial state cycle, there is a natural way to lift the Lee pair of one diagram to one of a related diagram. So, we examine a special case where we can guarantee the existence of such a nontrivial Lee pair.

Finally, we conclude the section by examining a relationship between state cycles based on the Seifert state, and Rasmussen's s-invariant, which comes from the Lee spectral sequence. In certain cases, we can use this information to conclude that the state cycle based on the Seifert state represents a nontrivial homology class.

5.1 Jacobsson Homomorphisms

The first maps we will work with are the Jacobsson homomorphisms. For every orientable cobordism between two links, Jacobsson [Jac04] constructed homomorphisms between the Khovanov homology of the two links and showed this construction is functorial up to sign, meaning that given two isotopic cobordisms Σ_1, Σ_2, the induced homomorphisms are the same up to sign. These homomorphisms preserve the homological grading, and shift the quantum grading by the Euler characteristic of the cobordism. In other words, a cobordism Σ induces a $(0, \chi(\Sigma))$ homomorphism Ψ_Σ on $Kh^{i,j}(L_1)$, as shown in Table 5.1.

$$\begin{array}{ccc} L_1 & \xrightarrow{\quad\Sigma\quad} & L_2 \\ Kh^{i,j}(L_1) & \xrightarrow{\quad\Psi_\Sigma\quad} & Kh^{i,j+\chi(\Sigma)}(L_2) \end{array}$$

Table 5.1: A cobordism Σ induces a $(0, \chi(\Sigma))$ homomorphism Ψ_Σ on $Kh^{i,j}(L_1)$.

The construction defines maps at the chain level for each Morse and Reidemeister move, and shows that these chain maps take cycles to cycles, inducing homomorphisms. Jacobsson's paper follows the Viro convention for marking enhanced states, so we will rewrite the homomorphisms used here in terms of the Bar-Natan generators, for convenience. The definitions are shown in terms of ET states: ET states not shown in the definitions will go to 0 under the homomorphisms. The light turquoise lines in the figures indicate how arcs in the local picture of the diagram complete to loops in the state in question.

Our goal now is to analyze situations in which these homomorphisms take state cycles to state cycles, so that we can have a way of "lifting" nontriviality of state cycles from one diagram to another. In this subsection we will describe a special case where this happens, which we will call *positive modification.* This construction will later be generalized to *quasipositive modification* in Chapter 6.

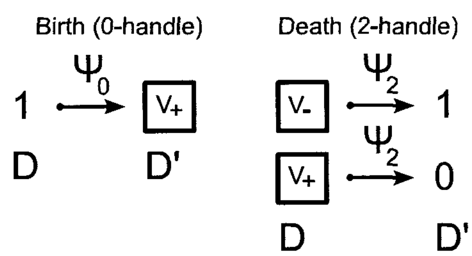

Figure 5.1: Simplest Jacobsson homomorphisms, coming from the Morse moves for birth and death of loops. Compare to Figure 15 on page 1226 of [Jac04].

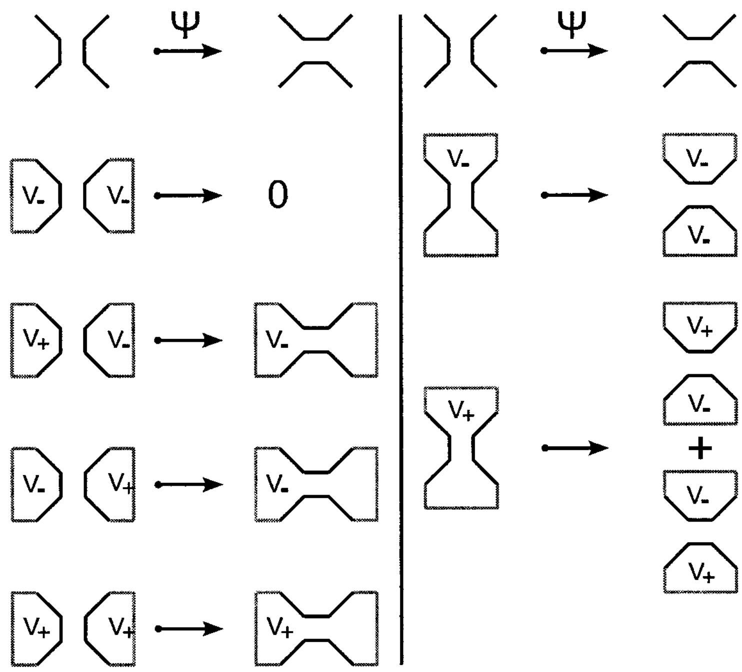

Figure 5.2: Jacobsson homomorphisms coming from the saddle Morse moves. Note that there are no traces here joining the loops. Compare to Figure 3 on page 1216 of [Jac04].

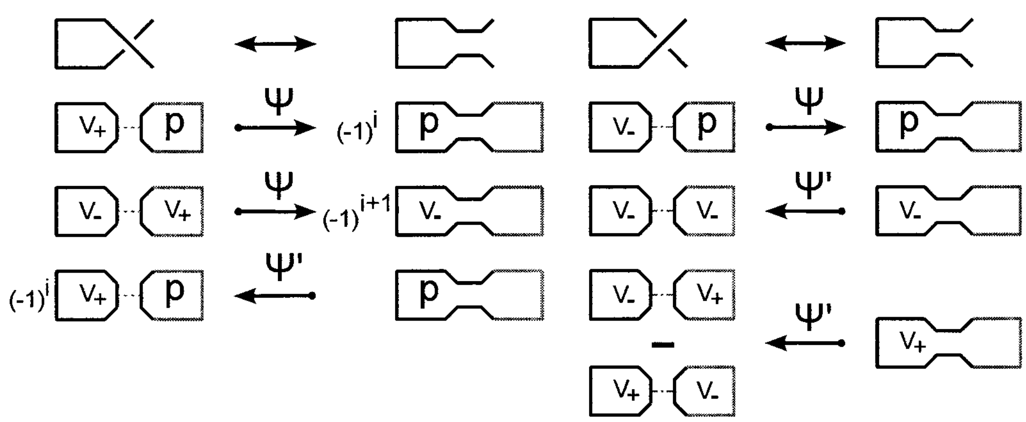

Figure 5.3: Jacobsson homomorphisms coming from negative and positive Reidemeister I, from left to right. Compare to Figure 16 on page 1228 of [Jac04]. The p here indicates that either v_- or v_+ can be marked on the loop marked by p.

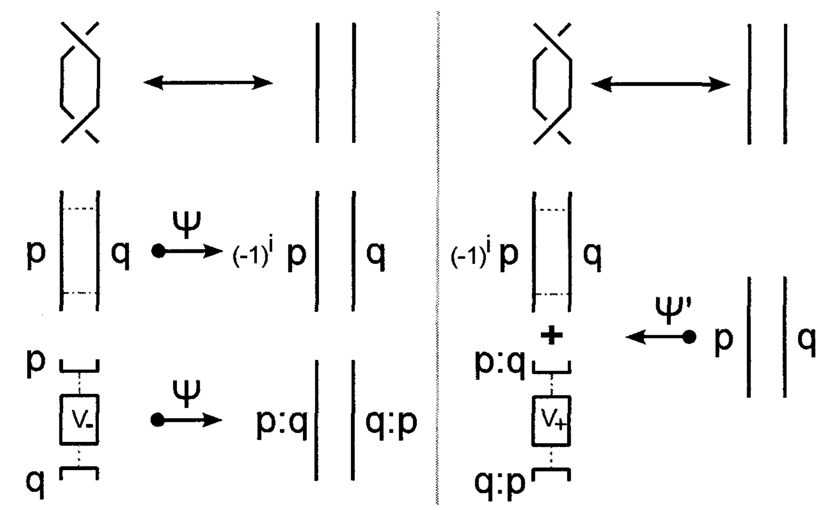

Figure 5.4: Jacobsson homomorphisms coming from Reidemeister II. The p:q, q:p convention indicates that locally there is a saddle cobordism from the pair of arcs (p, q) to the pair of arcs $(p : q, q : p)$. Given a choice of generators for p and q, the values of $p : q$ and $q : p$ match those of the saddle move from Figure 5.2 whose initial arcs match p and q. Compare to Figure 17 on page 1229 of [Jac04].

Definition 5.1. Let D be an oriented diagram, and α an ET state representing a cycle in $Kh(D)$. We say that an oriented diagram D' is obtained from D by *positive modification* on D compatible with α if there is a positive crossing b in D' so that 0-resolving b yields D as an oriented diagram and the 0-trace of b connects two distinct 0-tracing loops of α.

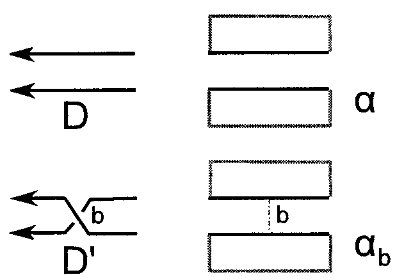

Figure 5.5: Illustration of positive modification at diagram and state level.

Lemma 5.2. *Let D be a diagram for a link, and α a state cycle based on state σ of D. Consider a diagram D' obtained from D by positive modification compatible with α. Let Ψ_J be the Jacobsson homomorphism induced by the cobordism of 0-resolving this positive crossing b associated to the positive modification. Then there exists a state cycle $\widetilde{\alpha}$ such that $\Psi_J(\widetilde{\alpha}) = \alpha$. In particular, if $[\alpha]$ is nontrivial, so is $[\widetilde{\alpha}]$.*

Proof. Let $\widetilde{\alpha}$ be the ET state in D' corresponding to α: the loops and markings match α, with the only difference being the added 0-mergetrace for the extra crossing b. Because the two loops joined by b are 0-tracing in α, they are marked by v_-. Note that since the new 0-trace for b joins two loops marked by v_- in $\widetilde{\alpha}$, the new edge differential for b will also be zero, so that $\widetilde{\alpha}$ is also a state cycle.

The cobordism between D' and D can be broken down into adding a 1-handle, followed by undoing a positive Reidemeister I twist - see Figure 5.6. When adding

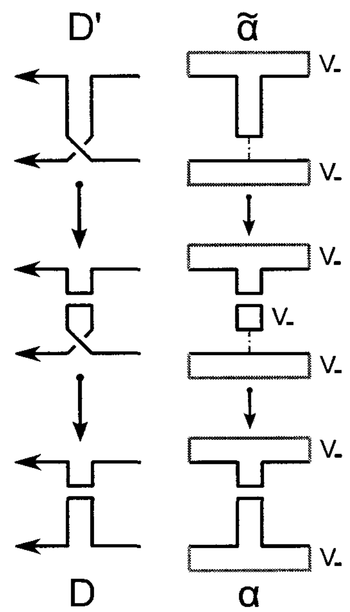

Figure 5.6: Illustration of cobordism involved in Lemma 5.2 at diagram and state level.

the 1-handle, we are splitting one loop into two, so the associated Jacobsson homomorphism takes the v_- on that loop to $v_- \otimes v_-$. Then, for undoing the Reidemeister one twist, the Jacobsson homomorphism takes the $v_- \otimes v_-$ pair of loops joined by the 0-trace of b to a single merged loop marked by v_-. So, writing this out in tensors, we have

$$\widetilde{\alpha} = (v_-) \otimes v_- \otimes \cdots \xrightarrow{\Psi_1} (v_- \otimes v_-) \otimes v_- \otimes \cdots \qquad \text{(1-handle)}$$

$$v_- \otimes (v_- \otimes v_-) \otimes \cdots \xrightarrow{\Psi_2} v_- \otimes (v_-) \otimes \cdots = \alpha \qquad \text{(RI+)}$$

See Figure 5.6 for an illustrated version of this. □

Remark 5.2.1. This is a generalization of Plamenevskaya's theorem about positive resolution (see Theorem 4 of [Pla06]) to state cycles, and gives the key idea for "lifting" state cycles by Jacobsson homomorphisms. Chapter 6 will extend Definition 5.1 to a slightly more general kind of modification, for which we will give an analogue of Lemma 5.2.

The Euler characteristic of the cobordism constructed in the lemma is -1, meaning that the quantum grading of $\widetilde{\alpha}$ is 1 higher than that of α. But, this shift is applied uniformly to every lift associated with this cobordism.

Corollary 5.3. *Suppose there are two state cycles, α and β, which are compatible with the same positive modification. Then, the Khovanov bigrading difference between the lifts, $\widetilde{\alpha}$ and $\widetilde{\beta}$, matches the bigrading difference between α and β. In particular, this means that if α and β lie on two distinct diagonals in $Kh(D)$, $\widetilde{\alpha}$ and $\widetilde{\beta}$ lie on distinct diagonals in $Kh(D')$, of the same relative separation.*

In theory one could try to check the nontriviality of a cycle by relating it to another known nontrivial state cycle by a more complicated series of Jacobsson homomorphisms. In practice, this is not so easy to produce. Even if one had a cobordism between two diagrams, the projection down of some cycle you want to investigate will probably not end up as a state cycle, limiting the option of iterating the process. It's also very possible for the induced homomorphism to be trivial on your cycle of interest, which gives no information.

