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Inventory Performance of the Damped Trend Forecasting Method

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Abstract

The Damped Trend (DT) forecasting method has been recognized for its superior accuracy. Li et al., (2014) show when DT forecasts are used within the order-up-to (OUT) policy, the bullwhip effect is avoided by using unconventional DT parameter settings. We extend this study in three directions. First, by investigating the relationship between the stability and invertibility, we show that stable DT parameter sets produce feasible forecasts. This further justifies the choice of unconventional DT parameter values. Second, we extend the bullwhip analysis from a lead-time of one to a general lead-time, identifying a stable and invertible region in the parameter space that possesses enviable bullwhip avoidance behavior. Third, we characterize the frequency response of the inventory levels maintained by the OUT policy with DT forecasts. For independently and identically distributed (i.i.d.) demand the net stock amplification ratio can be close to (but never smaller than) the lead-time and review period, with the intriguing benefit of bullwhip avoidance. For other demand patterns, the net stock amplification can be less than the i.i.d. lower bound. The bullwhip can also be reduced at the same time. Simulations of 62 sets of real demand time series verify our analytical results.

Keywords: Damped Trend Forecasting; Order-up-to; Inventory; Bullwhip; Invertibility; Frequency Response.

1. Introduction

Since damped trend (DT) forecasting method was proposed by Gardner and McKenzie (1985), its superior forecasting accuracy has been recognized in many empirical studies (Makridakis and Hibon, 2000; Gardner and Diaz-Saiz, 2008) and academic reviews (Armstrong, 2006; Fildes et al., 2008; Gardner and McKenzie, 2011). The economic benefits of DT, in terms of inventory, production, and shipping costs from real applications were reported in Gardner (1990) and Acar and Gardner (2012).

The bullwhip effect is an important phenomenon in supply chains; it describes how demand fluctuations are amplified echelon-to-echelon in a supply chain. It creates the undesirable consequences of excessive capacity investment, inefficient transport use, labor idling and over-time, and increased safety stock requirements. Li et al., (2014) study the DT method focusing on the supply chain dynamics induced by the order-up-to (OUT) replenishment policy with DT forecasting (OUT-DT), discovering unique bullwhip avoidance behavior under unit lead-times.

Although Li et al., (2014) found the DT method allows one to avoid the bullwhip effect, others report conflicting results. An empirical analysis of 10 forecasting methods based on the monthly unit sales of Chevrolet automobiles from 1991-2008 was conducted by Chiang et al., (2016). Here, the DT method generated bullwhip. Wang et al., (2017) also used empirical data and simulation to quantify the bullwhip effect and inventory variance produced by the OUT policy with various forecasting methods. They report that the DT method is not always superior to other simple forecasting methods, e.g., exponential smoothing and moving average.

The apparent contradiction between these results can be explained by the optimization objective and the allowable parameter space used. Li et al., (2014) optimized the forecasting parameters to minimize a utility (the combination of bullwhip and inventory variance), whereas Chiang et al., (2017) and Wang et al., (2017) minimized the one-period-ahead mean squared error. Furthermore, Li et al., (2014) used the complete stability region of forecasting parameters, whereas Chiang et al., (2017) and Wang et al., (2017) selected only parameters values in the $[0,1]$ range. Herein we show, for the DT forecasting, the invertible and stable regions are identical, indicating all stable DT forecasting model produces feasible forecasts. This allows us to study the complete DT parameter space.

We also investigate another important supply chain metric, net stock amplification (*NSAmp*). *NSAmp* is measured by the variance of inventory levels divided by the demand variance. It is related to the popular safety stock and fill-rate concepts (Zipkin, 2000; Disney et al., 2015). Both the bullwhip effect and *NSAmp* have been adopted to investigate the cost and supply chain dynamics (Costantino et al., 2016; Khosroshahi et al., 2016; Lin et al., 2016).

The paper is organized as follows: §2 introduces the model setup, §3 discusses DT invertibility, §4 investigates the *NSAmp* maintained by the OUT-DT system, §5 presents extensive simulation results, and §6 concludes.

2. Model Setup

The single source of error correction form of the DT forecasting method is given by

$$A_t = A_{t-1} + \phi B_{t-1} + \alpha\varepsilon_t,\ B_t = \phi B_{t-1} + \alpha\beta\varepsilon_t,\ \text{and}\ \hat{D}_{t+k|t} = A_t + \sum_{i=1}^{k}\phi^i B_t = A_t + \frac{\phi\left(1-\phi^k\right)}{1-\phi}B_t. \quad (1)$$

where $\varepsilon_t = \hat{D}_{t|t-1} - D_t$ is one step ahead forecast error, $\hat{D}_{t+k|t}$ is the forecast of the demand k periods ahead, D_{t+k}, made at time t. $\hat{D}_{t+k|t}$ is the sum of a level, A_t, and a trend, B_t, component. ϕ determines the shape of the future forecasts. $\{\alpha,\beta,\phi\}$ are the DT system parameters. Li et al., (2014) derived the following z-transfer functions of (1),

