Gamma (Co)homology of Commutative Algebras and Some Related Representations of the Symmetric Group

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I would like to dedicate this thesis to the memory of Janne Sholl, who did not live to see it completed.

Declaration

The material in this thesis is, to the best of my knowledge, original unless clearly stated otherwise.

Summary

This thesis covers two related subjects : homology of commutative algebras and certain representations of the symmetric group.

There are several different formulations of commutative algebra homology, all of which are known to agree when one works over a field of characteristic zero. During 1991–1992 my supervisor, Dr. Alan Robinson, motivated by homotopy-theoretic ideas, developed a new theory, Γ-homology [Rob,2]. This is a homology theory for commutative rings, and more generally rings commutative up to homotopy. We consider the algebraic version of the theory.

Chapter I covers background material and Chapter II describes Γ-homology. We arrive at a spectral sequence for Γ-homology, involving objects called tree spaces.

Chapter III is devoted to consideration of the case where we work over a field of characteristic zero. In this case the spectral sequence collapses. The tree space, T_n, which is used to describe Γ-homology has a natural action of the symmetric group S_n. We identify the representation of S_n on its only non-trivial homology group as that given by the first Eulerian idempotent $e_n(1)$ in $\mathbb{Q}S_n$. Using this, we prove that Γ-homology coincides with the existing theories over a field of characteristic zero.

In fact, the tree space, T_n, gives a representation of S_{n+1}. In Chapter IV we calculate the character of this representation. Moreover, we show that each Eulerian representation of S_n is the restriction of a representation of S_{n+1}. These Eulerian representations are given by idempotents $e_n(j)$, for $j = 1, ..., n$, in $\mathbb{Q}S_n$, and occur in the work of Barr [B], Gerstenhaber and Schack [G-S,1], Loday [L,1,2,3] and Hanlon [H]. They have been used to give decompositions of the Hochschild and cyclic homology of commutative algebras in characteristic zero. We describe our representations of S_{n+1} as virtual representations, and give some partial results on their decompositions into irreducible components.

In Chapter V we return to commutative algebra homology, now considered in prime characteristic. We give a corrected version of Gerstenhaber and Schack's [G-S,2] decomposition of Hochschild homology in this setting, and give the analagous decomposition of cyclic homology. Finally, we give a counterexample to a conjecture of Barr, which states that a certain modification of Harrison cohomology should coincide with André/Quillen cohomology.

Chapter 0 : Introduction

This introduction is given over to briefly summarising the existing cohomology theories for commutative algebras and some representation theory of the symmetric group.

Chapter I covers the Hochschild and Harrison theories in more detail. In particular, it gives Gerstenhaber and Schack's [G-S,1] decomposition of the Hochschild (co)homology of a commutative algebra in characteristic zero. This introduces the Eulerian idempotents $e_n(j)$ in $\mathbb{Q}S_n$. This chapter is entirely expository.

Chapter II provides the definition of Γ-(co)homology, which is described in terms of the tree spaces. Apart from the presentation of the Γ category, this material is due to my supervisor, Dr. Alan Robinson.

In Chapter III, by describing cycles in the tree space explicitly and investigating their connection with shuffles, it is shown that Γ-(co)homology coincides with Harrison (co)homology in characteristic zero.

Chapter IV describes the representation of the symmetric group given by the tree space, showing that it extends that given by the idempotent $e_n(1)$. This result is generalised to show that the representations $e_n(j)\mathbb{Q}S_n$ are all restrictions of representations of S_{n+1}. Character formulae are given for these representations.

Chapter V contains some remarks on commutative algebra cohomology in prime characteristics. It gives Gerstenhaber and Schack's [G-S,2] decomposition in this case, and clarifies some points about it. Then we give the analagous decomposition of cyclic cohomology. Finally, we give a counterexample to a conjecture of Barr.

Section 0.1 : (Co)homology Theories for Commutative Algebras

There is a standard cohomology theory for an associative algebra A over a commutative ground ring k, called Hochschild cohomology. This was introduced by Hochschild in 1945 [Ho], (for the case where k is a field), and is covered in the standard texts on homological algebra, such as Cartan and Eilenberg [C-E] and MacLane [Mc]. It fits into the general context of relative homological algebra, (see [Mc, Ch.IX.]). The n^{th} Hochschild cohomology group of A with coefficients in an A-bimodule M, $HH^n(A/k; M)$, is the relative Ext group, $\mathrm{Ext}^n_{A\otimes A^{op},k}(A; M)$. This consists of equivalence classes, under a standard equivalence relation, of those n-extensions of A by A-bimodules which are k-split. (See [Mc]). In the case where k is a field, requiring that the extension be k-split is no restriction, and the above is the same as the absolute Ext group, $\mathrm{Ext}^n_{A\otimes A^{op}}(A; M)$. Hochschild *homology* is defined in the obvious way. Hochschild (co)-homology is computed using a special case of the categorical bar resolution, ([Mc] IX.7). This is the standard Hochschild chain complex ($B_*(A)$, b′), which is a relatively projective resolution of A by A-bimodules.

The Hochschild cohomology theory has many nice properties, such as a dimension shifting technique [Ho] and Morita invariance [L,1]. The second Hochschild cohomology groups are closely related to algebra extensions [Ho]. Two examples are worth mentioning. The first, due to Connes [C], relates Hochschild homology to differential forms: if X is a smooth manifold and $A = C^\infty(X)$ is the algebra of smooth real-valued functions on X, then $HH_n(A; A)$ is isomorphic to $\Omega^n(X)$, the differential n-forms on X. The second relates Hochschild homology to standard group homology (see e.g. [Mc, Ch. IV]): if kG is the group algebra of a group G then $HH_n(kG; kG)$ is $H_n(G; kG_{ad})$, the group homology with coefficients in the G-module kG, with G acting by conjugation. Alternatively, the Hochschild homology of the group algebra can be seen as the homology of the free loop space on the classifying space of the group, $HH_n(kG; kG) \cong H_n(LBG; k)$ [Go].

For a commutative algebra A, we may consider the algebraic differential forms on A. The A-module of differential 1-forms $\Omega^1_{A/k}$ has generators da for a in A, satisfying d(ab) = a.db +b.da. Then $\Omega^1_{A/k}$ is isomorphic to I/I^2, where I is the kernel of the multiplication map $A\otimes A \to A$. The A-module of differential n-forms $\Omega^n_{A/k}$ is given by the exterior product over A, $\Lambda^n \Omega^1_{A/k}$. We have d: $\Omega^n_{A/k} \to \Omega^{n+1}_{A/k}$ given by $d(a_0 da_1 \ldots da_n) = da_0 da_1 \ldots da_n$ and we get the de Rham complex ($\Omega^*_{A/k}$, d), whose homology is the de Rham cohomology of the algebra, $H^*_{dR}(A)$. For A a smooth commutative algebra over k, and M a symmetric A-bimodule, a result of Hochschild, Kostant and Rosenberg [H-K-R] relates Hochschild homology to algebraic differential forms, $HH_n(A/k; M) \cong M\otimes_A \Omega^n_{A/k}$.

Cyclic cohomology was introduced by Connes [C] in 1985 in his paper 'Non-commutative differential geometry'. His original approach was through the theory of non-commutative differential forms on an algebra. Working over a field of characteristic zero, cyclic (co)homology can be defined by using the quotient of the Hochschild complex by the actions of the cyclic groups. Working over a general commutative ground ring, the definition can be given using a certain bicomplex, as described by Loday and Quillen [L-Q]. This gives rise to a spectral sequence, involving the homology of cyclic groups, converging to cyclic homology. An important property of cyclic homology is the long exact 'periodicity sequence' which links it to Hochschild homology, due to Connes [C]. Cyclic homology is Morita invariant. Returning to our two examples, for the algebra $A = C^\infty(X)$ of smooth functions on a manifold X, we have a result of Connes [C] relating the cyclic homology to de Rham cohomology:

$$HC_n(A) \cong \Omega^n(X)/d\Omega^{n-1}(X) \oplus \sum_{i\geq 1} H_{dR}^{n-2i}(X).$$

Loday and Quillen [L-Q] proved an algebraic version of this result. If A is a smooth commutative algebra over k containing $\mathbb{Q}$, then

$$HC_n(A) \cong \Omega^n_{A/k}/d\Omega^{n-1}_{A/k} \oplus \sum_{i\geq 1} H_{dR}^{n-2i}(A).$$

Secondly, the cyclic homology of a group algebra is given by the S^1-equivariant homology of the free loop space on the classifying space of the group, $HC_n(kG) \cong H_n^{S^1}(LBG; k)$ [Go]. Cyclic homology is also related to Lie algebra homology. Loday and Quillen [L-Q] show that, over a field of characteristic zero, cyclic homology is the primitive part of the homology of the Lie algebra of matrices.

There have been several attempts to construct a theory specifically suitable for commutative algebras. Firstly, there is Harrison cohomology, defined by Harrison [Ha] in 1962. In the case of a commutative algebra, the Hochschild complex has a shuffle product. The quotient of the Hochschild complex by the decomposable elements for this shuffle product gives a complex, whose (co)homology is the Harrison (co)homology. If one works over a ground ring containing $\mathbb{Q}$ then Harrison cohomology is a direct summand of Hochschild cohomology. In dimensions greater than two, it is also a direct summand of cyclic cohomology. The second Harrison cohomology groups can be related to commutative algebra extensions [Ha].

Another version of commutative algebra cohomology is that due independently to André [A] and to Quillen [Q]. This gives cohomology for A a commutative algebra over k a commutative ring, with coefficients in an A-module M. The cohomology groups, $D^*(A/k; M)$, are defined as certain derived functors of derivations on the category of k-algebras over A. In particular, $D^0(A/k; M) = Der_k(A, M)$. The derived functors are defined by applying the functor $X \mapsto Der_k(X, M)$ to a free simplicial k-algebra resolution of A, and taking cohomology. This theory is related to the more general cotriple cohomology, which is described in [B-B]. In particular, the symmetric algebra cotriple gives rise to another version of commutative algebra cohomology, which is described by Barr [B], and which is identical to the André/Quillen theory. When A is a projective k-module and k contains $\mathbb{Q}$ André/Quillen cohomology coincides with Harrison cohomology. The André/Quillen theory has the property of transitivity, that is to say there is a long exact sequence in cohomology for a triple of commutative rings. It also has a flat base change property, meaning that if $Tor^k_q(A, B) = 0$ for $q > 0$, where A and B are k-algebras, then we have isomorphisms:

$D^q(A \otimes_k B/B; M) \cong D^q(A/k; M)$

$D^q(A \otimes_k B/k; M) \cong D^q(A/k; M) \oplus D^q(B/k; M)$,

where M is a $A \otimes_k B$-module. If A is a smooth commutative k-algebra then $D^q(A/k; M) = 0$ for $q > 0$. There is a spectral sequence relating André/Quillen cohomology to Hochschild cohomology, which can be used to deduce a decomposition of Hochschild cohomology when k contains $\mathbb{Q}$.

Γ-homology is a new homology theory for commutative algebras, formulated by my supervisor, Dr. Alan Robinson [Rob,2]. In fact the theory extends to the topological analogue of commutative rings, E_∞-ring spectra [M]. Its introduction was motivated by consideration of obstructions to the existence of E_∞ structures. (Another theory for E_∞-ring spectra is the topological Hochschild homology of Bokstedt, Hsiang and Madsen [B-H-K]). The Γ-theory can be described by giving an explicit chain complex, closely related to the nerve of the category, Γ, of finite sets and surjective maps. If instead, we were to use *order preserving* surjective maps we would simply get the Hochschild complex. The idea is to describe an analogous complex, but, using the commutativity of the algebra, to build in the actions of the symmetric groups. We arrive at a spectral sequence for Γ-homology, involving the symmetric groups. Properties of the theory include a long exact sequence for triples of commutative algebras or E_∞-ring spectra. In this thesis, we shall only cover the strictly algebraic version of the theory. The main result of Chapter III is that Γ-homology coincides with the existing theories in characteristic zero.

Section 0.2 : Representation Theory of the Symmetric Group

The second subject covered by this thesis is representation theory of the symmetric group. Since Γ-homology is given by a spectral sequence involving the symmetric groups, it is important to understand the symmetric group representations which arise, the *tree representations.* This in turn led to the study of some closely related representations, those given by the Eulerian idempotents. For each n, in the rational group algebra of the symmetric group, $\mathbb{Q}S_n$, we have a collection of n mutually orthogonal idempotents $e_n(j)$, $j = 1, \ldots, n$, whose sum is 1. These were originally introduced by Gerstenhaber and Schack [G-S,1], who defined them as certain polynomials in the total shuffle operator, s_n. They showed that they are essentially the unique elements with the property of commuting with the Hochschild boundary. They therefore can be used to give a decomposition of the Hochschild (co)homology of a commutative algebra over a ground ring containing $\mathbb{Q}$. Natsume and Schack [N-S] generalised this result to give a similar decomposition of cyclic (co)homology for a commutative algebra over k containing $\mathbb{Q}$. An alternative approach, also giving the decompositions of Hochschild and cyclic (co)homology, is given by the λ-operations of Loday [L,1,2]. This involves the Eulerian partition of the symmetric group, according to the number of descents a permutation has. For each n, we have n representations $e_n(j)\mathbb{Q}S_n$ of S_n, giving a decomposition of the regular representation.

The main results on the symmetric group representations $e_n(j)\mathbb{Q}S_n$ given by these idempotents are due to Hanlon [H]. He gives a description in terms of sums of representations induced from wreath products, and thus provides character formulae. The general problem of giving descriptions of the decompositions of these representations into their irreducible components remains open. However, there is a description for the special case j = 1. This is due originally to Kraskiewicz and Weyman [K-W], and an alternative approach can be found in [S]. The same representations arise in the different context of free Lie algebras, and have here been studied by Reutenhauer [R] and by Garsia [G].

We show that each of the Eulerian representations of S_n is the restriction of a representation of S_{n+1}. We give a description of the new representations and character formulae. For certain cases, we give the decomposition of the representation into irreducible components.

Chapter I : Hochschild, Harrison and Cyclic (Co)homology

Introduction

This chapter outlines the background necessary for the subsequent work, describing some of the existing (co)homology theories for commutative algebras. It is entirely expository.

The first section describes Hochschild (co)homology. This is a standard cohomology theory for any associative algebra over a commutative ground ring with coefficients in a bimodule over the algebra. It was introduced by Hochschild in 1945 [Ho]. The cohomology groups can be described in terms of extensions of bimodules over the algebra. It is computed using a certain standard complex, which simplifies in the special case we are interested in, where the algebra is commutative and the coefficient bimodule is symmetric.

Next we introduce Harrison (co)homology. This is a theory for the case of a commutative algebra and a symmetric coefficient module. In this situation, Harrison (co)homology is computed from a certain quotient complex of the Hochschild complex, defined in terms of particular permutations called shuffles. This theory was developed by Harrison in 1962 [Ha].

The third section summarises further properties of Hochschild and Harrison (co)homology which hold in the special case where the ground ring contains $\mathbb{Q}$. By introducing an idempotent e_n in the group algebra $\mathbb{Q}S_n$, Barr [B] showed that the Harrison complex is a direct summand of the Hochschild complex, and hence Harrison (co)homology is a direct summand of Hochschild (co)homology. By defining further idempotents $e_n(j)$ in $\mathbb{Q}S_n$ for $j = 1, ..., n$, Gerstenhaber and Schack [G-S,1] generalised Barr's results. They give a decomposition of the n^{th} Hochschild (co)homology group as a direct sum of n parts, the first of which is the Harrison (co)homology.

We note the similar decomposition of cyclic (co)homology of a commutative algebra over a ground ring containing $\mathbb{Q}$, due to Natsume and Schack [N-S] and to Loday [L,1,2]. The standard periodicity sequence linking Hochschild and cyclic cohomology also decomposes as a sum. Other approaches to the decompositions can be found in [F-T], [B-V] and [V] (for differential graded commutative algebras), and [P]. The λ-operations and further properties of the Eulerian idempotents are covered in [L,1,2].

Finally, we give Hanlon's results [H] on the characters of the representations $e_n(j)\mathbb{Q}S_n$ of the symmetric group S_n.

Section I.1 : Hochschild (Co)homology

Let A be an associative algebra (with identity) over a commutative ring k and let $A^{\otimes n}$ denote $A\otimes A\otimes ...\otimes A$, the n-fold tensor product of A over k. The Hochschild (co)homology groups of A with coefficients in an A-bimodule M can be defined using a certain standard complex of A-bimodules, $B_*(A)$, where $B_n(A) = A\otimes A^{\otimes n}\otimes A$ and the boundary $b' : A^{\otimes(n+2)} \to A^{\otimes(n+1)}$ is given by

$$b'(a_0\otimes ... \otimes a_{n+1}) = \sum_{i=0}^{n}(-1)^i(a_0\otimes ... \otimes a_i a_{i+1}\otimes ... \otimes a_{n+1})$$

This complex is acyclic because of the contracting homotopy, $s : A^{\otimes(n+1)} \to A^{\otimes(n+2)}$ given by

$$s(a_0\otimes ... \otimes a_n) = (1\otimes a_0\otimes ... \otimes a_n),$$

satisfying $b's + sb' = 1$. Here s consists of k-module homomorphisms, providing k-splittings for b'. (Equivalently, s can be regarded as consisting of right A-module homomorphisms, providing right A-splittings).

Definition I.1.1.

The *Hochschild homology* groups of A with coefficients in an A-bimodule M are defined by

$$HH_*(A/k; M) = H(M\otimes_{A\otimes A^{op}}B_*(A)).$$

Similarly, the *Hochschild cohomology* groups are given by

$$HH^*(A/k; M) = H(\mathrm{Hom}_{A\otimes A^{op}}(B_*(A), M)).$$

When A is projective over k, $B_*(A)$ is a projective resolution of A as an $A\otimes A^{op}$-module. (See [C-E] p175). So,

$$HH^n(A/k; M) = \mathrm{Ext}^n_{A\otimes A^{op}}(A, M).$$

Thus $HH^n(A/k; M)$ consists of equivalence classes, under the usual relation, of n-extensions of A by A-bimodules. In particular, this holds when k is a field.

In general, $B_*(A)$ is a k-split resolution of the bimodule A by $(A\otimes A^{op}, k)$-relatively projective bimodules, (or an A-split resolution by $(A\otimes A^{op}, A)$-relatively projective bimodules). This means that Hochschild cohomology is actually an instance of a relative Ext group, and should more properly be written :

$$HH^n(A/k; M) = \mathrm{Ext}^n_{A\otimes A^{op},k}(A, M) \cong \mathrm{Ext}^n_{A\otimes A^{op},A}(A, M).$$

Thus $HH^n(A/k; M)$ consists of equivalence classes, under the usual relation, of those n-extensions of A by A-bimodules which are k-split, (or equivalently A-split). [See [Mc] p282].

Since we have

$$\mathrm{Hom}_{A\otimes A^{op}}(B_n(A), M) = \mathrm{Hom}_{A\otimes A^{op}}(A\otimes A^{op}\otimes A^{\otimes n}, M)$$
$$\cong \mathrm{Hom}_k(A^{\otimes n}, M),$$

we can use the universal property of the tensor product to describe the Hochschild complex in dimension n explicitly as the k-module of all k-multilinear functions f on the n-fold Cartesian product A x ... x A of A. It can be checked that this complex then has coboundary given by :

$$\delta f(a_1, ..., a_{n+1}) = a_1 f(a_2, ..., a_{n+1}) + \sum_{i=1}^{n} (-1)^i f(a_1, ..., a_i a_{i+1}, ..., a_{n+1})$$
$$+ (-1)^{n+1} f(a_1, ..., a_n) a_{n+1}.$$

This retrieves the original definition of Hochschild [Ho]. Similarly, for homology

$$M\otimes_{A\otimes A^{op}} B_n(A) \cong M\otimes A^{\otimes n},$$

and the boundary $b : M\otimes A^{\otimes n} \to M\otimes A^{\otimes n-1}$ is :

$$b (m\otimes a_1\otimes ... \otimes a_n) = ma_1\otimes a_2\otimes ... \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m\otimes a_1\otimes ... \otimes a_i a_{i+1}\otimes ... \otimes a_n$$
$$+ (-1)^n a_n m\otimes a_1\otimes ... \otimes a_{n-1}.$$

A zero cochain is a constant $m \in M$; its coboundary is the function $f : A \to M$ given by $fa = am - ma$. Hence, $HH^0(A, M)=\{ m \in M \mid ma = am \ \forall\, a\in A \}$, the invariant elements of M. A 1-cocycle is a k-module homomorphism $f : A \to M$ which satisfies $f(a_1a_2) = a_1f(a_2) + f(a_1)a_2 \ \forall\, a_1, a_2 \in A$. Such a functor is called a crossed homomorphism or *derivation* of A into M. It is a coboundary if it has the form $f_m(a) = am - ma$, and these are called principal crossed homomorphisms. Hence, $HH^1(A/k; M)$ consists of all the crossed homomorphisms $A \to M$, modulo the principal ones.

Now let A be a commutative k-algebra and let M be a symmetric A-bimodule i.e. $am = ma \ \forall\, a\in A, m\in M$. (Of course this is really the same as a left A-module). As Barr points out [B], using the fact that M is symmetric, we can work with a 'symmetrised' complex, since then we have

$$\mathrm{Hom}_{A\otimes A} (B_n(A), M) \simeq \mathrm{Hom}_A (C_n(A), M),$$

where $C_n(A) = A\otimes A^{\otimes n}$, viewed as a symmetric A-bimodule with A acting on the first factor.

Denote $(a_0\otimes ... \otimes a_n) \in C_n(A)$ by $a_0 [a_1, ..., a_n]$. The boundary $b : C_{n+1}(A) \to C_n(A)$ is given by the A-linear map such that :

$$b[a_1, ..., a_{n+1}] = a_1[a_2, ..., a_{n+1}] + \sum_{i=1}^{n}(-1)^i [a_1, ..., a_ia_{i+1}, ..., a_{n+1}] \\ + (-1)^{n+1} a_{n+1}[a_1, ..., a_n].$$

Then the coboundary of $f \in Hom_A(C_n(A), M)$ is given by the A-linear map such that:

$$\delta f[a_1, ..., a_{n+1}] = a_1 f[a_2, ..., a_{n+1}] + \sum_{i=1}^{n}(-1)^i f[a_1, ..., a_ia_{i+1}, ..., a_{n+1}] \\ + (-1)^{n+1} a_{n+1} f[a_1, ..., a_n].$$

Note that when M is symmetric $HH^0(A/k; M) = M$ and $HH^1(A/k; M) = Der_k(A, M)$, the derivations of A into M.

Section I.2 : Harrison (Co)homology and Shuffles

The (symmetrised) Hochschild chain complex ($C_n(A)$, b) for a commutative algebra has a product, known as the *shuffle* product.

Definition I.2.1.

A permutation π in the symmetric group S_n is called an *i-shuffle* if $\pi 1 < \pi 2 < ... < \pi i$ and $\pi(i+1) < \pi(i+2) < ... < \pi n$. Then the shuffle product $* : C_i(A) \otimes C_{n-i}(A) \to C_n(A)$ is given by

$$a[a_1, ..., a_i] * a'[a_{i+1}, ..., a_n] = \sum (\mathrm{sgn}\, \pi)\, aa'[a_{\pi^{-1}1}, ..., a_{\pi^{-1}n}]$$

where the sum is taken over all i-shuffles $\pi \in S_n$. It is easily seen that this product is associative and graded commutative. In fact the product passes to a map

$$* : C_i(A) \otimes_A C_{n-i}(A) \cong C_n(A) \to C_n(A).$$

Definition I.2.2.

We define elements $s_{i,n-i}$ of the group algebra kS_n.
Let $s_{i,n-i} = \sum (\mathrm{sgn}\, \pi)\, \pi$, where the sum is over all i-shuffles in S_n.

Let S_n act on on the left on $C_n(A)$ by $\pi\, a_0[a_1, ..., a_n] = a_0\, [a_{\pi^{-1}1}, ..., a_{\pi^{-1}n}]$.
Then $[a_1, ..., a_i] * [a_{i+1}, ..., a_n] = s_{i,n-i}[a_1, ..., a_n]$.

Definition I.2.3.

Let $Sh_n(A) = \sum_{i=1}^{n-1} \mathrm{Im}\, s_{i,n-i}$. Let $s_n = \sum_{i=1}^{n-1} s_{i,n-i}$. ($s_1 = s_0 = 0$).

The element s_n is called the *total shuffle operator*.
The shuffle product is well-behaved under the boundary :

Proposition I.2.4.

$$b([a_1, \dots, a_i] * [a_{i+1}, \dots, a_n]) = (b[a_1, \dots, a_i]) * [a_{i+1}, \dots, a_n] + (-1)^i [a_1, \dots, a_i] * (b[a_{i+1}, \dots, a_n])$$ □

It follows immediately that $Sh_*(A)$ is a subcomplex of $C_*(A)$. So we can consider the quotient complex $Ch_*(A) = C_*(A) / Sh_*(A)$. This is the Harrison complex.

Definition I.2.5.

For M a symmetric A-bimodule, the *Harrison (co)homology* groups of A, with coefficients in M are given by :

$Harr_*(A/k; M) = H(M \otimes_A Ch_*(A))$,

$Harr^*(A/k; M) = H(Hom_A(Ch_*(A), M))$.

Equivalently, Harrison cohomology can be regarded as the homology of the complex consisting of those Hochschild cochains which vanish on shuffles. i.e. A Hochschild n-cochain f is a Harrison cochain if $fs_{i,n-i} = 0$, for $i = 1, \dots, n-1$.

Section I.3 : The Decomposition of Hochschild (Co)homology

Throughout this section it is assumed that A is a commutative algebra over k containing $\mathbb{Q}$. The preliminary results used to prove the existence of a decomposition of Hochschild (co)homology in this situation are contained in Barr's paper 'Harrison homology, Hochschild homology, and triples' [B]. They are summarised below.

Let $sgn : \mathbb{Q}S_n \to \mathbb{Q}$ be the algebra homomorphism extending the usual alternating representation $sgn : S_n \to \mathbb{Q}$.

Definition I.3.1.

