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Spectral measures associated to rank two Lie groups and finite subgroups of $GL(2, \mathbb{Z})$

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Abstract

Spectral measures for fundamental representations of the rank two Lie groups $SU(3)$, $Sp(2)$ and G_2 have been studied. Since these groups have rank two, these spectral measures can be defined as measures over their maximal torus \mathbb{T}^2 and are invariant under an action of the corresponding Weyl group, which is a subgroup of $GL(2, \mathbb{Z})$. Here we consider spectral measures invariant under an action of the other finite subgroups of $GL(2, \mathbb{Z})$. These spectral measures are all associated with fundamental representations of other rank two Lie groups, namely $\mathbb{T}^2 = U(1) \times U(1)$, $U(1) \times SU(2)$, $U(2)$, $SU(2) \times SU(2)$, $SO(4)$ and $PSU(3)$.

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1 Introduction

In [10, 11, 12, 13, 14] the authors have studied spectral measures for fundamental representations of the rank two Lie groups $SU(3)$, $Sp(2)$ and G_2 . The spectral measure of an operator is a particular compactly supported probability measure on the spectrum of that operator. A representation graph of G is the fusion graph for an irreducible character of G . By a spectral measure for the Lie group G we will mean the spectral measure of (the adjacency matrix of) such a fusion graph. In particular, the joint spectral measure for the two fundamental representations of the rank two Lie group were studied. The joint spectral measure has support given by the joint spectrum $\sigma(\Delta_1, \Delta_2)$ of the operators given by the adjacency matrices Δ_1, Δ_2 of the representation graphs for the two fundamental representations ρ_1, ρ_2 of G . The pushforward of these joint spectral measures under the projection on the spectrum $\sigma(\Delta_j)$ of a single fundamental representation ρ_j then yields the spectral measure for ρ_j on $\sigma(\Delta_j)$.

Since these groups have rank two, these spectral measures can also be defined as measures over the two-torus \mathbb{T}^2 (which is isomorphic to the maximal torus of the rank two Lie group). The characters of the fundamental representations define a surjection from \mathbb{T}^2 to the joint spectrum $\sigma(\Delta_1, \Delta_2)$, which is invariant under the action of the Weyl group $W(G)$ of G on the torus. The Jacobian for the change of variables from \mathbb{T}^2 to $\sigma(\Delta_1, \Delta_2)$ clearly plays a key role in the spectral measures for G over both \mathbb{T}^2 and the joint spectrum. This approach fits with the spectral measure blowup philosophy of [2].

In conformal field theories built from these Lie groups, one considers the Verlinde algebra at a finite level k , represented by a non-degenerately braided system ${}_N\mathcal{X}_N$ of irreducible endomorphisms on a type III_1 factor N , whose fusion rules $\{\mathcal{N}_{\lambda\nu}^\mu\}$ reproduce exactly those of the positive energy representations of the loop group of the Lie group G at level k , $\mathcal{N}_\lambda\mathcal{N}_\mu = \sum_\nu \mathcal{N}_{\lambda\nu}^\mu\mathcal{N}_\nu$. The statistics generators S, T obtained from the braided tensor category ${}_N\mathcal{X}_N$ match exactly those of the Kač-Peterson modular S, T matrices which perform the conformal character transformations [28] (see also footnote 2 in [4]). The fusion graph for these irreducible endomorphisms are truncated versions of the representation graphs of G itself.

A braided subfactor is an inclusion $N \subset M$ where the dual canonical endomorphism decomposes as a finite combination of elements in ${}_N\mathcal{X}_N$, and yields a modular invariant

partition function through the procedure of α -induction [5, 3, 9]. The action of the N - N sectors ${}_N\mathcal{X}_N$ on the M - N sectors ${}_M\mathcal{X}_N$ and produces a nimrep (non-negative integer matrix representation of the original fusion rules) $\mathcal{G}_\lambda\mathcal{G}_\mu = \sum_\nu \mathcal{N}_{\lambda\nu}^\mu \mathcal{G}_\nu$ whose spectrum reproduces exactly the diagonal part of the modular invariant. In the case of the trivial embedding of N in itself, the nimrep \mathcal{G} is simply \mathcal{N} . The joint spectrum of the nimrep graphs for the fundamental generators of the system ${}_N\mathcal{X}_N$ is again contained in the joint spectrum $\sigma(\Delta_1, \Delta_2)$ of G . One can then determine the (joint) spectral measure for these nimrep graphs over both \mathbb{T}^2 and $\sigma(\Delta_1, \Delta_2)$ – see [10, 11, 12, 14] for more details in the cases of the rank two Lie groups $SU(3)$, $Sp(2)$ and G_2 .

For a rank two Lie group G , the spectrum of the McKay graphs (or representation graphs) of a finite subgroup $H \subset G$ are also contained in the joint spectrum $\sigma(\Delta_1, \Delta_2)$ of G . One can thus also determine the (joint) spectral measure for these graphs over both \mathbb{T}^2 and $\sigma(\Delta_1, \Delta_2)$ [13].

The compact semisimple rank two Lie groups are $SU(2) \times SU(2)$, $SU(3)$, $Sp(2)$ and G_2 (they are in fact simple, apart from $SU(2) \times SU(2)$), and are all connected. Their Weyl groups are the dihedral groups D_4 , $D_6 \equiv S_3$, D_8 and D_{12} respectively, where D_n is the dihedral group of order n , which are all subgroups of $GL(2, \mathbb{Z})$. Spectral measures for $SU(3)$, $Sp(2)$ and G_2 were considered in [10, 11, 12, 13, 14]. Spectral measures for $SU(2) \times SU(2)$ are products of the spectral measures for $SU(2)$, considered in [1, 10].

In this paper we consider spectral measures associated to all the finite subgroups Γ of $GL(2, \mathbb{Z})$ which are not also finite subgroups of $SL(2, \mathbb{Z})$. The spectral measures considered are invariant under an action of the group. We consider two types of representation graphs \mathcal{G}_ρ and \mathcal{H}_ρ , and hence determine spectral measures for two types of operators. The representation graph \mathcal{G}_ρ (with adjacency matrix denoted by Δ_ρ) is given by the fusion rules for characters with respect to multiplication by the character χ_ρ for the representation ρ . The representation graph \mathcal{H}_ρ (with adjacency matrix denoted by ${}^\Gamma\Delta_\rho$) is given by the fusion rules for characters of \mathbb{T}^2 , with respect to multiplication by the restriction of χ_ρ to \mathbb{T}^2 . (Joint) spectral measures for the representation graphs associated with fundamental representations of other rank two Lie groups, namely $\mathbb{T}^2 = U(1) \times U(1)$, $U(1) \times SU(2)$, $U(2)$, $SU(2) \times SU(2)$, $SO(4)$ and $PSU(3)$ are studied.