$$\begin{aligned}
\frac{A(z)}{\varepsilon(z)} &= \frac{z^2\alpha + z\alpha\phi(\beta-1)}{z^2 + z(\alpha-\phi-1+\alpha\beta\phi)+\phi(1-\alpha)},\\
\frac{B(z)}{\varepsilon(z)} &= \frac{z^2\alpha\beta - z\alpha\beta}{z^2 + z(\alpha-\phi-1+\alpha\beta\phi)+\phi(1-\alpha)},\\
\frac{\hat{D}_k(z)}{\varepsilon(z)} &= \frac{z^2\alpha\left(1+\beta\sum_{i=1}^{k}\phi^i\right) + z\alpha\left(\phi(\beta-1)-\beta\sum_{i=1}^{k}\phi^i\right)}{z^2+z(\alpha-1-\phi+\alpha\beta\phi)+\phi(1-\alpha)},
\end{aligned} \quad (2)$$

and showed when $\phi \neq 0$ that the following relations must be satisfied for stability

$$\begin{cases}\phi-1<\alpha\phi<\phi+1,\\ \alpha(\phi-1)<\alpha\beta\phi<(2-\alpha)(\phi+1);\end{cases} \quad (3)$$

when $\phi = 0$, $0<\alpha<2$ is required for stability.

The OUT replenishment policy is frequently applied in industry, especially in high volume settings (Cannella et al., 2017; Li and Disney 2017). We follow the same OUT-DT model and assumptions used in Li et al., (2014). In each period t, the manufacturer receives the raw materials, converts them into the finished products (with a replenishment lead-time of $T_p \in \mathbb{N}^0$

periods), and satisfies demand D_t from its on-hand inventory. The manufacturer sets his production targets/replenishment orders via

$$O_t = TNS + \hat{D}_{t+T_p+1|t} + \sum_{i=1}^{T_p}\left(\hat{D}_{t+i|t} - O_{t-i}\right) - NS_t. \tag{4}$$

Here, $\sum_{i=1}^{T_p} O_{t-i}$ is the work-in-progress (WIP). The time-varying target work-in-progress, $\sum_{i=1}^{T_p} \hat{D}_{t+i|t}$, is the sum of the demand forecasts made at time t for the periods from $t+1$ to $t+T_p$. *TNS* is a safety stock used to ensure a strategic level of inventory availability. The *TNS* can be set with

$$TNS^* = \sigma_{NS} z;\ z = F^{-1}\left[p_1\right]. \tag{5}$$

Here $F^{-1}\left[p_1\right]$ is the inverse of the c.d.f. of standard normal distribution evaluated at the target availability p_1 (Hosoda and Disney, 2009). The inventory balance equation,

$$NS_t = NS_{t-1} + O_{t-T_p-1} - D_t, \tag{6}$$

completes the model specification.

In order to preserve linearity of the system and allow for a tractable analysis, the following assumptions are made. Negative demand quantities indicate that customers are free to return products (or that the probability of negative demand is negligible). Negative orders are also permissible indicating that finished goods are disassembled into raw materials (or sent back to suppliers). There are no capacity constraints in the system. Unmet demand is backlogged.

We will later study the OUT-DT with frequency analysis. For this we will need the transfer function of the replenishment orders and the inventory levels:

$$\frac{O(z)}{\varepsilon(z)} = 1 + \frac{(z-1)\alpha\left(\left(1+T_p\right)\left(z+\phi(\beta-1)\right) + \beta(z-1)\left(\sum_{i=1}^{T_p+1}\phi^i + \sum_{j=1}^{T_p}\sum_{i=1}^{j}\phi^i\right)\right)}{z^2 + z\left(\alpha-1-\phi+\alpha\beta\phi\right) + \phi(1-\alpha)}, \tag{7}$$

$$\frac{NS(z)}{\varepsilon(z)} = \frac{\alpha\left(\left(1+T_p\right)\left(z+\phi(\beta-1)\right) + \beta(z-1)\left(\sum_{i=1}^{T_p+1}\phi^i + \sum_{j=1}^{T_p}\sum_{i=1}^{j}\phi^i\right)\right)}{z^{T_p+1}\left(z^2 + z\left(\alpha-1-\phi+\alpha\beta\phi\right) + \phi(1-\alpha)\right)} - \frac{\sum_{i=0}^{T_p} z^i}{z^{T_p+1}}. \tag{8}$$

We have omitted the details of how one arrives at (7) and (8) for brevity. We refer to a good textbook such as Nise (2013) for a basic introduction to discrete control theory and Li et al., (2014) for more information on the transfer function functions of the OUT-DT policy.