Let $\varepsilon_n = 1/n! \sum_{\pi \in S_n} (sgn\ \pi)\ \pi$

It is then clear that for all u in $\mathbb{Q}S_n$, $u\varepsilon_n = \varepsilon_n u = (sgn\ u)\ \varepsilon_n$.

Barr shows that this element, applied to any chain, vanishes under the boundary b, and that it is essentially the only such element :

Proposition I.3.2.

1). For all $[a_1, ..., a_n]$, $b\varepsilon_n[a_1, ..., a_n] = 0$.

2). If u in $\mathbb{Q}S_n$ satisfies $bu[a_1, ..., a_n] = 0$ for all $[a_1, ..., a_n]$ then $u = (\mathrm{sgn}\, u)\, \varepsilon_n$. □

The behaviour of the shuffle product under the boundary gives :

Proposition I.3.3.

For $n \geq 1$, $bs_n = s_{n-1}b$. ($s_1 = s_0 = 0$). □

Now we are in a position to introduce the idempotent e_n in $\mathbb{Q}S_n$, which was defined inductively by Barr.

Theorem I.3.4.

For each $n \geq 2$ there exists an element e_n of $\mathbb{Q}S_n$, ('Barr's idempotent'), satisfying :

1). e_n is a polynomial in s_n without constant term;

2). $\mathrm{sgn}\, e_n = 1$;

3). $be_n = e_{n-1}b$;

4). $e_n^2 = e_n$;

5). $e_n s_{i,n-i} = s_{i,n-i}$ for $1 \leq i \leq n-1$. □

Theorem I.3.5.

1). $e_n\mathbb{Q}S_n = \sum s_{i,n-i}\mathbb{Q}S_n$, so $Sh_n(A) = e_nC_n(A)$.

2). $C_n(A) = e_nC_n(A) \oplus (1-e_n)C_n(A)$, so $Ch_n(A) \simeq (1-e_n)\, C_n(A)$. □

So the Harrison complex $Ch_*(A)$ is a direct summand of the Hochschild complex $C_*(A)$, and hence Harrison homology, $Harr_*(A/k; M)$, is a direct summand of Hochschild homology, $HH_*(A/k; M)$.

Gerstenhaber and Schack [G-S,1] have shown how to extend the results of Barr to give a further decomposition of Hochschild homology in this situation. They show that in fact the Hochschild complex splits as a direct sum of sub-complexes :

$$C_n(A) = C_{1,n-1}(A) \oplus C_{2,n-2}(A) \oplus \ldots \oplus C_{n,0}(A),$$

where $C_{1,*-1}(A) = Ch_*(A)$, Harrison's complex.

Consider $s_n \in \mathbb{Q}S_n$. It is an element of a finite dimensional algebra over a field, and so must satisfy some monic polynomial with coefficients in that field. There is a unique one of lowest degree, called the *minimal polynomial* of s_n.

Theorem I.3.6.

The minimal polynomial of s_n is :

$$m_n(x) = \prod_{i=1}^{n} [\, x - (2^i - 2)\,] \qquad \square$$

Definition I.3.7.

Let $\mu_i = 2^i - 2$. Then $m_n(x) = (x-\mu_1)\ldots(x-\mu_n)$.

Then let

$$e_n(j) = \prod_{i \neq j} (s_n - \mu_i) \,/\, (\mu_j - \mu_i).$$

i.e. $e_n(j)$ is the value of the j^{th} Lagrange interpolation polynomial at s_n. The following is immediate:

Theorem I.3.8.

The $e_n(j)$'s are mutually orthogonal idempotents ('the Eulerian idempotents') satisfying :

$$\sum_{j=1}^{n} e_n(j) = 1, \text{ and } \sum_{j=1}^{n} \mu_j e_n(j) = s_n. \qquad \square$$

Notation

Let $e_n(j) = 0$ for $n < j$, and let $e_0(0) = 1$.

Theorem I.3.9.

1). $e_n = e_n(2) + \ldots + e_n(n)$;

2). $be_n(j) = e_{n-1}(j)b$; in particular $be_n(n) = 0$;

3). $\mathrm{sgn}\, e_n(j) = 0$ for $j \neq n$; $\mathrm{sgn}\, e_n(n) = 1$;

4). $e_n(n) = \varepsilon_n$. $\square$

Hence, the Hochschild chain complex $C_*(A)$ is a direct sum of the sub-complexes $e_*(j)C_*(A)$:

$$C_n(A) = \sum_{j=1}^{n} e_n(j)C_n(A).$$

This gives a decomposition of Hochschild (co)homology into a similar direct sum.

Definition I.3.10.

$HH_{j,n-j}(A/k; M)$ denotes the part of $HH_n(A/k;M)$ corresponding to $e_n(j)$.

Summarising the above we have :

Theorem I.3.11.

The Hochschild homology of a commutative algebra A over k containing $\mathbb{Q}$, with coefficients in a symmetric A-bimodule M, decomposes into a direct sum :

$HH_n(A/k; M) = HH_{1,n-1}(A/k; M) \oplus HH_{2,n-2}(A/k; M) \oplus \dots \oplus HH_{n,0}(A/k; M)$,

where $HH_{i,n-i}$ is the eigenspace for the eigenvalue 2^i-2 of the shuffle operator s_n.

In particular $HH_{1,n-1}$, the part corresponding to $e_n(1)$, is Harrison's n[th] homology group.

Similarly, for cohomology :

$HH^n(A/k; M) = HH^{1,n-1}(A/k; M) \oplus HH^{2,n-2}(A/k; M) \oplus \dots \oplus HH^{n,0}(A/k; M)$,

where $HH^{1,n-1}(A/k; M) = Harr^n(A/k; M)$. □

Gerstenhaber and Schack [G-S,2] also show that the n[th] component of the decomposition of $HH^n(A/k; M)$ consists of the 'skew multiderivations', where an n-cochain f is *skew* if $f\sigma = (\mathrm{sgn}\,\sigma)f$ for all $\sigma \in S_n$ and is a *multiderivation* if it is a derivation of each argument (all others being held fixed). For homology, the result is $HH_{n,0}(A/k; M) \cong M \otimes_A \Omega^n_{A/k}$. The decomposition coincides with the usual Hodge decomposition of $H^*(X, \mathbb{C})$, for X a smooth complex projective variety [G-S,3].

Gerstenhaber and Schack [G-S,1] also note that the $e_n(j)$'s are essentially the only operators which commute with the Hochschild boundary :

Theorem I.3.12.

If we have $t_n \in \mathbb{Q}S_n$ for n = 1, 2, ... such that $bt_n = t_{n-1}b$ for all n, then

$$t_n = \sum_{j=1}^{n} (\mathrm{sgn}\, t_j)\, e_n(j).$$ □

Section I.4 : The Decompositon of Cyclic (Co)homology

In the case where we work over a ground ring containing $\mathbb{Q}$, cyclic (co)homology is particularly simple to define. Essentially it is computed from the quotient of the Hochschild complex under the actions of the cyclic groups.

To calculate the Hochschild homology of A with coefficients A, we use the complex $(A^{\otimes(*+1)}, b)$ where $b : A^{\otimes(n+1)} \to A^{\otimes n}$ is given by :

$$b(a_0\otimes \dots \otimes a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0\otimes \dots \otimes a_i a_{i+1}\otimes \dots \otimes a_n) + (-1)^n(a_n a_0\otimes \dots \otimes a_{n-1}).$$

We denote its homology, $HH_*(A/k; A)$, simply by $HH_*(A)$.

We let the permutations of $\{0, 1, \dots, n\}$ act on $A^{\otimes(n+1)}$ by :

$$\pi(a_0\otimes \dots \otimes a_n) = (a_{\pi^{-1}0}\otimes \dots \otimes a_{\pi^{-1}n}).$$

Notation

Let λ_{n+1} denote the (n+1)-cycle $(0\ 1\ 2 \dots n)$ and let $t = (\operatorname{sgn}\lambda_{n+1})\lambda_{n+1}$.

We have the relation $b(1-t) = (1-t)b'$ (see [L–Q]). Hence we can consider the quotient complex $C^\lambda_*(A)$, where $C^\lambda_n(A) = A^{\otimes(n+1)}/(1-t)$, meaning that we divide out by the k-linear span of the elements of the form $(1-t)x$ in $A^{\otimes(n+1)}$. The boundary in this complex is again b.

Definition 1.4.1.

The homology of the complex $(C^\lambda_*(A), b)$ is denoted $H^\lambda_*(A)$. If the ground ring k contains $\mathbb{Q}$ this is the *cyclic homology* of A.

For cohomology, we use the dual complex $(C_\lambda^*(A), b)$, where $C_\lambda^n(A)$ consists of functions $f : A^{\otimes(n+1)} \to k$ which are cyclic, that is $f\lambda_{n+1} = (\operatorname{sgn}\lambda_{n+1})f = (-1)^n f$. The homology of this complex is denoted $H_\lambda^*(A)$, and is the cyclic cohomology of A if k contains $\mathbb{Q}$.

When A is commutative we have $be_n(j) = e_{n-1}(j)b$ as in the previous section. Now, when k contains $\mathbb{Q}$, all the arguments applied to Hochschild cohomology will apply to give a similar decomposition for cyclic cohomology provided that when f is a cyclic cochain so is each $fe_n(j)$. This has been proved by Natsume and Schack [N–S].

In order to define cyclic (co)homology in a characteristic free context, we introduce the (normalised) b–B double complex of Connes [C] (See [L–Q]) :

$$\begin{array}{ccccc}
\vdots & & \vdots & & \vdots \\
b\downarrow & B & b\downarrow & B & b\downarrow \\
A\otimes\overline{A}^{\otimes 2} & \longleftarrow & A\otimes\overline{A} & \longleftarrow & A \\
b\downarrow & & b\downarrow & & \\
A\otimes\overline{A} & \longleftarrow & A & & \\
b\downarrow & B & & & \\
A & & & &
\end{array}$$

Here, $\overline{A} = A/k$, regarding k as contained in A as multiples of the identity element. Each column of the double complex is a normalised Hochschild complex $(A\otimes\overline{A}^{\otimes *}, b)$, whose homology is still the Hochschild homology, $HH_*(A)$. The map $B: A\otimes\overline{A}^{\otimes(n-1)} \to A\otimes\overline{A}^{\otimes n}$ is given by

$$B(a_0\otimes a_1\otimes \ldots \otimes a_{n-1}) = \sum_{j=0}^{n-1}(-1)^{(n-1)j}(1\otimes a_j\otimes \ldots \otimes a_{n-1}\otimes a_0\otimes \ldots\otimes a_{j-1}).$$

One has $B^2 = 0$ and $Bb + bB = 0$ (See [L-Q]).

Definition I.4.2.

The *cyclic homology* of A, $HC_*(A)$, is given by the homology of the total complex corresponding to this double complex. For *cyclic cohomology*, $HC^*(A)$, we use the dual double complex.

Over k containing $\mathbb{Q}$ this definition coincides with the previous one, $HC_*(A) \cong H_*^{\lambda}(A)$. In this context, we need the additional relation $Be_{n-1}(i) = e_n(i+1)B$ to get the decomposition of cyclic homology for a commutative algebra, since then the double complex decomposes as the direct sum of the sub-double complexes [L,2]:

$$\begin{array}{ccccc}
\vdots & & \vdots & & \\
b\downarrow & B & b\downarrow & B & \\
e_n(j)(A\otimes\overline{A}^{\otimes n}) & \longleftarrow & e_{n-1}(j-1)(A\otimes\overline{A}^{\otimes(n-1)}) & \longleftarrow & \cdots \\
b\downarrow & B & b\downarrow & B & \\
e_{n-1}(j)(A\otimes\overline{A}^{\otimes(n-1)}) & \longleftarrow & e_{n-2}(j-1)(A\otimes\overline{A}^{\otimes(n-2)}) & \longleftarrow & \cdots \\
b\downarrow & & b\downarrow & & \\
\vdots & & \vdots & &
\end{array}$$

Denoting by $HC_{j,n-j}(A)$ the homology of the total complex corresponding to the above, we have :

Theorem I.4.3.

For A a commutative algebra over k containing $\mathbb{Q}$, $HC_{0,0}(A) = A$, and for $n \geq 1$,
$HC_n(A) = HC_{1,n-1}(A) \oplus \ldots \oplus HC_{n,0}(A)$. □

There is a long exact sequence linking Hochschild and cyclic homology, often refered to as the periodicity sequence :

$$\ldots \rightarrow HH_n(A) \xrightarrow{I} HC_n(A) \xrightarrow{S} HC_{n-2}(A) \xrightarrow{B} HH_{n-1}(A) \rightarrow \ldots$$

Here, I is induced by the inclusion of the first column in the b–B double complex and S, the periodicity operator, is induced by the map which shifts degrees in the total complex by 2. Natsume and Schack [N–S] and Loday [L,2] have shown that for a commutative algebra over a ground ring containing $\mathbb{Q}$ this sequence decomposes as a direct sum of long exact sequences :

$$\ldots \rightarrow HH_{r,n-r}(A) \xrightarrow{I_r} HC_{r,n-r}(A) \xrightarrow{S_r} HC_{r-1,n-r-1}(A) \xrightarrow{B_r} HH_{r,n-r-1}(A) \rightarrow \ldots$$

where I_r, B_r, S_r are the restictions of I, B, S.

For $n \geq 3$, Harrison homology $Harr_n(A/k; A)$ is the first part of the decomposition of cyclic homology $HC_n(A)$. For a smooth algebra or the algebra of smooth functions on a compact manifold the decomposition of $HC_*(A)$ coincides with those given by Loday and Quillen [L–Q] and Connes [C] in terms of de Rham homology [N–S], [L,2].

Section I.5 : The Symmetric Group Representations given by the Eulerian Idempotents

For each n = 1, 2, ... we have described mutually orthogonal idempotents $e_n(j)$ in $\mathbb{Q}S_n$, which give a decomposition of Hochschild (co)homology. These idempotents give representations $e_n(j)\mathbb{Q}S_n$ of the symmetric group S_n, which have been studied by Hanlon [H]. In this section we summarize his main results.

Let χ_n^j denote the character of the representation $e_n(j)\mathbb{Q}S_n$. Hanlon gives a formula for

χ_n^j as a certain direct sum of induced characters. These characters are induced from wreath product groups, which we shall describe briefly. See [J-K, Ch. 4] for more details.

Suppose we have a subgroup G of S_m of size g and a subgroup H of S_n of size h. Then the wreath product, HwrG, is a subgroup of S_{mn} of size $g.h^m$, consisting of (m+1)-tuples $(h_1, ..., h_m, g)$, with $h_i \in H$ for all i and $g \in G$. We think of HwrG as acting on $\{ (j, i) \mid 1 \leq j \leq n, 1 \leq i \leq m \}$ by $(h_1, ..., h_m, g)(j, i) = (h_{g(i)}(j), g(i))$.

If α is a linear character of H and β is a linear character of G, then there is a linear character $\alpha wr\beta$ of HwrG as follows. For each cycle $Y = (y_1, ..., y_s)$ of g, define A(Y) by $A(Y) = \alpha(h_{y_1} ... h_{y_s})$. Then define

$$(\alpha wr\beta) (h_1, ..., h_m, g) = \beta(g) \{ \prod_Y A(Y) \}.$$

The particular case we shall be interested in is where $H = C_n = < (1\ 2 ... n) >$, $G = S_m$. We may think of $C_n wr S_m$ as the group of mxm pseudo permutation matrices, where the non-zero entries are chosen from C_n. We consider the character $\alpha wr\beta$, where β is the trivial character of S_m, and α is the linear character of H given by $\alpha(1\ 2 ... n) = e^{2\pi i/n}$.

Now for any $\sigma \in S_n$ we describe a linear character ξ_σ of its centraliser, $Z(\sigma)$. Suppose σ has m_u u-cycles for each u. Then $Z(\sigma)$ is isomorphic to a direct product over u of the wreath products $C_u wr\ S_{m_u}$. The character ξ_σ is then a product of linear characters $\xi_\sigma^{(u)}$ of $C_u wr\ S_{m_u}$, where $\xi_\sigma^{(u)}$ is exactly the character $\alpha wr\beta$ above, with β the trivial character of S_{m_u} and α the linear character of C_u given by $\alpha(1\ 2 ... u) = e^{2\pi i/u}$.

Now we can state Hanlon's main result [H].

Theorem I.5.1.

$$\chi_n^j = \text{sgn} * \left(\oplus\ \text{ind}_{Z(\sigma_\mu)}^{S_n} (\xi_{\sigma_\mu}) \right),$$

where the sum is over all partitions μ of n with exactly j parts, σ_μ is any permutation in S_n with cycle type μ, and $*$ denotes product of characters. □

This determines the dimensions of the representations $e_n(j)\mathbb{Q}S_n$.

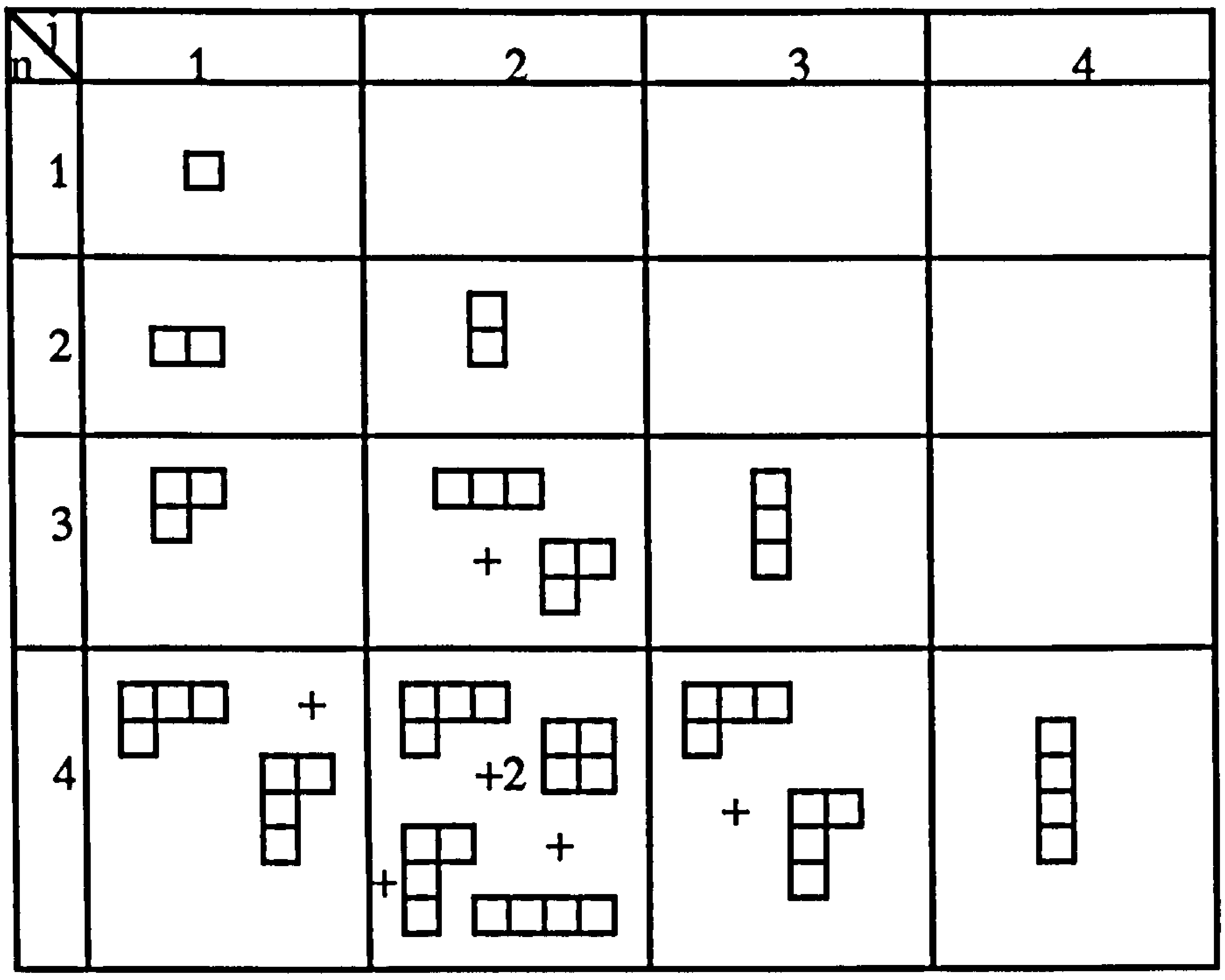
Corollary I.5.2.

The representation $e_n(j)\mathbb{Q}S_n$ has dimension equal to the number of permutations in S_n with exactly j cycles. □

Note that for j = 1 the formula above gives simply: $\chi_n^1 = \text{sgn} * \left(\text{ind}_{C_n}^{S_n}(\xi_{(12\ldots n)})\right).$

A general formula giving the decomposition of each representation $e_n(j)\mathbb{Q}S_n$ into irreducibles is not known, although Hanlon [H] states some results for particular values of j. For small values of n, Hanlon gives the following table of decompositions.

Table I.5.3.



For fixed n, the sum of the representations $e_n(j)\mathbb{Q}S_n$ is the regular representation of S_n. So, for μ a partition of n, the multiplicity of the irreducible representation [μ] in this sum is the number of standard Young tableaux of shape μ. The n^{th} component, $e_n(n)\mathbb{Q}S_n$, is always the sign representation. Hanlon gives explicit decompositions for the cases j = n−1 and j = n−2. He shows that the trivial representation appears in $e_n([\,(n+1)/2\,])\mathbb{Q}S_n$. For the case j = 1, we have the following result giving the decomposition of $e_n(1)\mathbb{Q}S_n$ into irreducibles, due originally to Kraskiewicz and Weyman [K−W]. Let T be a standard Young tableau. We call a number i an *ascent* of T if i+1 appears to the right of i in T. We let a(T) denote the sum of the ascents of T.

Theorem I.5.4.

Let μ be a partition of n. Then the multiplicity of [μ] in $e_n(1)\mathbb{Q}S_n$ is the number of standard Young tableaux T of shape μ with a(T) ≡ 1 (mod n). □

Stembridge [S] has also supplied a proof of this result.

Chapter II : Γ-(Co)homology

Introduction

This chapter is devoted to describing Dr. Alan Robinson's new (co)homology theory for commutative algebras, Γ-(co)homology. In fact this theory is applicable not just to strictly commutative algebras but also to those which are commutative up to homotopy. Here we will describe the algebraic version of the theory.

The first section describes the Γ category and gives a presentation.

From here on the material in this chapter is Dr. Robinson's, and we quote without proof the results we need to desribe the theory. First we cover tree spaces. These are particular building-like simplicial complexes, which are related to the nerve of the Γ category. Section 2 is devoted to a simple description, and a statement of the theorem on their homotopy type [Rob,1]. Section 3 relates them to the Γ category.

Finally, we define Γ-(co)homology of a commutative algebra, by giving a chain complex to compute it [Rob,2]. This involves the nerve of the Γ category, which can then be replaced by the tree space. The chain complex has a natural filtration, which gives rise to a spectral sequence in the usual way.

Section II.1 : The Category Γ

Definition II.1.1.

Γ is the category whose objects are the sets $[\, n\,] = \{ 1, ..., n \}$, $n \geq 1$, and whose morphisms are surjective maps.

We can give a presentation of this category as follows :

Let $\eta^i : [\,n+1\,] \to [\,n\,]$ be defined by $\eta^i(j) = \begin{cases} j & \text{if } j \leq i \\ j-1 & \text{if } j > i \end{cases}$, for $i = 1, ..., n$.

Then $\eta^i\eta^j = \eta^j\eta^{i+1}$ for $j \leq i$.

Each morphism $\gamma : [\,n\,] \to [\,n-k\,]$ of Γ may be expressed as a permutation $\sigma \in S_n$, followed by an order preserving surjection, i.e. as $\sigma \in S_n$ followed by a product of k η's, but not uniquely.

We use the standard presentation of S_n, with generators the transpositions $\sigma_i = (i\ i{+}1)$, and relations $\sigma_i^2 = 1$, $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$, $\sigma_i\sigma_j = \sigma_j\sigma_i$ for $|i-j| \geq 2$.

Then it is straightforward to check that :

$$\sigma_i \eta^j = \begin{cases} \eta^j \sigma_i & \text{if } i < j-1 \\ \eta^{j-1} \sigma_j \sigma_{j-1} & \text{if } i = j-1 \\ \eta^{j+1} \sigma_j \sigma_{j+1} & \text{if } i = j \\ \eta^j \sigma_{i+1} & \text{if } i > j \end{cases} \qquad \eta^j \sigma_j = \eta^j.$$

If $\gamma^{-1}(i)$ denotes the set of inverse images of i in order, and $p_i = | \gamma^{-1}(i) |$, then we can write each morphism γ of Γ uniquely as the permutation σ which is order preserving on inverse images, $\sigma : \{ \gamma^{-1}(1), \gamma^{-1}(2), \dots, \gamma^{-1}(n-k) \} \to \{ 1, 2, \dots, n \}$ in order, followed by $(\eta^{n-k})^{p_{n-k}-1} \dots (\eta^2)^{p_2-1}(\eta^1)^{p_1-1}$. Note that σ^{-1} is a $(p_1, \dots, p_{n-k})$-multishuffle of $\{ 1, \dots, n \}$.

If we drop the relations $\sigma_i^2 = 1$, we replace S_n by the braid group B_n, giving a braid group version of the Γ category.

Section II.2 : Tree Spaces

Definition II.2.1.

A *tree* is a compact contractible one-dimensional polyhedron. It is always triangulated so that each vertex is either an *end* (i.e. belongs to exactly one edge, called a *free edge*), or a *node* (i.e. belongs to at least three edges). Edges which are not free are called *internal edges.*

Let $n \geq 2$. An *n-tree* is a tree such that :

1). it has exactly n+1 ends, labelled by 0, 1, 2, ..., n;

2). each internal edge α has a *length* $l(\alpha)$, $0 < l(\alpha) \leq 1$.

Definition II.2.2.

An *isomorphism of n-trees* is a homeomorphism which is isometric on edges and which preserves the labelling of the ends.

