Pattern Formation and Travelling Waves in Reaction - Diffusion Equations

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Contents

List of Figures

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Declaration

I declare that to the best of my knowledge the contents of this thesis are original and my own work except where indicated otherwise.

Summary

This thesis is about pattern formation in reaction - diffusion equations, particularly Turing patterns and travelling waves. In chapter one we concentrate on Turing patterns. We give the classical approach to proving the existence of these patterns, and then our own, which uses the reversibility of the associated travelling wave equations when the wave speed is zero. We use a Lyapunov - Schmidt reduction to prove the existence of periodic solutions when there is a purely imaginary eigenvalue. We pay particular attention to the bifurcation point where these patterns arise, the 1:1 resonance. We prove the existence of steady patterns near a Hopf bifurcation and then include a similar result for dynamics close to a Takens - Bogdanov point.

Chapter two concentrates on travelling waves and looks for the existence of such in three different ways. Firstly we prove the conditions that are needed for the travelling wave equations to go through a Hopf bifurcation. Secondly, we look for the existence of travelling waves as the wave speed is perturbed from zero and prove when this occurs, again, using a Lyapunov - Schmidt reduction. Thirdly we describe a result proving the existence of periodic travelling waves when the wave speed is perturbed from infinity. In the last part of chapter two we prove the stability of such waves for $\lambda - \omega$ systems.

In chapter three we discuss computer simulations of the work done in the earlier chapters. We present the mappings used and prove that their behaviour is similar to the original partial differential equations. The two specific examples we give are a predator prey model and the complex Ginzburg - Landau equations.

Chapter 0

Introduction

This thesis is about pattern formation in reaction - diffusion equations. We study the existence of two types of patterns: those that are periodic in space but steady in time and those that are periodic in both space and time.

In 1952 two papers were published which have been very influential in mathematical modelling. They both presented models of biological phenomena. Turing [Turing] described a model for developmental pattern formation which explained the appearance of structure in a previously undifferentiated collection of cells in an embryo. Hodgkin and Huxley's work [Hodgkin] considered the propagation of an electrical pulse along a nerve axon. Although the two problems are very different, the models used belong to the same general class, that of reaction - diffusion equations. Turing showed that in some circumstances these equations can have solutions which are independent of time, but vary in space. Hodgkin and Huxley showed that under different conditions they can have travelling wave solutions.

In the following years many more phenomena ranging from contractions of the heart to the behaviour of the slime mold 'Dictyostelium discoideum' have been shown to exhibit similar patterns and have been modelled using the same class of equations (see [Murray] for example), see figure [0.1]. These phenomena have also been modelled using discrete methods, see for example

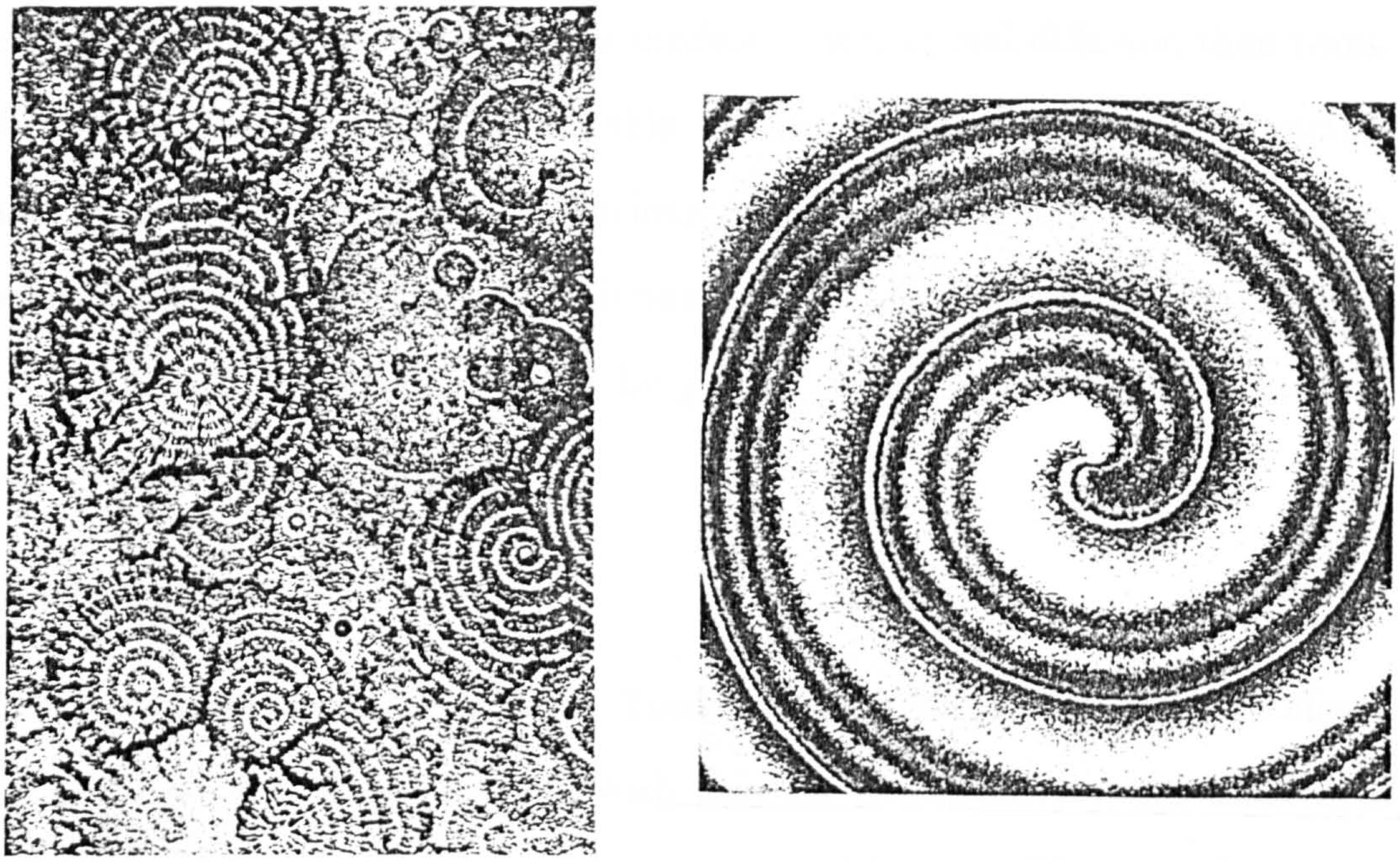


Figure 0.1: Patterns in slime mold and the B-Z reaction

[Hassell][Ives]. Travelling waves have also been seen in the 'CIMA' chemical reaction [Ouyang] [Winfree], but what is remarkable is that Turing patterns have also been seen there. Just about all the work that has been done on these two types of pattern has concentrated on either one or the other [Roberts].

In this thesis we include both in the same framework, or, in other words, we use the same set of equations to investigate the existence of these patterns. Instead of restricting ourselves to what might strictly be called 'Turing patterns' we consider any steady spatially periodic solutions. These are then just travelling waves with zero speed. A special type of model which we consider is those in which the reaction part of the system goes through a Hopf bifurcation. Kopell and Howard [Kopell & Howard] have already shown that travelling waves exist for such models, and we prove that Turing patterns also exist under certain conditions.

As much as possible we prove our results using general diffusion matrices as opposed to just diagonal diffusion which is the common approach. This is important since even if the original models have diagonal diffusion then transformation of variables will affect this. It also helps to give a wider picture and means that more varied behaviour is seen. Throughout we restrict ourselves to considering one spatial dimension. We give specific examples as we go along, and we finish the thesis by giving various results which have come out of computer simulations.

Overview

In chapter one we concentrate on Turing type patterns. We give the classical approach and then our own which uses the reversibility of the associated travelling wave equations when the wave speed is zero. We use a Lyapunov - Schmidt reduction to prove the existence of periodic solutions when there is a purely imaginary eigenvalue - a result that was first proved by Devaney [Devaney]. We pay particular attention to the bifurcation point where these patterns first arise. We prove the existence of patterns near a Hopf bifurcation and then describe the results of Pearson and Horsthemke [Pearson] who prove the same thing for dynamics close to a Takens - Bogdanov point.

In chapter two we look for the existence of travelling waves in three ways. Firstly we prove the conditions needed for the travelling wave equations to go through a Hopf bifurcation. Secondly we look for the existence of travelling waves as the wave speed c is perturbed from zero, again using Lyapunov - Schmidt reduction. This perturbation is the connection between steady periodic solutions and periodic travelling waves. Thirdly we describe a result by Kopell [Kopell] to prove the existence of periodic travelling waves when c is perturbed from infinity for two different types of local dynamics: periodic and near a Takens - Bogdanov point. The last part of chapter 2 proves the existence and stability of travelling waves in $\lambda - \omega$ systems, and extends a result in [Kopell & Howard].

In chapter 3 we discuss how we simulated some of the work of earlier chapters on computer. We present the mappings we used and prove their behaviour is similar to the original partial differential equations. We present two examples: a predator prey model and the complex Ginzburg - Landau equation for which we use the same approach as [Rand].

Reaction - Diffusion Systems

As already mentioned reaction - diffusion systems were proposed as models for biological phenomena in [Turing], and they have been widely studied since then. They are partial differential equations of the form

$$\frac{\partial u}{\partial t} = f(u) + D\frac{\partial^2 u}{\partial x^2}$$

where $u(x,t)$ is a vector of populations or densities, t is time and x is the spatial variable. As the name suggests these equations have two parts: a reaction part, given by f in the equation above, which describes the local dynamics of u; and a diffusion part which desaibes how u spreads out in space. Diffusion is based on the assumption of a random walk, and how individual populations affect each others movement is given by the matrix D. In all our work we take D to be constant.

Chapter 1

Turing Instability

In 1952 Alan Turing [Turing] suggested that reaction diffusion systems could exhibit steady state heterogeneous spatial patterns. His idea was that if, in the well mixed or homogeneous system (ie in the absence of diffusion), the system has a stable steady state then spatially inhomogeneous patterns may evolve by diffusion driven instability. For instance, in a two species model in one spatial dimension of the form

$$u_t = f(u, v) + d_1 u_{xx}$$

$$v_t = g(u, v) + d_2 v_{xx}$$

where f and g are the kinetics, Turing patterns may exist if certain conditions are met. These have been well documented for the case where the diffusion is diagonal [Murray]. We will be considering cases with general diffusion matrices.

In this chapter we put the idea of Turing instability in a broader setting by considering an alternative approach to the classical one. We use reversibility to prove the existence of steady, spatially periodic solutions of reaction diffusion equations over a much larger region of parameter space. We restrict ourselves to considering one spatial dimension in 'large' domains; where 'large' means that we may consider patterns of any wavelength. Throughout this chapter we consider the system of reaction diffusion equations

$$u_t = f(u) + Du_{xx} \tag{1.1}$$

where $u = u(x,t) \in \mathrm{R^n}, t \in \mathrm{R}, x \in \mathrm{R}$. We assume throughout that the diffusion matrix, D, is invertible, and that $f : \mathrm{R^n} \mapsto \mathrm{R^n}$ is C^∞.

Finally, the end of the chapter considers the existence of these patterns near certain singularities. Firstly we consider Hopf bifurcations and then, following [Pearson], consider a coalescence point of Hopf and saddle-node bifurcations.

1.1 The Classical Approach

The classical approach to proving the existence of Turing patterns is to look for homogeneous solutions which are stable to homogeneous perturbations, but unstable to a band of heterogeneous perturbations. For examples of simple models in which Turing patterns have been shown to exist see [Schnakenberg] and [Gierer]. We give a brief review of the main ideas restricting, for simplicity, to the case $n = 2$.

Let u_* be a homogeneous equilibrium solution to (1.1). Linearising about u_* gives the equation

$$w_t = d_u f(u_*)w + Dw_{xx}.$$

Let $A = d_u f(u_*)$. Then u_* is stable to perturbations of wavelength $2\pi/k$ if the eigenvalues of $A - k^2 D$ have negative real part. If $n = 2$ this is equivalent to the conditions:

$$tr(A - k^2 D) \ < \ 0 \tag{1.2}$$

$$det(A - k^2 D) \ > \ 0. \tag{1.3}$$

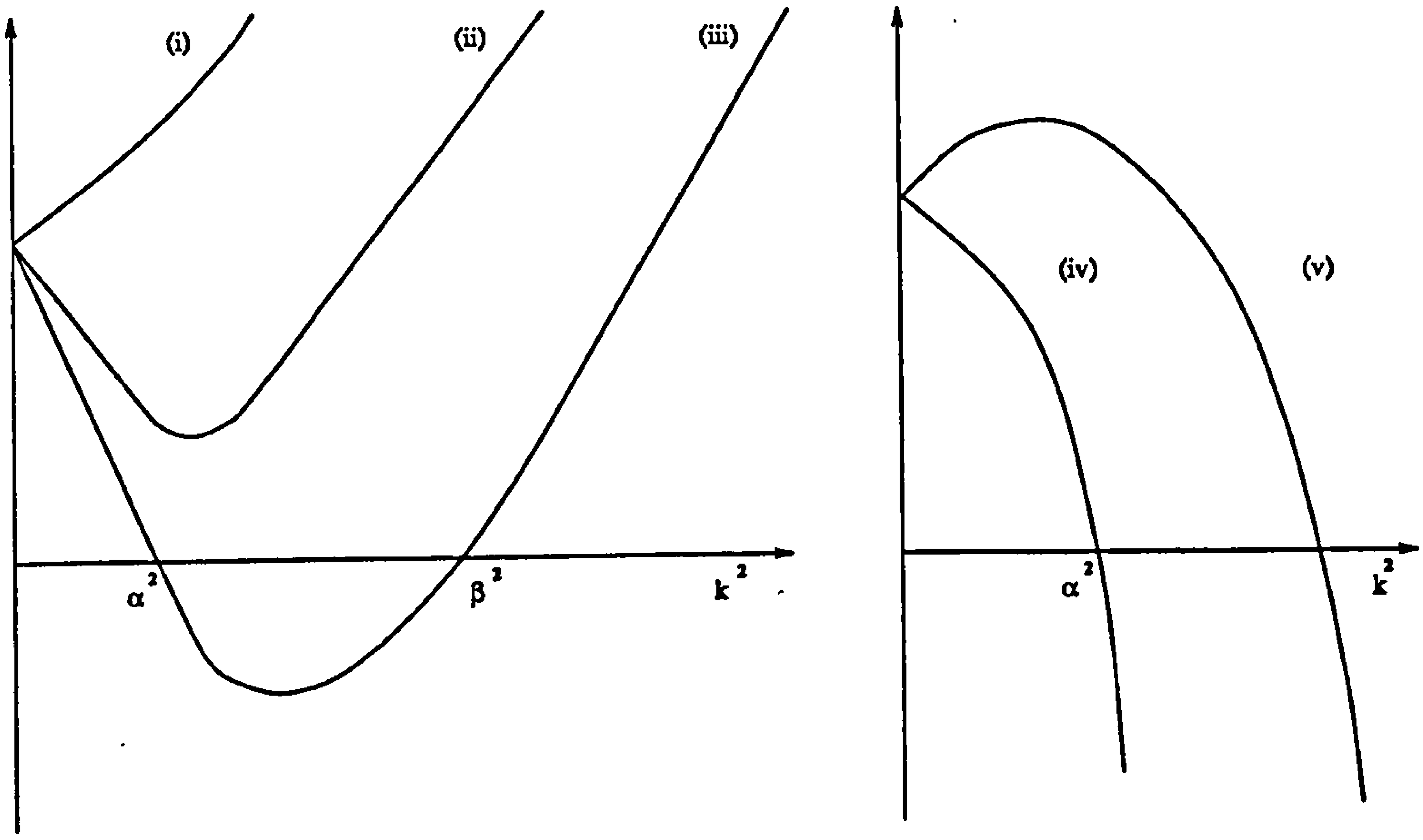


Figure 1.1: Graphs of $det(A - k^2D)$

where tr indicates the trace and det the determinant. We assume that $tr(A) < 0$ and $det(A) > 0$, so u_* is stable to homogeneous perturbations. We also assume that $d_{11} > 0$, $d_{22} > 0$ and $detD \neq 0$. Then (1.2) holds for all k. The left hand side of (1.3) is quadratic in k^2:

$$det(A - k^2D) = det(D)k^4 - Bk^2 + det(A)$$

where $B = a_{11}d_{22} + a_{22}d_{11} - a_{12}d_{21} - a_{21}d_{12}$. The possible graphs of this quadratic are shown in figure [1.1]. If $detD > 0$ and $B > 2\sqrt{detAdetD}$ then u_* is unstable to perturbations with wave numbers k in the finite interval $[\alpha, \beta]$, where α^2 and β^2 are the roots of the quadratic above. If $detD < 0$ then u_* is unstable to perturbations with k in the interval $[\alpha, \infty)$.

The existence of Turing patterns can be proved near the transition from case (ii) to case (iii) (see figure [1.2]) using bifurcation theory methods. Let ${\alpha_c}^2$ be the double root of the equation at the transition point. Bifurcation theory shows that for B sufficiently close to $2\sqrt{detAdetD}$ equation (1.1) will

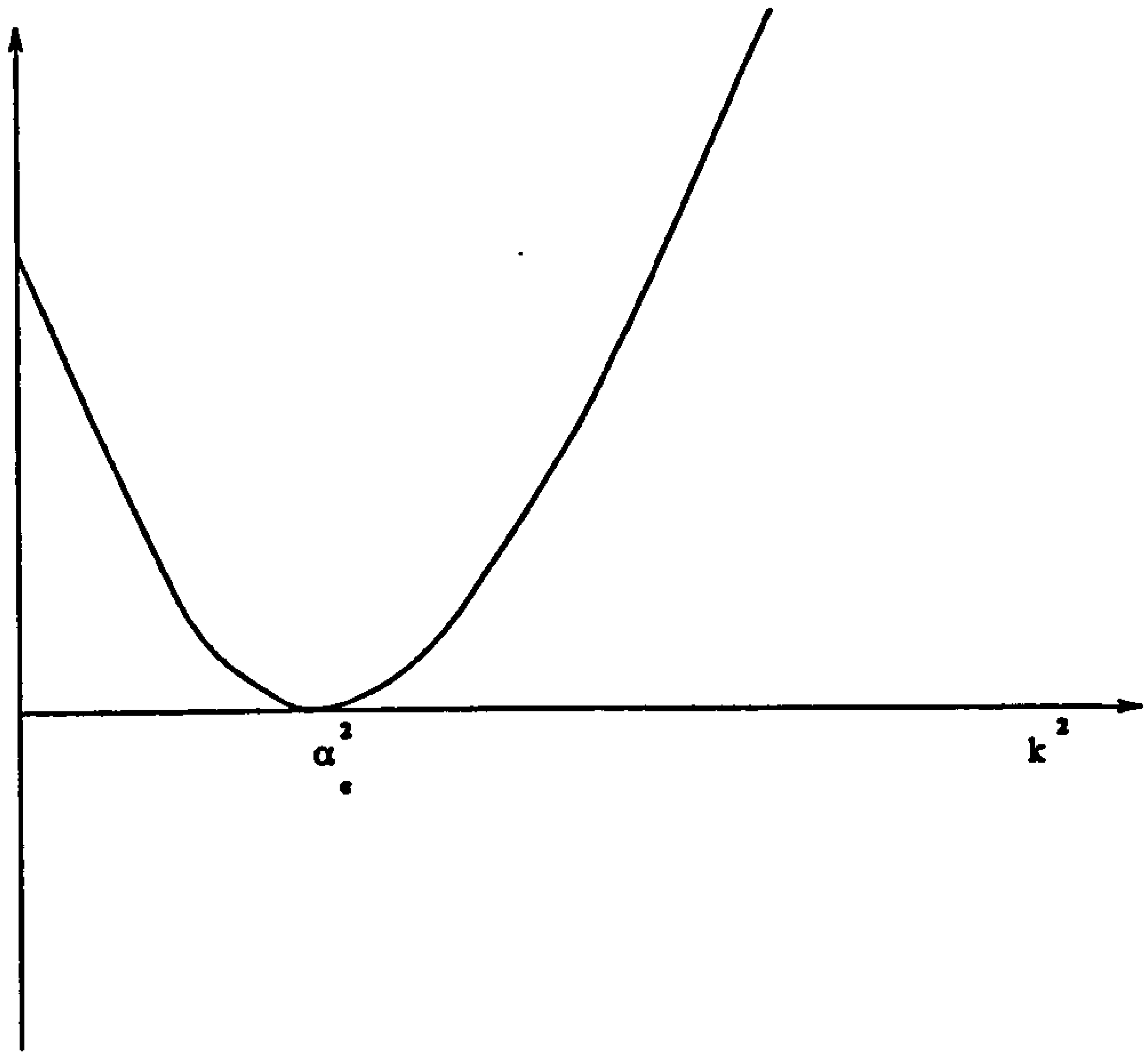


Figure 1.2: Graph of $det(A - k^2D)$ when there is a double root

have equilibrium solutions of wavelength $2\pi/\alpha_c$ which are close to u_*. If these solutions bifurcate supercritically, ie. they exist for $B > 2\sqrt{detAdetD}$, then they will be stable (at least to perturbations with the same wavelength).