3. Invertibility

The concept of invertibility refers to the feasibility of the identification of the demand process structure from past demand observations. Invertibility is related to linear moving average (MA) models or the MA part of autoregressive integrated moving average (ARIMA) models (Box et al., 2008). If an MA model (or an MA part in ARIMA models) can be expressed as an autoregressive (AR) model of infinite order, the model is deemed invertible and implies all relevant state variables are directly observable.

All linear exponential smoothing methods have equivalent ARIMA models (Box et al., 2008). The DT method is equivalent to the ARIMA(1,1,2) model at it can be written as

$$\left(1-B\right)\left(1-\phi B\right)d_t=\left(1-\theta_1 B-\theta_2 B^2\right)\varepsilon_t, \tag{9}$$

where B is backward shift operator (similar to, but not quite the same as z^{-1} in discrete control theory), $\theta_1=1+\phi-\alpha-\alpha\beta\phi$ and $\theta_2=\alpha\phi-\phi$ (Roberts, 1982). For the second-order MA part, it is invertible only if the roots of the characteristic equation

$$1-\theta_1 B-\theta_2 B^2=0 \tag{10}$$

lie *outside* the unit circle. From a control theory perspective, stability implies a system will return to a steady state in finite time when reacting to a finite input. An unstable system, on the other hand, will diverge exponentially or oscillate with ever increasing amplitude. In fact, the denominator of the DT forecasting transfer function (2) can be rewritten using θ_1 and θ_2 as $z^2-\theta_1 z-\theta_2$. The stability condition in control theory requires the roots of

$$z^2-\theta_1 z-\theta_2=0 \tag{11}$$

lie *inside* the unit circle. Notice, the roots of (10) are the inverse of the roots of (11), and the backward difference operator, *B*, is the inverse of the discrete delay operator *z*. The test for stability requires the roots of (11) to be inside the unit circle; invertibility requires the roots of (10) to be outside the unit circle. So the results for invertible/stable regions and stable regions are identical. Solving for the roots of (10) to find the invertibility criteria, we obtain

$$H_{\{1,2\}}=\left(\theta_1\pm\sqrt{\theta_1^2+4\theta_2}\right)\Big/2. \tag{12}$$

There are two possible pairs of values. Solving the simultaneous inequalities $\left|H_i\right|<1$ for $i=\{1,2\}$, both pairs give rise to identical calculations. The results are the same as our stability conditions in (3) when $\{\alpha,\beta,\phi\}\in\mathbb{R}$.

After investigating the relationship between DT stability and invertibility, we find the invertible regions are the same as the stable regions (though the implications of stability and invertibility are different). The consistency between the stable and invertible regions is important as we can be certain that stable DT parameter sets produce feasible forecasts. This allows us to investigate the DT performance over the complete stability region.

4. Net Stock Amplification with Bullwhip Avoidance Parameters

4.1 Desirable Parameter Values

Traditionally $\{\alpha,\beta,\phi\}$ is restricted to the $[0,1]$ interval. However, in the previous section, we found the DT forecasting is invertible and stable over a wider range of parameter values. In addition, Li et al., (2014) showed there exists a space in the DT parameter plane where the OUT-DT policy does not create bullwhip; that is, $\sigma_O^2<\sigma_D^2$. This bullwhip avoiding behavior only occurs with some unconventional $\{\alpha,\beta,\phi\}$ parameter settings. We extended their bullwhip analysis from $T_p = 1$ to a general lead-time (details omitted for brevity), and find four types of behavior[1]:

- The bullwhip avoiding areas in the parametrical plane when $\phi < -1$ disappear and reappear in sophisticated manners when the lead-time changes parity.
- The shapes of bullwhip avoidance region within the parametrical plane when $-1 < \phi < 0$ are different depending on the parity of the lead-time.
- There exists a region within the parametrical plane when $\phi > 1$ that also enables the OUT-DT system to avoid the bullwhip effect.
- When

$$0 < \phi < 1, \tag{13}$$

$$(\phi - 1)/\phi < \alpha < 0\text{, and} \tag{14}$$

$$\beta_{\min} \le \beta \le (\phi - 1)\phi^{-1};\ \beta_{\min} = \frac{-(1+T_p)(1+\phi)(1-\phi)^2}{\phi\left(1+T_p\left(1-\phi^2\right)+\phi\left(2\phi^{1+T_p}-\phi-2\right)\right)}, \tag{15}$$

  the OUT-DT policy is able to avoid creating bullwhip (see the dark area in Figure 1). When the constraints in (13), (14), and (15) are satisfied, we say the parameter set is a member of the bullwhip avoidance (*BA*) area, $\{\alpha, \beta, \phi\} \in BA$. The lower bound, $\beta_{\min}$ in (15), is increasing in the lead-time T_p.

