THE FINITE SECTION METHOD FOR DISSIPATIVE OPERATORS

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Abstract. We show that for self-adjoint Jacobi matrices and Schrödinger operators, peturbed by dissipative potentials in $\ell^1(\mathbb{N})$ and $L^1(0,\infty)$ respectively, the finite section method does not omit any points of the spectrum. In the Schrödinger case two different approaches are presented. Many aspects of the proofs can be expected to carry over to higher dimensions, particularly for a.c. spectrum.

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1. Introduction

In this paper we consider the finite section method for two classes of operator: dissipative Schrödinger operators on $(0,\infty)$ and dissipative Jacobi matrices in $\ell^2(\mathbb{N})$. More precisely, in the Schrödinger case, we start with a self-adjoint operator L_0 given by an expression

$$L_0 u = -u'' + q(x)u,$$

in which the potential q is in the limit-point case at infinity and integrable at 0; at 0 we impose, without loss of generality, a Dirichlet boundary condition. The domain of L_0 is thus

$$D(L_0) = \{u \in L^2(0,\infty) \mid -u'' + qu \in L^2(0,\infty),\ u(0) = 0\}$$

and L_0 is self-adjoint. We are interested in the dissipative operator

$$L = L_0 + is(x)\cdot,$$

in which s is an essentially bounded, non-negative element of $L^1(0,\infty)$. In applications using dissipative barrier methods s usually has compact support - say, $\mathrm{supp}(s) \subseteq [0,N]$ for some $N > 0$ - and is often the characteristic function of some finite interval. The finite section method involves considering the same differential expression but on a finite interval $[0,M]$ for some large values of M, with an artificial boundary condition at M.

In the case of Jacobi operators we start with a self-adjoint operator in $\ell^2(\mathbb{N})$ given formally by an infinite matrix

$$J_0 = \begin{pmatrix} b_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & 0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \end{pmatrix}$$

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in which (a_n) is a sequence of non-zero reals and (b_n) are is a real sequence. and we consider the spectra of the leading, large-but-finite square sub-matrices of

$$J = J_0 + i\text{diag}(s_1, s_2, \ldots),$$

in which $s = (s_j)_{j\in\mathbb{N}} \in \ell^1(\mathbb{N})$. In applications s is often finitely supported with, say, $s_{N-1} > 0$ and $s_j = 0$ for $j \geq N$. Very good reviews of the finite section method for infinite matrices may be found in [6, 14].

Already for the case in which there is no dissipative perturbation added to the self-adjoint part, we know that the finite section method will generally be responsible for spectral pollution. However using the results in Stolz and Weidmann [21] there is a simple argument, given in [1], which shows that for these one-dimensional operators there is at most one point of spectral pollution in each spectral gap[1]. Again for the self-adjoint case we know that, under the assumption that compactly supported functions form a core of the full operator, the finite section method will also approximate every point of the essential spectrum. This argument relies on the fact that if A is a self-adjoint operator and u is an element of its domain with norm 1, and if $\lambda \in \mathbb{C}$, then $\text{dist}(\text{Spec}(A), \lambda) \leq \|(A - \lambda I)u\|$; it is well known that this can fail spectacularly in the non-self-adjoint case.

Despite this, it remains relatively straightforward in the non-self-adjoint case (using, e.g., the arguments in [17] and [18]) to show that the finite section method will always manage to approximate isolated eigenvalues of the original operator, under fairly mild hypotheses. The question we address here is therefore a seemingly trivial one: can we be sure that the finite section method manages to approximate the essential spectrum, when it is applied directly to the dissipative operators rather than to the underlying self-adjoint one?

For certain classes of pseudo-ergodic Jacobi matrices, the results of Chandler-Wilde and Davies [5], using work of Lindner and Roch [15], show that the finite section method does not cause spectral pollution. At a talk in Cardiff University in January 2013 Lindner announced that, in collaboration with Chandler-Wilde, he has now established that for pseudo-ergodic random matrices the finite section method in fact approximates the whole spectrum, without pollution; hence, for samples from a distribution of random matrices, it generates approximations to the spectrum and nothing else, with probability 1.

We are dealing in this paper with operators which do not have the pseudo-ergodicity properties required to avoid spectral pollution. However we shall be able to show that spectral inclusion is indeed achieved under mild hypotheses: no spectral points are omitted by the finite section method.

The structure of the paper is as follows. Section 2 treats the case of Jacobi matrices; essential ingredients here are the Titchmarsh-Weyl m-functions for the truncated matrices and a simple trace formula which is valid uniformly for all the truncated operators. Section 3 treats the case of Schroödinger operators with compactly supported s, by a different method which comes close to an inverse moment problem approach; it assumes, in addition to the minimal hypotheses already introduced on the real potential q, that q is square integrable over finite intervals. Section 4 treats Schrödinger operators, now with essentially bounded $s \in L^1(0, \infty)$ and $\lim_{x\to\infty} s(x) = 0$, using what we call the 'Jacobi matrix method': in other words, Titchmarsh-Weyl m-functions and a uniform trace inequality. We assume in this section that, in addition to the minimal hypotheses already mentioned, the real potential q is essentially bounded below and square integrable over finite intervals. The reason for the lower-bound hypothesis on q is that the operator of multiplication by s, and its natural finite sections, are no longer trace class. We need to show that they are instead uniformly relatively trace class with respect to a family of finite sections of Schrödinger operators. The essential lower bound on q describes one convenient class of potentials for which this is true. There are certainly others, but they are not immediately amenable to the technique of [4] which we adopt, and a totally different approach would then be needed to establish Lemma 10.

We conclude our introduction with a caveat: this is not a paper about numerical methods. We avoid all discussion of the extensive literature on spectral pollution and quadratic relative spectra [13, 8, 22] and of

[1] The essence of the argument is that the finite section operators on any given interval are all self-adjoint extensions of one symmetric operator with deficiency indices (1,1)

methods based on generalizations of pseudospectra [10, 11]. Our interest in the problem treated stems from the fact that it arises when using the most naïve numerical approaches, yet appears to require a surprisingly elaborate analysis, compared to the self-adjoint case, for its solution.

2. Jacobi matrices

We start by considering the Jacobi operator J. Our main theorem is the following.

Theorem 1. *Suppose that $s \in \ell^1(\mathbb{N})$ and $s_j \geq 0$ for all j. Suppose that λ_{ess} is a point of essential spectrum of J. Then every open neighbourhood of λ_{ess} in $\mathbb{C}$ contains eigenvalues of the leading $K \times K$ submatrix of J, for all sufficiently large K.*

The proof of this result will be in several stages. The first case involves studying the zeros of a family of functions of the form $m(\lambda) - z$, where m is a Titchmarsh-Weyl coefficient of a self-adjoint Jacobi operator and $z \in \mathbb{C}^+$ is fixed. This reveals the simplicity of the underlying strategy and reveals the main features of the problem without the technical complications of the general cases. We then treat the case in which s is finitely supported: $s_{N-1} > 0$ and $s_j = 0$ for $j \geq N$. Finally we shall observe that the case of infinitely many sites with $s \in \ell^1(\mathbb{N})$ introduces no additional difficulties compared to the N-site case: indeed the crucial inequalities actually hold a fortiori.

We start with some preliminaries which establish our notation and review standard techniques. A point $\lambda \in \mathbb{C}$ is an eigenvalue of J if and only if there exists a corresponding eigenvector $u \in \ell^2(\mathbb{N})$ such that

$$a_{n-1}u_{n-1} + (b_n + is_n)u_n + a_n u_{n+1} = \lambda u_n, \quad n = 1, 2, \ldots, \tag{1}$$

in which we interpret the $n = 1$ equation by putting $u_0 = 0$ formally. Thus we can tell whether or not $\lambda \in \mathbb{C}$ is an eigenvalue by the Glazman decomposition trick [2, §126, p. 485]. Let $N \geq 2$ be fixed.

(1) Solve the equation

$$a_{n-1}v_{n-1} + (b_n + is_n)v_n + a_n v_{n+1} = \lambda v_n, \quad n = 1, 2, \ldots, N-1, \tag{2}$$

starting with the initial condition $v_0 = 0$ and $v_1 = 1$.

(2) Solve the equation

$$a_{n-1}\psi_{n-1} + (b_n + is_n)\psi_n + a_n\psi_{n+1} = \lambda\psi_n, \quad n \geq N,$$

with the condition $\sum_{n \geq N} |\psi_n|^2 < \infty$. For λ outside the essential spectrum there is (up to scalar multiples) just one such solution, by the limit-point hypothesis.

(3) Check the matching condition $\psi_{N-1}v_N - \psi_N v_{N-1} = 0$.

If we let

$$J_N^R = \begin{pmatrix} b_N + is_N & a_N & 0 & 0 & 0 & \cdots \\ a_N & b_{N+1} + is_{N+1} & a_{N+1} & 0 & 0 & \cdots \\ 0 & a_{N+1} & b_{N+2} + is_{N+2} & a_{N+2} & 0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & \cdots \end{pmatrix}$$

and

$$J_N^L = \begin{pmatrix} b_1 + is_1 & a_1 & 0 & 0 & 0 & \cdots \\ a_1 & b_2 + is_2 & a_2 & 0 & 0 & \cdots \\ 0 & a_2 & b_3 + is_3 & a_3 & 0 & \cdots \\ . & . & . & . & . & \cdots \\ . & . & . & . & . & a_{N-2} \\ . & . & . & . & a_{N-2} & b_{N-1} + is_{N-1} \end{pmatrix}$$

and define $e^L_{N-1} = (0, 0, \ldots, 0, 1)^T \in \mathbb{R}^{N-1}$, $e^R_1 = (1, 0, 0, \ldots)^T \in \ell^2(\mathbb{N})$ then a simple calculation shows that

$$\frac{v_{N-1}}{v_N} = -a_{N-1}\langle (J^L_N - \lambda)^{-1} e^L_{N-1}, e^L_{N-1}\rangle,$$

$$\frac{\psi_N}{\psi_{N-1}} = -a_{N-1}\langle (J^R_N - \lambda)^{-1} e^R_1, e^R_1\rangle;$$

our matching condition is therefore satisfied if

$$1 = a_{N-1}\langle (J^L_N - \lambda)^{-1} e^L_{N-1}, e^L_{N-1}\rangle a_{N-1}\langle (J^R_N - \lambda)^{-1} e^R_1, e^R_1\rangle.$$

Recalling that the a_n are all non-zero, we define

(3) $$m(\lambda) = \frac{-1}{a_{N-1}}\frac{\psi_N}{\psi_{N-1}} = \langle (J^R_N - \lambda)^{-1} e^R_1, e^R_1\rangle,$$

(4) $$f(\lambda) = \frac{-1}{a_{N-1}}\frac{v_N}{v_{N-1}} = \frac{1}{a^2_{N-1}\langle (J^L_N - \lambda)^{-1} e^L_{N-1}, e^L_{N-1}\rangle},$$

and we have the following lemma.