1.2 The Travelling Wave Equations

We now consider the same problem, but in more depth and using an alternative method. Later on we will be investigating the connection between the steady, spatially periodic solutions and travelling waves. To do this we consider the travelling wave equations of (1.1). Looking for solutions of the form $u = u(x-ct)$, where c is the wave speed, gives the system of second order O.D.E.'s :-

$$Du'' + cu' + f(u) = 0$$

or, equivalently, the system of first order O.D.E.'s

$$\begin{aligned} u' &= v \\ v' &= -D^{-1}[f(u) + cv], \end{aligned} \tag{1.4}$$

where $'$ denotes differentiation with respect to $s = x - ct$. When $c = 0$ (ie. when we have a stationary wave) the system of equations (1.4) becomes reversible under the involution $(s, u, v) \mapsto (-s, u, -v)$, where 'reversible' is defined as follows:-

Definition 1 *Consider the n-dimensional autonomous dynamical system*

$$\frac{dx}{dt} = f(x),$$

where $x \in \mathbf{R}^n$ and f is C^∞. This system is defined to be reversible if there exists a transformation R of the state variables such that R is an involution and

$$\frac{dx}{dt} = -Rf(Rx) = f(x).$$

(i.e. the equations are invariant under $x \mapsto Rx$ and $t \mapsto -t$.)

This symmetry can be used to look for periodic solutions of (1.4) when $c = 0$. These correspond to time independent, spatially heterogeneous, solutions of (1.1), ie. Turing patterns.

1.2.1 Linearisation

We will prove the existence of periodic solutions of (1.4) by using bifurcation theory methods. We begin by considering the linearisation of (1.4). Suppose $u = u_*$ is an equilibrium solution of $u_t = f(u)$, then $u = u_*$, $v = 0$ is an equilibrium of (1.4) and the linearisation at this point is

$$\begin{aligned} u' &= v \\ v' &= -D^{-1}[Au + cv], \end{aligned} \tag{1.5}$$

where $A = d_u f(u_*)$. We look for solutions of the form $(u, v) = e^{\lambda t}(\hat{u}, \hat{v})$ which, on substituting, gives

$$\begin{aligned} \lambda \hat{u} &= \hat{v} \\ \lambda \hat{v} &= -D^{-1}[A\hat{u} + c\hat{v}] \end{aligned}$$

and so

$$\lambda^2 \hat{u} = -D^{-1}[A + c\lambda I]\hat{u}.$$

For non-trivial solutions to exist we must have

$$\det[D^{-1}(A + c\lambda I) + \lambda^2 I] = 0$$

or, equivalently,

$$\det[A + c\lambda I + \lambda^2 D] = 0.$$

When $c = 0$, λ is an eigenvalue of (1.5) if and only if $-\lambda^2$ is an eigenvalue of $D^{-1}A$. So, if λ is an eigenvalue, then $\overline{\lambda}, -\lambda$ and $-\overline{\lambda}$ are also. This also follows from the reversibility of the equations [Sevryuk]. Thus the eigenvalues occur in the following groups: a double zero iff $D^{-1}A$ has a zero eigenvalue; two purely imaginary iff $D^{-1}A$ has a real positive eigenvalue; two real iff $D^{-1}A$ has a real negative eigenvalue; four non-real iff $D^{-1}A$ has a non-real eigenvalue.

1.3 Periodic Solutions in Reversible Systems

1.3.1 Liapunov - Devaney Theorem

For steady, spatially periodic solutions we will be concerned with when purely imaginary eigenvalues occur - i.e. when $D^{-1}A$ has real positive eigenvalues. In this case there is a result, analogous to Liapunov's Theorem for Hamiltonian systems, which applies to reversible systems and appears to have been presented first by Devaney [Devaney]. It states that if a reversible vector field has a purely imaginary eigenvalue at a symmetric equilibrium point p, then, subject to certain non-resonancy restrictions, there exists a one-parameter family of nested closed orbits of the vector field about p.

Definition 2 *An equilibrium point p of an R-reversible vector field is symmetric if, and only if, p is fixed by the involution R.*

Theorem 3 *Let X be a C^2 R-reversible vector field in a neighbourhood of a symmetric equilibrium point p. Let λ be an eigenvalue of $dX(p)$, and suppose λ is purely imaginary. Then, if no other principal eigenvalue is equal to λk for any +ve integer k, there exists a C^2, two-dimensional, invariant manifold M^λ containing p with the property that M^λ consists of a nested, one-parameter family of periodic orbits. Moreover, the periods of the closed orbits tend to $2\pi/|\lambda|$ as the initial conditions tend to p.*

Proof:

We give a sketch proof of this result as it applies to our problem. The proof uses Lyapunov - Schmidt reduction (see eg. [Golubitsky]) and follows a similar pattern to [Vanderbauwhede]. For an alternative proof see [Devaney].

Suppose we have the system

$$\frac{du}{dt} = f(u)$$

then we define

$$F : {C^1}_{2\pi} \times \mathbf{R} \mapsto C_{2\pi}$$

by

$$F(u(s),\tau) = (1+\tau)\frac{du}{ds} - f(u(s)).$$

where $C_{2\pi}$ and $C^1_{2\pi}$ are the spaces of continuous and once differentiable 2π periodic functions from $\mathbf{R}^n$ to $\mathbf{R}^n$. Then periodic solutions of period $\frac{2\pi}{1+\tau}$ are solutions of

$$F(u,\tau) = 0. \tag{1.6}$$

By rescaling time, if necessary, assume $d_u f(0)$ has simple eigenvalues $\pm i$ and no eigenvalues equal to $\pm ni$ for any integer n, $n \neq 1$. Then $L = d_u F(0,0)$ has a two dimensional kernel K and a two dimensional cokernel $C_{2\pi}/I$ where I is the image of L. Define an inner product on $C_{2\pi}$ by

$$\langle u(s), v(s) \rangle = \frac{1}{2\pi}\int_0^{2\pi} u(s)^T v(s) ds.$$

This restricts to an inner product on $C^1_{2\pi}$. Let M and N denote the orthogonal complements to K and I in $C^1_{2\pi}$ and $C_{2\pi}$ respectively:

$$C^1_{2\pi} = K \oplus M$$

$$C_{2\pi} = N \oplus I.$$

Let E denote the projection $E : C_{2\pi} \to I$ with $kerE = N$. Then equation (1.6) is equivalent to the pair of equations:

$$EF(u,\tau) = 0 \tag{1.7}$$

$$(I - E)F(u,\tau) = 0. \tag{1.8}$$

We can identify K and N with $\mathbf{C}$. Assume K and N are spanned over $\mathbf{C}$ by ϕ and ψ respectively; so that any element of K can be written as $z\phi$ for some z in $\mathbf{C}$ and, similarly, any element of N can be written as $w\psi$ for some $w \in \mathbf{C}$. Then we can write $u = z\phi + \hat{u}$ where $\hat{u} \in M$. Equation (1.7) becomes:

$$EF(z\phi + \hat{u}, \tau) = 0.$$

Since $Ed_uF(0,0) : C^1_{2\pi} \to I$ is surjective this equation can be solved implicitly for $\hat{u}$ to give

$$\hat{u} = W(z,\tau).$$

Where W is unique and defined for (z, τ) in some neighbourhood of $(0,0)$. This implies that

$$W(0, \tau) = 0 \ \forall \ \tau.$$

The projection $I - E : C_{2\pi} \to N$ is given explicitly by

$$(I - E)v = \langle \psi, v \rangle.$$

Substituting in W gives the following form for (1.8):

$$g(z, \tau) = \langle \psi, F(z\phi + W(z, \tau), \tau) \rangle = 0$$

where $g : \mathbf{C} \times \mathbf{R} \to \mathbf{C}$. Again, uniqueness of W implies that

$$g(0, \tau) = 0 \ \forall \ \tau.$$

There is a natural action of the circle group S^1 on $C_{2\pi}$ and $C^1_{2\pi}$ defined by

$$(\theta u)(s) = u(s + \theta).$$

The mapping F commutes with this action, and the inner product is invariant under it, hence

$$\langle \theta u, \theta v \rangle = \langle u, v \rangle.$$

It follows that the subspaces K, M, I and N are S^1 invariant; the projection E and the map W are equivariant and hence so is g, ie.

$$g(\theta \cdot z, \tau) = \theta \cdot g(z, \tau).$$

The action of S^1 on K, identified with $\mathbf{C}$, is given by

$$\theta \cdot z = e^{i\theta} z$$

and similarly for N. It follows that g must have the form (see [Golubitsky]

$$g(z, \tau) = [Q(|z|^2, \tau) + iP(|z|^2, \tau)]z.$$

So, non-trivial solutions of $g(z,\tau)=0$ are given by

$$P(|z|^2,\tau) = 0 \tag{1.9}$$

$$Q(|z|^2,\tau) = 0 \tag{1.10}$$

We now take into account the fact that the system is reversible, ie. there exists an orthogonal involution R such that $f(Ru) = -Rf(u)$. Therefore F is equivariant with respect to the actions

$$u(s) \mapsto Ru(-s) \ \ on\ C^1{}_{2\pi}$$

$$u(s) \mapsto -Ru(-s) \ \ on\ C_{2\pi}.$$

Again the inner product is invariant under this and so g will commute with the induced actions on K and N. This equivariance takes the form

$$g(\overline{z},\tau) = -\overline{g(z,\tau)}$$

which implies that

$$Q(|z|^2,\tau) \equiv 0.$$

Thus, periodic solutions are given by

$$P(|z|^2,\tau) = 0.$$

Lemma 4 $P_\tau(0,0) \neq 0$

Proof:

$$\begin{aligned} P_\tau(|z|^2,\tau) &= -ig_{z\tau}(z,\tau) \\ &= -i\langle\psi, d_uF_\tau(\phi + W_z) + d^2{}_uF(\phi + W_z, W_\tau) + d_uF(W_{\tau z})\rangle \end{aligned}$$

where we have missed out dependent variables for ease of presentation. We now evaluate this at $(z,\tau) = (0,0)$. We calculate derivatives of W by differentiating

(1.7) with the substitution $u = z\phi + W$. So to find W_z we differentiate with respect to z to get

$$EL(\phi + W_z) = 0 \quad at\ (0,0).$$

Now $\phi \in ker(L)$ and $EL = L$ gives

$$LW_z(0,0) = 0,$$

but $W_z(0,0) \in M$ and L is invertible on M so $W_z(0,0) = 0$. Similarly $W_\tau = L^{-1}EF_\tau$, but

$$F(0,\tau) \equiv 0 \Rightarrow F_\tau(0,0) = 0$$

and so $W_\tau(0,0) = 0$. Finally we may ignore the term involving d_uF since $L = d_uF(0,0)$ and ψ is chosen in the complement of the image of L. So the only term on the right hand side of the inner product to remain is

$$d_uF_\tau(0,0)\phi = \frac{d\phi}{ds}.$$

We may choose ϕ such that $\phi = e^{is}c$ where c is an eigenvector of $A = d_uf(0,0)$ with the properties:

$$Ac = -ic, \quad \overline{c}^t \cdot c = 1.$$

And, similarly, we may choose ψ such that $\psi = e^{is}d$ where d is an eigenvector with the properties:

$$A^t d = id, \quad \overline{d}^t c = 1 \quad d^t c = 0.$$

We know the latter two properties for d since

$$-id^t c = d^t(Ac) = (A^t d)c = id^t c \Rightarrow d^t c = 0,$$

and we know that $\overline{d}^t c \neq 0$ because otherwise c would be orthogonal to every eigenvector of A^t implying that $c = 0$ which is a contradiction. Plugging these values for ϕ and ψ into our equation for P_τ gives us

$$P_\tau(0,0) = 1.$$

This result implies that (1.9) can be solved for τ as a function of $|z|^2$, ie $\tau = \tau(|z|^2)$ in a neighbourhood of $z = 0$. Hence there is a one parameter family of periodic orbits near the origin with period near to 2π.

Applying this to system 1.1 gives the following corollary.

Corollary 5 *If $D^{-1}A$ has a real positive eigenvalue k^2 and m^2k^2 is not an eigenvalue for any integer m, then equation (1.1) has a one parameter family of steady spatially periodic solutions which converge to the homogeneous equlibrium u_*. As they approach u_* the wavelengths of the solutions converge to $2\pi/k$.*

If we return now to figure [1.1] then we see that this corollary gives us the existence of steady, spatially periodic solutions for all parameter values which fall into one of the cases (iii), (iv) or (v), not just those near the Turing bifurcation point. In cases (iv) and (v) there will be solutions with wavelength near $2\pi/\alpha$; in case (iii) there will be families of solutions with wavelengths near both $2\pi/\alpha$ and $2\pi/\beta$.

1.3.2 1:1 Resonance

The classical approach to finding Turing patterns (e.g. see [Murray]) amounts to discovering when (1.5) has two pairs of purely imaginary eigenvalues. It is therefore interesting to consider the bifurcations which occur in this case. What is known classically as a Turing bifurcation point coincides with what is called a 1:1 resonance, in the reversible system (1.4); that is to say there is a double eigenvalue $i\omega_0$ and a double eigenvalue $-i\omega_0$. The following results come from [Arnol'd].

We will suppose that our system depends on a parameter ϵ where $\epsilon = 0$ corresponds to the resonance and that the singular point is symmetric, at zero and is independent of ϵ. We may change the time scale if necessary so that $\omega_0 = 1$. So we have a double eigenvalue at i and at $-i$ when $\epsilon = 0$. We also assume that for $\epsilon < 0$ the eigenvalues of (1.5) do not lie on the imaginary axis, while for $\epsilon > 0$ they are $\pm\nu_1 i$, $\pm\nu_2 i$, where ν_1 and ν_2 are real and close to one. The difference $|\nu_2 - \nu_1|$ is of the order of $\sqrt{\epsilon}$. Since the origin is symmetric, each periodic solution of our system, in a neighbourhood of the origin, intersects the fixed involution plane at two points. These form a curve Γ_ϵ that depends on ϵ. On this curve the involution R permutes the points belonging to the same periodic solution.

Theorem 6 (Sevryuk) *If the terms of order ≤ 3 of the Taylor series satisfy certain non-degeneracy conditions, then by a suitable choice of a coordinate system Oxy, which depends smoothly on ϵ, for the fixed plane of the involution, and by a change in the sign of ϵ, if necessary, the equation of the family of curves Γ_ϵ is reduced to the form*

$$(\epsilon \pm x^2)x^2 = y^2.$$

The choice of sign is determined by the terms of order ≤ 3 in the Taylor series. With the $+$ sign the bifurcation is said to be hyperbolic, with the $-$ sign it is elliptic.

Remarks on the bifurcation diagrams - See figure [1.3]

(a) Elliptic mode: $\Gamma_\epsilon = \{x, y : (\epsilon - x^2)x^2 = y^2\}$.

For $\epsilon \leq 0$, Γ_ϵ is just the origin, i.e. there are no periodic solutions near the origin. For $\epsilon > 0$, Γ_ϵ has the form of a figure of eight. Thus, when the system passes through the 1:1 resonance in the elliptic mode, a two dimensional surface is created from the origin which is foliated into symmetric cycles and has the topology of a sphere with two points identified (corresponding to the

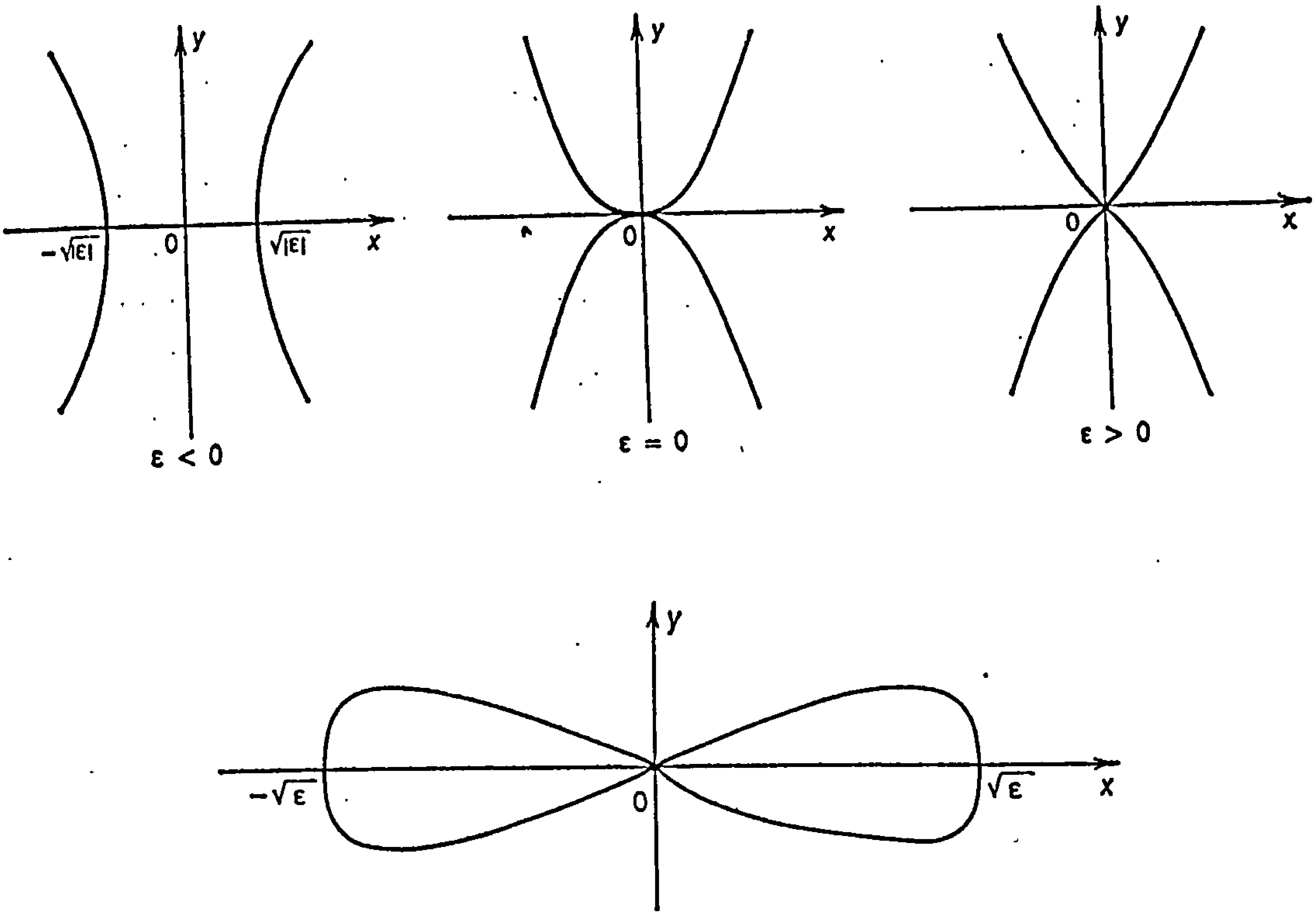


Figure 1.3: Bifurcation diagrams of periodic solutions near the 1:1 resonance

origin). The size of the surface is proportional to $\sqrt{\epsilon}$.

(b) Hyperbolic mode: $\Gamma_\epsilon = \{x, y : (\epsilon + x^2)x^2 = y^2\}$.

At $\epsilon = 0$ in the phase space of the system we have two two-dimensional C^1-surfaces which are tangential at zero and are foliated into symmetric cycles.

In order to relate the periodic solutions described above to the work we have already done we use a result from [Bridges]. Although, the work was done with Hamiltonian systems in mind, as is often the case, it can be shown to apply to reversible systems. We return to the case when we have the double eigenvalues $\pm i\omega_0$.

