Cardiff Economics Working Papers

Working Paper No. E2017/16

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December 2017

ISSN 1749-6010

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Repeated Implementation with Overlapping Generations of Agents

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December 15, 2017

Abstract

We study repeated implementation in a model with overlapping generations of agents. It is assumed that the preferences of agents do not change during their lifetime. A social choice function selects an alternative in each period as a function of the preferences of agents who are alive in that period. We show that any social choice function satisfying mild necessary conditions is repeatedly implementable in subgame perfect equilibrium if there are at least three agents and they live sufficiently long.

Keywords: Repeated Implementation, Subgame Perfect Implementation, Overlapping Generations, Necessary and Sufficient Conditions

JEL Classification Numbers: C72; C73; D71; D82

1 Introduction

Implementation theory studies what social choice rules (SCRs) can be implemented in various solution concepts. These SCRs explicitly or implicitly capture what the society considers desirable. Most of SCRs, however, are motivated by static problems; for example, how to redistribute initial endowments in static exchange economies. Consequently, the literature has also largely focused on one-shot implementation even if the mechanisms that are used to implement SCRs can themselves be dynamic.

However, if one considers issues like environmental protection or pension reform, it is natural to view them as dynamic problems that give rise to normative considerations like intergenerational equity and to corresponding SCRs that do not appear in static problems (see, for example, essays in Roemer and Suzumura, 2007). Therefore, our objective is to have a setup that allows for different

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generations of agents and we study what SCRs or, more precisely, social choice functions (SCFs) can be implemented in this setup.

Specifically, every period there are n agents alive. These agents have different ages. Every T periods the oldest agent passes away and a new agent is born. Thus, each agent lives for nT periods. It is assumed that the agent's type, which determines his preferences, does not change during his lifetime and the preferences are additively separable over time. In each period, an SCF specifies an alternative, which only depends on the types of agents who are alive in that period. Because a social designer never observes agents' types, she needs to design a sequence of mechanisms, called a regime, that allows her to elicit information about agents' types and, at the same time, to select the desired alternative in every period on the equilibrium path. If there exists such a regime, we say that the SCF is repeatedly implementable. We assume complete information among agents and use subgame perfect equilibrium (SPE) as our solution concept.

First, we derive two necessary conditions. They deal with two extreme cases. In one case, *all* agents can get their best possible lifetime utility, given their true preferences, while in the other, *all* agents must get their worst lifetime utility, given the reported preferences. If the premises of either of these conditions apply, we show that an SCF is repeatedly implementable only if it is a constant function. Intuitively, at these extremes, either the agents do not have incentives to reveal their true types because they already get the best, or they cannot be given such incentives because they are supposed to get the worst.

Second, we construct a regime and show that the two necessary conditions are also sufficient if there are at least three agents, $n \geq 3$, and they live sufficiently long, that is, T is sufficiently large. The regime borrows elements of the canonical mechanism that is used in one-shot subgame perfect implementation (Moore and Repullo, 1988). The connection between repeated implementation that we study and one-shot implementation in SPE is extensively discussed in Section 3, but intuitively the similarity between the two implementation problems arises because we assume that the agent's type does not change during his lifetime. Clearly, the two implementation problems also differ: in the repeated implementation problem, an alternative is selected in every period, while in the one-shot implementation problem, an alternative is selected only in the final period. Since the alternatives selected in earlier periods cannot be undone, repeated implementation is more restrictive. However, the larger is T, the less these restrictions matter. For this reason, we need a large T to establish the sufficiency result.

Repeated implementation has been studied by Kalai and Ledyard (1998); Chambers (2004); Lee and Sabourian (2011); Mezzetti and Renou (2017); Āzacis and Vida (2015). However, in all these papers, the same set of agents are alive in all periods. Thus, these papers do not consider the implementation of SCFs that can capture any intergenerational social choice considerations. Further, Kalai and Ledyard (1998) and Chambers (2004) assume that the state of the world is drawn only once and is kept fixed for all periods,[1] while Lee and Sabourian (2011); Mezzetti and Renou (2017); Āzacis and Vida (2015) assume that a new state is drawn in each period. Even though the state of the world is also changing over time in our setup, the setup is nevertheless closer to that in the first two papers because the type of agent is kept fixed throughout his lifetime.

The distinction between our setup and that in the latter three papers can also be explained from another angle. Salant (1991); Kandori (1992); Smith (1992) derive folk theorems for repeated games played by overlapping generations of players. The main message of these papers is that if the players are sufficiently long-lived, then the set of equilibrium payoffs is the same as in the repeated games with infinitely-lived players and discounting when the discount factor is sufficiently high. On the other hand, Lee and Sabourian (2011) show that if infinitely-lived agents are sufficiently patient, then only efficient SCFs are repeatedly implementable. Therefore, one might also expect that only efficient SCFs are repeatedly implementable in our setup when T is large. However, this is not the case because unlike Lee and Sabourian (2011), a new state is not drawn in every period in our setup.

Finally, as mentioned above, our setup shares commonalities with one-shot subgame perfect implementation that has been studied by Moore and Repullo (1988); Abreu and Sen (1990); Vartiainen (2007). We explain the connection in detail in Section 3.

The rest of the paper is organized as follows. Section 2 contains the description of the problem studied. Section 3 compares repeated implementation that we study with one-shot subgame perfect implementation that has already been studied in the literature. Section 4 derives the necessary conditions, while Section 5 proves also the sufficiency of these conditions for large T. Section 6 contains the discussion of several extensions to the model. Some of the proofs are relegated to the Appendix.

2 The Model

We consider a setup with overlapping generations of agents. Each generation consists of a single agent. Each agent lives for exactly nT periods where n and T are two positive integers. Let $\mathbb{Z}$ be the set of integers equal or greater than $-n+1$. The agent of generation $z \in \mathbb{Z}$, or agent z for short, is born at the beginning of period zT and dies at the end of period $(z+n)T-1$. Hence, there are exactly n agents alive at any moment.

Let A be a set of feasible alternatives that does not change over time. Let Θ be a finite set of possible agent's types, and it is the same for all generations.

[1] Hayashi and Lombardi (2016) also study implementation in a dynamic setup with persistent states, but in any given period, the socially desirable alternative depends not only on the state, but also on the history of alternatives that have been selected in the previous periods.

We assume that the types are drawn independently and identically across the generations according to the probability distribution p such that $p(\theta) > 0$ for each $\theta \in \Theta$. Agent's type does not change over his lifetime and we will write θ_z to denote the type of agent z. The payoff of agent z in period $t = zT, \ldots, (z+n)T-1$ is $u(a_t, \theta_z)$ if alternative $a_t \in A$ is implemented in that period,[2] and his lifetime payoff is simply

$$\sum_{t=zT}^{(z+n)T-1} u(a_t, \theta_z).$$

Throughout, we make the following assumptions about u. First, we assume that the agents have strict preferences over the alternatives in every state of the world:

Assumption A1 *$u(a, \theta) \neq u(b, \theta)$ for all $a, b \in A$ such that $a \neq b$, and for all $\theta \in \Theta$.*

Second, we assume that the change in the state leads to the change in the ordinal preferences:

Assumption A2 *For every $\theta, \phi \in \Theta$ such that $\theta \neq \phi$ there exists a pair $a, b \in A$ such that $u(a, \theta) > u(b, \theta)$ and $u(a, \phi) < u(b, \phi)$.*

Later it will be convenient to write the pair (a, b) as $(a(\theta, \phi), b(\theta, \phi))$ with the convention that the first alternative is more desirable in state θ and the second alternative is more desirable in state ϕ. We will discuss in Section 6 how these assumptions can be relaxed.

We consider the implementation of socially desirable alternatives from period 0 onwards. In period t, a social choice function f assigns an alternative in A as a function of types of all agents that are alive in that period, that is, $f(\theta_{z-n+1}, \ldots, \theta_z) \in A$ where $z = \lfloor t/T \rfloor$ is the quotient (the integer part) when dividing t by T. Note that the first argument of f denotes the type of the oldest agent who is alive in that period; the second argument denotes the type of the second oldest agent who is alive and so on. f does not need to be symmetric in its arguments. That is, the socially desirable alternative can change if we exchange the types of old and young generations. Note, however, that f is time independent. That is, $f(\theta_{z-n+1}, \ldots, \theta_z) = f(\theta_{k-n+1}, \ldots, \theta_k)$ if $(\theta_{z-n+1}, \ldots, \theta_z) = (\theta_{k-n+1}, \ldots, \theta_k)$ for any z and k. In particular, the socially desirable alternative remains the same during periods zT to $(z+1)T-1$ for any z.[3]

In principle, the socially desirable alternative of the current period could be allowed to depend on the entire history of the realized types up to and including

[2] At the cost of additional notation, we could allow u also to depend on the agent's identity, that is, to depend on z.