$\overline{T}_n$ is the space of isomorphism classes. We shall continue to refer to trees, although we actually mean isomorphism classes of trees.

$\overline{T}_n$ is a cubical complex. Two trees lie in the same cube if they differ only in internal edge lengths. These give the coordinates in the cube. It is a cone, since given any n-tree we can reach the tree without internal edges by contracting all the internal edges at once.

Definition II.2.3.

T_n, *the space of fully-grown n-trees,* is the base of the cone. It consists of trees that have at least one internal edge length equal to 1.

We quote results on the structure of T_n without proof. See Robinson [Rob,1] for details.

Proposition II.2.4.

1). T_n can be triangulated as a simplicial complex, such that every simplex is the face of an (n-3)-simplex.

2). Every (n-4)-simplex of T_n is a face of exactly three (n-3)-simplexes. □

A simplex of T_n corresponds to a *shape* of a fully-grown tree (i.e. an equivalence class under label-preserving homeomorphism) [Rob, 1]. Its faces correspond to those tree shapes obtained by shrinking an internal edge to zero. Again, we may use 'tree' to mean 'tree shape'.

Theorem II.2.5.

T_n is homotopy equivalent to a wedge of spheres, $T_n \simeq \bigvee_{(n-1)!} S^{n-3}$. □

Hence, T_n has only one non-zero (reduced) homology group, with coefficients K, $\overline{H}_{n-3}(T_n; K) = K^{(n-1)!}$. Since S_{n+1} acts on T_n, by permuting the labels 0, 1, ..., n on trees, this homology group gives a representation of S_{n+1}. In fact, in terms of Γ-cohomology we need only consider it as a representation of S_n, and we shall denote this KS_n-module by V_n. Robinson shows that the homology generators are regularly permuted by S_{n-1}.

Section II.3 : The Relationship between the Γ-category and the Tree Spaces

A standard construction gives, for any small category C, a simplicial set NC, the nerve of the category.

Definition II.3.1.

The *nerve* of a category C is a simplicial set NC with :
NC_0 = obj C, and for $k \geq 1$,

$$NC_k = \{ \text{composable strings of k morphisms in C} : C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \rightarrow \cdots \xrightarrow{f_k} C_k \}.$$

Denote such a string by $[f_k \mid f_{k-1} \mid ... \mid f_1]$.
Face maps $\partial_i : NC_k \rightarrow NC_{k-1}$ are given by :
$\partial_0 [f_k \mid f_{k-1} \mid ... \mid f_1] = [f_k \mid f_{k-1} \mid ... \mid f_2]$
$\partial_i [f_k \mid f_{k-1} \mid ... \mid f_1] = [f_k \mid ... \mid f_{i+1}f_i \mid ... \mid f_1]$ for $0 < i < k$,
$\partial_k [f_k \mid f_{k-1} \mid ... \mid f_1] = [f_{k-1} \mid ... \mid f_1]$.
Degeneracies $s_i : NC_k \rightarrow NC_{k+1}$, for i = 0, ..., k, are given by inserting identity maps :
$s_i [f_k \mid ... \mid f_{i+1} \mid f_i \mid ... \mid f_1] = [f_k \mid ... \mid f_{i+1} \mid 1 \mid f_i \mid ... \mid f_1]$.

We work with NΓ, and consider NΓ([n], [1]), consisting of the strings of morphisms in Γ, starting at [n] and ending at [1], and their faces. Intuitively, such a string of morphisms looks like an n-tree, where the end labelled 0 marks the end of the string.

We will also consider certain categories associated to the Γ-category. Thus [n]/Γ/[1] will denote the category of finite sets and surjections strictly under [n] and over [1]. This means that the category has objects given by a set [r], 1 < r < n, and two morphisms [n] → [r] → [1]. A morphism of the category is a surjection $[r_1] \rightarrow [r_2]$ such that the following diagram commutes:

$$\begin{array}{ccccc} [n] & \rightarrow & [r_1] & \rightarrow & [1] \\ & \searrow & \downarrow & \nearrow & \\ & & [r_2] & & \end{array}$$

The symmetric group S_n acts on this category, and hence on its nerve, by precomposition with permutations.

The relationship between the Γ category and the tree space is given by:

Proposition II.3.2.

There is an S_n-equivariant map, $\theta : | N([n]/\Gamma/[1]) | \rightarrow T_n$, which is a homotopy equivalence. This map is described in [Rob, 3]. □

$$\text{Hence, } \overline{H}_r(| N([n]/\Gamma/[1]) |; K) = \begin{cases} V_n & \text{if } r = n-3 \\ 0 & \text{otherwise} \end{cases}$$

Section II.4 : Γ-(Co)homology of Commutative Algebras

Let $g : [\,n\,] \to [\,m\,]$ be a morphism in Γ. By looking at the inverse image of each point in $[\,m\,]$, we may consider the components of g, $g^i : [\,n^i\,] \to [\,1\,]$, where $n^i = |\,g^{-1}(i)\,|$. Similarly, given a string of k morphisms of Γ ending at $[\,m\,]$, $[\,f_k \,|\, \ldots \,|\, f_1\,]$, (i.e. a k-simplex of NΓ), we decompose this into m strings of k morphisms each of which ends at $[\,1\,]$. We denote by $[\,f_k^{(i)} | \ldots | f_1^{(i)}\,]$ the i^{th} component of $[\,f_k \,|\, \ldots \,|\, f_1\,]$.

We are now in a position to define the chain complex giving Γ-homology. Let K be a commutative ground ring, and A a flat commutative algebra over K. We will define the Γ-homology of A, with coefficients in an A-module M. We have described $N\Gamma(\,[\,n\,],[\,1\,]\,)$. We denote by $KN\Gamma(\,[\,n\,],[\,1\,]\,)$ the corresponding free K-module. In what follows ⊗ denotes $\otimes_K$.

Definition II.4.1.

The *Γ-chain complex*, C_*^Γ, is defined as follows :

$$C_q^\Gamma(A/K;\ M) = \sum_{n\geq 1} KN\Gamma_q(\,[\,n\,],[\,1\,]\,) \otimes A^{\otimes n} \otimes M.$$

The boundary $d : C_q^\Gamma \to C_{q-1}^\Gamma$ is the alternating sum of face maps $\partial_i : C_q^\Gamma \to C_{q-1}^\Gamma$ for $0 \leq i \leq q$.

Note that a Γ-morphism $f : [\,n\,] \to [\,m\,]$ induces a map $f_* : A^{\otimes n} \to A^{\otimes m}$ by multiplying and permuting factors according to f in the obvious way.

Then the face maps are defined on generators $[\,f_q \,|\, \ldots \,|\, f_1\,] \otimes a \otimes m$ by :

$$\partial_0(\,[\,f_q | \ldots | f_1\,] \otimes a \otimes m\,) = [\,f_q | \ldots | f_2\,] \otimes f_{1*}(a) \otimes m$$
$$\partial_i(\,[\,f_q | \ldots | f_1\,] \otimes a \otimes m\,) = [\,f_q | \ldots | f_{i+1}f_i | \ldots | f_1\,] \otimes a \otimes m \text{ for } 0 < i < q$$
$$\partial_q(\,[\,f_q | \ldots | f_1\,] \otimes a \otimes m\,) =$$
$$\sum_{i=1}^{r} \left([\,f_{q-1}^{(i)} | \ldots | f_1^{(i)}\,] \otimes \left(a_{i_1} \otimes \ldots \otimes a_{i_k}\right) \otimes \left(\prod_{j \notin f_1^{-1} \ldots f_{q-1}^{-1}(i)} a_j \right) m \right)$$

where $f_{q-1} \ldots f_1 : [\,n\,] \to [\,r\,]$, and where $\{\,a_{i_1}, \ldots, a_{i_k}\}$ is the ordered preimage of i under $f_{q-1} \ldots f_1$.

Now, $d = \sum_{i=0}^{q}(-1)^i \partial_i$. It can be checked that these face operators satisfy the usual simplicial identities and so we get $d^2 = 0$. Hence we can define the homology of the chain complex in the usual way, giving the *Γ-homology* of A over K with coefficients in M, $H\Gamma_q(A/K; M)$. Of course, we can also define *Γ-cohomology*, using the corresponding cochain complex. For cohomology, A should be a projective K-algebra. (For a general commutative algebra the topological definition of Γ-(co)homology may be used [Rob,2], which coincides with the algebraic version described here when A is flat (projective)).

There is a natural filtration of the chain complex C_*^Γ by the size of the sets [n] in Γ :

$$F^pC_*^\Gamma(A/K; M) = \sum_{1\leq n\leq p} \left(K N\Gamma([n], [1]) \otimes A^{\otimes n} \otimes M\right).$$

This filtration of C_*^Γ gives rise to a spectral sequence in the usual way, and since $\bigcup_p F^pC_*^\Gamma(A/K; M) = C_*^\Gamma(A/K; M)$ this spectral sequence converges to Γ-homology:

$$E^1_{p,q} = H_{p+q-1}\left(\frac{F^pC_*^\Gamma(A/K; M)}{F^{p-1}C_*^\Gamma(A/K; M)}\right) \underset{p}{\Rightarrow} H\Gamma_{p+q-1}(A/K; M).$$

Now $F^p/F^{p-1} = KN\Gamma([p], [1]) \otimes A^{\otimes p} \otimes M$, with boundary induced from the boundary d described above. I.e. we get all the boundary terms involving $A^{\otimes p}$ from this boundary, other terms being zero.

We calculate the d^1 boundary in the spectral sequence :

$$d^1 : H_{p+q-1}(F^p/F^{p-1}) \rightarrow H_{p+q-2}(F^{p-1}/F^{p-2}).$$

This is the connecting homomorphism from the short exact sequence of chain complexes :

$$0 \rightarrow F^{p-1}/F^{p-2} \underset{i}{\rightarrow} F^p/F^{p-2} \underset{\pi}{\rightarrow} F^p/F^{p-1} \rightarrow 0.$$

By definition of the connecting homomorphism d^1 is given by $[z] \rightarrow [i^{-1}d\pi^{-1}z]$, where d is the boundary in the chain complex F^p/F^{p-2}. This is the same as the original boundary except terms in $A^{\otimes \leq p-2}$ become zero. Then d^1 is given by $[z] \rightarrow [\partial z]$ where ∂ is given by the terms of the original boundary which take $A^{\otimes p}$ terms to $A^{\otimes p-1}$ terms.

i.e. $\partial : KN\Gamma([p], [1]) \otimes A^{\otimes p} \otimes M \rightarrow KN\Gamma([p-1], [1]) \otimes A^{\otimes p-1} \otimes M$.

Then $\partial = \partial_0 + (-1)^{p+q-1}\partial_{p+q-1}$ where :

$$\partial_0\big([\ f_{p+q-1}|\ldots|\ f_1\ ]\otimes a\otimes m\big)=\begin{cases}[\ f_{p+q-1}|\ \ldots\ |\ f_2\ ]\otimes f_{1*}(a)\otimes m & \text{if } f_1\colon [p]\to[p-1]\\ 0 & \text{otherwise}\end{cases}$$

$$\partial_{p+q-1}([\ f_{p+q-1}|\ldots|\ f_1\ ]\otimes a\otimes m)=$$
$$\begin{cases}\text{component of } [\ f_{p+q-2}|\ \ldots\ |\ f_1\ ] \text{ which is } [p-1]\to[1] & \text{if such exists}\\ 0 & \text{otherwise}\end{cases}$$

In order to calculate the E^1-term, we use a second spectral sequence which converges to it. For each p, consider a term of the filtration quotient F^p/F^{p-1}, $[\ f_{p+q-1}\ |\ldots|\ f_1\ ]\otimes a\otimes m$. Firstly, we normalise the complex so that we may assume that f_{p+q-1} is not an isomorphism. On the other hand, our string of morphisms may begin with some number of isomorphisms, $f_1, \ldots, f_r$, say. We filter the filtration quotient according to this number r. A (p+q−1)-simplex, $[\ f_{p+q-1}\ |\ldots|\ f_1\ ]$, as above can now be seen as an element of:

$$N^+([\ p\ ]/\Gamma/[\ 1\ ])_{p+q-r-3}\otimes C_r(S_p),$$

where $C_r(S_p)$ denotes a string of r permutations in S_p (as in the standard bar construction for S_p - see [Mc] IV.5) and N^+ denotes the *augmented* nerve, i.e. there is an additional simplex of dimension −1 corresponding to the map $[\ p\ ]\to[\ 1\ ]$. For each p, the complex F^p/F^{p-1} with the r-filtration gives a second spectral sequence, converging to the E^1 term of the original spectral sequence:

$$F^1_{p,r,p+q-r-1}\underset{r}{\Rightarrow}E^1_{p,q}.$$

The filtration quotient for the second filtration now contains only those strings of morphisms which begin at [p] and begin with exactly r isomorphisms.

$$F^0_{p,r,p+q-r-1}=KN^+([\ p\ ]/\Gamma/[\ 1\ ])_{p+q-r-3}\otimes C_r(S_p)\otimes A^{\otimes p}\otimes M.$$

In this quotient the boundaries which decrease the r-filtration are zero, and so the differential is just the boundary operator in $N([\ p\ ]/\Gamma/[\ 1\ ])$. Thus,

$$F^1_{p,r,p+q-r-1}=\overline{H}_{p+q-r-3}(N([\ p\ ]/\Gamma/[\ 1\ ]);K)\otimes C_r(S_p)\otimes A^{\otimes p}\otimes M.$$

In the previous section, we stated results on the homology of $N([\ p\ ]/\Gamma/[\ 1\ ])$, which now allow us to see that the second spectral sequence collapses.

$$F^1_{p,r,p+q-r-1}=\begin{cases}V_p\otimes C_r(S_p)\otimes A^{\otimes p}\otimes M & \text{if } r=q\\ 0 & \text{otherwise.}\end{cases}$$

In F^1 the differential d^1 is induced by the differential in the original chain complex, and so is now just the differential in the two sided bar construction on S_p, acting on V_p on one side and on $A^{\otimes p}$ on the other. So the E^1 term of our original spectral sequence is given by

$$E^1_{p,q}=F^2_{p,q,p-1}=\mathrm{Tor}_q^{KS_p}\big(V_p,\ A^{\otimes p}\big)\otimes M.$$

Chapter III : Γ-(Co)homology in Characteristic Zero

Introduction

The main result of this chapter is Theorem III.4.2., which states that the Γ-homology of a flat commutative algebra coincides with the Harrison homology in the case where we work over k containing $\mathbb{Q}$. In Chapter I, Harrison homology was defined in terms of permutations called shuffles. Γ-homology was described in the previous chapter, in terms of tree spaces.

Firstly we need more information about the structure of the tree spaces. We already know their homology groups, but we need to explicitly identify the cycles. In particular, we identify a cycle in T_n, denoted c_n, which consists of 'trees with cyclic labelling'.

Once we have done this we can proceed to explain how shuffles act on the tree space. We show that the cycle described above, c_n, vanishes on shuffles. This enables us to describe the representation V_n of the symmetric group S_n given by the tree space T_n. We show that, working over k containing $\mathbb{Q}$, V_n is $e_n(1)kS_n$, where $e_n(1)$ is the idempotent in $\mathbb{Q}S_n$ described in Chapter I. (Definition I.3.9.). Since Harrison (co)homology is the part of the decomposition of Hochschild (co)homology corresponding to this idempotent $e_n(1)$, this gives us the connection with Harrison cohomology.

Next, we return to the spectral sequence giving Γ-homology which we had at the end of the last chapter. It turns out that this spectral sequence collapses when we work over k containing $\mathbb{Q}$, leaving us with a relatively simple chain complex to compute the Γ-homology. We show that this chain complex is isomorphic to the Harrison chain complex, and hence we get the result.

Finally, we consider what we can say without assuming characteristic zero. We show that the edge of the spectral sequence for Γ-homology still gives Harrison homology, and so in general we have a homomorphism between the Harrison and Γ theories.

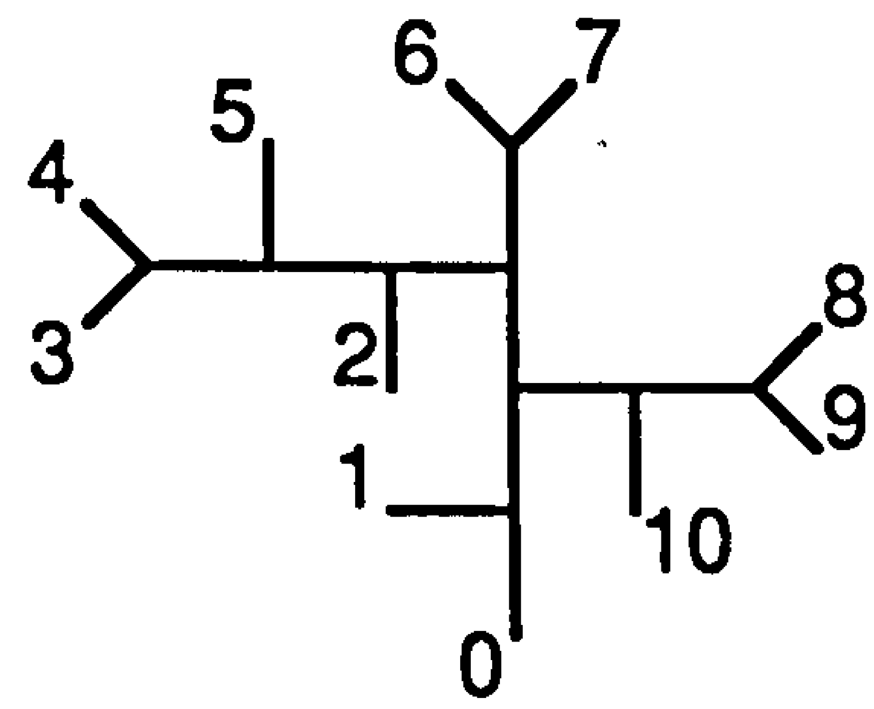
Section III.1 : Cycles in the Tree Space T_n

We have seen that $T_n \simeq \bigvee_{(n-1)!} S^{n-3}$. In this section we give a precise description of the generators of the only non-trivial homology group, $\overline{H}_{n-3}(T_n; k)$.

Definition III.1.1.

An n-tree has *cyclic labelling* if it can be drawn in such a way that the labels 0, 1, ..., n are encountered in order as you go around the tree.

Example

The 10-tree : 

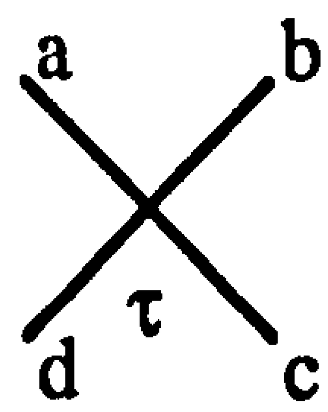
has cyclic labelling.

Proposition III.1.2.

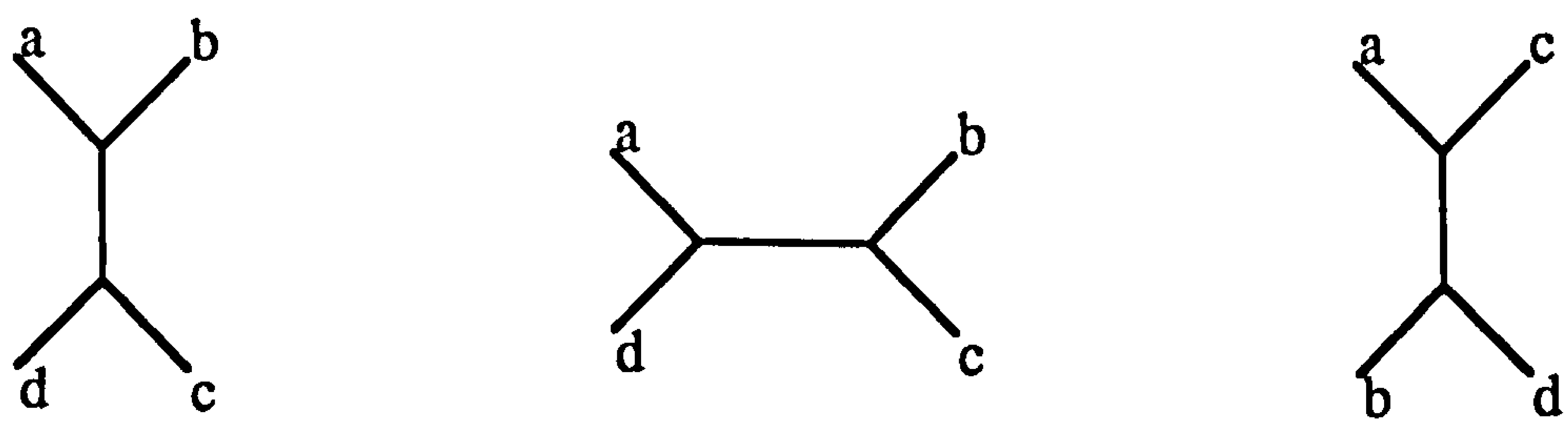
The (n-3)-dimensional simplices of T_n given by n-trees with cyclic labelling can be oriented so that they form a cycle.

Proof

Let t be such an (n-3)-simplex in T_n. So t is a cyclically-labelled n-tree with n-2 internal edges. Its boundaries are given by deleting internal edges. Since t has cyclic labelling, each component of its boundary is an (n-4)-simplex, also with cyclic labelling. An (n-4)-simplex is given by an n-tree with one node τ of order 4, all other nodes having order 3. A given (n-4)-simplex is the face of 3 (n-3)-simplices, since in general there are 3 ways of pulling apart the node of order 4 :

The general (n-4)-simplex s, 

is the face of the following 3 (n-3)-simplices :



However, if the (n-4)-simplex s has cyclic labelling, then so do the first two of the (n-3)-simplices shown, but the third does not. So each boundary component of the top-dimensional simplices with cyclic labelling occurs exactly twice. It remains to show that we can choose orientations so that these cancel, and so we have a cycle.

Claim

The trees with cyclic labelling in T_n can be given orientations so that they form a cycle.

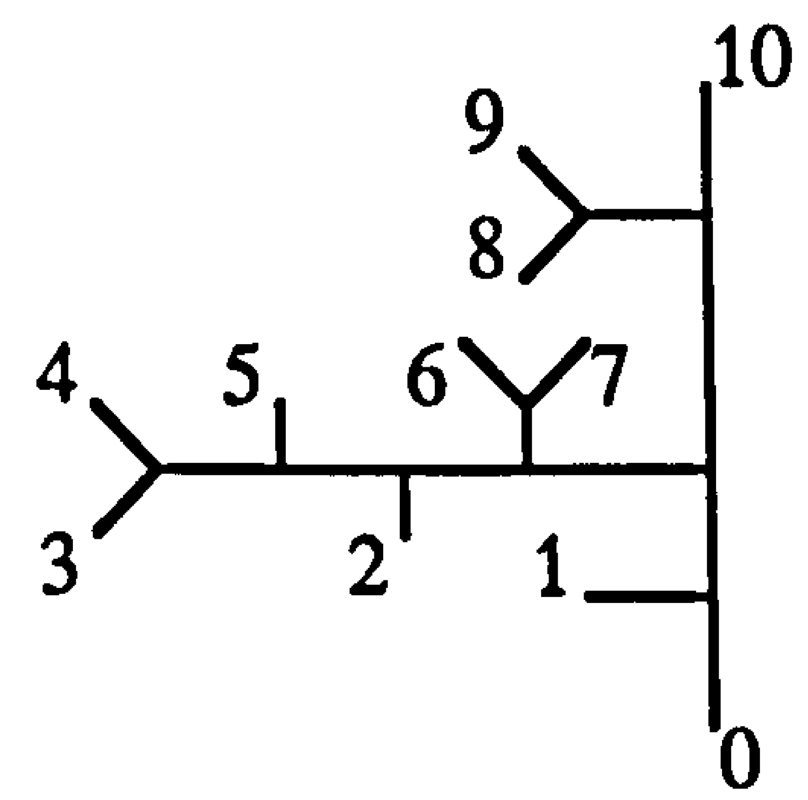
Proof of Claim

First we explain how we will specify an orientation for each tree with cyclic labelling. An orientation of a simplex is given by an ordering of its vertices, or equivalently an ordering of its faces. We explain how to order the internal edges of a given n-tree, t. There is a unique arc between any two ends of the tree (see [Lef]). Let a_n be the arc between n and n-1, a_{n-1} the arc between n-1 and n-2, ..., a_1 the arc between 1 and 0. Thus $a_n \ldots a_2 a_1$ is a path between n and 0. Since omitting an internal edge disconnects the tree into two components, such a path which visits every end must cover every internal edge. Now write this path as a sequence of internal edges, $e_0, \ldots, e_{n-3}$, omitting repetitions of edges already listed. This gives our ordering of the internal edges. From now on t will mean the tree t with this orientation and -t the same tree with the opposite orientation. The boundary ∂t is $\Sigma\,(-1)^j\, t(\hat{e}_j)$, where $t(\hat{e}_j)$ means the face of t given by deleting the edge e_j. The sign with which $t(\hat{e}_j)$ appears in the boundary of t is called the incidence number of $t(\hat{e}_j)$ in t, $i(t(\hat{e}_j), t)$.

Now we will choose to draw our trees with cyclic labelling with n at the top, 0 at the bottom, and everything else to the left of a straight trunk joining them.