Theorem 7 *Suppose the system has a 1:1 resonance as described above, and that the same non- degeneracy conditions are satisfied then the periodic solu-*

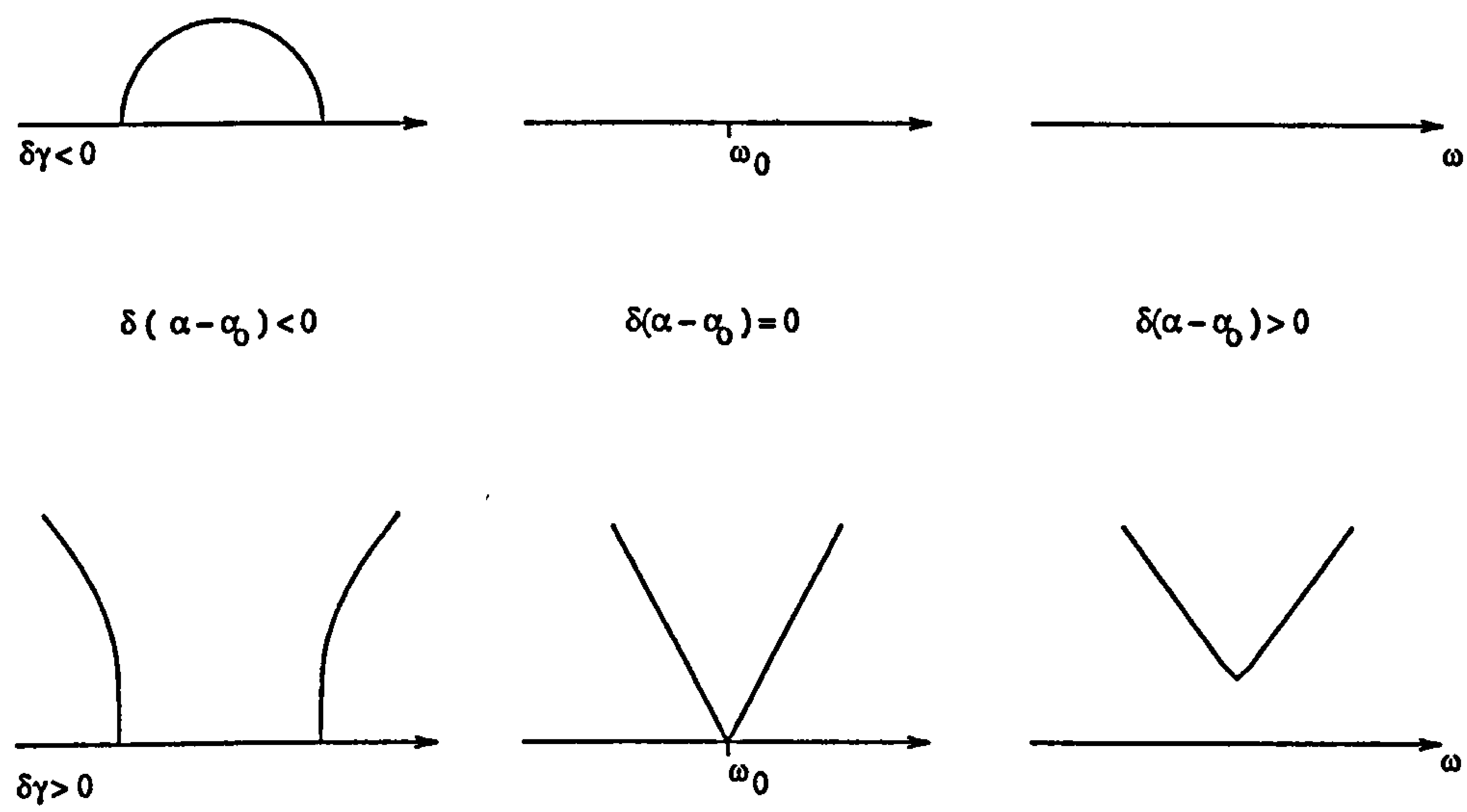


Figure 1.4: Branches of periodic solutions near the 1:1 resonance

tions in a neighbourhood of the bifurcation point are in one-to-one correspondence with zeroes of the mapping $g = 0$ *with*

$$g(x, \omega, \alpha) = x[-\delta(\omega - \omega_0)^2 + \gamma x^2 - (\alpha - \alpha_0)]$$

where x *is the amplitude of the periodic solutions;* ω *is the frequency of the solutions;* α *is the bifurcation parameter such that* $\alpha = \alpha_0$ *is the resonance point;* δ *and* γ *are either* $+1$ *or* -1 *and indicate whether the bifurcation is elliptic or hyperbolic.*

The branches of periodic solutions are shown in figure [1.4]. When $\delta\gamma < 0$ we have the classical Turing bifurcation and can relate figure [1.4] to figures [1.1] and [1.2]. The first picture correlates to case (iii) the second to figure [1.2] and the third to case (ii). Here the two branches of solutions are globally connected and vanish into the origin as $\alpha \to \alpha_0$. We must remember though, that we only expect these pictures to be true close to the bifurcation point. When $\delta\gamma > 0$ the related graphs will be the same only reflected in the k axis.

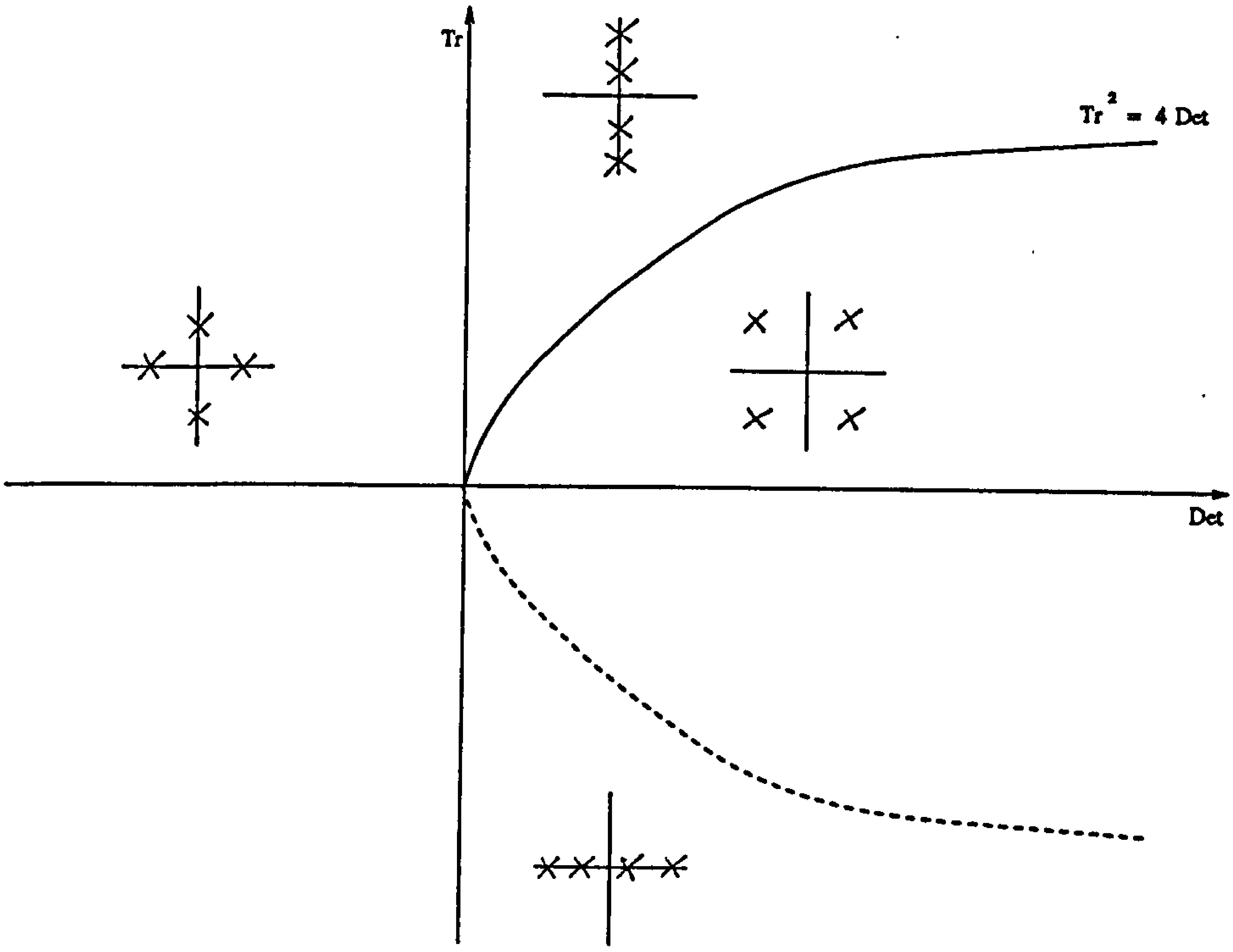


Figure 1.5: Position of eigenvalues for system (1.5)

Here the two branches come together, detach from the origin and persist into the unstable region.

1.4 Two Species Models

We can give a more detailed interpretation of the conditions for the existence of these patterns if we restrict ourselves to the case when $n = 2$, or, in other words, to two species models. When $n = 2$ (1.5) has four eigenvalues. The position of these is determined by the trace and determinant of $D^{-1}A$ as shown in figure [1.5]. Throughout we use $D = [d_{ij}]$ and $A = [a_{ij}]$. We have two cases to consider: firstly, when there is precisely one pair of purely imaginary eigenvalues, and secondly, when there are two pairs of purely imaginary eigenvalues.

1.4.1 One Pair of Imaginary Eigenvalues

The region of parameter space where we have precisely one pair of purely imaginary eigenvalues is defined by $\det D^{-1}A < 0$. In this case $D^{-1}A$ has a real positive eigenvalue and a real negative eigenvalue. There are two possibilities:

$\det D > 0$ and $\det A < 0$

Since $\det A < 0$ then u_* is a saddle point of $u_t = f(u)$, and so the periodic solutions here cannot be classed as Turing instabilities, strictly speaking, as u_* is not stable in the homogeneous system. If the 'self-diffusion' coefficients d_{11} and d_{22} are positive the condition $d_{11}d_{22} - d_{12}d_{21} > 0$ is satisfied either if the cross-diffusion terms have opposite signs, or if their product is small compared to that of the self-diffusion terms. In particular, it is satisfied for diagonal diffusion matrices.

$\det D < 0$ and $\det A > 0$

Here u_* is not a saddle point and could be stable. If we again assume that d_{11} and d_{22} are positive then the cross-diffusion terms must have the same sign and their product must now be large in comparison with that of the self-diffusion terms.

1.4.2 Two Pairs of Imaginary Eigenvalues

The region where there are two pairs of purely imaginary eigenvalues is given by

$$\det D^{-1}A > 0, \;\; (TrD^{-1}A)^2 > 4\det D^{-1}A$$

The first condition is just the opposite of that above, so if u_* is stable then D could be diagonal or have cross diffusion which is small or with coefficients of opposite signs. If u_* is a saddle point then we need strong positive cross diffusion. If we suppose that u_* is stable and that D is diagonal then the second condition becomes

$$\sqrt{\frac{d_{11}}{d_{22}}}a_{22} + \sqrt{\frac{d_{22}}{d_{11}}}a_{11} > 2\sqrt{\det A}. \quad (i)$$

The two conditions for u_* to be stable are

$$a_{11} + a_{22} < 0 \quad (ii)$$

and

$$a_{11}a_{22} - a_{12}a_{21} > 0. \quad (iii)$$

(i) and (ii) imply that a_{11} and a_{22} have opposite signs, and this with (iii) then gives that a_{12} and a_{21} must also have opposite signs. If we assume without loss of generality that $a_{11} > 0$ then $|a_{22}| > a_{11}$. The condition for the diffusion coefficients is $d_{22}/d_{11} > d^* > 1$ where d^* is the largest solution of

$$\sqrt{\frac{d_{11}}{d_{22}}}a_{22} + \sqrt{\frac{d_{22}}{d_{11}}}a_{11} - 2\sqrt{\det A} = 0.$$

These are the well known conditions for the classical Turing bifurcation.

1.5 Stability

Although we have proved the existence of steady, spatially periodic solutions over a larger region of parameter space, it seems likely that away from the actual Turing bifurcation point (ie. the 1:1 resonance) these solutions will be unstable. To see this let $u(x,t) = \hat{u}(x)$ be one of these solutions near u_*. Linearising (1.1) about $\hat{u}$ gives

$$w_t = d_u f(\hat{u})w + Dw_{xx}$$

We expect $\hat{u}$ to be stable if (and essentially only if) the spectrum of the operator on the right hand side of the equation lies in the left half plane. However, this operator is a perturbation of the operator at $u = u_*$, ie. $Aw + Dw_{xx}$, and we can only expect it to be stable near where $Aw + Dw_{xx}$ is stable - ie. near the Turing bifurcation point. Although only stable patterns are expected to be 'observable', information about unstable patterns is necessary in the construction of global bifurcation diagrams for Turing patterns.

1.6 Turing Instability Near a Hopf Bifurcation

In the next chapter we investigate what happens to the steady, spatially periodic solutions when the system $u_t = f(u)$ goes through a Hopf Bifurcation. Before we can do this we need to show that these solutions exist before the bifurcation takes place.

Proposition 8 *Consider the reaction-diffusion system*

$$u_t = f(u, \lambda) + Du_{xx} \ ,$$

where $u \in \mathrm{R}^n$ and $\lambda \in \mathrm{R}$. Suppose the homogeneous, system goes through a Hopf Bifurcation at $\lambda = \lambda_0$ from the stable steady state $u = u_s$. Then there exists a diffusion matrix D for which the reaction-diffusion system exhibits steady, spatially periodic solutions for λ close enough to λ_0.

Proof:

We define

$$J = \left.\frac{\partial f}{\partial u}\right|_{u=u_s} .$$

We may transform J into Jordan normal form, using a transformation Λ, to give

$$\Lambda J \Lambda^{-1} = \begin{pmatrix} \begin{pmatrix} \alpha(\lambda) & -\omega(\lambda) \\ \omega(\lambda) & \alpha(\lambda) \end{pmatrix} & 0 \\ 0 & [T_{ij}(\lambda)] \end{pmatrix} .$$

Here $\alpha(\lambda_0) = 0$, $\omega(\lambda_0) \neq 0$ and T is an (n-2)×(n-2) matrix. The diffusion matrix D may now be chosen so that $B = \Lambda(J - Dk^2)\Lambda^{-1}$ is of the form

$$B = \begin{pmatrix} \begin{pmatrix} \alpha - \delta_1 k^2 & -\omega - \delta_2 k^2 \\ \omega - \delta_3 k^2 & \alpha - \delta_4 k^2 \end{pmatrix} & 0 \\ 0 & [T_{ij} - \overline{D_{ij}} k^2] \end{pmatrix} .$$

We define the matrices E and F by

$$B = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}$$

This gives

$$|B| = |E||F|$$

where $|B| = det(B)$. If E satisfies the Turing condition $|E| = 0$, then so does B. We now consider the equation $|E| = 0$:

$$(\delta_1\delta_4 - \delta_2\delta_3)k^4 - (\alpha(\delta_1 + \delta_4) + \omega(\delta_3 - \delta_2))k^2 + \alpha^2 + \omega^2 = 0 .$$

If we assume that $\delta_1\delta_4 - \delta_2\delta_3 > 0$ then we need

$$\alpha(\delta_1 + \delta_4) + \omega(\delta_3 - \delta_2) > 0$$

and

$$(\alpha(\delta_1 + \delta_4) + \omega(\delta_3 - \delta_2))^2 - 4(\delta_1\delta_4 - \delta_2\delta_3)(\alpha^2 + \omega^2) > 0 .$$

For $u = u_s$ to be stable before the Hopf bifurcation we must have $\alpha < 0$. We may assume $\omega > 0$ for λ close to λ_0 and so if $\delta_3 - \delta_2 > 0$ then the first condition is satisfied near the bifurcation point since $\alpha \to 0$ as $\lambda \to \lambda_0$. In particular we note that the cross-diffusion terms cannot be zero in the Jordan normal form coordinates. The second condition may be rearranged to give

$$\alpha^2(\delta_1 - \delta_4)^2 + \omega^2(\delta_2 + \delta_3)^2 + 2\alpha\omega(\delta_1 + \delta_4)(\delta_3 - \delta_2) + 4\alpha^2\delta_2\delta_3 - 4\omega^2\delta_1\delta_4 > 0 .$$

This will be true close to the bifurcation point as long as $(\delta_2 + \delta_3)^2 > 4\delta_1\delta_4$.

Example

In order to give a clearer idea about what the diffusion coefficients must look like to satisfy the above conditions we consider a particular example. A common model used to describe a spatially extended system going through a Hopf Bifurcation is the complex Ginzburg - Landau (CGL) equation (see eg. [Newell]). Near a Hopf bifurcation a change of coordinates will put the terms of degree < 4 of the Taylor series of any vector field into the normal form of the CGL equation. Clearly, the change of coordinates will also affect any diffusion coefficients. Consequently, we use an extended version with four diffusion coefficients which, after rescaling the time and space variables to reduce the number of parameters by two, has the following form:

$$u_t = (\mu + i)u - (1 + \nu i)|u|^2 u + (1 + i\beta)u_{xx} + (\gamma + i\zeta)\overline{u}_{xx} \; .$$

Here μ is the bifurcation parameter. The origin is stable if $\mu < 0$ and unstable if $\mu > 0$. If this is written as a system of two equations for the real and imaginary parts of u then the self-diffusion coefficients are $d_{11} = 1 + \gamma$ and $d_{22} = 1 - \gamma$, and the cross-diffusion terms are $d_{12} = \zeta - \beta$ and $d_{21} = \zeta + \beta$. Linearising about the origin we have

$$B = J - Dk^2 = \begin{pmatrix} \mu - d_{11}k^2 & -1 - d_{12}k^2 \\ 1 - d_{21}k^2 & \mu - d_{22}k^2 \end{pmatrix}.$$

We now consider the equation $|B| = 0$ given by

$$(d_{11}d_{22} - d_{12}d_{21})k^4 - (\mu(d_{11} + d_{22}) + (d_{21} - d_{12}))k^2 + \mu^2 + 1 = 0 \; ,$$

and on substitution:

$$(1 + \beta^2 - \gamma^2 - \zeta^2)k^4 - 2(\mu + \beta)k^2 + (1 + \mu^2) = 0.$$

This equation has one positive solution for k^2 if

$$1 + \beta^2 < \gamma^2 + \zeta^2;$$

two positive solutions if

$$1+\beta^2 > \gamma^2+\zeta^2 > \frac{(1-\mu\beta)^2}{1+\mu^2}$$

and

$$\mu > -\beta;$$

and no positive solutions in the complement of the closure of the union of these two regions. See figure [1.6]. Remarks

1. One of the positive roots goes to ∞ as the boundary $1+\beta^2 = \gamma^2+\zeta^2$ is approached.

2. The 1:1 resonance occurs along the boundary $\gamma^2+\zeta^2 = (1-\mu\beta)^2/(1+\mu^2)$ when $\mu > -\beta$.

3. The 1:1 resonance occurs simultaneously with u_* being stable as a solution of the homogeneous equation when $\beta > 0$, $\mu < 0$ and $1+\beta^2 = \gamma^2+\zeta^2$. This situation corresponds to the classical Turing bifurcation.

The CGL equation will have steady, spatially periodic solutions whenever $\gamma^2+\zeta^2 > 1+\beta^2$, ie. whenever the self-diffusion terms are sufficiently different or the sum of the cross-diffusion terms is sufficiently great. However, these solutions are probably not stable as solutions of the P.D.E. Stable Turing patterns probably do result from the Turing bifurcation described in remark (3). For this to occur we must have $\beta > 0$, ie. the cross-diffusion terms must be sufficiently different, and $\gamma^2+\zeta^2$ must be in the relatively narrow band

$$\frac{(1-\mu\beta)^2}{1+\mu^2} < \gamma^2+\zeta^2 < 1+\beta^2.$$

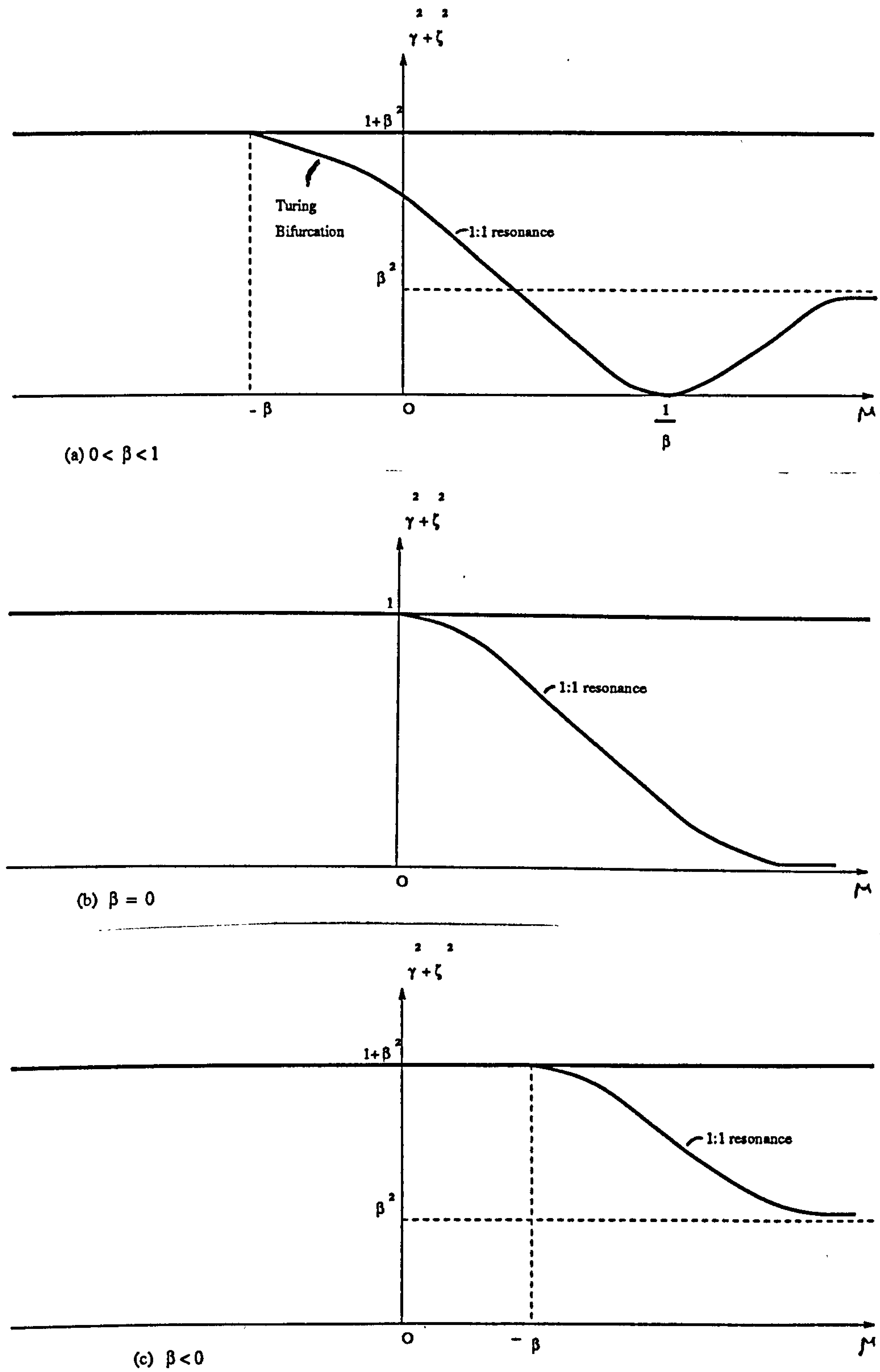


Figure 1.6: Bifurcation Diagrams for the CGL

1.7 Turing Instability Near a Takens - Bogdanov Point

In [Pearson] it is shown that if a Turing instability occurs in a reaction-diffusion system with a nearly scalar diffusion matrix, then the corresponding homogeneous or well mixed system has at least two eigenvalues near zero. Conversely, if the homogeneous system is sufficiently close to a Takens-Bogdanov point (a coalescence point of Hopf and saddle-node bifurcations), where two eigenvalues will be zero then there exists a nearly scalar diffusion matrix such that a Turing instability occurs. We give a condensed version of their results.