5.2 The Lee Homomorphism

Lee's homomorphism Φ_{Lee}, which serves as a differential on $Kh(L)$ and induces a spectral sequence [Lee05], is the next map we will analyze, to see what we can learn about state cycles. For knots, the Lee spectral sequence converges to $\mathbb{Q} \oplus \mathbb{Q}$; by work of Rasmussen [Ras04], there are two homology classes in Kh which survive to those two copies of $\mathbb{Q}$ of bigrading $(0, s \pm 1)$, where s is a smooth concordance invariant to be discussed in the next subsection. In practice, the quick convergence of this spectral sequence gives a (1,4) pairing of the other homology classes of Kh, which will prove of interest for state cycles.

Φ_{Lee} is defined analogously to the Khovanov differential: first Lee defines edge differentials, as shown in Table 5.2, then she pieces them together with signs based on the edges in exactly the same way as done in the Khovanov differential. So, Φ is defined on the same chain groups as Khovanov homology; it satisfies $\Phi \circ \Phi = 0$, anticommutes with the Khovanov differential, and is invariant under Reidemeister moves. This means that $\Phi + d$ gives a differential on the Khovanov chain complex CKh, and the resulting homology $H(CKh, d{+}\Phi)$ is called the Lee homology of a link, $Kh_{Lee}(L)$. Note also that since Φ anticommutes with d, Φ takes cycles of (CKh, d) to cycles of (CKh, d) and induces a (1,4) homomorphism on Kh.

$\bigcirc\ \bigcirc \xrightarrow{\mu_\Phi} \bigcirc$	$\bigcirc \xrightarrow{\Delta_\Phi} \bigcirc\ \bigcirc$
$v_+ \otimes v_+ \longmapsto 0$	$v_+ \longmapsto 0$
$v_+ \otimes v_- \longmapsto 0$	$v_- \longmapsto v_+ \otimes v_+$
$v_- \otimes v_+ \longmapsto 0$	
$v_- \otimes v_- \longmapsto v_+$	

Table 5.2: The edge differential for Φ shifts every nonzero result by $+4$ in quantum grading.

In terms of the Khovanov bigrading, Φ has bidegree $(1, 4)$, while the differential $\Phi + d$ either preserves the quantum degree or raises it by 4 for each monomial in its result. As Rasmussen [Ras04] observed, this means that quantum grading gives a filtration on the Khovanov chain complex which the differential $\Phi + d$ respects. This induces a spectral sequence converging to Kh_{Lee} whose E_1 page is given by $(Kh(L), \Phi)$.

Φ gives more information about knots whose Khovanov homology we have already calculated, particularly when this Khovanov homology has homological width less than 4. Because the bidegree of the differentials in the Lee spectral sequence, in terms of the Khovanov bigrading, is (1, 4i), one can often show that the spectral sequence converges at the E_2 page; in fact, it is conjectured that the spectral sequence always converges at the E_2 page. In such a case, Φ gives a (1,4) pairing between all nontrivial Khovanov homology classes which vanish in the Lee spectral sequence. This is not a bilinear pairing, but rather an enumeration of the homology classes by pairs of the form $(\alpha, \Phi(\alpha))$, plus the pair of homology classes which survive under the spectral sequence. It comes from the fact that convergence of the spectral sequence at the E_2 page means that:

$$Kh_{Lee} = \frac{\ker(\Phi)}{\mathrm{im}(\Phi)}$$

We will be interested in Lee pairs $(\alpha, \Phi(\alpha))$ where α is a nontrivial state cycle. Because of the (1,4) bigrading, α and $\Phi(\alpha)$ lie on adjacent diagonals, so one can interpret this Lee pair as a pair of diagonals related to the state cycle α. Theorem 6.6 will tell us later that we can then lift this pair of diagonals from one diagram to another via quasipositive modification. For now, we give a useful situation where this state cycle Lee pairing arises:

Proposition 5.4. *Suppose D is a + adequate diagram of a knot with homological width less than 4, and that its minimum homological dimension is less than 0 (i.e. D is not a positive diagram). Then $(\alpha_0, \Phi(\alpha_0))$ is a Lee pair, where α_0 is the ET state in which the all-0 state has every loop marked by v_-.*

Proof. Following Proposition 5.2 of [Ras04], the width condition on $Kh(D)$ forces the Lee spectral sequence to converge at the E_2 page. The E_1 page of the spectral sequence is given by $(Kh(D), \Phi)$: convergence means that the homology of $Kh(D)$ under this differential Φ is $\mathbb{Q} \oplus \mathbb{Q}$ in homological dimension 0, and trivial elsewhere.

Consider the ET state α_0, which marks every loop of the all-zero state σ_0 with v_-. As observed in Example 3.1, this represents a nontrivial homology class, in minimal quantum and homological grading. We claim that $[\Phi(\alpha_0)]$ cannot be 0.

α_0 definitely dies in the E_2 page of the spectral sequence, because its homological dimension is not 0 by hypothesis (α_0 has minimal homological grading, which we assumed was less than 0). But, because α_0 is in the minimal bigrading and Φ is a (1,4) homomorphism, there cannot be any β so that $\Phi(\beta) = \alpha_0$. So, since α_0 does not lie in the image of Φ, if it lay in the kernel of Φ it would survive to the E_2 page, a contradiction. It follows that $[\Phi(\alpha_0)]$ is nontrivial, so that $(\alpha_0, \Phi(\alpha_0))$ is a Lee pair. $\square$

Remark 5.4.1. Note that $\Phi(\alpha_0)$ will not be a state cycle, but its form is sufficiently simple that we can see how it interacts under a certain class of Jacobsson homomorphisms later (see Theorem 6.6).

5.3 The s-invariant

In this subsection, we will examine how to exploit the relationship of certain state cycles to the s-invariant to conclude nontriviality of those special state cycles. Rasmussen's s-invariant [Ras04] is defined roughly to be average of the quantum gradings of the generators which survive to the E_∞ page of the Lee spectral sequence. It turns out that s is a smooth concordance invariant, and it is intrisically linked to the Seifert state (the state smoothed according to the algorithm for the canonical Seifert surface of a diagram).

When a state cycle for the Seifert state has quantum grading equal to $s-1$, one can try to generalize an argument of Baldwin and Plamenevskaya [BP08] and prove that the state cycle is nontrivial. Recall that the Seifert state is always adequate, per the argument of Example 4.2.

Proposition 5.5. *Suppose α is a state cycle associated to the Seifert state, and its quantum grading is $s-1$. If α has no 1-block, then α represents a nontrivial homology class.*

Proof. Let $\mathfrak{s}_0$ be the Seifert state, where each loop is marked by $v_- \pm v_+$ in such a fashion that loops which share a crossing alternate in the sign for v_+. In other words, $\mathfrak{s}_0$ is given by the tensor $(v_- + v_+) \otimes (v_- - v_+) \otimes \cdots \otimes (v_- \pm v_+)$. By Corollary 3.6 of Rasmussen [Ras04], the smallest quantum grading term of $\mathfrak{s}_0$ that is nontrivial in the Lee homology has grading $s-1$. Following Rasmussen's notation, this means that $s(\mathfrak{s}_0) = s-1$.

When α has no 1-block, every loop is marked by v_-. So, if you multiply out the tensor for $\mathfrak{s}_0$, one sees that $\mathfrak{s}_0 = \alpha + \tau$, where every term in τ has higher quantum grading (since it has one or more v_+ factors).

Suppose that $[\alpha]$ is trivial. Then there is some γ so that $d\gamma = \alpha$. Note that since the Khovanov differential d preserves quantum grading, γ also has quantum grading $s - 1$.

Let $d' = d + \Phi$ be the Lee differential. By definition, Φ shifts quantum grading up by 4. So, $d'\gamma = d\gamma + \Phi\gamma = \alpha + v$, where the quantum grading of every term of v has strictly higher quantum grading than $s - 1$.

Consider $\mathfrak{s}_0 - d'\gamma$. This is homologous to $\mathfrak{s}_0$, so should have the same minimum nontrivial quantum grading as $\mathfrak{s}_0$. But, $\mathfrak{s}_0 - d'\gamma = \tau + v$, so that $s(\mathfrak{s}_0 - d'\gamma) > s - 1$, a contradiction. Therefore, no such γ exists and α represents a nontrivial homology class. $\square$

Remark 5.5.1. The argument for this case is essentially the same as that of Theorem 1.2 of Baldwin and Plamenevskaya [BP08]. We reorganize it slightly here since it will serve as a model for other state cycles.

Theorem 5.6. *Suppose α is a state cycle associated to the Seifert state, and its quantum grading is $s-1$. If α has a single loop in its 1-block, and that loop is marked by v_+, then α represents a nontrivial homology class.*

Proof. This time, we will show that $\mathfrak{s}_0 = \beta_0 + \beta_1 + \alpha + \tau$, where τ has quantum grading higher than α, and both β_0 and β_1 are trivial cycles in Khovanov homology, of quantum grading $\leq s - 1$. Then, if α is trivial, we will again look at something homologous to $\mathfrak{s}_0$ in Lee homology, and show that it has quantum grading strictly greater than $s-1$, getting a contradiction.

First, consider α. Since every loop outside of the 1-block must be marked by v_-, up to ordering of the loops $\alpha = v_+ \otimes v_- \otimes \cdots \otimes v_-$. Since the quantum grading of α is $s-1$, this means that any marking of the Seifert state with one v_+ and the rest of the loops v_- will also have quantum grading $s-1$; a state marked by only v_- will have quantum grading $s-3$.

Let β_0 be the Seifert state marked by v_- on every loop. By Proposition 4.2, we know that β_0 is trivial because the (only) loop in the 1-block is marked by v_-: it must be connected to some 0-tracing loop by only 1-traces, and that 0-tracing loop is also marked v_-. So, there is some γ_0 so that $d\gamma_0 = \beta_0$, and as before, $d'\gamma_0 = \beta_0 + v_0$ for v_0 of quantum grading 4 higher than the grading for β_0, namely $s+1$. In particular, v_0 has quantum grading higher than $s-1$.

Next, number the loops of the Seifert state other than the one in the 1-block by 1 through n, where the first loop is connected by a 1-smoothing to the loop marked by v_+ in α. Let δ_i be the Seifert state where loop i is marked by v_+ and all other loops are marked by v_-. Let ϵ_i denote the number of traces in a path from the loop marked by v_+ in α to loop i in the Seifert state, modulo 2. Note that this is well-defined, because the Seifert state is even, as discussed in Example 4.2; one can also view this sign choice simply as a choice of a 2-coloring for the associated state graph. Define β_1 by:

$$\beta_1 = \sum_{i=1}^{n} (-1)^{\epsilon_i} \delta_i$$

If we choose $\mathfrak{s}_0$ so that a $v_- + v_+$ term lies on the circle marked v_+ on α, then the sign of each term of β_1 will match up with the sign of the corresponding monomial in $\mathfrak{s}_0$, since loops in $\mathfrak{s}_0$ are marked by an alternation of $v_- + v_+$ and $v_- - v_+$ by Corollary 2.5 of Rasmussen [Ras04]. Either such choice satisfies $s(\mathfrak{s}_0) = s - 1$ by Corollary 3.6 of Rasmussen, so we are free to pick this convention for $\mathfrak{s}_0$. By construction, we now have that $\mathfrak{s}_0 = \beta_0 + \beta_1 + \alpha + \tau$, where τ are terms of quantum degree higher than $s - 1$.

We claim that β_1 is a cycle. To see this, consider first $d\delta_i$. The only nonzero edge differentials will come from 0-traces joining the loop marked v_+ with an adjacent v_- loop: the edge differential of such a 0-trace will result in a state where those two loops are merged, and every loop is marked by v_-. Suppose the adjacent v_- loop was numbered j; then δ_j will also have an edge differential from that 0-trace with the same output on that edge differential. In short, every such nonzero edge differential appears twice in β_1: the only question is whether they appear with opposite sign in the total differential of β_1.

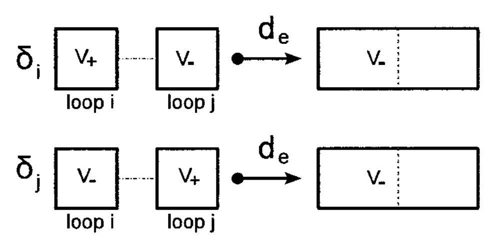

Figure 5.7: δ_i and δ_j sharing a common 0-smoothing.

