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Can the learnability criterion ensure determinacy in New Keynesian Models?

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Abstract

Forward-looking RE models such as the popular New Keynesian (NK) model do not provide a unique prediction about how the model economy behaves. We need some mechanism that ensures determinacy. McCallum (2011) says it is not needed because models are learnable only with the determinate solution and so the NK model, once learnt in this way, will be determinate. We agree: the only learnable solution that has agents converge on the true NK model is the bubble-free one. But once they have converged they must then understand the model and its full solution therefore including the bubble. Hence the learnability criterion still fails to pick a unique RE solution in NK models.

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1 Introduction

Determinacy is a longstanding issue in Rational Expectations (RE) models with forward-looking terms (the first to focus on it were Gourieroux et al., 1982). These terms enable a 'non-fundamental' or 'bubble' solution to be found besides the usual one only containing fundamentals. The usual practice to ignore these alternative solutions. For example, in the New Keynesian Taylor Rule (NK) model of inflation determination (which have been the centerpiece of monetary analysis over the past two decades), King (2000) and Woodford (2003) claim bubble paths are somehow impossible.[1] However, there needs to be a good reason to ignore these alternative solutions. As Cochrane (2011) has argued this is insufficient: a) these paths are 'possible' (nothing would stop them if they happened) but b) they are also incredible since they involve hyperinflation/hyperdeflation ('the Fed blowing up the world').

How much does all this matter? Models without determinacy (the absence of a unique RE solution) such as the popular NK model, are problematic as they stand and so do not rate as models of interest. They do not provide a unique prediction about how the model economy- and thus the actual economy being modeled- behaves. So there must be *some mechanism* that ensures determinacy (the Taylor Principle does not do it as argued by Cochrane, 2011).Consequently, a number of additional requirements have been proposed by various researchers in order to obtain a unique rational expectations equilibrium (REE).[2]

McCallum (2011) agrees with Cochrane's analytical point on the non-uniqueness of REE but goes on to defend the NK model: the bubble paths are 'not learnable' and learnability is a condition for a model to be well-founded. His thesis is that the bubble solution does not converge on the RE solution i.e., the bubble path is 'not learnable'. However, the stable solution is learnable: hence the NK model, when it is learnt, will have a unique stable solution. But it is hard to know what meaning to attach to the idea of a 'solution' (purely) being learnt.

In general in the learning literature the related question asked is whether agents when learning will converge on the rational expectations solution and so learn the RE model, so that henceforth it can operate as an RE model. Thus the convergence is to the RE model. So does McCallum mean by 'solutions are learnt' a) that people then know the model and it acts like a rational expectations model, having thus been learnt? or does he mean b) that people know the model and it is a rational expectations

[1] The NK approach to monetary economics provides the current standard model of inflation determination. By linking interest rate decisions directly to inflation and economic activity, the Taylor Rule offers a convenient tool for studying monetary policy while abstracting from a detailed analysis of the demand and supply of money. This change in the standard analytics is an understandable reflection of how most central banks now make monetary policy: by setting a short-term nominal interest rate, with little if any explicit role for money (see Friedman, 2003). Furthermore, econometric evidence supporting the stabilization properties of this rule (see Taylor, 1999) and its usefulness for understanding historical monetary policy (see Clarida et al., 2000) explains its popularity.

[2] These include a transversality condition on money supply behaviour that would rule out this explosive solution for the inflation rate (Minford and Srinivasan, 2011a, b) and non-Ricardian fiscal policy (Cochrane, 2011).

model, but these people are only aware of one solution i.e., the stable one? If a) then as we know they will know the general solution of an NK model which includes the bubble solution. If b) then they have not learnt the model since they will be unaware of the general solution that it implies! In this case, after 'learning', we would have some model with an autoregressive expectations process, and not the NK RE model which is supposed to be 'supported' by learnability. Thus the only intelligible statement that McCallum could be making seems to be a). But a) implies that if and when it is learnt the NK model has a bubble solution to be found besides the usual one only containing fundamentals.