Consider the general reaction-diffusion system

$$u_t = f(u; \lambda) + Du_{xx},$$

where u is an n-vector, λ is a p-vector of parameters, D is an n×n matrix and $x \in \mathbf{R}$. Suppose $u = u_s$ is a stable steady state of the homogeneous system so that the eigenvalues of

$$J = \left.\frac{\partial f}{\partial u}\right|_{u=u_s}$$

all have negative real part. In order to find Turing instabilities we analyse the determinant

$$\Delta \equiv |J - Dk^2|.$$

For the following results we need J to have at least two eigenvalues that are sufficiently close to zero. In order to find such a point we need two control parameters, and so we suppose $\lambda = (\lambda_1, \lambda_2)$ and without loss of generality we may assume the double-zero condition is at $\lambda = 0$. We also define D such that $D_{ij} = \alpha(\delta_{ij} + \varepsilon d_{ij})$, where α is a real number, δ_{ij} is the Kronecker delta and d is a real matrix. The determinant Δ may now be written as

$$\Delta(Dk^2, \varepsilon) = |J - \alpha k^2(I + \varepsilon d)|.$$

A well known result states that Turing instabilities cannot occur with scalar diffusion matrices. So in the system above there are no such instabilities when $\varepsilon = 0$. However, ε may be as close to zero as we like.

Lemma 9 *If, in the system described above, there exists a set of d_{ij} such that a Turing instability occurs for arbitrarily small but nonzero ε and the eigenvalues of J are bounded in modulus then*

$$\lim_{\varepsilon \to 0} \lambda_c = 0$$

where $\lambda = \lambda_c$ is the point where $\Delta = 0$.

The proof of this result comes from considering the characteristic equation and the position of the eigenvalues. For a Turing instability two of the eigenvalues must have surrounding neighbourhoods which contain part of the positive real axis. So, as ε decreases these two eigenvalues must near the origin.

Theorem 10 *For a system with kinetics sufficiently close to a Takens - Bogdanov point, there exists a set of d_{ij} such that a Turing instability occurs for arbitrarily small but nonzero ε.*

To prove this J is transformed to Jordan-Arnold form with transformation Λ to give

$$\Lambda J \Lambda^{-1} = \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ -\rho_1 & -\rho_2 \end{pmatrix} & 0 \\ 0 & (T_{ij}) \end{pmatrix}.$$

Here the Takens-Bogdanov point corresponds to $\rho_1 = \rho_2 = 0$ and the steady state is stable when ρ_1, $\rho_2 > 0$. T is an (n-2)×(n-2) matrix. The d_{ij} are now chosen such that $B = \Lambda(J - \alpha k^2)\Lambda^{-1}$ is of the form

$$B = \begin{pmatrix} \begin{pmatrix} -\alpha k^2 & 1 \\ -\rho_1 + \varepsilon c \alpha k^2 & -\rho_2 - \alpha k^2 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} T_{ij} - (\alpha + O(\varepsilon))k^2, i = j \\ T_{ij} - O(\varepsilon k^2), i \neq j \end{pmatrix} \end{pmatrix}.$$

The matrices E and F are defined by

$$B = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix} ,$$

and as before we note that if E satisfies the Turing condition then so does B. A quick calculation shows that this happens when

$$\varepsilon c - \rho_2 > 0 \ , \ \rho_1 = \frac{1}{4}(\rho_2 - \varepsilon c)^2 \ .$$

So, as long as we are close enough to the Takens - Bogdanov point there will be a Turing instability for small ε.

Chapter 2

Travelling Waves

In this chapter we investigate the periodic solutions of system (1.4) when $c \neq 0$. These correspond to periodic travelling wave solutions of (1.1). It is important to note that when $c \neq 0$ system (1.4) is no longer reversible and so the constraints on the eigenvalues that applied in the previous chapter no longer do so. We prove the existence of periodic solutions in three different ways. Firstly by looking for a Hopf bifurcation, secondly by considering what happens as c is perturbed from zero and lastly by looking at what happens when c is perturbed from infinity. The final section proves the existence of travelling waves and their stability for one class of reaction - diffusion systems, namely $\lambda - \omega$ systems.

2.1 Hopf Bifurcations

We know that system (1.4) exhibits periodic travelling wave solutions when it goes through a Hopf Bifurcation. This section investigates when this occurs.

2.1.1 Eigenvalues

For simplicity we only consider two species models, or in other words four dimensional systems of travelling wave equations. If we consider system (1.4) with linearisation (1.5) about the steady state $(u,v) = (u_*, 0)$ then the characteristic polynomial is

$$(detD)\lambda^4 + c(trD)\lambda^3 + ((detD)tr(D^{-1}A) + c^2)\lambda^2 + c(trA)\lambda + detA\ .$$

Descartes' rule of signs easily proves the following lemma.

Lemma 11 *If $trD > 0$, $trA < 0$ and $detA > 0$ then the four eigenvalues cannot all have strictly positive real parts, or all have strictly negative real parts.*

Note that the hypotheses of this lemma basically cover the case when u_* is a stable equilibrium point of $u_t = f(u)$, since $trD > 0$ is a reasonable assumption for any diffusion matrix.

Once c has been perturbed from zero, periodic travelling waves will occur in system (1.4) when there is a Hopf bifurcation.

Proposition 12 *For $c \neq 0$ system (1.4) will have a Hopf bifurcation precisely when $trA/trD > 0$ and*

$$c^2 = \frac{(trA)^2 detD + (trD)^2 detA - (trA)(trD)(detD)tr(D^{-1}A)}{(trA)(trD)}. \tag{2.1}$$

Proof:

We look for $(\lambda^2 + \alpha^2)$ to be a factor of the characteristic polynomial

$$|D|\lambda^4 + c(trD)\lambda^3 + (|D|tr(AD^{-1}) + c^2)\lambda^2 + c(trA)\lambda + |A|,$$

where $|M| = detM$. If we assume $(\lambda^2 + \alpha^2)$ is a factor, where α is a real number, then the polynomial will look like

$$|D|(\lambda^2 + \alpha^2)(\lambda^2 + b\lambda + d),$$

where b and d are real numbers, or on multiplying out

$$|D|(\lambda^4 + b\lambda^3 + (\alpha^2 + d)\lambda^2 + \alpha^2 b\lambda + \alpha^2 d).$$

Equating the coefficients of λ^3, λ and 1 in the two polynomials gives

$$\alpha^2 = \frac{trA}{trD}, \quad b = \frac{c(trD)}{|D|}, \quad d = \frac{|A|trD}{|D|trA}$$

giving the condition $trA/trD > 0$. Equating the coefficients for λ^2 gives

$$|D|(\alpha^2 + d) = |D|tr(D^{-1}A) + c^2$$

which, on substitution and rearrangement gives (2.1) and the result.

So, if $trD > 0$ as is usual in diffusion matrices, then a Hopf bifurcation can only occur if $trA > 0$, therefore, periodic travelling wave solutions of (1.1) can only bifurcate from an unstable equilibrium u^*. A Hopf bifurcation will occur for some value of c if the right hand side of (2.1) is greater than zero. To give us a better picture of what this result means we consider two examples.

Example 1: Diagonal Diffusion

We apply this result for the case where we have diagonal diffusion. By rescaling the spatial variable, if necessary, D becomes

$$D = \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix}.$$

If $(trA/trD) > 0$ then the denominator of the right hand side of (2.1) is greater than zero, and so if, in addition, the numerator is greater than zero then we will have a Hopf bifurcation for some value of c. On substitution this condition is

$$(a_{11} + a_{22})^2 d + (1 + d)^2(a_{11}a_{22} - a_{12}a_{21}) - (a_{11} + a_{22})(1 + d)(a_{22} + da_{11}) > 0$$

which, on rearranging becomes

$$-a_{12}a_{21}(1+d)^2 - (a_{22} - da_{11})^2 > 0$$

or

$$a_{12}a_{21} < -\frac{(a_{22} - da_{11})^2}{(1+d)^2} < 0.$$

In particular, this implies that the linearisation A must be of 'activator- inhibitor' (or 'predator-prey') type since a_{12} and a_{21} have opposite signs. To give us a clearer idea of what happens as d varies we rearrange the above condition to give us a polynomial in d:

$$p(d) = (-a_{11}^2 - a_{12}a_{21})d^2 + 2(a_{11}a_{22} - a_{12}a_{21})d - (a_{12}a_{21} + a_{22}^2) > 0.$$

The equation $p(d) = 0$ has two real roots if and only if

$$-a_{12}a_{21}(a_{11} + a_{22})^2 > 0.$$

So, if $a_{12}a_{21}$ is negative, then, for some value of d, there will always be a Hopf bifurcation, and hence periodic travelling waves. If we restrict ourselves to when $d > -1$ then we need $trA > 0$. Let $c^2 = q(d)$ then the asymptotic behaviour of q is given by

$$d \to -1^+ \Rightarrow q(d) \to -\infty \sim -\frac{1}{d}$$

and

$$d \to \infty \Rightarrow q(d) \to \pm\infty \sim \pm d.$$

The graph of q divides into two families depending on the sign of $-a_{11}^2 - a_{12}a_{21}$. These are shown in figure [2.1]. Note that if $-a_{11}^2 - a_{12}a_{21}$, $a_{11}a_{22} - a_{12}a_{21}$ and $-a_{22}^2 - a_{12}a_{21}$ are all negative then there are no Hopf bifurcation points for any $d > -1$.

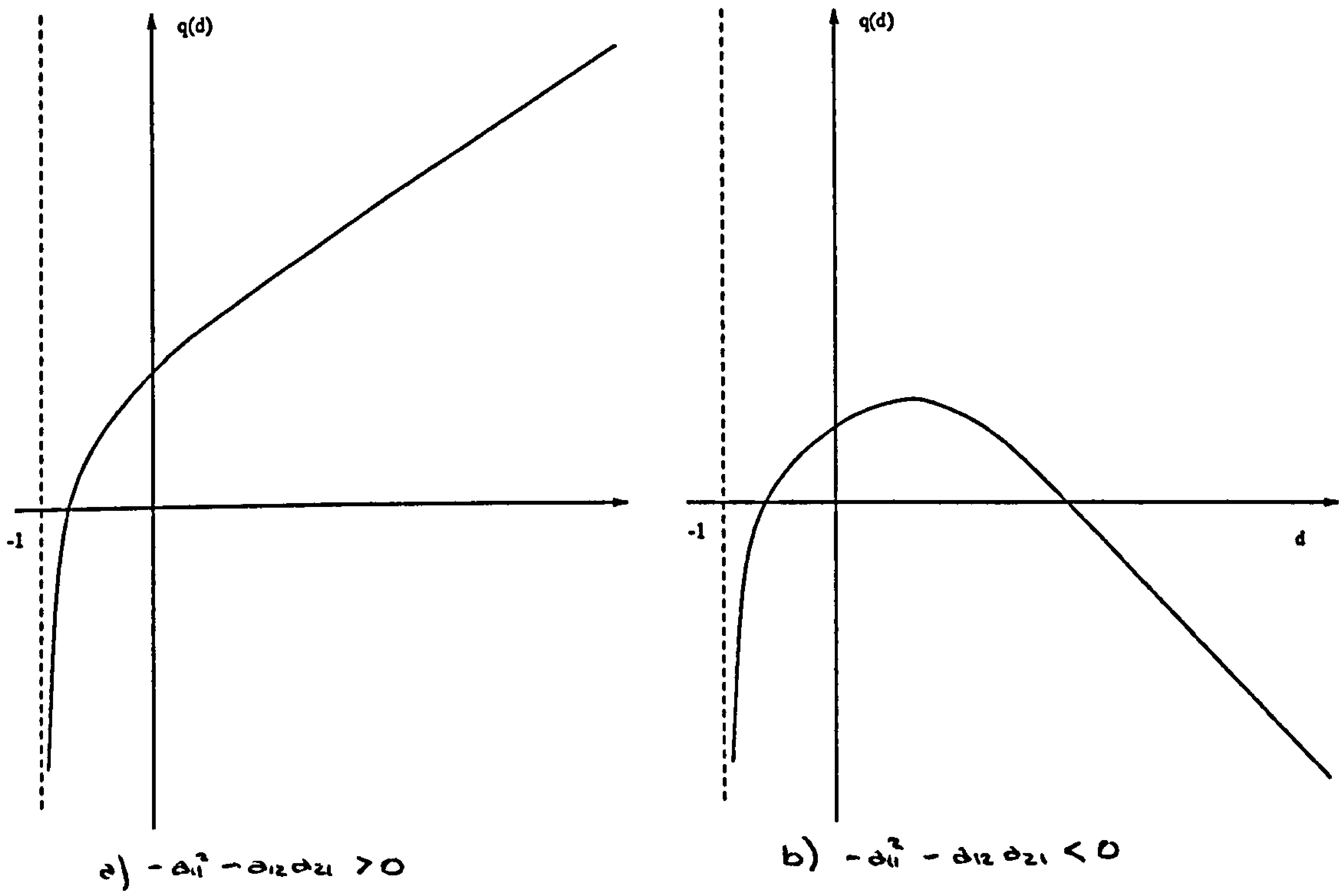


Figure 2.1: Graphs of $c^2 = q(d)$

Example 2: Oscillatory Linearisation

We now consider systems with oscillatory dynamics, ie. ones for which the linearisation at u_* of the local dynamics has the form

$$A = \begin{pmatrix} \lambda & -\omega \\ \omega & \lambda \end{pmatrix}$$

where, without loss of generality, $\omega > 0$. The system is linearly stable for $\lambda < 0$ and unstable for $\lambda > 0$. Substituting this into (2.1) gives

$$c^2 = \frac{\omega^2}{2trD}\frac{1}{\lambda}\left[(4|D| - (trD)^2)\left(\frac{\lambda}{\omega}\right)^2 + 2trD(d_{12} - d_{21})\left(\frac{\lambda}{\omega}\right) + (trD)^2\right].$$

which is an equation of the form

$$c^2 = k\frac{1}{\lambda}[A\lambda^2 + B\lambda + C].$$

Define $q(\lambda)$ to be the right hand side of this equation, then, if $trD > 0$ we have the following asymptotic behaviour

$$\lambda \to 0^+ \Rightarrow q(\lambda) \to \infty \sim \frac{1}{\lambda}$$

and

$$\lambda \to \infty \Rightarrow q(\lambda) \to \pm\infty \sim \pm\lambda.$$

So for λ small and positive there is always a Hopf bifurcation for c large enough.

Our assumption that $trD > 0$ implies that we are only considering $\lambda > 0$. Suppose $A > 0$ and $B > 0$. This means that for $A > 0$ we need

$$-(d_{11} - d_{22})^2 - 4d_{12}d_{21} > 0$$

and so $sign(d_{12}d_{21}) = -1$, and for $B > 0$ we need

$$d_{12} - d_{21} > 0.$$

These would both be satisfied if $d_{12} > 0$ and $d_{21} < 0$ and the values of the diagonal terms were close enough. In this case the graph of $q(\lambda)$ is given by figure [2.2]. In particular periodic solutions would only exist for $c > c_{min}$ for some $c_{min} > 0$. If, on the other hand $B < 0$ then

$$d_{12} - d_{21} < 0$$

and so we would need $d_{12} < 0$ and $d_{21} > 0$. The graphs in figure [2.3] show that in this case there are two Hopf Bifurcation loci which could mean that the periodic solutions could disappear for large λ. Note also that Hopf bifurcation points exist for all values of c.

Suppose now that $A < 0$ and $B = 0$. So, either we have a diagonal diffusion matrix, or the cross diffusion terms are equal. Here the graphs are given by figure [2.4]. Again there are Hopf bifurcation points for all values of c. If we