By construction, we already have that δ_i and δ_j have opposite sign in the sum for β_1, since loops i and j are adjacent. But note that δ_i and δ_j have the same underlying state, so the signs of the edge differentials for the same edge (coming from the 0-smoothing between loop i and loop j) will be the same. So, since the two enhanced states occur with opposite sign in β_1, each such edge pair will cancel out in the total differential, as desired. So, β_1 is a cycle.

Now, we want to construct γ_1 so that $d\gamma_1 = \beta_1$. Let c be the crossing associated to a 1-smoothing joining loop 1 to the 1-block. Let σ_c be the state which matches the Seifert state, except that c is now 0-smoothed. The result is that the loop of the 1-block is merged together with loop 1 in σ_c; by the definition of the 1-block, there were no 0-smoothings between loop 1 and the 1-block loop in the Seifert state, so there is only a single 0-pinchtrace in σ_c.

Let μ_1 be σ_c marked with a v_+ on the merged loop, and v_- on every other loop. Let μ_i for $i > 1$ be σ_c with the merged loop marked v_-, and the other loops marked as in δ_i. In other words, μ_i has a v_+ on loop i in σ_c and every other loop is marked by v_-.

Define μ by:

$$\mu = \sum_{i=1}^{n} (-1)^{\epsilon_i} \mu_i$$

We claim that $d\mu = \beta_1 \pm \alpha$, where the sign is determined by $(-1)^{\epsilon_1}$.

First, we will assume that the crossings are ordered so that the crossings 1-smoothed in the Seifert state come last, and that the first 1-smoothing which appears in the ordering is that associated to crossing c. We claim that this ordering of crossings guarantees that the signs associated to the edge differentials out of σ_c will all be positive.

Recall that the sign for an edge differential e is $(-1)^{|e|}$, where $|e|$ counts the number of crossings 1-smoothed which come before the $*$ crossing of edge e in the crossing ordering. The set of possible * positions for an edge differential out of σ_c corresponds to the set of 0-smoothings of the Seifert state, together with the extra crossing c which is 0-smoothed in σ_c but not the Seifert state. Thus $|e| = 0$ for every choice of *, and all edge differential signs are positive as claimed.

Consider the edge differential d_c for c, applied to μ. Since c is a 0-pinchtrace in σ_c, this edge differential takes the Δ form. Applied to μ_1, this differential returns $\alpha + \delta_1$ (this is the $v_+ \longrightarrow v_+ \otimes v_- + v_- \otimes v_+$ case); applied to μ_i for $i > 1$, this differential returns δ_i (this is the $v_- \longrightarrow v_- \otimes v_-$ case). So, $d_c\mu = \beta_1 \pm \alpha$.

Now we need to show that nonzero edge differentials from crossings other than c come as cancelling pairs in μ, as was the case for β_1. Other than c, all 0-traces are mergetraces; 0-smoothings between two v_- loops will thus have zero edge differentials. The nonzero edge differentials will come from a v_+ loop joining with v_- loop along a 0-smoothing; each such edge differential will appear exactly twice, once from marking v_- on one loop and v_+ on the other, and the second time from choosing the opposite markings. Since the two loops in question are connected by a 0-trace in the Seifert state, the edge differentials appear with opposite sign in μ, similar to the case we had before with edge differentials from β_1. So, $d\mu = \beta_1 \pm \alpha$ as claimed.

Assume that $[\alpha]$ is trivial; then there exists γ so that $d\gamma = \alpha$. Similarly, $\gamma_1 = \mu \mp \gamma$ now satisfies $d\gamma_1 = \beta_1$. Both α and β_1 have quantum grading $s-1$, so the same holds for γ and γ_1. It follows that $d'\gamma = \alpha + v$ and $d'\gamma_1 = \beta_1 + v_1$ for v, v_1 of quantum grading $s+3$.

Consider $\mathfrak{s}_0 - d'(\gamma_0 + \gamma_1 + \gamma)$. This is homologous to $\mathfrak{s}_0$, so $s(\mathfrak{s}_0 - d'(\gamma_0 + \gamma_1 + \gamma)) = s - 1$. But, $\mathfrak{s}_0 - d'(\gamma_0 + \gamma_1 + \gamma) = \tau - v_0 - v_1 - v$, so it has no monomials of degree

$s-1$; therefore, $s(\mathfrak{s}_0) > s-1$, a contradiction. It follows that α could not be a trivial cycle. □

Remark 5.6.1. Regarding the hypothesis on the loop in the 1-block, if that loop were marked by v_-, the associated state cycle would necessarily be trivial by Proposition 4.2. Spelling this out, since this is a knot, the loop of the 1-block must be connected to some 0-tracing loop. By definition, these two loops are only connected by 1-traces, because if any 0-trace touched our loop in the 1-block, it would not be a loop of the 1-block. The 0-tracing loop must also be marked by v_-, so with the 1-block loop marked by v_- the hypotheses of Proposition 4.2 apply and the state cycle must be trivial.

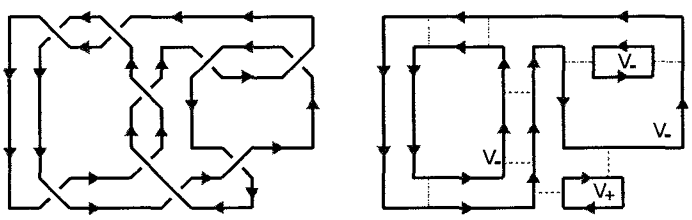

Figure 5.8: The Seifert state cycle for 9_{42} has a single loop in its 1-block, and quantum grading -1.

Example 5.1. The s invariant for 9_{42} is 0. For the usual Rolfsen diagram of 9_{42}, the state cycle associated to the Seifert state has quantum grading $-1 = s-1$ and a single loop in its 1-block, as seen in Figure 5.8. So, by Theorem 5.6, this state cycle is nontrivial.

Remark 5.6.2. Since the Seifert state is even, it is certainly 1-even, so there is some overlap between this result and Theorem 4.13. However, the assumption here on the quantum grading allows for state cycles which are not 1-isolated. For instance, in

Example 5.1, Λ_1 is connected and contains three vertices, two of which are marked by v_- (the top rightmost "island" in the state is the only loop which does not correspond to a vertex in Λ_1).

So, how about other state cycles based on the Seifert state with quantum grading $(0, s-1)$? There is a big jump in difficulty when one adds even a second loop marked by v_+, but we conjecture that this relationship holds in general:

Conjecture 5.7. *Suppose α is a state cycle associated to the Seifert state, and its quantum grading is $s-1$. Then α represents a nontrivial homology class.*

Chapter 6

Quasipositive Modification

In the last section, we saw a simple procedure, positive modification, that allowed one to lift a nontrivial state cycle of one diagram to a nontrivial state cycle of another diagram. A benefit of lifting from a state cycle to a state cycle is that the process can be iterated, allowing one to construct families of knots or links which each have nontrivial lifts of these cycles. This section will focus on a more interesting generalization of this procedure, quasipositive modification, which still offers the ability to lift state cycles to state cycles.

Roughly speaking, positive modification is the insertion of a positive crossing into a special place limited by the state cycle of interest. Similarly, quasipositive modification is the process of "gluing" in a quasipositive braid in a way that is compatible with a state cycle. A braid is *quasipositive* if, in terms of the braid group generators τ_i, it can be written in the form $\prod_k \omega_k \tau_{i_k} \omega_k^{-1}$, for some sequence of braid words ω_k and positive crossings τ_{i_k}. We will call the τ_{i_k} from such a presentation the *central positive crossings* of the braid word, to distinguish them from the positive crossings that might occur in the ω_i.

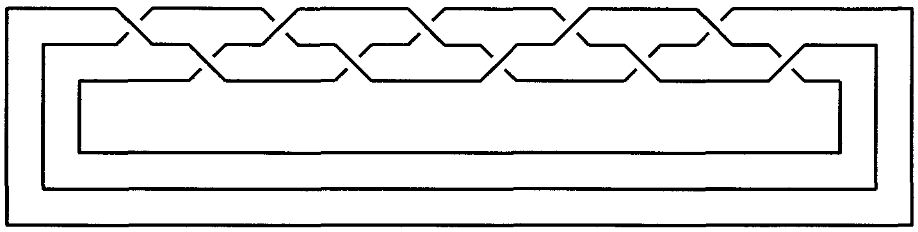

Figure 6.1: Closure of the quasipositive braid for the mirror image of 8_{20}.

Example 6.1. The mirror image of the knot 8_{20} has a diagram of the form $(\tau_1\tau_2\tau_1^{-1}\underline{\tau_2}\tau_1\tau_2^{-1}\tau_1^{-1})(\tau_2\underline{\tau_1}\tau_2^{-1})$. Here, the breakdown into each factor $\omega_i\tau_{u_k}\omega_i^{-1}$ is marked by parenthesis, with the central positive crossings underlined. See Figure 6.1.

For further examples, Baader has classified which prime knots up to 10 crossings can be realized as quasipositive braids, and Appendix A of [Baa05] has a full list of knots up to 10 crossings which are positive and quasipositive, with quasipositive braid words for those which are honestly quasipositive. Be aware that this table does not distinguish mirror images - usually a Jones polynomial calculation will establish that the braid in question is for the mirror image of the knot, as is the case here for 8_{20}.

When we glue in a quasipositive braid, orientation will be an issue, as we want the resulting diagram to keep the orientations of the crossings of the quasipositive braid. We will also want to place restrictions based on the state cycle we wish to lift. This leads to the next iteration of state-based definitions:

Definition 6.1. Let D be an oriented link diagram. An *Oriented Traced State*, or *OT State*, is a traced state for D, for which every arc between traces has been marked by

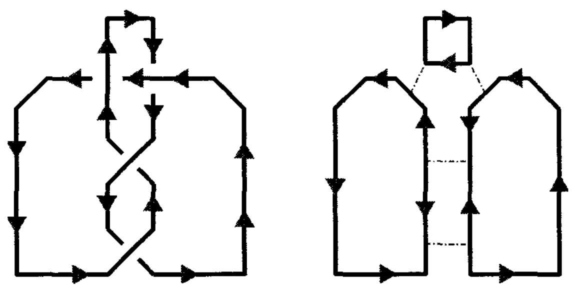

Figure 6.2: Oriented diagram for the figure 8, and the OT state for its all zero state.

the orientation of that arc in the original diagram. Similarly, an *Oriented Enhanced Traced State*, or *OET State*, is an ET state marked by this orientation information. If α is an ET state, denote the associated OET state by $< \alpha >$.

As shown in Figure 6.2, the orientations of the subarcs of a loop in an OT state will not in general give a consistent orientation for this loop. But, these oriented arcs will help specify the region we will be gluing in quasipositive braids, for quasipositive modification.

Definition 6.2. Let D be an oriented link diagram, and $< \sigma >$ an OET state for that diagram representing a nontrivial homology class. A collection of oriented arcs from $< \sigma >$, A_σ, is said to be *braid-parallel with respect to* $< \sigma >$ if:

1. Each of the arcs comes from separate loops in σ.
2. Each of the arcs is marked by v_- in σ.
3. The arcs can be joined by a line transverse to each which touches only the arcs of A_σ and meets each arc in the same orientation.

Roughly speaking, the braid-parallel arcs are what we will replace by a quasipositive braid, for quasipositive modification. The matching orientation of each arc agrees with how each braid strand is oriented in a braid, so that when replacing the braid-parallel arcs by the braid, positive and negative crossings in the braid will remain positive and negative in the new diagram. Alternatively, one can replace the transverse line by a "box" around the arcs, so that inside the box, the arcs look like the trivial braid on n strands. See Figure 6.3.

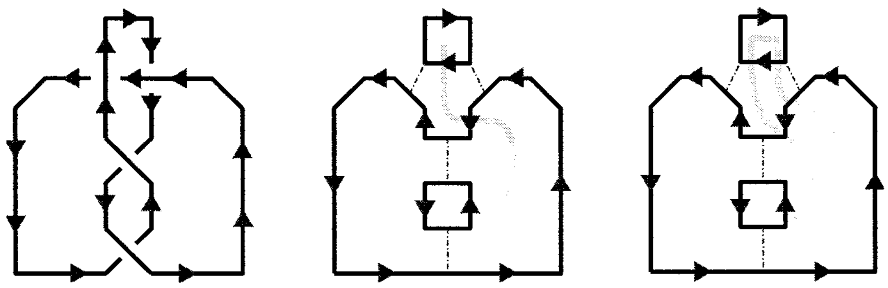

Figure 6.3: Oriented diagram for a figure-8 look-alike version of the positive trefoil, OT states for the all zero state, and a collection of 3 braid-parallel arcs marked by a transverse line, then a box.