Example

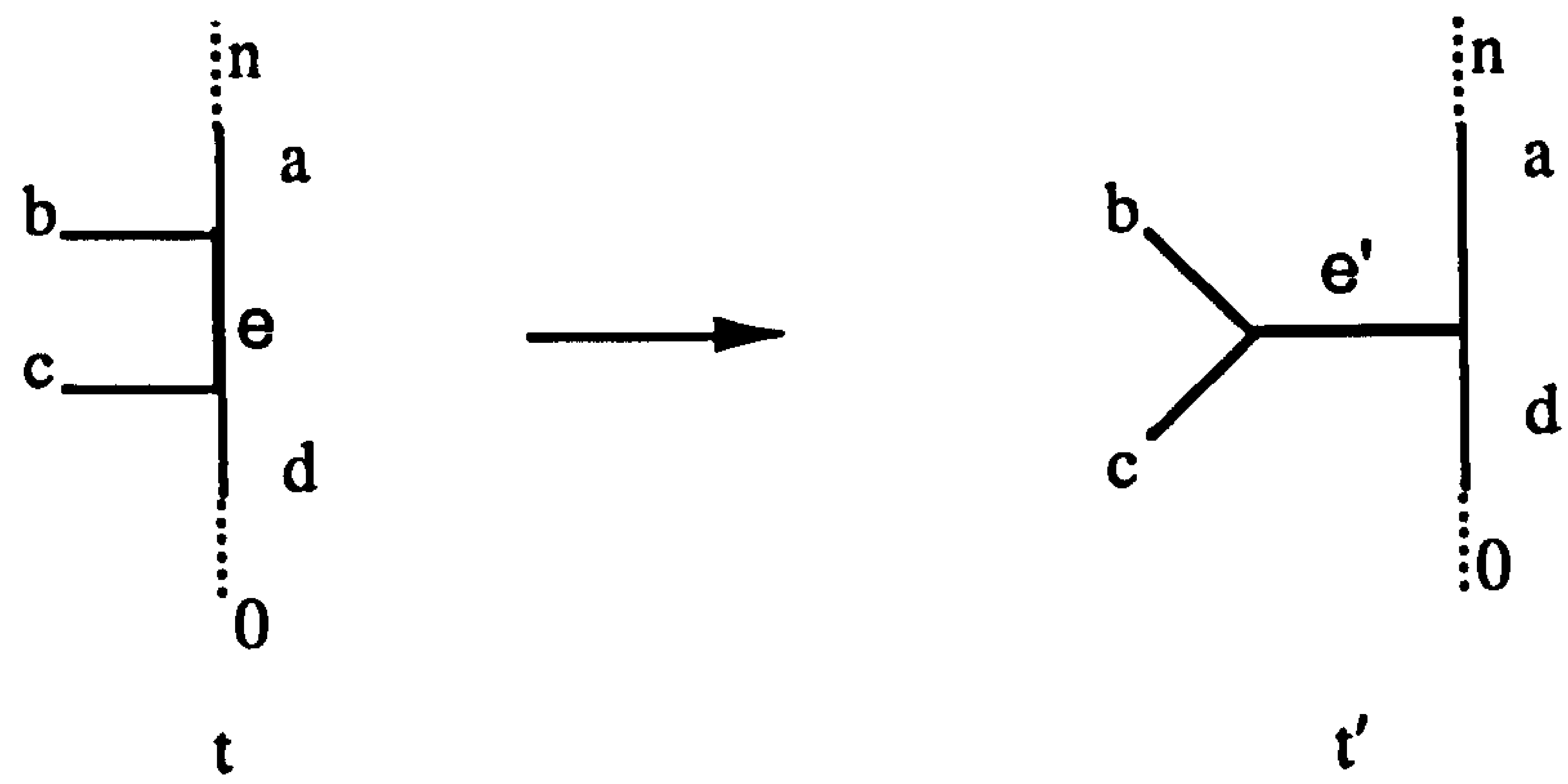
The 10-tree with cyclic labelling depicted earlier is now to be drawn :



Notation

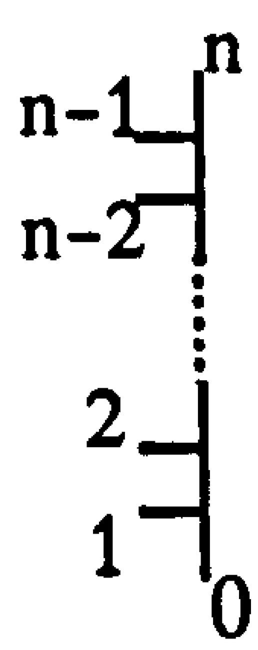
If a is some part of a tree, let | a | denote the number of internal edges of the tree in a, and let [a] denote $(-1)^{|a|}$.

The diagram below shows an operation, called *branching*, on a tree drawn as above, whereby two adjacent side branches are joined together :



Notice that the two trees, t and t', before and after branching share a common boundary component, (by deleting the edges e, e' indicated in the diagram), $t(\hat{e}) = t'(\hat{e}')$. We have $i(t(\hat{e}), t) = -[a][b]$ and $i(t'(\hat{e}'), t') = -[a]$. Given an orientation for the first tree t, (sgn t), we choose the orientation on the second, (sgn t'), so that (sgn t) $i(t(\hat{e}), t) = -$ (sgn t') $i(t'(\hat{e}'), t')$, making the common boundaries cancel. i.e. we choose (sgn t') $= -[b]$ (sgn t).

It is clear that if we start with the straight tree :



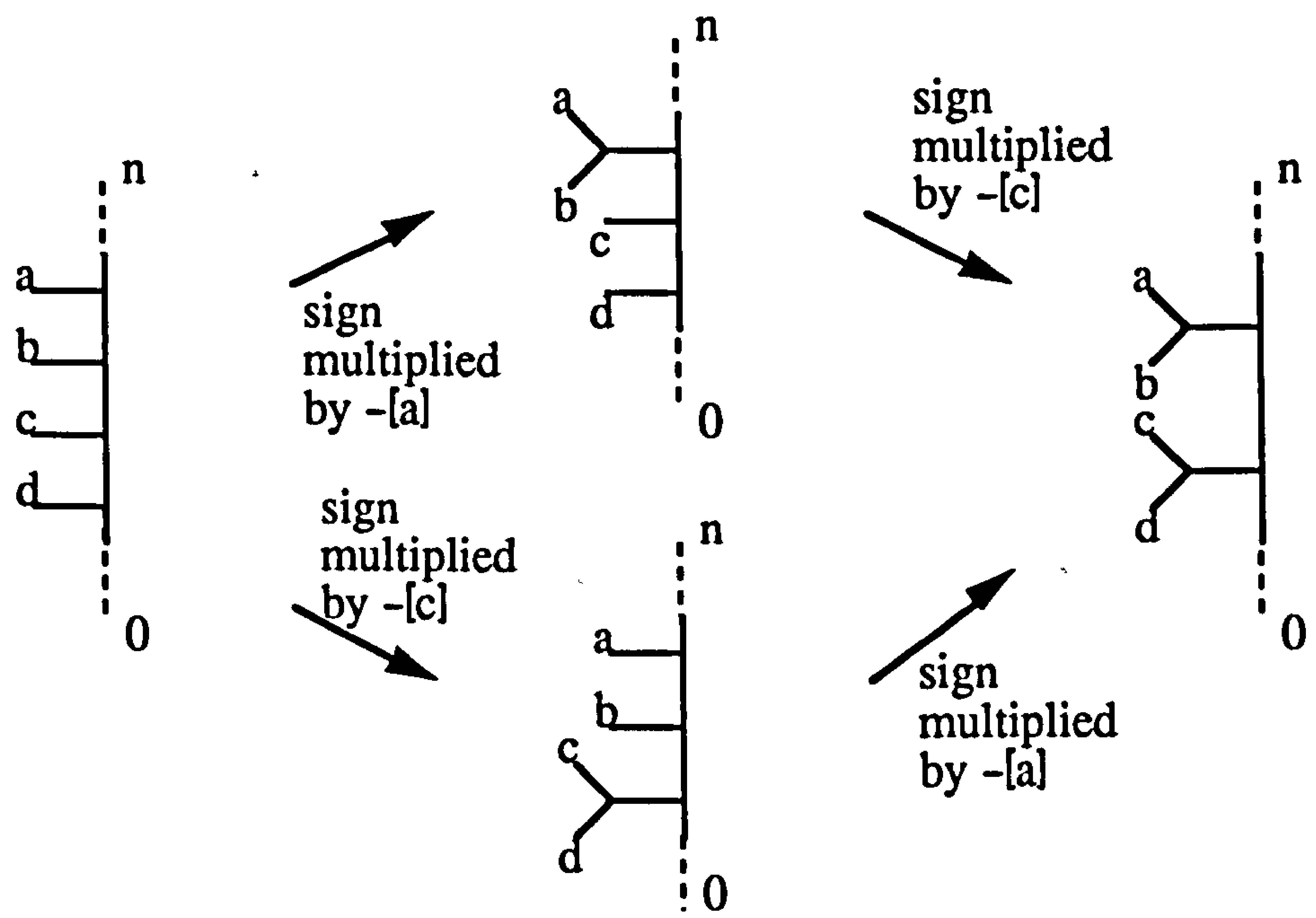
we can get every cyclic tree by a sequence, (not unique), of branchings. Now we start by choosing an orientation on the straight tree shown above and proceed with branchings until we have every cyclic tree, at each stage choosing the orientation determined by making the common boundary component cancel.

Claim

This gives a well-defined orientation to every cyclic tree.

Proof of Claim

Several sequences of branchings may lead to the same tree. However, they must then create all the same branches, so they can only differ by the order in which independent branchings are done. Hence it is sufficient to show that this does not affect the orientation assigned to the tree. This is shown by the following diagram, where the labelling of the arrows indicates the effect of each branching on the sign given to the tree.



Claim

With these orientations, trees with cyclic labelling form a cycle.

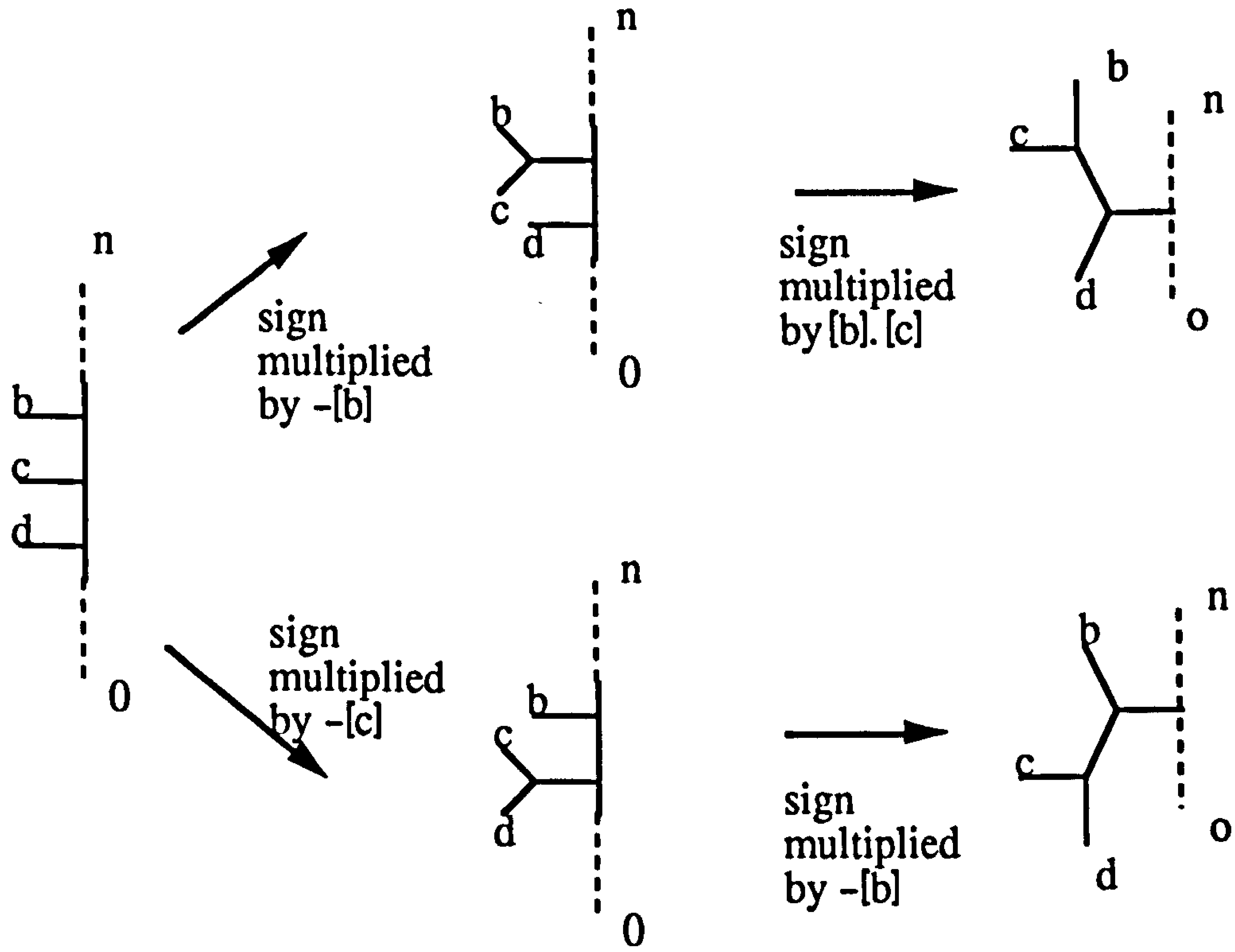
Proof of Claim

We have already seen that each boundary occurs exactly twice. Consider a general face s of a tree with cyclic labelling. As before we consider s as having four parts a, b, c, d, meeting at the node of order 4. Firstly, consider the case where the labels n and 0 appear on different parts of s, w.l.o.g. a and d. Then the two cyclic trees with this boundary may be drawn :

These differ precisely by a branching and by choice of signs their common boundaries cancel.

So suppose n and 0 appear on the same part of s, w.l.o.g. the part labelled a. Then we can draw the two cyclic trees , t and t′, with this boundary as :

Now i(t′(ê′), t′) = [b] i(t(ê), t). These two trees can be formed by the same sequence of branchings, except they differ by two moves at the stage of their 'common ancestor'.



So at this stage the signs on the two trees differ by -[b]. After this stage we do the same branchings to each tree, so these have the same effect on signs. Thus:

(sgn t′) i(t′(ê′), t′) = -[b] (sgn t) [b] i(t(ê), t) = - (sgn t) i(t(ê), t),

and the common boundaries cancel.

This completes the proof that trees with cyclic labelling form a cycle. □

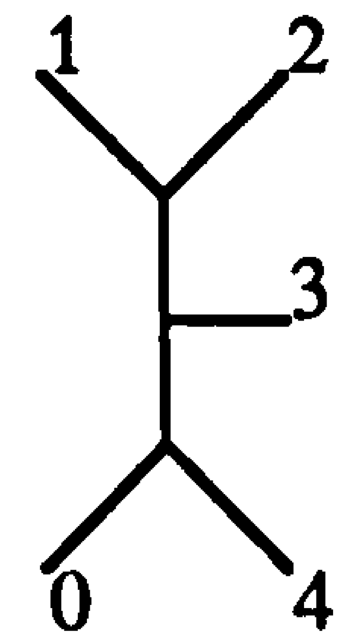
Notation

The cycle described above will be denoted by c_n. This is a formal sum, $\sum$(sgn t)t, over the n-trees t with (n-2) internal edges and with cyclic labelling, where (sgn t) is the orientation.

We let S_{n+1} act on the right on (oriented) n-trees : $t\sigma$ denotes the result of applying the permutation σ^{-1} to the labels of the tree t. Of course, $t\sigma$ comes with the orientation induced from that on t. This extends in the obvious way to formal sums of n-trees, such as the cycle c_n.

Remark

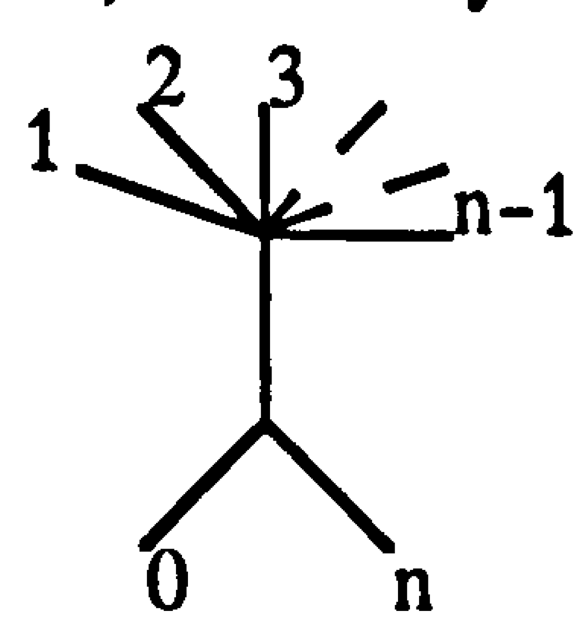
There is an obvious way of associating a bracketing of n objects to a cyclic tree in T_n.

For example we associate to the tree 

the bracketing $(((a_1a_2)a_3)a_4)$.

This gives a 1-1 correspondence between trees with cyclic labelling and bracketings. In fact, c_n is the dual of the boundary of the Stasheff polyhedron corresponding to bracketings of n objects. (See [St]).

Now consider $c_n\pi$, for $\pi \in S_{n-1}$, the permutations of { 1, 2, ..., n-1 }. Since c_n is a cycle so is each $c_n\pi$. These give (n-1)! distinct cycles, all passing through the vertex :



Now, in the proof of the homotopy type of T_n [Rob, 1], it is shown that the homology generators are the simplices given by the trees: 

(*), for $\pi \in S_{n-1}$, attached along their boundaries to a contractible space. Since 

is the only such 'straight tree' in c_n, the tree (*) above occurs in only $c_n\pi$, and so the $c_n\pi$, for $\pi \in S_{n-1}$, are homology generators, which are regularly permuted by S_{n-1}.

Notation

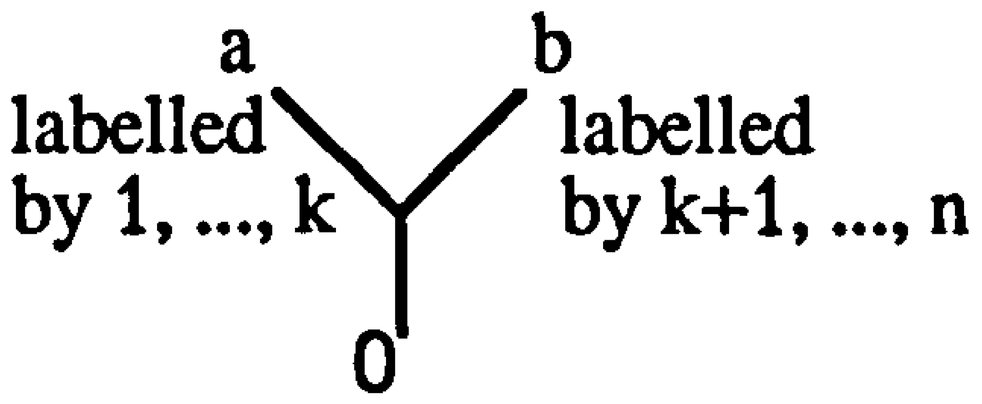
Let $\mu = (1\,2 \dots n) \in S_n$. Let $\Lambda_n = \sum_{i=0}^{n-1} (\mathrm{sgn}\ \mu^i)\mu^i$.

Proposition III.1.3.

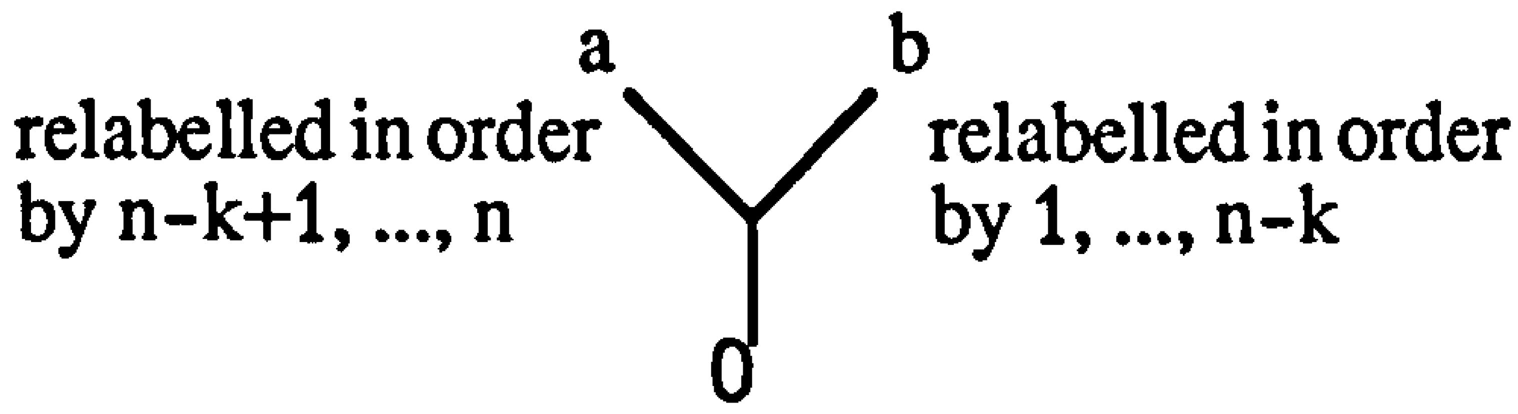
$c_n\Lambda_n = 0.$

Proof

Note that if a tree t with cyclic labelling has the form:



then $t\mu^k$ also has cyclic labelling:



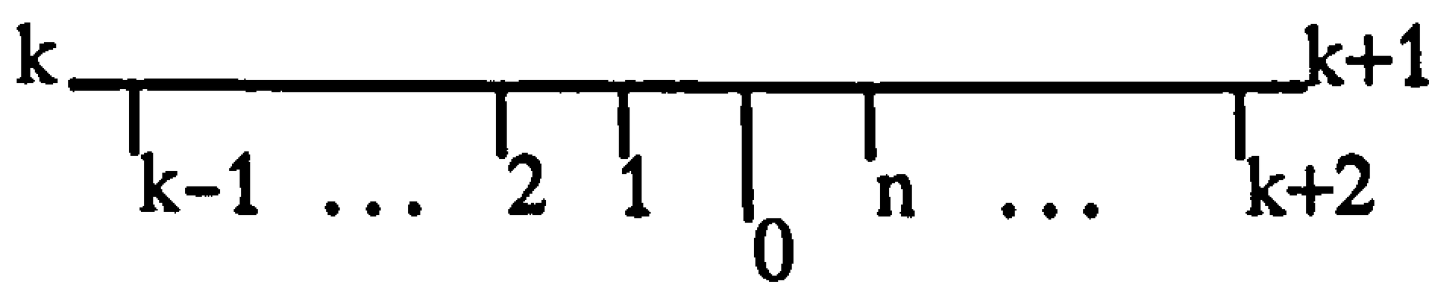
and k, n are the only powers of μ with this property.
So each tree $t\mu^i$ in $c_n\Lambda_n$ also appears as $(t\mu^k)\mu^{i-k}$.

Claim

We get opposite signs on these two trees, so all the terms cancel.

Proof of Claim

There are three ingredients here - the orientations of the cyclic trees, the effect of the permutations on these orientations, and the signs of the permutations. First consider the 'straight' cyclic tree t :



In this case it is fairly straightforward to show:

a). The signs of t and $t\mu^k$ in c_n differ by $(-1)^{(k-1)k/2\ +\ 1}$ for $k\neq n-1$, and by $(-1)^{(n-2)(n-1)/2}$ if $k=n-1$.

b). The difference between the sign on $t\mu^i$ induced from t and that on $(t\mu^k)\mu^{i-k}$ induced from $t\mu^k$ is $(-1)^{k(n-1)+(k-1)k/2}$ for $k\neq n-1$ and $(-1)^{(n-3)(n-2)/2}$ for $k=n-1$.

Since the signs of the permutations differ by $(-1)^{k(n-1)}$, this gives the result for straight cyclic trees.

Finally, we can deduce the result for a general cyclic tree from this by considering collections of adjacent simplices starting at the given tree and ending at a straight tree. We omit further details. □

Section III.2 : Tree Spaces and Shuffles

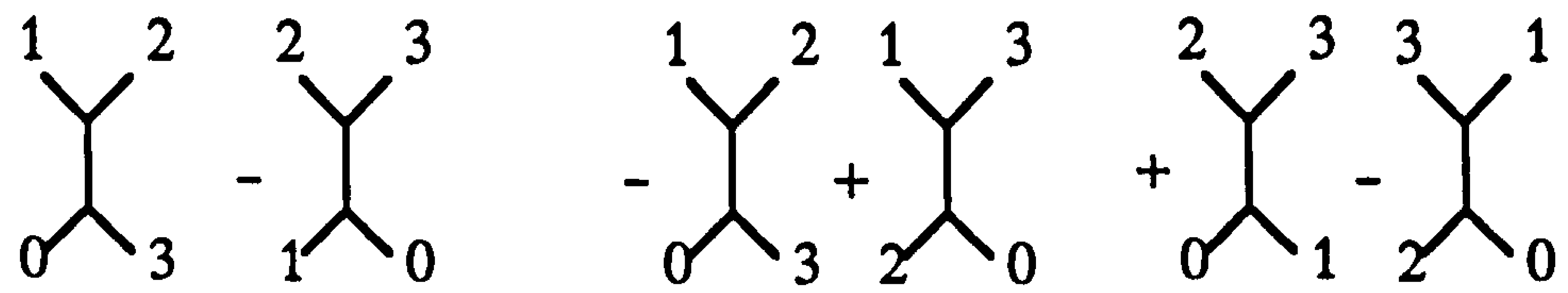
Definition III.2.1.

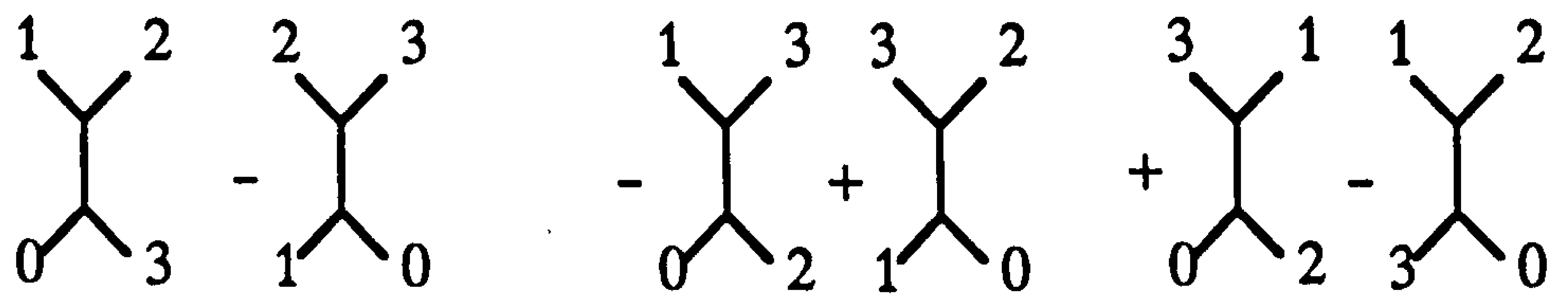
A pair of consecutive integers (i, i+1) is called a *descent* for the permutation $\pi \in S_n$ if $\pi i > \pi(i+1)$. The identity is the only permutation without any descents. Note that the identity permutation is an i-shuffle for each i. Apart from this, i-shuffles are those permutations with exactly 1 descent (i, i+1).

