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Sharp lower bounds on the fractional matching number

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Abstract

A *fractional matching* of a graph G is a function f from $E(G)$ to the interval $[0,1]$ such that $\sum_{e\in\Gamma(v)} f(e) \leq 1$ for each $v \in V(G)$, where $\Gamma(v)$ is the set of edges incident to v. The *fractional matching number* of G, written $\alpha'_*(G)$, is the maximum of $\sum_{e\in E(G)} f(e)$ over all fractional matchings f. For G with n vertices, m edges, positive minimum degree d, and maximum degree D, we prove $\alpha'_*(G) \geq \max\{\frac{m}{D}, n-\frac{m}{d}, \frac{d}{D+d}n\}$. For the first two bounds, equality holds if and only if each component of G is r-regular or is bipartite with all vertices in one part having degree r, where $r = D$ for the first bound and $r = d$ for the second. Equality holds in the third bound if and only if G is regular or is (d, D)-biregular.

1 Introduction

A *matching* in a graph is a set of disjoint edges. The *matching number* of a graph G, written $\alpha'(G)$, is the maximum size of a matching in G. A *perfect matching* in G is a matching in which each vertex has an incident edge; its size must be $n/2$, where $n = |V(G)|$ throughout this paper. A *fractional matching* of G is a function f from $E(G)$ to $[0,1]$ such that $\sum_{e\in\Gamma(v)} f(e) \leq 1$ for each $v \in V(G)$, where $\Gamma(v)$ denotes the set of edges incident to v. The *fractional matching number* of G, written $\alpha'_*(G)$, is the maximum of $\sum_{e\in E(G)} f(e)$ over all fractional matchings f.

Summing the inequality constraints for all vertices yields $2\sum_{e\in E(G)} f(e) \leq n$, so always $\alpha'_*(G) \leq n/2$. Since every matching can be viewed as a fractional matching, $\alpha'_*(G) \geq \alpha'(G)$ for every graph G, but equality need not hold. For example, $\alpha'_*(G) = n/2$ whenever G is k-regular (set each edge weight to $1/k$), but not every k-regular graph has a perfect matching.

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A *fractional perfect matching* is a fractional matching f such that $\sum_{e\in\Gamma(v)} f(e) = 1$ for every vertex v, which occurs if and only if $\alpha'_*(G) = n/2$.

In our discussion, we exclude isolated vertices, since doing so does not change the fractional matching number and eliminates possible divisions by 0 in the bounds. Hence our graphs also always have at least two vertices. We prove three lower bounds on $\alpha'_*(G)$ for every such graph G, and for each we characterize the graphs achieving equality. For G with n vertices, m edges, minimum degree d, and maximum degree D, we prove

$$\alpha'_*(G) \geq \max\left\{\frac{m}{D},\ n - \frac{m}{d},\ \frac{d}{D+d}n\right\}.$$

The first bound is trivial, obtained by setting each edge weight to $1/D$. Nevertheless, the three bounds are independent in the sense that none implies another. We illustrate this with two simple families of graphs: non-star trees and n-vertex wheels with $n \geq 5$. The n-vertex *wheel* is obtained from the cycle C_{n-1} by adding a vertex adjacent to all vertices on the cycle.

	$n - \frac{m}{d}$	$\frac{d}{d+D}n$	$\frac{m}{D}$	
non-star tree	1	$\frac{n}{D+1}$	$\frac{n-1}{D}$	increasing order
wheel with $n \geq 5$	$\frac{n+2}{3}$	$\frac{3n}{n+2}$	2	decreasing order

In fact, the third bound is always between the other two. This follows from the fact that it is a weighted average of the first two:

$$\frac{D}{d+D}\frac{m}{D} + \frac{d}{d+D}\left(n - \frac{m}{d}\right) = \frac{d}{d+D}n.$$

Thus proving the second bound will also prove the third, since the first is trivial.

When G is regular, $d = D = 2m/n$, so all three bounds equal $n/2$ and hold with equality. Equality can also hold when G has non-regular components. For either of the first two bounds, equality holds precisely when each regular component has degree r and each non-regular component is bipartite with all vertices of one part having degree r, where $r = D$ for the first bound and $r = d$ for the second.

Since the third bound is the weighted average of the first two, it holds with equality only when they both hold with equality. This requires that G is regular or that all components are (d, D)-biregular, where a graph is (a, b)-*biregular* if it is bipartite with the vertices of one part all having degree a and the others all having degree b. Note that for the first two bounds the condition is applied to each component independently, while for the third bound it is a global choice for all components at once.