Definition 6.3. Given oriented diagrams D and D', and an OET state $< \sigma >$ of D, we say that D' is a quasipositive modification on D compatible with $< \sigma >$ if there is a collection of arcs of D braid-parallel with respect to $< \sigma >$ and a quasipositive braid B on as many strands so that replacing the arcs in D with the quasipositive braid results in D'.

Example 6.2. Let D be the figure-8 look-alike diagram of the positive trefoil (same diagram as Figure 6.3). This is a + adequate diagram, so let α_0 be the ET state for the all-zero state where each loop is marked by v_-. Figure 6.3 has a collection of 3 arcs braid-parallel with respect to $< \alpha >$, so let D' be the diagram obtained by replacing those arcs with the quasipositive braid for the mirror of 8_{20}. Then D' is a quasipositive modification of D compatible with α. See Figure 6.4.

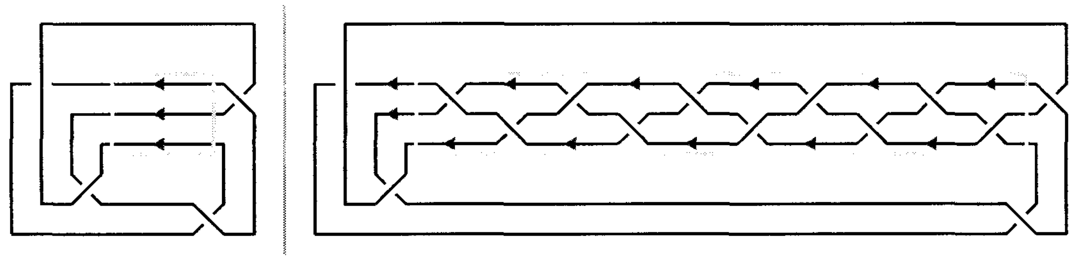

Figure 6.4: On the left, the figure-8 look-alike diagram of the positive trefoil is isotoped so that the braid-parallel arcs shown in Figure 6.3 are actually parallel. On the right is the quasipositive modification of this diagram by the mirror of 8_{20}.

Alternatively, one can view quasipositive modification as a procedure where one does a series of Reidemeister II moves followed by adding positive crossings, in such a way that removing the positive crossings yields a diagram that can be simplified, using Reidemeister II only, to the original diagram. The compatibility condition with a state cycle then limits where one can do the Reidemeister moves and add positive crossings; from this perspective, quasipositive modification is a series of positive modifications on a Reidemeister II modified diagram.

The first main benefit of quasipositive modification with respect to a nontrivial state cycle is that the state cycle can be lifted to a nontrivial state cycle for the new diagram, via an associated Jacobsson homomorphism.

Theorem 6.4. *Let D be an oriented diagram and $< \alpha >$ an OET state for a nontrivial state cycle. Suppose D' is gotten from D by quasipositive modification compatible with $< \alpha >$, and that Ψ is the associated Jacobsson homomorphism from $Kh(D')$ to $Kh(D)$. Then there exists a state cycle $\widetilde{\alpha}$ so that $\Psi(\widetilde{\alpha}) = \pm\alpha$. If B is the quasipositive braid associated to this modification, then $\widetilde{\alpha}$ is the ET state where:*

- *All crossings from D are smoothed as in α.*

- *Negative crossings from B are 1-smoothed, positive crossings from B are 0-smoothed.*

- *Every loop in $\widetilde{\alpha}$ is marked the same as α.*

Proof. The basic strategy here is to first "remove" the central positive crossings, via Lemma 5.2, then use Reidemeister II moves to get rid of the remaining conjugate pairs from the braid word. At each stage, the loops, and the markings on the loops, should remain the same: the only difference is that there are less traces from crossings, since we are progressively simplifying the diagram.

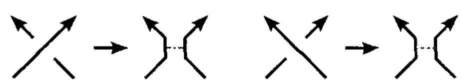

Figure 6.5: Illustration of how the oriented resolution looks at the ET state level for positive, negative crossings respectively.

The first thing to check is that $\widetilde{\alpha}$ makes sense: it needs to have the same underlying loop structure as α, with additional traces coming from all the extra crossings in D'. Because of the orientation condition on the collection of braid-parallel arcs, the braid segment from the quasipositive modification retains all the crossing signs of the original braid: positive crossings remain positive, negative crossings remain negative. Checking the cases shown in Figure 6.5, we see that the smoothing choice given for $\widetilde{\alpha}$ locally results in the original braid-parallel arcs, with extra traces coming from the additional crossings. So, the loops of $\widetilde{\alpha}$ are exactly the same as for α, and the marking choice gives us the entirety of an ET state.

All of the new crossings in D' go between the braid-parallel arcs, and the choice of resolutions has the traces of those crossings going between two arcs of the collection.

Since each arc comes from its own loop in α and $\widetilde{\alpha}$ has the same loop structure, each of the new traces are mergetraces in $\widetilde{\alpha}$ between loops marked by v_-. In particular, the new 0-traces are mergetraces between two loops marked by v_-, and the old 0-traces remain mergetraces between loops marked by v_-, since the underlying loop structure of $\widetilde{\alpha}$ is the same as that of α and none of the original traces have been altered. It follows from Proposition 3.2 that $\widetilde{\alpha}$ represents a state cycle.

At each step, whether we 0-resolve a central positive crossing, or perform a Reidemeister II move, the loop structure will remain the same: we will just be removing traces going from ET state to ET state. So, as long as the markings on the loops in question do not change, we will have a sequence of Jacobsson homomorphisms going from $\widetilde{\alpha}$ to $\pm\alpha$ as desired. The Positive Modification lemma (5.2) shows this holds for resolving the central positive crossings, so it suffices to check this situation holds for the Reidemeister II moves.

For the Reidemeister II moves, at each step we have a negative and positive crossing adjacent in the diagram. At the state level, these have been 1-smoothed and 0-smoothed, and the respective traces are both mergetraces going between the same two arcs. This puts us in the situation of the top left part of Figure 5.4, where we see that the markings of the two arcs remain the same after the Jacobsson homomorphism is applied, with a possible sign change on the resulting state cycle. So, the Reidemeister II homomorphisms act as claimed, and the resulting image of the last homomorphism will be $\pm\alpha$. $\square$

Remark 6.4.1. This is a generalization of Lemma 5.2; the corresponding result in

Plamenevskaya's case is that the contact invariant of a quasipositive braid is nontrivial (Corollary 1 of [Pla06]). A feature of our construction is that a quasipositive modification can be compatible with multiple nontrivial state cycles, a fact we will exploit in Chapter 7 to construct families of H-thick knots. Additionally, since it takes nontrivial state cycles to nontrivial state cycles, it is a process that can be iterated.

Suppose the braid associated to the quasipositive modification has k central positive crossings. Then the cobordism associated to this modification has Euler characteristic $-k$, so that the quantum grading of a lift is k higher than the quantum grading of the original state cycle. As was the case with positive modification, this shift is the same for every state cycle compatible with the quasipositive modification, because the Jacobsson homomorphism induced by cobordism Σ is a $(0, \chi(\Sigma))$ map:

Corollary 6.5. *Suppose $< \alpha >$ and $< \beta >$ are both compatible with the same quasipositive modification. Then the relative grading difference of α and β also holds for the lifts $\widetilde{\alpha}$ and $\widetilde{\beta}$. In particular, if α and β lie on distinct diagonals of $Kh(D)$, then $\widetilde{\alpha}$ and $\widetilde{\beta}$ lie on distinct diagonals of $Kh(D')$.*

In other words, if one can find nontrivial state cycles in D on n different diagonals which are compatible with a quasipositive modification, the lifts guarantee that D' will also have homological width at least n. Unfortunately, without more tools it can be difficult to find that many nontrivial state cycles.

However, another feature of quasipositive modification is that the associated Jacobsson homomorphism also lifts state cycle Lee pairs:

$$\begin{tikzcd}
 & \Phi(\widetilde{\alpha}) \arrow[dd, "\Psi_J"] \\
\widetilde{\alpha} \arrow[ur, "\Phi"] \arrow[dd, "\Psi_J"'] & \\
 & \Phi(\alpha) \\
\alpha \arrow[ur, "\Phi"] &
\end{tikzcd}$$

Theorem 6.6. *Let D be an oriented diagram, and $< \alpha >$ an OET state for a nontrivial state cycle of D. Suppose D' is a diagram gotten by quasipositive modification on D compatible with α, and that $\widetilde{\alpha}$ is the lift of α. Then $\Psi_J(\Phi_{Lee}(\widetilde{\alpha})) = \Phi_{Lee}(\Psi_J(\widetilde{\alpha})) = \Phi_{Lee}(\alpha)$.*

Proof. By Theorem 6.4 we have a state cycle lift of α, $\widetilde{\alpha}$. The first thing to consider is how Φ acts on these two state cycles. In both cases, every 0-trace is a mergetrace between two loops marked by v_-, so the edge maps from Φ will be of the form:

$$\mu_\Phi : \bigcirc \quad \bigcirc \longrightarrow \bigcirc$$

$$v_- \otimes v_- \longrightarrow v_+$$

Each of these edge maps targets a different state, so $\Phi(\alpha)$ and $\Phi(\widetilde{\alpha})$ will be a sum of ET states, each of which has a merged loop marked by v_+ corresponding to the associated 0-mergetrace.

Furthermore, regardless of whether the Lee spectral sequence converges at E_2 in D', $\Phi(\widetilde{\alpha})$ will still be a cycle, since Φ is a homomorphism on $Kh(D')$. We want to show that this maps down to $\pm\Phi(\alpha)$ under the Jacobsson homomorphism for the quasipositive modification. This can be broken down into three cases, by the kinds of terms in $\Phi(\widetilde{\alpha})$: image states coming from 0-traces of central positive crossings, image states coming from 0-traces of other positive crossings in the braid, and image states coming from 0-traces of α. The claim is that under the Jacobsson homomorphisms, the first two kinds of terms vanish, and the last terms survive to the equivalent terms of $\Phi(\alpha)$.

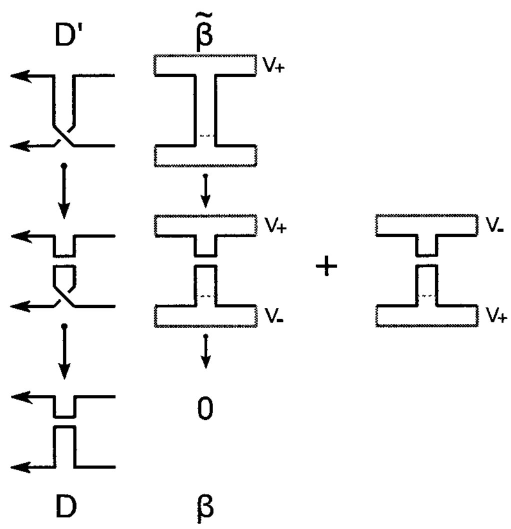

Figure 6.6: ET states and diagrams showing how a term coming from a central positive crossing vanishes under the Jacobsson homomorphism associated to 0-resolving that crossing.

Case 1: Suppose $\widetilde{\beta} \in Kh^{i,j}(D')$ is the Φ edge image of a 0-trace of a central positive crossing. Jacobsson homomorphisms commute up to sign, so it suffices to show $\widetilde{\beta}$ vanishes under the 0-resolution of this central positive crossing. As shown in Figure 6.6, this ET state survives the homomorphism for the 1-handle, but vanishes under the homomorphism for positive Reidemeister I. Consulting Figure 5.3, one sees that each state where the 1-trace of that positive crossing occurs is not listed, which means that it is sent to 0 under this Reidemeister I homomorphism.

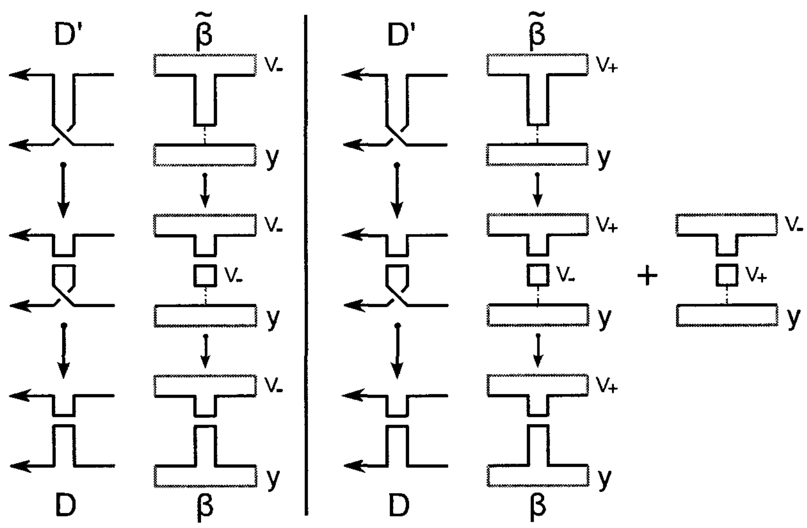

Figure 6.7: ET states and diagrams showing how ET states which have 0-traces for a positive crossing behave under 0-resolution of that positive crossing.