[3] From now on, zT should be understood as $\max\{zT, 0\}$.

that period. Therefore, the assumed specification of f can be thought as corresponding to the steady state. To illustrate it, consider the following example.[4]

Example: Let $n = 2$ and consider an exchange economy with two perishable goods X and Y that cannot be carried from one period to the next. The aggregate endowment is $\left(w^X, w^Y\right)$ in every period. The set of feasible allocations is $A = \left\{\left(\left(w^X - x, w^Y - y\right), (x, y)\right) | 0 < x < w^X, 0 < y < w^Y\right\}$, where (x, y) is the consumption bundle of the younger of two agents who are alive in that period. Let $\Theta = \{\theta, \theta'\}$ with $p(\theta) = \frac{1}{2}$. The per-period utility function is

$$u(x, y, \phi) = \log(x) + \phi \log(y)$$

for all $\phi \in \Theta$. A sequence of allocations (which are expressed in terms of the consumption bundle of the younger agent) is *stationary* if the period t allocation only depends on that period's state of the world. We write stationary allocations as $(x(\phi, \psi), y(\phi, \psi))$ for all $(\phi, \psi) \in \Theta^2$ with the convention that the first and second arguments refer to the types of the older and younger agents, respectively. The following maximization problem gives *stationary Pareto optimal* allocations (with equal weights attached to each state):

$$\begin{aligned} \max_{\{x(\phi,\psi),y(\phi,\psi)|(\phi,\psi)\in\Theta^2\}} \quad & \sum_{\phi\in\Theta}\sum_{\psi\in\Theta}\Bigg[u(x(\phi,\psi), y(\phi,\psi), \psi) \\ & + \frac{1}{2}\sum_{\eta\in\Theta} u\left(w^X - x(\psi,\eta), w^Y - y(\psi,\eta), \psi\right)\Bigg] \end{aligned}$$

The first order condition w.r.t. $x(\phi, \psi)$ is

$$\frac{1}{x(\phi,\psi)} - \frac{1}{w^X - x(\phi,\psi)} = 0,$$

implying $x(\phi, \psi) = \frac{1}{2}w^X$ for all $(\phi, \psi) \in \Theta^2$. The first order condition w.r.t. $y(\phi, \psi)$ is

$$\frac{\psi}{y(\phi,\psi)} - \frac{\phi}{w^Y - y(\phi,\psi)} = 0,$$

implying $y(\phi, \psi) = \frac{\psi}{\phi+\psi}w^Y$ for all $(\phi, \psi) \in \Theta^2$. Given these allocations, we can define a social choice function

$$f(\phi, \psi) = \left(\frac{1}{2}w^X, \frac{\psi}{\phi+\psi}w^Y\right)$$

for all $(\phi, \psi) \in \Theta^2$ that selects a stationary Pareto optimal allocation in every period and state. ■

[4]The example does not satisfy Assumption A1. We will comment on this in Section 6. Also, strictly speaking, A in the example changes over time since agents who are dead or who are not yet born, cannot receive positive consumption bundles. We can fix it by assuming free disposal.

A mechanism consists of messages that the agents can announce and an outcome function that selects a feasible alternative as a function of these messages. Let the message space of every agent in every period be M. It is without loss of generality, since we can always choose M to be sufficiently large. Hence, we can associate every mechanism with its outcome function. We restrict attention to deterministic mechanisms in which the agents announce their messages simultaneously. Let G be the set of all feasible mechanisms or, equivalently, outcome functions with a typical element g. Thus, given $g \in G$ and $m \in M^n$, alternative $a = g(m) \in A$ is implemented.

We assume that all agents that are alive in period t observe the entire history (to be defined below) up to period t, including the types of their opponents. That is, we are in a complete and perfect information environment. We also assume that the types of those agents who were born before period 0 are also known to a social designer. We will again relax these assumptions in Section 6.

A history of types at the start of period $t \geq 0$ is $\zeta_t = (\theta_{-n+1}, \ldots, \theta_0, \ldots, \theta_{\lfloor t/T \rfloor})$, while a history of messages at the start of period $t > 0$ is $\mu_t = (m^0, \ldots, m^{t-1})$ where $m^t = (m^t_{\lfloor t/T \rfloor - n+1}, \ldots, m^t_{\lfloor t/T \rfloor})$ is a profile of period t messages with m^t_z being the message of agent z. A history is $h_t = (\zeta_t, \mu_t)$ for $t > 0$ and $h_0 = \zeta_0$. Let H_t denote the space of all period t histories. We assume that the agents who are alive in period t, can distinguish between any two period t histories. In particular, it means that the agents learn at the start of each period what messages were announced in the previous period. On the other hand, the designer cannot distinguish between any two period t histories $h_t = (\zeta_t, \mu_t)$ and $h'_t = (\zeta'_t, \mu'_t)$ if $\mu_t = \mu'_t$, and ζ_t and ζ'_t share the same first $n-1$ elements $\theta_{-n+1}, \ldots, \theta_{-1}$.

A regime, r describes which mechanism is selected after each possible history: $g = r(h_t) \in G$ subject to the restriction that $r(h_t) = r(h'_t)$ if the designer cannot distinguish between histories h_t and h'_t. Note that we restrict attention to deterministic regimes. Because of that and also because the mechanisms are deterministic, it is fine to omit from the description of history h_t which mechanisms and alternatives have been selected in periods $0, \ldots, t-1$. We assume that the designer commits to a regime at the start of period 0 and that the agents know what regime the designer employs.

A pure strategy of agent z, s_z maps histories into messages: $s_z(h_t) \in M$ for all $t = zT, \ldots, (z+n)T - 1$ and $h_t \in H_t$. Let S_z be the space of agent z's strategies. Let s be a profile of strategies, one strategy for each $z \in \mathbb{Z}$. Also, let $s(h_t) = (s_{\lfloor t/T \rfloor - n+1}(h_t), \ldots, s_{\lfloor t/T \rfloor}(h_t))$. Given h_t and s, let $q(h_t|h_t, s) = 1$ and for any $\tau > t$, let

$$q(h_\tau|h_t, s) = \begin{cases} q(h_{\tau-1}|h_t, s) & \text{if } \tau/T \notin \mathbb{Z},\ \zeta_\tau = \zeta_{\tau-1},\ \mu_\tau = (\mu_{\tau-1}, s(h_{\tau-1})), \\ q(h_{\tau-1}|h_t, s)p(\theta) & \text{if } \tau/T \in \mathbb{Z},\ \zeta_\tau = (\zeta_{\tau-1}, \theta),\ \mu_\tau = (\mu_{\tau-1}, s(h_{\tau-1})), \\ 0 & \text{otherwise.} \end{cases}$$

For any $z \in \mathbb{Z}$, any $t = zT, \ldots, (z+n)T - 1$, and any h_t, the (expected) payoff of agent z for the rest of his life is

$$v_z(s|h_t, r) = \sum_{\tau=t}^{(z+n)T-1} \sum_{h_\tau \in H_\tau} q(h_\tau|h_t, s) u(g(s(h_\tau)), \theta_z),$$

where θ_z is the $z+n$-th element of ζ_t (since the first element of ζ_t is θ_{-n+1}) and $g = r(h_\tau)$ in $g(s(h_\tau))$.[5]

A strategy profile s is a subgame perfect equilibrium (SPE) of r if for all $z \in \mathbb{Z}$, all $t = zT, \ldots, (z+n)T-1$, all $h_t \in H_t$, and all $s'_z \in S_z$, it is true that $v_z(s|h_t, r) \geq v_z((s'_z, s_{-z})|h_t, r)$. A regime r repeatedly implements f in SPE if the set of SPE is non-empty and for each SPE s, we have that $g(s(h_t)) = f(\theta_{\lfloor t/T \rfloor - n+1}, \ldots, \theta_{\lfloor t/T \rfloor})$ for all t and h_t such that $q(h_t|h_0, s) > 0$, where $(\theta_{\lfloor t/T \rfloor - n+1}, \ldots, \theta_{\lfloor t/T \rfloor})$ are the last n elements of ζ_t. f is repeatedly implementable in SPE if there exists r that repeatedly implements it in SPE.

3 Comparison with One-shot Subgame Perfect Implementation

Our model shares similar features with one-shot subgame perfect implementation that has been studied by Moore and Repullo (1988); Abreu and Sen (1990); Vartiainen (2007). Therefore, we will now briefly sketch the problem of one-shot implementation in SPE (which we adapt to make comparable to our model) and then explain how it relates to the problem we study.

Thus, let the set of agents be $N = \{1, \ldots, n\}$, the set of states be Θ^n with a typical element $\vec{\theta} = (\theta_1, \ldots, \theta_n)$, and the set of alternatives be $\mathcal{A}$. The objective of the designer is to implement alternative $f(\vec{\theta}) \in \mathcal{A}$ when the state is $\vec{\theta}$. To do that, the designer employs a multi-stage mechanism.

Abreu and Sen (1990) show in their Theorem 1 that if f is one-shot implementable in SPE, then it satisfies the following Condition C1 with respect to some $B \subseteq \mathcal{A}$.

Condition C1 *For all $(\vec{\theta}, \vec{\phi}) \in \Theta^n \times \Theta^n$ such that $f(\vec{\theta}) \neq f(\vec{\phi})$, there exist a sequence of agents $z(1), \ldots, z(K) \in N$ and a sequence of alternatives $a_1, \ldots, a_{K+1} \in B$ with $a_1 = f(\vec{\theta})$ such that*

1. *$u(a_k, \theta_{z(k)}) > u(a_{k+1}, \theta_{z(k)})$ for $k = 1, \ldots, K$,*

2. *$u(a_K, \phi_{z(K)}) < u(a_{K+1}, \phi_{z(K)})$,*

3. *$a_k \neq \arg\max_{a \in B} u(a, \theta_{z(k)})$ for $k = 1, \ldots, K$.*

[5] Note that $g(m^t)$ need not coincide with $g(m^\tau)$ even if $m^t = m^\tau$ since the outcome functions can differ in periods t and τ.

Vartiainen (2007) shows in his Theorem 4 that Condition C1 together with unanimity are necessary and sufficient for subgame perfect implementation when $n \geq 3$ and the preferences are strict. The unanimity condition requires that if for some $\vec{\theta} \in \Theta^n$, there exists $b \in B$ such that $b = \arg\max_{a \in B} u(a, \theta_z)$ for all $z \in N$, then $f(\vec{\theta}) = b$.