When G is bipartite, it is easy to push weights to integers by alternately increasing and decreasing weights along paths or cycles of edges with non-integer weights. Thus $\alpha'_*(G) = \alpha'(G)$ when G is bipartite (see Theorem 2.1.3 of [3]). Hence any lower bound on $\alpha'_*(G)$ is

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also a lower bound on $\alpha'(G)$ when G is bipartite. Our third bound improves a classical such lower bound in many cases. It is well known that a bipartite graph G with parts X and Y and minimum degree d satisfies $\alpha'(G) \geq \min\{2d, |X|, |Y|\}$. When $D + d < n/2$, the lower bound $\frac{d}{D+d}n$ exceeds $2d$ and hence is stronger.

In the proof, we use a fractional analogue of the famous Berge–Tutte Formula [1] for the matching number. For a graph H, let $o(H)$ denote the number of components of H having an odd number of vertices. Given a graph G and $S \subseteq V(G)$, define the *deficiency* $\mathrm{def}(S)$ by $\mathrm{def}(S) = o(G - S) - |S|$, and let $\mathrm{def}(G) = \max_{S \subseteq V(G)} \mathrm{def}(S)$. The Berge–Tutte Formula is the equality $\alpha'(G) = \frac{1}{2}(n - \mathrm{def}(G))$. The special case $\alpha'(G) = n/2$ reduces to Tutte's 1-Factor Theorem [4]: a graph G has a perfect matching if and only if $o(G - S) \leq |S|$ for all $S \subseteq V(G)$. (A 1-*factor* of G is a subgraph whose edges form a perfect matching.)

For the fractional analogue, let $i(H)$ denote the number of isolated vertices in H. Let $\mathrm{def}_*(S) = i(G-S)-|S|$ and $\mathrm{def}_*(G) = \max_{S \subseteq V(G)} \mathrm{def}_*(S)$. The fractional analogue of Tutte's 1-Factor Theorem is that G has a fractional perfect matching if and only if $i(G - S) \leq |S|$ for all $S \subseteq V(G)$ (implicit in Pulleyblank [2]), and the fractional version of the Berge–Tutte Formula is $\alpha'_*(G) = \frac{1}{2}(n - \mathrm{def}_*(G))$ (see [3], pages 19–20).

2 Sharp Lower Bounds for $\alpha'_*(G)$

Although the lower bound m/D is trivial, we give another proof of it by the same fractional Berge–Tutte method as for the bound $n - \frac{m}{d}$ in order to obtain a short proof of the characterization of equality. Later we provide another approach to that characterization.

The exclusion of isolated vertices in Theorem 1 is used implicitly at several points in the proof. It ensures that d and D are both nonzero and that both parts are nonempty when the graph is bipartite. An *independent set* is a set of vertices inducing no edges. Let $[A, B]$ denote the set of edges in G with endpoints in A and B.

Theorem 1. *For an n-vertex graph G with m edges, positive minimum degree d, and maximum degree D,*

$$\alpha'_*(G) \geq n - \frac{m}{d} \quad \text{and} \quad \alpha'_*(G) \geq \frac{m}{D}. \tag{1}$$

For each bound, equality holds if and only if each component of G is r-regular or is a bipartite graph with all vertices in one part having degree r, where r is the relevant member of $\{d, D\}$.

Proof. The number of vertices, number of edges, and fractional matching number are additive over components, and the claimed lower bounds become stronger when the minimum degree increases or the maximum degree decreases. Hence it suffices to prove the claims when G is connected. We have also observed the bounds and equalities for regular graphs, so we may assume $d < 2m/n < D$.

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By the fractional Berge–Tutte Formula, $\alpha'_*(G) = \frac{1}{2}(n - \text{def}_*(S))$ for some $S \subseteq V(G)$. If $S = \varnothing$, then $\alpha'_*(G) = n/2$ and the bounds hold, but not with equality, since G is not regular. Hence we may assume $S \neq \varnothing$ and $\text{def}_*(S) > \text{def}_*(\varnothing) = 0$. Let Q be the set of isolated vertices in $G - S$, so $\text{def}_*(S) = |Q| - |S|$ and Q is nonempty. Let $R = V(G) - S - Q$.