Lemma 1. *If $\lambda \in \mathbb{C}$ is a point at which $m(\cdot)$ and f are analytic and is such that*

$$m(\lambda) = f(\lambda),$$

then λ is an eigenvalue of J.

Remark 1. *The function $m(\cdot)$ is the usual Titchmarsh-Weyl m-function for the infinite Jacobi matrix J^R_N.*

When the finite section method is employed, the matrix J^R_N is truncated to become

$$J^M_N = \begin{pmatrix} b_N + is_N & a_N & 0 & 0 & 0 & \cdots \\ a_N & b_{N+1} + is_{N+1} & a_{N+1} & 0 & 0 & \cdots \\ 0 & a_{N+1} & b_{N+2} + is_{N+2} & a_{N+2} & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & a_{N+M-1} \\ \cdot & \cdot & \cdot & \cdot & a_{N+M-1} & b_{N+M} + is_{N+M} \end{pmatrix}$$

The function $m(\lambda)$ is correspondingly approximated by a new function

(5) $$m_M(\lambda) = \langle (J^M_N - \lambda)^{-1} e_1, e_1\rangle,$$

in which $e_1 = (1, 0, 0, \ldots, 0) \in \mathbb{R}^M$; and Lemma 1 becomes:

Lemma 2. *If $\lambda \in \mathbb{C}$ is a point at which $m_M(\cdot)$ and $f(\cdot)$ are analytic and if*

(6) $$m_M(\lambda) = f(\lambda),$$

then λ is an eigenvalue of the leading $(N + M) \times (N + M)$ finite section of J.

Remark 2. *Observe the following Nevanlinna properties of m_M, m and f, which follow from their definitions in terms of resolvents, (3) and (5).*

(1) *$\Im(f(\lambda)) \geq 0$ for $\Im\lambda \leq 0$;*
(2) *if $s_j = 0$ for $j \geq N$ then $\Im(m(\lambda))$ and $\Im(m_M(\lambda))$ have the same sign as $\Im(\lambda)$;*
(3) *if $s_j \geq 0$ for $j \geq N$ then $\Im(m_M(\lambda)) \leq 0$ and $\Im(m_M(\lambda)) \leq 0$ for $\Im(\lambda) \leq 0$.*

Our question is: do the roots of the equation (6) approximate every point of $\sigma_{\text{ess}}(J) = \sigma_{\text{ess}}(J_0)$?

Proposition 1. *Suppose that $s_j = 0$ for $j \geq N$. Let $z \in \mathbb{C}^+$ be fixed. Then the roots of the equation $m_M(\lambda) = z$ cannot accummulate, as $M \to \infty$, at any point off the real axis, unless that point is a root of the equation $m(\lambda) = z$. As a consequence there are only finitely many such accummulation points in any strip $a \leq \Re(\lambda) \leq b$.*

Proof. Off the real axis, since our operators J_N^R and J_N^M are self-adjoint, Titchmarsh-Weyl nesting analysis (see Gesztesy and Clarke [7]) shows that $m_M(\lambda) \to m(\lambda)$ locally uniformly, where $m(\lambda)$ is the Titchmarsh-Weyl function for the self-adjoint Jacobi operator J_N^R. Any zeros of $m_M(\lambda) - z$ which converge to a point off the real axis therefore converge to a zero of $m(\lambda) - z$. $\square$

Proposition 2. *Denote the zeros of $m_M(\lambda) - z$ explicitly in terms of their real and imaginary parts by $(\mu_k + i\nu_k)$. Then*

$$\sum_k \nu_k = -\Im(1/z).$$

Proof. Using the formula for $m_M(\lambda)$ we expand the resolvent $(J_N^M - \lambda)^{-1}$ in eigenvectors (w_k) of the Hermitian matrix J_N^M,

$$(J_N^M - \lambda)^{-1} e_1 = \sum_{k=1}^{M+1} \frac{\langle e_1, w_k \rangle}{\lambda_k - \lambda} w_k,$$

and deduce that

$$m_M(\lambda) = \sum_{k=1}^{M+1} \frac{|\langle e_1, w_k \rangle|^2}{\lambda_k - \lambda}. \tag{7}$$

The equation $m_M(\lambda) = z$ can be rearranged as a polynomial equation of order $M + 1$ in λ:

$$\prod_{k=1}^{M+1} (\lambda_k - \lambda) = \frac{1}{z} \sum_{k=1}^{M+1} |\langle e_1, u_k \rangle|^2 \prod_{j \neq k} (\lambda_j - \lambda).$$

The sum of the roots is given by the coefficient of $(-\lambda)^M$ in this equation:

$$\sum_{k=1}^{M+1} (\mu_k + i\nu_k) = \sum_{k=1}^{M+1} \lambda_k - \frac{1}{z}.$$

The result follows upon taking imaginary parts. $\square$

Proposition 3. *Let $z \in \mathbb{C}^+$ be fixed. Then the roots of the equation $m_M(\lambda) = z$ approximate every point of $\sigma_{\mathrm{ess}}(J) = \sigma_{\mathrm{ess}}(J_0)$.*

Proof. Suppose for a contradiction that there is some point $\lambda_{\mathrm{ess}} \in \sigma_{\mathrm{ess}}(J)$ which has an open neighbourhood U containing no zeros of $m_M(\lambda) - z$ for some subsequence of integers M tending to infinity. Choose an open interval $(a, b) \ni \lambda_{\mathrm{ess}}$ such that $U \supset [a, b]$. Let $\mu_k + i\nu_k$ be the zeros of $m_M(\lambda) - z$; these all lie strictly in $\mathbb{C}^+$ since $m(\lambda)$ is real-valued for real λ and $\Im(z) > 0$, and there are precisely $M + 1$ of them since they are the zeros of a degree $M + 1$ polynomial equation. We may assume that they all lie at a distance at least $\delta > 0$ from the line segment $[a, b]$. Consider the function

$$g_M(\lambda) = \frac{1}{m_M(\lambda) - z} P_M(\lambda) \tag{8}$$

in which P_M the given by the Blaschke product

$$P_M(\lambda) = \prod_{k=1}^{M+1} \left(\frac{\lambda - (\mu_k + i\nu_k)}{\lambda - (\mu_k - i\nu_k)} \right) = \prod_{k=1}^{M+1} \left(1 - \frac{2i\nu_k}{\lambda - (\mu_k - i\nu_k)} \right).$$

Evidently $g_M(\lambda)$ is analytic in the semi-strip $a \leq \Re(\lambda) \leq b$, $\Im(\lambda) > -\delta$.

We know that among the zeros of $m_M(\lambda) - z$ there are those which converge to zeros of $m(\lambda) - z$, and others. In the strip $a \leq \Re(\lambda) \leq b$ there are only finitely many zeros of $m(\lambda) - z$, by Proposition 1. We may therefore surround the zeros of $m(\lambda) - z$ in the strip by an open set V which is sufficiently small to ensure that it contains, for all sufficiently large M, only those zeros of $m_M(\lambda) - z$ which converge to zeros of $m(\lambda) - z$, and the number of these will be fixed for all sufficiently large M. By re-numbering the zeros of $m_M(\lambda) - z$ if necessary we may suppose that it is precisely the zeros $\mu_k + i\nu_k$, $k = 1, \ldots, \nu$, of $m_M(\lambda) - z$ which lie in V, and hence converge to zeros of $m(\lambda) - z$. We separate the product for P_M correspondingly:

$$P_M(\lambda) = p_M(\lambda)\tilde{P}_M(\lambda), \;\; p_M(\lambda) = \prod_{k=1}^{\nu}\left(1 - \frac{2i\nu_k}{\lambda - (\mu_k - i\nu_k)}\right); \;\; \tilde{P}_M(\lambda) = \prod_{k=\nu+1}^{M+1}\left(1 - \frac{2i\nu_k}{\lambda - (\mu_k - i\nu_k)}\right).$$

We consider the behaviour of $\tilde{P}_M$ in the strip $a \leq \Re(\lambda) \leq b$. Firstly, $\tilde{P}_M$ has no zeros at all here, for sufficiently large M, because of the way we have sub-divided our set of zeros of $m_M(\lambda) - z$: those which remain outside a neighbourhood of the real axis necessarily converge to infinity or to zeros of $m(\lambda) - z$ off the real axis, and we have assumed that no zeros of $m_M(\lambda) - z$ converge to a neighbourhood of $[a, b]$, so all zeros of $m_M(\lambda) - z$ in the strip $a \leq \Re(\lambda) \leq b$ are enumerated among $\mu_k + i\nu_k$, $k = 1, \ldots, \nu$. Thus in the strip $a \leq \Re(\lambda) \leq b$ we have, for all sufficiently large M, the inequality $|\lambda - (\mu_k + i\nu_k)| \geq \delta > 0$, $k = \nu + 1, \ldots, M + 1$, where δ is independent of M, whence also $|\lambda - (\mu_k - i\nu_k)| \geq \delta > 0$, $k = \nu + 1, \ldots, M + 1$. Hence $\tilde{P}_M$ has no poles either, for $a \leq \Re(\lambda) \leq b$. From Proposition 2 we have, a fortiori,

$$\sum_{k=\nu+1}^{M+1} \nu_k \leq -\Im(1/z).$$

It follows that for all $a \leq \Re(\lambda) \leq b$,

$$\exp(2\delta^{-1}\Im(1/z)) \leq |\tilde{P}_M(\lambda)| \leq \exp(-2\delta^{-1}\Im(1/z)). \tag{9}$$

The behaviour of $p_M(\lambda)$ for large M is simple: since the number of factors is constant for sufficiently large M and since each factor converges, there exists a rational function $p(z)$ having ν zeros and ν poles in the strip $a \leq \Re(\lambda) \leq b$, all off the real axis, such that $p_M(\lambda) \to p(\lambda)$ locally uniformly away from the poles of p.