Condition C1 requires the existence of agent $z(K)$ who has a preference reversal between a_K and a_{K+1} when the state changes from $\vec{\theta}$ to $\vec{\phi}$. This is ensured by Assumption A2. Further, alternatives $f(\vec{\theta})$ and a_K are connected in a particular way. One can think about this connection as follows. Suppose that at stage 1, agents announce that the state is $\vec{\theta}$. Now, in stage k, agent $z(k)$ has a choice either to agree that the state is $\vec{\theta}$ and alternative a_k is implemented, or to claim that the state is $\vec{\phi}$, in which case we proceed to stage $k+1$ where agent $z(k+1)$ gets to choose. If the true state is $\vec{\theta}$ but agent $z(k)$ claims that it is $\vec{\phi}$, then in equilibrium, he expects alternative a_{k+1}. Since he prefers a_k to a_{k+1} in state $\vec{\theta}$ according to C1.1, he will not want to falsely claim the state to be ϕ. Since it is true for all k, alternative $a_1 = f(\vec{\theta})$ is implemented as desired. If, however, the true state is $\vec{\phi}$ and agent $z(k)$ claims it to be $\vec{\phi}$, then he will get *his best alternative* in B rather than a_{k+1} in stage $k+1$. Therefore, unless a_k is already his best alternative, agent $z(k)$ has incentives to announce the true state. C1.3 ensures that the agent has this incentive. This eliminates an equilibrium in which agents claim that the state is $\vec{\theta}$, when it is $\vec{\phi}$.

Suppose that the set of alternatives is $\mathcal{A} = A^T$ with a typical element $\vec{a} = (a_1, \ldots, a_T)$ and the utility function is additively separable, $u(\vec{a}, \theta) = \sum_{t=1}^{T} u(a_t, \theta)$. The major difference between the repeated implementation studied here and the one-shot implementation is that in the former case, alternative a_1 is implemented in period 1, alternative a_2 in period 2, and so on, while in the latter case, all the alternatives in $\vec{a}$ are implemented at the end once all the communication is done. The sequential nature of repeated implementation imposes further restrictions beyond those specified in Condition C1. For example, under repeated implementation, it must be that $\vec{a}_k$ and $\vec{a}_{k+1}$ share the first $k-1$ elements.[6] Also, agent $z(k)$ may not anymore be able to obtain his best profile of alternatives in B when challenging a lie. For example, agent $z(1)$ is only guaranteed to get his best in $\{(a_1, \ldots, a_T) \in B | a_1 = \hat{a}_1\}$ where $\hat{a}_1$ is the alternative that is implemented in period 1. We avoid these complications by assuming that T is large. With large T, alternatives that are implemented in the initial K periods while the state of the world is being elicited, have negligible impact on agent's lifetime utility. Therefore, one can think that the communication by the agents with the mechanism is effectively over before any implementation of alternatives takes places, which is exactly what happens in the one-shot implementation.

Even more, when T is large, not only there are no further necessary conditions

[6] The sequence $\vec{a}_1, \ldots, \vec{a}_{K+1}$ would look as follows: $\vec{a}_1 = (a_1, a_2, a_3, \ldots, a_T)$, $\vec{a}_2 = (b_1, b_2, b_3, \ldots, b_T)$, $\vec{a}_3 = (b_1, c_2, c_3, \ldots, c_T)$, $\vec{a}_4 = (b_1, c_2, d_3, \ldots, d_T)$ and so on.

beyond those that arise under one-shot implementation in SPE, but even these latter conditions can be sharpened further. First, one can set $K = 2$ in Condition C1. Suppose when the state changes from $\vec{\theta}$ to $\vec{\phi}$, the type of, say, agent 1 changes, that is, $\theta_1 \neq \phi_1$. Then, because of Assumption A2, C1.2 is satisfied if we set $z(2) = 1$. Second, we can choose as $z(1)$ any agent for whom $\vec{a}_1 = f(\vec{\theta})$ does not result either in his lowest or highest possible lifetime utility. If $\vec{a}_1$ gives the lowest possible lifetime utility to agent $z(1)$, then we cannot find $\vec{a}_2$ that gives even lower utility than $\vec{a}_1$, that is, C1.1 is violated. In turn, if $\vec{a}_1$ gives the highest possible lifetime utility to agent $z(1)$, then C1.3 is violated.[7] Third, as mentioned before, Vartiainen (2007) shows that the unanimity condition is also necessary for subgame perfect implementation. If all agents agree that $b \in B$ is the best profile of alternatives and b belongs to the range of f, then this case is already covered by C1.3. If b does not belong to the range of f, then we can exploit the fact that the agents are born and pass away in different periods in order to modify B so that there is no agreement anymore between the agents on what is best (even if all agents have identical preferences). In either case, we can ignore the unanimity condition.

Thus, the only necessary conditions for repeated implementation of f in SPE are the ones that correspond to a situation where either $f(\vec{\theta})$ gives the worst possible utility to every agent or it gives the best possible utility to every agent in state $\vec{\theta}$. These conditions will be derived in the next section where we will also take into account the overlapping pattern of agents' lifes.

4 The Necessary Conditions

Now we return back to our original setup. Let $f(\Theta) = \{a \in A | \exists(\theta_1, \dots, \theta_n) \in \Theta^n \text{ s.t. } f(\theta_1, \dots, \theta_n) = a\}$ denote the range of f. Let $\overline{a}(\theta) = \arg\max_{a \in A} u(a, \theta)$ and $\underline{a}(\theta) = \arg\min_{a \in A} u(a, \theta)$, which are assumed to exist.[8] Let $\mathbb{Z}_+$ be the set of non-negative integers.

Condition C2 *If there exist $a \in A$ and an infinite sequence of types $\theta_0, \theta_1, \dots$ such that*

1. *$\overline{a}(\theta) = a$ for all $\theta \in \Theta$, and*
2. *$f(\theta_z, \dots, \theta_{z+n-1}) = a$ for all $z \in \mathbb{Z}_+$,*

then $f(\theta_1, \dots, \theta_n) = a$ for all $(\theta_1, \dots, \theta_n) \in \Theta^n$, that is, f is constant.

[7] Therefore, in general, $z(1) \neq z(2)$ and one cannot set $K = 1$, that is, Maskin monotonicity of f is not guaranteed.

[8] If, say, the maximum did not exist for some θ, then there must be $a \in A \backslash f(\Theta)$ such that $u(a, \theta) > u(b, \theta)$ for all $b \in f(\Theta)$. Then, let $\overline{a}(\theta) = a$. $\underline{a}(\theta)$ can be defined similarly. This will not affect the necessary conditions as their premises only apply when either $\overline{a}(\theta) \in f(\Theta)$ or $\underline{a}(\theta) \in f(\Theta)$.

The first premise of Condition C2 says that all types agree on the best alternative, while the second premise ensures that a can be selected by f in every period. In particular, there exists an infinite sequence of types such that if one evaluates the social choice function for any n adjacent types in this sequence, the selected alternative according to f will be a. Further, the statement of the second premise can be strengthened by noting that this infinite sequence of types consists of repetitions of the same finite sequence. This follows from the fact that Θ is finite and f only depends on the types of n consecutive generations. Hence, there must exist k and z in $\mathbb{Z}_+$ such that $(\theta_k, \ldots, \theta_{k+n-1}) = (\theta_z, \ldots, \theta_{z+n-1})$. But then we can construct another sequence that consists of repetitions of $\theta_k, \ldots, \theta_{z-1}$.

We claim that if the above two premises hold, then there exists an equilibrium in which a is selected in every period irrespective of the realized types. Hence, if f is repeatedly implementable, then it must be a constant function that selects a for all possible realizations of types. The intuition is simple: a is the best alternative for all types and they can ensure that it is selected in every period by pretending to have types as defined by the sequence $\theta_0, \theta_1, \ldots$. Clearly, no agent will have incentives to deviate. Therefore, there exists an equilibrium in which a is implemented in every period. Thus, f must be constant. Formally,

Proposition 1 *If f is repeatedly implementable in SPE, then Condition C2 must hold.*

The proof of Proposition 1 appears in the Appendix.

Condition C3 *If there exists $\theta \in \Theta$ such that $f(\theta, \theta_2, \ldots, \theta_n) = \underline{a}(\theta)$ for all $(\theta_2, \ldots, \theta_n) \in \Theta^{n-1}$, then $f(\theta_1, \ldots, \theta_n) = \underline{a}(\theta)$ for all $(\theta_1, \ldots, \theta_n) \in \Theta^n$, that is, f is constant.*

Note that if Condition C3 also holds for $\theta' \neq \theta$, then it must be that $\underline{a}(\theta') = \underline{a}(\theta)$ since f is assumed to be a function. More importantly, if the premise of Condition C3 is not true for type θ, then this type never expects his lowest possible lifetime payoff of $nTu(\underline{a}(\theta), \theta)$ when f is implemented.

We claim that if the premise of Condition C3 holds, then there exists an equilibrium in which $\underline{a}(\theta)$ is selected in every period irrespective of the realized types. That is, there is an equilibrium in which every agent is claimed to be of type θ. Hence, if f is repeatedly implementable, then it must be a constant function that selects $\underline{a}(\theta)$ for all possible realizations of types. If in the case of Condition C2, no type wanted to deviate because he received his best alternative in every period, now no type can deviate because he must receive the worst alternative of type θ in every period. Formally,

Proposition 2 *If f is repeatedly implementable in SPE, then Condition C3 must hold.*

The proof of Proposition 2 is again in the Appendix.

5 The Sufficient Conditions

Our main result is that Conditions C2 and C3 are not only necessary but also sufficient if there are at least three agents alive at any moment and they live sufficiently long.