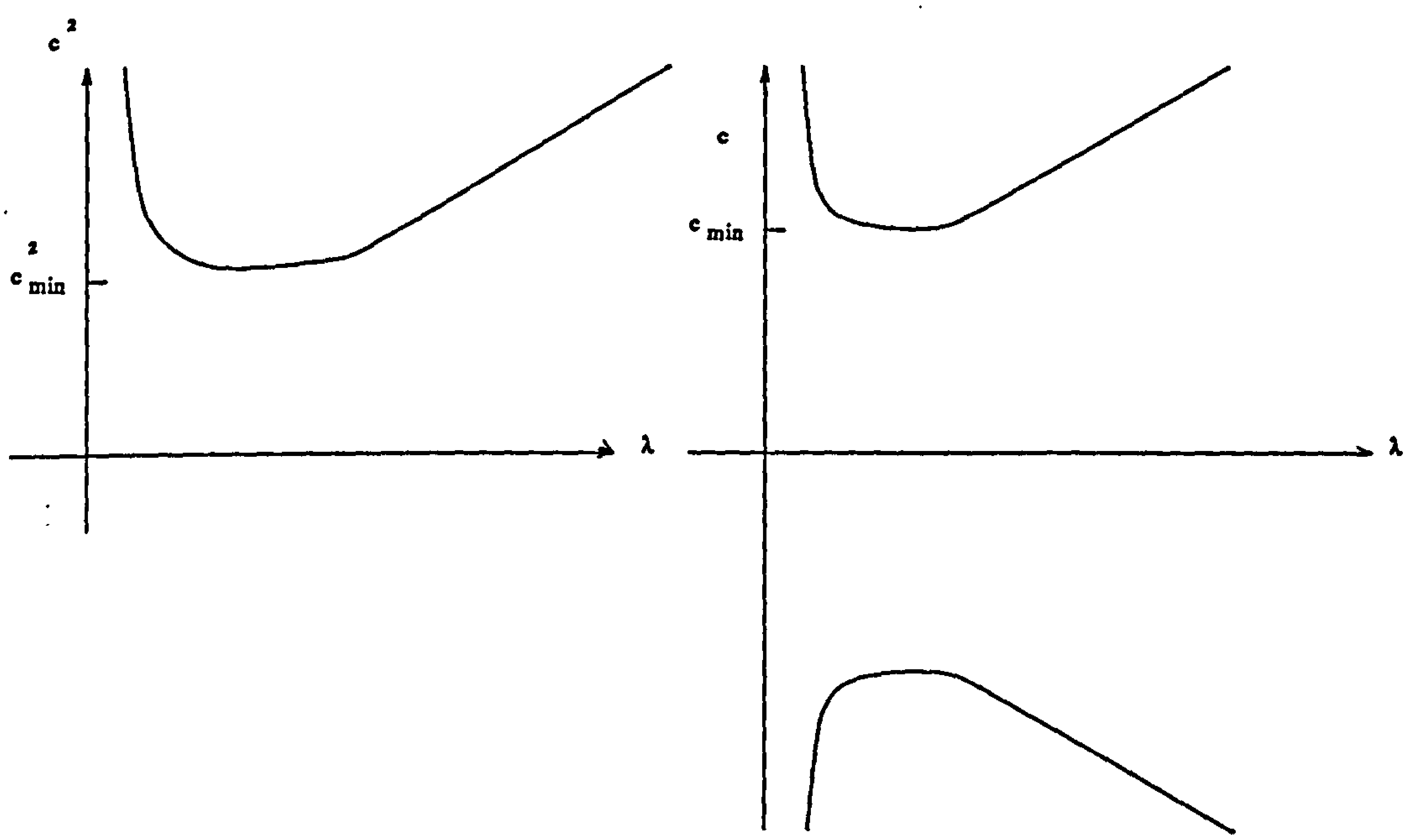


Figure 2.2: Graph of $c^2 = q(\lambda)$ when $A > 0$, $B > 0$

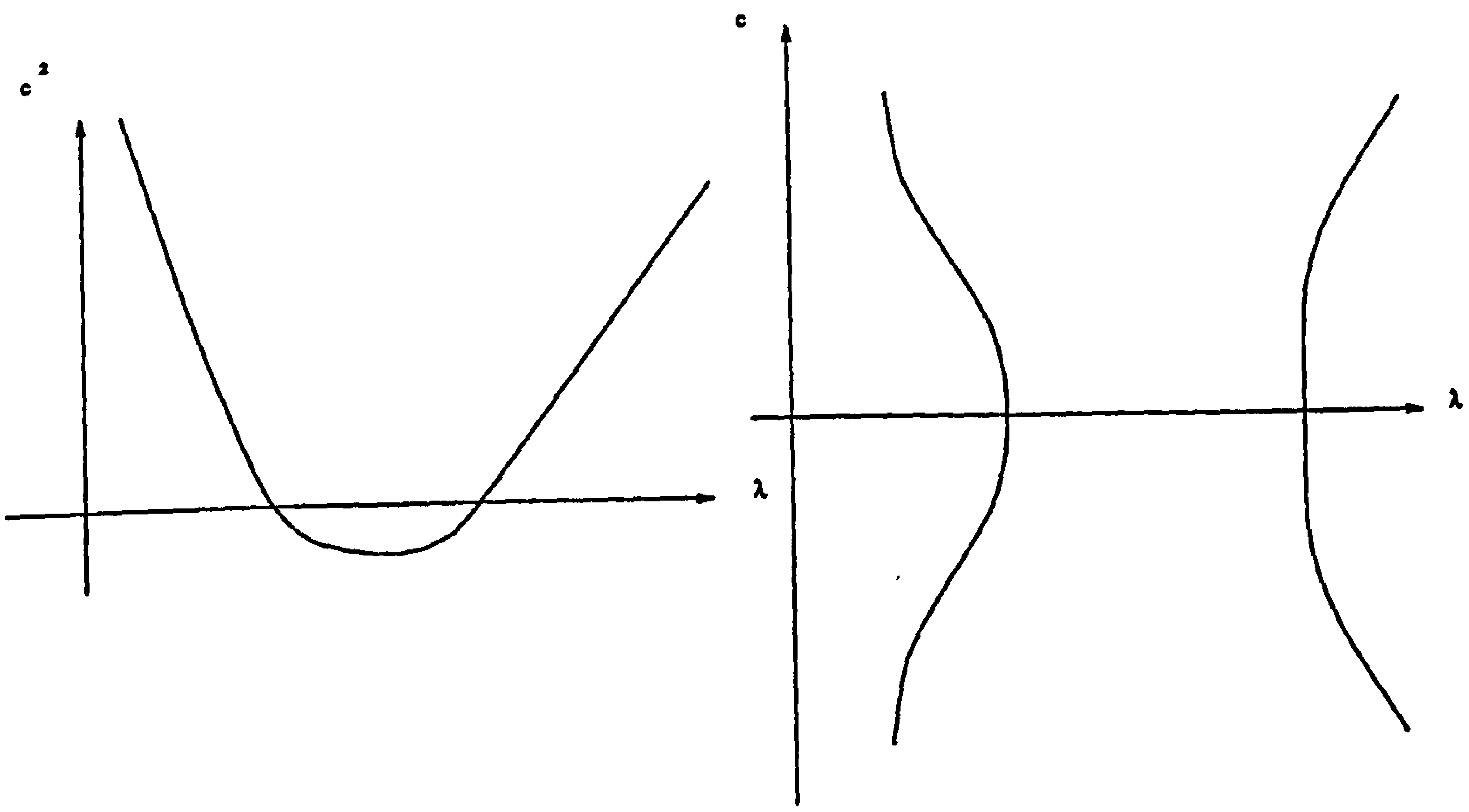


Figure 2.3: Graph of $c^2 = q(\lambda)$ when $A > 0$, $B < 0$

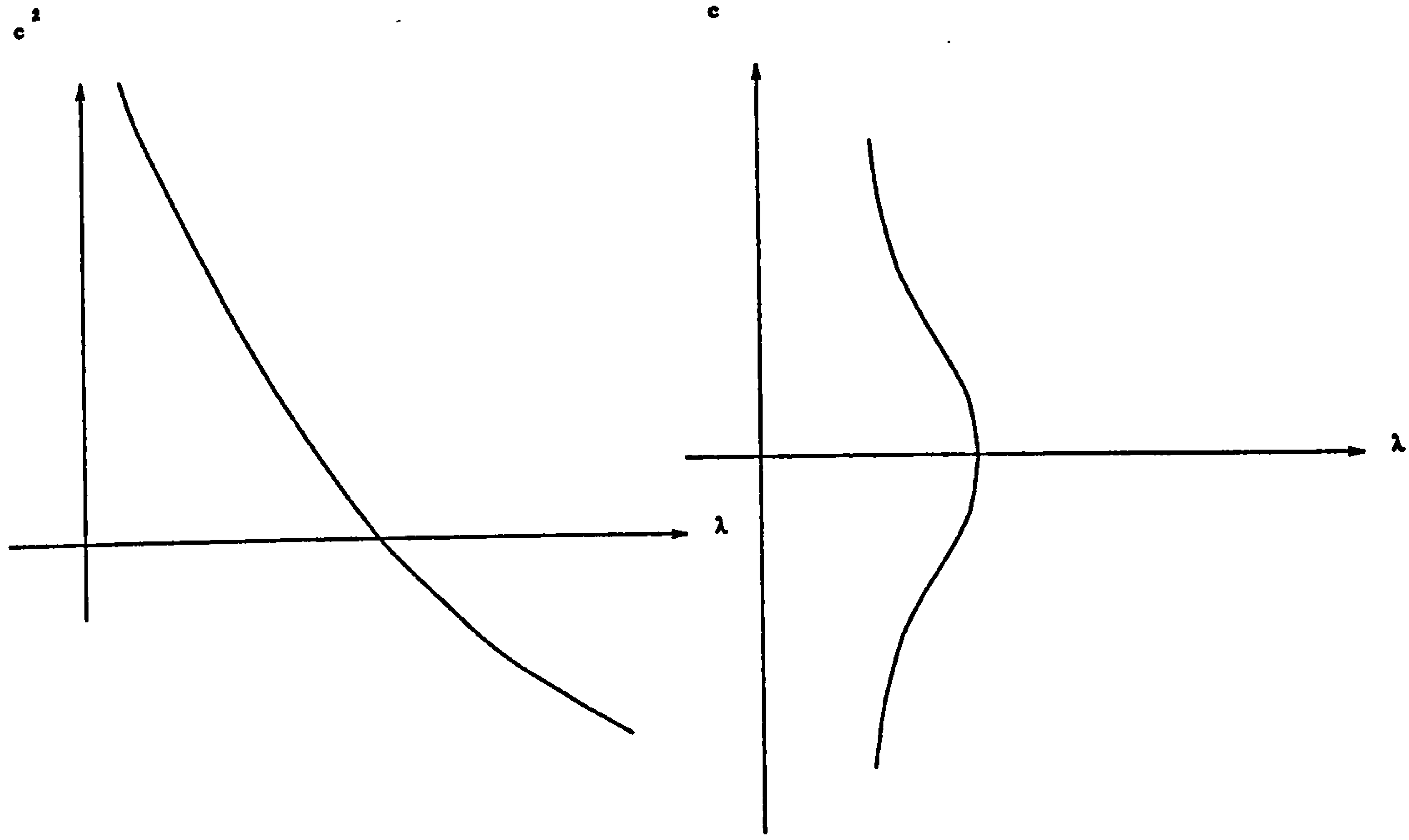


Figure 2.4: Graph of $c^2 = q(\lambda)$ when $A < 0$

assume the diffusion matrix is diagonal then $q(\lambda) = 0$ when $\lambda = \left|\frac{d_{11}+d_{22}}{d_{11}-d_{22}}\right| \omega$.

The graphs for the cases $A < 0$, $B < 0$ and $A < 0$, $B > 0$ are similar to the ones in figure [2.4] for when $B = 0$.

2.2 Perturbation from $c = 0$

This section investigates what happens to the Turing patterns as the wave speed is perturbed from zero and whether any travelling waves result.

2.2.1 Lyapunov - Schmidt Reduction

We know from chapter 1 that system (1.4) has periodic solutions when $c = 0$ if there is a pair of eigenvalues on the imaginary axis. Using Lyapunov - Schmidt reduction we investigate if there is a branch of periodic solutions near these when $c \neq 0$. Suppose we write system (1.4) as $u' = f(u, c)$ where $u \in \mathbf{R}^4$.

To find periodic solutions we follow the same steps as already described in chapter 1, only this time we have the extra parameter, c. Let

$$F : C^1_{2\pi} \times \mathbf{R} \times \mathbf{R} \mapsto C_{2\pi}$$

be defined by

$$F(u(s), \tau, c) = (1+\tau)\frac{du}{ds} - f(u(s), c).$$

We restrict the discussion to the case when we have one pair of imaginary eigenvalues. Firstly we may rescale the equations so that the imaginary eigenvalues have the values $\pm i$. The kernel of $dF_{(0,0,0)}$ is two dimensional and so the reduced equation is

$$g : \mathbf{C} \times \mathbf{R} \times \mathbf{R} \mapsto \mathbf{C}$$

and solving $g(z, c, \tau) = 0$ yields small amplitude periodic solutions. The group actions are as in Chapter 1, which yields that g has the form

$$g(z, \tau, c) = [Q(|z|^2, \tau, c) + iP(|z|^2, \tau, c)]z,$$

and so non-zero solutions of g are given by $P = 0$, $Q = 0$. A similar proof to that given in Chapter 1 may be used to show that $P_\tau(0,0,0) \neq 0$ and so $P = 0$ can be solved for τ as a function of $|z|^2$ and c. So periodic solutions are now given by

$$Q(|z|^2, \tau(|z|^2, c), c) = 0.$$

As $Q(|z|^2, \tau, 0) \equiv 0$ by our analysis in Chapter 1 then we may write

$$Q(|z|^2, \tau(|z|^2, c), c) = q(|z|^2, \tau(|z|^2, c), c)c$$

for some real valued function q. Now, solutions are given by $c = 0$ (the time reversible solutions) and

$$h(|z|^2, c) \equiv q(|z|^2, \tau(|z|^2, c), c) = 0.$$

The Taylor series for h is

$$h(|z|^2, c) = h(0,0) + h_c(0,0)c + h_{|z|^2}(0,0)|z|^2 + h.o.t.$$

where $h.o.t.$ stands for 'higher order terms'. If $h(0,0) \neq 0$ then there will not be any periodic orbits near $z = 0$ for $c \neq 0$. However, if $h(0,0) = 0$ then a branch of periodic solutions, parameterised by c, bifurcates from $z = 0$, $c = 0$. We would expect this to happen when the Hopf locus described earlier in the chapter, intersects the $c = 0$ axis.

Proposition 13 *A branch of periodic solutions bifurcates from $z = 0$, $c = 0$ precisely when the conditions given in Proposition 2 are met for $c = 0$, ie $tr(A)/tr(D) > 0$ and*

$$\frac{(trA)^2 detD + (trD)^2 detA - (trA)(trD)(detD)tr(D^{-1}A)}{(trA)(trD)} = 0 \quad .$$

Proof:

In order to find $h(0,0)$ we must calculate $Re(g_{zc}(0,0,0))$. Using the same notation as Chapter 1 we note that

$$g(z,\tau,c) = \langle \psi, F(z\phi + W(z,\tau,c),\tau,c)\rangle$$

where ϕ and ψ are the basis for K and N respectively. For ease of presentation we leave out dependance on variables. Differentiating gives

$$g_z = \langle \psi, d_u F(\phi + W_z)\rangle \ ,$$

and

$$g_{zc} = \langle \psi, d_u F_c(\phi + W_z) + d_u F(W_{cz}) + {d_u}^2 F(\phi + W_z, W_c)\rangle.$$

As in Lemma 4 in Chapter 1 two of the terms on the right hand side disappear to leave us with

$$g_{zc}(0,0,0) = \langle \psi, d_u F_c(0,0,0)\phi\rangle.$$

If we let $\mathbf{u} = (u, v)$, then in the notation of (1.4) we have

$$F(\mathbf{u}, \tau, c) = \begin{pmatrix} (1+\tau)u' \\ (1+\tau)v' \end{pmatrix} - \begin{pmatrix} v \\ -D^{-1}(f(u) + cv) \end{pmatrix}$$

and so

$$d_{\mathbf{u}} F_c(0,0,0)\phi = \begin{pmatrix} 0 \\ -D^{-1} \end{pmatrix} \phi.$$

Let

$$\phi = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} e^{is}$$

where (α, β) is an eigenvector for the eigenvalue $-i$, which gives us

$$-i \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 0 & I \\ -D^{-1}A & -cD^{-1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$$

where each entry is a (2×2) matrix, with 0 being the zero matrix and I the identity matrix. When $c = 0$ we get the following

$$\begin{aligned} -i\alpha &= \beta \\ \beta &= D^{-1}A\beta. \end{aligned}$$

Similarly if $\psi = (\gamma, \xi)^T e^{is}$ then we have

$$i \begin{pmatrix} \gamma \\ \xi \end{pmatrix} = \begin{pmatrix} 0 & -(D^{-1}A)^T \\ I & -c(D^{-1})^T \end{pmatrix} \begin{pmatrix} \gamma \\ \xi \end{pmatrix}$$

which, again at $c = 0$, gives us

$$\begin{aligned} i\xi &= \gamma \\ \xi &= (D^{-1}A)^T \xi. \end{aligned}$$

Now

$$\begin{aligned} g_{zc}(0,0,0) &= \langle \psi, (0, -D^{-1})^T \cdot \phi \rangle \\ &= \langle (\gamma, \xi)^T e^{is}, (0, -D^{-1}\beta)^T e^{is} \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \begin{pmatrix} \overline{\gamma} \\ \overline{\xi} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -D^{-1}\beta \end{pmatrix} ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} -D^{-1}\beta \cdot \overline{\xi} \, ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{-1}{|D|} \begin{pmatrix} d_{22}\beta_1 - d_{12}\beta_2 \\ d_{11}\beta_2 - d_{21}\beta_1 \end{pmatrix} \cdot \begin{pmatrix} \overline{\xi_1} \\ \overline{\xi_2} \end{pmatrix} ds \end{aligned}$$

where $|D| = det(D)$ and $\beta = (\beta_1, \beta_2)$, $\xi = (\xi_1, \xi_2)$. Remember that for $h(0,0)$ to be equal to zero we need $Re(g_{zc}(0,0,0))$ to be equal to zero, which is equivalent to the condition

$$Re(\overline{\xi_1}(d_{22}\beta_1 - d_{12}\beta_2) + \overline{\xi_2}(d_{11}\beta_2 - d_{21}\beta_1)) = 0 \ .$$

For ease of presentation we restrict ourselves to the case when D is diagonal for the remainder of the proof. The same method is used in proving the result for the most general case. The condition now becomes:

$$Re(\overline{\xi_1}d_{22}\beta_1 + \overline{\xi_2}d_{11}\beta_2) = 0 \ .$$

As shown below we can choose ξ and β so that their entries are real which gives the condition

$$\xi_1 d_{22}\beta_1 + \xi_2 d_{11}\beta_2 = 0 \ . \tag{2.2}$$

Now, $\beta = D^{-1}A\beta$ and so

$$\begin{pmatrix} a_{11}d_{22} - d_{11}d_{22} & a_{12}d_{22} \\ a_{21}d_{11} & a_{22}d_{11} - d_{11}d_{22} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

We may choose β such that

$$\beta_1 = a_{12}d_{22} \ , \ \beta_2 = d_{11}d_{22} - a_{11}d_{22} \ .$$

Similarly, since $\xi = (D^{-1}A)^T\xi$, we may choose ξ such that

$$\xi_1 = a_{21}d_{11} \ , \ \xi_2 = d_{11}d_{22} - a_{11}d_{22} \ .$$

Substituting these into (2.2) gives

$$a_{12}a_{21}d_{22}^2 + a_{11}^2d_{22}^2 - 2a_{11}d_{11}d_{22}^2 + d_{11}^2d_{22}^2 = 0 \ .$$

Multiplying out the condition that $det(D^{-1}A - I) = 0$ gives the equation

$$a_{11}a_{22}d_{11}d_{22} - a_{11}d_{11}d_{22}^2 - a_{22}d_{11}^2d_{22} + d_{11}^2d_{22}^2 - a_{12}a_{21}d_{11}d_{22} = 0 \ .$$

Subtracting these two equations we get

$$a_{12}a_{21}d_{22}^2+a_{11}^2d_{22}^2-a_{11}d_{11}d_{22}^2-a_{11}a_{22}d_{11}d_{22}+a_{22}d_{11}^2d_{22}+a_{12}a_{21}d_{11}d_{22}=0\,. \quad (2.3)$$

We may also have chosen β and ξ such that

$$\beta_1 = d_{11}d_{22} - a_{22}d_{11}\ ,\ \beta_2 = a_{21}d_{11}\ ,$$

$$\xi_1 = d_{11}d_{22} - a_{22}d_{11}\ ,\ \xi_2 = a_{12}d_{22}\ .$$

If we again substitute these values into (2.2) and substitute in the equation $det(D^{-1}A - I) = 0$ then we have

$$a_{12}a_{21}d_{11}d_{22}-a_{11}a_{22}d_{11}d_{22}+a_{11}d_{11}d_{22}^2+a_{22}^2d_{11}^2-a_{22}d_{11}^2d_{22}+a_{12}a_{21}d_{11}^2=0\,. \quad (2.4)$$

Subtracting (2.3) from (2.4) gives

$$2a_{12}a_{21}d_{11}d_{22} - 2a_{11}a_{22}d_{11}d_{22} + a_{12}a_{21}d_{11}^2 + a_{12}a_{21}d_{22}^2 + a_{22}^2d_{11}^2 + a_{11}^2d_{22}^2 = 0$$

which is the same as equation (2.1) when $c = 0$.

Example

If we refer back to Example 2 in the previous section for the case when the diffusion was diagonal, then there exists a branch of small amplitude periodic solutions bifurcating from $c = 0$ where the graph in figure [2.4] crosses the axis. This will happen exactly when $\lambda = \left|\frac{d_{11}+d_{22}}{d_{11}-d_{22}}\right|\omega$.

2.3 Perturbation from $c = \infty$

We now consider what happens as c is perturbed from ∞. To do this we look at two different types of local dynamics.

2.3.1 Periodic Dynamics

This section uses a result in [Kopell]. Suppose, as before, we have the system

$$u_t = f(u) + Du_{xx}$$

where f is such that we have periodic local dynamics. We show that for wave speed, c, large enough there always exist periodic travelling waves. Recall that the first order O.D.E. travelling wave equations are

$$\begin{aligned} u' &= v \\ v' &= -D^{-1}[f(u) + cv]. \end{aligned}$$

Let $d = \|D\|$ and $\hat{D} = \frac{1}{d}D$. We now scale the travelling wave equations by using the substitutions $\tau = \frac{c}{d}s$, $\hat{u} = u$, $\hat{v} = cv$ so that they become:

$$\begin{aligned} \hat{u}' &= \frac{d}{c^2}\hat{v} \\ \hat{v}' &= -\hat{D}^{-1}[\hat{v} + f(\hat{u})]. \end{aligned}$$

When $\frac{d}{c^2} = 0$ the system has an invariant manifold of equilibria given by $\hat{v} = f(\hat{u})$. This manifold is normally hyperbolic and so persists when $\frac{d}{c^2}$ is perturbed from zero. ie. for $\frac{d}{c^2} < \epsilon$ for some $\epsilon > 0$. To approximate the dynamics on the invariant manifold we again look at a scaling of the original equations, but this time using $\tau = \frac{1}{c}s$, $\hat{u} = u$, $\hat{v} = cv$ to give

$$\begin{aligned} \hat{u}' &= \hat{v} \\ \frac{d}{c^2}\hat{v}' &= -\hat{D}^{-1}[\hat{v} + f(\hat{u})] \end{aligned}$$

which, on setting $\frac{d}{c^2}$ to zero, reduces to

$$\hat{u}' = -f(\hat{u}).$$

This is just the original local dynamics with time reversed. Hence the flow on the invariant submanifold above is a perturbation of the flow of the original

O.D.E. with time reversed. In particular, if the O.D.E. has normally hyperbolic periodic solutions, then so does the travelling wave system for $\frac{d}{c^2} < \epsilon$. For a fixed diffusion matrix, D, there exists a family of periodic travelling waves with wavespeeds $c > \sqrt{d/\epsilon}$. As $c \to \infty$ these waves converge to homogeneous oscillations.

If D is close to a scalar matrix (eg. $D = dI$ for some $d \in \mathbf{R}$) then it can be shown that the periodic travelling waves are stable if $\frac{d}{c^2}$ is small enough [Kopell & Howard]. In particular the homogeneous oscillations are stable. Both the travelling waves and the homogeneous oscillations can be destabilised by unequal diffusion coefficients or by the presence of cross-diffusion. An example which demonstrates this for cross diffusion is given in section (2.4).

2.3.2 Takens - Bogdanov Dynamics

Suppose we have a reaction - diffusion system which has the codimension two singularity described in section (1.7); ie a coalescence point of Hopf and saddle - node bifurcations. Near the singularity it can be shown that, for certain parameter values, the two dimensional system has a homoclinic orbit (see [Guckenheimer]). The phase portrait will resemble that given by figure [2.5] with the homoclinic orbit indicated by the arrow.

A similar proof to the previous section may be applied to the full reaction - diffusion system to show the existence of travelling waves for large wave speed c, obtained by perturbing the homoclinic solution. The difference in this case is that instead of periodic travelling waves this system has travelling pulses as portrayed in figure [2.6]. This result has been proved using a different method in [Schneider].