Case 2: Suppose $\widetilde{\beta} \in Kh^{i,j}(D')$ is the Φ edge image of a 0-trace of a noncentral positive crossing from the braid. Figure 6.7 shows that $\widetilde{\beta}$ is "preserved" under 0-resolution of every central positive crossing, in the sense that the underlying loop markings and state remain the same under each such Jacobsson homomorphism. To see that it vanishes under the Reidemeister II move which cancels this 1-smoothed crossing, consider the lefthand side of Figure 5.4. $\widetilde{\beta}$ will have 1-smoothed both the positive and negative crossing of this Reidemeister II pair, but the only nonzero images under the Jacobsson homomorphism come from 0-smoothing one of the two crossings and 1-smoothing the other. So, $\widetilde{\beta}$ vanishes under this Jacobsson homomorphism as claimed.

Case 3: Suppose $\widetilde{\beta} \in Kh^{i,j}(D')$ is the Φ edge image of a 0-trace not coming from the braid. As in Case 2, $\widetilde{\beta}$ is preserved under the 0-resolution of every central positive crossing. As for the Reidemeister II moves, since the 1-smoothed crossing does not occur on any of the Reidemeister II pairs, we fall into the top lefthand case of Figure 5.4, which preserves the markings of the loops involved each time the homomorphism is applied, up to sign. So, the net result is $\pm\Phi_e(\alpha)$, where e is the associated edge.

The last thing one must deal with is whether there are any global problems with all the sign changes that are taking place in the Jacobsson homomorphisms. But, each of the homomorphisms will apply the same sign change to all of the surviving terms, so that we really do have $\Psi(\Phi(\widetilde{\alpha})) = \pm\Phi(\alpha)$ as claimed. $\square$

Example 6.3. Suppose D is a quasipositive diagram for a knot of homological width less than 4, with at least one negative crossing. As seen in Proposition 5.4, if α_0 is the usual + adequate all-zero cycle, then α_0 and $\Phi(\alpha_0)$ are nontrivial homology classes. Therefore, $\widetilde{\alpha_0}$ and $\Phi(\widetilde{\alpha_0})$ are nontrivial homology classes in D'. This theorem thus provides a method of lifting the two adjacent diagonals these homology classes lie on, so that there is a diagonal that lies above the diagonal for $\widetilde{\alpha_0}$.

If D' remained + adequate, this would follow immediately from Khovanov's analysis of the Krull-Schmidt decomposition of the Khovanov chain complex in [Kho03], but D' as a diagram does not remain + adequate under quasipositive modification. Without knowing that $\widetilde{\alpha_0}$ is minimal in $Kh(D')$, one cannot *a priori* tell whether the second diagonal lies above or below the diagonal for $\widetilde{\alpha_0}$.

Furthermore, this process of lifting such Lee pairs can be iterated even when the resulting D' is not + adequate: for nontriviality of the lifts, all we require is that $[\alpha]$ and $[\Phi(\alpha)]$ are both nontrivial homology classes.

Chapter 7

Families of H-thick knots via Quasipositive Modification

In this section, we examine the state cycles of a diagram of the knot 9_{42} and exhibit how to perform several compatible quasipositive modifications. In this way, we construct several families of H-thick knots which cannot be detected by Khovanov's thickness criterion, and a sequence of prime knots and links related by quasipositive modification for which width is increasing. Finally, we discuss other potential sources of base knots for quasipositive modification.

7.1 The Base Knot 9_{42}

9_{42} is an H-thick knot, and this subsection will show three homology classes, one in each of its diagonals, which we can lift by quasipositive modification. Table 7.1 lists its rational Khovanov homology according to the Knot Atlas [BNM], with the three homology classes we will lift marked by shading.

	-4	-3	-2	-1	0	1	2
7							1
5							
3					1	1	
1				1	1		
-1				1	1		
-3		1	1				
-5							
-7	1						

Table 7.1: Rational Khovanov homology of 9_{42}. Homology classes which will be lifted under quasipositive modification have been shaded.

The first thing to note is that the "usual" diagram from Bar-Natan's Knot Atlas [BNM] for 9_{42} is + adequate; an isotoped form of this diagram is shown in Figure 7.1. This gives us a state cycle representative, α_0, for the minimal quantum and homological graded entry of its rational Khovanov homology, (-4, -7). Following Example 6.3, this gives us a second nontrivial, liftable homology class from its Lee pair at bigrading (-3,-3), since 9_{42} has width 3.

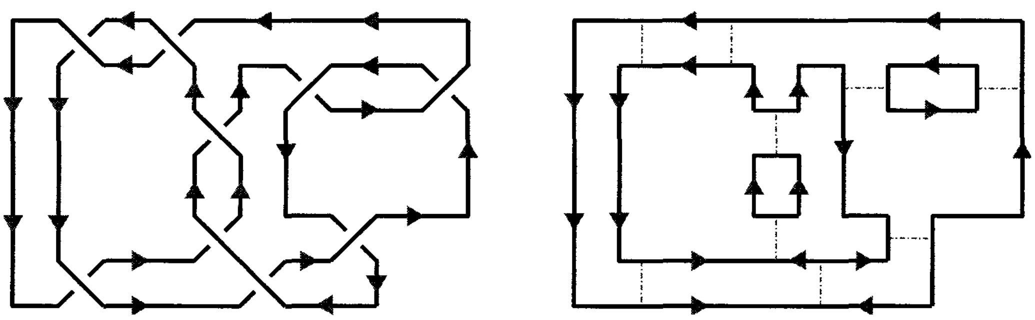

Figure 7.1: Usual diagram for 9_{42} and the all-zero OT state, showing this diagram is + adequate.

For the third homology class, we would like to find a state cycle representative for the "thick" diagonal of 9_{42}'s homology. There is only one nontrivial homology entry for that diagonal, at bigrading (0, -1). The s-invariant of 9_{42} is 0, which would give us a representative for this class if we had an explicit form of this generator. But, we lack a general form for this generator in terms of the Khovanov chain generators.

However, the "Seifert state", gotten by smoothing the diagram according to the rules for constructing the canonical Seifert surface, is an adequate state. And, marking it as a state cycle, with v_- on the 0-tracing loops and v_+ on the 1-block, gives a state cycle α_s of the desired bigrading, (0, -1). Since there is only one loop in its 1-block, and its quantum grading is $s - 1$, Theorem 5.6 guarantees this state cycle is nontrivial. Figure 7.2 shows the associated OET state for this state cycle. The ET state has marking $v_- \otimes v_- \otimes v_- \otimes v_+$, 4 loops, and 4 1-smoothings in a 9 crossing diagram with 4 negative and 5 positive crossings, for those wanting to verify the bigrading.

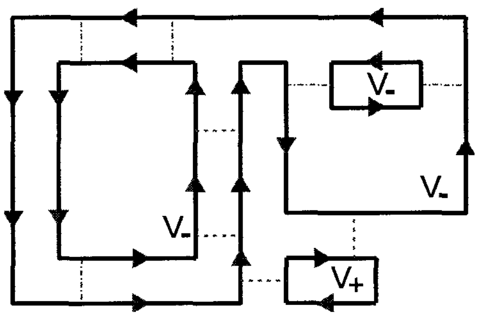

Figure 7.2: OET for α_s, the Seifert state cycle for 9_{42}.

To lift these three homology classes, we need to choose a set of braid-parallel arcs compatible with both state cycles, and a quasipositive braid. The next two subsections will do this for two variants of this diagram, and analyze two families based on these choices of braid-parallel arcs.

7.2 Adding Positive Twists

Looking over the OET states for α_0 and α_s, it is easy to find pairs of braid-parallel arcs compatible with both states: Figure 7.3 shows the pair we will examine here. Since there are only two arcs, whatever quasipositive braid we would use for the modification will in fact be Reidemeister equivalent to the braid word τ_1^n: n positive half-twists between those two strands. By the quasipositive modification theorems, this gives a simple family of thick links K_n, where n is the number of half-twists.

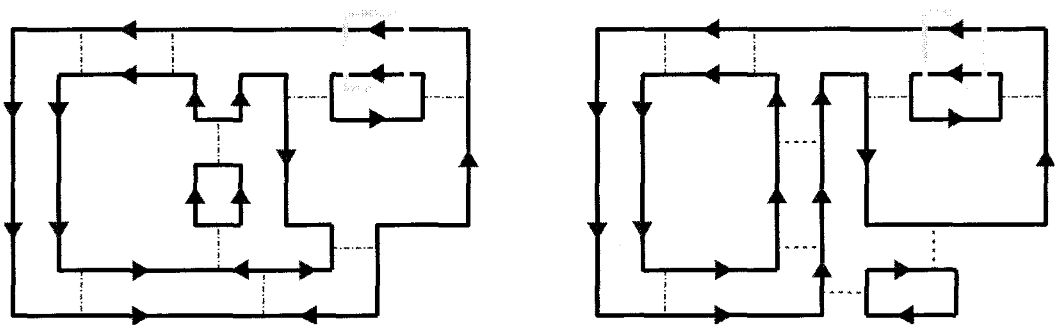

Figure 7.3: OET states for α_0 and α_s, with a pair of compatible braid parallel arcs marked.

Using SnapPea [Wee], one can check that the surgery link associated to this positive twisting is hyperbolic. Using Thurston's hyperbolic Dehn surgery theorem [Thu02], it follows that all but a finite number of such K_n must be hyperbolic. So, all but a finite number of the K_{2n} must be prime knots, showing that this family cannot be obtained by taking connect sums of something with a thick knot.

Furthermore, adding positive twists lends itself well to a recursive formula for the Jones polynomial of these knots. Recall that the skein formula for the Jones polynomial $V(L)$ is given by:

$$q^{-1}V(L_+) - qV(L_-) = (q^{1/2} - q^{-1/2})V(L_o) \tag{7.1}$$

Here, L_+, L_- and L_o are the link where a crossing is circled and replaced by a positive crossing, a negative crossing, and the oriented smoothing, respectively. To apply this to our K_n, note that changing one of the added positive half-twists to a negative half-twist will cancel the next positive half-twist, leaving K_{n-2}, while the oriented resolution of a positive half-twist just reduces the number of positive half-twists by 1, yielding K_{n-1}. So, we can rewrite (7.1) as the following recursion:

$$V(K_n) = q^2 V(K_{n-2}) + (q^{3/2} - q^{1/2})V(K_{n-1}) \tag{7.2}$$

Using KnotTheory' [BNMG], one can verify the following calculations, which begin the recursion and suggest that the Jones polynomials of the K_{2n} knots is alternating:

$$V(K_0) = q^3 - q^2 + q - 1 + q^{-1} - q^{-2} + q^{-3}$$

$$V(K_1) = q^{9/2} - q^{7/2} + q^{5/2} - 2q^{3/2} + q^{1/2} - 2q^{-1/2} + q^{-3/2} - q^{-5/2}$$

$$V(K_2) = q^6 - q^5 + q^4 - 2q^3 + 2q^2 - 2q + 2 - q^{-1} + q^{-2}$$

$$V(K_3) = q^{15/2} - q^{13/2} + q^{11/2} - 2q^{9/2} + 2q^{7/2} - 3q^{5/2} + 2q^{3/2} - 2q^{1/2} + q^{-1/2} - q^{-3/2}$$

$$V(K_4) = q^9 - q^8 + q^7 - 2q^6 + q^5 - 3q^4 + 3q^3 - 2q^2 + q - 1 + q^{-1}$$

In fact, this pattern continues, leading to:

Theorem 7.1. *The Jones polynomial of the knots K_{2n} is alternating, in the following sense: if $V(K_{2n}) = \sum a_t q^t$, then either all the a_{2k} terms are positive and the a_{2k+1} terms negative, or vice versa, with no zero coefficients within* $\mathrm{Span}(V(K_{2n}))$.