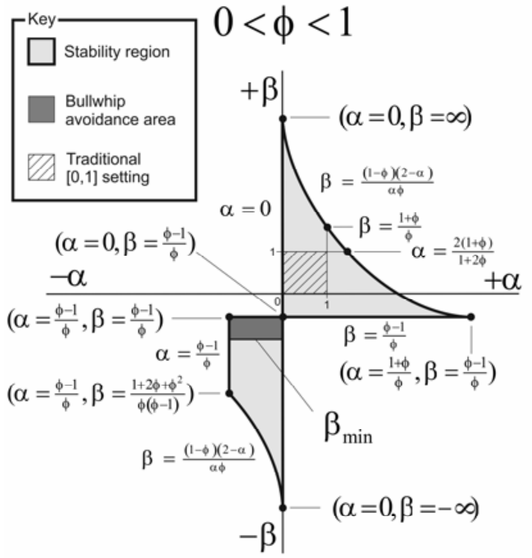


Figure 1. Selected parameter region within the stability boundary

The influence of the lead-time on bullwhip when $\phi > 0$ is less dramatic than when $\phi < 0$ making it a good candidate for general applications. The parametrical plane $0 < \phi < 1$ enables the amplification of bullwhip for only a very few frequencies, outperforming the case when $\phi > 1$ (Li et al., 2014). Therefore, we focus on the performance when the value of DT parameters are selected from the bullwhip avoidance area when $0 < \phi < 1$.

4.2 The Case of i.i.d. Demand

This section studies the performance of the OUT-DT system when i.i.d. demand is present. A

[1] We refer to Li et al., (2014) for more information on the first three areas of the bulleted list; herein we focus on the most desirable area described by (13), (14), and (15).

general demand case will be investigated in the next section. We first consider the case when $T_p = 0$ before moving on the alter lead-times. When $T_p = 0$, the boundaries of the *BA* region can be written as

$$0 < \phi < 1,\ (\phi - 1)\phi^{-1} < \alpha < 0,\ \text{and}\ (-\phi - 1)\phi^{-1} \le \beta \le (\phi - 1)\phi^{-1}. \tag{16}$$

The *Bullwhip* ratio can be expressed as

$$Bullwhip = \frac{\sigma_O^2}{\sigma_D^2} = \frac{2 - 2\phi^2 + \alpha^2\phi(1 + \phi + \beta\phi)(3 + 2\beta\phi) + \alpha\left(3 + \phi\left(2 - \phi + \beta\left(3 + \phi - 4\phi^2\right)\right)\right)}{\left(1 - (1 - \alpha)\phi\right)\left(2 - \alpha + (2 - \alpha - \alpha\beta)\phi\right)}. \tag{17}$$

Net stock amplification (*NSAmp*) is

$$NSAmp = \frac{\sigma_{NS}^2}{\sigma_D^2} = 1 + \frac{\alpha\left(1 + (\alpha + \beta)\phi - (1 - \alpha - \beta - \alpha\beta)\phi^2\right)}{\left(1 - (1 - \alpha)\phi\right)\left(2 - \alpha + (2 - \alpha - \alpha\beta)\phi\right)}. \tag{18}$$

The bullwhip ratio (17) has no stationary points within the stability region. Any local minima and maxima must exist on the boundaries and/or at non-differentiable points. Eq. (17) is always differentiable within (16). Taking each boundary into consideration, we find[2]:

- when $\beta \uparrow (\phi - 1)\phi^{-1}$ and $\alpha \uparrow 0$, $Bullwhip \to 1$,
- when $\beta \uparrow (\phi - 1)\phi^{-1}$ and $\alpha \downarrow (\phi - 1)\phi^{-1}$, $Bullwhip \to \phi^2$, which is between 0 and 1 when $\phi \in (0,1)$,
- when $\beta \downarrow (-\phi - 1)\phi^{-1}$ and $\alpha \uparrow 0$, $Bullwhip \to 1$,
- when $\beta \downarrow (-\phi - 1)\phi^{-1}$ and $\alpha \downarrow (\phi - 1)\phi^{-1}$, $Bullwhip \to \infty$.

It is easy to conclude that $Bullwhip \to 0$ when $\beta \uparrow (\phi - 1)\phi^{-1}$, $\alpha \downarrow (\phi - 1)\phi^{-1}$, and $\phi \downarrow 0$. Therefore, when capacity costs dominate, a large negative α, a small negative β, and a ϕ close to zero should be used.

