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An edge colouring of multigraphs

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Abstract

We consider a strict *k-colouring* of a multigraph $\boldsymbol{G}$ as a surjection f from the vertex set of $\boldsymbol{G}$ into a set of colours $\{1,2,\ldots,k\}$ such that, for every non-pendant vertex x of $\boldsymbol{G}$, there exist at least two edges incident to x and coloured by the same colour. The maximum number of colours in a strict edge colouring of $\boldsymbol{G}$ is called the *upper chromatic index* of $\boldsymbol{G}$ and is denoted by $\overline{\chi}(\boldsymbol{G})$. In this paper we prove some results about it.

1 Introduction

Let $\boldsymbol{G}$=(X,$\mathbf{E}$) be an arbitrary multigraph. A *strict edge k-colouring* of $\boldsymbol{G}$ is a surjection f from the edge set $\mathbf{E}$ into a set of colours $\{1,2,\ldots,k\}$ such that, for every non-pendant vertex x of $\boldsymbol{G}$, there exist at least two edges incident to x and coloured by f with the same colour.

Following the definition, the minimum number of colours in a strict edge colouring of a multigraph is one. This is a complementary fashion of the fact that, in the classical edge colouring, the maximum number of colours is trivially equal to the number of edges of the multigraph.

The maximum number k for which there exists a strict edge k-colouring of a multigraph $\boldsymbol{G}$ is called the *upper chromatic index* of $\boldsymbol{G}$ and is denoted by $\overline{\chi}(\boldsymbol{G})$. An edge colouring of $\boldsymbol{G}$ which uses exactly $\overline{\chi}(\boldsymbol{G})$ colours is called a *maximal edge colouring.*

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2 Main results

Theorem 2.1 - *Let* $\boldsymbol{G}_1, \boldsymbol{G}_2$ *be two disjointed multigraphs,* x *a vertex of* $\boldsymbol{G}_1$ *such that* $d(x) \geq 2$, y *a vertex of* $\boldsymbol{G}_2$ *such that* $d(y) \geq 2$, σ *a simple path from* x to y *with no edge in common with* $\boldsymbol{G}_1$ *and* $\boldsymbol{G}_2$, $\boldsymbol{G} = \boldsymbol{G}_1 \cup \boldsymbol{G}_2 \cup \sigma$. *Then* $\overline{\chi}(\boldsymbol{G}) = \overline{\chi}(\boldsymbol{G}_1) + \overline{\chi}(\boldsymbol{G}_2) + 1$.

Proof. Let be $h = \overline{\chi}(\boldsymbol{G}_1)$, $k = \overline{\chi}(\boldsymbol{G}_2)$ and let f be a strict edge h-colouring of $\boldsymbol{G}_1$, g a strict edge k-colouring of $\boldsymbol{G}_2$ with no colour in common. Since we can obtain a strict edge $h+k+1$-colouring of $\boldsymbol{G}$ simply by giving to all the edges of the path jointing x and y a colour distinct from all the colours of f and g, then $\overline{\chi}(\boldsymbol{G}) \geq h+k+1$.

Suppose that $\overline{\chi}(\boldsymbol{G}) \geq h+k+2$. Then there exists an edge p-colouring f of $\boldsymbol{G}$, with $p \geq h+k+2$. Since the edges of σ must be coloured with the same colour, the number of colours of f in the multigraph $\boldsymbol{G}_1$ is not less than $h+1$ or the number of colours of f in the multigraph $\boldsymbol{G}_2$ is not less than $k+1$, that's false. So $\overline{\chi}(\boldsymbol{G}) = h+k+1$. □

Theorem 2.2 - *If* $\boldsymbol{G}$ *is an eulerian multigraph and* P *is an edge partition of* $\boldsymbol{G}$ *in cycles, then* $\overline{\chi}(\boldsymbol{G}) \geq |P|$.

Proof. For an eulerian connected multigraph, there exists, as it is well known, a partition as P. Observe that it is possible to give the same colour to all the edges of every cycle of P and colours pairwise distinct to every cycles of P. □

Remarks

1) Considering theorem 2.2, there exist cases in which $\overline{\chi}(\boldsymbol{G}) > |P|$. It suffice to examine a simple graph $\boldsymbol{G}$ with 6 vertices formed by two cycles of length 4 having two vertices and no edge in common: since every vertex has even degree, this graph is eulerian and $\overline{\chi}(\boldsymbol{G}) = 3$.

2) Observe that, if every cycle of the partition P has exactly one vertex in common with exactly one other cycle of P, then the graph is simple and $\overline{\chi}(\boldsymbol{G}) = |P|$.