The main result of this section is that the cycle c_n vanishes on shuffles.
i.e. $c_n s_{i,n-i} = 0$ for i = 1, ..., n-1.

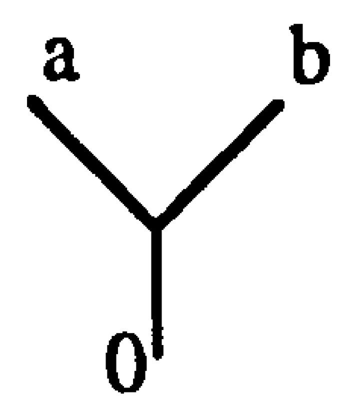
Example

n = 3.
$s_{1,2} = 1 - (1\ 2) + (1\ 3\ 2)$.
$s_{2,1} = 1 - (2\ 3) + (1\ 2\ 3)$.

$c_3 s_{1,2} =$  = 0.

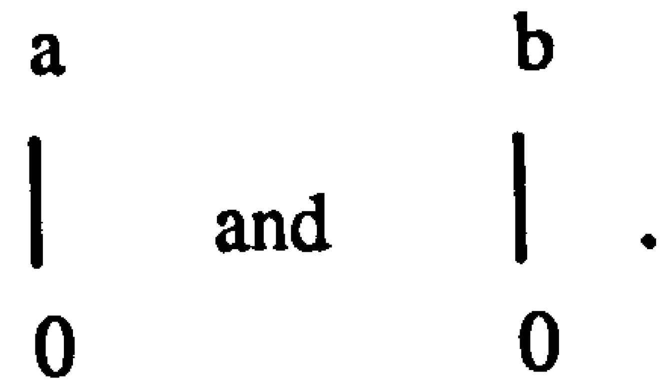
$c_3 s_{2,1} =$  = 0.

The proof is based upon the following observation :
Let t be a top-dimensional tree in T_n with cyclic labelling, and consider the node where the free edge labelled 0 is attached :



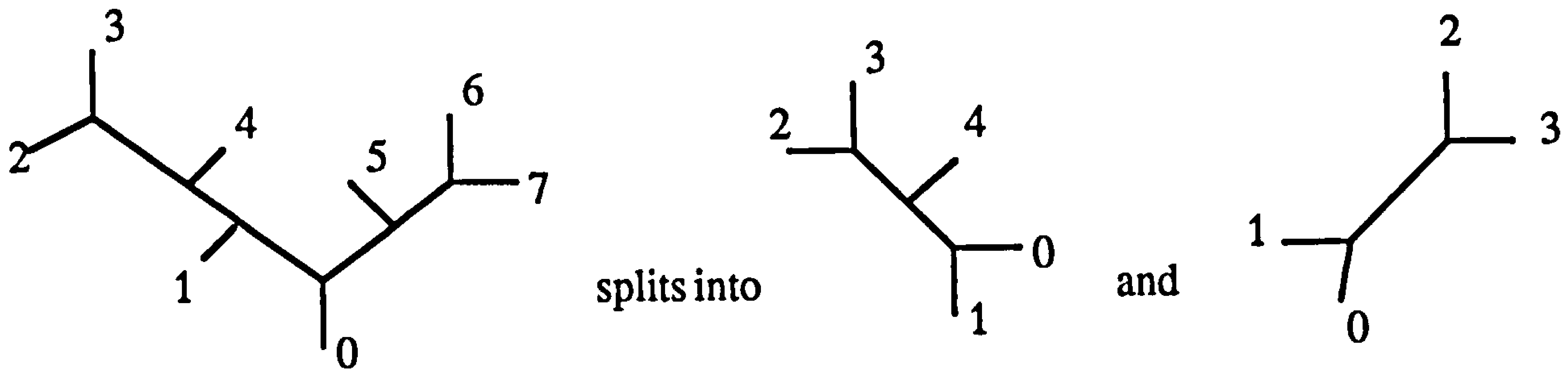
Since t has cyclic labelling we may assume w.l.o.g. that the subtrees a and b are labelled by

the sets { 1, ..., k } and { k+1, ..., n } respectively.
Imagine t split into the two trees :



We will denote these by a′ and b′. It is clear that a′ is a tree with cyclic labelling in T_k, and that, by relabelling b′ in order, b′ may be thought of as a tree with cyclic labelling in T_{n-k}.

Example



The above observation allows the use of induction arguments.

We will also consider the reverse of this process, whereby we graft together a tree in c_k, a′, and a tree in c_{n-k}, b′, to obtain a tree in c_n. Later we will need to consider how this process affects signs.

Lemma III.2.2.

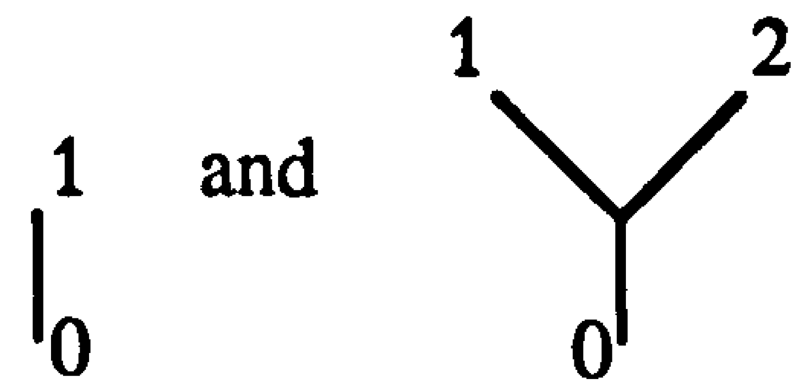
Suppose that the two trees a and b in c_k differ in sign by ε (= ±1). Let c be a tree in c_{n-k}. Then the two trees in c_n, obtained by grafting c to a and to b, also differ in sign by ε.

Proof

We write down a collection of adjacent simplices forming a path from a to b. i.e. we have a collection of trees in c_k, { a_1, ..., a_r } such that a_1 = a, a_r = b, and such that a_i and a_{i+1} share a common boundary, for each i. Signs are such that these common boundaries cancel, so consideration of this collection of trees gives us the difference in sign of a and b, ε.

But then consider the collection of trees in c_n, obtained by attaching c to each of the above trees. Clearly this is a collection of adjacent simplices, forming a path between the two trees that we are interested in. Signs are such that the common boundaries cancel as above, and so the two trees differ in sign by ε just as do a and b. □

Note that c_n is first non-empty for n = 3. However we formally extend the definitions to n = 1, 2. Let c_1 and c_2 be :



Then $c_1 s_1 = 0$ trivially, since $s_1 = 0$, and note that $c_2 s_2 = c_2 s_{1,1} = 0$ since $s_2 = s_{1,1} = 1 - (1\ 2)$. We adopt these conventions to cover the cases where one of a or b is labelled by a set of only one or two elements.

Proposition III.2.3.

$c_n s_{i,n-i} = 0$ for i = 1, ..., n-1.

Proof

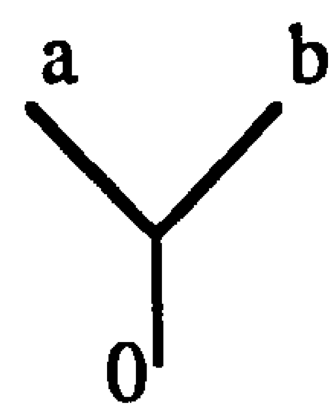
This will be proved by induction on n.

The example above demonstrates the result for n=3. Assume the result for k < n.

Fix some i = 1, ..., n-1.

Recall that $t\pi$ means apply the permutation π^{-1} to the labels of the tree t. So $c_n s_{i,n-i}$ is a collection of trees got by applying the inverses of i-shuffles to top-dimensional trees in T_n with cyclic labelling.

Let t be a top-dimensional tree in T_n, and as above draw t as :



We consider which such trees will give a tree with cyclic labelling upon applying some i-shuffle π. (This is equivalent to considering which trees occur in $c_n s_{i,n-i}$).

Suppose a and b are labelled by the sets S_1 and S_2 respectively; w.l.o.g. we may assume that $1 \in S_1$. Suppose $|S_1| = k$, for some $k \in \{1, ..., n-1\}$, $|S_2| = n-k$.

We break the proof down into several stages.

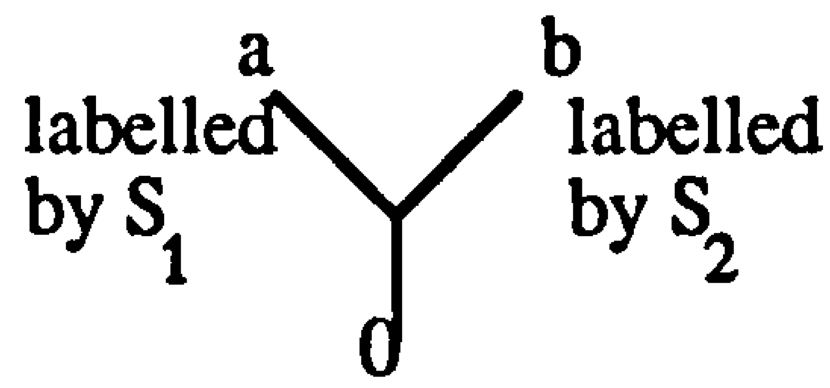
Lemma III.2.4.

If a tree t, as above, occurs in $c_n s_{i,n-i}$ it must satisfy one of the following conditions

I. $S_1 = \{1, ..., k\}$, $S_2 = \{k+1, ..., n\}$.
II. $S_1 = \{1, ..., j, j+n-k+1, ..., n\}$, $S_2 = \{j+1, ..., j+n-k\}$, where $j = i$ or $j = i+k-n$.
III. $S_1 = \{1, ..., j, i+1, ..., k+i-j\}$, $S_2 = \{j+1, ..., i, k+i-j+1, ..., n\}$, for some $j < i$.

Proof

Consider a general tree t as described above :



If we are to get a tree with cyclic labelling when we apply some i-shuffle π to t, we must have one of the following :

A. $\pi(S_1) = \{1, ..., k\}$ and $\pi(S_2) = \{k+1, ..., n\}$.
or B. $\pi(S_1) = \{n-k+1, ..., n\}$ and $\pi(S_2) = \{1, ..., n-k\}$.

In case A, if $s \in S_2$, $s+1 \in S_1$ then $\pi s > \pi(s+1)$, and so $(s, s+1)$ is a descent for π. Now in $s_{i,n-i}$ we have permutations with at most one descent, at $(i, i+1)$. It follows that one of the conditions I, II with $j = i+k-n$, or III must hold.

In case B, if $s \in S_1$, $s+1 \in S_2$ then $\pi s > \pi(s+1)$, and so $(s, s+1)$ is a descent for π. Again, this only happens for $s=i$. So the only possibilities are I if $k = i$, and II with $j = i$.

I.e. A tree t only appears in $c_n s_{i,n-i}$ if it satisfies one of the conditions above. □

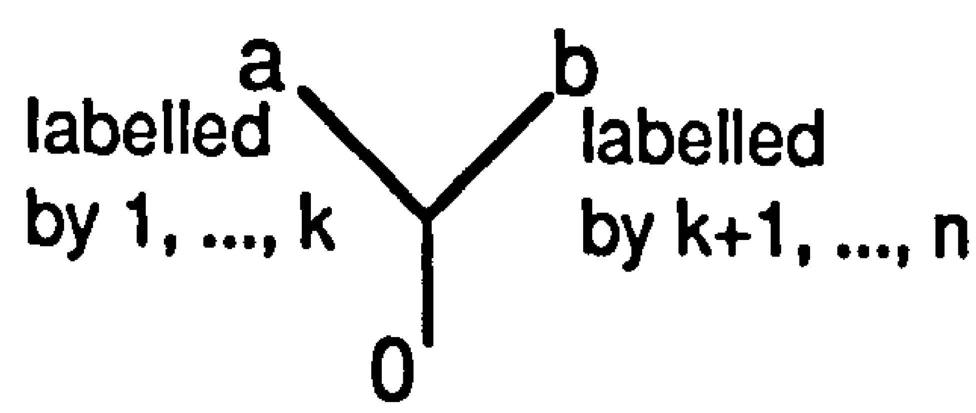
We treat each case separately.

Lemma III.2.5.

Each tree t which occurs in $c_n s_{i,n-i}$ and which satisfies condition I, occurs an even number of times with cancelling signs.

Proof

t has form :



We treat cases A and B separately.

A. $\pi\{1, ..., k\} = \{1, ..., k\}$, $\pi\{k+1, ..., n\} = \{k+1, ..., n\}$.
A non-trivial permutation of either set will introduce a descent. A shuffle has at most one descent, so π must be the identity on one of the sets.
If $i < k$, then π must fix $\{k+1, ..., n\}$, and we can think of π as an i-shuffle of $\{1, ..., k\}$. I.e. π fixes b, and shuffles a; we think of π acting on the corresponding tree a' in T_k. So, i-shuffles π of $\{1, ..., n\}$ such that $t\pi^{-1}$ has cyclic labelling correspond to i-shuffles π of $\{1, ..., k\}$ such that $a'\pi^{-1}$ has cyclic labelling. By the induction hypothesis there are an even number of such i-shuffles, $\{\pi_r\}$ say, and the signs on the copies of a' cancel. I.e.

$$\sum_r (\text{sgn } \pi_r)(\text{sgn from orientation of } (a'\pi_r^{-1})\pi_r) = 0.$$

Hence, regarding the permutations π_r as in S_n, there are an even number of such, so that $t\pi_r^{-1}$ has cyclic labelling, and by Lemma III.2.2 :

$$\sum_r (\text{sgn } \pi_r)(\text{sgn from orientation of } (t\pi_r^{-1})\pi_r)$$
$$= \varepsilon \sum_r (\text{sgn } \pi_r)(\text{sgn from orientation of } (a'\pi_r^{-1})\pi_r) = 0.$$

If $i > k$, then π must fix $\{1, ..., k\}$, and we may think of π as an (i-k)-shuffle of $\{k+1, ..., n\}$. Similarly to the first case we consider (i-k)-shuffles σ which take b' to $b'\sigma^{-1}$ with cyclic labelling. By induction, $\sum_{\text{such } \sigma} (\text{sgn } \sigma)(\text{sgn } (b'\sigma^{-1})\sigma) = 0.$
To take into account relabelling b by 1, ..., n-k in order, we want i-shuffles $\pi = \mu^k\sigma\mu^{n-k}$, where $\mu = (1\ 2\ ...\ n)$. So sgn π = sgn σ.
Again Lemma III.2.2 gives us $\sum_{\text{such } \pi} (\text{sgn } \pi)(\text{sgn } (t\pi^{-1})\pi) = 0.$

If $i = k$, the shuffle π must be the identity, and we get one copy of the tree t if it is a tree with cyclic labelling. However this will cancel with :

B. $\pi\{1, ..., k\} = \{n-k+1, ..., n\}$, $\pi\{k+1, ..., n\} = \{1, ..., n-k\}$.
There is only one such shuffle, where $i = k$, namely

$$\mu^{n-k} = \begin{pmatrix} 1 & 2 & \dots k & k+1 & k+2 \dots n \\ n-k+1 & n-k+2 & \dots n & 1 & 2 \dots n-k \end{pmatrix}.$$

It is clear that this k-shuffle will only give a tree with cyclic labelling if t has cyclic labelling. Of course in this case we also get a copy of the tree from the identity permutation as above. Now in the proof that $c_n\Lambda_n = 0$ we saw that the signs on these two copies of t differ by $(-1)^{k(n-1)+1}$; the sign of μ^{n-k} is $(-1)^{k(n-1)}$, so the two copies of t do appear with opposite signs and cancel. □

Since the other cases are similar we shall give less details.

Lemma III.2.6.

Each tree t which occurs in $c_n s_{i,n-i}$ and which satisfies condition II, occurs an even number of times with cancelling signs.

Proof

A. $\pi\{1, ..., i+k-n, i+1, ..., n\} = \{1, ..., k\}$, $\pi\{i+k-n+1, ..., i\} = \{k+1, ..., n\}$.
We have a descent at (i, i+1), and this must be the only one.
So $\{i+k-n+1, ..., i\} \to \{k+1, ..., n\}$ in order. i.e. π simply relabels b in order.
Now the i-shuffles π we require correspond to (i+k-n)-shuffles σ on a. So consider (i+k-n)-shuffles σ such that $a'\sigma^{-1}$ has cyclic labelling, and use induction to conclude that there are an even number of such, and that $\sum_{\text{such } \sigma} (\text{sgn } \sigma)(\text{sgn } (a'\sigma^{-1})\sigma = 0$.
A similar argument to above allows us to conclude that:

$$\sum_{\substack{\pi \text{ i-shuffle} \\ \text{s.t. } t\pi^{-1} \text{ cyclic}}} (\text{sgn } \pi)(\text{sgn } (t\pi^{-1})\pi) = 0.$$

B. $\pi\{1, ..., i, i+n-k+1, ..., n\} = \{n-k+1, ..., n\}$, $\pi\{i+1, ..., i+n-k\} = \{1, ..., n-k\}$.
We have a descent at (i, i+1) and for a shuffle π this must be the only one.
So $\{i+1, ..., i+n-k\} \to \{1, ..., n-k\}$ in order. Again π simply relabels b in order and is an i-shuffle on a. By induction, using similar arguments to above, there are an even number of shuffles with the required property and signs cancel. □

Lemma III.2.7.

Each tree t which occurs in $c_n s_{i,n-i}$ and which satisfies condition III, occurs an even number of times with cancelling signs.

Proof

Only case A is possible.
$\pi\{1, ..., j, i+1, ..., k+i-j\} = \{1, ..., k\}$, $\pi\{j+1, ..., i, k+i-j+1, ..., n\} = \{k+1, ..., n\}$.
π has a descent at (i, i+1). Its restriction to a' is a j-shuffle such that $a'\pi^{-1}$ has cyclic labelling, and its restriction to b' is an (i-j)-shuffle such that $b'\pi^{-1}$ has cyclic labelling. Fix

such a j-shuffle on a'. Now apply induction to conclude that there are an even number of such (i-j)-shuffles and the signs on the copies of b' cancel. Hence, as above, the same is true for the copies of t. Repeating this for each j-shuffle on a' we have the result. □

So in each case, all copies of a given tree cancel. This completes the proof that $c_n s_{i,n-i} = 0$. □

Section III.3 : The Representation of S_n given by the Tree Space T_n

The aim of this section is to show that, working over K containing $\mathbb{Q}$, V_n is the same representation of S_n as that given by the right KS_n-module $e_n(1)KS_n$. The idempotent $e_n(1)$ in $\mathbb{Q}S_n$ was defined in Chapter I, as a certain polynomial in the 'total shuffle operator', s_n, as indeed were idempotents $e_n(j)$ for j = 1, ..., n. As we saw in Chapter I, these idempotents are used to give a decomposition of the Hochschild (co)homology of a commutative algebra over a ground ring containing $\mathbb{Q}$. It is an immediate consequence of the definition of $e_n(j)$ that :

$$e_n(j)s_n = (2^j-2)e_n(j).$$

In particular, $e_n(1)s_n = 0$. In fact, $e_n(1)s_{k,n-k} = 0$ for k = 1, ..., n-1 [B].

Notation

Let $p_{n,k} = (n\ k)(n-1\ k+1)(n-2\ k+2) \ldots \in S_n$, for k = 1, ..., n.
Let $\rho_{n,k} = (-1)^{([n-k+1][n-k+2]/2\ -1)} p_{n,k} \in \mathbb{Q}S_n$, for k = 1, ..., n.

Note that if π is a k-shuffle in S_n, either $\pi n = n$ or $\pi k = n$.
Let $s^{(n)}_{k,n-k} = \sum (\text{sgn}\ \pi)\pi$, where the sum is over k-shuffles fixing n.
Let $s^{(k)}_{k,n-k} = \sum (\text{sgn}\ \pi)\pi$, where the sum is over k-shuffles such that $\pi k = n$.
So $s_{k,n-k} = s^{(k)}_{k,n-k} + s^{(n)}_{k,n-k}$. Note that we may think of $s^{(n)}_{k,n-k}$ as in $\mathbb{Q}S_{n-1}$:
$s^{(n)}_{k,n-k} = s_{k,n-1-k}$, for k = 1, ..., n-2; $s^{(n)}_{n-1,1} = 1$.

Proposition III.3.1.

$$c_n\rho_{n,k} = c_n s^{(n)}_{k-1,n-k+1}\ , \text{ for } k = 1, \ldots, n.$$

[we regard 1 as the only 0-shuffle].

Proof

Use downward induction on k.

When k = n, the result is trivial, since both sides equal c_n.

Assume the result for k+1, and deduce it for k.

We use $c_n s_{k,n-k} = 0$. (Proposition III.2.3.)

Now as above, $s_{k,n-k} = s^{(k)}_{k,n-k} + s^{(n)}_{k,n-k}$.

So,

$$c_n s^{(k)}_{k,n-k} = - c_n s^{(n)}_{k,n-k} = - c_n \rho_{n,k+1} \qquad (*)$$

by the induction hypothesis.

Note that $\rho_{n,k+1}(\ k\ k+1 \ldots n\ ) = (-1)^{n-k+1}\rho_{n,k}$.

Also, π is a k-shuffle such that $\pi k = n$ if and only if $\pi(\ k\ k+1 \ldots n\ )$ is a (k-1)-shuffle fixing n.

i.e. $s^{(k)}_{k,n-k}(\ k\ k+1 \ldots n\ ) = \operatorname{sgn}(\ (\ k\ k+1 \ldots n\ )\ )\, s^{(n)}_{k-1,n-k+1} = (-1)^{n-k}\, s^{(n)}_{k-1,n-k+1}$.

Then composing each side of the above equation (*) with (k k+1 ... n) gives the result. □

Remark

The above proof uses only $c_n s_{k,n-k} = 0$, for k = 1, ..., n-1. Since $e_n(1)$ satisfies $e_n(1)s_{k,n-k} = 0$, for k = 1, ..., n-1, the same proof gives :

$$e_n(1)\rho_{n,k} = e_n(1)s^{(n)}_{k-1,n-k+1}.$$

Proposition III.3.2.

Working over K containing $\mathbb{Q}$, the tree representation V_n of S_n coincides with $e_n(1)KS_n$. i.e. $V_n \cong e_n(1)KS_n$, as right KS_n-modules.

Proof

We have $e_n(1)\rho_{n,k} = e_n(1)s^{(n)}_{k-1,n-k+1}$.

Then for $\alpha \in S_n$, $\alpha = (\ (n\ k)\ (n-1\ k+1) \ldots\ )\alpha'$, for some k = 1, ..., n and some $\alpha' \in S_{n-1}$.

So, $e_n(1)\alpha = e_n(1)(\ (n\ k)\ (n-1\ k+1) \ldots\ )\alpha'$

$$= (-1)^{(\ [n-k+1]\,[n-k+2]/2\ -\ 1\ )}e_n(1)s^{(n)}_{k-1,n-k+1}\,\alpha'.$$

The right-hand side contains only terms $e_n(1)\pi$ for $\pi \in S_{n-1}$, so this shows that $e_n(1)KS_n$ is generated by $e_n(1)\pi$ for $\pi \in S_{n-1}$. Since Hanlon [H] has shown that $e_n(1)KS_n$ has dimension (n-1)!, the $e_n(1)\pi$'s for $\pi \in S_{n-1}$ must form a basis.

Similarly we have : $c_n\alpha = (-1)^{(\ [n-k+1]\,[n-k+2]/2\ -\ 1\ )}c_n s^{(n)}_{k-1,n-k+1}\,\alpha'$.

So $\theta : V_n \to e_n(1)KS_n$ defined by $\theta(c_n\pi) = e_n(1)\pi$, and extending linearly, gives a well-defined KS_n-module isomorphism. □

Remark

Since S_{n+1} acts on T_n, V_n is actually an S_{n+1}-module, giving a representation of S_{n+1} which restricts to $e_n(1)KS_n$. We shall return to this and discuss the representation theory further in Chapter IV. Note also that although the idempotent $e_n(1)$ is only defined rationally, the representation V_n makes sense over any ground ring.

Section III.4 : Γ-(co)homology in Characteristic Zero

We saw in the previous chapter that the Γ-homology of a flat commutative k-algebra A, where k is a commutative ground ring, is given by a spectral sequence :

$$E^1_{p,q} = \mathrm{Tor}^{kS_p}_q(V_p,\ A^{\otimes p}) \otimes M \underset{p}{\Rightarrow} H\Gamma_{p+q-1}(A/k;\ M),$$

where $d^1_{p,q} : E^1_{p,q} \to E^1_{p-1,q}$ is induced from the original differential in the Γ chain complex; and where $A^{\otimes p}$ is a left kS_p-module as before via $\pi^{-1}(a_1\otimes \dots \otimes a_n) = (a_{\pi 1}\otimes \dots \otimes a_{\pi n})$, and V_p is a right kS_p-module as explained above.