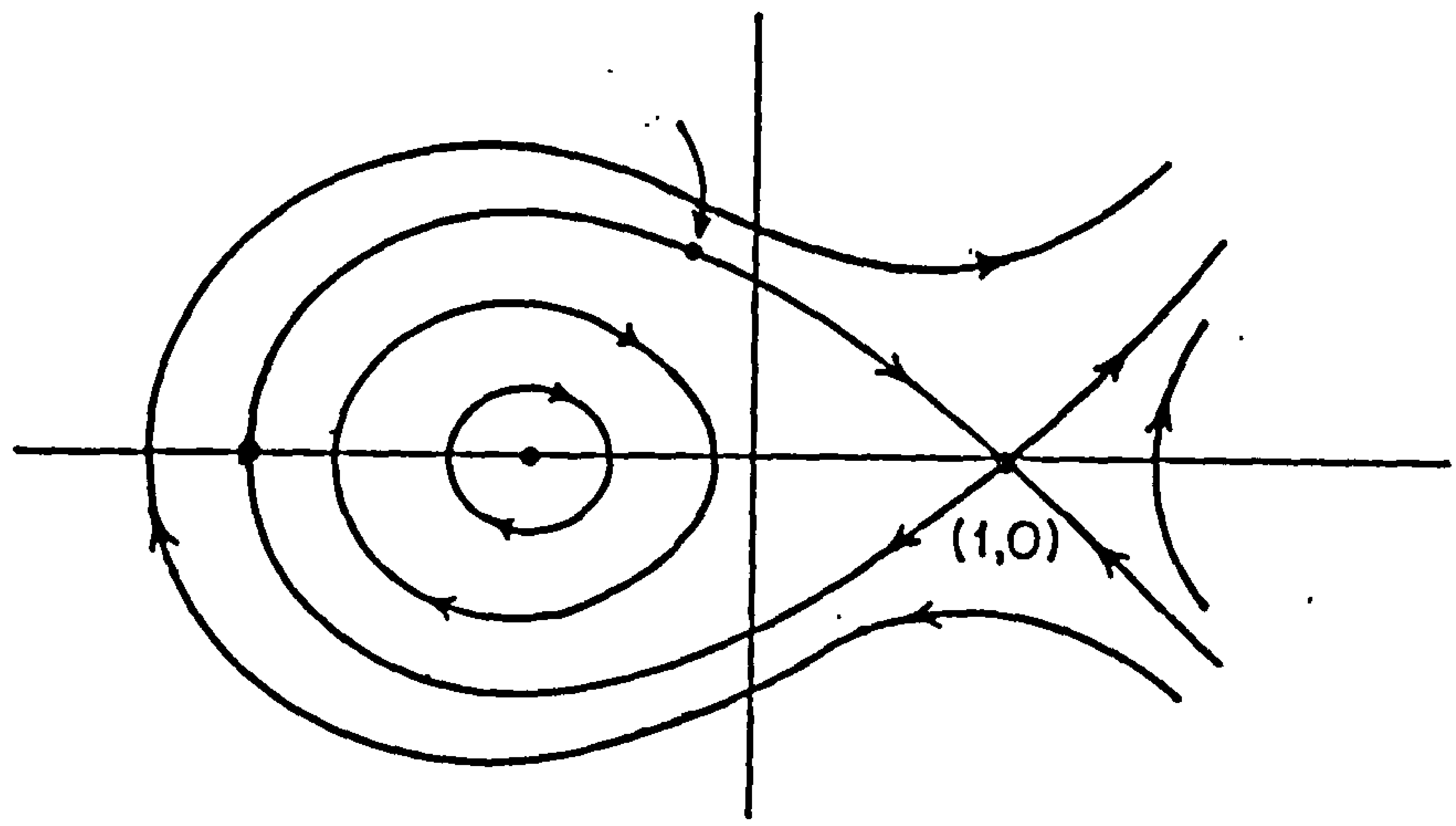


Figure 2.5: Phase portrait showing homoclinic orbit

Figure 2.6: Pulse travelling wave

2.4 $\lambda - \omega$ Systems

In this section we analyse in more detail a specific class of reaction - diffusion system with oscillatory kinetics, namely, $\lambda-\omega$ systems. We prove the existence of both homogeneous and travelling wave solutions and the conditions needed for their stability. This is an extension of a result in [Kopell & Howard]; the difference being that our work includes cross diffusion in the system and analyses its affect. An example of such a system is the complex Ginzburg - Landau (CGL) equation as seen in chapter 1. In fact the CGL gave us the idea for the diffusion coefficients used in this section. If we take γ and ζ to equal zero in the CGL equation given in chapter 1, then the diffusion is the same as that given in (2.5). The system with two reactants is

$$\begin{aligned} \dot{u} &= u\lambda(r) + v\omega(r) + \nabla^2(u - \beta v) \\ \dot{v} &= v\lambda(r) - u\omega(r) + \nabla^2(v + \beta u) \end{aligned} \tag{2.5}$$

where $r^2 = u^2 + v^2$, and $\beta \in \mathbb{R}$. Here λ and ω are real functions of r, and are chosen so that the system goes through a Hopf bifurcation depending on some parameter. If $r_0 > 0$ is an isolated zero of λ then the spatially homogeneous system exhibits periodic behaviour. If, in addition, $\lambda'(r_0) < 0$ and $\omega(r_0) \neq 0$ then a limit cycle exists. We start by changing system (2.5) into (r, θ) form, defined by

$$u = r\cos\theta, \quad v = r\sin\theta$$

giving the transformed equations

$$\begin{aligned} \dot{r} &= r\lambda(r) + r_{xx} - r\theta_x^2 + \beta(-2r_x\theta_x - r\theta_{xx}) \\ \dot{\theta} &= \omega(r) + \frac{2r_x\theta_x}{r} + \theta_{xx} + \beta(\frac{r_{xx}}{r} - \theta_x^2). \end{aligned} \tag{2.6}$$

We are looking for travelling wave solutions of the form:

$$r = \alpha; \ \theta = \sigma t - kx, \tag{2.7}$$

and on substituting these into (2.6) we find the following necessary and sufficient conditions for them to exist, namely

$$\begin{aligned} k^2 &= \lambda(\alpha) \\ \sigma &= \omega(\alpha) - \beta k^2 \end{aligned}$$

So, using α as the parameter, there is a one parameter family of travelling wave solutions of (2.5) given by

$$\begin{aligned} u &= \alpha \cos[(\omega(\alpha) - \beta\lambda(\alpha))t - \lambda^{\frac{1}{2}}(\alpha)x] \\ v &= \alpha \sin[(\omega(\alpha) - \beta\lambda(\alpha))t - \lambda^{\frac{1}{2}}(\alpha)x] \end{aligned} \tag{2.8}$$

Note that, in the notation of the previous section, the wave speed $c = \sigma/k$. Therefore, the waves described in section (2.3) are those detailed above when k is small. Because of the simplicity of the wave solutions in their polar form, linear stability of these waves is considered by using small perturbations ρ and ϕ and setting

$$r = \alpha + \rho(x,t); \quad \theta = \sigma t - kx + \phi(x,t),$$

where $|\rho| \ll 1, |\phi| \ll 1$. Substituting these into (2.6) gives

$$\begin{aligned} \rho_t &= (\alpha+\rho)\lambda(r) + \rho_{xx} - (\alpha+\rho)(-k+\phi_x)^2 \\ &\quad +\beta[-2\rho_x(-k+\phi_x) - (\alpha+\rho)\phi_{xx}] \\ \phi_t + \sigma &= \omega(\alpha+\rho) + \frac{2\rho_x(-k+\phi_x)}{(\alpha+\rho)} + \phi_{xx} + \beta[\frac{\rho_{xx}}{\rho+\alpha} - (-k+\phi_x)^2]. \end{aligned}$$

Linearising these equations, we get

$$\begin{aligned} \rho_t &= \alpha[\rho\lambda'(\alpha) + 2k\phi_x] + \rho_{xx} + \beta[2k\rho_x - \alpha\phi_{xx}] \\ \phi_t &= \rho\omega'(\alpha) - \frac{2k\rho_x}{\alpha} + \phi_{xx} + \beta[\frac{\rho_{xx}}{\alpha} + 2k\phi_x]. \end{aligned} \tag{2.9}$$

We need to find conditions on k and σ such that ρ and $\phi \to 0$ as $t \to \infty$.

Proposition 14 *Travelling wave solutions of (2.5) given by (2.7) and (2.8) with amplitude α and wave number k are linearly stable if and only if*

$$4k^2\left[1 + \left(\frac{\omega'}{\lambda'}\right)^2\right] + \alpha\beta\omega' + \alpha\lambda' < 0.$$

Proof:

The coefficients in (2.9) are constants so we look for solutions in the usual fourier form by setting

$$\begin{pmatrix} \rho \\ \phi \end{pmatrix} = \begin{pmatrix} \rho_0 \\ \phi_0 \end{pmatrix} \exp(st + iqx),$$

where q is the perturbation wave number and ρ_0, ϕ_0 are constants. For stability we want $Re(s) < 0$. Substituting this form into (2.9) above gives

$$\begin{pmatrix} s + q^2 - \alpha\lambda' - 2i\beta kq & -2i\alpha kq - \alpha\beta q^2 \\ -w' + \frac{2ikq}{\alpha} + \frac{\beta q^2}{\alpha} & s + q^2 - 2i\beta kq \end{pmatrix} \begin{pmatrix} \rho_0 \\ \phi_0 \end{pmatrix} = 0, \qquad (2.10)$$

where we have used the notation $\lambda'(\alpha) = \lambda'$ and $\omega'(\alpha) = \omega'$ for ease of presentation. In order that we have a nontrivial solution we need the determinant of the 2×2 matrix in (2.10) to be zero, ie

$$(s+q^2-\alpha\lambda'-2i\beta kq)(s+q^2-2i\beta kq)-(-2i\alpha kq-\alpha\beta q^2)(-\omega'+\frac{2ikq}{\alpha}+\frac{\beta q^2}{\alpha}) = 0,$$

which gives us the quadratic equation in s

$$\begin{aligned} s^2 + (2q^2 - \alpha\lambda - 4i\beta kq)s + q^4 - \alpha\lambda' q^2 + 2i\alpha\beta\lambda' kq \\ -4\beta^2 kq^2 - [2i\alpha\omega' kq + 4k^2q^2 + \alpha\beta\omega' q^2 - \beta^2 q^4] &= 0, \end{aligned}$$

with the solutions

$$\begin{aligned} s_{1,2} &= -q^2 + \tfrac{\alpha\lambda'}{2} + 2i\beta kq \\ &\pm\sqrt{\left(\tfrac{\alpha\lambda'}{2}\right)^2 + 4k^2q^2 + 2i\alpha\omega' kq + \alpha\beta\omega' q^2 - \beta^2 q^4 - 4i\beta kq^3}. \end{aligned}$$

If either s_1 or s_2 has a positive real part for any q then the wave solutions (2.7) are linearly unstable. If $q = 0$ then $s_1 = 0, s_2 = \alpha\lambda'$. When $q > 0$ then $Re(s_1) > Re(s_2)$ so for stability we need only consider s_1. Using

$$Re(z^{\frac{1}{2}}) = \frac{1}{\sqrt{2}}(Re(z) + |z|)^{\frac{1}{2}} \quad \forall\ z \ in\ \mathbf{C}$$

we have

$$\begin{aligned} Re(s_1) = -q^2 + \tfrac{\alpha\lambda'}{2} + \tfrac{1}{\sqrt{2}}\Big[\left(\tfrac{\alpha\lambda'}{2}\right)^2 + 4k^2q^2 + \alpha\beta\omega' q^2 - \beta^2 q^4 \\ + \left[\left(\left(\tfrac{\alpha\lambda'}{2}\right)^2 + 4k^2q^2 + \alpha\beta\omega' q^2 - \beta^2 q^4\right)^2 + (2\alpha\omega' kq - 4\beta kq^3)^2\right]^{\frac{1}{2}}\Big]^{\frac{1}{2}}. \end{aligned} \qquad (2.11)$$

We know that $Re(s_1) = 0$ at $q = 0$, and so if the differential of $Re(s_1)$ with respect to q^2 is negative at $q^2 = 0$ then $Re(s_1) < 0$ for small q^2.

$$\left[\frac{dRe(s_1)}{dq^2}\right]_{q=0} = -1 + \frac{1}{\alpha|\lambda'|}\left[4k^2 + \alpha\beta\omega' + \frac{4\alpha^2\omega'^2k^2}{(\alpha\lambda')^2}\right]$$

and, since, for the periodic solutions in the spatially homogeneous system to be stable, $\lambda' < 0$ our condition for stability for small q^2 becomes

$$4k^2\left[1 + \left(\frac{\omega'}{\lambda'}\right)^2\right] + \alpha\beta\omega' + \alpha\lambda' < 0. \tag{2.12}$$

We assume this is true for the rest of the calculation. The condition $Re(s_1) < 0$ is equivalent to:-

$$q^2 - \frac{\alpha\lambda'}{2} > \frac{1}{\sqrt{2}}\left[\left(\frac{\alpha\lambda'}{2}\right)^2 + 4k^2q^2 + \alpha\beta\omega' q^2 - \beta^2q^4 \right.$$
$$\left. + \left[\left(\left(\frac{\alpha\lambda'}{2}\right)^2 + 4k^2q^2 + \alpha\beta\omega' q^2 - \beta^2q^4\right)^2 + (2\alpha\omega' kq - 4\beta kq^3)^2\right]^{\frac{1}{2}}\right]^{\frac{1}{2}}.$$

Now $\lambda' < 0$ so we can square both sides giving

$$(\beta^2+2)q^4 - 4q^2\left(k^2 + \frac{\alpha\lambda'}{2} + \frac{\alpha\beta\omega'}{4}\right) + \left(\frac{\alpha\lambda'}{2}\right)^2 >$$
$$\left[\left(\left(\frac{\alpha\lambda'}{2}\right)^2 + 4k^2q^2 + \alpha\beta\omega' q^2 - \beta^2q^4\right)^2 + (2\alpha\omega' kq - 4\beta kq^3)^2\right]^{\frac{1}{2}}.$$

Using (2.12) we know that $k^2 + \frac{\alpha\beta\omega'}{4} + \frac{\alpha\lambda'}{2} < 0$ and so we can again square both sides, and after a lot of algebra and rearranging the condition becomes $Re(s_1) < 0\ \forall q$ if and only if

$$k^2 < \left[1 - \frac{4\beta^2q^4 - 4\alpha\beta\omega' q^2 + \alpha^2\omega'^2}{4\left(q^2 - \frac{\alpha\lambda'}{2}\right)^2 + 4\beta^2q^4 - 4\alpha\beta\omega' q^2 + \alpha^2\omega'^2}\right]\frac{(\beta^2+1)q^2 - \alpha\lambda' - \alpha\beta\omega'}{4}$$

Since $\lambda' < 0$ the right hand side is an increasing function in q, and, therefore, for the condition to hold for all q it must be true for q^2 small. If we assume q^2 is small then the condition becomes

$$k^2(\alpha^2\omega'^2 + \alpha^2\lambda^2) < -(\alpha\lambda' + \alpha\beta\omega')\left(\frac{\alpha\lambda'}{2}\right)^2,$$

which, on rearranging gives us equation (2.12). Therefore it is a necessary and sufficient condition for the travelling waves to be stable which gives us the result.

This result shows that the effect of the cross diffusion term, β, on travelling wave solutions of (2.5) depends on the sign of $\omega'(\alpha)$. If we suppose that for $\beta = 0$ the waves are stable, and that $\omega'(\alpha) < 0$, then if we perturb β from 0 such that $\beta > 0$ then the region of stability for the waves becomes smaller and for β large enough the waves will become unstable. If, on the other hand, $\beta < 0$ then the region of stability becomes larger. Similarly if $\omega'(\alpha) > 0$. So, as long as $\omega'(\alpha) \neq 0$, then adjusting the cross diffusion term may stabilise or destabilise the travelling wave solutions (2.8).

For the homogeneous oscillations $k = 0$ and so the stability condition (2.12) becomes

$$\alpha(\beta\omega' + \lambda') < 0.$$

We note that if $\beta = 0$ then this condition is always true, and so the homogeneous oscillations are always stable. Again these can be destabilised as long as $\beta\omega' > 0$ and large enough. A simple calculation shows that when $k = 0$, then $Re(s_1) < 0$ for q^2 sufficiently large as $Re(s_1) \simeq -q^2$ in this case. This means that when the homogeneous solution goes unstable, it is to waves with a small wavenumber q, that is to say they will have a long wavelength.

Steady spatially periodic patterns may be investigated for these systems by considering when the wave speed is zero. Note that the wavespeed $c = \sigma/k$ and so for $c = 0$ we need $\sigma = 0$. This is equivalent to

$$\omega(\alpha) - \beta k^2 = \omega(\alpha) - \beta\lambda(\alpha) = 0 \,.$$

Example

As already mentioned an example of a $\lambda - \omega$ system is the Complex Ginzburg Landau equation. The stability of the travelling waves that exist for this system has already been investigated, for example see [Newell]. Since the diffusion is as described above we may use the transformation $\hat{u} = e^{i\mu_i t}u$ to give us the following form of the equation:

$$\dot{u} = \mu u - (1 + i\nu)|u|^2 u + (1 + i\beta)\, \nabla^2 u$$

where $u(x,t)$ is a complex variable, and μ, ν and β are all real. In the above notation $\lambda(r) = \mu - r^2$ and $\omega(r) = -\nu r^2$. Differentiating with respect to r gives $\lambda'(\alpha) = -2\alpha$ and $\omega'(\alpha) = -2\alpha\nu$ which means that in the spatially homogeneous system there is a stable limit cycle. Substituting into (2.12) and using $k^2 = \mu - \alpha^2$ produces the condition for the travelling waves to be linearly stable:

$$4(\mu - \alpha^2)(1 + \nu^2) - 2\alpha^2\beta\nu - 2\alpha^2 < 0,$$

which rearranges to give the condition

$$\alpha^2 > \frac{2\mu(1 + \nu^2)}{3 + \beta\nu + 2\nu^2}.$$

Since $k^2 = \mu - \alpha^2 \geq 0$ we know that $\sqrt{\mu} \geq \alpha$, and so from the one parameter family of wave solutions given by (2.8) the ones that are stable are those such that

$$\sqrt{\mu} \geq \alpha > \sqrt{\frac{2\mu(1 + \nu^2)}{3 + \beta\nu + 2\nu^2}}.$$

See figure [2.7]. This picture reinforces the result in section (2.3). Travelling waves exist and are stable near to when $c = \infty$, ie. the curve $\alpha = \sqrt{\mu}$. Finally, the stability condition for the homogeneous oscillations is $1 + \beta\nu > 0$.