M. Gionfriddo, L. Milazzo, V. Voloshin proved [4] the following theorems:

Theorem 2.3 - *Let* $\boldsymbol{G}$=(X,**E**) *be an arbitrary multigraph*, *c the maximum number of disjoint cycles*, *p the number of pendant vertices of* $\boldsymbol{G}$. *Then*

$$\overline{\chi}(\boldsymbol{G}) = c + |\mathbf{E}| - |\mathrm{X}| + p$$

Corollary 2.4 - *For a graph* $\mathbf{K}_n$ *with* $n \geq 3$, *we have*:

$$\begin{cases} \overline{\chi}(\mathbf{K}_n) = \frac{9k^2-7k}{3} & \text{if} \quad n = 3k \\ \overline{\chi}(\mathbf{K}_n) = \frac{9k^2+k-2}{2} & \text{if} \quad n = 3k+1 \\ \overline{\chi}(\mathbf{K}_n) = \frac{9k^2+5k-2}{2} & \text{if} \quad n = 3k+2 \end{cases}$$

Now we can prove the following

Corollary 2.5 - *For a graph* $\mathbf{K}_{m,n}$ *with* $1 < m \leq n$, *we have*:

$$\begin{cases} \overline{\chi}(\mathbf{K}_{m,n}) = \frac{2mn-2n-m}{2} & \text{if } m \text{ is even} \\ \overline{\chi}(\mathbf{K}_{m,n}) = \frac{2mn-2n-m-1}{2} & \text{if } m \text{ is odd} \end{cases}$$

Proof. Observe that the maximum number of disjoint cycles of $\mathbf{K}_{m,n}$ is $\frac{m}{2}$ if m is even and $\frac{m-1}{2}$ if m is odd. Since $\mathbf{K}_{m,n}$ has $m+n$ non pendant vertices and mn edges, the statement follows by simple calculating from theorem 2.3. □

In the case $m=n$, we have:

$$\begin{cases} \overline{\chi}(\mathbf{K}_{n,n}) = \frac{2n^2-3n}{2} & \text{if } n \text{ is even} \\ \overline{\chi}(\mathbf{K}_{n,n}) = \frac{2n^2-3n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Theorem 2.6 - *For every tree* $\boldsymbol{A}$=(X,**E**), *we have*

$$\overline{\chi}(\boldsymbol{A}) = \sum_{x \in X} (d(x) \div 2) + 1$$

where, for every $m,n \in \mathbf{N}$, $m \div n = m\text{-}n$ *if* $m \geq n$ *and zero otherwise.*

Proof. Let r be a root of the tree. Observe that every maximal edge-colouring f of $\boldsymbol{A}$ has the property that, for every vertex x of $\boldsymbol{A}$ with $x \neq r$ and $d(x)>2$, there exist exactly $d(x)$-2 edges incident to x coloured by f with colours pairwise distinct. Since for every pendant vertex y of $\boldsymbol{A}$, $d(y) \div 2 = 0$, the statement of theorem follows. □

Before introducing theorem 2.7, we will call *p-tree of height h* a tree defined by induction in the following way:

1) A vertex is a p-tree of height 0;

2) A p-star is a p-tree of height 1;

3) For $h \geq 2$, we call a p-tree of height h a tree obtained from a p-tree $\boldsymbol{A}$ of height h-1 by connecting every pendant vertex of $\boldsymbol{A}$ with p other vertices.

Theorem 2.7 - *For a p-tree* $\boldsymbol{A}$ *of height h with* $p \geq 2$, *we have* $\overline{\chi}(\boldsymbol{A}) = p^h$-1.

Proof. By induction. If $h=1$, the statement is trivially true. Let be $h>1$ and suppose the statement true for every p-tree of height h-1. From a p-tree $\boldsymbol{A}'$ of height h-1 and from a maximal edge colouring f of $\boldsymbol{A}'$ we can obtain a p-tree $\boldsymbol{A}$ of height h and a maximal edge colouring g of $\boldsymbol{A}$ by adding p^h vertices and p^h edges, from which at most $(p\text{-}1)p^{h-1}$ can be coloured by colours pairwise distinct from the colours used by f. Therefore $\overline{\chi}(\boldsymbol{A}) = \overline{\chi}(\boldsymbol{A}') + (p\text{-}1)p^{h-1} = p^{h-1}\text{-}1 + (p\text{-}1)p^{h-1} = p^h\text{-}1$, and so the assertion follows. □

Remarks

1) It is possible to prove theorem 2.7 starting from theorem 2.6. In fact, in a p-tree of height h with $p \geq 2$, there are 1 vertex with degree p, $\frac{p^h-1}{p-1}$-1 vertices with degree $p+1$ and p^h pendant vertices, so that:

$$\overline{\chi}(\boldsymbol{A}) = \sum_{x \in A} (d(x) \div 2) + 1 = (\frac{p^h - 1}{p - 1} - 1)(p - 1) + p - 2 + 1 = p^h - 1$$

2) If we apply theorem 2.3, we obtain simply the statement of the-

orem 2.7 by observing that a tree is acyclic and the number of pendant vertices of a p-tree of height h is p^h.

Corollary 2.8 - *For a p-tree $\boldsymbol{A}$ with $p \geq 2$ and n vertices, we have* $\overline{\chi}(\boldsymbol{A})=(n\text{-}1)(1\text{-}\frac{1}{p})$.

Proof. Let h be the height of $\boldsymbol{A}$. Since $n=\frac{p^{h+1}-1}{p-1}$, we obtain $n=\frac{p(\overline{\chi}(\boldsymbol{A})+1)-1}{p-1}$, from which, by a simple calculation, the statement follows. □

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