In sum we can all agree: NK models have serious problems and are not 'proper models'. Also they are proper models if the indeterminacy is somehow removed. But this does not tell us what mechanism inside the model can remove indeterminacy: note learnability is not a mechanism inside the model. It is a desirable final attribute of a good model. Once the model has been learnt, it is impossible to stop agents from knowing the true RE model and discovering the general solution with bubbles. McCallum's analysis gives us no remedy for this. Unfortunately it is still a model with a non-uniqueness problem. Hence we agree with Cochrane that imposing 'desirable attributes' on the model does not answer the root theoretical question: how can we modify this model internally to make it determinate?

This article is organized as follows. In Section 2 we study determinacy in the standard three-equation NK model. We explain how researchers deal with multiple equilibria in these models. In Section 3 we review the concept of E-stability, explain how this criterion is alleged to select the economically relevant RE solution in cases in which multiple equilibria obtain, and we show that it fails to do so. In Section 4 we argue that a terminal condition on monetary behaviour justified by welfare can modify the NK model internally to make it determinate. Section 5 provides concluding remarks.

2 Eliminating multiple equilibria in New-Keynesian models-the role of the Taylor Principle

Now let us consider a standard NK model with frictions (for example, see Clarida et al., (1999), Bullard and Mitra (2002) and Woodford (2003)). For determinacy questions, we can work with a stripped-down model without constants or shocks.

$$\pi_t = \beta E_t \pi_{t+1} + \lambda z_t, \quad 0 < \beta < 1,\ \lambda > 0 \tag{2.1}$$

$$z_t = E_t z_{t+1} - \frac{1}{\sigma}\left(r_t - E_t \pi_{t+1}\right), \quad \sigma > 0 \tag{2.2}$$

where π_t = inflation, z_t = the output gap and r_t = the nominal interest rate. This representation can represent deviations from a specific equilibrium of a model with shocks (see Cochrane, 2011). The first equation is the NK Phillips curve (NKPC). It is derived from the first order conditions of intertemporallyoptimizing firms that set prices subject to costs.[3] The second equation is a log-linear approximation to an Euler equation for the timing of aggregate expenditure, sometimes called an "intertemporal IS relation". This is the one that indicates how monetary policy affects aggregate expenditure: the expected short-term real rate of return determines the incentive for intertemporal substitution between expenditure in periods t and $t+1$.

As it stands this is a 2 equation, 3 unknown (π_t, z_t, r_t) model. The remaining equation required to close the system is a specification of monetary policy. We might, for example, close the model by assuming $r_t = \overline{r}$, a constant. Substituting for $r_t = \overline{r}$ in (2.2), the model (2.1-2.2) can be written in the form,

$$\begin{bmatrix} \beta & 0 \\ \frac{1}{\sigma} & 1 \end{bmatrix} \begin{matrix} E_t\pi_{t+1} \\ E_t z_{t+1} \end{matrix} = \begin{bmatrix} 1 & -\lambda \\ 0 & 1 \end{bmatrix} \begin{matrix} \pi_t \\ z_t \end{matrix} + \begin{bmatrix} 0 \\ \frac{1}{\sigma} \end{bmatrix} \overline{r}, \text{or:}$$

$$\begin{matrix} E_t\pi_{t+1} \\ E_t z_{t+1} \end{matrix} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\lambda}{\beta} \\ -\frac{1}{\beta\sigma} & \frac{\lambda+\beta\sigma}{\beta\sigma} \end{bmatrix} \begin{matrix} \pi_t \\ z_t \end{matrix} + \begin{bmatrix} 0 \\ \frac{1}{\sigma} \end{bmatrix} \overline{r}.$$

The stability/instability of the equilibrium is predicted solely on the make up of the said Jacobian - J_E. Determinacy of the equilibrium requires that we have just enough stable roots as there are predetermined variables.