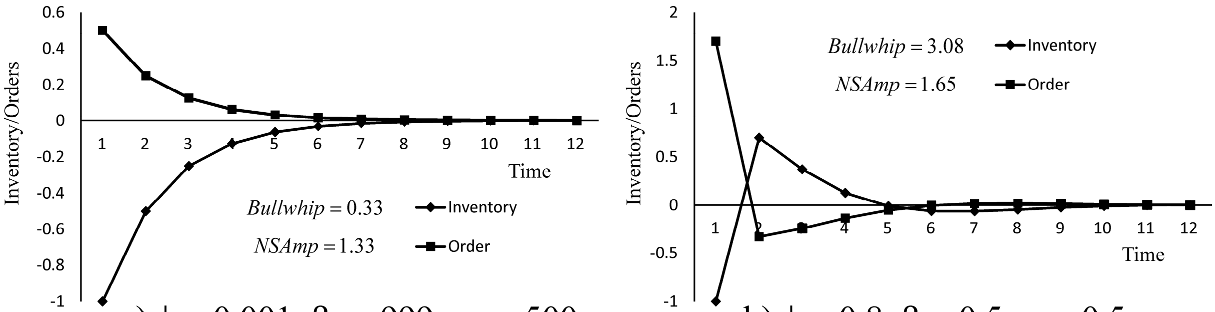
Following the same procedure we find that the minimal *NSAmp* = 1 when $\alpha \uparrow 0$. For other lead-times, $Bullwhip = 0$ still exists at $\beta \uparrow (\phi - 1)\phi^{-1}$, $\alpha \downarrow (\phi - 1)\phi^{-1}$ and $\phi \downarrow 0$, and a minimal *NSAmp* of $1 + T_p$ is allowed when $\alpha \uparrow 0$. These results mean when i.i.d. customer demand is present, $NSAmp \ge 1 + T_p$ in the OUT-DT system; when a constant forecast is used *NSAmp* $= 1 + T_p$. This is similar to Disney and Towill's (2003) finding that the OUT policy (with other forecasting methods) always induces an $NSAmp \ge 1 + T_p$ under i.i.d. demand. Further, when the inventory variance is minimized, *Bullwhip* = 1. We conclude that $\alpha \to 0$ in situations where inventory costs are more significant than capacity costs.

$NSAmp \to \infty$ and the bullwhip effect is minimized when $\beta \uparrow (\phi - 1)\phi^{-1}$, $\alpha \downarrow (\phi - 1)\phi^{-1}$, and $\phi \downarrow 0$. A zero-order variance for i.i.d. demand is gained at the cost of net stock amplification, indicating that a trade-off exists between order and inventory variance. When the business

[2] $\uparrow$ means *approach from below*, $\downarrow$ *approach from below*, and $\to$ *tends to*.

objective is to reduce both capacity and inventory cost, a ϕ close to zero and a small negative β are recommended as it guarantees the elimination of the bullwhip effect; then we only need to optimize α based on the balance between inventory and capacity costs.

When $\phi \downarrow 0$, $\beta = (\phi - 1)\phi^{-1}$, and $(\phi - 1)\phi^{-1} < \alpha < 0$, the impulse responses of orders and inventory (see Fig. 2a) are quite different to those from the conventional $[0,1]$ parameter region (see Fig. 2b). Our recommendation produces a smoothed, damped, and exponential increasing (or decreasing) impulse response, rather than an oscillatory response. This further strengthens our argument that unconventional OUT-DT parameters result in a better dynamics.



a) $\phi = 0.001$, $\beta = -999$, $\alpha = -500$ b) $\phi = 0.8$, $\beta = 0.5$, $\alpha = 0.5$

Figure 2. The impulse response of orders and inventory in the OUT-DT system when $T_p = 0$

4.3 The Case of General Demand

We will now conduct a frequency response analysis of the OUT-DT policy to study its inventory dynamics for a general demand process. This demand process could be stationary or non-stationary, correlated or not, heteroskedastic or not; there are no restrictions. Let AR_{NS} be the net stock *amplitude ratio*. It can be obtained by letting $z = e^{j\omega}$ in the transfer function (8) and taking the modulus, $AR_{NS} = \left| NS\left(e^{j\omega}\right) / \varepsilon\left(e^{j\omega}\right) \right|$. AR_{NS} describes how the net stock reacts to a single sine wave demand with frequency ω. This is an important measure as *NSAmp* can then be found from

$$NSAmp = \frac{\sigma_{NS}^2}{\sigma_D^2} = \left(\frac{1}{\pi} \int_0^{\pi} \left| \frac{NS\left(e^{j\omega}\right)}{\varepsilon\left(e^{j\omega}\right)} \right|^2 \left| \frac{D\left(e^{j\omega}\right)}{\varepsilon\left(e^{j\omega}\right)} \right|^2 d\omega \right) \Bigg/ \left(\frac{1}{\pi} \int_0^{\pi} \left| \frac{D\left(e^{j\omega}\right)}{\varepsilon\left(e^{j\omega}\right)} \right|^2 d\omega \right), \tag{19}$$