Theorem 1 *Suppose f satisfies Conditions C2 and C3 and $n \geq 3$. Then, there exists T^* such that for all $T \geq T^*$, f is repeatedly implementable in SPE.*

We will define a regime that implements f in SPE for sufficiently large T. (In particular, the description of the regime assumes that $T \geq 2$.) Although the definition of the regime is rather involved, it shares similarities with the canonical mechanism of Moore and Repullo (1988) that is used for one-shot subgame perfect implementation. (See, for example, Appendix B in Abreu and Sen (1990) for its description.) Before we describe it formally, we give some intuition. The regime has five parts. Part I applies as long as agents send unanimous messages. This part is used to elicit the type of the newly-born agent, say, agent z in period zT. Although in the equilibrium, the type of agent $z-1$ is elicited in period $(n-1)T$, we must give the opportunity to the agents to "confess" in period zT that they have lied about his type. This is done in part II. If we omitted this part, we would have subgames that do not have any Nash equilibrium. Thus, the role of part II is to ensure that the extensive game, which is induced by the regime, has a Nash equilibrium in every subgame. Part III gives incentives to agent z to deviate if the other agents unanimously lie about his type in period zT. Similarly, part IV gives incentives to agent z to deviate if the other agents lie about the type of agent $z-1$. In all other cases, part V applies, which is the so-called integer game.

We exploit the fact that the periods, in which different agents live, do not coincide and design parts III-V in such a way so as to ensure that the agents disagree about the most preferred sequence of alternatives that is obtainable and always have incentives to deviate. Therefore, parts III-V are not played on the equilibrium path. Part II is also not played on the equilibrium path because agent $z-1$ prefers to deviate in period $(z-1)T$ rather than to confess in period zT. Thus, only part I is played on the equilibrium path in which the agents truthfully report the type of the newly-born agent.

Let $M = \Theta \times \Theta \times \mathbb{Z}_+$. We fix two distinct, arbitrary alternatives $c, d \in A$. Given any t, let $z = \lfloor t/T \rfloor$ be the youngest agent who is alive in period t. Also, let $N_t = \{z-n+1, \ldots, z\}$ denote the set of agents who are alive in period t. Let $\theta_{-n+1}, \ldots, \theta_{-1}$ be given.

The regime r is defined as follows. We start in period $t = 0$ and apply the following algorithm:

I. If there is $\theta \in \Theta$ s.t. $m_k^t = (\theta_{z-1}, \theta, \cdot)$ for at least $n-1$ agents $k \in N_t$ including agent z, then set $\theta_z = \theta$ and let $g(m^t) = g(m^\tau) = f(\theta_{z-n+1}, \ldots, \theta_z)$

for all $\tau = t+1, \ldots, t+T-1$ and all m^τ. (Thus, the messages in periods $\tau = t+1, \ldots, t+T-1$ play no role.) Once period $t = (z+1)T$ arrives, we go back to the beginning of the algorithm and start it over again for $t = (z+1)T$.

II. If there is $(\eta, \theta) \in \Theta \backslash \{\theta_{z-1}\} \times \Theta$ s.t. $m_k^t = (\eta, \theta, \cdot)$ for at least $n-1$ agents $k \in N_t$ including agent z, then $g(m^t) = g(m^{t+2j+2}) = c$ and $g(m^{t+2j+1}) = d$ for all $j \in \mathbb{Z}_+$ and all m^{t+j+1}.

III. If there is $(\eta, \theta) \in \Theta^2$ such that $m_k^t = (\eta, \theta, \cdot)$ for all $k \in N_t \backslash \{z\}$ and $m_z^t = (\cdot, \phi, \cdot)$ where $\phi \neq \theta$, then $g(m^t) = a(\theta, \phi)$ and we proceed to period $t+1$ where[9]

  (a) If $m_k^{t+1} = (\cdot, \cdot, 0)$ for at least $n-1$ agents $k \in N_t$, then $g(m^{t+1}) = b(\theta, \phi)$ and $g(m^\tau) = \overline{a}(\phi)$ for all $\tau > t+1$ and all m^τ where ϕ and θ are given by m^t.

  (b) If $m_k^{t+1} = (\cdot, \cdot, 1)$ for at least $n-1$ agents $k \in N_t$, including agent z, then $g(m^{t+1}) = g(m^{t+2}) = a(\theta, \phi)$ and $g(m^\tau) = \underline{a}(\theta)$ for all $\tau > t+2$ and all m^τ.

  (c) If $m_k^{t+1} = (\cdot, \cdot, 1)$ for all $k \in N_t \backslash \{z\}$ while $m_z^{t+1} = (\cdot, \cdot, l)$ where $l \neq 1$, then $g(m^{t+1}) = b(\theta, \phi)$, $g(m^{t+2}) = a(\theta, \phi)$, and $g(m^\tau) = \underline{a}(\theta)$ for all $\tau > t+2$ and all m^τ.

  (d) For all other messages, let k^* be the agent who has announced the highest integer and suppose $m_{k^*}^{t+1} = (\psi, \cdot, l)$. (Ties are always broken in favour of the agent with the highest index.) Then, $g(m^{t+1}) = b(\theta, \phi)$ and $g(m^\tau) = \overline{a}(\psi)$ for all $\tau > t+1$ and all m^τ.

IV. If there is $(\eta, \phi) \in \Theta^2$ such that $m_k^t = (\eta, \phi, \cdot)$ for all $k \in N_t \backslash \{z\}$ and $m_z^t = (\theta, \phi, \cdot)$ where $\theta \neq \eta$, then $g(m^t) = a(\phi, \theta)$ and we proceed to period $t+1$ where

  (a) If $m_k^{t+1} = (\cdot, \cdot, 0)$ for at least $n-1$ agents $k \in N_t$, then $g(m^{t+1}) = b(\phi, \theta)$ and $g(m^\tau) = \overline{a}(\phi)$ for all $\tau > t+1$ and all m^τ.

  (b) If $m_k^{t+1} = (\cdot, \cdot, 1)$ for at least $n-1$ agents $k \in N_t$, including agent $z-1$, then $g(m^{t+1}) = a(\eta, \theta)$ and $g(m^\tau) = \underline{a}(\phi)$ for all $\tau > t+1$ and all m^τ.

  (c) If $m_k^{t+1} = (\cdot, \cdot, 1)$ for all $k \in N_t \backslash \{z-1\}$ while $m_{z-1}^{t+1} = (\cdot, \cdot, l)$ where $l \neq 1$, then $g(m^{t+1}) = b(\eta, \theta)$ and $g(m^\tau) = \underline{a}(\phi)$ for all $\tau > t+1$ and all m^τ.

  (d) For all other messages, let k^* be the agent who has announced the highest integer and suppose $m_{k^*}^{t+1} = (\psi, \cdot, l)$. Then, $g(m^{t+1}) = b(\phi, \theta)$ and $g(m^\tau) = \overline{a}(\psi)$ for all $\tau > t+1$ and all m^τ.

[9] Recall our convention that type θ agent prefers $a(\theta, \phi)$ to $b(\theta, \phi)$, while type ϕ agent prefers $b(\theta, \phi)$ to $a(\theta, \phi)$.

V. For all other messages, let k^* be the agent who has announced the highest integer and suppose $m^t_{k^*} = (\theta, \cdot, l)$. Then,

(a) If $k^* = z$, then $g(m^t) = c$, $g(m^{t+1}) = d$, and $g(m^\tau) = \overline{a}(\theta)$ for all $\tau > t+1$ and all m^τ.

(b) If $k^* \neq z$, then $g(m^t) = g(m^\tau) = \overline{a}(\theta)$ for all $\tau = t+1, \ldots, t+T-1$ and all m^τ, $g(m^{t+T+2j}) = c$ and $g(m^{t+T+2j+1}) = d$ for all $j \in \mathbb{Z}_+$ and all m^{t+T+j}.

If the messages fall under part I in period zT, we will say that they are *unanimous*. Thus, note that if the messages are unanimous in period zT, then they must also have been unanimous in periods kT for all $0 \leq k < z$. We will say that the unanimous messages are *truthful* if the agents report the true types of agents $z-1$ and z.

Any constant f is trivially implementable. Therefore, we assume that f is not constant. But then, the necessary conditions will only be satisfied if their premises are empty. In particular, it means that for every $\theta_1 \in \Theta$ there exists $(\theta_2, \ldots, \theta_n) \in \Theta^{n-1}$ such that $f(\theta_1, \theta_2, \ldots, \theta_n) \neq \underline{a}(\theta_1)$. Therefore, if f is implemented, no type expects his lowest possible lifetime payoff when he is born.

We first construct equilibrium strategies that implement f in SPE when T is sufficiently large. We only specify what the strategies prescribe agents to do on the equilibrium path and after unilateral deviations. (At the end of the proof, we argue that there is a Nash equilibrium in every subgame.) Let θ_k denote the true type of agent k for all $k \in \mathbb{Z}$. Let in period $t = 0$, $m^0_k = (\theta_{-1}, \theta_0, 0)$ for all $k \in N_0$. For any $t > 0$, let $z = \lfloor (t-1)/T \rfloor$ now denote the youngest agent that was alive in the previous period, and $m^t_k = (\theta_{\lfloor t/T \rfloor - 1}, \theta_{\lfloor t/T \rfloor}, 0)$ for all $k \in N_{\lfloor t/T \rfloor}$ if $m^{zT}_k = (\theta_{z-1}, \theta_z, 0)$ for at least $n-1$ agents $k \in N_{zT}$ including agent z; otherwise, $m^t_k = (\cdot, \cdot, 1)$ for all $k \in N_{\lfloor t/T \rfloor}$. That is, the strategies require every period to announce the true types of two youngest agents who are alive in that period and integer 0 as long as the last time when a new agent was born, at least $n-1$ agents including the newborn did exactly this; otherwise, keep announcing integer 1 forever. If they follow these strategies, f is clearly implemented in every period. Agent $z \in \mathbb{Z}$ can unilaterally change the outcome only if he deviates in period zT. (Hence, agents $-n+1, \ldots, -1$ cannot unilaterally change the outcome at all.) The deviation will trigger either part IIIb, IIIc, or IVb. Since type θ (we now drop the subscript z) prefers $a(\theta, \phi)$ to $b(\theta, \phi)$ for any $\phi \neq \theta$, agent z strictly prefers the outcome in IIIb to that in IIIc. It remains to check that the deviation to either IIIb or IVb is not profitable, and that the defined strategies form Nash equilibria in the subgames corresponding to IIIb and IVb.