The paper is organised as follows. In Section 2 we discuss some preliminary material, beginning with the finite subgroups of $GL(2, \mathbb{Z})$. Then in Section 2.3 we discuss orbit functions [19, 20] which have been the objects of much attention in the last decade, see e.g. [22]. For an irreducible representation of a compact semisimple Lie group G with Weyl group $W(G)$, these orbit functions are the contribution to the character from a single orbit under the action of $W(G)$. We present an analogous definition in the case of an orbit under the action on \mathbb{T}^2 of any reflection group which is a finite subgroup Γ of $GL(2, \mathbb{Z})$. The orbit functions are used to define formal characters (which coincide with the characters of G for Γ the Weyl group $W(G)$), which in turn are used in Section 2.4 to define families of representation graphs associated with the finite subgroup Γ . The first family is given by the fusion graphs \mathcal{G} for these formal characters, and the second by the fusion graphs \mathcal{H} for the action of these formal characters on \mathbb{T}^2 . A discussion on the relation between spectral measures over certain different domains is given in Section 2.5.

In Section 3 we discuss the spectral measure for the representation graphs \mathcal{H} , \mathcal{G} over \mathbb{T}^2 (for all finite subgroups of $GL(2, \mathbb{Z})$). Then in Sections 4-9 we determine the (joint) spectral measures for the representation graphs over the (joint) spectrum for each finite

subgroup of $GL(2, \mathbb{Z})$. This will essentially be obtained by determining the Jacobian $J = J_\Gamma$ of a particular change of variable for each group Γ . As one consequence, we prove a conjecture from the Online Encyclopedia of Integer Sequences (OEIS) [24], namely that the number of walks on \mathbb{N}^2 starting and ending at $(0, 0)$ and consisting of $2n$ steps taken from $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$, is given by the squared Catalan numbers A001246 (see Remark 8.5).

2 Preliminaries

2.1 Finite subgroups of $GL(2, \mathbb{Z})$

There are 13 finite subgroups $\Gamma \subset GL(2, \mathbb{Z})$, up to conjugacy in $GL(2, \mathbb{Z})$ [23]:

$$(i) \quad \Gamma \subset SL(2, \mathbb{Z}) : \quad \mathbb{Z}_0, \mathbb{Z}_2^{(1)}, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_6,$$

$$(ii) \quad \Gamma \not\subset SL(2, \mathbb{Z}) : \quad \mathbb{Z}_2^{(2)}, \mathbb{Z}_2^{(3)}, D_4^{(1)}, D_4^{(2)}, D_6^{(1)}, D_6^{(2)}, D_8, D_{12},$$

where $\Gamma^{(i)}, \Gamma^{(j)}$ denote non-conjugate embeddings of Γ in $GL(2, \mathbb{Z})$ for $i \neq j$. The first five subgroups are also finite subgroups of $SL(2, \mathbb{Z})$. We denote by \mathfrak{G} the set of all finite subgroups of $GL(2, \mathbb{Z})$ (up to conjugacy in $GL(2, \mathbb{Z})$) which are not finite subgroups of $SL(2, \mathbb{Z})$, i.e. those listed in (ii). Generators for these groups are given below:

$$\begin{aligned} \mathbb{Z}_0 : I, & \quad \mathbb{Z}_2^{(1)} : -I, & \quad \mathbb{Z}_3 : T_3 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, & \quad \mathbb{Z}_4 : T_4 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\ \mathbb{Z}_6 : T_6 = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, & \quad \mathbb{Z}_2^{(2)} : T_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & \quad \mathbb{Z}_2^{(3)} : T'_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ D_4^{(1)} : -I, T_2, & \quad D_4^{(2)} : -I, T'_2, & \quad D_6^{(1)} : T_3, T'_2, \\ D_6^{(2)} : T_3, -T'_2, & \quad D_8 : T_4, T'_2, & \quad D_{12} : T_6, T'_2. \end{aligned}$$

There is an obvious action of $\Gamma \subset GL(2, \mathbb{Z})$ on \mathbb{R}^2 , that is, $T(m, n) = (a_{11}m + a_{12}n, a_{21}m + a_{22}n)$, for $m, n \in \mathbb{R}$, which drops to the quotient $\mathbb{R}^2/\mathbb{Z}^2 \cong \mathbb{T}^2$.

For a finite subgroup $\Gamma \in \mathfrak{G}$, we denote by P_+ the fundamental domain of the quotient \mathbb{Z}^2/Γ such that $(\lambda_1, \lambda_2) \in P_+$ for all $0 \leq \lambda_2 \leq \lambda_1$, and by P_{++} the set $P_{++} = \{\lambda \in P_+ \mid \gamma\lambda \neq \lambda \text{ for any } \gamma \in \Gamma\}$, i.e. if $\lambda \in P_+ \setminus P_{++}$ then λ lies on the boundary of P_+ .

We have inclusions of these subgroups as illustrated in Figure 2.1. The lines between these subgroups indicate the inclusions of one subgroup in another, where a double line denotes that one is a normal subgroup of the other. Above each subgroup Γ is a diagram illustrating the lines of reflection (the solid lines in each diagram) given by the action of Γ on \mathbb{T}^2 , and the shaded region indicates a fundamental domain of \mathbb{T}^2/Γ . The individual diagrams are given in more detail in Sections 5-9.

We are interested in the finite subgroups of $GL(2, \mathbb{Z})$ which could appear as the Weyl group for a rank two Lie group. The finite subgroups of $SL(2, \mathbb{Z})$ are generated by rotations of \mathbb{Z}^2 , whilst the subgroups in \mathfrak{G} are generated by reflections of \mathbb{Z}^2 . As Weyl groups are finite reflection groups, we will restrict our attention to the groups in \mathfrak{G} , i.e. the finite subgroups of $GL(2, \mathbb{Z})$ that are not subgroups of $SL(2, \mathbb{Z})$. We note however that $\mathbb{Z}_n \in SL(2, \mathbb{Z})$ is a normal subgroup of a dihedral group $D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2 \in GL(2, \mathbb{Z})$.