Considering the function $g_M(\lambda)$ we observe that, for real λ, $m_M(\lambda)$ is real or infinite, and hence $\left|\frac{1}{m_M(\lambda)-z}\right| \leq \frac{1}{\Im(z)}$ for $\lambda \in \mathbb{R}$. This bound holds trivially for $\lambda \in \mathbb{C}^-$ as there $\Im(m_M(\lambda))$ and $\Im(z)$ are of opposite signs, so

$$\left|\frac{1}{m_M(\lambda) - z}\right| \leq \frac{1}{\Im(z)}, \quad \Im(\lambda) \leq 0. \tag{10}$$

The function $g_M(\lambda)$ is rational with no poles in $\mathbb{C}^+$ and, on the real axis, it is also bounded above by $1/\Im(z)$ since $|P_M(\lambda)| = 1$ for real λ. By the Phragmen-Lindelöf Theorem, it follows that

$$|g_M(\lambda)| \leq \frac{1}{\Im(z)}, \quad \Im(\lambda) \geq 0. \tag{11}$$

The term $\tilde{P}_M(\lambda)$ has absolute value 1 on the real axis and, off the real axis, is bounded in modulus for $a \leq \Re(\lambda) \leq b$ by (9). The term $p_M(\lambda)$ has absolute value 1 on the real axis and is bounded in a neighbourhood of the real axis, since all its poles, of which there are boundedly many, lie in the lower half-plane and converge to points with $\Im(\lambda) \leq -\delta$. Combining this with the bound (10) we see that the functions $g_M(\lambda)$ are uniformly bounded in the half-strip $a \leq \Re(\lambda) \leq b$, $\Im(\lambda) \geq -\delta/2$. Since the $\tilde{P}_M$ have bounded inverses by (9) it follows that the family of analytic functions given by $p_M(\lambda)/(m_M(\lambda) - z)$, as M tends to infinity on the chosen subsequence, is uniformly bounded in the half-strip $a \leq \Re(\lambda) \leq b$, $\Im(\lambda) \geq -\delta/2$, and analytic in its interior. This means that its limit as $M \to \infty$ on the subsequence is analytic in a neighbourhood of $[a, b]$, which means that $1/(m(\lambda)-z)$ is analytic in a neighbourhood of $[a,b]$ and hence that $m(\lambda)$ is holomorphic in a neighbourhood of $[a,b]$. However (a,b) contains essential spectrum of J_N^R, so this is impossible by Weyl's Theorem. □

In order to generalize this result we need to replace $m(\lambda)-z$ by $m(\lambda)-f(\lambda)$, which means we shall need a bound away from zero for $\Im(f(\lambda))$, valid on the real axis. For large $|\lambda|$ we can write (4) as

$$f(\lambda) = \frac{-\lambda}{a_{N-1}^2\left\langle\left(I+\frac{1}{\lambda}J_N^L+\frac{1}{\lambda}^2(J_N^L)^2+\cdots\right)e_{N-1}^L, e_{N-1}^L\right\rangle}$$

from which, for large real λ,

$$\Im(f(\lambda)) = \frac{1}{a_{N-1}^2}\Im\langle J_N^L e_{N-1}^L, e_{N-1}^L\rangle + O(\lambda^{-1}) = \frac{s_{N-1}}{a_{N-1}^2} + O(\lambda^{-1}). \tag{12}$$

Here we assume without loss of generality that $s_{N-1}>0$. If we also know that $s_{N-2}\neq 0$ then it may be shown that $\Im(f(\lambda)) >$ const. >0 for all real λ, though in general $\Im(f(\lambda))$ may have zeros on the real axis. However we do know that for real λ, the function $\Im(f(\lambda))$ is a rational function of λ. Equation (12) implies that $\Im(f(\lambda))$ is a quotient of two polynomials of equal order:

Lemma 3. *Suppose $s_{N-1}>0$. Then there exist polynomials $q(\lambda)$ and $\tilde{q}(\lambda)$ of equal order such that*

$$\Im(f(\lambda)) = \frac{q(\lambda)}{\tilde{q}(\lambda)}, \quad \lambda\in\mathbb{R}.$$

We are now in a position to prove our main result for Jacobi matrices with finitely supported dissipative perturbation s.

Theorem 2. *Suppose that $s_j\geq 0$ for all j, that $s_{N-1}>0$ and that $s_j=0$ for $j\geq N$. Let f be as in (4). Then every point of $\sigma_{\rm ess}(J)$ is approximated, for large M, by roots of the equation $m_M(\lambda)=f(\lambda)$. In view of Lemmas 1 and 2 this means that every point of $\sigma_{\rm ess}(J)$ is approximated by eigenvalues of finite section truncations of J.*

Proof. The proof is very similar to that of Proposition 3 in strategy; however the details are slightly more involved. Suppose that the result is false: there exists an interval (a,b) containing points of essential spectrum of J_0 such that the functions $m_M(\lambda)-f(\lambda)$ fail to have zeros in a neighbourhood of the line segment $[a,b]$ in $\mathbb{C}$ for some sequence of integers M tending to infinity. This means that the function $1/(m_M(\lambda)-f(\lambda))$ has no singularities in a neighbourhood of the line segment $[a,b]$. The first departure from the method of proof used for Proposition 3 is that we shall re-define the function g_M previously given by (8). Using the polynomial q from Lemma 3 we consider

$$g_M(\lambda) = \frac{q(\lambda)}{(\lambda+i)^d(m_M(\lambda)-f(\lambda))}P_M(\lambda), \tag{13}$$

in which d is the degree of $q(\lambda)$ and $P_M(\lambda)$ is the Blaschke product associated with non-real zeros of the denominator $m_M(\lambda)-f(\lambda)$, which all lie in the upper half-plane. The term $(\lambda+i)^d$ in the denominator cancels the growth of $q(\lambda)$ at infinity. The inclusion of the polynomial $q(\lambda)$ in the numerator, which cancels any real zeros of the denominator, together with the result of Lemma 3, imply that on the real axis, where $m_M(\lambda)$ is real-valued, g_M is uniformly bounded: there exists a constant C independent of M such that

$$|g_M(\lambda)|\leq C, \quad \lambda\in\mathbb{R}. \tag{14}$$

This replaces the bound (10) which we used before.

In order to estimate the behaviour of the Blaschke factors P_M in the upper half plane we re-write the expression for g_M in the form

$$g_M(\lambda) = \frac{q(\lambda)}{(\lambda+i)^d}\left\{\frac{a_{N-1}v_{N-1}(\lambda)}{a_{N-1}v_{N-1}(\lambda)m_M(\lambda)+v_N(\lambda)}\right\}P_M(\lambda)$$

and observe that the Blaschke factors will involve the non-real zeros $\mu_k + i\nu_k$, $\mu_k \in \mathbb{R}$, $\nu_k > 0$, of the denominator

$$d_M(\lambda) := a_{N-1}v_{N-1}(\lambda)m_M(\lambda) + v_N(\lambda).$$

We seek bounds similar to those in Proposition 2. Recall that $v_j(\lambda)$ is a polynomial of degree $j-1$ in λ, independent of M. Writing the equation $d_M(\lambda) = 0$ using (7) as

$$\frac{1}{c}a_{N-1}v_{N-1}(\lambda)\sum_k |\langle e_1, w_k\rangle|^2 \prod_{j\neq k}(\lambda_j - \lambda) + \prod_{k=1}^{N-1}(\alpha_k - \lambda)\prod_{k=1}^{M+1}(\lambda_k - \lambda) = 0,$$

in which $v_N(\lambda) = c\prod_{k=1}^{N-1}(\alpha_k - \lambda)$, $c \neq 0$, we see that d_M has $M+N$ zeros $\mu_k + i\nu_k$ and they satisfy

$$\sum_{k=1}^{M+N}(\mu_k + i\nu_k) = \sum_{k=1}^{M+1}\lambda_k + \sum_{k=1}^{N-1}\alpha_k, \tag{15}$$

whence, since the λ_k are all real,

$$\sum_{k=1}^{M+N}\nu_k = \sum_{k=1}^{N-1}\Im(\alpha_k), \tag{16}$$

independently of M – despite the fact that the λ_k do, generally, depend on M – because the α_k are determined completely by $v_N(\lambda)$.

Factoring $P_M(\lambda) = p_M(\lambda)\tilde{P}_M(\lambda)$ into the part p_M involving those zeros which converge to the (finitely many, say ν) eigenvalues off the real axis in the strip $a \le \Re(\lambda) \le b$ and the remaining term $\tilde{P}_M(\lambda)$ which has no zeros in the strip $a \le \Re(\lambda) \le b$, say

$$\tilde{P}_M(\lambda) = \prod_{k=\nu+1}^{M_N}\left(1 - \frac{2i\nu_k}{\lambda - (\mu_k - i\nu_k)}\right),$$

we obtain for $\tilde{P}_M$ the uniform bounds

$$\exp(-2A\delta^{-1}) \le |\tilde{P}_M(\lambda)| \le \exp(2A\delta^{-1}), \quad A = \sum_k \alpha_k < \infty, \tag{17}$$

where $\delta > 0$ is chosen so that $|\lambda - (\mu_k + i\nu_k)| > \delta$ for $k = \nu+1, \dots, M+N$ and $a \le \Re(\lambda) \le b$. This replaces (9). Note that we also have, trivially,

$$|P_M(\lambda)| \le 1, \quad |\tilde{P}_M(\lambda)| \le 1, \quad \Im(\lambda) \ge 0. \tag{18}$$

We can now deduce that the functions $g_M(\lambda)$ are rational and analytic in the upper half-plane where, by the Phragmen-Lindelöf Principle and (14,18), they are uniformly bounded; moreover for $a \le \Re(\lambda) \le b$ they are bounded away from zero except at a bounded number of isolated points (the zeros of $q(\lambda)p_M(\lambda)$). In the lower half-plane, since the singularities of the Blaschke factors do not approach the real axis in the strip $a \le \Re(\lambda) \le b$, we have similar properties in some M-independent rectangle with top $[a, b]$, bounded below by $\Im(\lambda) \ge \max(-1, -\delta)$.

On a subsequence of M tending to infinity the Blaschke factors, being a normal family thanks to (17), converge to non-trivial functions analytic in an open neighbourhood of the line segment $[a,b]$. The same is true of the g_M. Thus their quotient

$$\frac{q(\lambda)}{(\lambda+i)^d(m(\lambda)-f(\lambda))}$$

is analytic in an open neighbourhood of $[a,b]$, which means that m is meromorphic in a neighbourhood of $[a,b]$. However (a,b) contains essential spectrum, so the Titchmarsh-Weyl function m of the self-adjoint operator J_N^R cannot have this property. The proof is complete. □

Remark 3. *The result (15) is, in fact, nothing more than the obvious statement that the trace of the $(N+M)\times(N+M)$ finite section of J is equal to trace(J_N^L) + trace(J_N^M). To see this one needs only note that trace$(J_N^M)=\sum_{k=1}^{M+1}\lambda_k$ and that the α_k, $k=1,\dots,N-1$, which are the zeros of $v_N(\lambda)$, are precisely the eigenvalues of the $(N-1)\times(N-1)$ matrix J_N^L.*

We are now ready to prove Theorem 1, which was stated at the start of this section.

Proof of Theorem 1. If $s=0$ then there is nothing to prove as the result follows in this self-adjoint case by using discrete versions of the approach of Stolz and Weidmann [21]. Suppose then that there exists some $N\geq 2$ such that $s_{N-1}>0$. The only question which remains to be answered is whether or not the proof of Theorem 2 still works in this case. In fact a careful examination of the proof shows that there are only three points to check.