Proposition 1 *If the number of eigenvalues of J_E outside the unit circle is equal to the number of non-predetermined variables (or forward-looking variables), then there exists a unique stable solution. Blanchard and Kahn (1980)*

Proposition 2 *Let θ_1, θ_2 lie in the complex plane, then: the θ_i's ($i = 1, 2$) are both outside the unit circle if and only if the following conditions are satisfied:*

$$\begin{aligned} |Tr\,(J_E)| &< |1 + Det\,(J_E)| \\ |Det\,(J_E)| &> 1. \end{aligned}$$

In the NK model set out above both π_t and z_t are non-predetermined. Therefore, we need both of the eigenvalues of J_E:

$$J_E = \begin{bmatrix} \frac{1}{\beta} & -\frac{\lambda}{\beta} \\ -\frac{1}{\beta\sigma} & \frac{\lambda+\beta\sigma}{\beta\sigma} \end{bmatrix},$$

to lie outside the unit circle. The eigenvalues of J_E that is, θ_1 and θ_2, are computed by setting

[3] This equation represents a log-linear approximation to the dynamics of aggregate inflation in a model of staggered price-setting of the kind first proposed by Calvo (1983).

$\det(J_E - \theta I) = 0$. This gives a second-order polynomial in θ:

$$p(\theta) = \theta^2 - \theta Tr(J_E) + Det(J_E),$$

where $Tr(J_E) = (\lambda + \sigma + \beta\sigma) / (\beta\sigma)$ and $Det(J_E) = 1/\beta$. For the usual parameter values in NK models ($0 < \beta < 1$, $\sigma > 0$and $\lambda > 0$) the proposition $|Tr(J_E)| < |1 + Det(J_E)|$ is not satisfied.

Proposition 3 *If the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables, there is an infinity of stable solutions. Blanchard and Kahn (1980)*

The system does not provide a unique solution for π_t and z_t. For a fixed nominal interest rate ($r_t = \overline{r}$), the model economy will feature infinitely many non-explosive ouput and inflation paths — the 'non-uniqueness' problem (Taylor, 1977). That is, one could choose any value for π_t different from $\pi^\star$, and the solution describes a path that eventually takes the system back to steady state (i.e., $\pi_t \longrightarrow \pi^\star$, as $t \longrightarrow \infty$). Because there is an uncountable number of such paths, each of which follows a path back to steady state, it follows that there is a multiplicity of stable equilibria. In principle any of these stable paths could be selected. The model does not restrict our choice.

2.1 The Taylor Principle and Determinacy of the Equilibrium

Suppose instead of a fixed interest rate rule we close our system (2.1-2.2) above by specifying a policy rule of the kind proposed by Taylor (1993) for the central bank's operating target for the short-term nominal interest rate,

$$r_t = \phi_\pi \pi_t, \quad \phi_\pi > 1. \tag{2.3}$$

Substituting this feedback rule (2.3) in (2.2) for r_t, the model (2.1-2.3) can be written in the form,

$$\begin{matrix} E_t\pi_{t+1} \\ E_t z_{t+1} \end{matrix} = \begin{bmatrix} \frac{1}{\beta} & -\frac{\lambda}{\beta} \\ \frac{\phi_\pi\beta - 1}{\beta\sigma} & \frac{\lambda+\beta\sigma}{\beta\sigma} \end{bmatrix} \begin{matrix} \pi_t \\ z_t \end{matrix}.$$

As before the eigenvalues of J_E that is, θ_1 and θ_2, are computed by setting $\det(J_E - \theta I) = 0$. This gives a second-order polynomial in θ:

$$p(\theta) = \theta^2 - \theta Tr(J_E) + Det(J_E),$$

where $Tr(J_E) = (\lambda + \sigma + \beta\sigma) / (\beta\sigma)$ and $Det(J_E) = (\sigma + \lambda\phi_\pi) / (\beta\sigma)$. For a unique stable solution we need both of the eigenvalues of J_E to lie outside the unit circle. Clearly, proposition 2 is satisfied provided, $\phi_\pi > 1$.

The crucial question is how does the Fed plan to stabilise inflation in this model? In this model,

$E_t z_{t+i}$ and $E_t \pi_{t+i}$ explode in any equilibrium other than $z = 0, \ \ \pi = 0$. According to Bullard and Mitra (2002) and other NK modellers, $\phi_\pi > 1$ (the Taylor Principle), would stabilize inflation. But how does it rule out the unstable path? According to Bullard and Mitra (2002) the intuition for this result is that any deviation of private sector expected inflation from the rational expectations value leads to an increase in the real interest rate when the Taylor principle is satisfied. This reduces the output gap through equation (2.2) which in turn reduces inflation through equation (2.1). Such a policy, therefore, succeeds in guiding initially non-rational private sector expectations towards the RE value.[4]