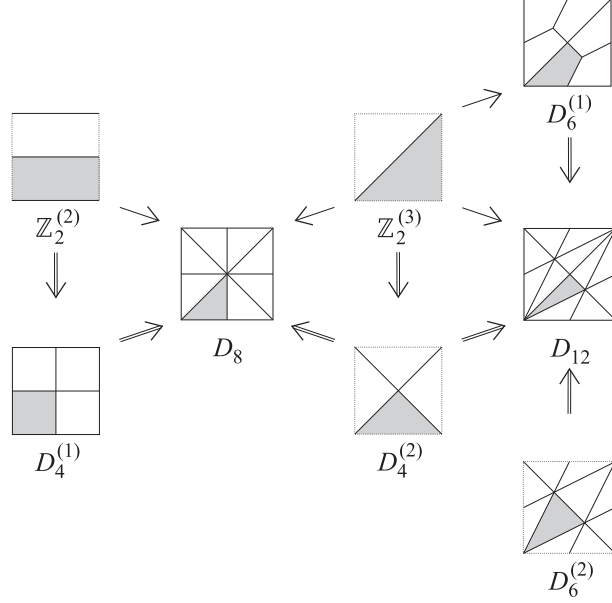


Figure 1: Inclusions of finite subgroups of $GL(2, \mathbb{Z})$

More precisely, $\mathbb{Z}_2^{(1)} \times \mathbb{Z}_2^{(j)} \cong D_4^{(j-1)}$ for $j = 2, 3$, $\mathbb{Z}_3 \times \mathbb{Z}_2^{(3)} \cong D_6^{(1)}$, $\mathbb{Z}_4 \times \mathbb{Z}_2^{(j)} \cong D_8$ for $j = 2, 3$, and $\mathbb{Z}_6 \times \mathbb{Z}_2^{(3)} \cong D_{12}$.

2.2 Rank two Lie groups

The simply-connected compact simple rank two Lie groups are $SU(3)$, $Sp(2)$ (the second order symplectic group, the set of 4×4 unitary matrices U such that $U^T J U = J$, where $J = I_2 \otimes T_4$, for I_2 the 2×2 identity matrix), and G_2 , whilst the only other simply-connected compact semisimple rank two Lie group is $SU(2) \times SU(2)$. Their Weyl groups are the dihedral groups $D_6 \equiv S_3$, D_8 , D_{12} and D_4 respectively, where D_n is the dihedral group of order n , which are all subgroups of $GL(2, \mathbb{Z})$. All compact connected rank two Lie groups are given by quotients of $SU(3)$, $Sp(2)$, G_2 or of products of $SU(2)$, $\mathbb{T} = U(1)$, by finite subgroups of their centers. The centers of $SU(3)$, $Sp(2)$, G_2 , $SU(2)$ are \mathbb{Z}_3 , \mathbb{Z}_2 , \mathbb{Z}_0 , \mathbb{Z}_2 respectively. Thus all connected but non-simply-connected compact rank two Lie groups are given by $\mathbb{T}^2 = U(1) \times U(1)$, $\mathbb{T} \times SU(2) = U(1) \times SU(2)$, the double covered groups $(\mathbb{T} \times SU(2))/\mathbb{Z}_2 = U(2)$, $(SU(2) \times SU(2))/\mathbb{Z}_2 = SO(4)$, $\mathbb{T} \times (SU(2)/\mathbb{Z}_2) = U(1) \times SO(3)$, $SU(2) \times (SU(2)/\mathbb{Z}_2) = SU(2) \times SO(3)$, $Sp(2)/\mathbb{Z}_2 = SO(5)$, the triple covered group $SU(3)/\mathbb{Z}_3 = PSU(3)$, and the quadruple covered group $(SU(2)/\mathbb{Z}_2) \times (SU(2)/\mathbb{Z}_2) = SO(3) \times SO(3)$. These Lie groups have Weyl groups 0 , \mathbb{Z}_2 , \mathbb{Z}_2 , D_4 , \mathbb{Z}_2 , D_4 , D_8 , D_6 and D_4 respectively, which are also all finite subgroups of $GL(2, \mathbb{Z})$. There are two non-conjugate embeddings each of \mathbb{Z}_2 , D_4 and D_6 in $GL(2, \mathbb{Z})$, as described in Section 2.1. By comparing the action of the Weyl group on the maximal torus with the action of $\Gamma \subset GL(2, \mathbb{Z})$ on \mathbb{T}^2 described in Section 2.1, we find that both embeddings in $GL(2, \mathbb{Z})$ of each of \mathbb{Z}_2 , D_4 and D_6 appear, as described in Table 1. Thus the Weyl groups for the connected compact rank two Lie groups listed above in fact exhaust all finite subgroups of $GL(2, \mathbb{Z})$.

Compact Lie group G	Lie algebra	W_G	$\pi_0(G)$	$\pi_1(G)$
$SU(3)$	$A_2 = \mathfrak{su}(3)$	$D_6^{(2)}$	0	0
$Sp(2)$	$C_2 = \mathfrak{sp}(2)$	D_8	0	0
G_2	\mathfrak{g}_2	D_{12}	0	0
$SU(2) \times SU(2)$	$A_1 \times A_1 = \mathfrak{su}(2) \times \mathfrak{su}(2)$	$D_4^{(1)}$	0	0
$\mathbb{T}^2 = U(1) \times U(1)$	\mathfrak{t}^2	0	0	\mathbb{Z}^2
$\mathbb{T} \times SU(2)$	$\mathfrak{t} \times \mathfrak{su}(2)$	$\mathbb{Z}_2^{(2)}$	0	\mathbb{Z}
$U(2)$	$\mathfrak{u}(2) \cong \mathfrak{t} \times \mathfrak{su}(2)$	$\mathbb{Z}_2^{(3)}$	0	\mathbb{Z}
$SO(4)$	$D_2 = \mathfrak{so}(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(2)$	$D_4^{(2)}$	0	\mathbb{Z}_2
$PSU(3)$	$\mathfrak{su}(3)$	$D_6^{(1)}$	0	\mathbb{Z}_3
$\mathbb{T} \times SO(3)$	$\mathfrak{t} \times \mathfrak{su}(2)$	$\mathbb{Z}_2^{(2)}$	0	$\mathbb{Z} \times \mathbb{Z}_2$
$SU(2) \times SO(3)$	$\mathfrak{su}(2) \times \mathfrak{su}(2)$	$D_4^{(1)}$	0	\mathbb{Z}_2
$SO(3) \times SO(3)$	$\mathfrak{su}(2) \times \mathfrak{su}(2)$	$D_4^{(1)}$	0	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$SO(5)$	$B_2 = \mathfrak{so}(5) \cong \mathfrak{sp}(2)$	D_8	0	\mathbb{Z}_2
$\mathbb{T} \times O(2) = \mathbb{T} \times (\mathbb{T} \rtimes \mathbb{Z}_2)$	\mathfrak{t}^2	$\mathbb{Z}_2^{(2)}$	\mathbb{Z}_2	-
$O(2) \times O(2)$	\mathfrak{t}^2	$D_4^{(1)}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	-
$O(2) \times SU(2)$	$\mathfrak{t} \times \mathfrak{su}(2)$	$D_4^{(1)}$	\mathbb{Z}_2	-
$O(2) \times SO(3)$	$\mathfrak{t} \times \mathfrak{su}(2)$	$D_4^{(1)}$	\mathbb{Z}_2	-
$\mathbb{T}^2 \rtimes \mathbb{Z}_2^{(3)}$	\mathfrak{t}^2	$\mathbb{Z}_2^{(3)}$	\mathbb{Z}_2	-
$(SU(2) \times SU(2)) \rtimes \mathbb{Z}_2$	$\mathfrak{su}(2) \times \mathfrak{su}(2)$	D_8	\mathbb{Z}_2	-
$(O(2) \times O(2)) \rtimes \mathbb{Z}_2$	\mathfrak{t}^2	D_8	D_8	-

Table 1: Compact Lie groups G of rank two and their corresponding Lie algebra, Weyl group $W_G \subset GL(2, \mathbb{Z})$, group of components $\pi_0(G)$ and fundamental group $\pi_1(G)$.