We consider the case where k contains $\mathbb{Q}$. Then $kS_p = e_p(1)kS_p \oplus (1-e_p(1))kS_p$, and so $V_p \cong e_p(1)kS_p$ is a projective kS_p-module. Hence

$$E^1_{p,q} = \mathrm{Tor}^{kS_p}_q(V_p,\ A^{\otimes p}) \otimes M = \begin{cases} V_p \otimes_{kS_p} A^{\otimes p} \otimes M & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

Thus the spectral sequence collapses, E^1 having non-zero terms along the line q = 0 only. So over a ground ring containing $\mathbb{Q}$, Γ-homology is given by the homology of the chain complex:

$$(\ V_p \otimes_{kS_p} A^{\otimes p} \otimes M,\ d^1\ ).$$

We will now describe the boundary map

$$d^1 : V_n \otimes_{kS_n} A^{\otimes n} \otimes M \to V_{n-1} \otimes_{kS_{n-1}} A^{\otimes n-1} \otimes M.$$

This map is induced from those parts of the original boundary map d which take terms in $A^{\otimes n}$ to terms in $A^{\otimes n-1}$. We will denote by $[\ v\otimes a\otimes m\ ]$ an element of $V_n \otimes_{kS_n} A^{\otimes n} \otimes M$. Now we have seen that V_n has basis $c_n\pi$, for $\pi \in S_{n-1}$.

Recall from Chapter I that we may describe the Hochschild complex for a commutative algebra A with coefficients in a symmetric A-bimodule M as ($A^{\otimes *}\otimes M$, b), where the Hochschild boundary b : $A^{\otimes n}\otimes M \to A^{\otimes n-1}\otimes M$ is given by

$$b(a_1\otimes...\otimes a_n\otimes m) = a_2\otimes...\otimes a_n\otimes a_1 m + \sum_{i=1}^{n-1}(-1)^i a_1\otimes...\otimes a_i a_{i+1}\otimes...\otimes a_n\otimes m$$
$$+ (-1)^n a_1\otimes...\otimes a_{n-1}\otimes a_n m$$

(Note that the formula here is modified to take into account that we are writing the coefficients M on the right.)

Lemma III.4.1.

$d^1 [c_n\pi\otimes a\otimes m] = [c_{n-1}\otimes b(\pi a\otimes m)]$,

where $\pi \in S_{n-1}$, $a \in A^{\otimes n}$, $m \in M$, and where b is the Hochschild boundary.

Proof

First let us consider $d^1 [c_n\otimes a_1\otimes...\otimes a_n \otimes m]$. Only the first and last terms of the original boundary contribute to d^1. We must interpret these in terms of trees. (I.e. we transfer the boundary from N([n]/Γ/[1]) to T_n using the map θ of p23). Let us consider the first term ∂_0. In the original chain complex this was given by

$\partial_0 ([f_{n-1}| ... | f_1] \otimes a \otimes m) = [f_{n-1}| ... | f_2] \otimes f_{1*}a \otimes m$,

and this contributes to d^1 only if $f_1 : [n] \to [n-1]$.

We can see that for each tree in c_n this operation corresponds to wherever possible replacing a pair of ends labelled by i and i+1 by a single end labelled by i, and then relabelling the rest of the tree in order; thus $a_1\otimes...\otimes a_n\otimes m$ becomes $a_1\otimes...\otimes a_i a_{i+1}\otimes...\otimes a_n\otimes m$. It is clear that for each i = 1, ..., n−1, replacing a pair i, i+1 with i on each tree where such a pair occurs gives us a copy of c_{n-1}.

Now consider the final term of the original boundary ∂_{n-1}. In the original chain complex this was given by summing over components of $[f_{n-2}| ... | f_1]$. Here we only have the case where there are exactly two components, [n−1] → [1] and [1] → [1], and we get the first of these. For c_n this operation corresponds to omiting a 1 or an n wherever such appears in a pair together with 0 on a tree, (after omiting 1 you relabel the resulting tree as appropriate); thus $a_1\otimes...\otimes a_n\otimes m$ becomes $a_2\otimes...\otimes a_n\otimes a_1 m$ or $a_1\otimes...\otimes a_{n-1}\otimes a_n m$. Again it is clear that in each case you are left with a copy of c_{n-1}.

So we have seen that the terms which occur are precisely those in

$$[c_{n-1}\otimes b (a_1\otimes...\otimes a_n \otimes m)].$$

It remains to check that the signs are correct, but this follows from $d^1 d^1 = 0$. (For example, considering $x_1\otimes...\otimes x_n$, where the x_i's are generators of the polynomial algebra $k[x_1,...,x_n]$, then the standard cancelling of terms under b^2 is the only one possible). Hence we have $d^1 [c_n \otimes a_1 \otimes ... \otimes a_n \otimes m] = [c_{n-1} \otimes b (a_1 \otimes ... \otimes a_n \otimes m)]$.

So $d^1 [c_n \pi \otimes a \otimes m] = d^1 [c_n \otimes \pi a \otimes m] = c_{n-1} \otimes b [\pi a \otimes m]$ as required. □

We are now in a position to prove the main theorem.

Theorem III.4.2.

For A a flat (projective) commutative algebra over k containing $\mathbb{Q}$, and M an A-module, Γ-(co)homology coincides with Harrison (co)homology :

$$H\Gamma_{n-1}(A/k; M) = Harr_n(A/k; M),$$
$$H\Gamma^{n-1}(A/k; M) = Harr^n(A/k; M).$$

Proof

We will prove the result in homology. We have a chain complex $(V_n \otimes_{kS_n} A^{\otimes n} \otimes M, d^1)$. Now we have seen that, for k containing $\mathbb{Q}$, $V_n \cong e_n(1)kS_n$, by $c_n x \mapsto e_n(1)x$, for $x \in kS_n$. (Proposition III.3.2.).

Hence, $V_n \otimes_{kS_n} A^{\otimes n} \otimes M \cong e_n(1)kS_n \otimes_{kS_n} A^{\otimes n} \otimes M \cong e_n(1)A^{\otimes n} \otimes M$.

Explicitly we have

$\theta_n : V_n \otimes_{kS_n} A^{\otimes n} \otimes M \to e_n(1)A^{\otimes n} \otimes M$ given by $\theta_n [c_n x \otimes a \otimes m] = e_n(1)xa \otimes m$,

for $x \in kS_n$, and

$\psi_n : e_n(1)A^{\otimes n} \otimes M \to V_n \otimes_{kS_n} A^{\otimes n} \otimes M$ by $\psi_n(e_n(1)a \otimes m) = [c_n \otimes a \otimes m]$.

It is easily checked that these maps are well-defined and mutually inverse.

Now recall (from I.3) that Harrison homology is precisely the homology of the complex $(e_n(1)A^{\otimes n} \otimes M, b)$, where $be_n(1) = e_{n-1}(1)b$. We wish to show that we have an isomorphism of chain complexes :

$(V_* \otimes A^{\otimes *} \otimes M, d^1) \cong (e_*(1)A^{\otimes *} \otimes M, b)$.

Now $d^1 \psi_n(e_n(1)a \otimes m) = d^1 [c_n \otimes a \otimes m] = [c_{n-1} \otimes b (a \otimes m)]$, by Lemma III.4.1. and

$\psi_{n-1} b(e_n(1)a \otimes m) = \psi_{n-1} e_{n-1}(1)b(a \otimes m) = [c_{n-1} \otimes b (a \otimes m)]$.

So $d^1 \psi_n = \psi_{n-1} b$.

Also $\theta_{n-1} d^1 [c_n x \otimes a \otimes m] = \theta_{n-1} [c_{n-1} \otimes b (xa \otimes m)] = e_{n-1}(1)b(xa \otimes m)$, and

$b\theta_n [c_n x \otimes a \otimes m] = b (e_n(1)xa \otimes m) = e_{n-1}(1)b(xa \otimes m)$.

So $\theta_{n-1} d^1 = b\theta_n$.

Hence, we have an isomorphism of chain complexes, giving the result. □

Section III.5 : The Relationship between the Harrison and Γ Theories in General

The idempotent $e_n(1)$ is only defined in characteristic zero, but the representation V_n makes sense over a general ground ring, and a simple modification of the above arguments shows that the edge of the spectral sequence for Γ-homology is still Harrison homology.

Proposition III.5.1.

$$V_n \otimes_{kS_n} A^{\otimes n} \cong A^{\otimes n} / \left(\sum_{i=1}^{n-1} s_{i,n-i} A^{\otimes n} \right)$$

Proof

Define $\theta : A^{\otimes n} \to V_n \otimes_{kS_n} A^{\otimes n}$ by $\theta(a) = c_n \otimes a$. If $\sum_i c_n \alpha_i \otimes a_i \in V_n \otimes_{kS_n} A^{\otimes n}$, for $\alpha_i \in kS_n$, $a_i \in A^{\otimes n}$ then $\sum_i c_n \alpha_i \otimes a_i = c_n \otimes \sum_i \alpha_i a_i = \theta(\sum_i \alpha_i a_i)$, so θ is surjective.

Now $\sum_{i=1}^{n-1} s_{i,n-i} A^{\otimes n} \subset \mathrm{Ker}\ \theta$ since $\theta(s_{i,n-i}a) = c_n \otimes s_{i,n-i}a = c_n s_{i,n-i} \otimes a = 0$, by Proposition III.2.3. In fact, in Section III.3 we saw that the relations $c_n s_{i,n-i} = 0$ determine the S_n-module structure of V_n, so that $c_n \alpha = 0$ for $\alpha \in kS_n$ if and only if $\alpha \in \sum_{i=1}^{n-1} s_{i,n-i} kS_n$. Hence $\mathrm{Ker}\ \theta = \sum_{i=1}^{n-1} s_{i,n-i} A^{\otimes n}$. □

Proposition III.5.2.

For A a flat (projective) commutative algebra over any commutative ring k, and M an A-module, we have homomorphisms :

$$H\Gamma_{n-1}(A/k; M) \to \mathrm{Harr}_n(A/k; M),$$
$$H\Gamma^{n-1}(A/k; M) \leftarrow \mathrm{Harr}^n(A/k; M).$$

Proof

The above proposition says that in the spectral sequence for Γ-homology we always have $E^1_{n,0} \cong A^{\otimes n} / (\sum_{i=1}^{n-1} s_{i,n-i} A^{\otimes n}) \otimes M$, Harrison's n^{th} chain group. As above, we can identify the boundary $d^1 : E^1_{n,0} \to E^1_{n-1,0}$ with the Hochschild boundary. Thus

$$E^2_{n,0} \cong \mathrm{Harr}_n(A/k; M).$$

So the required homomorphism is the edge map of the spectral sequence. Similarly for cohomology. □

Chapter IV : Extensions of the Eulerian Representations of the Symmetric Groups

Introduction

In this chapter we consider some representation theory of the symmetric group. From Chapter III, the tree space T_n is a simplicial complex satisfying $T_n \simeq \bigvee_{(n-1)!} S^{n-3}$. Since the symmetric group S_{n+1} acts on T_n by permuting the labels 0, 1, ..., n on trees, the only non-trivial homology group $\overline{H}_{n-3}(T_n; k)$ gives a representation of S_{n+1}. We denote this representation by V_n', and its restriction to the subgroup S_n (which keeps 0 fixed) by V_n. From Section III.3, working over $k = \mathbb{Q}$, the representation V_n is that given by the first Eulerian idempotent, $e_n(1)\mathbb{Q}S_n$.

Working over $\mathbb{Q}$, we begin by identifying V_n' as $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$, where $\Lambda_{n+1}e_n(1)$ is an idempotent described below. Next we show that each Eulerian representation, $e_n(j)\mathbb{Q}S_n$, is in fact a restriction of a representation of S_{n+1}, given by an idempotent $\Lambda_{n+1}e_n(j)$ in $\mathbb{Q}S_{n+1}$. Then we provide a description of the representation $\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$ as a virtual representation, by first proving a certain relation between the idempotents $e_n(j)$ and $e_{n+1}(j)$. For the tree representation, the result is that V_n' is given by V_n induced to S_{n+1} modulo V_{n+1}. This description leads to a character formula, using the results obtained by Hanlon [H] for the Eulerian representations. Finally, we state some partial results on the decompositions of these representations into irreducibles. A formula for the decomposition of the $e_n(j)$ representation into irreducible components is not known in general. It may be productive to approach this problem by considering how the $\Lambda_{n+1}e_n(j)$ representation decomposes, and then restricting. However, we have not made much progress in this direction. We give a table of decompositions for small n; the results for j = 1, n-2, n-1, n; and the relationship between our representations and the trivial representation.

Section IV.1 : The Representation of S_{n+1} given by the Tree Space T_n

In this section it is shown that V_n' is the same representation of S_{n+1} as that given by the right S_{n+1}-module $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$, where $\Lambda_{n+1}e_n(1)$ is an idempotent in $\mathbb{Q}S_{n+1}$ described below.

Recall that c_n is the cycle in the tree space T_n consisting of trees with cyclic labelling.

Proposition IV.1.1.

$c_n (0\ 1 \dots n) = (-1)^n c_n$.

Proof

Obviously (0 1 ... n) takes a tree with cyclic labelling to another such tree. So it is just a question of checking that (0 1 ... n) has the stated effect on orientations. This is straightforward. □

Notation

Let $\lambda_{n+1} = (0\ 1 \dots n) \in S_{n+1}$.

Let $\Lambda_{n+1} = \frac{1}{n+1} \sum_{i=0}^{n} \left(\operatorname{sgn} \lambda_{n+1}^i\right) \lambda_{n+1}^i \in \mathbb{Q}S_{n+1}$.

Clearly, Λ_{n+1} is an idempotent in $\mathbb{Q}S_{n+1}$.

Proposition IV.1.2.

$\Lambda_{n+1} s_n = s_n \Lambda_{n+1}$.

Proof

A typical term on the right-hand side of this equation is $\pi\lambda_{n+1}^j$, appearing with sign, where π is some shuffle in S_n. Now $\pi\lambda_{n+1}^j(0) = \pi(j)$. So $\pi\lambda_{n+1}^j = \lambda_{n+1}^{\pi(j)}\pi'$, for $\pi' = \lambda_{n+1}^{-\pi(j)}\pi\lambda_{n+1}^j$ in S_n.

Claim

$\pi \to \pi' = \lambda_{n+1}^{-\pi(j)}\pi\lambda_{n+1}^j$ is a bijection $S_n \to S_n$, taking shuffles to shuffles, for j = 0, 1, ..., n.

Proof of Claim

For j = n, this was proved by Natsume and Schack [N-S Lemma 9]. It is straightforward to show that the claim follows by iterating their result.

So each term of the right-hand side, $\pi\lambda_{n+1}^j$ with sign, appears in the left-hand side as $\lambda_{n+1}^{\pi(j)}\pi'$, with π' a shuffle, also with sign. □

Corollary IV.1.3.

$\Lambda_{n+1}e_n(j) = e_n(j)\Lambda_{n+1}$ for $j = 1, \ldots, n$.

Proof

Each $e_n(j)$ is a polynomial in s_n. Since Λ_{n+1} commutes with s_n, it commutes with each $e_n(j)$. □

Corollary IV.1.4.

$\Lambda_{n+1}e_n(j)$ is an idempotent in $\mathbb{Q}S_{n+1}$.

Proof

$(\Lambda_{n+1}e_n(j))^2 = \Lambda_{n+1}e_n(j)\Lambda_{n+1}e_n(j) = \Lambda_{n+1}{}^2e_n(j)^2 = \Lambda_{n+1}e_n(j)$. □

So in particular $\Lambda_{n+1}e_n(1)$ is an idempotent in $\mathbb{Q}S_{n+1}$, giving a representation of S_{n+1}, $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$.

Proposition IV.1.5.

Working over $\mathbb{Q}$, the tree representation V_n' of S_{n+1} coincides with $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$. i.e. $V_n' \cong \Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$ as right $\mathbb{Q}S_{n+1}$-modules.

Proof

Since V_n' restricts to $V_n \cong e_n(1)\mathbb{Q}S_n$ (Proposition III.3.2.), we first show that $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$ restricts to $e_n(1)\mathbb{Q}S_n$. Consider the homomorphism of right $\mathbb{Q}S_n$-modules $\theta : e_n(1)\mathbb{Q}S_n \to \Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$ given by left multiplication by Λ_{n+1}. Now since Λ_{n+1} and $e_n(1)$ commute, and since we may write $\pi \in S_{n+1}$ uniquely as $\lambda^i_{n+1}\pi'$ for some i and some $\pi' \in S_n$, we have

$\Lambda_{n+1}e_n(1)\pi = e_n(1)\Lambda_{n+1}\lambda^i_{n+1}\pi' = (\text{sgn } \lambda^i_{n+1})\, e_n(1)\Lambda_{n+1}\pi' = (\text{sgn } \lambda^i_{n+1})\, \Lambda_{n+1}e_n(1)\pi'$.

Hence, the homomorphism of right $\mathbb{Q}S_n$-modules $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1} \to e_n(1)\mathbb{Q}S_n$ which is given by $\Lambda_{n+1}e_n(1)\pi \mapsto (\text{sgn } \lambda^i_{n+1})\, e_n(1)\pi'$ for $\pi \in S_{n+1}$ is an inverse for θ. So $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$ and $e_n(1)\mathbb{Q}S_n$ are isomorphic as $\mathbb{Q}S_n$-modules as required.

Now we have $c_n\lambda^i_{n+1} = (\text{sgn } \lambda^i_{n+1})c_n$ (Proposition IV.1.1.). So the action of the (n+1)-cycles λ^i_{n+1} is the same on c_n as on $\Lambda_{n+1}e_n(1)$, hence the result. □

So the tree representation V_n' is a representation of S_{n+1} restricting to the first Eulerian representation of S_n, $e_n(1)\mathbb{Q}S_n$. More generally, we have defined idempotents $\Lambda_{n+1}e_n(j)$ for $j = 1, ..., n$ and we show these give representations of S_{n+1} restricting to the Eulerian representations of S_n, $e_n(j)\mathbb{Q}S_n$.

Proposition IV.1.6.

$\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$ restricts to $e_n(j)\mathbb{Q}S_n$, for $j = 1, ..., n$.

Proof

The proof is exactly as for the case $j = 1$ given above. □

Proposition IV.1.7.

The sum of the representations $\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$ for $j = 1, ..., n$ is given by taking the sign representation of $C_{n+1} = \langle \lambda_{n+1} \rangle \subset S_{n+1}$, and inducing up to S_{n+1}.

Proof

$$\sum_{j=1}^{n} \Lambda_{n+1}e_n(j) = \Lambda_{n+1}\sum_{j=1}^{n} e_n(j) = \Lambda_{n+1}.$$

So the sum of the representations $\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$ is $\Lambda_{n+1}\mathbb{Q}S_{n+1}$. It is easily seen that this representation is as claimed. □

Section IV.2 : A Relation between $e_n(j)$ and $e_{n+1}(j)$

In this section we prove certain relations between our idempotents, which will be needed in the following section to give descriptions of our representations. The main result is Proposition IV.2.5., giving a simplification of the product $e_n(j)e_{n+1}(j)$.

From now on we shall revert to standard notation where S_{n+1} denotes permutations of $\{ 1, ..., n+1 \}$ rather than $\{ 0, 1, ..., n \}$. So λ_{n+1} now denotes $(1\ 2\ ...\ n\ n+1)$. The symmetric group S_n is contained in S_{n+1} as the permutations fixing n+1, and similarly $\mathbb{Q}S_n \subset \mathbb{Q}S_{n+1}$.

Lemma IV.2.1.

$\Lambda_{n+1}e_n(j-1) = e_{n+1}(j)\,\Lambda_{n+1}$, for $j = 1, ..., n+1$.

Proof

This follows from Loday's relation $Be_n(j-1) = e_{n+1}(j)B$ [L,1; Theorem 4.6.6, p150], where B is Connes' boundary operator in the normalised setting. (The formula for B in this normalised situation is just $(n+1)\, s\, \Lambda_{n+1}$ - see p15). □

Corollary IV.2.2.

$\Lambda_{n+1}e_n(j-1)\mathbb{Q}S_{n+1} \subset e_{n+1}(j)\mathbb{Q}S_{n+1}$, for $j = 2, ..., n+1$. □

Hence, since the group algebra $\mathbb{Q}S_{n+1}$ is semi-simple, we may write:

$$e_{n+1}(j)\mathbb{Q}S_{n+1} = \Lambda_{n+1}e_n(j-1)\mathbb{Q}S_{n+1} \oplus y\mathbb{Q}S_{n+1},$$

for some $y \in \mathbb{Q}S_{n+1}$.

Notation

Let $p_n = (1\ n)(2\ n-1)(3\ n-2) \ldots \in S_n$ and let $op_n = (-1)^{n(n+1)/2}p_n = (-1)^n(\mathrm{sgn}\ p_n)\ p_n \in \mathbb{Q}S_n$. Now consider the idempotent $\sigma_n(j) = 1/2\ (1+(-1)^j op_n)$ in $\mathbb{Q}S_n$.

Gerstenhaber and Schack [G-S,1] show that these idempotents correspond to the even and odd parts or the Eulerian decomposition: $\sigma_n(j) = \sum_{j\ \mathrm{even}} e_n(j)$ if j is even, and $\sigma_n(j) = \sum_{j\ \mathrm{odd}} e_n(j)$ if j is odd. In particular $\sigma_n(j)$ (and hence p_n) is a polynomial in s_n and so commutes with s_n. Of course, $e_n(i)\sigma_n(j) = \sigma_n(j)e_n(i) = \begin{cases} e_n(i) & \text{if } i \equiv j \bmod 2 \\ 0 & \text{otherwise} \end{cases}$.

Notation

Let $s^*_{n+1} = \sum (\mathrm{sgn}\ \pi)\ \pi$, where the sum is over shuffles in S_{n+1} which do not fix n+1. So we may write $s_{n+1} = 1 + s_n + s^*_{n+1}$.

Lemma IV.2.3.

$$(1 + s^*_{n+1})\ \sigma_{n+1}(j) = \sigma_n(j+1)\ (1 + s^*_{n+1}).$$

Proof

Equivalently, we show $p_n\ (1 + s^*_{n+1})\ p_{n+1} = (-1)^n\ (1 + s^*_{n+1})$. First consider the term $p_n \cdot p_{n+1}$ on the left-hand side. Note that $p_n \cdot p_{n+1} = (\,n+1\ n \ldots 2\ 1\,) = \lambda^{-1}_{n+1}$, and since this is a 1-shuffle with $\lambda^{-1}_{n+1}(1) = n+1$ it appears in the right-hand side and the factor $(-1)^n$ cancels its sign. Now let π be a k-shuffle in S_{n+1}, not fixing n+1, so we must have $\pi k =$

n+1. Consider $p_n \pi p_{n+1}$. Then it is easy to see that $p_n \pi p_{n+1}$ is an (n+2-k)-shuffle such that (n+2-k) $\mapsto$ n+1. (If k = 1, we must have $\pi = \lambda_{n+1}^{-1}$, and we get $p_n \pi p_{n+1} = 1$). □

Recall that $s_n e_n(j) = e_n(j) s_n = \mu_j e_n(j)$, where $\mu_j = 2j-2$.

Lemma IV.2.4.

$e_n(j)\, \sigma_{n+1}(j)\, (s_{n+1} - \mu_j) = 0.$

Proof

$$\begin{aligned} e_n(j)\, \sigma_{n+1}(j)\, (s_{n+1} - \mu_j) &= e_n(j)\, (s_{n+1} - \mu_j)\, \sigma_{n+1}(j), \text{ since } \sigma_{n+1}(j) \text{ is a polynomial in } s_{n+1} \\ &= e_n(j)\, (1 + s_n + s_{n+1}^* - \mu_j)\, \sigma_{n+1}(j) \\ &= e_n(j)\, (s_n - \mu_j)\, \sigma_{n+1}(j) + e_n(j)\, (1 + s_{n+1}^*)\, \sigma_{n+1}(j) \\ &= e_n(j)\, (1 + s_{n+1}^*)\, \sigma_{n+1}(j), \text{ since } e_n(j) s_n = \mu_j e_n(j) \\ &= e_n(j)\, \sigma_n(j+1)\, (1 + s_{n+1}^*) \text{ by Lemma IV.2.3.} \\ &= 0. \end{aligned}$$
□

Proposition IV.2.5.

$e_n(j)\, e_{n+1}(j) = e_n(j)\, \sigma_{n+1}(j).$

Proof

It follows from Lemma IV.2.4. that $e_n(j)\sigma_{n+1}(j)$ is contained in the left ideal $\mathbb{Q}S_{n+1}(e_{n+1}(j))$. Hence, $e_n(j)\, \sigma_{n+1}(j) = e_n(j)\, \sigma_{n+1}(j)\, e_{n+1}(j)$. But $\sigma_{n+1}(j)\, e_{n+1}(j) = e_{n+1}(j)$. So, $e_n(j)\, \sigma_{n+1}(j) = e_n(j)\, e_{n+1}(j)$. □

In fact, using the same methods, we have:

$e_n(j)\, e_{n+1}(j) = e_n(j)\, \sigma_{n+1}(j) = \sigma_n(j)\, e_{n+1}(j).$

Section IV.3 : $\Lambda_{n+1} e_n(j) \mathbb{Q}S_{n+1}$ as a Virtual Representation

The main result of this section is Theorem IV.3.3., giving a description of the representation $\Lambda_{n+1} e_n(j) \mathbb{Q}S_{n+1}$. In order to prove this we first need a proposition.

Definition IV.3.1.

We define certain elements of the group algebra $\mathbb{Q}S_{n+1}$:

$$x_{n+1}(j) = \frac{2}{n+1}\left[(n-1)+(-1)^j op_n + \sum_{i=3}^{n} (i-2)(\mathrm{sgn}\ \lambda_{n+1}^i)\lambda_{n+1}^i \right].$$

Proposition IV.3.2.

$$e_n(j)\, e_{n+1}(j)\, x_{n+1}(j) = (1 - \Lambda_{n+1})\, e_n(j).$$

Proof

We will use $e_n(j) = (-1)^j e_n(j) op_n$; also $op_n op_{n+1} = -(\mathrm{sgn}\ \lambda_{n+1}^{-1})\ \lambda_{n+1}^{-1}$.