First, observe that the bound (14) is valid a fortiori because, by Remark 2, $\Im(m_M(\lambda))$ and $\Im(f(\lambda))$ have opposite signs for real λ. The bound (14) was obtained by throwing away the contribution of m_M on the real axis; including this term only improves the bound.

Secondly, in view of Remark 3, the trace formulae (15,16) become

$$\sum_{k=1}^{M+N}(\mu_k+i\nu_k)=\sum_{k=1}^{M+N}(a_k+is_k);\qquad 0\leq\sum_{k=1}^{M+N}\nu_k=\sum_{k=1}^{M+N}s_k\leq\|s\|_{\ell^1(\mathbb{N})}.$$

Thirdly (and finally) we need to examine whether or not it is still true that in the case $s\in\ell^1(\mathbb{N})$ the Titchmarsh-Weyl functions m_M converge locally uniformly to m off the real axis, at points of analyticity of m. (The reader should bear in mind that m corresponds to a dissipative operator now.) One way to do this is to follow through the analysis in [7] and observe that since $s_k\to 0$ as $k\to\infty$ the important points in all the proofs still work. More formally, we invoke the work of Monaquel and Schmidt [19]. Since $s\in\ell^1(\mathbb{N})$ we know that $s_k\to 0$ as $k\to\infty$. Given $\epsilon>0$ we may choose a Glazman decomposition point N_ϵ such that $s_k<\epsilon$ for all $k\geq N_\epsilon$. We indicate explicitly the dependence of the Titchmarsh-Weyl coefficients on N_ϵ with the notations $m_M^{N_\epsilon}$ for those arising from the finite matrices and and m^{N_ϵ} for those arising from the infinite matrix. Inspecting equations (33) and (34) in [19] we see that invoking Theorem 4.1 of that paper shows that

$$m_M^{N_\epsilon}(\lambda)\to m^{N_\epsilon}(\lambda),\quad M\to\infty;\ \text{loc. unif. w.r.t. } |\Im\lambda|\geq 2\epsilon.$$

We next observe the well known fact that our original ϵ-independent Titchmarsh-Weyl coefficients with N fixed can be expressed in terms of the new coefficients by the same fractional linear transformation whose coefficients do not depend on M:

$$m_M(\lambda)=\frac{-1}{a_{N-1}}\left(\frac{a_{N_\epsilon-1}m_M^{N_\epsilon}(\lambda)\phi_N^{N_\epsilon}(\lambda)+\theta_N^{N_\epsilon}(\lambda)}{a_{N_\epsilon-1}m_M^{N_\epsilon}(\lambda)\phi_{N-1}^{N_\epsilon}(\lambda)+\theta_{N-1}^{N_\epsilon}(\lambda)}\right), \tag{19}$$

$$m(\lambda)=\frac{-1}{a_{N-1}}\left(\frac{a_{N_\epsilon-1}m^{N_\epsilon}(\lambda)\phi_N^{N_\epsilon}(\lambda)+\theta_N^{N_\epsilon}(\lambda)}{a_{N_\epsilon-1}m^{N_\epsilon}(\lambda)\phi_{N-1}^{N_\epsilon}(\lambda)+\theta_{N-1}^{N_\epsilon}(\lambda)}\right), \tag{20}$$

where ϕ^{N_ϵ} and θ^{N_ϵ} are solutions of the three-term recurrence relation (1) with initial conditions

$$\phi_j^{N_\epsilon} = \begin{cases} 1 & (j = N_\epsilon) \\ 0 & (j = N_\epsilon - 1) \end{cases} \quad ; \qquad \theta_j^{N_\epsilon} = \begin{cases} 0 & (j = N_\epsilon) \\ 1 & (j = N_\epsilon - 1) \end{cases} \quad .$$

The fractional linear transformations allow us to deduce that $m_M(\lambda) \to m(\lambda)$ locally uniformly for $|\Im\lambda| \geq 2\epsilon$, away from the poles of m. However since the m_M and m do not depend on λ, we obtain the desired convergence of m_M to m locally uniformly for $\Im\lambda \neq 0$, away from the poles of m.

Every other element of the proof carries over from the case of finitely supported s without change. □

3. Schrödinger operators on the half-line

In this section we treat Schrödinger operators for which the imaginary part of the potential is compactly supported and has one sign. The case in which the imaginary part of the potential need not be compactly supported will be treated in Section 4. The method which we develop in the present section could be applied equally well to the Jacobi operator case of Section 2. However the results which it yields would not improve those of Section 2. Throughout this section q is assumed to be square integrable over finite intervals.

We consider truncations of the operators L and L_0 of Section 1. L_0 will be replaced by the family of operators L_0^M in $L^2(0, M)$, $M \gg 1$, given by

$$D(L_0^M) = \{u \in L^2(0, M) \; -u'' + qu \in L^2(0, M);\; u(0) = 0 = u(M)\},$$

and we take $L^M = L_0^M + is(x)\cdot$, for $M > N := \sup(\text{supp}(s))$; since $s \in L^\infty$ we choose $D(L^M) = D(L_0^M)$.

We shall require corresponding Titchmarsh-Weyl functions. For $\Im(\lambda) \neq 0$ there is, up to scalar multiples, a unique non-trivial solution of the differential equation $-u'' + qu = \lambda u$, which we denote $\psi(x;\lambda)$; the corresponding Weyl-Titchmarsh function is

$$m(\lambda) = \frac{\psi'(N;\lambda)}{\psi(N;\lambda)}. \tag{21}$$

There is also a unique-up-to scalar multiples non-zero solution of $-u'' + qu = \lambda u$ in the interval $[N, M]$ which satisfies $u(M) = 0$; denoting this solution by $\psi_M(x;\lambda)$ we define the corresponding Weyl-Titchmarsh coefficient by

$$m_M(\lambda) = \frac{\psi_M'(N;\lambda)}{\psi_M(N;\lambda)}. \tag{22}$$

Finally we define a solution $v(x;\lambda)$ of $-v'' + (q + is)v = \lambda v$ on $(0, \infty)$ by the initial condition $v(0;\lambda) = 0$, $v'(0;\lambda) = 1$. By a standard Glazman decomposition argument a point λ is an eigenvalue of L if and only if

$$m(\lambda) - \frac{v'(N;\lambda)}{v(N;\lambda)} = 0; \tag{23}$$

similarly, a point λ is an eigenvalue of L^M if and only if

$$m_M(\lambda) - \frac{v'(N;\lambda)}{v(N;\lambda)} = 0. \tag{24}$$

Note that any eigenvalue λ of L or L^M lies in $\mathbb{C}^+$ since, if u is the corresponding normalized eigenfunction, then $\Im(\lambda) = \int_0^N s|u|^2$.

We wish to prove that for large M, every point of $\sigma_{\text{ess}}(L)$ can be approximated to arbitrary accuracy by eigenvalues of L^M, i.e. by zeros of the function $m_M(\lambda) - \frac{v'(N;\lambda)}{v(N;\lambda)}$.

Our first step is to obtain a bound on the $m_M(\lambda)$, in terms of the resolvents of the self-adjoint operators L_N^M defined by:

$$D(L_N^M) = \{u \in L^2(N, M) \mid -u'' + qu \in L^2(N, M),\; u(N) = 0 = u(M)\}, \tag{25}$$

(26)
$$L_N^M u = -u'' + qu.$$

To this end we express the solution $\psi_M(x;\lambda)$ in terms of the resolvent of L_N^M. Let w be a compactly supported smooth function with $w(N) = 1$, $w'(N) = 0$ and $-w'' + qw \in L^2(0,\infty)$, the last of these conditions holding since q is square integrable over finite intervals. Evidently w does not depend on M. We assume without loss of generality that $M \geq \sup(\text{supp}(w))$, so in particular $\|(-D^2 + (q-\lambda))w\|_{L^2(N,M)}$ does not depend on M. An elementary calculation shows that the function $\phi_M(x;\lambda) = \psi_M(x;\lambda)/\psi(N;\lambda) - w(x)$ lies in the domain of L_N^M and satisfies

(27)
$$(L_N^M - \lambda)\phi_M = -(-D^2 + (q-\lambda))w \in L^2(N,M).$$

Consequently for λ in the resolvent set of L_N^M,

(28)
$$\phi_M = -(L_N^M - \lambda)^{-1}(-D^2 + (q-\lambda))w.$$

Moreover $\phi'(N;\lambda) = \psi'(N;\lambda)/\psi(N;\lambda)$ since $w'(N) = 0$, and so

(29)
$$m_M(\lambda) = \phi'(N;\lambda).$$

Lemma 4. *Suppose q is square integrable over finite intervals. Then for λ in any compact set there exists a constant $C > 0$ independent of M such that*

$$|m_M(\lambda)| \leq \frac{C}{\mathit{dist}(\lambda, \sigma(L_N^M))}.$$

Proof. From eqn. (28) and the M-independent bound on $\|(-D^2+(q-\lambda))w\|_{L^2(N,M)}$ we obtain immediately a bound of the form

(30)
$$\|\phi_M(\cdot;\lambda)\|_{L^2(N,M)} \leq \frac{C}{\text{dist}(\lambda, \sigma(L_N^M))}.$$

Since $w(N) = 1$ and $w'(N) = 0$, two integrations by parts together with (27) and (29) yield

$$m_M(\lambda) = \phi_M'(N;\lambda)w(N) - \phi_M(N;\lambda)w'(N) = \int_N^M \left\{w(-D^2 + (q-\lambda))\phi_M - \phi_M(-D^2 + (q-\lambda)w)\right\}$$
$$= -\int_N^M (\phi_M + w)(-D^2 + (q-\lambda))w,$$

whence the required bound on $m_M(\lambda)$, for λ in any compact set, follows from (30). □

Lemma 5. *Under the conditions of Lemma 4,*

$$|m_M'(\lambda)| \leq \frac{C}{|\mathit{dist}(\lambda, \sigma(L_N^M))|^2}.$$

Proof. Define $\upsilon_M(x;\lambda) = \frac{\partial}{\partial\lambda}\phi_M(x;\lambda)$. From the expression (28) we have

$$\upsilon_M = -(L_N^M - \lambda)^{-2}(-D^2 + (q-\lambda))w + (L_N^M - \lambda)^{-1}w.$$

This immediately yields, for λ in any compact set, a bound of the form

(31)
$$\|\upsilon_M(\cdot;\lambda)\|_{L^2(N,M)} \leq \frac{C}{|\text{dist}(\lambda, \sigma(L_N^M))|^2},$$

and the rest of the proof is similar to Lemma 4. □

Remark 4. *The fact that bounds of the forms in Lemma 4 and Lemma 5 must hold with some constants C which may depend on M follows straightforwardly from the Riesz-Herglotz representation theorem for the Nevanlinna function m_M in the upper half-plane and the fact that the representing measure therein is supported on the spectrum of the corresponding operator L_N^M. The important point in these lemmata is that the constant C can be chosen to be independently of M under the additional hypotheses used here.*

Lemma 6. *Let $f(\lambda) = v'(N;\lambda)/v(N;\lambda)$ where v is as in (23,24). Suppose that I_1 and I_2 are two disjoint finite intervals on the real axis neither of which intersects $\sigma_{\rm ess}(L)$ (i.e. both are contained in gaps in the essential spectrum). Then for each sufficiently large M there exists a closed rectangular contour Γ_M intersecting the real axis twice, once in I_1 and once in I_2, with length bounded independently of M, such that*

$$\sup_{z\in\Gamma_M}\left|\frac{1}{m_M(\lambda)-f(\lambda)}\right|$$

is bounded uniformly in M, and such that there are no poles of f on or inside Γ_M.