Unfortunately as Cochrane (2009, 2011) has pointed out the standard NK model logic works differently. The Taylor principle destabilizes the economy. If inflation rises, the Fed commits to raising future inflation, and leads us into a nominal explosion. That is, if current inflation misbehaves the Fed threatens to implement such paths (hyperinflation or hyperdeflation). Thus the threat is to 'blow up the world' — and this threat is supposed to be so terrifying that private agents expect the stable path instead. No economic consideration rules out the explosive solutions.[5]

This example makes it crystal-clear that inflation determination comes from a threat to increase future inflation if current inflation gets too high. If inflation takes off along a bubble path what is there to stop it in this model? The NK answer is: just the dreadful thought that this might happen. This is because in this model the monetary authority is absolutely committed to raising interest rates more than one for one with inflation, for all values of inflation. For only one value of inflation today will we fail to see inflation that explodes. NK modellers thus conclude that inflation today jumps to this unique value.

But how do they rule out the explosive equilibria? Here NK authors become vague, saying that such paths would be 'inconceivable' and hence 'ruled out by private agents'.[6] The problem as pointed out by Minford and Srinivasan (2011a, b) is twofold: first, that these threats are not credible. The reason is that, once inflation or deflation happens, carrying through on the threat is a disastrous policy. As a result self-destructive threats are less likely to be carried out *ex-post*, and thus less likely to be believed *ex-ante*. The second problem with these threats is that even if they were credible and did actually happen, there seems to be nothing to stop people following the implied paths. Clearly they will prefer the stable path;

[4] On the other hand, if the policy rule does not satisfy this principle, a deviation of private sector expected inflation from the RE value leads to a decrease in the real interest rate which increases the output gap through (2.2) and increases inflation through (2.1). Over time, this leads to upward revisions of both expected inflation and expected output gap. The interest rate rule is unable to offset this tendency and the economy moves further away from the RE equilibrium.

[5] One might argue that there is some boundary on the output gap such as zero output. However, the output gap is a log which can tend to minus infinity- thus its antilog tends to zero but never actually gets to it. Thus in this model all variables, inflation, the output gap, the nominal interest rate and the real interest rate, would follow the same bubble paths without bound. If inflation is exploding positively, then from the Phillips Curve the output gap must explode negatively ($z_t = \frac{1}{\lambda}[\pi_t - E_t \pi_{t+1}]$) and the real interest rate must explode positively from the IS curve ($r_t - E_t \pi_{t+1} = \frac{1}{\sigma}[z_t - E_t z_{t+1}]$; note that z_t is less negative than $E_t z_{t+1}$). Hence every variable is consistently destabilised.

[6] Thus for example King (2000, p. 58–59, cited in Cochrane, 2011) writes: "By specifying [$\phi > 1$] then, the monetary authority would be saying, 'if inflation deviates from the neutral level, then the nominal interest rate will be increased relative to the level which it would be at under a neutral monetary policy.' If this statement is believed, then it may be enough to convince the private sector that the inflation and output will actually take on its neutral level."

Similarly Woodford (2003, p.128, cited in Cochrane, 2009) writes: "The equilibrium.....is nonetheless locally unique, which may be enough to allow expectations to coordinate upon that equilibrium rather than on one of the others."

but how can they be sure it will happen, given that all the paths are feasible.While undesirable from a social viewpoint, they do not appear to be *impossible.* Hence there is nothing to make them infeasible. McCallum (2009a, 2011) agrees about the existence of this problem and proposes to rule these paths out by the 'learnability criterion' to which we now turn.

3 Ruling out unstable equilibria in New-Keynesian models - the Learnability Criterion

In this section we review the concept of E-stability (learnability) and explain how McCallum (2011) uses this criterion for "selection" of the economically relevant RE solution in cases in which the unstable (or bubble) path obtains. Recall the NK model we developed earlier:

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \lambda z_t, \quad 0 < \beta < 1, \lambda > 0 && (3.1) \\ z_t &= E_t z_{t+1} - \frac{1}{\sigma}\left(r_t - r_t^n - E_t \pi_{t+1}\right), \quad \sigma > 0 && (3.2) \end{aligned}$$

where following Bullard and Mitra (2002) we have replaced i_t with $r_t - r_t^n$, where r_t is the nominal interest rate and r_t^n is the natural rate of interest, assumed to obey:

$$r_t^n = \rho r_{t-1}^n + \epsilon_t, \quad 0 < \rho < 1.$$

As this model has two equations but three unknowns (π_t, z_t, r_t), we need a further equation for the nominal interest rate to close the model. As before consider a contemporaneous version of the Taylor rule :[7]

$$r_t = \phi_\pi \pi_t, \quad \phi_\pi > 1. \tag{3.3}$$

Substituting (3.3) into the expectations IS equation and using the NKPC we can write the system as:

$$x_t = B E_t x_{t+1} + \gamma r_t^n, \tag{3.4}$$

where $x_t = [\pi_t,\, z_t]^/$ and

$$B = \frac{1}{\sigma + \phi_\pi \lambda} \begin{bmatrix} \beta\sigma + \lambda & \lambda\sigma \\ 1 - \phi_\pi \beta & \sigma \end{bmatrix},$$

where the form of γ is omitted since it is not needed in what follows. Since both π_t and z_t in the system are free, we need both of the eigenvalues of B to be inside the unit circle for determinacy. As we

[7] Bullard and Mitra (2002) consider various versions of this rule and identify necessary and sufficient conditions under which agents with the correct perceived law of motion (PLM) might learn the RE solution.

have already seen this is satisfied for the model in question provided, $\phi_\pi > 1$.[8] But how does the Fed plan to stabilise inflation in this model? As we have seen $E_t z_{t+i}$ and $E_t \pi_{t+i}$ explode in any equilibrium other than $z = 0,\ \pi = 0$. How to rule out such explosive equilibria?

3.1 E-Stability and Learnability

Let us recast agents in our model as econometricians and ask whether, if endowed with the correct reduced form model for x_t, these agents could learn the parameterization of this model $(\beta,\, \sigma,\, \lambda,\, \phi_\pi,\, \rho)$ which we assume is unknown to them. That is, we assume that agents have the correct perceived law of motion (PLM) and posit that by running regressions each period, as new data becomes available, they might learn the model parameters, i.e. they would learn to have rational expectations. So the central question is: if agents estimate a statistical model which is a correct specification of an REE, under what circumstances will the estimates converge to that REE?

We now define precisely the concept of E-stability. For the study of learning, we endow agents in our model with a PLM of the form,

$$x_t = a + c r_t^n, \tag{3.5}$$

which corresponds to the unique stable solution (also the Minimum State Variable (MSV) solution in McCallum's terminology). Here a and c are the undetermined coefficients. This more general form allows for a constant term for this model.

Learning agents would use the PLM to form expectations of x_{t+1}:

$$E_t x_{t+1} = a + c\rho r_t^n.$$

Substituting the learning agent's forecast into equation (3.4) we obtain the actual law of motion (ALM) implied by the PLM,

$$x_t = B\left(a + c\rho r_t^n\right) + \gamma r_t^n \equiv Ba + \left(Bc\rho + \gamma\right) r_t^n. \tag{3.6}$$

Using (3.5) and (3.6), we can define a map, T, from the PLM to the ALM as

$$T(\theta) = \begin{pmatrix} Ba \\ Bc\rho + \gamma \end{pmatrix},$$

where $\theta = [a,\, c]^{/}$. Expectational-stability (E-stability) is determined by the following matrix differ-

[8] A similar result holds if weight is given to the output gap i.e., rules of the form: $i_t = \phi_\pi \pi_t + \phi_z z_t$. In this case, Bullard and Mitra (2002) show that the condition for determinacy of equilibrium is: $\phi_\pi + \frac{1-\beta}{\lambda}\phi_z > 1$.

ential equation

$$\frac{d\theta}{d\tau} = T(\theta) - \theta, \tag{3.7}$$

where τ denotes artificial or notional time. The fixed points of equation (3.7) give us the MSV solution. We say that a particular MSV solution $(\overline{a}, \overline{c})$ is E-stable (learnable) if the fixed point of the differential equation (3.7) is locally asymptotically stable at that point. The conditions for E-stability of the MSV solution $(\overline{a}, \overline{c})$ are given in Proposition 10.3 of Evans and Honkapohja (2001). Component by component we have

$$\begin{aligned} \frac{da}{d\tau} &= (B-1)\,a \\ \frac{dc}{d\tau} &= \gamma + (B\rho - 1)\,c. \end{aligned}$$