Note that $2m$ counts each edge twice; we group contributions by vertices. At least $d|Q|$ edges are incident to Q. At most $D|S|$ edges are incident to S, including all of $[Q, S]$, and the remaining edges are incident to R. We thus have

$$2d|Q| + d|R| \leq 2m \leq 2D|S| + D|R|. \tag{2}$$

Using $n = |Q| + |S| + |R|$ and $\text{def}_*(S) = |Q| - |S|$, we obtain

$$d(n + \text{def}_*(S)) \leq 2m \leq D(n - \text{def}_*(S)), \tag{3}$$

and then $\alpha'_*(G) = \frac{1}{2}(n - \text{def}_*(S)$ yields

$$d(n - \alpha'_*(G)) \leq m \leq D\alpha'_*(G), \tag{4}$$

which are the desired bounds.

When G is not regular, equality in either bound requires equality in the appropriate part of (2). If $2d|Q| + d|R|$ counts each edge twice, then only edges of $[S, Q]$ are incident to S. Since $[Q, R] = \varnothing$, and we have reduced to G being connected, we conclude that $R = \varnothing$ and that S is independent. Now G is bipartite, and every vertex of Q has degree d.

Similarly, if every edge contributes exactly twice to $2D|S| + D|R|$, then again only edges of $[S, Q]$ are incident to S (edges from S to R would contribute 3 to the count, and edges within S would contribute 4). As before, we conclude that $R = \varnothing$ and that S is independent. Thus G is bipartite, and every vertex of S has degree D.

Finally, let G be connected and bipartite with parts X and Y such that each vertex of X has degree r. If $r = d$, then to show equality in $\alpha'_*(G) \geq n - m/d$ we find S such that $\text{def}_*(S) = \frac{2m}{d} - n$. Let $S = Y$, so $i(G - S) = |X|$, and X plays the role of Q above. Since $m = d|X|$, we have $\text{def}_*(S) = i(G - S) - |S| = |X| - (n - |X|) = \frac{2m}{d} - n$, as desired.

When G is bipartite, $\alpha'_*(G)$ is bounded above by the size of the smaller part, since those vertices group the edges into disjoint sets that receive weight at most 1. Hence when $r = D$ and the vertices in X all have degree D, we have $|Y| \geq |X| = \frac{m}{D}$. $\square$

Corollary 2. *If G has n vertices, positive minimum degree d, and maximum degree D, then*

$$\alpha'_*(G) \geq \frac{d}{D + d}n,$$

with equality if and only if G is regular or is (d, D)-biregular.

Proof. We have noted that this inequality is the sum of the inequalities in (1), weighted by $\frac{d}{d+D}$ and $\frac{D}{d+D}$. Thus $\frac{dn}{d+D}$ is between the minimum and the maximum of those two bounds and equals one of them if and only if it equals both. Furthermore, equality holds in this bound if and only if there is no better lower bound on $\alpha'_*(G)$, so equality requires one of the other bounds (and hence both of them) to hold with equality. Requiring G to satisfy the characterizations of equality in those two bounds requires G to be regular or (d, D)-biregular. Conversely, such a graph G satisfies both of those bounds with equality. $\qquad\square$

Finally, we note that the condition for equality in $\alpha'_*(G) \geq \frac{m}{D}$ can also be obtained directly by improving the uniform weighting when G is not as described. We may assume that G is connected and not regular.

The set T of vertices with non-maximum degree must be independent, because there is slack at vertices of T under this edge-weighting. More generally, if two vertices of T are connected by a trail of odd length, the weights along the trail can be alternately increased and decreased (by the same amount) to increase the total weight. If G is bipartite, it therefore cannot have vertices of non-maximum degree in opposite parts, so a bipartite such graph has degree D at all vertices in one part.

It remains to consider non-bipartite G having an odd cycle C through vertices of maximum degree. Let P be a shortest path to C from a vertex v of T, reaching C at w. Mark the edges of a trail following the union of P and C alternately positive and negative, starting positive at v. On C the two edges at w have the same sign. Change the weights on $P \cup C$ in the indicated direction, by ϵ on P and by $\epsilon/2$ on C. The total weight at each vertex other than v remains unchanged. Whether P has even length or odd length, the total weight increases by $\epsilon/2$. Hence $\alpha'_*(G)$ cannot equal $\frac{m}{D}$ when G is connected, non-bipartite, and not regular.

References

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