Disney and Towill (2003). The $AR_{NS}\big|_{\omega \to 0} = 0$, before climbing towards its (first) local maximum. The initial value of AR_{NS} suggests that there is an opportunity to avoid excessive *NSAmp* when the demand pattern is dominated by low-frequency harmonics. This may occur when the demand has long-term trends. The high-frequency response of the OUT-DT inventory system is described by,

$$AR_{NS}\big|_{\omega=\pi} = \frac{\left| 2\left(1+(-1)^{T_p}\right)(1+\phi) + \alpha \begin{pmatrix} 4\beta\left(\sum_{i=1}^{T_p+1} \phi^i + \sum_{j=1}^{T_p} \sum_{i=1}^{j} \phi^i\right) - \beta\phi\left(3+(-1)^{T_p}\right) + \\ \left(1-(-1)^{T_p}\right)(1+\phi) + 2T_p\left(1+\phi(1-\beta)\right) \end{pmatrix} \right|}{2\left|2-\alpha+\phi(2-\alpha(1-\beta))\right|}. \tag{20}$$

For $T_p = 0$, (20) reduces to

$$AR_{NS}\big|_{\omega=\pi} = \frac{2(1+\phi)}{2-\alpha+\left(2-\alpha(1-\beta)\right)\phi}. \tag{21}$$

If the DT parameters are selected from the *BA* region, $AR_{NS}\big|_{\omega=\pi, T_p=0} < 1$. However, if the DT parameters are selected from the traditional $[0,1]$ region, $1 \le AR_{NS}\big|_{\omega=\pi, T_p=0} \le 4$. The *BA* region produces a better high-frequency response for inventory. When $\beta = (\phi-1)\phi^{-1}$,

$$AR_{NS}\big|_{\omega=\pi/2} = \sqrt{\frac{1+\phi^2}{1+(1-\alpha)^2\phi^2}} < 1, \tag{22}$$

for $0<\phi<1$ and $(\phi-1)\phi^{-1} < \alpha < 0$. By studying the monotonicity of AR_{NS} for $T_p = 0$ and $\beta = (\phi-1)\phi^{-1}$ we find that AR_{NS} is strictly decreasing in ω when $\omega \in (\pi/2, \pi)$. We conclude when $T_p = 0$, the OUT-DT system is able to dampen inventory variability over at least half of the frequency spectrum when the DT parameters are selected from the desirable region. This may lead to $NSAmp < 1$ for some demand patterns.

For $T_p = 1$, the *BA* region reduces to

$$0<\phi<1,\ (\phi-1)\phi^{-1} < \alpha < 0,\ -\phi^{-1} \le \beta \le (\phi-1)\phi^{-1}. \tag{23}$$

(20) is simplified to

$$AR_{NS}\big|_{\omega=\pi} = \frac{2(1+\phi)\left|\alpha(1+\beta\phi)\right|}{2-\alpha+\left(2-\alpha(1-\beta)\right)\phi}, \tag{24}$$

and $AR_{NS}\big|_{\omega=\pi} < \sqrt{2}$ if the DT parameters are drawn from the *BA* region in (23). In terms of AR_{NS} at $\pi/2$, if $\beta = (\phi-1)\phi^{-1}$, then

$$AR_{NS}\big|_{\omega=\pi/2} = \sqrt{\frac{(1+\phi^2)(2+2\alpha\phi+\alpha^2\phi^2)}{1+(1-\alpha)^2\phi^2}} < \sqrt{2}. \tag{25}$$

AR_{NS} is also strictly decreasing in ω when $\omega \in (\pi/2, \pi)$, $T_p = 1$, and $\{\alpha, \beta, \phi\} \in BA$. Therefore, if the demand is dominated by the high-frequency harmonics (between $\pi/2$ and π), it is possible for the OUT-DT system to achieve $NSAmp < 2$ when $T_p = 1$.

For other lead-times, when $\beta \uparrow (\phi-1)\phi^{-1}$ and $\alpha \downarrow (\phi-1)\phi^{-1}$,

$$AR_{NS}\big|_{\omega=\frac{\pi}{2}}=\sqrt{\frac{1+\phi^{2(1+T_p)}+2\phi^{1+T_p}\sin\left[T_p\,\pi/2\right]}{2}}. \tag{26}$$

When $\phi\downarrow 0$, (26) reduces to $\sqrt{2}/2$. It means the frequency harmonics at $\pi/2$ will not be amplified using this set of parameter values.

The final value of *AR* is also intriguing. Provided that the parameter set $\{\alpha,\beta,\phi\}$ is selected from the *BA* region, the final value of AR_{NS} as $\omega\to\pi$ is

$$AR_{NS}\big|_{\omega=\pi}<1,\ \text{for}\ T_p=\{2x:x\in\mathbb{N}_0\};\ \text{and}\ AR_{NS}\big|_{\omega=\pi}<\sqrt{1+T_p},\ \ \forall T_p. \tag{27}$$

Due to the existence of multiple stationary points in the inventory frequency response for $T_p\geq 2$, its monotonicity for general lead-time is too complex to study analytically. Based on our findings for the $T_p=0$ and 1, as well as (26) and (27), we conjecture that $NSAmp<1+T_p$ is possible for the OUT-DT system for a suitable demand pattern (and any lead-time).