Proof. The first essential observation is that f is a meromorphic function with the property that $\Im(f(\lambda)) > 0$ for $\Im(\lambda) \leq 0$ and all its poles in $\mathbb{C}^+$. To see this we first observe that the poles of f are the zeros of $v(N;\lambda)$, which in turn are the eigenvalues of the differential expression $-v'' + (q+is)v = \lambda v$ on $[0,N]$ equipped with Dirichlet boundary conditions. These eigenvalues satisfy

$$\Im(\lambda) = \frac{\int_0^N s|v|^2}{\int_0^N |v|^2};$$

since s is positive on a set of positive measure and since non-trivial solutions of a second order linear equation cannot vanish on a set of positive measure, this means that the poles of f satisfy $\Im(\lambda) > 0$. For $\Im(\lambda) \leq 0$ we use a simple integration by parts to show that

$$\frac{v'(N;\lambda)}{v(N;\lambda)}|v(N;\lambda)|^2 = \int_0^N \{|v'(x;\lambda)|^2 + (q(x)+is(x)-\lambda)|v(x;\lambda)|^2\}dx,$$

which immediately yields $\Im(f(\lambda)) > 0$ for $\Im(\lambda) \leq 0$.

For the rest of the proof we use the locally uniform convergence $m_M(\lambda) \to m(\lambda)$ in the upper half plane; since f is meromorphic in the upper half-plane the zeros of $m(\lambda) - f(\lambda)$ can only accummulate at infinity or on the real axis. In the lower half-plane there are no zeros since $\Im m_M(\lambda) < 0$ there while $\Im f(\lambda) > 0$ there. Thus for a.e. $r > 0$ we can choose two finite horizontal line-segments, one in $\mathbb{C}^+$ and one in $\mathbb{C}^-$, at distance $r > 0$ from the real line, which extend past I_1 and I_2 in both directions and on which $m_M(\lambda) - f(\lambda)$ is bounded away from zero for all sufficiently large M. Moreover since all the poles of f lie strictly in $\mathbb{C}^+$ we can choose the two line segments such that there are no poles of f on them or in the rectangle between them.

To form our closed contour Γ_M we connect the two horizontal line segments by vertical line-segments passing through I_1 and I_2 respectively. In each interval I_1, I_2 there is at most one spectral point of L_M^N because each spectral gap contains at most one point of spectral pollution, as we mentioned in the introduction (Section 1). Thus we can always choose, for each M, the vertical line segments to be such that $\mathrm{dist}(\lambda,\sigma(L_N^M)) \geq \delta > 0$ there, for some small δ independent of M, and hence on those vertical line segments we have from Lemma 5 the bound

$$|m_M'(\lambda)| \leq C/\delta^2$$

for some $C > 0$ independent of M. At the points where the vertical segments cut the real line, because $m_M(\cdot)$ takes values in $\mathbb{R}\cup\{\infty\}$, we have the bound

$$|m_M(\lambda) - f(\lambda)| \geq |\Im(f(\lambda))| > c > 0,$$

for some constant c. Combining this with the bound on m'_M we see that if

$$|\Im(\lambda)| < \frac{c}{2C}\delta^2 \tag{32}$$

on the vertical line segments, then we shall have $|\Im(m_M(\lambda))| < \frac{c}{2}$ on those segments and hence

$$|m_M(\lambda) - f(\lambda)| \geq \frac{c}{2}.$$

The inequality (32) will clearly be satisfied provided $r < \frac{c}{2C}\delta^2$, which completes the proof. $\square$

We introduce the self-adjoint operator L_N defined by

$$D(L_N) = \{u \in L^2(N,\infty) \,|\, -u'' + qu \in L^2(N,\infty),\ u(N) = 0\}, \tag{33}$$

$$L_N u = -u'' + qu. \tag{34}$$

Let I be an open interval containing a point of $\sigma_{\rm ess}(L_0) = \sigma_{\rm ess}(L_N)$, which is such that there are gaps in the essential spectrum immediately to the left and immediately to the right of I. We shall see below that every neighbourhood of I must contain an an arbitrarily large number of eigenvalues of L^M as M increases. However we would also like to know how these eigenvalues are distributed. If $\tilde{I}$ is any open sub-interval of I containing a point of $\sigma_{\rm ess}(L_0)$ and if the integrated density of states

$$\rho_N(\tilde{I}) = \lim_{M\to\infty} \frac{1}{\sqrt{M-N}}\{\text{number of eigenvalues of } L_N^M \text{ in } \tilde{I}\}$$

of the underlying self-adjoint operator L_N, exists and is non-zero, then the number of eigenvalues of L_N^M in $\tilde{I}$ will be $O(\sqrt{M})$ for large M. We shall use this fact below to obtain information about the distribution of eigenvalues of L^M in a neighbourhood of $\tilde{I}$. In fact we shall not even need as strong a hypothesis as the existence of the integrated density of states for L_N to prove our result.

Proposition 4. *Let I be an open interval containing points of $\sigma_{\rm ess}(L_0) = \sigma_{\rm ess}(L_N)$ and suppose that there are gaps I_1 and I_2 in the essential spectrum, immediately to the left and to the right of I respectively. Let f be as in Lemma 6 and let Γ_M be, for each M, a rectangular closed contour chosen to satisfy the outcome of Lemma 6. Let g be any function which is analytic in an open set containing all the curves Γ_M and the regions they enclose. Then there exists a constant C independent of M and g such that*

$$\left|\sum_j g(n_j) - \sum_j g(p_j)\right| \leq C\|g\|_{L^2(\Gamma_M)}, \tag{35}$$

where $\{n_j\}$ are the zeros of $m_M(\lambda) - f(\lambda)$ inside Γ_M (i.e. the eigenvalues of L_N^M inside Γ_M) and $\{p_j\}$ are the poles of m_M inside Γ_M (i.e. the eigenvalues inside Γ_M of the Hermitian problem with Dirichlet boundary conditions at N and M).

Proof. This proof follows immediately by applying the results of Lemmata 5 and 6 to the identity

$$\sum_j g(n_j) - \sum_j g(p_j) = \frac{1}{2\pi i}\int_{\Gamma_M} g(\lambda)\frac{m'_M(\lambda) - f'(\lambda)}{m_M(\lambda) - f(\lambda)}d\lambda,$$

noting that the Γ_M are chosen not to enclose or pass through any poles of f. $\square$

Before proceeding with the rest of this section we make a brief digression which yields an interesting aside.

Corollary 1. *Under the conditions of Proposition 4 we have*

$$\sum_{k=0}^{\infty} R^{-2k} \left| \sum_j (n_j)^k - \sum_j (p_j)^k \right|^2 \leq 2\pi R C^2, \tag{36}$$

uniformly in M. Here we assume without loss of generality that the contour Γ_M is replaced by a circle C_R of radius R with centre at some real point.

Proof. The estimate (35) can be rewritten in the form

$$\left\langle g, \left(\sum_j \frac{1}{\overline{\lambda} - \overline{n_j}} - \sum_j \frac{1}{\overline{\lambda} - p_j} \right) \right\rangle_{L^2(C_R)} \leq 2\pi C \|g\|_{L^2(C_R)}. \tag{37}$$

Let $\mathbb{D}_R$ be the interior of C_R and let $H^2(\mathbb{D}_R)$ be the Hardy class of functions analytic in $\mathbb{D}_R$. Then (37) means that

$$\left\| P_+ \left(\sum_j \frac{1}{\overline{\lambda} - \overline{n_j}} - \sum_j \frac{1}{\overline{\lambda} - p_j} \right) \right\|_{H^2(\mathbb{D}_R)} \leq 2\pi C,$$

where P_+ denotes the Riesz projection onto $H^2(\mathbb{D}_R)$. By a change of variables if necessary we may assume without loss of generality that C_R has centre at 0. Making the change of variable $z = \lambda/R$ so $|z| = 1$ and $\overline{\lambda} = R/z$ on C_R we obtain, by direct calculation,

$$\left\| P_+ \left(\sum_j \frac{z}{1 - z\overline{n_j}/R} - \sum_j \frac{z}{1 - z p_j/R} \right) \right\|_{H^2} \leq 2\pi C R^{1/2},$$

where P_+ now denotes the canonical Riesz projection onto the Hardy space in the unit disc. This projection can be omitted since the poles of $z/(1 - z\overline{n_j}/R)$ and of $z/(1 - z p_j/R)$ lie outside the unit disc. We can now calculate the L^2 norm on the unit circle using the Fourier coefficients and obtain

$$\sum_{k=0}^{\infty} \left| \sum_j \left(\frac{\overline{n_j}}{R} \right)^k - \sum_j \left(\frac{\overline{p_j}}{R} \right)^k \right|^2 \leq 2\pi C^2 R,$$

which is equivalent to the required result. □

Remark 5. *The $\{n_j\}$ are the eigenvalues of L^M inside Γ_M and the $\{p_j\}$ are the eigenvalues of the (self-adjoint) Dirichlet problem on (N, M) inside Γ_M. The number of $\{p_j\}$ in a fixed interval grows like $\sqrt{M}$ as $M \to \infty$ if the problem has an integrated density of states. Taking just the $k = 0$ term in (36) shows that the difference between the number of $\{n_j\}$ and the number of $\{p_j\}$ inside Γ_M remains bounded, so the number of $\{n_j\}$ also grows like $\sqrt{M}$ for large M. More generally, we can deduce a high level of closeness* in average *of the eigenvalue spacings for two 'cut' problems, one self-adjoint and the other not. We could apply self-adjoint techniques to analyze the spacing behaviour for the eigenvalues of the non-self-adjoint problem, in average.*

Returning to our main theme, we wish to show that if I is as in Proposition 4 and if $\tilde{I}$ is any open subinterval of I containing a point of $\sigma_{\mathrm{ess}}(L_0) = \sigma_{\mathrm{ess}}(L_N)$ then there will be arbitrarily many eigenvalues of L^M in a neighbourhood of $\tilde{I}$, for sufficiently large M. To this end we need an appropriate choice of g.