Proof. The proof is by induction on the number of half-twists. For convenience, we will consider everything in terms of the renormalization $P(K_n)(x) = V(K_n)(x^2)$: this way we can describe the odd n case without worrying about all the half-powers. In this renormalization, (7.2) becomes:

$$P(K_n) = x^4 P(K_{n-2}) + (x^3 - x)P(K_{n-1}) \tag{7.3}$$

Rewriting the base examples in terms of this renormalization, we have:

$$P(K_0) = x^6 - x^4 + x^2 - 1 + x^{-2} - x^{-4} + x^{-6}$$

$$P(K_1) = x^9 - x^7 + x^5 - 2x^3 + x - 2x^{-1} + x^{-3} - x^{-5}$$

$$P(K_2) = x^{12} - x^{10} + x^8 - 2x^6 + 2x^4 - 2x^2 + 2 - x^{-2} + x^{-4}$$

$$P(K_3) = x^{15} - x^{13} + x^{11} - 2x^9 + 2x^7 - 3x^5 + 2x^3 - 2x + x^{-1} - x^{-3}$$

$$P(K_4) = x^{18} - x^{16} + x^{14} - 2x^{12} + x^{10} - 3x^8 + 3x^6 - 2x^4 + x^2 - 1 + x^{-2}$$

Furthermore, as a consequence of the skein relations for the Jones polynomial, we know that $P(K_{2n})$ will have only even powers of x, while $P(K_{2n+1})$ will only have odd powers, since the respective link familes have an odd and even number of components, respectively. To account for these two cases at once, we will consider the following induction hypothesis:

Induction Hypothesis. For all $n \leq k$, $P(K_n)$ has leading coefficient $+1$; if the leading power is t, then coefficients a_j^n of $P(K_n)$ will be positive for $j \equiv t \pmod 4$, and negative for $j \equiv t+2 \pmod 4$. Coefficients of powers i within $\mathrm{Span}(P(K_{k+1}))$ so that $i \equiv t \pmod 2$ are nonzero. The leading power t of $P(K_n)$ will be $3(n+2)$, and the trailing power will be $n-6$.

We will build this up by a few other inductive claims first.

Claim 7.1.1. $P(K_n)$ has leading power $3(n+2)$ and leading coefficient $+1$.

By inspection, it is clear this holds for the base cases $P(K_0)$ and $P(K_1)$. So, assume everything holds for $n \leq k$, and consider $P(K_{k+1})$. The leading powers of $P(K_k)$ and $P(K_{k-1})$ are $3k+6$ and $3k+3$ respectively, so by the skein formula, the highest powers each will contribute to $P(K_{k+1})$ are $3+(3k+6) = 3k+9$ and $4+(3k+3) = 3k+7$ respectively. Clearly $3k+9$ will be the highest power of the result, and because the sole such term comes from $P(K_k)$ and is multiplied by $+1$, the resulting coefficient will remain $+1$ by induction.

Claim 7.1.2. $P(K_n)$ has trailing power $n-6$ and trailing coefficient $(-1)^n$.

By inspection, this holds for $P(K_0)$ and $P(K_1)$. Now, assume this holds for all $n \leq k$. Observe that $-xP(K_k)$ will contribute lowest power $k-5$, while $x^4P(K_{k-1})$ will contribute lowest power $k-3$. So, the trailing term will come from $-xP(K_k)$, and the coefficient will be -1 multiplied by the trailing coefficient of $P(K_k)$, namely $(-1) * (-1)^k = (-1)^{k+1}$.

Claim 7.1.3. Denote the coefficient of x^j in the expansion of $P(K_n)$ by a_j^n. Then

$\|a_j^n\| \geq \|a_{j-3}^{n-1}\|$.

Proof. Inspecting the first few examples, it is clear this holds for the base cases. To account for the general case, we need to rewrite the recursion in terms of the coefficients. For a fixed power x^j and polynomial $P(K_n)$, (7.3) becomes:

$$a_j^n = a_{j-4}^{n-2} + a_{j-3}^{n-1} - a_{j-1}^{n-1} \tag{7.4}$$

So, assume $\|a_j^n\| \geq \|a_{j-3}^{n-1}\|$ holds for $n < k$. Expanding by (7.4), we get:

$$\|a_j^k\| = \|a_{j-4}^{k-2} + a_{j-3}^{k-1} - a_{j-1}^{k-1}\| \tag{7.5}$$

We want to compare this to $\|a_{j-3}^{k-1}\|$, which appears as a summand in (7.5). Let's first analyze the relative sign distribution inside the absolute value sign: we will normalize so that a_{j-3}^{k-1} is positive, multiplying the interior of the absolute value by -1 if needed. a_{j-1}^{k-1} is a coefficient from the same polynomial which differs by an x^2, so it must appear with the opposite sign from a_{j-3}^{k-1}. So, $-a_{j-1}^{k-1}$ is positive when a_{j-3}^{k-1} is positive.

On the other hand, a_{j-4}^{k-2} comes from $P(K_{k-2})$, an x^3 factor lower term than a_{j-1}^{k-1}, which is negative. We know from Claim 7.1.1 that the leading power of $P(K_{k-1})$ is $3k+3$, while that of $P(K_{k-2})$ is $3k$. So, a_{j-4}^{k-2} must match the negative sign of a_{j-1}^{k-1}. This means that within the absolute value sign of (7.5), when a_{j-3}^{k-1} and $-a_{j-1}^{k-1}$ are both positive terms, a_{j-4}^{k-2} will be negative.

The upshot is that if $\|a_{j-1}^{k-1}\| \geq \|a_{j-4}^{k-2}\|$, then $a_{j-4}^{k-2} - a_{j-1}^{k-1} \geq 0$. In such a case,

$a^{k-2}_{j-4} - a^{k-1}_{j-1}$ and a^{k-1}_{j-3} have the same sign under the absolute value sign, giving us the following equality:

$$||a^{k-2}_{j-4} + a^{k-1}_{j-3} - a^{k-1}_{j-1}|| = ||a^{k-2}_{j-4} - a^{k-1}_{j-1}|| + ||a^{k-1}_{j-3}|| \tag{7.6}$$

$$\geq ||a^{k-1}_{j-3}|| \tag{7.7}$$

But, $||a^{k-1}_{j-1}|| \geq ||a^{k-2}_{j-4}||$ is simply an index shift of the induction hypothesis, completing the proof of this claim. □

We're now ready to tackle the last part of the induction. The claims have already dealt with the leading and trailing coefficients and powers, so we have only to show the alternating signs of the coefficients.

Note first that all the terms from the $x^3P(K_k)$ expansion will match the parity of the terms from the $-xP(K_k)$ expansion, since x^3 and $-x$ differ by an even power. However, all terms from the expansion of $x^4P(K_{k-1})$ will have the opposite sign parity, since the leading power is $3k+7$, which differs by 2 from the lead power of $x^3P(K_k)$.

To account for this discrepancy, we need to show that, for each power, we get a larger coefficient sum from $(x^3-x)P(K_k)$ than from $x^2P(K_{k-1})$. As in Claim 7.1.3, we will show this on a coefficient by coefficient level. We want to show that the coefficient of x^j in the expression $(x^3-x)P(K_k)$ is greater in absolute value than that of x^k in the expansion of $x^2P(K_{k-1})$. At the coefficient level, this becomes:

$$||a^{k}_{j-3} - a^{k}_{j-1}|| > ||a^{k-1}_{j-4}|| \tag{7.8}$$

We then expand the lefthand side of this using the recursion relation, getting:

$$\|a^k_{j-3} - a^k_{j-1}\| \tag{7.9}$$

$$=\|(a^{k-2}_{j-7} + a^{k-1}_{j-6} - a^{k-1}_{j-4}) - (a^{k-2}_{j-5} + a^{k-1}_{j-4} - a^{k-1}_{j-2})\| \tag{7.10}$$

$$=\|a^{k-1}_{j-6} - 2a^{k-1}_{j-4} + a^{k-1}_{j-2} + a^{k-2}_{j-7} - a^{k-2}_{j-5}\| \tag{7.11}$$

Now, consider the relative parity of each term in (7.11). We would like to compare this to a^{k-1}_{j-4}, so let's assume that coefficient is negative (if it were positive, we could multiply the whole sum in the absolute value by -1). Clearly, $-2a^{k-1}_{j-4}$ will be positive with this sign choice, and ultimately we would like to break off one of these copies as a separate absolute value part, as we did in Claim 7.1.3.

a^{k-1}_{j-6} differs in power from this by 2, so it has the opposite sign, becoming positive. The same thing happens to a^{k-1}_{j-2}. In contrast, a^{k-2}_{j-7} comes from the next lower polynomial, and differs by a power of 3, so it has the same negative sign as a^{k-1}_{j-4}. On the other hand, a^{k-2}_{j-5} differs from a^{k-2}_{j-7} by a power of 2, so must be positive: this makes $-a^{k-2}_{j-5}$ a negative number. In summary, our sign choice yields the following parity in the absolute value sign:

$$\|\underbrace{a^{k-1}_{j-6}}_{+} - \underbrace{2a^{k-1}_{j-4}}_{-} + \underbrace{a^{k-1}_{j-2}}_{+} + \underbrace{a^{k-2}_{j-7}}_{-} - \underbrace{a^{k-2}_{j-5}}_{+}\| \tag{7.12}$$

Termwise, the left three terms become positive, while the right two are negative. If we are to use our earlier trick of separating out one of the "positive" terms, $-a^{k-1}_{j-4}$, we need to check that the remaining difference remains "positive". Namely, we need to show:

$$||a_{j-6}^{k-1} - a_{j-4}^{k-1} + a_{j-2}^{k-1}|| \geq ||a_{j-7}^{k-2} - a_{j-5}^{k-2}|| \tag{7.13}$$

But, by induction using (7.8), we already know that:

$$||a_{j-6}^{k-1} - a_{j-4}^{k-1}|| > ||a_{j-7}^{k-2}|| \tag{7.14}$$

And, Claim 7.1.3 gives us the remaining piece, suitably shifted:

$$||a_{j-2}^{k-1}|| \geq ||a_{j-5}^{k-2}|| \tag{7.15}$$

Now, thanks to all the lefthand terms of (7.13) being positive, we can break apart the absolute value with equality, and then apply (7.14) and (7.15) to verify the inequality of (7.13):

$$||a_{j-6}^{k-1} - a_{j-4}^{k-1} + a_{j-2}^{k-1}|| \tag{7.16}$$

$$=||a_{j-6}^{k-1} - a_{j-4}^{k-1}|| + ||a_{j-2}^{k-1}|| \tag{7.17}$$

$$>||a_{j-7}^{k-2}|| + ||a_{j-5}^{k-2}|| \tag{7.18}$$

$$=||a_{j-7}^{k-2} - a_{j-5}^{k-2}|| \tag{7.19}$$

The very last equality comes, again, from our earlier sign analysis in (7.12). Now, since (7.13) holds, we can break up our earlier absolute value from (7.11) with equality, obtaining:

$$||a^k_{j-3} - a^k_{j-1}|| \tag{7.20}$$

$$=||a^{k-1}_{j-6} - 2a^{k-1}_{j-4} + a^{k-1}_{j-2} + a^{k-2}_{j-7} - a^{k-2}_{j-5}|| \tag{7.21}$$

$$=||a^{k-1}_{j-6} - a^{k-1}_{j-4} + a^{k-1}_{j-2} + a^{k-2}_{j-7} - a^{k-2}_{j-5}|| + ||a^{k-1}_{j-4}|| \tag{7.22}$$

$$>||a^{k-1}_{j-4}|| \tag{7.23}$$

This verifies (7.8), completing the claim that the coefficients of $P(K_{k+1})$ alternate in the appropriate fashion. The remaining claim to verify is that the appropriate coefficients within $\mathrm{Span}(P(K_{k+1}))$ are nonzero. This follows largely from Claim 7.1.3: this tells us that coefficients of powers 3 higher than nonzero coefficients of $P(K_k)$ will be nonzero, since their magnitude is greater than that of nonzero coefficients of $P(K_k)$. This covers coefficients of powers $(3k+6)+3) = 3k+9$ down to $(k-6)+3 = k-3$, leaving only the trailing coefficient, of x^{k-5} in question. But by Claim 7.1.2, we know this coefficient is nonzero. □

Corollary 7.2. *K_{2n} cannot be detected as thick using the alternating Jones polynomial test of Khovanov.*

Corollary 7.3. $\mathrm{Span}(V(K_{2n})) = 2n + 6$

Proof. In terms of x coefficients, we know the leading power of $P(K_{2n})$ is $6n + 6$ and the trailing coefficient is $2n - 6$, so $\mathrm{Span}(P(K_{2n})) = 4n + 12$. It follows from the change of variables that $\mathrm{Span}(V(K_{2n})) = 2n + 6$. □

Theorem 7.4. *For each n, K_{2n} is not an adequate knot.*

Proof. Since the Jones polynomial of these knots begins and ends with ± 1, we need to analyze the Kauffman polynomial to determine adequacy conditions. What we will show is that K_{2n} is not - adequate, and hence admits no adequate diagram.