We argued before that type θ does not expect his lowest possible lifetime payoff if f is implemented. Let $\overline{b}(\theta) = \arg\max_{a \in f(\Theta) \setminus \{\overline{a}(\theta)\}} u(a, \theta)$ and $\underline{b}(\theta) = \arg\min_{a \in f(\Theta) \setminus \{\underline{a}(\theta)\}} u(a, \theta)$ denote the best and worst alternatives for type θ in the range of f, which are different from $\overline{a}(\theta)$ and $\underline{a}(\theta)$, respectively. Also, let

$\pi = \min_{\theta \in \Theta} p(\theta)$. We can lower-bound the lifetime payoff of type θ, when f is implemented, by $nTu(\underline{a}(\theta), \theta) + \pi T(u(\underline{b}(\theta), \theta) - u(\underline{a}(\theta), \theta))$. (Note that the same alternative is selected during periods $kT \ldots, (k+1)T - 1$ if the agents send unanimous messages in period kT.) On the other hand, the payoffs of agent z from parts IIIb and IVb can be upper-bounded by $(nT-3)u(\underline{a}(\theta), \theta) + 3u(\overline{a}(\theta), \theta)$. Thus, agent z will not want to trigger either IIIb or IVb if this payoff is less than the lower bound on his lifetime payoff when f is implemented, which implies:

$$\pi T(u(\underline{b}(\theta), \theta) - u(\underline{a}(\theta), \theta)) \geq 3(u(\overline{a}(\theta), \theta) - u(\underline{a}(\theta), \theta)). \tag{1}$$

(1) will be satisfied if T is sufficiently large.

Also, it is easy to verify that the defined strategies form Nash equilibria in the subgames corresponding to IIIb and IVb. Only agent z can unilaterally deviate from IIIb and trigger IIIc, but we have already argued that it is not profitable. Similarly, only agent $z-1$ can unilaterally deviate from IVb and trigger IVc, but again it is not profitable because he strictly prefers $a(\theta_{z-1}, \phi)$ to $b(\theta_{z-1}, \phi)$ for any $\phi \neq \theta_{z-1}$. This completes the proof that the constructed strategies indeed form an SPE if T satisfies (1) for all θ.

In the remainder of the proof, we will argue that there do not exist undesirable equilibria. Thus, suppose we are in period $t = zT$ for some $z \in \mathbb{Z}_+$ and suppose that up to that period, agents had been sending unanimous messages in periods $0, T, 2T, \ldots (z-1)T$, that is, part I of the regime has applied so far. Suppose, however, the messages of period t fall under part V. These messages cannot be part of SPE: even if all agents in N_t had the same type θ, there is a disagreement between agents z and $z - n + 1$ because of the overlapping nature of their life spans. Agent z strictly prefers the sequence given in Va, while agent $z - n + 1$ strictly prefers the sequence given in Vb. Hence, each of them will want to win the integer game in part V.

Likewise, period $t = zT$ messages cannot fall under part III in any SPE. If this part is triggered, agent $z - n + 1$ does not get his highest possible continuation payoff because $a(\theta, \phi) \neq b(\theta, \phi)$ and $a(\theta, \phi) \neq \underline{a}(\theta)$ (the latter is true because type θ strictly prefers $a(\theta, \phi)$ to $b(\theta, \phi)$). On the other hand, if this agent deviates and triggers part V, he can get his highest possible continuation payoff. Hence, such a deviation is profitable. A similar argument implies that part IV cannot be triggered in any SPE. Later we will also rule out equilibria in which part II is played on the equilibrium path, but now we study equilibria, in which the agents are sending unanimous messages in periods $t = zT$ for all $z \in \mathbb{Z}_+$.

We first prove the following lemma:

Lemma 1 *Suppose there exists an equilibrium in unanimous but untruthful messages. Then, there always exists an agent, say, agent z and type of this agent, say, ϕ such that he expects less than his maximal lifetime payoff of $nTu(\overline{a}(\phi), \phi)$ and either the agents have sent untruthful messages about his type or the type of agent $z-1$.*

Proof of Lemma 1: Consider an equilibrium in unanimous but untruthful messages.

Because the premises of Condition C2 are empty, either there is no $a \in A$ that is the best alternative for all types, or there is no sequence of types $\theta_0, \theta_1, \dots$ such that $f(\theta_z, \dots, \theta_{z+n-1}) = a$ for all $z \in \mathbb{Z}$. We will consider each of these two cases separately.

Suppose first there does not exist $a \in A$ that is the best alternative for all types. Suppose that the agents in period zT lie about the true type, say, θ of agent z. If this agent does not receive his maximal lifetime payoff, we are done with the claim. If he receives the maximal payoff, it must be that alternative $\overline{a}(\theta)$ is implemented in every period during his lifetime. But by the assumption, there exists a type of agent $z+1$ for whom $\overline{a}(\theta)$ is not the best alternative, which again establishes the claim.

Suppose now that there exists $a \in A$ that is the best alternative for all types, but there does not exist an infinite sequence of types $\theta_0, \theta_1, \dots$ such that $f(\theta_z, \dots, \theta_{z+n-1}) = a$ for all $z \in \mathbb{Z}$. Let $\phi_0, \phi_1, \dots$ denote the true types and $\theta_0, \theta_1, \dots$ denote the reported types of agents $0, 1, \dots$. Suppose that $\theta_z \neq \phi_z$, but agent z expects the maximal possible lifetime payoff, that is, alternative a is implemented (at least) during periods $zT, \dots, (z+n)T-1$. We claim that there exists a profile of types $(\phi_{z+1}, \dots, \phi_{z+n-1}) \in \Theta^{n-1}$ such that the type of at least one of the agents among $z+1, \dots, z+n-1$ is misreported. Suppose not. Then, it must be that $f(\theta_z, \phi_{z+1}, \dots, \phi_{z+n-1}) = a$ for all $(\phi_{z+1}, \dots, \phi_{z+n-1}) \in \Theta^{n-1}$. But this contradicts the assumption that there does not exist an infinite sequence of types $\theta_0, \theta_1, \dots$ such that $f(\theta_k, \dots, \theta_{k+n-1}) = a$ for all $k \in \mathbb{Z}$: if every agent (starting agent $z+1$), irrespective of his true type, reported that his type is θ_z, the alternative a would be selected forever. Thus, there exists a realization of types $\phi_{z+1}, \dots, \phi_{z+n-1}$ such that the type of some agent among $z+1, \dots, z+n-1$ is misreported. Pick this agent (if there exist several, then pick the one with the highest index), and repeat now the above argument for this agent. Since we have assumed that there does not exist an infinite sequence of types $\theta_0, \theta_1, \dots$ that would allow the agents to obtain alternative a forever, ultimately there must be an agent whose type is misreported and who does not receive his highest possible lifetime payoff. This completes the proof of the lemma. ■

Thus, in any equilibrium with unanimous, but untruthful messages, there is an agent, say, z who expects less than his highest possible lifetime payoff and either the agents send untruthful messages about his type or the type of agent $z-1$. We will show that this agent can obtain higher payoff by triggering either part III or IV in period zT. To do that, we first analyse the equilibria of subgames after either part III or IV has been triggered.

Suppose first that period $t = zT$ messages fall under part III, that is, all agents in $N_t \backslash \{z\}$ announce $(\eta, \theta, \cdot)$, while agent z announces $(\cdot, \phi, \cdot)$ with $\phi \neq \theta$ and ϕ is his true type. First, there is an equilibrium that falls under part IIIa, in which everyone in N_t announces 0. There could also be an equilibrium that falls under part IIId if all agents in N_t agree on the best alternative. There cannot be an equilibrium falling under part IIIb because agent z would prefer to deviate and trigger part IIIc. Neither there can be an equilibrium falling under part IIIc because any agent in $N_t \backslash \{z\}$ would prefer to deviate and trigger part IIId. To summarize, in any equilibrium after part III has been triggered, the sequence $a(\theta, \phi), b(\theta, \phi), \overline{a}(\phi), \overline{a}(\phi), \ldots$ is implemented.

Suppose now that period $t = zT$ messages fall under part IV, that is, agents in $N_t \backslash \{z\}$ all announce $(\eta, \phi, \cdot)$, while agent z announces $(\theta, \phi, \cdot)$ where $\theta \neq \eta$, ϕ is the true type of agent z, and θ is the true type of agent $z-1$. Like above, there is an equilibrium that falls under part IVa and possibly there is an equilibrium that falls under part IVd. There cannot be an equilibrium falling under part IVb because agent $z-1$ would prefer to trigger part IVc. Neither there can be an equilibrium falling under part IVc because agent z is better off by triggering part IVd: even if $b(\eta, \theta) = \overline{a}(\phi)$ and $b(\phi, \theta) = \underline{a}(\phi)$, agent z receives his best alternative for more periods in part IVd. To summarize, in any equilibrium after part IV has been triggered, the sequence $a(\phi, \theta), b(\phi, \theta), \overline{a}(\phi), \overline{a}(\phi), \ldots$ is implemented.