The Lie group G is connected if and only if its group of components $\pi_0(G)$ is trivial, and simply-connected if and only if its fundamental group $\pi_1(G)$ is trivial. The compact rank two Lie Groups in Table 1 are grouped into blocks as follows. The first block of four Lie groups are all semi-simple, connected, simply-connected compact Lie groups – these are the only simply-connected compact rank two Lie groups. The next block consists of the products of \mathbb{T} and $SU(2)$ (excluding $SU(2) \times SU(2)$ which was included in the first block). Note that $\mathbb{T} \times SU(2)$ is semisimple, but \mathbb{T}^2 is not usually regarded as semisimple since it is abelian. The groups listed in the third block are all compact connected rank two Lie groups which are (double-, triple-, or quadruple-)covered by groups in the first two blocks. The dashed line separates the Lie groups for which the Weyl group $W_G \subset GL(2, \mathbb{Z})$ do not already appear in the first two blocks, and those for which W_G does appear. The latter Lie groups are not considered in this paper.

In the final block we have listed non-connected compact rank two Lie groups. We do not list Lie groups which are products $G \times H$ where G is a compact, connected rank two Lie group and H is an arbitrary finite group, as the Weyl group in this case is just W_G , the Weyl group for G , and its group of connected components is H . The only semidirect product $\mathbb{T}^2 \rtimes \Gamma$, $\Gamma \subset GL(2, \mathbb{Z})$, which appears in this list is for $\Gamma = \mathbb{Z}_2^{(3)} \subset GL(2, \mathbb{Z})$. This is because all other subgroups $\Gamma \subset GL(2, \mathbb{Z})$ either act trivially by conjugation or

the only non-trivial action of Γ is that given by $\mathbb{Z}_2^{(3)}$ (as is the case for \mathbb{Z}_4 and $D_4^{(2)}$). Thus the semidirect product $\mathbb{T}^2 \rtimes \Gamma$ reduces to a product of \mathbb{T}^2 (or $\mathbb{T}^2 \rtimes \mathbb{Z}_2^{(3)}$) by some $\Gamma' \subset GL(2, \mathbb{Z})$. Finally, for the last two groups listed, the \mathbb{Z}_2 -action interchanges the two components in the product.

In this paper we determine spectral measures associated to the compact connected rank two Lie groups listed in the first three blocks, as these are sufficient to exhaust all finite subgroups of $GL(2, \mathbb{Z})$, up to conjugation in $GL(2, \mathbb{Z})$. For our purposes it is sufficient only to know the embedding of the Weyl group as a subgroup of $GL(2, \mathbb{Z})$. The irreducible characters and corresponding representation graphs will be constructed in the proceeding sections from knowledge of $\Gamma \subset GL(2, \mathbb{Z})$. For one of the groups G contained in the fourth block in Table 1, one could obtain the spectral measures in this case by considering its corresponding covering group H in the first two blocks, and determining the spectral measures for representation graphs corresponding to those irreducible representations of H which are fundamental generators (see Section 2.4) of G . Such spectral measures will not be determined in this paper.

The framework used here, which is described in Sections 2.3-2.5, cannot be used for subgroups $\Gamma \subset SL(2, \mathbb{Z})$, since a fundamental domain P_+ of \mathbb{Z}^2/Γ is not uniquely defined. It is still possible to associate a graph \mathcal{G} , and hence a spectral measure, to Γ . One way of doing this is described in Remark 7.2 for the case of $\mathbb{Z}_2^{(1)}$. However the joint spectrum \mathfrak{D} of the graph \mathcal{G} is equal to the joint spectrum of $D_4^{(1)} \cong \mathbb{Z}_2^{(1)} \rtimes \mathbb{Z}_2^{(2)}$, and the spectral measure over \mathfrak{D} is twice that for $D_4^{(1)}$. Similar statements can be made for the other subgroups $\Gamma \subset SL(2, \mathbb{Z})$, with the spectral measure over the joint spectrum \mathfrak{D} for Γ being twice that for the corresponding dihedral group $D \cong \Gamma \rtimes \mathbb{Z}_2$. Thus from this perspective the finite subgroups of $SL(2, \mathbb{Z})$ do not give anything new compared with the subgroups in \mathfrak{G} .

2.3 Orbit Functions and characters

Symmetric, anti-symmetric orbit functions C_λ, S_λ respectively are defined for a compact, semisimple Lie group G and are closely related to the Weyl group W_G of G [19, 20] (see also [22]). They are also called C -, S -functions respectively, since when defined for $SU(2)$ these functions coincide with cosine, sine functions respectively. When G is a Lie group of rank n , then C_λ, S_λ are functions of n variables which are the set of distinct points in \mathbb{R}^n generated by the action of W_G on λ . The n -tuples λ are usually taken to be in the set $P_+ = \{\sum_{i=1}^n \lambda_i \Lambda_i \mid 0 \leq \lambda_i \in \mathbb{Z}\}$, where Λ_i are the fundamental weights of G . However, the definition extends to $\lambda \in P = \{\sum_{i=1}^n \lambda_i \Lambda_i \mid \lambda_i \in \mathbb{Z}\} = \mathbb{Z}^n$.

We will define C -, S -functions for any finite subgroup Γ of $GL(2, \mathbb{Z})$, by replacing the Weyl group W_G with Γ . As discussed in Section 2.2, such a Γ is in fact the Weyl group for a compact, connected rank two Lie group G , although the existence of such a Lie group is not necessary for the definition. Such a Lie group G will not necessarily be semisimple, and thus extends the definitions of [19, 20] of orbit functions to non-semisimple compact, connected Lie groups. For $\lambda \in \mathbb{Z}^n$ and $\theta \in \mathbb{R}^n$,

$$C_\lambda(\theta) := \sum_{\gamma \in \Gamma} e^{2\pi i \langle \gamma \lambda, \theta \rangle}, \quad S_\lambda(\theta) := \sum_{\gamma \in \Gamma} \det(\gamma) e^{2\pi i \langle \gamma \lambda, \theta \rangle}, \quad (1)$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean inner product on \mathbb{R}^n . For $\Gamma \subset SL(2, \mathbb{Z})$, the definitions of C_λ and S_λ coincide, since $\det(\gamma) = 1$ for all $\gamma \in \Gamma$.