Following the conventions of Thistlethwaite and Stoimenow, the Kauffman polynomial we consider will differ from that listed on Bar-Natan's Knot Atlas by the substitution $a \to a^{-1}$. Recall that the Kauffman polynomial $F(K)(a, z)$ is defined by the following relations, where D is the chosen diagram for K, D_+ is a diagram where a circled crossing is positive, D_- is that same diagram with the crossing replaced by a negative crossing, D_0 replaces the crossing by a 0-smoothing, and D_1 replaces it by a 1-smoothing:

$$F(K)(a, z) = a^{w(D)}\Lambda(D) \tag{7.24}$$

$$\Lambda(D_+) + \Lambda(D_-) = z(\Lambda(D_0) + \Lambda(D_1)) \tag{7.25}$$

$$\Lambda(\text{positive RI twist}) = a^{-1}\Lambda(|) \tag{7.26}$$

$$\Lambda(\text{negative RI twist}) = a\Lambda(|) \tag{7.27}$$

$$\Lambda(\bigcirc) = 1 \tag{7.28}$$

Looking over the ingredients of (7.25), we know that if D_+ focuses on one of the positive half-twists of K_n, then as with the Jones recursion, D_- is K_{n-2}, and D_0 is

K_{n-1}. D_1 is the new quantity to understand, but it turns out that 1-smoothing the leftmost half-twist results in the mirror of 8_{19} with n negative Reidemeister I twists added. This leads to the following Λ recursion:

$$\Lambda(K_n) = -\Lambda(K_{n-2}) + z\Lambda(K_{n-1}) + za^n\Lambda(8_{19}!) \tag{7.29}$$

The writhe of our diagram for K_n is easily seen to be $1+n$, as 9_{42} has 4 negative crossings, and 5 positive crossings in its usual diagram. We will be concerned with coefficients of $F(D)$, since this is a knot invariant and can give obstructions to the knot being - adequate. So, translating (7.29) into the Kauffman polynomial, we obtain:

$$F(K_n) = -a^{1+n}\Lambda(K_{n-2}) + za^{1+n}\Lambda(K_{n-1}) + za^{1+2n}\Lambda(8_{19}!) \tag{7.30}$$

Let $[F(D)]_{(m,l)}$ denote the coefficient of $z^m a^l$ in $F(D)$. By (2.10) of Stoimenow [Sto07], if $l - m$ is the maximum integer so that $[F(D)]_{(m,l)}$ is nonzero, then if any such coefficient is negative, K cannot be - adequate (note that Stoimenow's A-semiadequate corresponds to our convention for - adequate). So, for the recursion, we just need to see how these maximal terms carry over for each term.

For our base case, KnotTheory' [BNMG] can be used to show that $F(K_0)$ is given by:

$$
\begin{aligned}
F(K_0) =& a^{-1}z^7 + z^7a + a^-2z^6 + z^6a^2 + 2z^6 - 5a^{-1}z^5 - 5z^5a - 5a^{-2}z^4 \\
& - 5z^4a^2 - 10z^4 + 6a^{-1}z^3 + 6z^3a + 6a^{-2}z^2 + 6z^2a^2 + 12z^2 - 2a^{-1}z \\
& - 2za - 2a^{-2} - 2a^2 - 3
\end{aligned}
$$

So, the maximum $l - m$ term here is $-2a^2$, and the knot is not - adequate. The writhe for the standard diagram is $+1$, so dividing by a, one gets that the maximum $l - m$ term for $\Lambda(K_0)$ is $-2a$. This will prove useful for the recursion.

Next, the Kauffman bracket for K_1 is:

$$
\begin{aligned}
F(K_1) =& 3 + 3a^2 + a^4 - 2a^{-1}z^{-1} - 3az^{-1} - a^3z^{-1} + 8a^{-1}z + 15az \\
& + 8a^3z + a^5z - 2z^2 - 7a^2z^2 - 5a^4z^2 - 11a^{-1}z^3 - 27az^3 - 16a^3z^3 \\
& - 5z^4 + 5a^4z^4 + 6a^{-1}z^5 + 17az^5 + 11a^3z^5 + 5z^6 + 4a^2z^6 - a^4z^6 \\
& - a^{-1}z^7 - 3az^7 - 2a^3z^7 - z^8 - a^2z^8
\end{aligned}
$$

The maximum $l - m$ terms are $a^4 - a^3z^{-1} + a^5z$. One of the terms is negative, so this link is also not - adequate. The corresponding maximum $l - m$ terms from $\Lambda(K_1)$ are given by $a^2 - az^-1 + a^3z$.

For the recursion, one can calculate that $\Lambda(8_{19}!)$ is:

$$\Lambda(8_{19}!) = -5 - 5a^{-2} - a^2 + 5a^{-1}z + 5az + 10z^2 + 10a^{-2}z^2 - 5a^{-1}z^3$$
$$- 5az^3 - 6z^4 - 6a^{-2}z^4 + a^{-1}z^5 + az^5 + z^6 + a^{-2}z^6$$

Thus, the term for which $l-m$ is maximal is $-a^2$, which has a negative coefficient. We can now use these values to examine how the maximal $l-m$ terms behave under the recursion, for both Λ and F:

Claim 7.4.1. For $n \geq 2$, the maximal $l-m$ term for $F(K_n)$ is $-za^{2n+3}$, and the maximal $l-m$ term for $\Lambda(K_n)$ is $-za^{n+2}$.

Proof. Proof of the claim is a simple induction, using the recursion and the base values established above. For our base case $n=2$, the recursion tells us that:

$$\begin{aligned}\max_{l-m} F(K_2) &\geq -a^3 \max_{l-m} \Lambda(K_0) + za^3 \max_{l-m} \Lambda(K_1) + za^5 \max_{l-m} \Lambda(8_{19}!) \\ &= -a^3(-2a) + za^3(za^3 - az^{-1} + a^2) + za^5(-a^2) \\ &= 2a^4 + z^2a^6 - a^4 + za^5 - za^7\end{aligned}$$

So, the maximum $l-m$ term for $F(K_2)$ is $-za^7$, and dividing by the writhe, the maximum $l-m$ term for $\Lambda(K_2)$ is $-za^4$, which matches the claim. We need to analyze K_3 also before the induction:

$$\max_{l-m} F(K_3) \geq -a^4 \max_{l-m} \Lambda(K_1) + za^4 \max_{l-m} \Lambda(K_2) + za^7 \max_{l-m} \Lambda(8_{19}!)$$
$$= -a^4(a^2 - az^{-1} + a^3 z) + za^4(-za^4) + za^7(-a^2)$$
$$= -a^6 + a^5 z^{-1} - a^7 z - z^2 a^8 - za^9$$

Again, the $-za^9$ term is maximal for $F(K_3)$, and correspondingly $-za^5$ is maximal for $\Lambda(K_3)$. We can now catch the rest of the cases by induction:

$$\max_{l-m} F(K_n) \geq -a^{1+n} \max_{l-m} \Lambda(K_{n-2}) + za^{1+n} \max_{l-m} \Lambda(K_{n-1}) - za^{2n+3}$$
$$= -a^{1+n}(-za^n) + za^{1+n}(-za^{n+1}) - za^{2n+3}$$
$$= za^{2n+1} - z^2 a^{2n+2} - za^{2n+3}$$

So, the maximal $l - m$ term for $F(K_n)$ is $-za^{2n+3}$; dividing by the writhe $(1+n)$, we get that the maximal $l - m$ term for $\Lambda(K_n)$ is $-za^{n+2}$ as claimed. □

The claim tells us that the maximal $l - m$ term for the Kauffman polynomial of every K_n has a negative coefficient, so the corresponding knot or link cannot admit a - adequate, and hence adequate, diagram. □

7.3 Setup for modification by a 3-braid

For a more complicated example of quasipositive modification, we need to find a trio of braid-parallel arcs. While there is no such trio in the previous diagram, we can do a positive stabilization to get a new + adequate diagram for 9_{42} which now has a set of three braid-parallel arcs compatible with the two state cycles. See Figure 7.4.

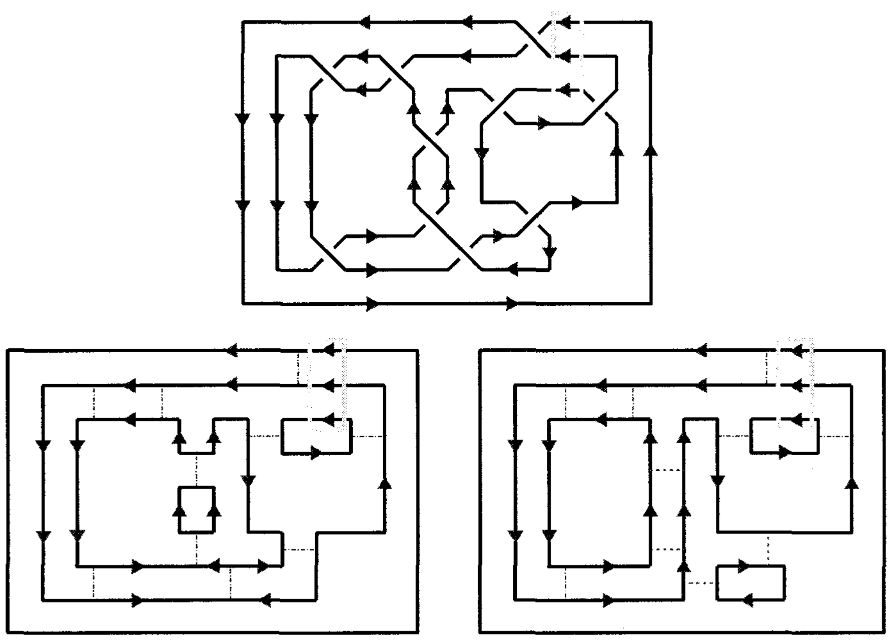

Figure 7.4: Positive stabilization of 9_{42} and the new OET states for α_0 and α_s, with a trio of braid parallel arcs marked.

Any quasipositive 3-braid can now be glued in for a quasipositive modification. This gives a lot of variety in kinds of H-thick knots and links one can construct by modification of 9_{42}; in the next two subsections, we will consider two such families.

7.4 Conjugating a Positive Crossing

For another family of thick knots which are prime, not adequate, and have alternating Jones polynomial, we can simply conjugate a positive crossing with a 3-braid word of choice. For this example, we will conjugate by $w_1 = \tau_1\tau_2\tau_1^{-1}$, to modify by the quasi-positive braid word $q_1 = \tau_1\tau_2\tau_1^{-1}\tau_2\tau_1\tau_2^{-1}\tau_1^{-1}$, choosing the same 3 braid-parallel arcs as in Figure 7.4. Note that inserting multiple copies of this braid word consecutively is isotopic by Reidemeister II cancellation to conjugating multiple positive crossings by w_1.

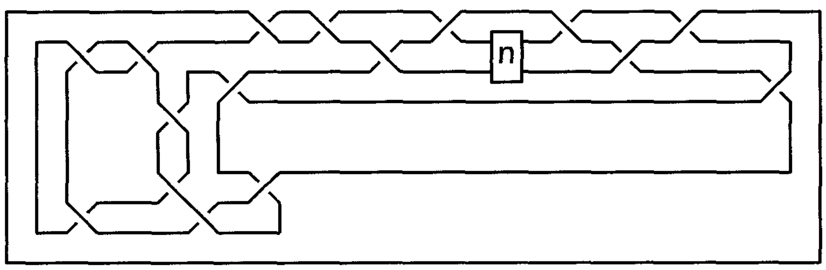

Figure 7.5: Diagram of $K(9_{42}, q_1, n)$; the n represents n positive half-twists.

So, consider the family formed by successively doing quasipositive modification of 9_{42} by the 3-braid word q_1, denoted $K(9_{42}, q_1, n)$. It turns out that this family is closely related to that of Subsection 7.2, though comparison of the Jones polynomial suggests these families are distinct.