We may also look for steady, spatially periodic solutions by considering when the wave speed is zero. Remember that $c = \sigma/k$ and so for $c = 0$ we

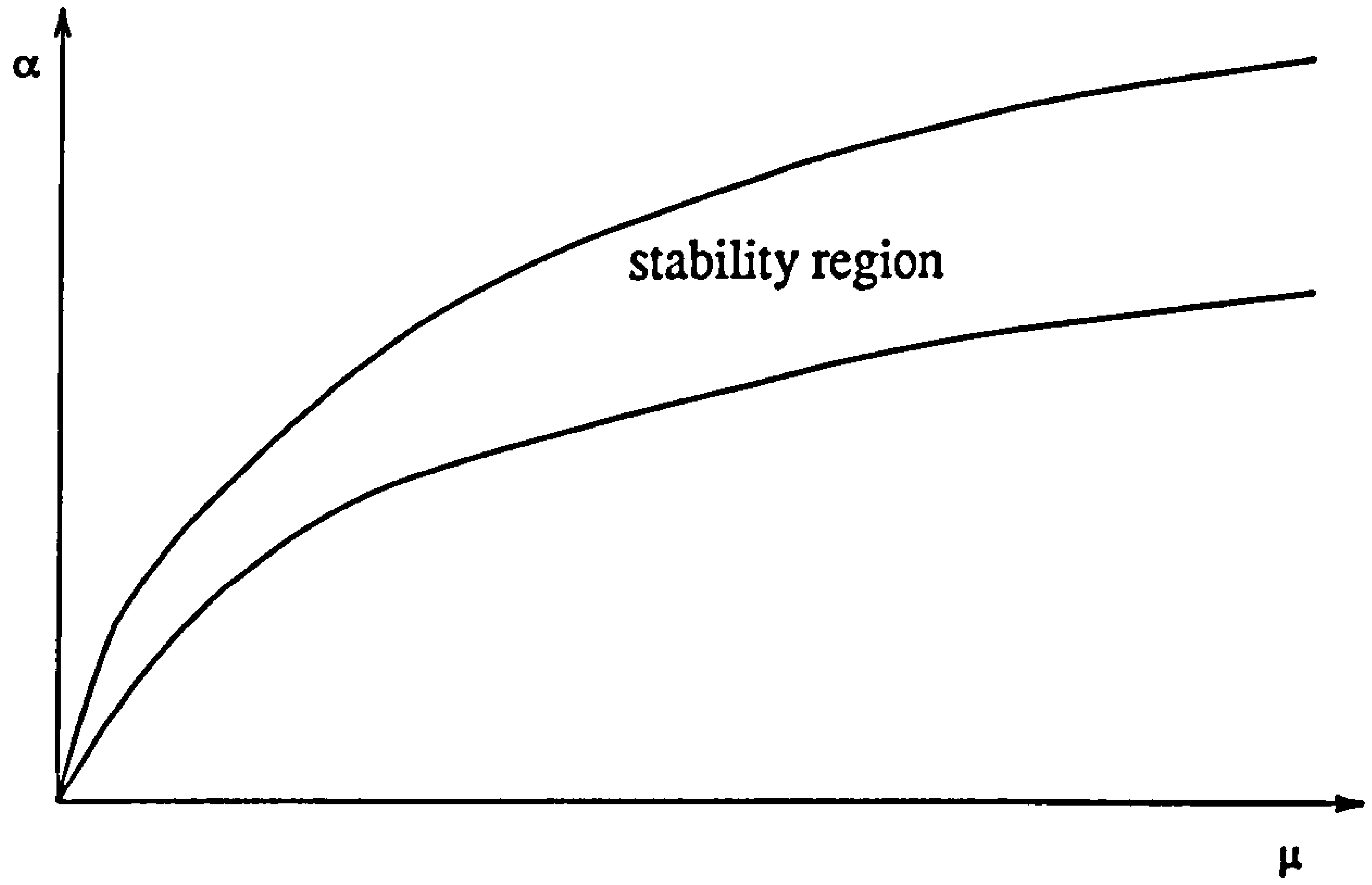


Figure 2.7: Stability region for travelling waves in the CGL

need $\sigma = 0$. Recall

$$\sigma = \omega(\alpha) - \beta k^2 .$$

Since the variable transformation we used puts the equation into a rotating reference frame, when looking for stationary waves we need to reinsert the imaginary part of the first coefficient ie μ_i. So for $\sigma = 0$ we have

$$\mu_i - \nu\alpha^2 - \beta k^2 = 0$$

which, on substituting $k^2 = \mu - \alpha^2$ and rearranging we get the condition

$$\mu = \frac{1}{\beta}[\mu_i + (\beta - \nu)\alpha^2].$$

This curve crosses the curve where we have homogeneous oscillations at the point where

$$\frac{1}{\beta}[\mu_i + (\beta - \nu)\alpha^2] = \alpha^2$$

or when

$$\mu_i - \nu\alpha^2 = 0 .$$

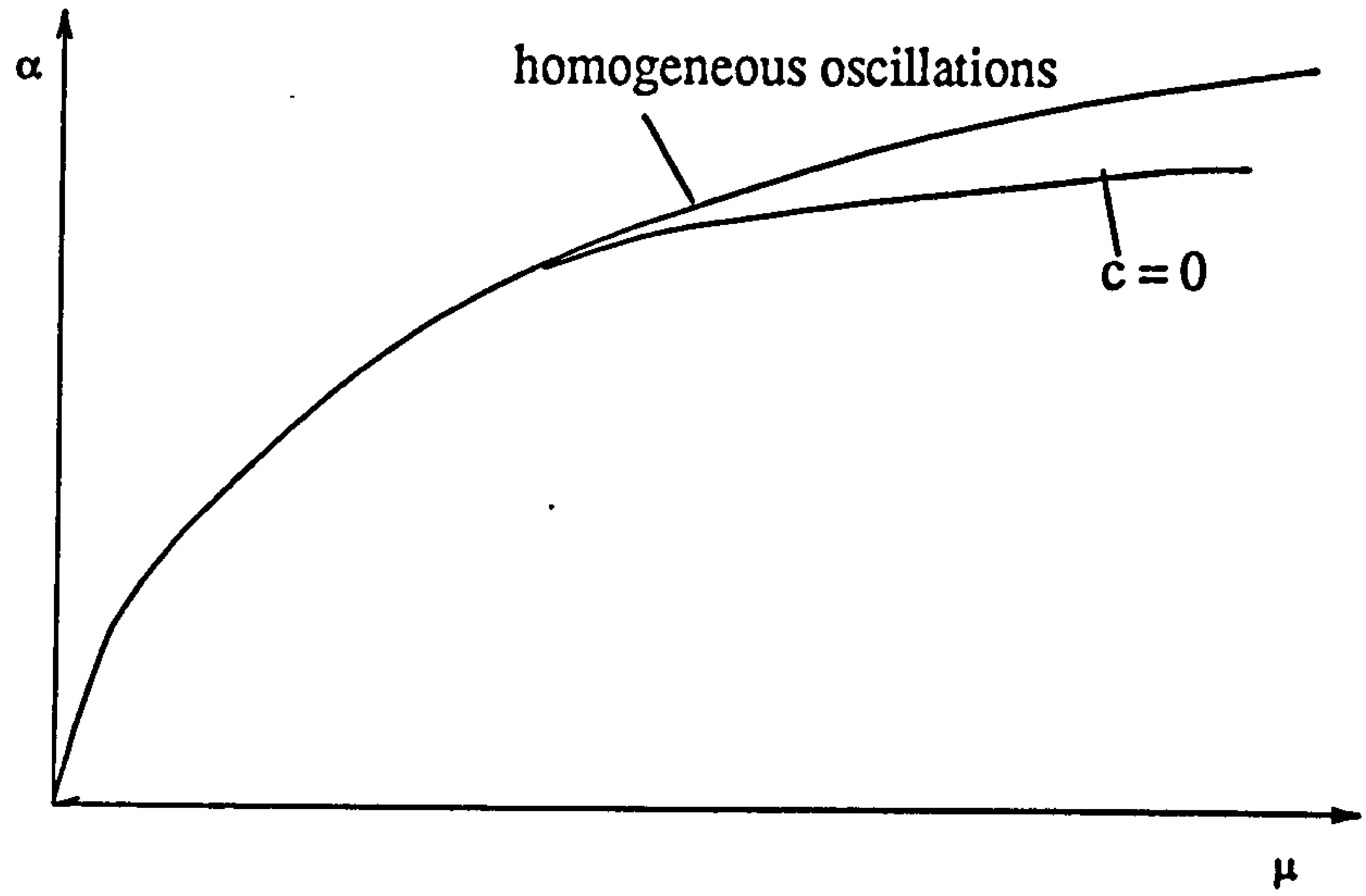


Figure 2.8: Turing patterns in the CGL

So the two graphs meet if $sign(\mu_i\nu) = 1$. If we assume $sign(\mu_i\beta) < 0$ then the graphs will be as portrayed in figure [2.8]. Note that between the two stability curves we would expect these patterns to be stable. If $\mu_i = 0$ then small amplitude periodic spatial patterns branch out from the origin. This seems to contradict the results of section (1.6) except in that case we had scaled μ_i to equal one and in so doing assumed that it was not equal to zero.

Chapter 3

Simulations

3.1 Mappings

The usual way to look at the behaviour of a dynamical system on a computer is by using a mapping. We now discuss some definitions and results which we needed to do computer simulations of systems that have appeared in earlier chapters.

We start with a mapping that describes a local reaction

$$u_{t+1} = f(u_t)$$

where $f : \mathrm{R}^n \to \mathrm{R}^n$. We use periodic boundary conditions in all of our numerical work, so we may think of the spatial domain as being S^1. Let $\mathcal{F}(S^1, \mathrm{R}^n)$ be the set of C^∞ functions from S^1 to R^n, then f lifts to an operator on $\mathcal{F}$ by

$$u_t(x) \mapsto f(u_t(x)).$$

Define $\Delta : \mathcal{F}(S^1, \mathrm{R}^n) \to \mathcal{F}(S^1, \mathrm{R}^n)$ to be the Laplacian acting as

$$\Delta(u_1, \ldots, u_k) = (\Delta u_1, \ldots, \Delta u_k) ,$$

and then $\exp(\delta\Delta) : \mathcal{F}(S^1, \mathrm{R}^n) \to \mathcal{F}(S^1, \mathrm{R}^n)$ is the diffusion operator where δ is a matrix of diffusion coefficients.

Definition 15 *We define a reaction - diffusion mapping on $\mathcal{F}$ to be a map of the form*

$$\Phi = e^{\delta\Delta} \circ f \ .$$

Let $R_\theta : S^1 \to S^1$ be defined by $R_\theta(\phi) = \phi + \theta$. Then R lifts to a map on $\mathcal{F}$ in the obvious way. For a travelling wave we want u such that $u_{t+1}(x) = u_t(R_\theta x)$ for some $\theta \in S^1$ by analogy with the continuous case. In the system above we have

$$u_{t+1}(x) = e^{\delta\Delta} \circ f(u_t(x))$$

which, on substituting $u_{t+1}(x) = u_t(x + \theta)$, gives

$$u_t(x + \theta) = e^{\delta\Delta} \circ f(u_t(x)).$$

If we now let $y = x + \theta$ and $u = u_t$ then

$$\begin{aligned} u(y) &= e^{\delta\Delta} \circ f(u(y - \theta)) \\ &= e^{\delta\Delta} \circ f(R_{-\theta} u(y)). \end{aligned}$$

Therefore, for travelling waves, we are looking for fixed points of the operator $e^{\delta\Delta} \circ f \circ R_{-\theta}$.

3.1.1 Turing Instability

Suppose we have the local two dimensional mapping $u_{t+1} = f(u_t)$, and the Jacobian at some fixed point $\hat{u}$ is given by $A = [a_{ij}]$ for $i, j = 1, 2$. Then if $\gamma = det(A)$ and $\beta = tr(A)$ the system is linearly stable about this point if all three of the following conditions hold:

(a) $\beta - \gamma < 1$

(b) $\beta + \gamma > -1$

(c) $\gamma < 1$.

Our reaction - diffusion system is

$$u_{t+1} = e^{\delta\Delta} \circ f(u_t) \ . \tag{3.1}$$

If we let $w = u - \hat{u}$ then linearising gives

$$w_{t+1} = e^{\delta\Delta} \circ A w_t.$$

Let W be a time independent solution to the spatial eigenvalue problem. ie. $\Delta W + k^2 W = 0$ where k is the eigenvalue. Let W_k be the eigenfunction corresponding to the wave number k. We look for solutions of the form

$$w_t(x) = \sum_k c_k \lambda^t W_k$$

where λ is the eigenvalue which determines growth with respect to time, and the c_k are constants. Substituting this into the linearised equation above gives for each k

$$\lambda^{t+1} W_k = e^{\delta\Delta} \circ A \lambda^t W_k$$

which, on cancelling the λ's, gives

$$\lambda W_k = e^{\delta\Delta} \circ A W_k\ .$$

The eigenspace corresponding to W_k is spanned by W_k times the basis vectors. Therefore both A and $e^{\delta\Delta}$ leave the eigenspace corresponding to W_k invariant. We need to calculate the linear operator A_k such that

$$\lambda W_k = A_k W_k\ .$$

For Turing instability we need $|\lambda| > 1$ for some $k \neq 0$. In other words we need one of the three conditions (a), (b) or (c) not to hold for A_k. For simplicity we now suppose that δ is a diagonal matrix with $\delta_{11} = \delta_1 > 0$ and $\delta_{22} = \delta_2 > 0$ which, after a simple calculation, gives us

$$A_k = \begin{pmatrix} a_{11}e^{-\delta_1 k^2} & a_{12}e^{-\delta_1 k^2} \\ a_{21}e^{-\delta_2 k^2} & a_{22}e^{-\delta_2 k^2} \end{pmatrix}$$

and so

$$\begin{aligned} \gamma' &= \gamma e^{-(\delta_1+\delta_2)k^2} \\ \beta' &= a_{11}e^{-\delta_1 k^2} + a_{22}e^{-\delta_2 k^2}. \end{aligned}$$

Firstly condition (c) is always satisfied since $\gamma' < \gamma < 1$ and so we either need $\beta' - \gamma' > 1$ or $\beta' + \gamma' < -1$. It is easily seen that for diffusion driven instability δ_1 and δ_2 cannot be equal.

Lemma 16 *If, in the notation above, a_{11} or a_{22} are either greater than one or less than minus one then diffusion coefficients δ_1, δ_2 can be chosen such that (3.1) will display Turing instability for some values of k.*

Proof:

Without loss of generality we need only consider a_{11}. Firstly we suppose $a_{11} > 1$. Then, if we choose δ_1 small enough and δ_2 such that $\delta_2 \gg \delta_1$, we have

$$\beta' - \gamma' \simeq e^{-\delta_2 k^2}(a_{22} - a_{11}a_{22} + a_{12}a_{21}) + e^{-\delta_1 k^2} a_{11}.$$

If δ_1 and δ_2 have been chosen correctly then it is easy to see that $\beta' - \gamma' > 1$ for some k. Similarly, if we assume that $a_{11} < -1$, and we choose δ_1 and δ_2 in the same way then

$$\beta' + \gamma' \simeq e^{-\delta_1 k^2} a_{11} + e^{-\delta_2 k^2}(a_{22} + a_{11}a_{22} - a_{12}a_{21}).$$

Again, if δ_1 and δ_2 have been chosen correctly, then $\beta' + \gamma' < -1$ for some k.

3.1.2 Mappings from Continuous Systems

Suppose we have the continuous two dimensional reaction system

$$\begin{aligned} \dot{u} &= uf(u,v) \\ \dot{v} &= vg(u,v) \end{aligned} \tag{3.2}$$

then we consider the mapping

$$\begin{aligned} u_{t+1} &= u_t \exp(\epsilon f(u_t, v_t)) \\ v_{t+1} &= v_t \exp(\epsilon g(u_t, v_t)) \end{aligned} \tag{3.3}$$

where $\epsilon > 0$ is small. We propose that (3.3) displays similar dynamics to (3.2) away from the axes $u = 0$ and $v = 0$. This condition is not as bad as it sounds as we are mainly applying this to biological models where we do not expect either of the species to die out. Clearly, if $(\hat{u}, \hat{v})$ is an equilibrium point for (3.2) then it is also an equilibrium point for (3.3). Also if $(\hat{u}, \hat{v})$ is an equilibrium point for (3.3) and $\hat{u} \neq 0$, $\hat{v} \neq 0$ then it is also an equilibrium point for (3.2).

Lemma 17 *If $(\hat{u}, \hat{v})$ is a coexistence equilibrium point of (3.2), (ie $\hat{u} \neq 0$, $\hat{v} \neq 0$) and if A denotes the Jacobian of (3.2) at this point and B denotes the Jacobian of (3.3) at the same point then if $A = [a_{ij}]$ for $i, j = 1, 2$ then*

$$B = \begin{pmatrix} 1 + \epsilon a_{11} & \epsilon a_{12} \\ \epsilon a_{21} & 1 + \epsilon a_{22} \end{pmatrix} .$$

Proof:

$$\begin{aligned} A(\hat{u}, \hat{v}) &= \begin{pmatrix} u f_u(\hat{u}, \hat{v}) & u f_v(\hat{u}, \hat{v}) \\ v g_u(\hat{u}, \hat{v}) & v g_v(\hat{u}, \hat{v}) \end{pmatrix} \\ B(\hat{u}, \hat{v}) &= \begin{pmatrix} 1 + \epsilon \hat{u} f_u(\hat{u}, \hat{v}) & \epsilon \hat{u} f_v(\hat{u}, \hat{v}) \\ \epsilon \hat{v} g_u(\hat{u}, \hat{v}) & 1 + \epsilon \hat{v} g_v(\hat{u}, \hat{v}) \end{pmatrix} . \end{aligned}$$

Lemma 18 *Suppose we have a coexistence equilibrium point $(\hat{u}, \hat{v})$. Then, $\exists \epsilon_* > 0$ such that if $\epsilon < \epsilon_*$ then $(\hat{u}, \hat{v})$ is linearly stable in (3.2) if and only if it is linearly stable in (3.3). Furthermore if $\epsilon < \epsilon_*$ then (3.2) exhibits diffusion driven instability at this point if and only if (3.3) also displays diffusion driven instability.*

Proof:

We will use the notation of the previous lemma and note that γ and β refer to B. Since $(\hat{u}, \hat{v})$ is stable in (3.2) then $det(A) > 0$ and $tr(A) < 0$. Now

$$\gamma = 1 + \epsilon tr(A) + \epsilon^2 det(A),$$

$$\beta = 2 + \epsilon tr(A)$$

and so we need to check conditions (a), (b) and (c) detailed above. Firstly

$$\beta - \gamma = 1 - \epsilon^2 det(A) < 1$$

which is true for all ϵ. Secondly

$$\beta + \gamma = 3 + \epsilon tr(A) + \epsilon^2 det(A)$$

which can obviously be made to be greater than -1 for $\epsilon < \epsilon_1$ for some $\epsilon_1 > 0$. Lastly

$$\gamma = 1 + \epsilon tr(A) + \epsilon^2 det(A)$$

and $\gamma < 1$ for $\epsilon < \epsilon_2$ where $\epsilon_2 = -tr(A)/det(A)$. Now let $\epsilon^* = min(\epsilon_1, \epsilon_2)$. We omit the only if part of the proof as it is just as easy.

It can be shown that if (3.2) displays diffusion driven instability then without loss of generality A is of the form:

$$A(\hat{u}, \hat{v}) = \begin{pmatrix} + & + \\ - & - \end{pmatrix} \; or \; \begin{pmatrix} + & - \\ + & - \end{pmatrix}$$

and since we know that

$$B = \begin{pmatrix} 1 + \epsilon a_{11} & \epsilon a_{12} \\ \epsilon a_{21} & 1 + \epsilon a_{22} \end{pmatrix}$$

then by our earlier analysis (3.3) exhibits diffusion driven instability since $b_{11} > 1$.

If we now assume that (3.3) has a Turing instability at $(\hat{u}, \hat{v})$ then, since ϵ is small we may assume that either b_{11} or b_{22} is greater than one. Without loss of generality we assume that $b_{11} > 1$. This implies that $a_{11} > 0$. Now we know that $tr(A) < 0$, therefore $a_{22} < 0$. This along with the fact that $det(A) > 0$ gives us that a_{12} and a_{21} must have opposite signs. Hence A is of the form:

$$A(\hat{u}, \hat{v}) = \begin{pmatrix} + & + \\ - & - \end{pmatrix} \; or \; \begin{pmatrix} + & - \\ + & - \end{pmatrix}$$

and so (3.2) displays Turing instability.

Diffusion

In order to implement the diffusion operator numerically we discretise S^1 and use

$$u_{t+1} = e^{d\Delta u_t}$$

where Δu_t is an average over neighbours, and d is a scaled version of the diffusion matrix. In practice we use

$$u_{t+1} = (1 + \frac{d}{M}\Delta)^M u_t$$

where, as $M \to \infty$ we approach the exponential above. In our simulations a typical value for M is three.

3.2 Examples

We now use the results above in some specific examples.

3.2.1 Predator-Prey Model

Consider the following reaction diffusion system

$$\dot{u} = u(1-u) - \frac{auv}{u+b} + d_1 u_{xx}$$

$$\dot{v} = rv(1 - \frac{v}{u}) + d_2 v_{xx}.$$

This is a predator - prey system where the prey (u) has logistic growth in the absence of predators and the predator (v) response is of type II [Murray]. The equilibrium points of the system are $(1, 0)$ and $(\hat{u}, \hat{v})$ where $\hat{u} = \hat{v}$ is given by solving the equation

$$\frac{a\hat{u}}{\hat{u}+b} = 1 - \hat{u}.$$

This system exhibits Turing instability when $(\hat{u}, \hat{v})$ is stable locally and also goes through a Hopf bifurcation by decreasing r. When r is very small there exists relaxation oscillations.

The corresponding mapping has the form

$$\begin{aligned} u_{t+1} &= e^{\delta_1 \Delta} \circ \exp \epsilon(1 - u_t - \tfrac{a v_t}{u_t + b}) \\ v_{t+1} &= e^{\delta_2 \Delta} \circ \exp \epsilon r(1 - \tfrac{v}{u}) \end{aligned} .$$

Linearising the reaction part gives us

$$B(\hat{u}, \hat{v}) = \begin{pmatrix} 1 + \epsilon \hat{u}(\frac{a\hat{u}}{(\hat{u}+b)^2} - 1) & -\epsilon(1 - \hat{u}) \\ \epsilon r & 1 - \epsilon r \end{pmatrix}.$$

For Turing instability we need

$$\frac{a\hat{u}}{(\hat{u} + b)^2} > 1$$

or alternatively

$$\hat{u} < \frac{1}{2}(1 - b).$$

We also need d_1 and d_2 chosen as outlined in section (3.1.1) (ie. $d_2 > d_1$). These patterns can be seen in figure [3.1]. The height and the number of waves may be changed by altering the parameters. As r is decreased and the systems goes through a Hopf bifurcation the pattern starts to oscillate and then degenerates to a homogeneous standing wave. Although travelling waves do exist near the Hopf bifurcation point (see Chapter 2) they were not observed on the computer.

We note at this point that $(\hat{u}, \hat{v})$ is stable if

$$r > \frac{\hat{u}(1 - b - 2\hat{u})}{\hat{u} + b}$$

as long as ϵ is small enough. For a particular value of b we get the bifurcation diagram given in figure [3.2].