The lifetime payoff of agent z of type ϕ if he deviates when the other agents have sent untruthful messages either about his type or the type of agent $z-1$ can be lower-bounded by $(nT-2)u(\overline{a}(\phi), \phi) + u(\underline{a}(\phi), \phi) + u(\underline{b}(\phi), \phi)$ (where we use the fact that $a(\theta, \phi) \neq b(\theta, \phi)$ and $a(\phi, \theta) \neq b(\phi, \theta)$). On the other hand, if his lifetime payoff from sticking to unanimous, but untruthful messages is less than maximal (and we know that there exists such an agent), this payoff can be upper-bounded by $nTu(\overline{a}(\phi), \phi) + \pi T(u(\overline{b}(\phi), \phi) - u(\overline{a}(\phi), \phi))$. The agent will have incentives to deviate if the former payoff exceeds the latter, which implies that the following inequality must hold:

$$2u(\overline{a}(\phi), \phi) - u(\underline{a}(\phi), \phi) - u(\underline{b}(\phi), \phi) < \pi T(u(\overline{a}(\phi), \phi) - u(\overline{b}(\phi), \phi)). \tag{2}$$

(2) will be satisfied if T is sufficiently large. We conclude that there are no equilibria in unanimous, but untruthful messages if T satisfies (2) for all ϕ.

It remains to consider period zT messages that fall under part II. Clearly, neither agent $z-1$, nor agent z expects his maximal lifetime payoff in this case. Then depending whether the type of agent $z-1$ is misreported in period $(z-1)T$ or in zT, either agent $z-1$ will deviate in period $(z-1)T$ or agent z will deviate in period zT if T satisfies (2). Hence, part II will never be triggered in the equilibrium.

Finally, we need to verify that there is a Nash equilibrium in every subgame. Suppose first agents have sent unanimous and truthful messages about the type of agent $z-1$ in period $(z-1)T$. Consider a subgame starting in period zT. We can construct strategies for this subgame similar to those that we constructed to show that the regime implements f. Basically, the agents announce unanimous and truthful messages in periods kT for all $k \geq z$. Agent k can unilaterally change the outcome only in period kT to that given either by part IIIb or IVb.

The inequality in (1) still ensures that such a deviation is not profitable. (Since the types of agents $z-n+1,\ldots,z-2$ need not be truthfully announced in this subgame, f might not be implemented during periods $zT,\ldots,(z+n-2)T-1$ while agents $z-n+1,\ldots,z-2$ are alive. However, the lower bound on lifetime payoffs for agents $k \geq z$ is still $nTu(\underline{a}(\theta),\theta)+\pi T(u(\underline{b}(\theta),\theta)-u(\underline{a}(\theta),\theta))$.)

Suppose next agents have sent unanimous but untruthful messages about the type of agent $z-1$ in period $(z-1)T$. Consider again a subgame starting in period zT. Then, there exists an equilibrium in which agents announce the true types of agents $z-1$ and z in that period. These messages fall under part II of the regime. If agent z deviates, then the outcome is given either by part IIIb or IVb. Again, (1) ensures that such a deviation is not profitable. This proves that there are Nash equilibria in subgames starting in periods zT for all z. We already know that there are also Nash equilibria in subgames starting in periods $zT+1$ and they correspond to parts IIIa or IVa. And, finally, there exist equilibria in subgames starting in all other periods simply because the outcome of the regime does not depend on the messages of those periods.

To complete the proof, define T^* as the smallest $T \in \mathbb{Z}_+$ that satisfies (1) and (2) for all $\theta \in \Theta$. We have shown that the regime r implements f in SPE for any $T \geq T^*$.

6 Discussion

Relaxing Assumption A1. If we allowed for weak preferences, we would need to generalize Conditions C2 and C3, but we would still expect them to be not only necessary, but also sufficient for T large enough. Though, we might also need to adjust the regime to ensure that parts II-V are still not played on the equilibrium path. We do not attempt to generalize Conditions C2 and C3, but note that they remain necessary and sufficient in their current form as long as the best and worst alternatives, $\overline{a}(\theta)$ and $\underline{a}(\theta)$, are unique for all θ. Thus, Assumption A1 can already be relaxed quite substantially.

Also, consider the example in Section 2. Even though Assumption A1 is not satisfied, f in the example and, in fact, any f vacuously satisfies Conditions C2 and C3. Since it is an environment with private goods, there does not exist an allocation that is best for everyone. Also, since every feasible allocation gives strictly positive quantities of both goods to every agent who is alive in a given period, f never selects the worst allocation for any type.

Relaxing Assumption A2. Assumption A2 can also be substantially relaxed at the cost of requiring even larger T. In particular, it is enough if the utility function of one type is not an affine transformation of the utility function of another type:

Assumption A3 *For every $\theta, \phi \in \Theta$, there do not exist $\alpha > 0$ and β such that $u(a,\theta) = \alpha u(a,\phi) + \beta$ for all $a \in A$.*

Assumption A3 says that for every pair $\theta, \phi \in \Theta$, we can always find two lotteries over alternatives in A (where the probabilities are rational numbers) such that type θ prefers one lottery over the other, while it is opposite for type ϕ. But with each lottery we can associate a (finite) sequence of alternatives where the frequency of each alternative in the sequence is equal to the probability that this alternative is chosen in the corresponding lottery. Then, in the regime, we can everywhere replace alternatives $a(\theta, \phi)$ and $b(\theta, \phi)$ with the constructed sequences.[10]

Stochastic regimes. We have assumed that the designer can only employ deterministic regimes. If T exceeds T^* given in the proof of the theorem, then allowing for stochastic regimes will not expand the set of social choice functions that are implementable in SPE. However, the use of stochastic regimes can reduce T^* that is required for a function to be implementable. In fact, we will now argue that as long as the necessary conditions are satisfied, f is implementable with a stochastic regime for $T = 2$ (even if we replace Assumption A2 with A3). For that, we modify parts III and IV in our original regime as follows:

III' If there is $(\eta, \theta) \in \Theta^2$ such that $m_k^t = (\eta, \theta, \cdot)$ for all $k \in N_t \backslash \{z\}$ and $m_z^t = (\cdot, \phi, \cdot)$ where $\phi \neq \theta$, then $g(m^t) = \tilde{e}$ and we proceed to period $t+1$ where

- (a) If $m_k^{t+1} = (\cdot, \cdot, 0)$ for at least $n-1$ agents $k \in N_t$, then $g(m^{t+1}) = g(m^\tau) = \overline{a}(\phi)$ for all $\tau > t+1$ and all m^τ.
- (b) If $m_k^{t+1} = (\cdot, \cdot, 1)$ for at least $n-1$ agents $k \in N_t$, including agent z, then $g(m^{t+1}) = \tilde{a}(\theta, \phi)$ and $g(m^\tau) = \underline{a}(\theta)$ for all $\tau > t+1$ and all m^τ.
- (c) If $m_k^{t+1} = (\cdot, \cdot, 1)$ for all $k \in N_t \backslash \{z\}$ while $m_z^{t+1} = (\cdot, \cdot, l)$ where $l \neq 1$, then $g(m^{t+1}) = \tilde{b}(\theta, \phi)$ and $g(m^\tau) = \underline{a}(\theta)$ for all $\tau > t+1$ and all m^τ.
- (d) For all other messages, let k^* be the agent who has announced the highest integer and suppose $m_{k^*}^{t+1} = (\psi, \cdot, l)$. Then, $g(m^{t+1}) = g(m^\tau) = \overline{a}(\psi)$ for all $\tau > t+1$ and all m^τ.

IV' If there is $(\eta, \phi) \in \Theta^2$ such that $m_k^t = (\eta, \phi, \cdot)$ for all $k \in N_t \backslash \{z\}$ and $m_z^t = (\theta, \phi, \cdot)$ where $\theta \neq \eta$, then $g(m^t) = \tilde{e}$ and we proceed to period $t+1$ where

- (a) If $m_k^{t+1} = (\cdot, \cdot, 0)$ for at least $n-1$ agents $k \in N_t$, then $g(m^{t+1}) = g(m^\tau) = \overline{a}(\phi)$ for all $\tau > t+1$ and all m^τ.
- (b) If $m_k^{t+1} = (\cdot, \cdot, 1)$ for at least $n-1$ agents $k \in N_t$, including agent $z-1$, then $g(m^{t+1}) = \tilde{a}(\eta, \theta)$ and $g(m^\tau) = \underline{a}(\phi)$ for all $\tau > t+1$ and all m^τ.

[10] Although we can, we do not need to replace *everywhere*. For example, if agent z triggers either part III or IV of the regime, then in period $t = zT$, it is enough to give some fixed alternative.

(c) If $m_k^{t+1} = (\cdot,\cdot,1)$ for all $k \in N_t \backslash \{z-1\}$ while $m_{z-1}^{t+1} = (\cdot,\cdot,l)$ where $l \neq 1$, then $g(m^{t+1}) = \tilde{b}(\eta,\theta)$ and $g(m^\tau) = \underline{a}(\phi)$ for all $\tau > t+1$ and all m^τ.

(d) For all other messages, let k^* be the agent who has announced the highest integer and suppose $m_{k^*}^{t+1} = (\psi,\cdot,l)$. Then, $g(m^{t+1}) = g(m^\tau) = \overline{a}(\psi)$ for all $\tau > t+1$ and all m^τ.

In the modified regime, $\tilde{e}$ denotes a lottery that chooses the alternative that would be selected if agent z had sent the same message as everyone else (that is, $f(\theta_{z-n+1},\ldots,\theta_{z-1},\theta)$ if the messages of others correspond to part I or c if the messages correspond to part II) with a very high probability and a different alternative with the remaining probability. $\tilde{a}(\theta,\phi)$ (resp., $\tilde{a}(\eta,\theta)$) is a lottery that chooses $\underline{a}(\theta)$ (resp., $\underline{a}(\phi)$) with a very high probability and $a(\theta,\phi)$ (resp., $a(\eta,\theta)$) with the remaining probability. $\tilde{b}(\theta,\phi)$ and $\tilde{b}(\eta,\theta)$ are defined similarly. (If we make Assumption A3, $a(\theta,\phi),\ldots,b(\eta,\phi)$ themselves are lotteries.)