Using SnapPea [Wee], one can again check that the surgery link associated to this positive twisting is hyperbolic. Using Thurston's hyperbolic Dehn surgery theorem [Thu02], it follows that all but a finite number of such $K(9_{42}, q_1, n)$ must be hyperbolic. Hence almost every member is prime. Next, calculating the Jones polynomial via KnotTheory' [BNMG] for the first three members of the family, one sees that their

Jones polynomials are alternating:

$$V(K(9_{42},q_1,1)) = -2q^{-5/2} + 3q^{-3/2} - 5q^{-1/2} + 5q^{1/2} - 6q^{3/2} + 6q^{5/2} - 5q^{7/2} + 4q^{9/2} - 2q^{11/2} + q^{13/2} - q^{15/2}$$

$$V(K(9_{42},q_1,2)) = 2q^{-2} - 4q^{-1} + 7 - 9q + 10q^2 - 11q^3 + 10q^4 - 8q^5 + 6q^6 - 3q^7 + 2q^8 - q^9$$

$$V(K(9_{42},q_1,3)) = -2q^{-3/2} + 4q^{-1/2} - 8q^{1/2} + 11q^{3/2} - 14q^{5/2} + 15q^{7/2} - 15q^{9/2} + 13q^{11/2} - 10q^{13/2} + 7q^{15/2} - 4q^{17/2} + 2q^{19/2} - q^{21/2}$$

This pattern holds in general, yielding:

Theorem 7.5. *The Jones polynomial of $K(9_{42},q_1,n)$ is alternating, with no gaps.*

Proof. The proof is almost the same as that of Theorem 7.1. Doing the change of variables, the Jones polynomials of the first three members can be renormalized to:

$$P(K(9_{42},q_1,1)) = -2x^{-5} + 3x^{-3} - 5x^{-1} + 5x^1 - 6x^3 + 6x^5 - 5x^7 + 4x^9 - 2x^{11} + x^{13} - x^{15}$$

$$P(K(9_{42},q_1,2)) = 2x^{-4} - 4x^{-2} + 7 - 9x^2 + 10x^4 - 11x^6 + 10x^8 - 8x^{10} + 6x^{12} - 3x^{14} + 2x^{16} - x^{18}$$

$$P(K(9_{42}, q_1, 3)) = -2x^{-3} + 4x^{-1} - 8x^1 + 11x^3 - 14x^5 + 15x^7 - 15x^9 + 13x^{11} - 10x^{13}$$
$$+ 7x^{15} - 4x^{17} + 2x^{19} - x^{21}$$

The only difference of note is the recursion for the leading and trailing powers and coefficients: in this family, the leading power is $12 + 3n$ and leading coefficient is -1; the trailing power is $n - 6$ and trailing coefficient is $(-1)^n * 2$. Otherwise, the proofs of the corresponding claims from Theorem 7.1 follow without change. □

Corollary 7.6. *$K(9_{42}, q_1, n)$ is not adequate.*

Proof. Because the trailing coefficient is $(-1)^n * 2$, $K(9_{42}, q_1, n)$ admits no adequate diagram by Proposition 1 of Lickorish and Thistlethwaite [LT88]. □

So, $K(9_{42}, q_1, 2n)$ gives another infinite family of H-thick knots which is not detected by Khovanov's thickness criterion. Presumably doing other such conjugations of a positive crossing will yield more families of this kind.

7.5 Modification by the Mirror of 8_{20}

Now we will consider the case of gluing in multiple copies of the quasipositive mirror of 8_{20}, which will form a different family of H-thick knots. Call such a knot with n copies of the mirror of 8_{20} glued in via this quasipositive modification $K(9_{42}, 8_{20}, n)$. In actuality, this will not always return a knot: in the case that $n \equiv 2 \pmod 3$, the permutation of the strands gives a 3 component link, but otherwise $K(9_{42}, 8_{20}, n)$ will be a knot.

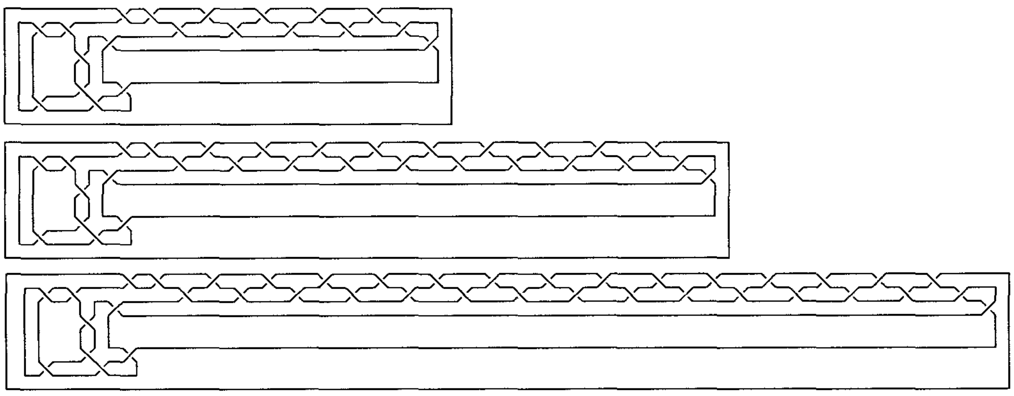

Figure 7.6: First three members of family $K(9_{42}, 8_{20}, n)$. Notice that the second member is a link, while the first and third are knots.

An interesting pattern in this family of H-thick links is that the width of successive members seems to be increasing. Using JavaKh-v2, an update of Jeremy Green's program by Scott Morrison written for [FGMW09], one can calculate that the width of the first four members is 3, 4, 5, and 6 respectively. Furthermore, this increasing width is not coming from some well-hidden connect-sum operation hiding in the quasipositive modification:

Proposition 7.7. *For $n = \{1, 2, 3, 4\}$, $K(9_{42}, 8_{20}, n)$ is a prime link. In particular, it is* not *a nontrivial connect sum with a thick link as a summand.*

Proof. The method of proof is to use prime tangle decomposition, following Lickorish [Lic81]. We will first examine the case of $K(9_{42}, 8_{20}, 1)$; primeness for the other links will follow the same general pattern.

Figure 7.7 illustrates a breakdown of $K(9_{42}, 8_{20}, 1)$ into two tangles. If we can show that both tangles are prime, then the original knot is prime by Theorem 1 of

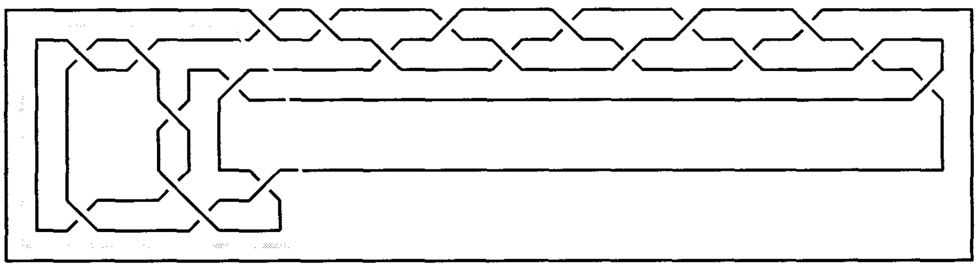

Figure 7.7: $K(9_{42}, 8_{20}, 1)$ broken into prime tangles. The enclosed tangle is Tangle A; the exterior is Tangle B.

[Lic81].

Tangle A is a variant on prime tangle (a) of Lickorish, which just has two extra positive half-twists added. Lickorish's same argument shows Tangle A is prime - we add the untangle as shown in Figure 7.8, and get an unknot with no nontrivial summand. As in (a), Tangle A itself is not the untangle, because one of its arcs is a knotted spanning arc of the ball; primeness of Tangle A follows.

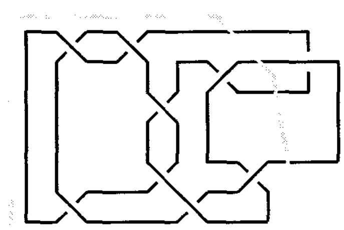

Figure 7.8: If we add the untangle to Tangle A as shown here, we get an unknot with no nontrivial summand. To see the unknotting, look at rightmost part and just start undoing twists via Reidemeister I.

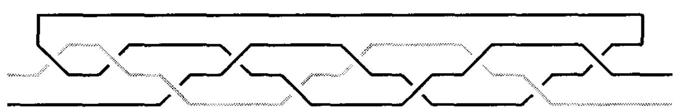

Figure 7.9: An isotoped version of Tangle B. The grey and black parts are both unknotted arcs, when viewed by themselves.

Primeness of Tangle B follows Lickorish's example (c). Each of the two arcs of the tangle are unknotted, seen by examining Figure 7.9. And, by adding an untangle to

Tangle B as shown in Figure 7.10, we get a 3-bridge knot, the mirror of 8_{21} (the Jones polynomial is enough to determine this identification). Because the bridge number of this composition is higher than 2, Tangle B could not have been untangled. So, Tangle B is prime.

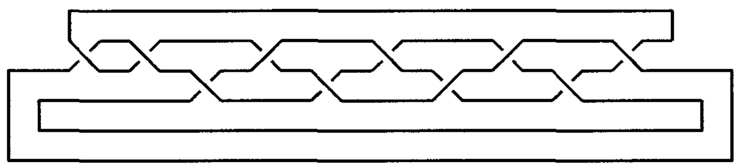

Figure 7.10: Tangle B composed with an untangle. This is a 10 crossing diagram, and only one knot of 10 crossings or less matches its Jones polynomial, so it is the mirror of 8_{21}.

Now let's consider similar tangle decompositions to prove the primeness in the cases $n = 2, 3, 4$. The idea is to make Tangle A the same, and consider what happens to the more complicated Tangle B cases. The only new thing one must check is that adding an untangle to the new variants of Tangle B are not 2-bridge. To show this, one can calculate the Khovanov homology for these links. In each case, the Khovanov homology is H-thick, of width 3, 4, and 5 respectively. But, 2-bridge links have alternating diagrams, which means their Khovanov homology must be H-thin. So, the bridge index of these two links is higher than 2, and the new Tangle B's are prime, as desired. $\square$

Remark 7.7.1. Note that via this prime tangle decomposition, we end up using H-thickness of some subtangles to prove these knots and links are prime. A better understanding of why the width seems to increase when adding the mirror of 8_{20} would also give a proof that this full family of links is prime.

On the other hand, these links have nonalternating Jones polynomial, so their thickness (but not increasing width) can be detected by Khovanov's thickness criteria. For reference, the Jones polynomials of the first four in this family are listed below, calculated using KnotTheory' [BNMG]:

$$\begin{aligned}
V(K(9_{42}, 8_{20}, 1)) &= 2q^{-2} - 3q^{-1} + 5 - 5q + 4q^2 - 4q^3 + 2q^4 - q^5 + q^7 \\
V(K(9_{42}, 8_{20}, 2)) &= -q^{-3} + 3q^{-2} - 5q^{-1} + 9 - 11q + 16q^2 \\
&\quad - 17q^3 + 19q^4 - 17q^5 + 14q^6 - 10q^7 + 6q^8 \\
&\quad - 2q^9 + q^{11} - q^{12} \\
V(K(9_{42}, 8_{20}, 3)) &= q^{-4} - 4q^{-3} + 10q^{-2} - 20q^{-1} + 34 - 51q + 69q^2 - 85q^3 + 98q^4 \\
&\quad - 104q^5 + 105q^6 - 97q^7 + 84q^8 - 67q^9 + 48q^{10} - 31q^{11} + 16q^{12} \\
&\quad - 6q^{13} + 2q^{15} - 2q^{16} + q^{17} \\
V(K(9_{42}, 8_{20}, 4)) &= -q^{-5} + 5q^{-4} - 15q^{-3} + 35q^{-2} - 68q^{-1} + 118 - 183q + 264q^2 \\
&\quad - 353q^3 + 444q^4 - 526q^5 + 587q^6 - 620q^7 + 619q^8 - 585q^9 + 522q^{10} \\
&\quad - 438q^{11} + 343q^{12} - 250q^{13} + 166q^{14} - 98q^{15} + 49q^{16} - 17q^{17} \\
&\quad + q^{18} + 5q^{19} - 5q^{20} + 3q^{21} - q^{22}
\end{aligned}$$

For completeness, one might ask whether these links have adequate diagrams. Unfortunately, the Kauffman polynomial, which provides most of the obstructions to being adequate, is too computationally intensive to calculate even for the 20 crossing diagram of the $n = 1$ case. For $n = 1$, the last coefficient of the Jones polynomial is not ± 1, guaranteeing there is no adequate diagram for this first knot by Proposition 1 of Lickorish and Thistlethwaite [LT88]. But, subsequent Jones polynomials for the next 3 cases all have first and last Jones coefficient ± 1, leaving only the Kauffman polynomial obstructions.

Conjecture 7.8. *For each natural number n, $K(9_{42}, 8_{20}, n)$ admits no adequate diagram.*

7.6 Other Base Knots

9_{42} is not the only valid base knot for constructing H-thick knots by quasipositive modification: 10_{132}, 10_{136}, and 10_{145} all have the same setup of a + adequate diagram, and a "thick" state cycle representative in the third diagonal, for a knot of width 3. Note that for all these knots, 9_{42} included, the "thick" state cycle representative has bigrading $(0, s-1)$, where s is Rasmussen's s invariant. Also, in each case, the s invariant is smaller than the signature of the knot.

One can also find examples with a slightly different setup, by direct computation. 10_{161} has a - adequate diagram for its standard minimal diagram, but there is a state cycle representative for its lowermost corner homology class at $(-9, -23)$. Its width is 3, so even though the diagram is not + adequate, the same methods will give us 2 diagonals from the Lee pairing of this data. And, there is a third off diagonal state cycle which is nontrivial, at bigrading $(-3, -13)$. For these state cycles, nontriviality was checked by a Java program written by the author.

In general, though, this program is limited to calculating nontriviality for diagrams of 10 crossings or less, restricting the number of H-thick base knots that can be examined by direct computation. Better methods of checking nontriviality for state cycles are needed, if one is to extend the selection of base knots for such thick families.

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