In the case where $\Gamma = W_G$ is the Weyl group for some compact, semisimple Lie group G , and $\lambda \in P_+$, the definition of C_λ given above is not the usual definition of C -function used by Patera et al., which is $\tilde{C}_\lambda(x) = |\text{Stab}_\lambda|^{-1} \sum_{w \in W_G} e^{2\pi i \langle w\lambda, x \rangle}$. Here $\text{Stab}_\lambda = \{w \in W_G | w\lambda = \lambda\}$ is the stabilizer group of λ . Note however that for $\lambda \in P_{++} = \{\lambda = \sum_{i=1}^n \lambda_i \Lambda_i | 0 < \lambda_i \in \mathbb{Z}\}$, i.e. for λ in the interior of P_+ , $|\text{Stab}_\lambda| = 1$ so that in that case $\tilde{C}_\lambda = C_\lambda$.

For any $\Gamma \subset GL(2, \mathbb{Z})$ these orbit functions are orthogonal over \mathbb{T}^n (c.f. [21] in the case where Γ is the Weyl group for some compact, semisimple Lie group):

$$\int_{\mathbb{T}^n} C_\lambda(\theta) \overline{C_\mu(\theta)} d\theta = \delta_{\lambda, \mu} |\Gamma| = \int_{\mathbb{T}^n} S_\lambda(\theta) \overline{S_\mu(\theta)} d\theta, \quad (2)$$

for $\lambda, \mu \in P_+$, where $\theta = (\theta_1, \theta_2, \dots, \theta_n)$, $e^{2\pi i \theta} \in \mathbb{T}^n$ and $d\theta = d\theta_1 \cdots d\theta_n$ for $d\theta_i$ the uniform Lebesgue measure over \mathbb{T} . These equalities follow from the orthogonality $\int_{\mathbb{T}} u^m du = \delta_{m,0}$ of \mathbb{T} , since

$$\int_{\mathbb{T}^n} C_\lambda(\theta) \overline{C_\mu(\theta)} d\theta = \sum_{\gamma, \gamma' \in \Gamma} \int_{\mathbb{T}^n} e^{2\pi i \langle \gamma\lambda - \gamma'\mu, \theta \rangle} d\theta = \sum_{\gamma, \gamma' \in \Gamma} \delta_{\gamma\lambda, \gamma'\mu} = |\Gamma| \delta_{\lambda, \mu},$$

and

$$\begin{aligned} \int_{\mathbb{T}^n} S_\lambda(\theta) \overline{S_\mu(\theta)} d\theta &= \sum_{\gamma \in \Gamma} \det(\gamma) \sum_{\gamma' \in \Gamma} \det(\gamma') \int_{\mathbb{T}^n} e^{2\pi i \langle \gamma\lambda - \gamma'\mu, \theta \rangle} d\theta \\ &= \sum_{\gamma \in \Gamma} \det(\gamma) \sum_{\gamma' \in \Gamma} \det(\gamma') \delta_{\gamma\lambda, \gamma'\mu} = |\Gamma| \delta_{\lambda, \mu}, \end{aligned}$$

where we also use the fact that for $\lambda, \mu \in P_+$, $\gamma\lambda = \gamma'\mu \Leftrightarrow \lambda = \mu$ and $\gamma = \gamma'$.

For $\Gamma \in \mathfrak{G}$, we have $S_\lambda = 0$ for $\lambda \in P_+ \setminus P_{++}$, i.e. for λ lying on the boundary of P_+ , since for every element γ with $\det(\gamma) = 1$ (i.e. γ is a rotation), there exists an element γ' with $\det(\gamma') = -1$ (i.e. γ' is a reflection) such that $\gamma'\gamma\lambda = \gamma\lambda$. Thus in the summation for S_λ the terms for $\gamma'\gamma$ and γ have opposite signs and thus cancel.

When $\Gamma \neq \mathbb{Z}_2^{(3)}$, we denote by ϱ the point in P_{++} such that $\langle \varrho, \varrho \rangle \leq \langle \lambda, \lambda \rangle$ for all $\lambda \in P_{++}$. For $\mathbb{Z}_2^{(3)}$ there are two points $(1, 0)$, $(0, -1)$ in P_{++} which satisfy this condition, and we take ϱ to be $(1, 0)$. Then we define a formal character χ_λ for $\lambda \in P_+$ by

$$\chi_\lambda(\omega_1, \omega_2) = S_{\lambda + \varrho}(\theta_1, \theta_2) / S_\varrho(\theta_1, \theta_2), \quad (3)$$

where $\omega_j = e^{2\pi i \theta_j} \in \mathbb{T}$, $j = 1, 2$. Note that χ_λ is non-zero only for $\lambda \in P_{++}$, since for $\lambda' \in P_+ \setminus P_{++}$, $\chi_{\lambda'} = 0$ (since $S_{\lambda'} = 0$).

We observe that since $\Gamma = W_G$ is the Weyl group for a compact, connected Lie group G listed in the first three blocks of Table 1, equation (3) is just the Weyl character formula for G . In the case of the other connected Lie groups G listed in Table 1, since these groups are each covered by a group H from the first two blocks, (3) still yields the character for irreducible representations of the Lie group, but only for the subset of P_+ corresponding to those irreducible representations of H which are irreducible representations of G .

For non-connected Lie groups, which are all semi-direct products $G = N \rtimes \mathbb{Z}_2$ where N is one of the groups listed in the first two blocks of Table 1 and is normal in G , the irreducible representations λ and $\nu(\lambda)$ (for ν the non-trivial element of \mathbb{Z}_2) are either isomorphic or not. In the case where they are, one obtains two irreducible representation $(\lambda, 0)$, $(\lambda, 1)$ of G whose characters on \mathbb{T}^2 are both given by $\chi_{(\lambda,0)}(t) = \chi_{(\lambda,1)}(t) = \chi_\lambda(t)$ for $t \in \mathbb{T}^2$. In the case where they are not isomorphic, one obtains a single irreducible representation λ' of G whose character on \mathbb{T}^2 is given by $\chi_{\lambda'}(t) = \chi_\lambda(t) + \chi_{\nu(\lambda)}(t)$. Thus equation (3) is the Weyl character formula in the case where $\lambda \cong \nu(\lambda)$, whilst for $\lambda \not\cong \nu(\lambda)$, the Weyl character formula reads $\chi_{\lambda'}(\omega_1, \omega_2) = (S_{\lambda+\varrho}(\theta_1, \theta_2) + S_{\nu(\lambda+\varrho)}(\theta_1, \theta_2))/S_\varrho(\theta_1, \theta_2)$.