One can verify that inequalities similar to (1) and (2) are satisfied for $T = 2$. Roughly speaking, (1) is satisfied for $T = 2$ because $u(\overline{a}(\theta),\theta)$ on the right hand side is replaced with a payoff that is arbitrarily close to $u(\underline{a}(\theta),\theta)$. Similarly, (2) is satisfied for $T = 2$ because $u(\underline{a}(\phi),\phi) + u(\underline{b}(\phi),\phi)$ on the left hand side is replaced with a payoff that is arbitrarily close to $2u(\overline{a}(\phi),\phi)$. Also, because $\tilde{e}$ is not the best alternative for any agent, there still cannot be an SPE in which either part III' or IV' is triggered on the equilibrium path.

(Re)starting the implementation. It is assumed that the designer knows the types of agents $-n+1,\ldots,-1$. Suppose that she does not know them. One possibility is to fix arbitrary types for these agents. Then, if there exists an equilibrium in unanimous messages, we know from the proof of the theorem that these messages must be truthful. Although f will not be implemented from period 0, it will be implemented from period $(n-1)T$, once agents $-n+1,\ldots,-1$ will have passed away. The problem, however, is that this equilibrium might not exist: because the type of agent -1 is chosen arbitrarily, agent 0 might prefer to trigger part IV. Instead, the equilibrium strategies would involve period 0 messages that fall under part II of the regime. To avoid this outcome, we need to elicit the true type of agent -1 before we apply the regime r.

Suppose we are in period $-T$. The designer can fix arbitrary types for agents $-n+1,\ldots,-2$ but ask agent -1 to announce his type and select the best alternative for that type during periods $-T,\ldots,-1$. The agent, however, might still prefer to lie and receive the outcome corresponding to part II. Therefore, we modify this part for $t = 0$ as follows:

II' If there is $(\eta,\theta) \in \Theta\backslash\{\theta_{-1}\} \times \Theta$ s.t. $m_k^0 = (\eta,\theta,\cdot)$ for at least $n-1$ agents $k \in N_0$ including agent 0, then $g(m^0) = g(m^\tau) = \underline{a}(\eta)$ for $\tau = 1,\ldots,(n-1)T-1$, and $g(m^\tau) = \overline{a}(\theta)$ if $\overline{a}(\theta) \neq \underline{a}(\eta)$; otherwise, $g(m^\tau) = \underline{a}(\theta)$ for all $\tau \geq (n-1)T$ and all m^τ.

With this modification, it is still true that agent 0 does not want to trigger either part IIIb or IVb if the agents are truthful about the types of agents -1 and 0 in part II',[11] but he has the incentives to trigger either part IIIa or IVa if the agents are not truthful. Furthermore, agent -1 does not gain by lying about his type in period $-T$. If he lies, he cannot improve his payoff during periods $-T, \ldots, -1$, while he gets his lowest possible payoff in the equilibrium from period 0 on. On the other hand, since f is not constant, agent -1 does not expect his lowest payoff from period 0 on in the equilibrium with unanimous messages if he announces his type honestly. Hence, he strictly prefers to announce his true type in period $-T$.

The regime also has the property that once the messages differ from those in part I, one of the predetermined infinite sequences is implemented. This can be costly from the perspective of the social designer. However, we can modify the regime in a way that allows to restart the implementation of f if a deviation from part I has occurred. Specifically, if such a deviation occurs in period zT, we can terminate all those sequences that are defined in the regime by the end of period $(z+n)T-1$, when all agents who were alive in period zT, have passed away. The designer can then fix arbitrary types of agents $z+1, \ldots, z+n-1$, elicit the type of agent $z+n$ as above, and restart r from period $(z+n+1)T$. This will ensure that f is again implemented from period $(z+2n-1)T$ onwards (once agents $z+1, \ldots, z+n-1$ have passed away). Thus, the regime has certain robustness against mistakes of agents.

Less than complete/perfect information. The game induced by the regime r is a game with perfect information and simultaneous moves. From the proof of the theorem, it is clear that we can relax the assumption of complete information about agents' types. It is enough if the type of agent z is common knowledge between the agents in $N_{zT} \cup \{z+1\}$. Further, it is enough if only period zT messages are observable for all $z \in \mathbb{Z}_+$ and only by the agents in N_{zT}. With such less than perfect information, any f satisfying the necessary conditions is implemented for sufficiently large T by the regime r in *extended subgame perfect equilibrium* (for the definition, see page 877 in Kreps and Wilson, 1982) or in any of its refinements such as sequential equilibrium or belief-free equilibrium.[12]

[11] The worst outcome for agent 0 is if he receives $\underline{a}(\theta) = \underline{a}(\eta)$ for $(n-1)T$ periods and receives $\overline{a}(\theta)$ for T periods. It is strictly better than the outcome in part IIIb if $T > 3$. On the other hand, since $\pi \leq \frac{1}{2}$ and $u(\underline{b}(\theta), \theta) - u(\underline{a}(\theta), \theta) \leq u(\overline{a}(\theta), \theta) - u(\underline{a}(\theta), \theta)$, it follows that (1) is satisfied if $T \geq 6$.

[12] We choose extended SPE because our view is that the weaker is the equilibrium concept, the more robust is the implementation. If f is implementable in extended SPE, then it is also implementable in any of its refinements provided that they exist. The converse obviously need not be true. Furthermore, we can even relax the assumption about Bayesian updating in the definition of extended SPE. For example, agents do not need to believe that the types of older generations are drawn according to p.

Appendix

Proof of Proposition 1: Suppose f is repeatedly implementable in SPE. Let s denote an SPE. Suppose that the premises of Condition C2 hold. Using s, we will construct another equilibrium strategy profile s' that implements alternative a forever after some history where a is defined in Condition C2.

Let $\theta_{-n+1}, \ldots, \theta_{-1}$ be given. Let $\theta_0, \theta_1, \ldots$ be the sequence defined in Condition C2. Let $\zeta_t = (\theta_{-n+1}, \ldots, \theta_{\lfloor t/T \rfloor})$ be period t history of types that is obtained from these two sequences. Let $\mu_t = (\mu_{t-1}, s(h_{t-1}))$ be period t history of messages and $h_t = (\zeta_t, \mu_t)$ be period t history (which are obtained recursively) that would arise if the agents followed strategies s and the history of types was ζ_t. We reserve notation ζ_t, μ_t, h_t to these particular histories. Also note that these histories occur with a strictly positive probability if the agents follow strategies s.

For any period $t \geq nT$, let Θ_t denote the set of period t histories of types that has ζ_{nT} as a sub-history and let $H'_t = \{h'_t = (\zeta'_t, \mu_t) | \zeta'_t \in \Theta_t\}$. Thus, H'_t consists of histories that only differ from h_t in their history of types after period nT. Let $H'_t = \{h_t\}$ for all $t = 0, \ldots, nT-1$. Let s' be defined as follows: $s'(h'_t) = s(h_t)$ for all $h'_t \in H'_t$ and $s'(h''_t) = s(h''_t)$ for all $h''_t \in H_t \backslash H'_t$. Thus, after any $h'_t \in H'_t$, the strategies s' tell the agents to send the same messages as after history h_t.

If the agents follow strategies s', alternative a is implemented forever after history h_{nT} irrespective of the realised types. We verify that s' is an SPE. Since s is an SPE, then by definition, s' implies Nash equilibrium play in subgames starting after any $h''_t \in H_t \backslash H'_t$. Consider a subgame after any $h'_t \in H'_t$ for $t \geq nT$. Since the agents receive their best alternative in every period, they do not have incentives to deviate. Finally, consider a subgame after h_t for $t = 0, \ldots, nT-1$. The payoff of any agent who is alive in period t and deviates from s', is the same as his payoff if he deviated from s. On the other hand, his payoff if he does not deviate from s' is higher than his payoff if he follows s (because he obtains his best alternative in future periods with higher probability). Since he did not have incentives to deviate from s, he does not have incentives to deviate from s'. Thus, s' is an SPE. But then, if f is repeatedly implementable in SPE, it must be a constant function that always selects a. ■

Proof of Proposition 2: The proof is somewhat involved. Therefore, we first try to provide a verbal explanation. Given an equilibrium strategy profile, we construct another strategy profile with the following properties. Consider a period zT history such that agents $z-n+1, \ldots, z-1$ are of type θ, where θ is defined in Condition C3. Starting from period zT, agents play according to the original strategies but as if the types of agents $z, z+1, \ldots$ were also θ, even if they are not. If there is ever a unilateral deviation, say, in period t, then the agents continue to play according to the original strategies but as if the types of those who were born between periods zT and t were θ, while the types of those who are born after period t correspond to their true types. (There can be further deviations, but they all must be unilateral. If a multilateral deviation occurs, then the agents revert back to the original strategies.)

Now, if the agents follow the new strategies, then $\underline{a}(\theta)$ is implemented forever from period zT onwards. We claim that no agent has incentives to unilaterally deviate. If an agent could obtain anything different than $\underline{a}(\theta)$ in any of the periods during his lifetime, then this agent would have incentives to deviate from the original strategies when his true type was indeed θ, thus contradicting the assumption that the original strategies formed an SPE. One can further argue that the new strategies also form Nash equilibria in subgames after a unilateral deviation has occurred. If, say, a deviation occurs in period t, then it must still be the case that the agents who are born before or in period t cannot profitably deviate in periods $t+1, t+2, \ldots$. If they could, then they could also profitably deviate from the original strategies when their true types were indeed θ. On the other hand, the agents who are born after period t do not have a profitable deviation because their play under the new strategies is still conditioned on their true type, that is, they do not pretend about their type under the new strategies. (And they only care what actions their opponents take, not whether their opponents pretend or not.)