Now $e_n(j)\, e_{n+1}(j)\, x_{n+1}(j) = e_n(j)\, \sigma_{n+1}(j)\, x_{n+1}(j)$, by Proposition IV.2.5.

$$= 1/2\, e_n(j)\, (1 + (-1)^j op_{n+1})\, x_{n+1}(j)$$

So $e_n(j)\, e_{n+1}(j)\, x_{n+1}(j) = 1/2\, e_n(j)\, (1 + op_n op_{n+1})\, x_{n+1}(j)$ since $e_n(j) = (-1)^j e_n(j) op_n$

$$= 1/2\, e_n(j)\, (1 - (\mathrm{sgn}\ \lambda_{n+1}^{-1})\ \lambda_{n+1}^{-1})\, x_{n+1}(j)$$

Now :

$- (\mathrm{sgn}\ \lambda_{n+1}^{-1})\ \lambda_{n+1}^{-1}\ x_{n+1}(j)$

$$= \frac{-2}{n+1}\left[(n-1)(\mathrm{sgn}\,\lambda_{n+1}^{-1})\lambda_{n+1}^{-1} + (-1)^j(\mathrm{sgn}\,\lambda_{n+1}^{-1})\lambda_{n+1}^{-1} op_n + \sum_{i=2}^{n-1} (i-1)(\mathrm{sgn}\ \lambda_{n+1}^i)\lambda_{n+1}^i \right]$$

$$= \frac{-2}{n+1}\left[(-1)^j(\mathrm{sgn}\,\lambda_{n+1}^{-1})\lambda_{n+1}^{-1} op_n + \sum_{i=2}^{n} (i-1)(\mathrm{sgn}\ \lambda_{n+1}^i)\lambda_{n+1}^i \right].$$

So :

$(1 - (\mathrm{sgn}\ \lambda_{n+1}^{-1})\ \lambda_{n+1}^{-1})\, x_{n+1}(j)$

$$= \frac{2}{n+1}\left[(n-1)+(-1)^j op_n - (-1)^j(\mathrm{sgn}\,\lambda_{n+1}^{-1})\lambda_{n+1}^{-1} op_n - \sum_{i=2}^{n} (\mathrm{sgn}\ \lambda_{n+1}^i)\lambda_{n+1}^i \right]$$

Hence,

$e_n(j)\, e_{n+1}(j)\, x_{n+1}(j)$

$$= \frac{e_n(j)}{n+1}\left[(n-1)+(-1)^j op_n - (-1)^j(\mathrm{sgn}\,\lambda_{n+1}^{-1})\lambda_{n+1}^{-1} op_n - \sum_{i=2}^{n} (\mathrm{sgn}\ \lambda_{n+1}^i)\lambda_{n+1}^i \right]$$

$$= \frac{e_n(j)}{n+1}\left[n - (\mathrm{sgn}\,\lambda_{n+1}^{-1}) op_n \lambda_{n+1}^{-1} op_n - \sum_{i=2}^{n} (\mathrm{sgn}\ \lambda_{n+1}^i)\lambda_{n+1}^i \right]$$

$$= \frac{e_n(j)}{n+1}\left[n - (\mathrm{sgn}\,\lambda_{n+1})\lambda_{n+1} - \sum_{i=2}^{n} (\mathrm{sgn}\ \lambda_{n+1}^i)\lambda_{n+1}^i \right]$$

$$\text{Thus } e_n(j)\, e_{n+1}(j)\, x_{n+1}(j) = \frac{e_n(j)}{n+1}\left[\; n - \sum_{i=1}^{n} (\operatorname{sgn}\ \lambda^i_{n+1})\lambda^i_{n+1}\right]$$

$$= e_n(j)\,(1 - \Lambda_{n+1}) = (1 - \Lambda_{n+1})\, e_n(j)\,. \qquad \square$$

Now we can prove the main result. First note that given an idempotent $e \in \mathbb{Q}S_n$, giving a representation $e\mathbb{Q}S_n$ of S_n, then the induced representation of S_{n+1} is given by $e\mathbb{Q}S_{n+1}$.

Theorem IV.3.3.

The representation $\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$ is given as a virtual representation by inducing the sum from i = 1 to j of the representations $e_n(i)\mathbb{Q}S_n$ to S_{n+1} and subtracting the sum from i = 1 to j of the representations $e_{n+1}(i)\mathbb{Q}S_{n+1}$.
i.e. in terms of $\mathbb{Q}S_{n+1}$- modules :

$$\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1} \oplus \bigoplus_{i=1}^{j} e_{n+1}(i)\mathbb{Q}S_{n+1} \cong \bigoplus_{i=1}^{j} e_n(i)\mathbb{Q}S_{n+1}.$$

Proof

The result will be proved by induction on j. First we consider the case j = 1. Here we need to show that :

$$\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1} \oplus e_{n+1}(1)\mathbb{Q}S_{n+1} \cong e_n(1)\mathbb{Q}S_{n+1}.$$

Now it is clear that $e_n(1)\mathbb{Q}S_{n+1} = \Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1} \oplus (1 - \Lambda_{n+1})\, e_n(1)\mathbb{Q}S_{n+1}$.
So we must show that :

$$e_{n+1}(1)\mathbb{Q}S_{n+1} \cong (1 - \Lambda_{n+1})\, e_n(1)\mathbb{Q}S_{n+1}.$$

Note that these modules both have dimension n!.
We define $\theta : e_{n+1}(1)\mathbb{Q}S_{n+1} \to (1 - \Lambda_{n+1})\, e_n(1)\mathbb{Q}S_{n+1}$ to be the homomorphism of right $\mathbb{Q}S_{n+1}$-modules given by left multiplication by the element $(1 - \Lambda_{n+1})\, e_n(1)$.
Then $(1 - \Lambda_{n+1})e_n(1) = e_n(1)e_{n+1}(1)x_{n+1}(1) = (1 - \Lambda_{n+1})e_n(1)e_{n+1}(1)x_{n+1}(1) = \theta(e_{n+1}(1)x_{n+1}(1))$, by IV.3.2. Thus θ is surjective, and so an isomorphism, giving the result for n = 1.

Now we assume the result for j-1, and consider j. Using the induction hypothesis it is sufficient to show that :

$$\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1} \oplus e_{n+1}(j)\mathbb{Q}S_{n+1} \cong e_n(j)\mathbb{Q}S_{n+1} \oplus \Lambda_{n+1}e_n(j-1)\mathbb{Q}S_{n+1}.$$

Now we clearly have : $e_n(j)\mathbb{Q}S_{n+1} = \Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1} \oplus (1 - \Lambda_{n+1})\, e_n(j)\mathbb{Q}S_{n+1}$, so we must show that :

$$e_{n+1}(j)\mathbb{Q}S_{n+1} \cong \Lambda_{n+1}e_n(j-1)\mathbb{Q}S_{n+1} \oplus (1 - \Lambda_{n+1})e_n(j)\mathbb{Q}S_{n+1}.$$

By Corollary IV.2.2,

$$e_{n+1}(j)\mathbb{Q}S_{n+1} = \Lambda_{n+1}e_n(j-1)\mathbb{Q}S_{n+1} \oplus y\mathbb{Q}S_{n+1}.$$

Hence, the above simplifies to showing that :

$$y\mathbb{Q}S_{n+1} \cong (1 - \Lambda_{n+1})\, e_n(j)\mathbb{Q}S_{n+1}.$$

We define $\theta : e_{n+1}(j)\mathbb{Q}S_{n+1} \to (1 - \Lambda_{n+1})\, e_n(j)\mathbb{Q}S_{n+1}$ to be the homomorphism of right $\mathbb{Q}S_{n+1}$-modules given by left multiplication by $(1 - \Lambda_{n+1})e_n(j)$.
Now $\Lambda_{n+1}e_n(j-1)\mathbb{Q}S_{n+1} \subset \text{Ker}\,\theta$, since $(1 - \Lambda_{n+1})\, e_n(j)\Lambda_{n+1}e_n(j-1) = 0$.
Hence, θ induces a $\mathbb{Q}S_{n+1}$-module homomorphism :

$$\theta' : y\mathbb{Q}S_{n+1} \to (1 - \Lambda_{n+1})\, e_n(j)\mathbb{Q}S_{n+1}.$$

Now we check the dimensions of these $\mathbb{Q}S_{n+1}$-modules.
Recall that Hanlon [H] gave the dimension of $e_n(j)\mathbb{Q}S_n$ as the number of permutations in S_n with exactly j cycles, s(n, j). Also, we have already seen that $\Lambda_{n+1}e_n(j)\mathbb{Q}S_n$ does restrict to $e_n(j)\mathbb{Q}S_n$, so has the same dimension. So, $y\mathbb{Q}S_{n+1}$ has dimension s(n+1, j) -s(n, j-1), and $(1 - \Lambda_{n+1})e_n(j)\mathbb{Q}S_{n+1}$ has dimension (n+1)s(n, j) - s(n, j) = ns(n, j). Since an easy calculation gives s(n+1, j) = s(n, j-1) + ns(n, j), the two modules do have the same dimension.
Hence, it is sufficient to show that θ' is surjective to conclude that it is a $\mathbb{Q}S_{n+1}$-module isomorphism.

$$\begin{aligned} \text{But,}\quad (1 - \Lambda_{n+1})e_n(j) &= e_n(j)\, e_{n+1}(j)\, x_{n+1}(j), \text{ by Proposition IV.3.2.} \\ &= (1 - \Lambda_{n+1})e_n(j)\, e_{n+1}(j)\, x_{n+1}(j) \\ &= \theta'(\, e_{n+1}(j)\, x_{n+1}(j)\,). \end{aligned}$$

Hence θ' is surjective. □

Notation

Let Ψ^j_{n+1} denote the character of the representation $\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$ of S_{n+1}.
Let χ^j_n denote the character of the representation $e_n(j)\mathbb{Q}S_n$ of S_n. (This is Hanlon's notation).

Corollary IV.3.4.

$$\Psi^j_{n+1} = \sum_{i=1}^{j} \mathrm{ind}_{S_n}^{S_{n+1}}(\chi^i_n) - \sum_{i=1}^{j} \chi^i_{n+1}.$$

Proof

Immediate from the above. □

We give the formula for the character Ψ^1_{n+1} of the tree representation V_n' explicitly.

Corollary IV.3.5.

$$\Psi^1_{n+1}(\pi) = \begin{cases} \mathrm{sgn}(\pi)\frac{1}{n}p^{n/p}(n/p)!\mu(p) & \text{if } \pi \text{ has cycle type } p^{n/p}.1 \\ -\mathrm{sgn}(\pi)\frac{1}{n+1}q^{n+1/q}((n+1)/q)!\mu(q) & \text{if } \pi \text{ has cycle type } q^{n+1/q}, q \neq 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $\pi \in S_{n+1}$, and where μ denotes the classical Moebius function.
(The *cycle type* of $\pi \in S_n$ is $1^{\alpha_1}2^{\alpha_2}\ldots$ if when π is written as the product of disjoint cycles there are α_1 1-cycles, α_2 2-cycles and so on).

Proof

Firstly, we have shown that Ψ^1_{n+1} restricts to χ^1_n. So any π with at least one fixed point, being conjugate to an element of S_n, has :

$$\Psi^1_{n+1}(\pi) = \begin{cases} \mathrm{sgn}(\pi)\frac{1}{n}p^{n/p}(n/p)!\mu(p) & \text{if } \pi \text{ has cycle type } p^{n/p}.1 \\ 0 & \text{otherwise,} \end{cases}$$

since this holds for χ^1_n - see [H] or [G]. (This is a straightforward calculation from Hanlon's result: $\chi^1_n = \mathrm{sgn} * \left(\mathrm{ind}_{C_n}^{S_n}(\xi_{(12\ldots n)})\right)$). Now consider $\pi \in S_{n+1}$ without fixed points. We have shown that $\Psi^1_{n+1} = \mathrm{ind}_{S_n}^{S_{n+1}}(\chi^1_n) - \chi^1_{n+1}$. The standard formula for an induced character gives :

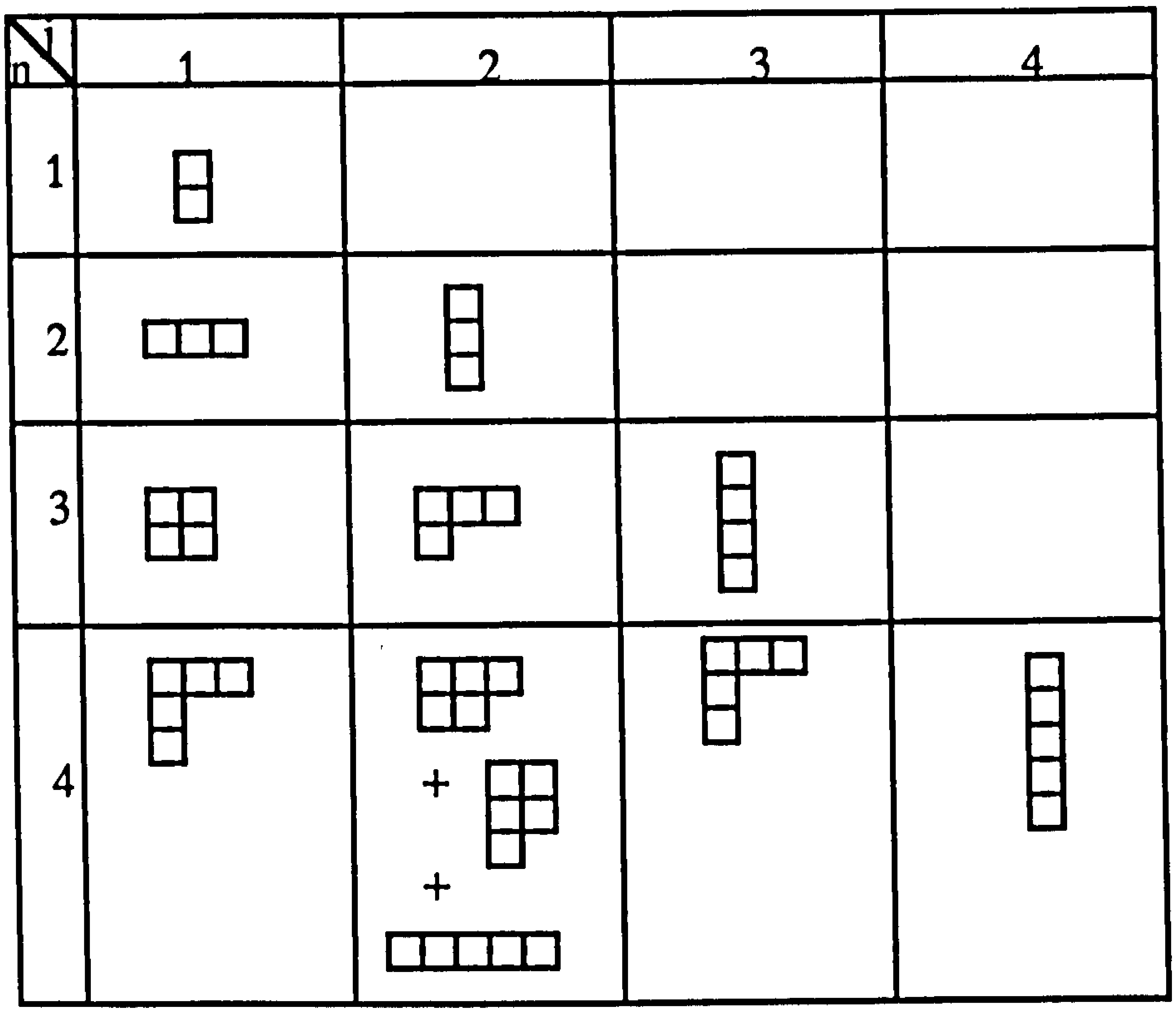
$$\mathrm{ind}_{S_n}^{S_{n+1}}(\chi^1_n)(\pi) = 1/n! \sum_{\sigma \in S_{n+1}} \chi^1_n(\sigma\pi\sigma^{-1}).$$

Conjugates of π have the same cycle type, and hence do not lie in S_n, so this expression is zero. So, for π without fixed points, $\Psi^1_{n+1} = -\chi^1_{n+1}$, and the result follows from [H]. □

Section IV.4 : Some Results on Decompositions

The following table lists the decompositions of the representations of S_{n+1} corresponding to the idempotents $\Lambda_{n+1}e_n(j)$, for n = 1, ..., 4 and j = 1, ..., n.

Table IV.4.1.



The first column gives the representation of S_{n+1} which comes from the tree space T_n. (We have seen that this is $\Lambda_{n+1}e_n(1)\mathbb{Q}S_{n+1}$). The above restrict to the representations listed by Hanlon [H] for the $e_n(j)\mathbb{Q}S_n$'s (see Table I.5.3.).

In the diagram above, the sum along the n^{th} row gives the representation $\Lambda_{n+1}\mathbb{Q}S_{n+1}$, i.e. the sign representation of C_{n+1} induced to S_{n+1}, which restricts to the regular representation of S_n – the sum along the n^{th} row in Table I.5.3.

For some values of j, it is possible to describe the decomposition of the representation $\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$ into irreducible components. Let ω^λ denote the irreducible character of the symmetric group S_{n+1} corresponding to the partition λ of n+1.

Proposition IV.4.2.

1). (j=1) The multiplicity of ω^λ in Ψ^1_{n+1} is the number of standard tableaux T of shape λ such that $a(T) \equiv 1 \pmod n$ minus the number such that $a(T) \equiv 1 \pmod{n+1}$. (Recall from I.5. that a(T) denotes the sum of ascents of a tableau T).

2). (j=n−2) $\Psi^{n-2}_{n+1} = \omega^{2^2 1^{n-3}} \oplus \omega^{321^{n-4}} \oplus \omega^{3^2 1^{n-5}} \oplus \omega^{51^{n-4}}$.

3). (j=n−1) $\Psi^{n-1}_{n+1} = \omega^{31^{n-2}}$.

4). (j=n) $\Psi^{n}_{n+1} = \omega^{1^{n+1}}$.

5). The multiplicity of ω^λ in the sum of characters $\sum_{j=1}^{n} \Psi^j_{n+1}$ is the number of standard tableau T of shape λ such that $a(T) \equiv 0 \pmod{n+1}$.

Proof

1). *Claim :* T is a standard tableau for S_{n+1} such that $a(T) \equiv 1 \pmod n$ if and only if it is obtained from a standard tableau T′ for S_n, with $a(T') \equiv 1 \pmod n$, by attaching n+1 to the end of some row/column.

Proof of Claim : Given such a tableau T, n+1 must appear at the end of a row or column since T is standard. Removing it gives a standard tableau T′ for S_n and the only possible change to the ascents is that n may cease to be one. So $a(T) \equiv a(T') \pmod n$.

Now since in χ^1_n ω^μ has multiplicity the number of standard tableau T′ of shape μ for S_n such that $a(T') \equiv 1 \pmod n$ [K−W], the claim shows that in $\mathrm{ind}_{S_n}^{S_{n+1}}(\chi^1_n)$, ω^λ has multiplicity the number of standard tableau T of shape λ for S_{n+1} such that $a(T) \equiv 1$ (mod n). Since, $\Psi^1_{n+1} = \mathrm{ind}_{S_n}^{S_{n+1}}(\chi^1_n) - \chi^1_{n+1}$ the result follows.

2).,3). and 4). These results follow directly from Hanlon's for χ^{n-2}_n, χ^{n-1}_n and χ^n_n. i.e. For i = n−2, n−1, n, the decomposition of Ψ^i_{n+1} given above is the only one which will restrict back to give the correct decomposition of χ^i_n. Of course, in the case j = n, we have $\Lambda_{n+1} e_n(n) = \Lambda_{n+1}\varepsilon_n = \varepsilon_{n+1}$, and we can see directly that we have the sign representation.

5). We have seen that this sum of characters is just the sign character of C_{n+1} induced to

S_{n+1}. The formula for the decomposition of this is given by Stembridge [S]. □

We also give the relationship between our characters and the trivial character.

Proposition IV.4.3.

The trivial character $\omega 1^{n+1}$ appears only in $\Psi^{n/2}_{n+1}$ if n is even and does not appear in any Ψ^{i}_{n+1} if n is odd.

Proof

Let $e_{n+1} = 1/(n+1)! \sum_{\pi \in S_{n+1}} \pi$.

It is easily checked that $\Lambda_{n+1} e_{n+1} = \begin{cases} e_{n+1} & \text{if n is even} \\ 0 & \text{if n is odd.} \end{cases}$

Hence, the trivial representation does not appear in $\Lambda_{n+1}\mathbb{Q}S_{n+1}$ when n is odd. When n is even it appears once, and this must be in $\Lambda_{n+1}e_n(n/2)\mathbb{Q}S_{n+1}$, since Hanlon [H] shows that the trivial representation of S_n appears in the restriction $e_n([(n+1)/2])\mathbb{Q}S_n$. □

Corollary IV.4.4.

The character ω^{n1} does not appear in any Ψ^{i}_{n+1} if n is even and appears only in $\Psi^{(n+1)/2}_{n+1}$ if n is odd.

Proof

The irreducible character ω^{n1} of S_{n+1} is the only one apart from ω^{n+1} which gives the trivial character of S_n on restriction. Hence the result follows from the above. □

The even/odd parts of the representation can be described explicitly. Consider the idempotent

$$\Lambda_{n+1}.(1 + (\mathrm{sgn}\, p_n)p_n)/2 = \Lambda_{n+1}\sigma_n(n) = \begin{cases} \sum_{j \text{ even}} \Lambda_{n+1} e_n(j) & \text{if n is even} \\ \sum_{j \text{ odd}} \Lambda_{n+1} e_n(j) & \text{if n is odd} \end{cases}.$$

This gives the even or odd part of the decomposition according to whether n is even or odd. It clearly gives the representation of S_{n+1} induced from the sign representation of the dihedral subgroup generated by λ_{n+1} and p_n.

Proposition IV.4.5.

The character of the representation $\Lambda_{n+1}\sigma_n(n)\mathbb{Q}S_{n+1}$ is given by, for n even

$$\sigma \mapsto \begin{cases} n!/2 & \text{if } \sigma = 1 \\ (n/2)!\ 2^{n/2-1}\ (\operatorname{sgn}\ \sigma) & \text{if } \sigma \text{ has cycle type } 2^{n/2}.1 \\ \dfrac{(n+1/q)!\ q^{n+1/q}\varphi(q)(\operatorname{sgn}\ \sigma)}{2(n+1)} & \text{if } \sigma \text{ has cycle type } q^{n+1/q} \text{ for } q \text{ dividing } n+1 \\ 0 & \text{otherwise} \end{cases}$$

and for n odd by

$$\sigma \mapsto \begin{cases} n!/2 & \text{if } \sigma = 1 \\ ((n-1)/2)!\ 2^{(n-3)/2}\ (\operatorname{sgn}\ \sigma) & \text{if } \sigma \text{ has cycle type } 2^{(n-1)/2}.1^2 \\ \dfrac{(n+1/q)!\ q^{n+1/q}\varphi(q)(\operatorname{sgn}\ \sigma)}{2(n+1)} & \text{if } \sigma \text{ has cycle type } q^{n+1/q} \text{ for } q \text{ dividing } n+1, q \neq 2 \\ \dfrac{2^{(n-1)/2}\ ((n+3)/2)!\ (\operatorname{sgn}\ \sigma)}{(n+1)} & \text{if } \sigma \text{ has cycle type } 2^{(n+1)/2} \\ 0 & \text{otherwise} \end{cases}$$

where φ is Euler's function.

Proof

We work out which cycle types occur in the subgroup $D_{2(n+1)} = \langle (1\ n)(2\ n-1)\ldots, (1\ 2 \ldots n+1) \rangle = \langle p_n, \lambda_{n+1} \rangle$. The elements are λ^i_{n+1} and $p_n\lambda^i_{n+1}$ for i = 1, ..., n+1. We have the elements of the cyclic group, with cycle type $q^{(n+1)/q}$ occuring $\varphi(q)$ times for each q dividing n+1. Now the elements $p_n\lambda^i_{n+1}$ all have order 2, and consideration of fixed points shows that if n is even they all have cycle type $2^{n/2}.1$, whereas if n is odd half have type $2^{(n-1)/2}.1^2$ and the other half have type $2^{(n+1)/2}$. Now the result follows from the standard formula for an induced character. □

We can give formulae of sorts for the decomposition of this representation into irreducibles, for example:

Proposition IV.4.6.

When $n \equiv 2 \pmod 4$ the multiplicity of ω^λ in $\sum_{j\ \text{even}} \Psi^j_{n+1}$ is given by:

(the number of standard tableaux T of shape λ such that $a(T) \equiv 0 \pmod 2$))
+ 1/2 [the number of standard tableaux T of shape λ such that $a(T) \equiv 0 \pmod{n+1}$ - $\omega^{\lambda}(1)$]

Proof

By Frobenius reciprocity,

$$< \omega^{\lambda}, \operatorname{ind}_{D_{2(n+1)}}^{S_{n+1}}(\operatorname{sgn}) >$$
$$= < \omega^{\lambda}{}_{D_{2(n+1)}}, \operatorname{sgn} >_{D_{2(n+1)}}$$
$$= \frac{1}{2(n+1)} \sum_{\sigma \in D_{2(n+1)}} \omega^{\lambda}(\sigma) \operatorname{sgn}(\sigma)$$
$$= \frac{1}{2(n+1)} \sum_{i=0}^{n} \left(\omega^{\lambda}(\lambda_{n+1}^{i}) \operatorname{sgn}(\lambda_{n+1}^{i}) + \omega^{\lambda}(p_n \lambda_{n+1}^{i}) \operatorname{sgn}(p_n \lambda_{n+1}^{i}) \right).$$

As above, for n even all the $p_n \lambda_{n+1}^{i}$'s have cycle type $2^{n/2}.1$. So we get:

$$\frac{1}{2(n+1)} \left((n+1) \left\langle \omega^{\lambda}, \operatorname{ind}_{C_{n+1}}^{S_{n+1}}(\operatorname{sgn}) \right\rangle + (n+1)\, \omega^{\lambda}(\sigma) \operatorname{sgn}(\sigma) \right),$$ where σ has type $2^{n/2}.1$,

$$= \frac{1}{2} \left(\left\langle \omega^{\lambda}, \operatorname{ind}_{C_{n+1}}^{S_{n+1}}(\operatorname{sgn}) \right\rangle + \omega^{\lambda}(\sigma) \operatorname{sgn}(\sigma) \right)$$
$$= \frac{1}{2} \left(\left\langle \omega^{\lambda}, \operatorname{ind}_{C_{n+1}}^{S_{n+1}}(\operatorname{sgn}) \right\rangle + 2 \left\langle \omega^{\lambda}, \operatorname{ind}_{<p_n>}^{S_{n+1}}(\operatorname{sgn}) \right\rangle - \omega^{\lambda}(1) \right).$$

This formula involves representations induced from the two cyclic groups C_{n+1} and $<p_n>$, and the result follows from the results of Stembridge [S] on such representations. □

Chapter V : Some Remarks on Commutative Algebra (Co)homology in Prime Characteristics.