Using the results of Evans and Honkapohja (2001), we need the real parts of the eigenvalues of $D\left(T\left(\overline{\theta}\right) - \overline{\theta}\right)$ to be less than zero i.e., the eigenvalues of both B and $B\rho$ to have real parts less than one for E-stability. The eigenvalues of $B\rho$ are given by the product of the eigenvalues of B and ρ, and since $0 < \rho < 1$, it suffices to have only the eigenvalues of B to have real parts less than one for E-stability. As shown above the characteristic polynomial of B is given by,

$$p(\theta) = \theta^2 - \theta Tr(B) + Det(B),$$

where $Tr(B) = (\beta\sigma + \lambda + \sigma) / (\sigma + \phi_\pi \lambda)$ and $Det(B) = (\sigma\beta) / (\sigma + \phi_\pi \lambda)$.

Both eigenvalues of B are inside the unit circle if and only if both of the following conditions hold (see Bullard and Mitra, 2002)

$$|Det(B)| < 1 \tag{3.8}$$

$$|Tr(B)| < |1 + Det(B)|. \tag{3.9}$$

Condition (3.8) implies the inequality $\phi_\pi \lambda > -(1-\beta)\,\sigma$, which is trivially satisfied since, $0 < \beta < 1$. Condition (3.9) is satisfied provided, $\phi_\pi > 1$. Recall that the Taylor principle was also a necessary condition for determinacy of the REE. It turns out that the condition that ensures that the MSV solution is E-stable (learnable) is identical to the condition that guarantees uniqueness of REE, i.e., the Taylor principle.

3.2 The Learnability Criterion for selection of the economically relevant RE solution

McCallum in a series of articles (2003, 2004, 2007, 2009a, b, 2011) has proposed E-Stability and learnability criterion for "selection" of the economically relevant RE solution in cases in which the unstable, or bubble, path obtains. He has also suggested that this condition acts generally in rational expectations models as the support for ruling out bubble paths and getting the MSV solution. His main point is that the bubble solution does not converge on the RE solution i.e., the bubble path is 'not learnable'. However, the stable solution is learnable: hence the NK model, when it is learnt, will have a unique stable solution

This point can be briefly reviewed. For clarity, we shall concentrate on a frictionless NK model used by both Cochrane (2009) and McCallum (2011). This model has the advantage of transparency and so the least risk of confusion for the general argument.[9] In this model our semi-reduced form solution is

$$0 = b_1[(1 + \mu_1)\pi_t + e_t - E_t\pi_{t+1}],$$

where the monetary policy error, $e_t = \rho e_{t-1} + \varepsilon_t$ with ε_t being white noise and with $|\rho| < 1$), and $\mu_1 > 0$ is the Taylor principle.

This model has a bubble-free or MSV solution (1) $\pi_t = \frac{-1}{1+\mu_1-\rho}e_t$. It also has a bubble solution (2) $\pi_t = \frac{-1}{1+\mu_1-\rho}e_t + B_t$ where B_t is a sunspot which is expected to explode at the rate $(1 + \mu_1)$, so that $E_t\pi_{t+1} = \frac{-\rho}{1+\mu_1-\rho}e_t + (1+\mu_1)B_t$; or equivalently $\pi_t = \frac{1}{\rho}e_t + (1+\mu_1)\pi_{t-1}$. Notice that *the general solution of the model is (2).* That is to say, (2) expresses in one expression 'what the model implies' about the path(s) of inflation, the endogenous variable. (1) is only a solution if B_t is ignored: but *according to the model it cannot be.* McCallum's point is that, while solution (1) above is E-stable, solution (2) is not. He goes on to say that as a consequence, solution (2) cannot occur in practice (because if a solution cannot be learnt, then it has no way of coming into being)- and thus that solution (1) is in fact the *only* outcome that can be predicted by the model.