By a suitable change of spectral variable, if necessary, we may assume that the interval $\tilde{I}$ has endpoints ± 1. We choose

$$g(\lambda) = \int_{-1+\delta}^{1-\delta} \exp(-\alpha(\lambda - k)^2) dk, \tag{38}$$

in which the parameter $\alpha > 0$ and $\delta \in (0,1)$ will be chosen for convenience.

Lemma 7. *For $-1+\delta \leq \mu \leq 1-\delta$, $g(\mu) \geq \sqrt{\pi/(4\alpha)}$ for all sufficiently large α, for $\delta \in (0, 1/4)$.*

Proof. We observe that

$$g(\mu) = \int_{-1+\delta}^{1-\delta} \exp(-\alpha(\mu - k)^2) dk \geq \int_0^{2-2\delta} \exp(-\alpha t^2) dt$$
$$= \frac{1}{\sqrt{\alpha}} \int_0^{\sqrt{\alpha}(2-2\delta)} \exp(-t^2) dt \geq \sqrt{\frac{\pi}{4\alpha}},$$

for all sufficiently large α. $\square$

Lemma 8. *Suppose $|\Re(\lambda)| \geq 1$ and $|\Im(\lambda)| \leq \nu$, where $\nu > 0$ is fixed. Then*

$$|g(\lambda)| \leq \frac{\exp(\alpha(\nu^2 - \delta^2))}{2\alpha\delta}.$$

Proof. Writing λ in terms of its real and imaginary parts as $\lambda = \mu + it$, we have $|\exp(-\alpha(\lambda - k)^2)| = \exp(-\alpha(\mu - k)^2)\exp(\alpha t^2)$ and hence, for $|t| \leq \nu$,

$$|g(\lambda)| \leq \exp(\alpha\nu^2) \int_{-1+\delta}^{1-\delta} \exp(-\alpha(\mu - k)^2) dk \leq \exp(\alpha\nu^2) \int_\delta^\infty \exp(-\alpha x^2) dx$$
$$= \frac{\exp(\alpha\nu^2)}{\sqrt{\alpha}} \int_{\delta\sqrt{\alpha}}^\infty \frac{1}{2x} 2x \exp(-x^2) dx = \frac{\exp(\alpha\nu^2)}{\sqrt{\alpha}} \left\{ \left[\frac{\exp(-x^2)}{2x} \right]_\infty^{\delta\sqrt{\alpha}} - \int_{\delta\sqrt{\alpha}}^\infty \frac{1}{4x^2} \exp(-x^2) dx \right\}$$
$$\leq \frac{\exp(\alpha(\nu^2 - \delta^2))}{2\alpha\delta}.$$

$\square$

Lemma 9. *For $|\Im(\lambda)| \leq \nu$,*

$$|g(\lambda)| \leq \frac{\sqrt{\pi}\exp(\alpha\nu^2)}{\sqrt{\alpha}}.$$

Proof. Using the fact that if $\lambda = \mu + it$ then $|\exp(-\alpha(\lambda - k)^2)| = \exp(-\alpha(\mu - k)^2)\exp(\alpha t^2)$ we obtain

$$|g(\lambda)| \leq \exp(\alpha\nu^2) \int_{-1+\delta}^{1-\delta} \exp(-\alpha(\mu - k)^2) dk \leq \exp(\alpha\nu^2) \int_{-\infty}^\infty \exp(-\alpha x^2) dx = \frac{\sqrt{\pi}\exp(\alpha\nu^2)}{\sqrt{\alpha}}.$$

$\square$

We now use (35) with the bounds just proved. Suppose that $m_M(\lambda) - f(\lambda)$ has no zeros in the strip $-1 \leq \Re(\lambda) \leq 1$ containing $\tilde{I}$. Then taking ν to be the maximum distance of the contour Γ_M from the real axis we obtain, due to the positivity of g and Lemma 9,

$$\sum_{-1+\delta \leq p_j \leq 1-\delta} g(p_j) \leq \sum_j g(n_j) + \frac{C_\nu \exp(\alpha\nu^2)}{\sqrt{\alpha}},$$

in which we have indicated that the constant C in (35) certainly depends on ν. Define $\tilde{I}_\delta$ to be the interval $[-1+\delta, 1-\delta]$ and let

$$N_P(M;\tilde{I}_\delta) = \#\{-1+\delta \leq p_j \leq 1-\delta\}; \quad N_Z(M) = \#\{n_j \text{ inside } \Gamma_M\}.$$

Using the lower bounds on the $g(p_j)$ provided by Lemma 7 and the upper bounds on the $g(n_j)$ provided by Lemma 8 we obtain

$$\sqrt{\frac{\pi}{4\alpha}} N_P(M;\tilde{I}_\delta) \leq \frac{\exp(\alpha(\nu^2-\delta^2))}{2\alpha\delta} N_Z(M) + \frac{C_\nu \exp(\alpha\nu^2)}{\sqrt{\alpha}}.$$

We now choose

$$\alpha \sim \frac{1}{2\nu^2}\log N_P(M;\tilde{I}_\delta).$$

Note that for sufficiently small $\delta > 0$, $N_P(M;\tilde{I}_\delta) \to \infty$ as $M \to \infty$, by the self-adjoint theory of finite sections and the fact that $\tilde{I}_\delta$ contains a point of essential spectrum. Thus there exists a new constant $\tilde{C}_\nu$ such that, for all sufficiently large M,

$$N_P(M;\tilde{I}_\delta) \leq \frac{\sqrt{N_P(M;\tilde{I}_\delta)}(N_P(M;\tilde{I}_\delta))^{-\delta^2/(2\nu^2)} N_Z(M)}{\delta\sqrt{\pi\alpha}} + \tilde{C}_\nu\sqrt{N_P(M;\tilde{I}_\delta)},$$

whence

$$(\sqrt{N_P(M;\tilde{I}_\delta)} - \tilde{C}_\nu)^{1+\delta^2/(2\nu^2)}\sqrt{\log(N_P(M))} \leq \frac{\sqrt{2}\nu N_Z(M)}{\delta}. \tag{39}$$

Theorem 3. *Suppose that q is square integrable over compact intervals and that s is compactly supported, essentially bounded and non-negative. Let I be any open interval containing a point of $\sigma_{\rm ess}(L_0) = \sigma_{\rm ess}(L_N)$, such that there are gaps in the essential spectrum immediately to the left and to the right of I. Let $\tilde{I}$ be any open sub-interval of I containing a point of essential spectrum. If there is a neighbourhood of the closure of $\tilde{I}$ which contains no zeros of $m_M(\lambda) - f(\lambda)$ (i.e. no eigenvalues of L^M) for some sequence of values of M tending to infinity, then the self-adjoint Dirichlet problems on the intervals $[N, M]$ have eigenvalue counts which satisfy the following inequality: for all sufficiently small $\delta > 0$, for all sufficiently small $\nu > 0$ there exists a constant $\tilde{C}_\nu > 0$ such that for all sufficiently large M in the sequence,*

$$(\sqrt{N_P(M;\tilde{I}_\delta)} - \tilde{C}_\nu)^{1+\delta^2/(2\nu^2)}\sqrt{\log(N_P(M;\tilde{I}_\delta))} \leq \frac{\sqrt{2}\nu N_P(M;I)}{\delta}; \tag{40}$$

here $N_P(M;I)$ is the number of eigenvalues of L_N^M in I.

Proof. The proof follows from (39) by noting that in view of (35), $N_P(M;I)$ differs from $N_Z(M)$ by at most a ν-dependent constant, which is therefore negligible for large M. □

Corollary 2. *1) Let I, $\tilde{I}$ and $\tilde{I}_\delta$ be as in Theorem 3. Suppose that the self-adjoint operator L_N has a non-zero integrated density of states on $\tilde{I}_\delta$ for some $\delta > 0$ and a finite integrated density of states on I. Then the situation of Theorem 3 is impossible; consequently there must be, for all sufficiently large M, zeros of $m_M(\lambda) - f(\lambda)$ (i.e. eigenvalues of L^M) in every neighbourhood of every interval $\tilde{I} \subseteq I$ of $\sigma_{\rm ess}(L_0) = \sigma_{\rm ess}(L_N)$ whose interior contains an interval with non-trivial integrated density of states.*

2) The same conclusion holds if, instead of the integrated-density-of-states hypothesis, we assume merely that $\liminf_{M\to\infty}\{N_P(M;\tilde{I}_\delta)/N_P(M,I)\}$ is non-zero.

Proof. 1) If the hypotheses of the Corollary are satisfied then $N_P(M;\tilde{I}_\delta)$ and $N_P(M;I)$ can be estimated for large M in terms of the integrated densities of states:

$$N_P(M;\tilde{I}_\delta) \sim \sqrt{M-N}\rho_N(\tilde{I}_\delta), \quad N_P(M;I) \sim \sqrt{M-N}\rho_N(I),$$

in which $\rho_N(\tilde{I}_\delta)$ and $\rho_N(I)$ are non-zero constants. This means that the inequality (40) cannot hold since one can choose $\delta \geq \nu\sqrt{2}$. Consequently every neighbourhood of $\tilde{I}$ must contain a zero of $m_M(\lambda) - f(\lambda)$.

2) This follows by a very similar argument. □

4. Schrödinger operators by Jacobi matrix methods

In the previous section we used a different approach from that taken for Jacobi operators, and obtained the spectral approximation result, Corollary 2, which uses a weak but technical hypothesis on the integrated density of states for the underlying self-adjoint operator. We now adapt the Jacobi matrix approach and prove the following result.

Theorem 4. *Suppose that L is the operator of Section 1, that q is essentially bounded below and square integrable over finite intervals, and that $s \in L^1(0,\infty) \cap L^\infty(0,\infty)$ satisfies $s(x) \to 0$ as $x \to \infty$. Let L^M be the operators introduced in Section 3. Let $\lambda_{\rm ess}$ be a point of $\sigma_{\rm ess}(L)$. Then every open neighbourhood of $\lambda_{\rm ess}$ in $\mathbb{C}$ contains, for all sufficiently large M, an eigenvalue of L^M.*

Our first step towards a proof of this result requires a trace estimate to replace the trace formula (15) for the zeros of the function $m_M(\lambda) - f(\lambda)$, which are the eigenvalues of L^M. The proof of this result is based on estimates of a Green's function on the diagonal, adapted from [4].