We now turn to a formal proof. Suppose f is repeatedly implementable in SPE. Let s denote an SPE. Suppose that the premise of Condition C3 holds. Using s, we will construct another equilibrium strategy profile s' that implements alternative $\underline{a}(\theta)$ forever after some history where θ is defined in Condition C3.

Let $\theta_{-n+1}, \ldots, \theta_{-1}$ be given. For all $t \geq 0$, let ζ_t denote a history of types such that $\theta_z = \theta$ for all $z = 0, \ldots, \lfloor t/T \rfloor$, that is, every agent who was born between periods 0 and t, is of type θ. Let $\mu_t = (\mu_{t-1}, s(h_{t-1}))$ be period t history of messages and $h_t = (\zeta_t, \mu_t)$ be period t history (which are obtained recursively) that would arise if the agents followed strategies s and the history of types was ζ_t. We again reserve notation ζ_t, μ_t, h_t to these particular histories. Also note that these histories occur with a strictly positive probability if the agents follow strategies s.

Let $H'_t = \{h_t\}$ for all $t = 0, \ldots, 2nT-1$. Let $N_t = \{\lfloor t/T \rfloor - n + 1, \ldots, \lfloor t/T \rfloor\}$ denote the set of agents who are alive in period t. For any period $t \geq 2nT$, we define H'_t recursively as follows:

$$H'_t = \left\{ h'_t = (\zeta'_t, \mu'_t) \;\middle|\; \begin{array}{l} \exists h'_{t-1} = (\zeta'_{t-1}, \mu'_{t-1}) \in H'_{t-1}, \phi \in \Theta, z \in N_{t-1}, m_z^{t-1} \in M \\ \text{s.t. } \zeta'_t = (\zeta'_{t-1}, \phi) \text{ if } t/T \in \mathbb{Z}, \text{ otherwise } \zeta'_t = \zeta'_{t-1}, \\ \text{and } \mu'_t = (\mu'_{t-1}, (m_z^{t-1}, s_{-z}(h'_{t-1}))) \end{array} \right\}.$$

In words, if $h'_t \in H'_t$, then the message history μ'_t is such that in every period, at most one agent sends a message different from what is prescribed by s. We narrow the set H'_t to

$$H''_t = \left\{ h'_t \in H'_t \;\middle|\; \begin{array}{l} \text{if for some } \tau < t, h_\tau \text{ is a subhistory of } h'_t, \text{ but } h_{\tau+1} \text{ is not,} \\ \text{then } \exists z \in N_\tau, m_z^\tau \in M \text{ s.t. } m_z^\tau \neq s_z(h_\tau) \end{array} \right\}.$$

That is, the reason why h'_t diverged from h_t is because in some period $\tau < t$, an agent deviated from s, and not because the type of agent $(\tau+1)/T$ is different from θ, provided that $(\tau+1)/T \in \mathbb{Z}$. (However, once a deviation takes place, the types of agents that are born after period τ can differ from θ.) For every $h'_t \in H''_t$, let us denote τ that appears in the definition of H''_t as $\tau(h'_t)$. If no such τ exists, then set $\tau(h'_t) = t$. (Note that h_{2nT-1} is a subhistory for all $t \geq 2nT$ and all $h'_t \in H''_t$. Therefore, the only reason why such τ does not exist, is because $h'_t = h_t$.) Let $z(h'_t) = \lfloor \tau(h'_t)/T \rfloor$ denote the youngest agent who is alive in period $\tau(h'_t)$.

Below, for all $z = -n+1, \ldots, \lfloor t/T \rfloor$, θ'_z (resp., $\hat{\theta}_z$) will refer to the $z+n$-th coordinate of ζ'_t (resp., $\hat{\zeta}_t$). Let $\hat{H}_t$ be defined as follows:

$$\hat{H}_t = \left\{ \hat{h}_t = (\hat{\zeta}_t, \hat{\mu}_t) \,\middle|\, \begin{array}{l} \exists h'_t = (\zeta'_t, \mu'_t) \in H''_t \text{ s.t. } \hat{\mu}_t = \mu'_t \text{ and } \hat{\theta}_z = \theta'_z \\ \forall z = -n+1, \ldots, 2n-1 \text{ and } z = z(h'_t)+1, \ldots, \lfloor t/T \rfloor \end{array} \right\}.$$

Thus, every history $\hat{h} \in \hat{H}_t$ only differs from some $h'_t \in H''_t$ in the types of agents $2n, \ldots, z(h'_t)$.

Finally, let σ be a mapping that associates for all $t \geq 0$, every history $\hat{h}_t = (\hat{\zeta}_t, \hat{\mu}_t) \in \hat{H}_t$ with a history $h'_t = (\zeta'_t, \mu'_t) \in H''_t$ as follows: $\sigma(\hat{h}_t) = h'_t$ if $\hat{\mu}_t = \mu'_t$ and $\hat{\theta}_z = \theta'_z$ for all $z = -n+1, \ldots, 2n-1$ and all $z = z(h'_t)+1, \ldots, \lfloor t/T \rfloor$.

Let s' be defined as follows: $s'(h'_t) = s(\sigma(h'_t))$ for all $h'_t \in \hat{H}_t$ and $s'(h'_t) = s(h'_t)$ for all $h'_t \in H_t \backslash \hat{H}_t$. In words, as long as no deviation occurs, agents play according to s but as if their types were θ. If a unilateral deviation occurs, they still play according to s but only as if the types of agents who were born before the deviation occurred (except possibly types $-n+1, \ldots, -1$), were θ.

If the agents follow strategies s', alternative $\underline{a}(\theta)$ is implemented forever after history h_{2nT-1} irrespective of the realised types. We verify that s' is an SPE. Since s is an SPE, then by definition, s' implies Nash equilibrium play in subgames starting after any $h'_t \in H_t \backslash \hat{H}_t$. The same is true for subgames after h_t for all $t = 0, \ldots, (n+1)T-1$ since agents $z = -n+1, \ldots, n$ face exactly the same incentives under both s and s'. (They will not be alive by period $2nT$ when s and s' start to differ.) Consider a subgame after any $h'_t \in \hat{H}_t$ for any $t \geq (n+1)T$. Suppose agent $z > z(h'_t)$ is alive in period t. The messages that his opponents send in the subgame after history h'_t if they follow s' are exactly the same as the messages that they would send in the subgame after history $\sigma(h'_t)$ if they followed s. Further, the type of agent z is also exactly the same. Since he does not have incentives to deviate from s in subgame $\sigma(h'_t)$, then he also does not have incentives to deviate from s' in subgame h'_t. (Note that the regime only depends on the history of messages.)

Suppose agent $n+1 \leq z \leq z(h'_t)$ is alive in period t. We claim that this agent cannot obtain anything else than alternative $\underline{a}(\theta)$ in every period that he is alive, starting period t, by unilaterally deviating from s'. If he could obtain a different sequence of alternatives, then he could also obtain the same sequence if the history was $\sigma(h'_t)$ and, hence, he was of type θ, and the agents followed strategies s. But since any sequence that is different from receiving alternative $\underline{a}(\theta)$ in every period is strictly preferred by type θ, this agent has a profitable deviation from s after history $\sigma(h'_t)$, contradicting that s is an SPE. Thus, s' is also an SPE. But then, if f is repeatedly implementable in SPE, it must be a constant function that always selects $\underline{a}(\theta)$. ■

References

Abreu, D., and A. Sen (1990): "Subgame Perfect Implementation: A Necessary and Almost Sufficient Condition," *Journal of Economic Theory*, 50(2), 285 – 299.

Āzacis, H., and P. Vida (2015): "Repeated Implementation," SFB/TR 15 Discussion Paper 518.

Chambers, C. P. (2004): "Virtual Repeated Implementation," *Economics Letters*, 83(2), 263 – 268.

Hayashi, T., and M. Lombardi (2016): "Temporary Implementation," Available at SSRN: https://ssrn.com/abstract=2786946.

Kalai, E., and J. O. Ledyard (1998): "Repeated Implementation," *Journal of Economic Theory*, 83(2), 308 – 317.

Kandori, M. (1992): "Repeated Games Played by Overlapping Generations of Players," *The Review of Economic Studies*, 59(1), 81–92.

Kreps, D. M., and R. Wilson (1982): "Sequential Equilibria," *Econometrica*, 50(4), 863–894.

Lee, J., and H. Sabourian (2011): "Efficient Repeated Implementation," *Econometrica*, 79(6), 1967–1994.

Mezzetti, C., and L. Renou (2017): "Repeated Nash implementation," *Theoretical Economics*, 12(1), 249–285.

Moore, J., and R. Repullo (1988): "Subgame Perfect Implementation," *Econometrica*, 56(5), 1191–1220.

Roemer, J., and K. Suzumura (eds.) (2007): *Intergenerational Equity and Sustainability*, International Economic Association Series. Palgrave Macmillan.

Salant, D. J. (1991): "A Repeated Game with Finitely Lived Overlapping Generations of Players," *Games and Economic Behavior*, 3(2), 244 – 259.

Smith, L. (1992): "Folk Theorems in Overlapping Generations Games," *Games and Economic Behavior*, 4(3), 426 – 449.

Vartiainen, H. (2007): "Subgame Perfect Implementation: A Full Characterization," *Journal of Economic Theory*, 133(1), 111 – 126.