To see this point it is convenient to express our model in the form (3.4) above. Thus we have,

$$\pi_t = BE_t\pi_{t+1} + \gamma e_t, \tag{3.4.1}$$

[9] McCallum (2011) takes the standard three-equation NK model and simplifies it by assuming full price flexibility so that output equals the natural rate in each period. This eliminates the Calvo Phillips curve and the output gap term in the standard Taylor rule. He also assumes that the natural rate of output is a constant which yields:

$$\begin{aligned} 0 &= b_0 + b_1(R_t - E_t\pi_{t+1}) + v_t, \\ R_t &= \mu_0 + (1 + \mu_1)\pi_t + e_t. \end{aligned}$$

We can combine these two relations to yield

$$0 = b_0 + b_1[\mu_0 + (1 + \mu_1)\pi_t + e_t - E_t\pi_{t+1}] + v_t.$$

Then if the shock term (v_t) is neglected and $\mu_0 = -b_0/b_1$ is recognized as a constant real rate of interest, we get the semi-reduced form used in the text.

where $B = 1/1 + \mu_1$ and $\gamma = -(1/1 + \mu_1)$. Is the bubble solution learnable? Suppose we endow agents in our model with a PLM of the form,

$$\pi_t = ae_t + c\pi_{t-1}, \tag{3.5.1}$$

which *does not* correspond to the MSV or fundamental solution (solution (1) above). Recall that agents assume that data is being generated by the process $\pi_t = ae_t + c\pi_{t-1}$, but that they do not know the parameters a and c. At time t they have estimates (a_t, c_t) which they use to make their forecasts, so that $E_t\pi_{t+1}$ is given by

$$E_t\pi_{t+1} = a(\rho + c)e_t + c^2\pi_{t-1}.$$

Substituting the learning agent's forecast into equation (3.4.1) we obtain the actual law of motion (ALM) implied by the PLM,

$$\pi_t = [Ba(\rho + c) + \gamma]e_t + Bc^2\pi_{t-1}. \tag{3.6.1}$$

Using (3.5.1) and (3.6.1), we can define a map, T, from the PLM to the ALM as

$$T(\theta) = \begin{pmatrix} Ba(\rho + c) + \gamma \\ Bc^2 \end{pmatrix},$$

where $\theta = [a, c]^{\prime}$. E-stability is determined by the following matrix differential equation $\frac{d\theta}{d\tau} = T(\theta) - \theta$. Component by component we have

$$\begin{aligned} \frac{da}{d\tau} &= (B(\rho + c) - 1)a + \gamma \\ \frac{dc}{d\tau} &= (Bc - 1)c. \end{aligned}$$

Using the results of Evans and Honkapohja (2001), we need the real parts of the eigenvalues of $D\left(T\left(\overline{\theta}\right) - \overline{\theta}\right)$ to be less than zero, i.e. the eigenvalues of both $B(\rho + c)$ and Bc must have real parts less than one for E-stability. Notice that the stability condition is satisfied if agents use the bubble-free or stable solution ($c = 0$). In this case it suffices to have only the eigenvalues of $B\rho$ to have real parts less than one for E-stability. Since $0 < \rho < 1$, it suffices to have only the eigenvalues of $B = (1/1 + \mu_1)$ to have real parts less than one for E-stability which is satisfied provided, $\mu_1 > 0$.

By contrast if agents in our model are endowed with a PLM of the form, $\pi_t = ae_t + c\pi_{t-1}$, where π_{t-1} is an *extraneous* state variable which is expected to explode at the rate $c = 1 + \mu_1$, then convergence to (3.5.1) occurs with probability zero. Notice in this case the eigenvalues of both $B(\rho + c) = (1 + \mu_1 + \rho/(1 + \mu_1)) > 1$ and $Bc = 1$ do not satisfy the stability requirement. This example makes it clear that if agents do least-squares learning (as in Evans and Honkapohja, 2001) assuming any of the solution expressions that are not bubble-free, they will not converge on the NK RE model. What is the intuition behind this result? The reason the PLM for the 'unique stable' solution is learnable is because, as a 'backward looking solution' it is stable in the usual way (all eigenvalues inside the unit circle), whereas the 'unstable' PLM (for the rogue solutions) is simply unstable backwards and so presumably cannot be learnt because it fluctuates wildly, forming no patterns in the data that converge on a unique steady equilibrium path. Thus agents would give up on the unstable formula, realising they were not learning. We can see no reason in broad terms to question this conclusion.