Lemma 10. *Suppose that s and q satisfy the hyptheses of Theorem 4. Let $(\lambda_k^M)_{k\in\mathbb{N}}$ be the eigenvalues of L^M. Suppose that there is a point $\lambda_{\rm ess}$ of $\sigma_{\rm ess}(L)$ and an open set U containing $\lambda_{\rm ess}$ such that, on some subsequence of integers M tending to infinity, none of the λ_k^M or $\overline{\lambda_k^M}$ lie in U. Then for every set V with $\overline{V} \subset U$, there exists a constant C independent of M and a constant $\delta \in \mathbb{R}$ such that for all sufficiently large M in the subsequence and all $\lambda \in \overline{V}$,*

$$\left| \sum_{k\in\mathbb{N}} \frac{\Im\left(\frac{-1}{\lambda_k^M+\delta}\right)}{(\lambda+\delta)^{-1} - (\overline{\lambda_k^M}+\delta)^{-1}} \right| \leq C, \quad \left| \sum_{k\in\mathbb{N}} \frac{\Im\left(\frac{-1}{\lambda_k^M+\delta}\right)}{(\lambda+\delta)^{-1} - (\lambda_k^M+\delta)^{-1}} \right| \leq C.$$

Proof. Since the $\lambda_k^M - \lambda$ and $\overline{\lambda_k^M} - \lambda$ are uniformly bounded away from 0 for $\lambda \in \overline{V}$, an elementary calculation shows that it suffices to prove that for some $\delta > 0$,

$$\sum_{k\in\mathbb{N}} \Im\left(\frac{-1}{\lambda_K^M + \delta}\right)$$

is bounded independently of M. Since the operators L^M are all dissipative with trace-class resolvents and since the real axis lies in the resolvent set of L^M for $s \not\equiv 0$ we can invoke Lidskii's Theorem [9, Thm. 8.4, p. 101]:

$$\sum_{k\in\mathbb{N}} \Im\left(\frac{-1}{\lambda_K^M + \delta}\right) = \text{trace}(-\Im(L^M + \delta I)^{-1}). \tag{41}$$

We must therefore obtain an M-independent bound on the right hand side of (41). We use the Hilbert identity to write

$$\begin{aligned} -\Im(L^M + \delta I)^{-1} = \frac{1}{2i}\left\{((L^M)^* + \delta I)^{-1} - (L^M + \delta I)^{-1}\right\} &= ((L^M)^* + \delta I)^{-1} s (L^M + \delta I)^{-1} \\ &= (\sqrt{s}(L^M + \delta I)^{-1})^*(\sqrt{s}(L^M + \delta I)^{-1}), \end{aligned}$$

whence we see that

$$\text{trace}(-\Im(L^M + \delta I)^{-1}) = \|\sqrt{s}(L^M + \delta I)^{-1}\|_2^2,$$

in which $\|\cdot\|_2$ denotes the Hilbert-Schmidt norm. Thus it suffices to obtain a bound

$$\|\sqrt{s}(L^M+\delta I)^{-1}\|_2 \leq C, \tag{42}$$

in which C does not depend on M.

In order to simplify the estimates, we would like to replace $\sqrt{s}(L^M+\delta I)^{-1}$ by $\sqrt{s}(L_0^M+\delta I)^{-1}$, in which $L_0^M = L^M - is$ is the self-adjoint Schrödinger operator on $(0,M)$ with Dirichlet boundary conditions and potential q which was introduced in Section 3. To this end we again use the Hilbert identity to write

$$\sqrt{s}(L^M+\delta I)^{-1} - \sqrt{s}(L_0^M+\delta I)^{-1} = -\sqrt{s}(L_0^M+\delta I)^{-1}(is)(L^M+\delta I)^{-1},$$

whence, using the inequality $\|AB\|_2 \leq \|A\|_2\|B\|$, with $\|\cdot\|$ denoting the natural operator norm,

$$\|\sqrt{s}(L^M+\delta I)^{-1}\|_2 \leq \|\sqrt{s}(L_0^M+\delta I)^{-1}\|_2(1+\|s\|_\infty\|(L^M+\delta I)^{-1}\|).$$

Since q is assumed to be essentially bounded below, the numerical ranges of the operators L^M are bounded below in real part, so there exists a choice of $\delta > 0$ such that the norms $\|(L^M+\delta I)^{-1}\|$ are bounded independently of M. Thus to obtain the bound (42) it is enough to obtain a bound

$$\|\sqrt{s}(L_0^M+\delta I)^{-1}\|_2 \leq C. \tag{43}$$

Let $(\phi_k^M)_{k=1}^\infty$ be the orthonormal eigenfunctions of L_0^M, with eigenvalues $(\lambda_k^M)_{k=1}^\infty$. Since q is essentially bounded below we may assume that $\delta > 0$ is sufficiently large to ensure $\lambda_k^M + \delta \geq 1$ for all k and M: for instance, if we choose δ so that $q(x)+\delta \geq 1$ a.e. then this result will hold. Now

$$\begin{aligned}\|\sqrt{s}(L_0^M+\delta I)^{-1}\|_2^2 &= \sum_{k=1}^\infty \|\sqrt{s}(L_0^M+\delta I)^{-1}\phi_k^M\|^2 = \sum_{k=1}^\infty \int_0^M \frac{s(x)(\phi_k^M(x))^2dx}{\left(\lambda_k^M+\delta\right)^2} \\ &\leq \sum_{k=1}^\infty \int_0^M \frac{s(x)(\phi_k^M(x))^2dx}{\lambda_k^M+\delta} = \int_0^M s(x)G_M(x,x)dx,\end{aligned}$$

where G_M is the Green's kernel of the resolvent $(L_0^M+\delta I)^{-1}$ The proof will therefore be complete if we can bound $G_M(x,x)$ uniformly in x and M.

To do this we shall invoke the results of Chernyavskaya and Shuster [4]. A small amount of additional work is required since the results there are for problems posed on the whole of $\mathbb{R}$ rather than on a finite interval or on a semi-axis.

We have already assumed that $q+\delta \geq 1$ a.e. on $[0,\infty)$. Extend q to the negative semi-axis in such a way that this inequality continues to hold there. Let ψ_L and ψ_R be the solutions[2] of $-y''+(q+\delta)y = 0$ which are square summable in $(-\infty,0)$ and $(0,\infty)$ respectively; these are unique up to scalar multiples and we normalize them so that their Wronskian $W(\psi_R,\psi_L)$ is 1. Define

$$\phi(x) = \frac{\psi_R(x)}{\psi_R(0)} - \frac{\psi_L(x)}{\psi_L(0)}.$$

A standard calculation now shows that the Green's function G_M for our problem is given for $t \leq x$ by

$$G_M(x,t) = \psi_L(0)\phi(t)\left(\psi_R(x) - \frac{\psi_R(M)}{\phi(M)}\phi(x)\right).$$

To show that this is bounded we invoke the expressions for ψ_L and ψ_R in [4, Theorem 1]:

$$\psi_L(x) = \sqrt{\rho(x)}\exp\left(\frac{1}{2}\int_{x_0}^x \frac{d\xi}{\rho(\xi)}\right), \quad \psi_R(x) = \sqrt{\rho(x)}\exp\left(-\frac{1}{2}\int_{x_0}^x \frac{d\xi}{\rho(\xi)}\right),$$

[2]Chernyavskaya and Shuster denote ψ_L by v and ψ_R by u.

in which the function $\rho(x) > 0$ is $G(x,x)$, where G is the Green's kernel for the problem on $(-\infty,\infty)$. Using these formulae we find that

$$\phi(x) = -2\sqrt{\frac{\rho(x)}{\rho(0)}}\sinh\left(\frac{1}{2}\int_0^x \frac{d\xi}{\rho(\xi)}\right),$$

$$\phi(x)\psi_R(x) = -\sqrt{\frac{\rho(x)}{\rho(0)}}\left[1-\exp\left(-\int_0^x \frac{d\xi}{\rho(\xi)}\right)\right]\exp\left(-\frac{1}{2}\int_0^{x_0}\frac{d\xi}{\rho(\xi)}\right),$$

and

$$\phi(x)^2\frac{\psi_R(M)}{\phi(M)} = -2\frac{\rho(x)}{\rho(0)}\left[\frac{\sinh^2\left(\frac{1}{2}\int_0^x \frac{d\xi}{\rho(\xi)}\right)}{\sinh\left(\frac{1}{2}\int_0^M \frac{d\xi}{\rho(\xi)}\right)}\right]\exp\left(-\frac{1}{2}\int_{x_0}^M \frac{d\xi}{\rho(\xi)}\right).$$

It is therefore clear that $G_M(x,x)$ is uniformly bounded in $x \geq 0$ and $M \gg 1$ if $\rho(x) > 0$ is uniformly bounded in x. However by [4, Theorem 2] we have $\frac{d(x)}{4} < \rho(x) < \frac{3d(x)}{2}$, where $d(x)$ is, for each x, the solution of

$$2 = d\int_{x-d}^{x+d}(q(t)+\delta)dt.$$

As we have chosen δ such that $q(t)+\delta \geq 1$, we have $2 \geq 2d^2$, so $0 < d(x) \leq 1$. This completes the proof. □

Let $\delta > 0$ be as in Lemma 10. Define the convergent Blaschke product

$$P_M(\lambda) = \prod_{k\in\mathbb{N}}\left(\frac{(\lambda+\delta)^{-1}-(\lambda_k^M+\delta)^{-1}}{(\lambda+\delta)^{-1}-(\overline{\lambda_k^M}+\delta)^{-1}}\right) = \prod_{k\in\mathbb{N}}\left(\frac{\lambda_k^M-\lambda}{\overline{\lambda_k^M}-\lambda}\right)\left(\frac{\overline{\lambda_k^M}+\delta}{\lambda_k^M+\delta}\right). \tag{44}$$

Corollary 3. *Each function P_M is well defined for $\Im(\lambda) \geq 0$, and for $\Im\lambda < 0$ provided $\lambda \notin \bigcup_{k\in\mathbb{N}}\{\overline{\lambda_k^M}\}$. More generally, given any compact set $K \subseteq \mathbb{C}\setminus\bigcap_{P\in\mathbb{N}}\overline{\bigcup_{M>P}\bigcup_{k\in\mathbb{N}}\{\lambda_k^M\}}$, the functions $1/P_M$ are uniformly bounded in K while the P_M are uniformly bounded in K^* (the complex conjugates of elements of K).*

Proof. A typical term in the Blaschke product (44) has the form

$$\frac{(\lambda+\delta)^{-1}-(\lambda_k^M+\delta)^{-1}}{(\lambda+\delta)^{-1}-(\overline{\lambda_k^M}+\delta)^{-1}} = 1 + \frac{2i\Im\left(\frac{-1}{\lambda_k^M+\delta}\right)}{(\lambda+\delta)^{-1}-(\overline{\lambda_k^M}+\delta)^{-1}}.$$

The required upper bound on P_M therefore follows using the standard exponential bounds for each term, as in the proofs of (9,17), together with the result of Lemma 10. The bound from below follows by considering the reciprocal terms, which involves swapping λ_k^M and $\overline{\lambda_k^M}$ in all the calculations. □

Lemma 11. *Let $f(\lambda) = v'(N;\lambda)/v(N;\lambda)$ be as in Lemma 6. Then $\Im(f(\lambda))$ is nonzero for all real λ; moreover,*

$$\Im(f(\lambda)) \geq \frac{1}{2}\int_0^N s(t)dt - o(\lambda^{-1}), \quad \lambda\to+\infty; \tag{45}$$

$$\Im(f(\lambda)) \geq C\exp(-2(N-\sigma)\sqrt{-\lambda}), \quad \lambda\to-\infty, \tag{46}$$

where C is a positive constant and $\sigma \in (0,N)$ is chosen such that $\int_\sigma^N s(t)dt > 0$.