Introduction

In this chapter we consider some aspects of commutative algebra (co)homology in prime characteristics. We begin by describing the work of Gerstenhaber and Schack [G-S,2], giving a modified decomposition of the Hochschild (co)homology valid in this situation. In fact their reasoning is slightly wrong and we make the necessary corrections. We also correct their statement that the first part of the decomposition is still the null space of the total shuffle operator or Harrison's cohomology.

In the next section, we explain how to give the analagous decomposition of cyclic (co)homology in prime characteristics.

Finally, we give a counterexample to a conjecture of Barr (see [G-S,1] p232), which states that a certain modification of Harrison cohomology, taking into account torsion, should coincide with André/Quillen cohomology in prime characteristics.

Section V.1 : The Decomposition of Hochschild (Co)homology in Prime Characteristics.

We have seen that when we work over k containing $\mathbb{Q}$, there is a direct sum decomposition of Hochschild (co)homology. The first part of this decomposition is Harrison (co)homology. Now suppose that k contains $\mathbb{Z}_p$. In this situation Gerstenhaber and Schack [G-S,2] give a modified version of the decomposition. They show that for $0 < i < p$, the idempotents $\bar{e}_n(i) = \sum_{m \geq 0} e_n(i+(p-1)m)$ are defined in characteristic p. Thus, setting $\overline{HH}_{i,n-i}(A/k; M) = \bar{e}_n(i)HH_n(A/k; M)$, we have a direct sum decomposition of Hochschild homology into p-1 parts :

$$HH_n(A/k; M) = \overline{HH}_{1,n-1}(A/k; M) \oplus \dots \oplus \overline{HH}_{p-1,n-p+1}(A/k; M),$$

and similarly for cohomology.

In fact, as we show, although it is true that these idempotents are defined in characteristic p, the reasoning of Gerstenhaber and Schack is slightly wrong. Below we make the necessary corrections to their argument.

First we will summarise some more results over $\mathbb{Q}$.

Definition V.1.1.

Let $s_n^{(q)} = \sum_{p \in P(n,q)} \sum_{\sigma \in Sh_p} (\operatorname{sgn} \sigma)\sigma \in \mathbb{Z}S_n$,

where P(n,q) denotes the set of all ordered partitions $p = (p_1, ..., p_q)$ of n into q parts with $\Sigma p_i = n$ and all $p_i \geq 0$, and for $p \in P(n,q)$ the set of p-multishuffles, Sh_p, consists of $\sigma \in S_n$ such that $\sigma(1) < ... < \sigma(p_1)$, $\sigma(p_1+1) < ... < \sigma(p_1+p_2)$, ..., $\sigma(p_1+...+p_{q-1}+1) < ... < \sigma(n)$.

Hence, $s_n^{(1)} = 1$ and $s_n^{(2)} = s_n+2$, where s_n is the total shuffle operator. The elements $s_n^{(q)}$ come from the q^{th} characteristic endomorphism of the shuffle bialgebra, and differ from Loday's λ-operations only by a sign, $s_n^{(q)} = (-1)^{q-1} \lambda_n^q$ [G-S,2], [L,2]. Now, Gerstenhaber and Schack show that $b\, s_n^{(q)} = s_{n-1}^{(q)} b$ for each q, so that using the universal property of the idempotents $e_n(j)$ with respect to commuting with the Hochschild boundary we have :

Proposition V.1.2. ([G-S,2], [L,2])

$s_n^{(q)} = \sum_{i=1}^{n} q^i e_n(i)$ for all $q \geq 1$. □

In particular, $e_n(i) s_n^{(q)} = q^i e_n(i)$ for $i = 1, ..., n$, $q \geq 1$.

Now we wish to translate the above to characteristic p.

Proposition V.1.3.

The idempotents $\bar{e}_n(i) = \sum_{m \geq 0} e_n(i+(p-1)m)$ are defined in kS_n, where k contains $\mathbb{Z}_p$, for p prime.

Proof

Over $\mathbb{Q}$, we have $e_n(i+(p-1)m)\ (s_n^{(q)} - q^i) = (q^{i+(p-1)m} - q^i)\ e_n(i+(p-1)m)$, for i, $q = 1, ..., p-1$, $m \geq 0$.

Now although $q^{i+(p-1)m} - q^i$ is zero mod p, the denominator of $e_n(i+(p-1)m)$ may also contain multiples of p. However, by choosing a sufficiently large r, we have for each i:

$$e_n(i+(p-1)m)\ (s_n^{(q)} - q^i)^r = (q^{i+(p-1)m} - q^i)^r\ e_n(i+(p-1)m) \equiv 0 \bmod p.$$

Hence, $\bar{e}_n(i)(s_n^{(q)} - q^i)^r \equiv 0 \bmod p$, for $i, q = 1, \dots, p-1$.

Now let's also choose r to be a power of the prime p, so that $(s_n^{(q)} - q^i)^r \equiv \left(s_n^{(q)}\right)^r - q^{ir} \equiv \left(s_n^{(q)}\right)^r - q^i \bmod p$. Hence, we have $\bar{e}_n(i)\left(s_n^{(q)}\right)^r \equiv q^i \bar{e}_n(i) \bmod p$. So the idempotents $\bar{e}_n(i)$ satisfy the equations:

$$\left(s_n^{(q)}\right)^r \equiv \sum_{i=1}^{p-1} q^i \bar{e}_n(i) \pmod p \qquad (*)$$

The matrix of coefficients (q^i) is Vandermonde and invertible over $\mathbb{Z}_p$, so the idempotents $\bar{e}_n(i)$ are defined in characteristic p. □

Applying the same argument, we can deduce $b\bar{e}_n(i) = \bar{e}_{n-1}(i)b$ from $bs_n^{(q)} = s_{n-1}^{(q)}b$ and (*), so that these idempotents do decompose Hochschild homology.

The equation (*) replaces the equation $s_n^{(q)} \equiv \sum_{i=1}^{p-1} q^i \bar{e}_n(i) \pmod p$ of Gerstenhaber and Schack, which is wrong.

Now, Gerstenhaber and Schack incorrectly assert that the first part of the cohomology decomposition, $\overline{HH}^{1,n-1}(A/k; M)$ corresponding to $\bar{e}_n(1)$, is the null space of the shuffle operator s_n and thus Harrison's n^{th} cohomology group. In fact, we show that the three objects, the null space of s_n, Harrison's n^{th} cohomology group and $\overline{HH}^{1,n-1}(A/k; M)$ are all different.

We begin by showing that in characteristic p, Harrison cohomology is not the null space of the shuffle operator. A Hochschild n-cochain f is a Harrison cochain if it vanishes on shuffles. However, this means that we must have $fs_{i,n-i} = 0$ for $i = 1, \dots, n-1$. In characteristic zero this condition is equivalent to $fs_n = 0$, and the Harrison n^{th} cohomology is the null space of s_n. However, in characteristic p, these two conditions are not equivalent. For example, we have:

$s_{1,2} = 1 - (1\ 2) + (1\ 3\ 2)$

$s_3 = 2 - (1\ 2) - (2\ 3) + (1\ 2\ 3) + (1\ 3\ 2)$.

If we work in characteristic 2 and consider those Hochschild 3-cocycles, f, which are invariant under permutations, then $fs_{1,2} = 0 \Leftrightarrow f = 0$, so there are no such Harrison cocycles. However, $fs_3 = 0 \Leftrightarrow 2f = 0$, so any such cocycle lies in the null space of s_3. A specific example is given by taking the polynomial algebra $\mathbb{Z}_2[x]$, with coefficients in $\mathbb{Z}_2$, on which x acts trivially. Define a cochain $f: \mathbb{Z}_2[x]^{\otimes 3} \to \mathbb{Z}_2$ by

$$f[x^{i_1}, x^{i_2}, x^{i_3}] = \begin{cases} 1 & \text{if } i_1 = i_2 = i_3 = 1 \\ 0 & \text{otherwise} \end{cases}$$

and extending linearly. Then f is not a Harrison cocycle. However, f is a Hochschild 3-cocycle which is in the null space of s_3, and it is not the coboundary of any 2-cocycle, g, in the null space of s_2, since :

$\delta g[x, x, x] = g[x^2, x] + g[x, x^2] = gs_2[x, x^2]$.

Secondly, we note that Harrison cohomology is not $\overline{HH}^{1,*-1}$. We consider the example of a polynomial algebra. If A = k [x], a polynomial algebra on one indeterminate, then $HH^n(A/k; k) = 0$ for $n \geq 2$. For André/Quillen homology we have $D^n(A/k; k) = 0$ for $n \geq 1$. Now Barr [B] gives an example to show that in characteristic p, the Harrison cohomology groups of the polynomial algebra k [x] are non-zero in dimensions $2p^m$, for any integer $m > 0$. This shows that the Harrison and André/Quillen theories are different in characteristic p. It also shows that the Harrison theory is not a direct summand of the Hochschild theory in prime characteristics. So $\overline{HH}^{1,n-1}(A/k; M) \neq Harr^n(A/k; M)$.

Thirdly, one can check directly that $\overline{HH}^{1,n-1}(A/k; M)$, the part corresponding to the idempotent $\bar{e}_n(1)$, is not the null space of s_n. For example, in characteristic 3:

$\bar{e}_3(1) = e_3(1) + e_3(3) = 2 + (1\ 3)$,

and one easily checks that $\bar{e}_3(1)s_3 \not\equiv 0 \pmod 3$. As above, we only have that $\bar{e}_n(1)(s_n)^r \equiv 0 \pmod p$ for some sufficiently large r. In the above example, we have $\bar{e}_3(1)(s_3)^3 \equiv 0 \pmod 3$.

We should consider the homology theory $\overline{HH}_{1,*-1}(A/k, M)$, since it is another commutative algebra homology theory agreeing with Harrison theory in characteristic zero. Firstly, it is not the same as Γ-homology: it is a direct summand of Hochschild homology and therefore the higher groups are zero for the polynomial algebra k [x], whereas Robinson [Rob,2] has calculated that we get some non-zero Γ-homology groups. Secondly, it is not the same as André/Quillen homology. To see this, consider the polynomial algebra $k[x_1, ..., x_r]$, whose higher André/Quillen groups are all zero.

We show that $\overline{HH}_{1,n-1}(k[x_1, ..., x_r]; k) \neq 0$, when $n \equiv 1 \pmod{p-1}$ and $n \leq r$. Here k is a $k[x_1, ..., x_r]$-module via the map which sends each x_i to zero.

Notation

Let $\Lambda_k^n[dx_1, ..., dx_r]$ denote the degree n elements of $\Lambda_k[dx_1, ..., dx_r]$, the exterior k-algebra on symbols $dx_1, ..., dx_r$ of degree 1.

Proposition V.1.4.

If k contains $\mathbb{Z}_p$,

$$\overline{HH}_{1,n-1}(k[x_1, ..., x_r]; k) = \begin{cases} \Lambda_k^n [dx_1, ..., dx_r] & \text{if } n = m(p-1)+1 \text{ for } m \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

Proof

$$\overline{HH}_{1,n-1}(k[x_1, ..., x_r]; k) = \bar{e}_n(1)HH_n(k[x_1, ..., x_r]; k) = \bar{e}_n(1)\Lambda_k^n[dx_1, ..., dx_r].$$

Now $(\operatorname{sgn} \bar{e}_n(1)) = (\operatorname{sgn} \sum_{m\geq 0} e_n(1+(p-1)m))$

$$= \begin{cases} 1 & \text{if } n = m(p-1)+1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the result. □

So, $\overline{HH}_{1,*-1}$ gives a theory somewhere between the André/Quillen and Γ theories.

Section V.2 : The Decomposition of Cyclic (Co)homology in Prime Characteristics.

We note that we can give an analagous decomposition of cyclic (co)homology over k containing $\mathbb{Z}_p$. As above, the idempotents $\bar{e}_n(i)$, for i = 1, ..., p-1, are defined in characteristic p, and we have $b\bar{e}_n(i) = \bar{e}_{n-1}(i)b$.

Proposition V.2.1.

$B\bar{e}_{n-1}(i-1) = \bar{e}_n(i)B$, for i = 2, ..., p-1,

$B\bar{e}_{n-1}(p-1) = \bar{e}_n(1)B$,

where B is Connes' boundary operator in the normalised setting.

Proof

Now Loday shows that $s_n^{(q)}B = Bq\,s_{n-1}^{(q)}$ [L,1,2]. Thus $\left(s_n^{(q)}\right)^r B = Bq^r\left(s_{n-1}^{(q)}\right)^r$. Then by (*), for r some sufficiently large power of p, we have

$$\sum_{i=1}^{p-1} q^i \bar{e}_n(i)\, B \equiv Bq^r \sum_{i=1}^{p-1} q^i \bar{e}_{n-1}(i) \equiv Bq \sum_{i=1}^{p-1} q^i \bar{e}_{n-1}(i)$$

$$\equiv \sum_{i=2}^{p} q^i B\bar{e}_{n-1}(i-1) \equiv qB\,\bar{e}_{n-1}(p-1) + \sum_{i=2}^{p-1} q^i B\bar{e}_{n-1}(i-1) \quad (\text{mod } p)$$

The result follows by inverting the matrix (q^i). □

It follows that the (b-B) bicomplex for cyclic homology is a direct sum of the sub-bicomplexes, for i = 0, 1, ..., p-1, where $\overline{C}_*^{(i)}$ denotes $\bar{e}_*(i)(A\otimes\overline{A}^{\otimes *})$:

$$\begin{array}{cccccccccccc}
 & \vdots & & \vdots & & & \vdots & & \vdots & & \vdots & \\
 & b\downarrow & B & b\downarrow & & & b\downarrow & & b\downarrow & & b\downarrow & \\
n+1 & \overline{C}_{n+1}^{(i)} & \leftarrow & \overline{C}_{n}^{(i-1)} & \leftarrow \cdots \leftarrow & & \overline{C}_{n-i+2}^{(1)} & \leftarrow & \overline{C}_{n-i+1}^{(p-1)} & \leftarrow & \overline{C}_{n-i}^{(p-2)} & \leftarrow\cdots \\
 & b\downarrow & B & b\downarrow & & & b\downarrow & & b\downarrow & & b\downarrow & \\
n & \overline{C}_{n}^{(i)} & \leftarrow & \overline{C}_{n-1}^{(i-1)} & \leftarrow \cdots \leftarrow & & \overline{C}_{n-i+1}^{(1)} & \leftarrow & \overline{C}_{n-i}^{(p-1)} & \leftarrow & \overline{C}_{n-i-1}^{(p-2)} & \leftarrow\cdots \\
 & b\downarrow & & b\downarrow & & & b\downarrow & & b\downarrow & & b\downarrow &
\end{array}$$

(Here we adopt the convention $\bar{e}_0(0) = 1$. For i = 0, the bicomplex simply contains A, concentrated in bidegree (0, 0).)
So, denoting by $\overline{HC}_{i,*-i}(A)$ the homology of the total complex of the above, we have:

Proposition V.2.2.

If A is a commutative algebra over k containing $\mathbb{Z}_p$, we have a decomposition of cyclic homology into p-1 parts :

$HC_n(A) = \overline{HC}_{1,n-1}(A) \oplus \ldots \oplus \overline{HC}_{p-1,n-p+1}(A)$, for n > 0,

$HC_0(A) = \overline{HC}_{0,0}(A) = A$. □

For p = 2, the decomposition is trivial. For p = 3, we have a decompostion of $HC_n(A)$ into two parts, and this is $HC_n(A) = HD'_n(A) \oplus HD_n(A)$, Loday's splitting in terms of dihedral homology, where A has the trivial involution [L,4].

The above also shows that the periodicity sequence linking Hochschild and cyclic homology respects their decompositions.

Proposition V.2.3.

For $i = 2, \ldots, p-1$, there are long exact sequences :

$$\ldots \to \overline{HH}_{i,n-i}(A) \xrightarrow[I]{} \overline{HC}_{i,n-i}(A) \xrightarrow[S]{} \overline{HC}_{i-1,n-i-1}(A) \xrightarrow[B]{} \overline{HH}_{i,n-i-1}(A) \to \ldots$$

and for $i = 1$:

$$\ldots \to \overline{HH}_{1,n-1}(A) \xrightarrow[I]{} \overline{HC}_{1,n-1}(A) \xrightarrow[S]{} \overline{HC}_{p-1,n-p-1}(A) \xrightarrow[B]{} \overline{HH}_{1,n-2}(A) \to \ldots$$

□

Corollary V.2.4.

$\overline{HH}_{1,n-1}(A)$ is isomorphic to a direct summand of $HC_n(A)$ for $2 < n < p$.

Proof

For $i = 1$, the bicomplex is:

$$\begin{array}{llllllll}
 & \vdots & & \vdots & & \vdots & & \\
 & b\downarrow & B & b\downarrow & & b\downarrow & & \\
p & \overline{C}_p^{(1)} & \leftarrow & \overline{C}_{p-1}^{(p-1)} & \leftarrow & \overline{C}_{p-2}^{(p-2)} & \leftarrow & \cdots \\
 & b\downarrow & B & b\downarrow & & b\downarrow & & \\
p-1 & \overline{C}_{p-1}^{(1)} & \leftarrow & 0 & \leftarrow & 0 & \leftarrow & \cdots \\
 & b\downarrow & & \downarrow & & \downarrow & & \\
 & \vdots & & & & & & \\
 & \overline{C}_2^{(1)} & & & & & & \\
 & \downarrow & & & & & & \\
 & \overline{C}_1^{(1)} & \leftarrow & \overline{C}_0^{(0)} & & & & \\
 & \downarrow & & & & & & \\
 & 0 & & & & & &
\end{array}$$

It is immediate that $\overline{HH}_{1,n-1}(A) \cong \overline{HC}_{1,n-1}(A)$ for $2 < n < p$. □

Corollary V.2.5.

When k contains $\mathbb{Z}_p$, Harrison homology, $Harr_n(A/k, A)$, is a direct summand of Hochschild homology, $HH_n(A)$, in dimensions $n < p$, and of cyclic homology, $HC_n(A)$, in dimensions $2 < n < p$.

Proof

Note that for $n < p$, $\bar{e}_n(1) = e_n(1)$ and so $\overline{HH}_{1,n-1}(A) = HH_{1,n-1}(A) = Harr_n(A/k, A)$. □

Remark

Let k be $\mathbb{Z}_p$. The idempotents $\bar{e}_n(i)$, for $i = 1, ..., p-1$, are defined in characteristic p, so they give rise to p-modular representations $\bar{e}_n(i)kS_n$ of the symmetric group S_n. Since $\sum_i \bar{e}_n(i) = 1$, these decompose the regular representation into p-1 parts.

Section V.3 : A counterexample to a conjecture of Barr.

We note that, if T is the tensor algebra of a commutative algebra A, then T/T*T may have torsion. For example, [a, b]*[a, b] = 2[a, b, a, b], so that when the ground ring k contains $\mathbb{Q}$ a Harrison 4-cochain must vanish on [a, b, a, b], but not when k has characteristic 2. Let us denote by HB^* the homology of the subcomplex of all Hochschild cochains vanishing not only on shuffles but also on those elements some multiple of which is a shuffle. Barr conjectures that this gives the triple cohomology (André/Quillen cohomology). (See [G-S,1] p232). We give a counterexample.

Proposition V.3.1.

$HB^5(\mathbb{Z}_2[x]; \mathbb{Z}_2) \neq 0$, where $\mathbb{Z}_2$ is a $\mathbb{Z}_2[x]$-module via the trivial x-action.

Proof

Define a Hochschild 4-cochain g by letting g[1, x, 1, x] = g[x, 1, x, 1] = g[1, x, x, 1] = g[x, 1, 1, x] = 1, letting g be zero on any other chain of the form $[x^{i1}, x^{i2}, x^{i3}, x^{i4}]$, and extending linearly.

Let $f = \delta g$. Now $f \neq 0$, since for example f [1, 1, x, 1, x] = 1. So, of course, f is a non-zero Hochschild 5-cocycle.

Next we check that g vanishes on shuffles, i.e. $gs_{i,4-i} = 0$ for i = 1, 2, 3. It is clearly sufficient to check that g vanishes on shuffles which involve [1, x, 1, x], [x, 1, x, 1], [1, x, x, 1] or [x, 1, 1, x]. For example,

g(1 * [x, 1, x]) = g[1, x, 1, x] - g[x, 1, 1, x] + g[x, 1, 1, x] - g[x, 1, x, 1] = 0.

Similar calculations show that g does indeed vanish on all such shuffles. Hence g is a Harrison cochain, and so $f = \delta g$ is a Harrison 5-cocycle.

Notice, however, that $2[1, x, 1, x] = [1, x]*[1, x]$, so $[1, x, 1, x]$ has a multiple which is a shuffle and yet $g[1, x, 1, x] \neq 0$. i.e. g does not satisfy Barr's condition.
In fact, if $f = \delta h$, then:

$$1 = f[1, 1, x, 1, x] = \delta h[1, 1, x, 1, x] = h[1, x, 1, x].$$

So f is not the coboundary of any cochain satisfying Barr's condition. Hence, if we can show that f itself does vanish on elements some multiple of which is a shuffle, then $[f]$ is a non-zero element of $HB^5(\mathbb{Z}_2[x]; \mathbb{Z}_2)$.
Now one easily checks that the only chains of the form $[x^{i_1}, x^{i_2}, x^{i_3}, x^{i_4}, x^{i_5}]$ on which f is non-zero are:

$a_1 = [1, 1, x, 1, x]$, $\quad a_4 = [x, 1, x, 1, 1]$,
$a_2 = [1, x, 1, 1, x]$, $\quad a_5 = [x, 1, 1, x, 1]$,
$a_3 = [1, 1, x, x, 1]$, $\quad a_6 = [1, x, x, 1, 1]$.

We have $f(a_i) = 1$, for $i = 1, ..., 6$. We need only check for shuffles which are multiples of expressions involving these terms. We introduce the following notation:

$b_1 = [1, 1, 1, x, x]$, $\quad b_2 = [x, 1, 1, 1, x]$, $\quad b_3 = [x, x, 1, 1, 1]$,
$b_4 = [1, x, 1, x, 1]$.

We want to consider all possible shuffles involving three 1's and 2 x's. Routine calculations yield that these have the following form:

$$x_1(a_1+a_2) + x_2(a_1+a_5) + x_3(a_2+a_4) + x_4(b_1+a_2+a_3) + x_5(b_1+a_2+a_6) + x_6(b_2+a_1+a_4) + x_7(b_3+a_3+a_5) + x_8(b_3+a_2+a_6) + x_9 b_4, \text{ where } x_i \in \mathbb{Z}.$$

(For example, $a_1 + a_2 = [1, x] * [1, 1, x]$).
Now consider when the above expression has the form $q(\Sigma k_i a_i + \Sigma l_j b_j)$, for $q, k_i, l_j \in \mathbb{Z}$. Firstly, the coefficient of each a_i and b_j is a multiple of q, and adding the coefficients of a_2, a_5 and b_2 gives $\sum_{i=1}^{8} x_i$. So $\sum_{i=1}^{8} x_i \in q\mathbb{Z}$. Secondly, the sum of the coefficients of the a_i's is $q\Sigma k_i = 2\sum_{i=1}^{8} x_i$. Hence, $q\Sigma k_i \in 2q\mathbb{Z}$, and $\Sigma k_i \equiv 0 \pmod 2$.
But then $f(\Sigma k_i a_i + \Sigma l_j b_j) = \Sigma k_i \equiv 0 \pmod 2$.
Hence, f does vanish on any element some multiple of which is a shuffle. So $0 \neq [f] \in HB^5(\mathbb{Z}_2[x]; \mathbb{Z}_2)$. □

Notation

Cb^* denotes the complex of Hochschild cochains satisfying Barr's condition; as before Ch^* denotes Harrison's cochain complex; Cs^* denotes the complex of cochains vanishing on the shuffle operator s_* and HS^* its homology.

Now if f is a Hochschild n-cochain,

(f satisfies Barr's condition) $\Rightarrow$ ($fs_{i,n-i} = 0$ for $i = 1, \ldots, n-1$) $\Rightarrow$ ($fs_n = 0$),

so we have inclusion maps at the level of cochain complexes :

$$Cb^* (A/k; M) \to Ch^* (A/k; M) \to Cs^* (A/k; M),$$

giving induced maps in homology :

$$HB^*(A/k; M) \to Harr^*(A/k; M) \to HS^*(A/k; M).$$

These maps are isomorphisms when k contains $\mathbb{Q}$, but not in general.

Index of Notation

$\mathrm{ind}_H^G(X)$	representation X induced from H to G	17
λ_{n+1}	$(0\ 1\ ...\ n) \in S_{n+1}$, or $(1\ 2\ ...\ n+1) \in S_{n+1}$	14 50
$\Lambda_{n+1}e_n(j)$	new idempotents, in $\mathbb{Q}S_{n+1}$	49
$p_{n,k}$	$(n\ k)(n-1\ k+1)(n-2\ k+2) ... \in S_{n+1}$	41
$\rho_{n,k}$	$(-1)^{([n-k+1][n-k+2]/2\ -\ 1)}\, p_{n,k}$	41
$s_{i,n-i}$	sum, with sign, over i-shuffles in S_n	9
s_n	total shuffle operator in $\mathbb{Q}S_n$	9
$Sh_*(A)$	subcomplex of $C_*(A)$ generated by shuffles	9
T_n	space of fully-grown n-trees	22
V_n	representation of S_n on T_n	22
V'_n	representation of S_{n+1} on T_n	47
χ_n^j	character of representation $e_n(j)\mathbb{Q}S_n$	17
Ψ_{n+1}^i	character of representation $\Lambda_{n+1}e_n(j)\mathbb{Q}S_{n+1}$	55

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