If demand is dominated by high-frequency harmonics, (for instance, if negatively correlated AR(1) demand was present, Box et al., 1994) the OUT-DT system avoids creating the bullwhip and simultaneously reduces the inventory variance. When $T_p=0$, our results imply $\sigma_{NS}^2<\sigma_D^2$; that is, a near zero inventory policy is possible. In summary, the *BA* area has an enviable frequency response in terms of both orders and inventory levels for any lead-time.

5. Numerical Investigations

In this section, 62 sets of real-world demand series are simulated in Excel to verify our analytical results. These demand series came from previous research and projects conducted by the authors. There are low-volume products as well as high volume products. The source of the data ranges from retailers, manufacturers, to logistics companies, and distributors. Intermittent demand series were excluded. We assume the lead-time $T_p=1$. We compared performance based on three objectives: minimizing the standard deviation of net stock, minimizing the standard deviation of order, and minimizing the sum of the order and inventory standard deviation. We optimized the objective function by choosing DT parameters from the traditional [0,1] region and from the *BA* region defined by (13), (14), and (15).

Figure 3 summarizes the results for 62 real demand time series when parameter values are selected from *BA* region. Consider first the case of minimizing σ_{NS}. σ_{NS} is closely related to the safety stock a company must hold in inventory to minimize holding and backlog costs, Hosoda and Disney (2009). A minimized σ_{NS} is able to reduce inventory costs and improve the service level achieved with given safety stock. This objective often exists in businesses where they care more about inventory costs and service level than excessive capacity investment, labor idling, and over-time. The majority of *NSAmp* ratios are around two. In 11 of the 62 time series, the OUT-DT model enabled *NSAmp* < 2, verifying our prediction that it is possible to achieve $NSAmp<1+T_p$. There were also 35 instances when *Bullwhip* < 1, even though minimizing σ_{NS} was the sole objective. For those scenarios which cannot avoid bullwhip, the majority of the bullwhip ratios were maintained near one. This shows that the OUT-DT system can keep good control of inventory costs and service levels without inducing significant capacity costs. Managers only need to consider minimizing the standard deviation of inventory if inventory costs are more significant than capacity costs in their supply chains; this will lead to the best inventory performance and relatively good bullwhip behavior.

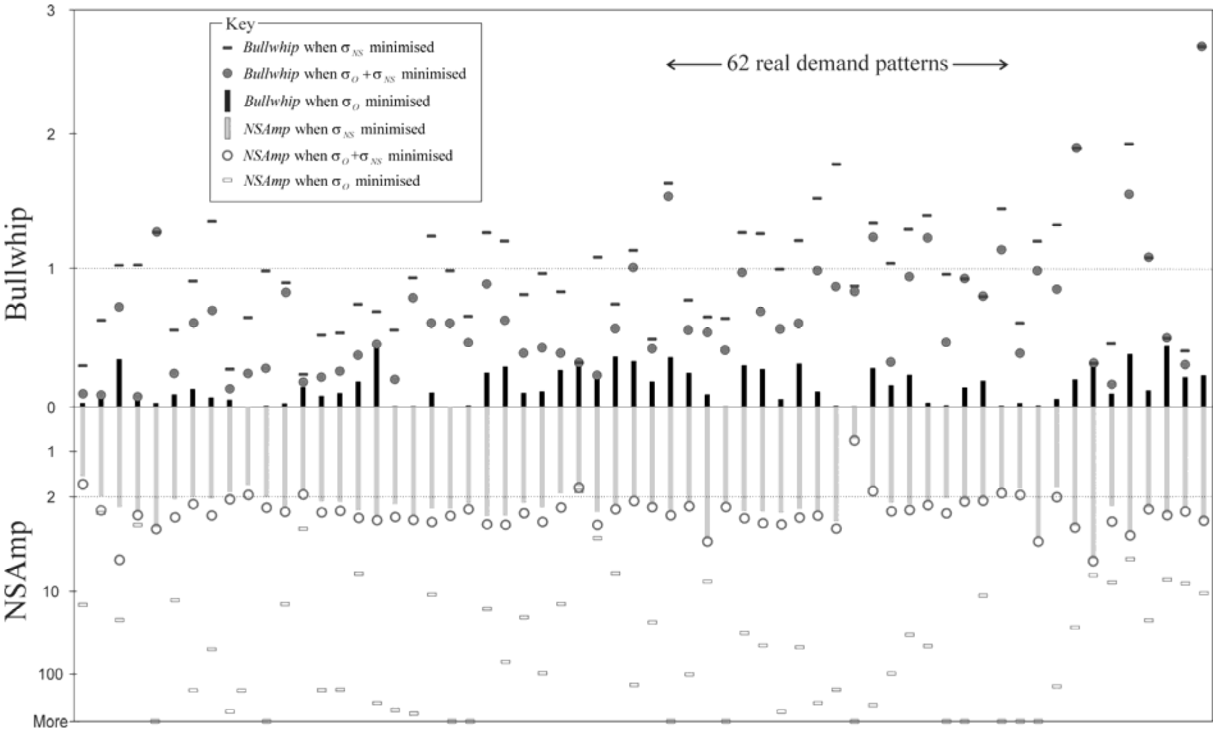


Figure 3. The bullwhip and NSAmp of 62 real-world data sets when $\sigma_O + \sigma_{NS}$ is minimized, together with the optimized DT parameters