Proof. We use transformator kernels (see, e.g., [12]) to represent v in terms of the solution of the free problem:

$$v(x;\lambda) = \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}} + \int_0^x K(x,t)\frac{\sin(t\sqrt{\lambda})}{\sqrt{\lambda}}dt;$$

the kernel K is bounded and has bounded derivatives since $q + is \in L^2(0,N)$. Using an integration by parts to gain an extra power of $\sqrt{\lambda}$ in the denominator of the integrand we obtain

$$v(x;\lambda) = \frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}} + r_1(x;\lambda), \quad v'(x;\lambda) = \cos(x\sqrt{\lambda}) + r_2(x;\lambda), \tag{47}$$

in which

$$|r_1(x;\lambda)| \le C|\lambda|^{-1}\exp(x\Im(\sqrt{\lambda})), \quad |r_2(x;\lambda)| \le C|\lambda|^{-1/2}\exp(x\Im(\sqrt{\lambda})), \tag{48}$$

for some positive constant C. Now an integration by parts shows that for real λ,

$$\Im(f(\lambda)) = \Im\left(\frac{v'(N;\lambda)}{v(N;\lambda)}\right) = \frac{1}{|v(N;\lambda)|^2}\int_0^N s(t)|v(t;\lambda)|^2 dt.$$

Using the asymptotic formula (47) we obtain, when $\lambda \to -\infty$,

$$\Im(f(\lambda)) \sim \exp(-2N\sqrt{-\lambda})\int_0^N s(t)\exp(2t\sqrt{-\lambda})dt \ge \exp(-2(N-\sigma)\sqrt{-\lambda})\int_\sigma^N s(t)dt,$$

which proves (46). To obtain (45) we observe from (47) that $|v(N;\ \lambda)|^2 \le \lambda^{-1}(1 + O(\lambda^{-1/2}))$ for all sufficiently large, positive λ, and hence

$$\Im(f(\lambda)) \ge \lambda(1+O(\lambda^{-1/2}))\int_0^N s(t)|v(t;\lambda)|^2dt = (1+O(\lambda^{-1/2}))\left\{\int_0^N s(t)\sin^2(t\sqrt{\lambda})dt + O(\lambda^{-1/2})\right\}$$

$$\to \frac{1}{2}\int_0^N s(t)dt, \quad \lambda \to +\infty,$$

by a trigonometric identity and the Riemann-Lebesgue lemma. This completes the proof. □

Let $\sigma \in (0,N)$ be as in Lemma 11; let P_M be the Blaschke product (44) and let

$$g_M(\lambda) = \frac{\exp(-2(N-\sigma)\sqrt{-\lambda})}{m_M(\lambda) - f(\lambda)}P_M(\lambda), \tag{49}$$

in which the square root is chosen which has positive imaginary part for λ in the upper half plane. We wish to examine the large-M behaviour of the functions g_M. First, however, we wish to examine these functions for fixed M; in particular we wish to show that they take their maximum values on the real axis. This will follow from the strong form of the Phragmen-Lindelöf Principle [16, Theorem 7.6] due to the following result.

Lemma 12. *There exists a sequence of positive reals $(r_n)_{n\in\mathbb{N}}$ such that $r_n \nearrow \infty$ as $n \nearrow \infty$ and a constant $C_M > 0$ such that*

$$\sup_{|\lambda|=r_n, \Im\lambda\ge 0} |g_M(\lambda)| \le C_M r_n^{-1/2}. \tag{50}$$

In fact, $r_n = (n+1/4)^2\pi^2/M^2$.

Proof. We recall the formula for the Titchmarsh-Weyl function (22). In this formula we shall use asymptotic formulae for $\psi_M(x;\lambda)$ and $\psi_M'(x;\lambda)$ similar to (47):

$$\psi_M(x;\lambda) = \frac{\sin((x-M)\sqrt{\lambda})}{\sqrt{\lambda}} + r_{1,M}(x;\lambda), \quad \psi_M'(x;\lambda) = \cos((x-M)\sqrt{\lambda}) + r_{2,M}(x;\lambda), \tag{51}$$

in which

$$|r_{1,M}(x;\lambda)| \leq C|\lambda|^{-1}\exp((M-x)\Im(\sqrt{\lambda})), \quad |r_{2,M}(x;\lambda)| \leq C|\lambda|^{-1/2}\exp((M-x)\Im(\sqrt{\lambda})). \tag{52}$$

Since the Blaschke factor P_M and the exponential term have modulus ≤ 1 in the upper half plane, we need only consider the term

$$\frac{1}{m_M(\lambda)-f(\lambda)} = \frac{v(N;\lambda)\psi_M(N;\lambda)}{\psi'_M(N;\lambda)v(N;\lambda)-\psi_M(N;\lambda)v'(N;\lambda)},$$

which, upon use of the asymptotic formulae, admits a bound

$$\left|\frac{1}{m_M(\lambda)-f(\lambda)}\right| \leq \frac{|\lambda|^{-1}\exp(M\Im\sqrt{\lambda})(1/4+O(\lambda^{-1/2}))}{|\lambda^{-1/2}\sin(M\sqrt{\lambda})|-C\exp(M\Im\sqrt{\lambda})|\lambda|^{-1}}, \tag{53}$$

provided the denominator is positive. We examine the term $|\lambda|^{-1/2}\sin(M\sqrt{\lambda})$ in the denominator, on the semi-circle $\lambda = (n+1/4)^2\pi^2\exp(2i\theta)/M^2$, $\theta \in [0,\pi/2]$. An elementary calculation shows that

$$|\sin(M\sqrt{\lambda})| = \sqrt{\cosh^2((n+1/4)\pi\sin\theta)-\cos^2((n+1/4)\pi\cos\theta)} \geq \frac{1}{2}\cosh((n+1/4)\pi\sin\theta)$$

for all $\theta \in [0,\pi/2]$, for all sufficiently large n. The other term in the denominator of the right hand side of (53) admits the bound

$$|\lambda|^{-1}\exp(M\Im\sqrt{\lambda}) \leq \frac{2M^2}{(n+1/4)^2\pi^2}\cosh((n+1/4)\pi\sin(\theta)),$$

and is therefore of lower order, for large n. Thus the denominator in (53) is of order $\lambda^{-1/2}\exp(M\Im\sqrt{\lambda})$ on the chosen semi-circles, and gives us the bound we seek. □

Proposition 5. *The functions g_M defined in (49) admit a uniform-in-M bound*

$$\sup_{\Im\lambda\geq 0}|g_M(\lambda)| \leq C. \tag{54}$$

Proof. The Blaschke factors P_M remove the singularities due to the zeros of $m(\lambda)-f(\lambda)$ in the upper half plane. This, together with Lemma 12, shows that the Strong Phragmen-Lindelöf Principle is applicable to the g_M, so we need only check that the required bound holds for $\lambda \in \mathbb{R}$. On the real axis we have $|P_M(\lambda)| = 1$, $\Im f(\lambda) > 0$ and $\Im m_M(\lambda) \leq 0$ so it suffices to have a bound on

$$\frac{|\exp(-2(N-\sigma)\sqrt{-\lambda})|}{\Im(f(\lambda))}.$$

This is immediate from Lemma 11, eqns. (45,46). □

We are now ready to prove Theorem 4.

Proof of Theorem 4. Suppose that the result is false: there exists an open neighbourhood U of $\lambda_{\rm ess}$ in $\mathbb{C}$ which does not contain eigenvalues of L^M for some sequence of values of M which tends to infinity. By re-labelling the operators of this subsequence if necessary, we may assume without loss of generality that U does not contain an eigenvalue of L^M for any M. Let K be any compact subset of U with non-empty interior which contains $\lambda_{\rm ess}$ in its interior. In particular, the interior of K intersects both the upper and lower half-planes in $\mathbb{C}$. Without loss of generality we can choose K to be symmetric across the real line, so that $K = K^*$. Corollary 3 can now be invoked to show that both P_M and $1/P_M$ are uniformly bounded on K. From Proposition 5 it now follows that the functions $1/(m_M(\lambda)-f(\lambda))$, $M \in \mathbb{N}$, are uniformly bounded on K. Being a normal family they have a convergent subsequence whose limit is analytic in the interior of K, and in particular at a point of $\sigma_{\rm ess}(L)$.

We now require a dissipative version of the classical Titchmarsh-Weyl nesting analysis in order to assert that $\lim_{M\to\infty} m_M(\lambda) = m(\lambda)$. In the lower half-plane $\Im\lambda < 0$ the functions m_M have no poles and this result is proved by Sims [20], see also Brown et al. [3]. For $\lambda \in K \bigcap \mathbb{C}^+$, recall the Weyl solutions $\psi(x;\lambda)$ and $\psi_M(x;\lambda)$ used to define $m(\lambda)$ and $m(\lambda)$ in (21,22). Choose X such that for all $x \geq X$, $s(x) < \frac{1}{2}\Im\lambda$. Then applying the Sims nesting analysis on the intervals $[X, M]$, $M \to \infty$, we deduce that

$$\lim_{M\to\infty} \frac{\psi_M'(X;\lambda)}{\psi_M(X;\lambda)} = \frac{\psi'(X;\lambda)}{\psi(X;\lambda)}.$$

Since $m_M(\lambda)$ and $m(\lambda)$ can be written in terms of $\frac{\psi_M'(X;\lambda)}{\psi_M(X;\lambda)}$ and $\frac{\psi'(X;\lambda)}{\psi(X;\lambda)}$ by means of fractional linear transformations similar to (19,20), it follows that provided $\lambda \in K \bigcap \mathbb{C}^+$ is not a pole of m then $m_M(\lambda) \to m(\lambda)$ as $M \to \infty$. We thus have

$$\frac{1}{m_M(\lambda) - f(\lambda)} \to \frac{1}{m(\lambda) - f(\lambda)} \text{ a.e. in } K. \tag{55}$$

However we have already shown that the limit of the quantity on the left of (55) is analytic in K, and we know that f is meromorphic. Thus m is meromorphic in K. However this is impossible as K contains essential spectrum of L. The proof is complete. □

Remark 6. *The hypothesis that $s(x) \to 0$ as $x \to \infty$ can almost certainly be removed by a careful re-examination of the convergence analysis for the Titchmarsh-Weyl coefficients.*

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