It was noted in [12, §5] in the context of the Lie group G_2 that there is a connection between the orbit function $S_{\lambda+\varrho}(x)$ and the modular S -matrix for the conformal field theory associated to G_2 at finite level k . More precisely, if we let $x = ((\mu_1 + 1)/3(k + 4), -(\mu_2 + 1)/(k + 4))$, then up to a common scalar multiple, $S_{\lambda+\varrho}(x) = S_{\lambda,\mu}$. Here we are instead using Dynkin labels for $\lambda, \mu \in P_+$. Similarly, if we let $x = ((\mu_1 + 1)/2(k + 2), (\mu_2 + 1)/2(k + 2))$, $((2\mu_1 + \mu_2 + 3)/3(k + 3), (\mu_1 + 2\mu_2 + 3)/3(k + 4))$, $((\mu_1 + \mu_2 + 2)/2(k + 3), (\mu_1 + 2\mu_2 + 3)/2(k + 3))$ respectively, then up to a common scalar multiple, $S_{\lambda+\varrho}(x) = S_{\lambda,\mu}$ for $SU(2) \times SU(2)$, $SU(3)$, $Sp(2)$ respectively, at finite level k , again using Dynkin labels for $\lambda, \mu \in P_+$. Thus for all the semi-simple, connected, simply-connected compact rank two Lie groups the modular S -matrix is given by the orbit S -function, up to some common scalar multiple which ensures that the S -matrix has norm 1.

2.4 Representation graphs

We construct two families of graphs for each subgroup $\Gamma \subset \mathfrak{G}$. One family is the McKay graphs \mathcal{G}^Γ for the irreducible representations of a compact, connected Lie group G with Weyl group Γ , the other is the McKay graphs \mathcal{H}^Γ for the action of the irreducible representations of G on the irreducible representations of the torus \mathbb{T}^2 .

For any $\lambda, \mu \in P_+$, $\chi_\lambda \chi_\mu$ decomposes into a finite sum of characters χ_ν for $\nu \in P_+$ since these are characters of a compact, connected Lie group G . More explicitly,

$$\chi_\lambda \chi_\mu = \sum_{\nu} m_{\mu\nu}^\lambda \chi_\nu, \quad (4)$$

where $m_{\mu\nu}^\lambda \in \mathbb{N}$ such that $\sum_{\nu} m_{\mu\nu}^\lambda < \infty$ for all $\lambda, \mu \in P_+$.

Thus we may form the (infinite) graphs $\mathcal{G}_\lambda^\Gamma$, the McKay graphs for the irreducible representations λ of the compact, connected Lie group G , whose vertices correspond to the characters and edges correspond to multiplication by χ_λ . The normal matrices $\Delta_\lambda = (m_{\mu\nu}^\lambda)_{\mu,\nu}$, the adjacency matrices of the graphs $\mathcal{G}_\lambda^\Gamma$, commute since the characters χ_λ do. In the case where $\chi_\lambda(x)$ is real-valued for all $x \in \mathbb{T}^2$, the matrix Δ_λ is self-adjoint.

Now let $\{\sigma_{(\mu_1, \mu_2)}\}_{\mu_1, \mu_2 \in \mathbb{Z}}$ be the irreducible characters of \mathbb{T}^2 , where $\sigma_{(\mu_1, \mu_2)}(t_1, t_2) = t_1^{\mu_1} t_2^{\mu_2}$, for $t_i \in \mathbb{T}$, $\mu_1, \mu_2 \in \mathbb{Z}$. The characters χ_λ of G decompose as a finite sum of characters σ_ν of \mathbb{T}^2 for $\nu \in \mathbb{Z}^2$, and hence $\chi_\lambda \sigma_\mu$ decomposes into a finite sum of characters σ_ν , $\nu \in \mathbb{Z}^2$,

$$\chi_\lambda \sigma_\mu = \sum_{\nu} n_{\mu\nu}^\lambda \sigma_\nu, \quad (5)$$

where $n_{\mu\nu}^\lambda \in \mathbb{N}$ such that $\sum_{\nu} n_{\mu\nu}^\lambda < \infty$ for all $\lambda, \mu \in \mathbb{Z}^2$. By considering the action of χ_λ on the σ_μ we obtain a second pair of (infinite) graphs, $\mathcal{H}_\lambda^\Gamma$, whose vertices correspond to the

irreducible characters of \mathbb{T}^2 , and edges correspond to multiplication by χ_λ . We will label the vertex corresponding to σ_ν by $\nu \in \mathbb{Z}^2$. Again, the normal matrices ${}^\Gamma\Delta_\lambda = (n_{\mu\nu}^\lambda)_{\mu,\nu}$, the adjacency matrices of the graphs $\mathcal{H}_\lambda^\Gamma$, commute, and in the case where $\chi_\lambda(x)$ is real-valued for all $x \in \mathbb{T}^2$, the matrix ${}^\Gamma\Delta_\lambda$ is self-adjoint.

We will specify a pair of points $\rho_1, \rho_2 \in P_+$ which we will call fundamental generators, since their characters are generators in the sense that χ_λ for any other $\lambda \in P_+$ appears in the decomposition of the product of some powers of χ_{ρ_1} and χ_{ρ_2} . In fact, $\{\chi_{\rho_1}, \chi_{\rho_2}\}$ is a system of generators for the algebra of characters in all cases, except for the groups $\mathbb{Z}_2^{(3)}$ and $D_6^{(1)}$. For $\mathbb{Z}_2^{(3)}$ one needs to also use the additional information that $\chi_{\bar{\lambda}}(\omega_1, \omega_2) = \chi_\lambda(\bar{\omega}_1, \bar{\omega}_2)$ ($= \chi_\lambda(\omega_1, \bar{\omega}_2)$), where $\bar{\lambda} = (\lambda_1, -\lambda_2)$ for $\lambda = (\lambda_1, \lambda_2)$, which corresponds to the automorphism of the graph $\mathcal{G}_\mu^{\mathbb{Z}_2^{(3)}}$ given by reflecting the graph about the line $y = -x$ and reversing all orientations (see Figure 6 for the cases where $\mu = \rho_1, \rho_2$ respectively). For $D_6^{(1)}$ one needs to also use the additional information that $\chi_{\bar{\lambda}}(\omega_1, \omega_2) = \chi_\lambda(\bar{\omega}_1, \bar{\omega}_2)$, where $\bar{\lambda} = (\lambda_1 + \lambda_2, -\lambda_2)$, which corresponds to the automorphism of the graph $\mathcal{G}_\mu^{D_6^{(1)}}$ given by reflecting the graph about the x -axis and reversing all orientations (see Figure 22 for the cases where $\mu = \rho_1, \rho_2$ respectively).

We will study (joint) spectral measures for the pair of graphs $(\mathcal{G}_{\rho_1}^\Gamma, \mathcal{G}_{\rho_2}^\Gamma)$, and similarly for the pair of graphs $(\mathcal{H}_{\rho_1}^\Gamma, \mathcal{H}_{\rho_2}^\Gamma)$. Equation (4) can be interpreted as meaning that the matrix Δ_{ρ_i} has eigenvector $(\chi_\nu(\theta))_\nu$ for eigenvalue $\chi_{\rho_i}(\theta)$, $\theta \in [0, 2\pi]^2$. Thus the spectrum of Δ_{ρ_i} is given by $\chi_{\rho_i}(\mathbb{T}^2)$. Similarly, from equation (5) we see that the spectrum of ${}^\Gamma\Delta_{\rho_i}$ is also given by $\chi_{\rho_i}(\mathbb{T}^2)$.