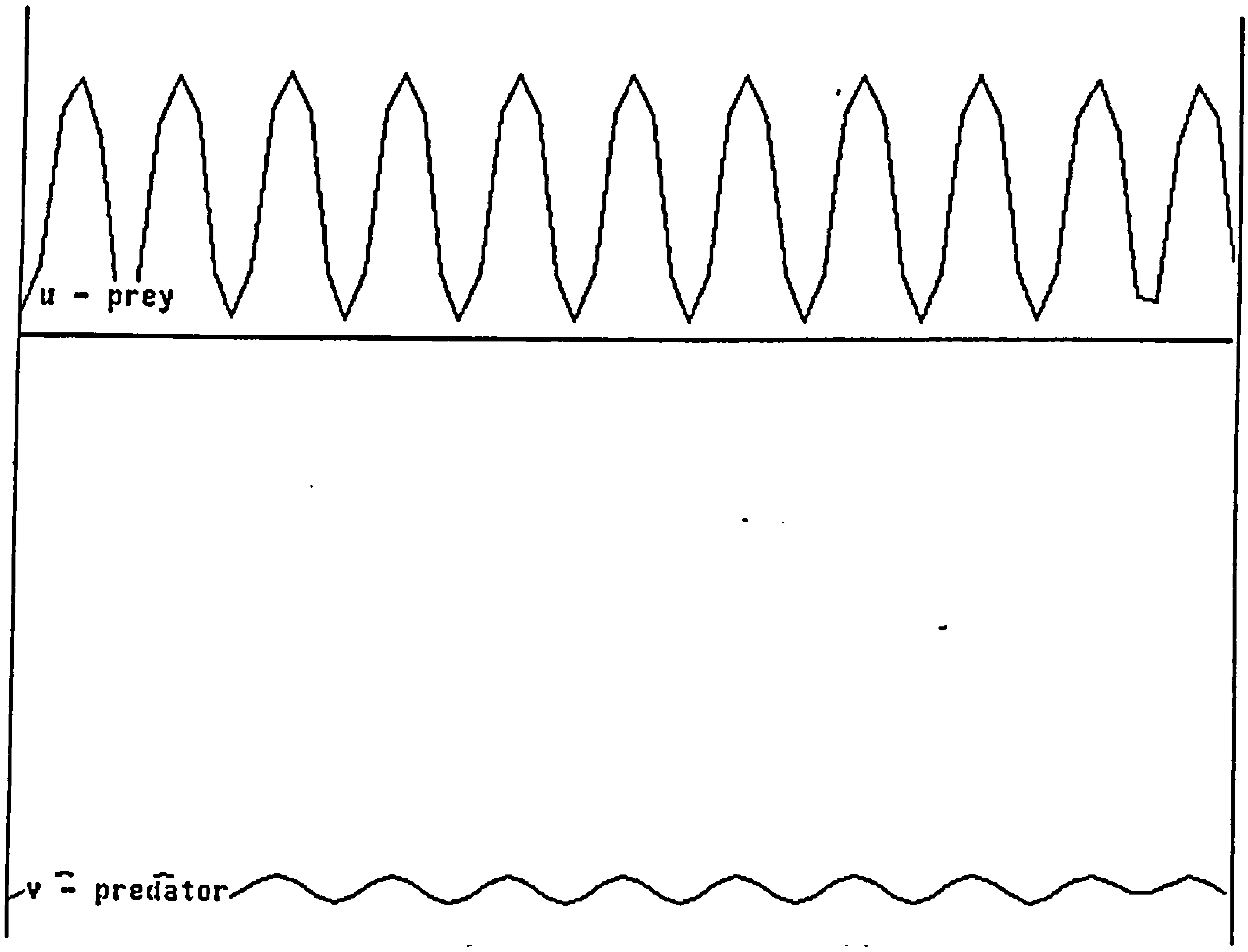


Figure 3.1: Turing patterns for predator - prey model

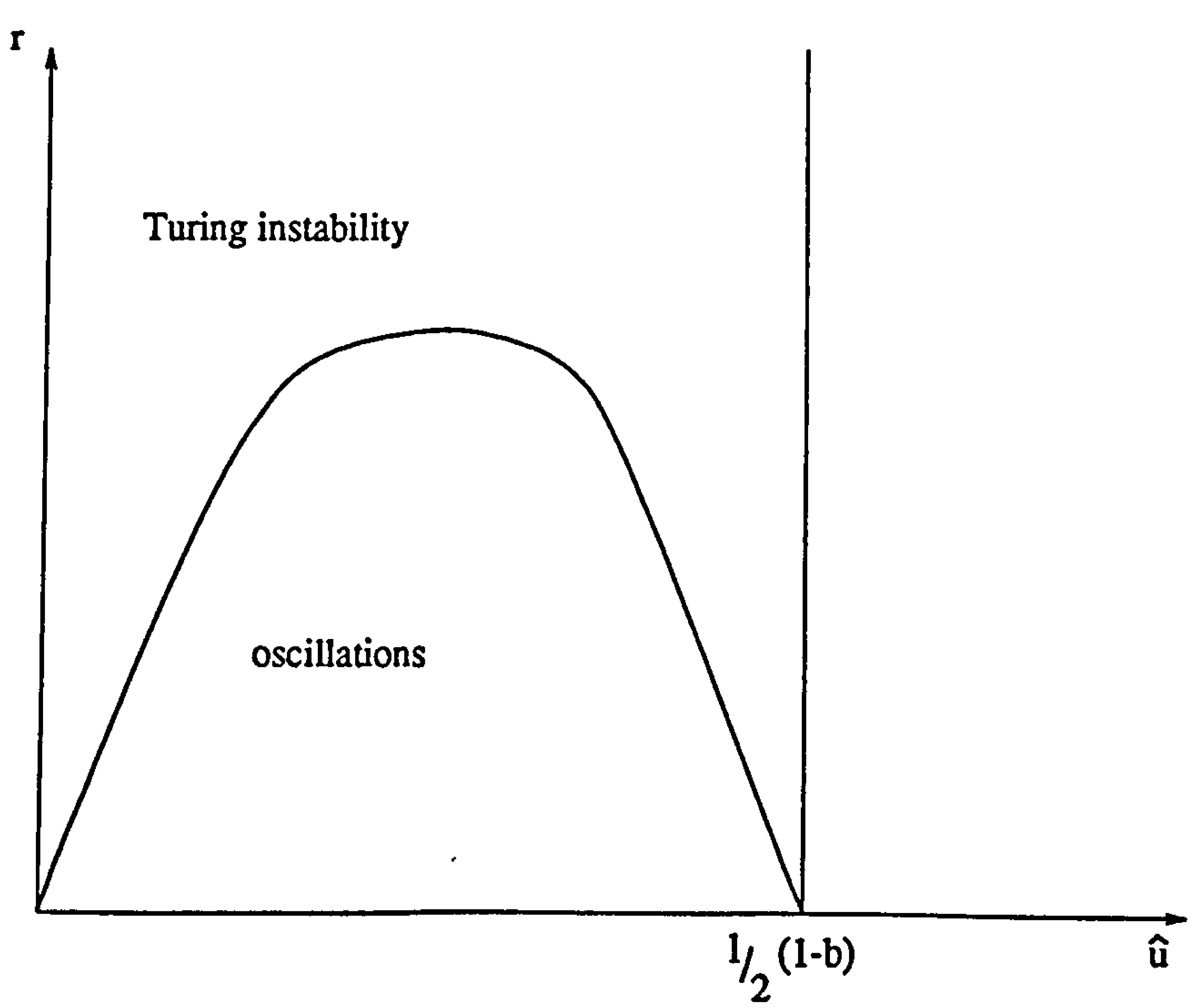


Figure 3.2: Bifurcation diagram when b is constant

3.2.2 Complex Ginzburg Landau Equation

The Complex Ginzburg-Landau partial differential equation appears in many interesting dynamical systems. It describes a system close to a global Hopf bifurcation. For other simulations work see for example [Doering]. We start with the equation

$$z_t = \mu z - (1 + i\alpha)|z|^2 z + (1 + i\beta) z_{xx} \tag{3.4}$$

where $z \in \mathbf{C}$ and μ, α, $\beta \in \mathbf{R}$. When the CGL goes through the Hopf bifurcation it oscillates about zero, and so we do not use the mapping described in section (3.1), but for the computer simulation we use the mapping described in [Rand]. We split the mapping into two parts: the local dynamics and the diffusion. If we look at (3.4) without the last term it can be written in (r, θ) form as

$$\begin{aligned} r' &= \mu r - r^3 \\ \theta' &= -\alpha r^2. \end{aligned}$$

We integrate the first of these for a time interval τ to give

$$r(t+\tau) = \frac{\sqrt{\mu} r(t)}{\sqrt{\lambda\mu + (1 - \lambda r(t)^2)}}$$

where $\lambda = e^{-2\mu\tau}$. Using this we can derive the local dynamics which are given by

$$z_{n+1} = F(z_n) = \frac{\sqrt{\mu}}{\sqrt{\mu\lambda + (1-\lambda)|z_n|^2}} e^{-i\tau\alpha|z_n|^2 z_n}.$$

To add the diffusion we have as before

$$z_{n+1} = F(e^{\tau_0(1+i\beta)\triangle z_n})$$

where $\triangle z_n$ is an average over neighbours. Again, in practice we use

$$\hat{z}_n = (1 + \frac{\tau_0}{M}(1 + i\beta)\triangle)^M z_n$$

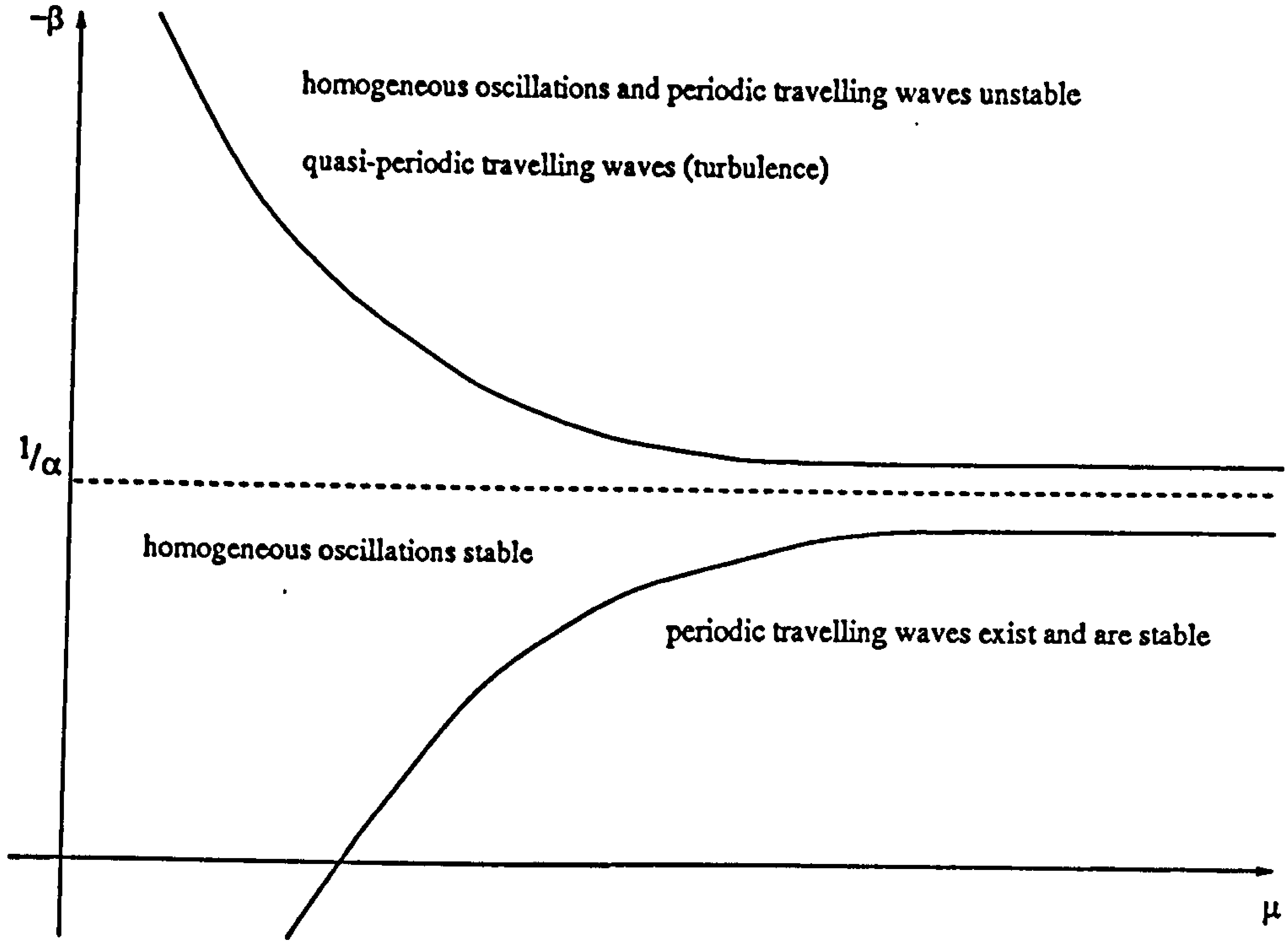


Figure 3.3: Bifurcation diagram for the CGL mapping

where, as $M \to \infty$ we approach the exponential above. This full mapping has properties which closely resemble equation (3.4) for τ small.

The homogeneous solution for equation (3.4) is linearly stable if $1+\alpha\beta > 0$. The equivalent condition for the mapping is

$$1 + 2\frac{\mu}{1-\lambda}\tau\alpha\beta > 0.$$

This condition tends to the one for the P.D.E. as $\tau \to 0$.

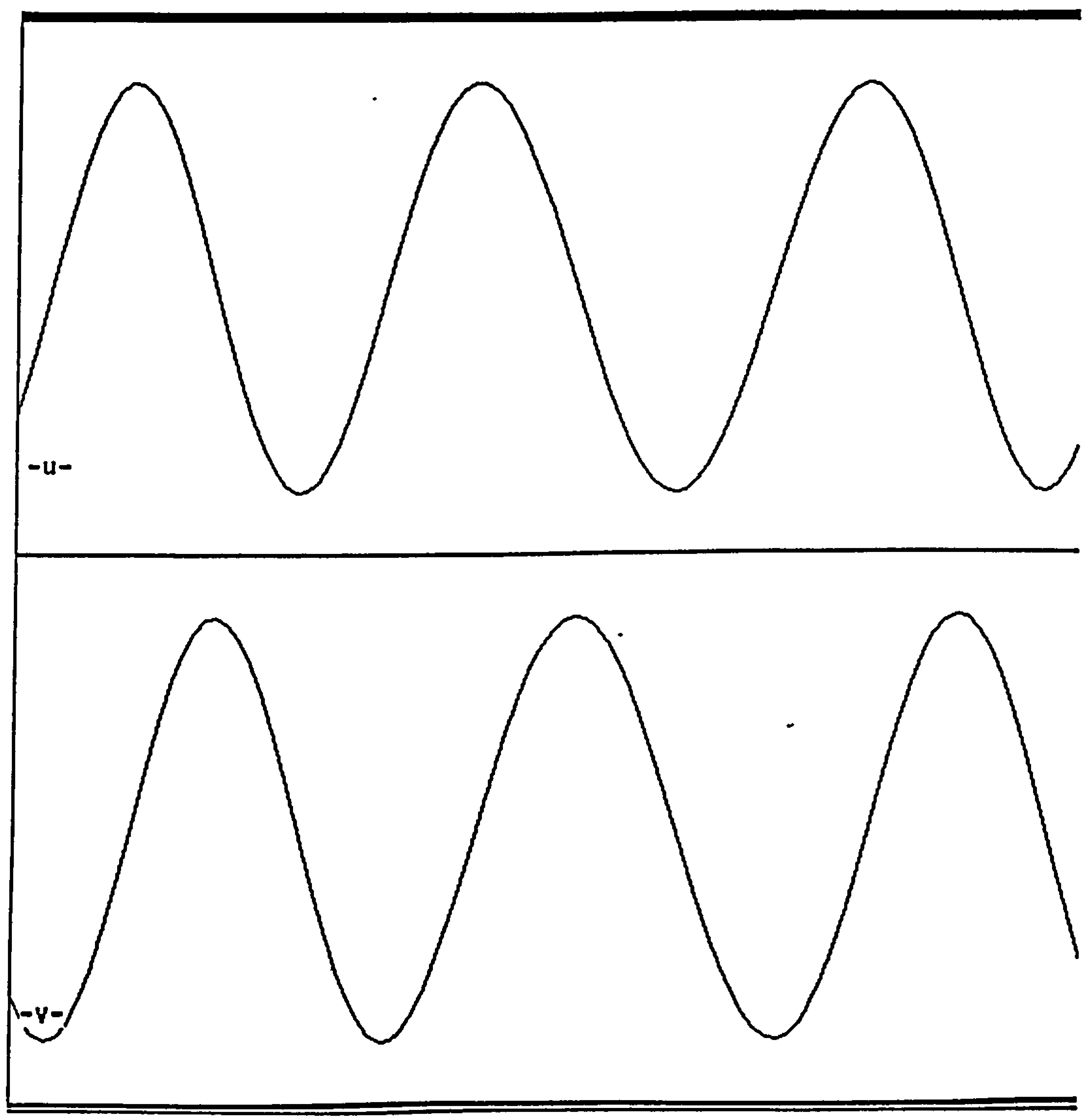
If we fix $\alpha > 0$ and assume $\beta < 0$ then computer simulations suggest the bifurcation diagram given in figure [3.3]. On the computer we split the mapping into its real and imaginary parts. Let $u = Re(z)$ and $v = Im(z)$ then the local dynamics of u and v are given by

$$\begin{aligned} u_{t+1} &= f(u_t^2 + v_t^2)[\cos(\tau\alpha(u_t^2 + v_t^2))u_t + \sin(\tau\alpha(u_t^2 + v_t^2))v_t] \\ v_{t+1} &= f(u_t^2 + v_t^2)[\cos(\tau\alpha(u_t^2 + v_t^2))v_t - \sin(\tau\alpha(u_t^2 + v_t^2))u_t] \end{aligned}$$

where

$$f(x) = \frac{\mu}{\sqrt{\mu\lambda + (1-\lambda)x}}$$

and the diffusion is added as before. Examples of the periodic travelling waves exhibited by the mapping are given in figure [3.4]. Although there is a band of stable travelling waves for any particular value of μ, using periodic boundary conditions means that only a finite number of these will be seen in practice.



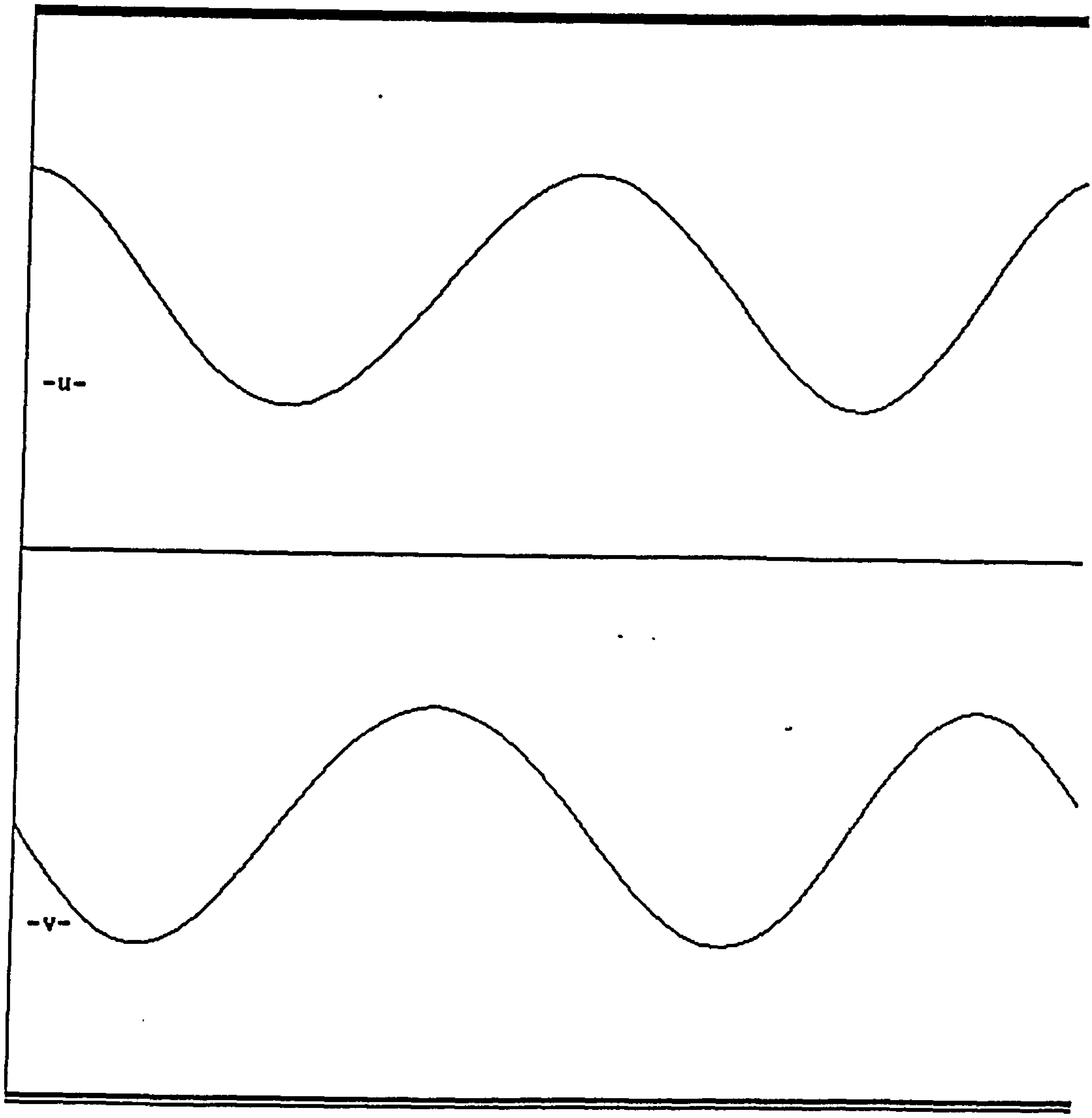


Figure 3.4: Travelling waves of the CGL mapping

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