The second objective is to minimize the standard deviation of orders σ_O; that is, to minimize the bullwhip effect. This objective to relevant in situations when the capacity costs are much more important than inventory costs. Figure 3 shows that the OUT policy with DT forecasting mechanism successfully avoids the bullwhip effect in all 62 time series. This is consistent with Li et al., (2014) who showed that the OUT-DT system is able to avoid the bullwhip effect when $\{\alpha, \beta, \phi\} \in BA$. As expected, bullwhip avoidance sometimes comes at the cost of a large *NSAmp*. Despite the fact that some bullwhip ratios are close to zero, the OUT-DT policy cannot achieve the same bullwhip performance as the model advocated by Dejonckheere et al., (2003), where a proportional OUT policy could reduce the order variance to zero.

The last objective is to minimize $\sigma_O + \sigma_{NS}$; that is, both capacity and inventory costs are present. We assume the standard deviation of inventory and of orders are equally costly. The empirical results show that the OUT-DT system successfully eliminates the bullwhip effect in 52 of the 62 time series and maintained the *NSAmp* ratios between one and three in the majority of the cases (Figure 3). Observing the optimized parameters, we notice that only a few of the optimized ϕ values are close to unity. The majority of the optimized α and β take on small negative values. These results concur with the findings of Li et al., (2014).

Table 1 compares the results of using *BA* region and conventional [0,1] region. In most cases, *BA* region generates less bullwhip no matter the objective function. The results show that the *BA* region produces less *NSAmp* in 53 out of 62 cases, suggesting *BA* region produces more accurate forecasts (when measured as the variance of the forecast errors over the lead-time and review period) than the traditional recommendation; including the popular exponential smoothing. For the second objective, there is a large amplification in inventory variability because of the significant reduction on bullwhip in the *BA* region. When optimizing $\sigma_O + \sigma_{NS}$, the *BA* region generally performs better than the [0,1] interval.

Objective: minimizing	σ_{NS}	σ_O	$\sigma_O + \sigma_{NS}$
No. of datasets with *Bullwhip* improvements	61	62	62
No. of datasets with *NSAmp* improvements	53	20	28
Average % reduction/increase on σ_O	21.56%	70.36%	35.25%
Average % reduction/increase on σ_{NS}	5.85%	-950%	2%
Average % reduction/increase on $\sigma_O + \sigma_{NS}$	12.72%	-556%	16.89%

Table 1. Comparing the performance of the BA region over the conventional [0,1] *region*

6. Conclusions

By proving the invertibility and the stability regions of the DT forecasting mechanism are identical, we have offered theoretical support for exploring the performance of the OUT-DT system over a wider range of parameter values than conventional recommendations. While others involved in evaluating the utility of DT forecasts have chosen the parameter values from the $[0,\ 1]$ interval, our work shows that if unconventional $\{\alpha, \beta, \phi\}$ values are selected *Bullwhip* can be avoided, *NSAmp* can be minimized, and our results hold for general lead-times.

We have characterized the inventory performance of the OUT-DT system. For i.i.d. customer demand, its inventory performance is similar to other forecasting methods and $NSAmp \geq 1 + T_p$. However, for non-i.i.d. general demand processes, the OUT-DT policy with $\{\alpha, \beta, \phi\} \in BA$ exhibits some remarkable performance. The OUT-DT system works particularly well when the demand series is dominated by low-frequency harmonics (such as trend) and/or the harmonics at frequencies between $\frac{\pi}{2}$ and π. This is because the OUT-DT policy acts as a low-pass filter where half of the frequency harmonics in the demand signal can be dampened in its inventory response.

We have verified our analytical results by simulating the response to 62 real demand patterns. Using $\{\alpha, \beta, \phi\} \in BA$, the OUT-DT system is also likely to reduce the net stock amplification. Our simulations have confirmed that the OUT-DT policy can avoid the bullwhip effect for general lead-times. The bullwhip avoiding character of the OUT-DT system exists even with different objectives. The superior inventory control that the OUT-DT system exhibits, and its ability to avoid inducing significant capacity costs, makes it a good choice for managing inventory and setting production/distribution targets in supply chains.

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