2.5 Spectral measures over different domains

Suppose A is a unital C^* -algebra with state φ . If $b \in A$ is a normal operator then there exists a compactly supported probability measure ν_b on the spectrum $\sigma(b) \subset \mathbb{C}$ of b , uniquely determined by its moments

$$\varphi(b^m b^{*n}) = \int_{\sigma(b)} z^m \bar{z}^n d\nu_b(z), \quad (6)$$

for non-negative integers m, n . If a is self-adjoint (6) reduces to

$$\varphi(a^m) = \int_{\sigma(a)} x^m d\nu_a(x), \quad (7)$$

with $\sigma(a) \subset \mathbb{R}$, for any non-negative integer m .

One can also consider more general measures over the joint spectrum $\sigma(a, b) \subset \sigma(a) \times \sigma(b) \subset \mathbb{C}^2$ of commuting normal operators a and b . The abelian C^* -algebra B generated by a, b and the identity 1 is isomorphic to $C(X)$, where X is the spectrum of B . The joint spectrum is defined as $\sigma(a, b) = \{(a(x), b(x)) \mid x \in X\}$. In fact, one can identify the spectrum X with its image $\sigma(a, b)$ in \mathbb{C}^2 , since the map $x \mapsto (a(x), b(x))$ is continuous and injective, and hence a homeomorphism since X is compact [26]. In the case where the operators a, b act on a finite-dimensional Hilbert space, this is the set of all pairs of real numbers (λ_a, λ_b) for which there exists a non-zero vector ϕ such that $a\phi = \lambda_a\phi$, $b\phi = \lambda_b\phi$. Then there exists a compactly supported probability measure $\tilde{\nu}_{a,b}$ on $\sigma(a, b)$,

the joint spectral measure of a, b , which, for $a \neq b$, is uniquely determined by its cross moments

$$\varphi(a^{m_1} a^{*n_1} b^{m_2} b^{*n_2}) = \int_{\sigma(a,b)} w^{m_1} \overline{w}^{n_1} z^{m_2} \overline{z}^{n_2} d\tilde{\nu}_{a,b}(w, z), \quad (8)$$

for all non-negative integers m_i, n_i . In the case where a or b is self-adjoint, it is sufficient to consider the cross moments with $n_1 = 0$ or $n_2 = 0$ respectively in (8) to determine the measure $\tilde{\nu}_{a,b}$. For a, b both self-adjoint, the spectral measure for a is given by the pushforward $(p_a)_*(\tilde{\nu}_{a,b})$ of the joint spectral measure $\tilde{\nu}_{a,b}$ under the orthogonal projection p_a onto the spectrum $\sigma(a)$. In particular, ν_a, ν_b can be determined by additionally setting $m_2 = n_2 = 0, m_1 = n_1 = 0$ respectively.

Let $x_\lambda = \chi_\lambda(\omega_1, \omega_2)$ and let $\Psi_{\lambda,\mu}$ be the map $(\omega_1, \omega_2) \mapsto (x_\lambda, x_\mu)$. We denote by $\mathfrak{D}_{\lambda,\mu}$ the image of $\Psi_{\lambda,\mu}(\mathbb{T}^2)$. Note that $\mathfrak{D}_{\lambda,\mu} \cong \mathfrak{D}_{\mu,\lambda}$. Then any Γ -invariant measure $\varepsilon_{\lambda,\mu}$ on \mathbb{T}^2 produces a probability measure $\tilde{\nu}_{\lambda,\mu}$ on \mathfrak{D} by

$$\int_{\mathfrak{D}_{\lambda,\mu}} \psi(x_\lambda, x_\mu) d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu) = \int_{\mathbb{T}^2} \psi(\chi_\lambda(\omega_1, \omega_2), \chi_\mu(\omega_1, \omega_2)) d\varepsilon_{\lambda,\mu}(\omega_1, \omega_2), \quad (9)$$

for any continuous function $\psi : \mathfrak{D}_{\lambda,\mu} \rightarrow \mathbb{C}$. Any such measure is uniquely determined by its cross moments $s_{m_1, n_1, m_2, n_2} = \int_{\mathfrak{D}_{\lambda,\mu}} x_\lambda^{m_1} \overline{x_\lambda}^{n_1} x_\mu^{m_2} \overline{x_\mu}^{n_2} d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu)$. Since the χ_λ are invariant under the action of Γ on \mathbb{T}^2 , $\mathfrak{D}_{\lambda,\mu}$ is isomorphic to a quotient of \mathbb{T}^2 by Γ . We denote by C a fundamental domain of \mathbb{T}^2 under the action of Γ . The torus then contains $|\Gamma|$ copies of C , so that

$$\int_{\mathbb{T}^2} \phi(\omega_1, \omega_2) d\varepsilon_{\lambda,\mu}(\omega_1, \omega_2) = |\Gamma| \int_C \phi(\omega_1, \omega_2) d\varepsilon_{\lambda,\mu}(\omega_1, \omega_2), \quad (10)$$

for any Γ -invariant function $\phi : \mathbb{T}^2 \rightarrow \mathbb{C}$.

As discussed above any probability measure on $\mathfrak{D}_{\lambda,\mu}$ yields a probability measure on σ_λ , given by the pushforward $(p_\lambda)_*(\tilde{\nu}_{\lambda,\mu})$ of the joint spectral measure $\tilde{\nu}_{\lambda,\mu}$ under the orthogonal projection p_λ onto the spectrum $\sigma(\lambda)$. In particular, when $\psi(x_\lambda, x_\mu) = \tilde{\psi}(x_\lambda)$ is only a function of one variable x_λ , then

$$\int_{\mathfrak{D}_{\lambda,\mu}} \tilde{\psi}(x_\lambda) d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu) = \int_{\sigma_\lambda} \tilde{\psi}(x_\lambda) \int_{\mathfrak{D}_{\lambda,\mu}(x_\lambda)} d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu) = \int_{\sigma_\lambda} \tilde{\psi}(x_\lambda) d\nu_\lambda(x_\lambda)$$

where the measure $d\nu_\lambda(x_\lambda) = \int_{x_\mu \in \mathfrak{D}_{\lambda,\mu}(x_\lambda)} d\tilde{\nu}_{\lambda,\mu}(x_\lambda, x_\mu)$ is given by the integral over $x_\mu \in \mathfrak{D}_{\lambda,\mu}(x_\lambda) = \{x_\mu \in \sigma_\mu \mid (x_\lambda, x_\mu) \in \mathfrak{D}_{\lambda,\mu}\}$. Note that $\mathfrak{D}_{\lambda,\bar{\lambda}} = \sigma_\lambda$, thus for non-self-adjoint λ the joint spectral measure of $\lambda, \bar{\lambda}$ (over $\mathfrak{D}_{\lambda,\bar{\lambda}}$) is in fact the spectral measure of both λ and $\bar{\lambda}$.