One could add that these models are *also not testable* (at least in any normal way). Since the NK model asserts that 'anything can happen' (due to the sunspot), it must be in principle consistent with anything and everything. Thus just as such models are not learnable, so too they are also not testable. Also since these models have infinite error variances (of the sunspots) they also have infinite standard errors of their estimated parameters; thus they are *not estimatable* either (at least with finite standard errors).

Nevertheless, the fundamental problem with McCallum's thesis is that while the only learnable solution that has agents converge on the true NK model is the bubble-free one, once they have converged they must then understand the model and its full solution which includes the bubble. What have these agents in the NK model learnt? They have learnt the structural parameters of the model including the Fed's reaction function, i.e. a response coefficient on inflation greater than one. They have also learnt that in this model if inflation takes off along a bubble path there is nothing to stop it! Agents have learnt that in the NK world with the Taylor rule the monetary authority is absolutely committed to raising interest rates more than one for one with inflation, for all values of inflation. If inflation rises, the Fed commits to raise future inflation, and leads us into a nominal explosion. Therefore, the bubble solution is a legitimate solution in this model.

Having figured out the structural parameters of the model they will then realise that the general solution of an NK model includes bubbles. Like Adam and Eve after eating the apple, they will then know too much and will be tormented by the general solution. Thus the learnability criterion cannot stop these agents in the NK model, once convergence has occurred and it has thereby been learnt, from reverting to the true general solution. Hence the learnability criterion still fails to pick a unique RE solution in NK models. To prevent them from doing this, we would have to assume that they did not know the model, they just had 'simple-minded' autoregressive expectations. But then this is no longer the NK model with RE. For example it would not satisfy the Lucas Critique because expectations would not respond to changes in the model forcing processes.[10] This would certainly not be the NK model that learnability purports to support.

[10] It could of course be a pure learning model, so that when policy changed, agents would relearn the newly appropriate autoregressive solution. But this is not the same as the NK RE model, since agents would not have rational expectations. Instead there would be periods of learning whenever policy or other exogenous processes changed.

4 What mechanism inside the model can remove indeterminacy in NK models?

Here we refer to two papers we recently wrote (Minford and Srinivasan, 2011a, b) which McCallum (2011) cites: these argue that we can rule out bubbles by providing an internal mechanism to the model. Our idea is similar to that of Obstfeld and Rogoff (1983), and see Cochrane (2011) for a comprehensive survey of other somewhat similar approaches. Also one can see in a loose way that the ECB's second (money) pillar could be interpreted as a mechanism of this sort.

The idea is to use transversality conditions for nominal variables analogous to those for real variables; we posit a money demand and money supply function. The latter mimics the Taylor Rule in 'normal times' (i.e., money is supplied to meet the Taylor Rule interest rate setting). However, if a bubble path for inflation were to occur then the money supply would revert to a 'fixed-inflation' rule- similar to a 'fixed exchange rate' rule- in which money supply would be whatever was needed to enforce the constancy of inflation. This terminal condition acts to terminate any bubble prospectively: hence no bubble path can occur and the normal Taylor Rule is always observed.

This also deals with indeterminacy when there is no unique stable path, the 'non-uniqueness' problem- an example is the Taylor Rule before the 1980s according to Clarida et al. (2000) when they argue the Taylor principle did not hold ($\phi_\pi < 1$). The terminal condition also disables these bubble paths (though here these are implosive or stable, the variance of inflation is still unbounded).

5 Conclusion

Models without determinacy are problematic as they stand and so do not rate as models of interest. On this we can all agree, including McCallum and Cochrane. So there must be some mechanism that ensures determinacy. McCallum says it is not needed because models are learnable only with the determinate solution and so the NK model, once learnt in this way, will be determinate. We agree: the only learnable solution that has agents converge on the true NK model is the bubble-free one. But once they have converged they must then understand the model and its full solution therefore including the bubble. Hence the learnability condition still fails to select the determinate solution. So the problem remains. Terminal conditions on monetary behaviour justified by welfare can provide the mechanism, converting NK models into proper NK models that can be used by economists in the usual way.

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