3 Joint spectral measures for rank two Lie groups

3.1 Joint spectral measure for ${}^\Gamma\Delta_{\rho_i}$

The adjacency matrices ${}^\Gamma\Delta_{\rho_i}$ can be identified with operators on $\ell^2(\mathbb{Z}) \otimes \ell^2(\mathbb{Z})$, where if χ_{ρ_i} decomposes into irreducible characters of \mathbb{T}^2 as $\chi_{\rho_i} = \sum_\nu p_\nu^i \sigma_\nu$, then ${}^\Gamma\Delta_{\rho_i} = \sum_\nu p_\nu^i s^{\nu_1} \otimes s^{\nu_2}$,

where $\nu = (\nu_1, \nu_2)$, s is the bilateral shift on $\ell^2(\mathbb{Z})$. For $\Omega = (\delta_{j,0})_{j \in \mathbb{Z}}$, we regard $\Omega \otimes \Omega$ as corresponding to the vertex $(0, 0)$ whilst $(s^{\mu_1} \otimes s^{\mu_2})(\Omega \otimes \Omega)$ corresponds to the vertex (μ_1, μ_2) of $\mathcal{H}_{\rho_i}^\Gamma$. We define a state φ on $C^*(v_Z^1, v_Z^2)$ by $\varphi(\cdot) = \langle \cdot, (\Omega \otimes \Omega), \Omega \otimes \Omega \rangle$. Then $\varphi(s^{\lambda_1} \otimes s^{\lambda_2}) = \langle (s^{\lambda_1} \otimes s^{\lambda_2})(\Omega \otimes \Omega), \Omega \otimes \Omega \rangle = \delta_{\lambda_1, 0} \delta_{\lambda_2, 0}$.

Then we have the following result for the joint spectral measure over \mathbb{T}^2 of $(\Gamma\Delta_{\rho_1}, \Gamma\Delta_{\rho_2})$:

Theorem 3.1. The joint spectral measure ε (over \mathbb{T}^2) for the pair of graphs $(\mathcal{H}_{\rho_1}^\Gamma, \mathcal{H}_{\rho_2}^\Gamma)$ is given by the uniform Lebesgue measure $d\varepsilon(\omega_1, \omega_2) = d\omega_1 d\omega_2$.

Proof: The result follows from the fact that $\int_{\mathbb{T}} \omega^\lambda d\omega := \int_0^1 e^{2\pi i \theta \lambda} d\theta = \delta_{\lambda, 0}$, and thus

$$\begin{aligned} \int_{\mathbb{T}^2} \chi_{\rho_1}(\omega_1, \omega_2)^{m_1} \overline{\chi_{\rho_1}(\omega_1, \omega_2)^{n_1}} \chi_{\rho_2}(\omega_1, \omega_2)^{m_2} \overline{\chi_{\rho_2}(\omega_1, \omega_2)^{n_2}} d\omega_1 d\omega_2 \\ = \varphi((\Gamma\Delta_{\rho_1})^{m_1} (\Gamma\Delta_{\rho_1}^T)^{n_1} (\Gamma\Delta_{\rho_2})^{m_2} (\Gamma\Delta_{\rho_2}^T)^{n_2}). \end{aligned}$$

See e.g. [10, Theorem 2] and [12, Theorem 3.1] for explicit details in the cases of $SU(3)$ and G_2 respectively. \square

In fact, by a similar proof the measure ε given in Theorem 3.1 is the joint spectral measure over \mathbb{T}^2 for the pair of representation graphs $(\mathcal{H}_\lambda^\Gamma, \mathcal{H}_\mu^\Gamma)$ for any pair $\lambda, \mu \in P_+$ (note that λ, μ are irreducible representations of a connected Lie group G from the first three blocks of Table 1 such that the Weyl group of G is Γ). Thus the spectral measure over \mathbb{T}^2 is independent of the choice of $\lambda, \mu \in P_+$.

We now consider the joint spectral measure $\tilde{\nu}$ over $\mathfrak{D} := \mathfrak{D}_{\rho_1, \rho_2}$, the joint spectrum of the commuting normal operators $\Gamma\Delta_{\rho_1}, \Gamma\Delta_{\rho_2}$. We denote by J_Γ be the Jacobian $J_\Gamma = \det(\partial(x, y)/\partial(\theta_1, \theta_2))$ for the change of variables $x := x_{\rho_1}$, $y := x_{\rho_2}$. Over \mathfrak{D} , we thus obtain

$$\int_C \psi(\chi_{\rho_1}(\omega_1, \omega_2), \chi_{\rho_2}(\omega_1, \omega_2)) d\omega_1 d\omega_2 = \int_{\mathfrak{D}} \psi(x, y) |J_\Gamma(x, y)|^{-1} dx dy, \quad (11)$$

Then from Theorem 3.1 and (10) we obtain

Theorem 3.2. The joint spectral measure $\tilde{\nu}$ (over \mathfrak{D}) for the pair of graphs $(\mathcal{H}_{\rho_1}^\Gamma, \mathcal{H}_{\rho_2}^\Gamma)$ is

$$d\tilde{\nu}(x, y) = \frac{|\Gamma|}{|J_\Gamma(x, y)|} dx dy.$$

We determine the Jacobian $J = J_\Gamma$ for each group Γ in the following sections.

3.2 Joint spectral measure for Δ_{ρ_i}

The joint spectral measure for Δ_{ρ_i} (over \mathbb{T}^2) for $SU(3)$, $Sp(2)$, G_2 was shown to be given by $d\varepsilon(\omega_1, \omega_2) = a_\Gamma |J_\Gamma(\theta_1, \theta_2)|^2 / 16\pi^4 d\omega_1 d\omega_2$, where $d\omega$ is the uniform Lebesgue measure over \mathbb{T} and $a_\Gamma = 1$ for $\Gamma = D_8, D_{12}$ (the Weyl groups of $Sp(2), G_2$ respectively), and $a_{D_6^{(2)}} = 2$ for $SU(3)$ [10, 11, 12, 13, 14]. Over \mathfrak{D} , the joint spectral measure is thus given by $d\tilde{\nu}(x, y) = a_\Gamma |J_\Gamma(x, y)| / 16\pi^4 dx dy$. The same results hold for $SU(2) \times SU(2)$, where $a_\Gamma = 1$ [1, 10]. It can be shown from Sections 4-6 that the same results hold for the rank two Lie groups $\mathbb{T}^2 = U(1) \times U(1)$, $\mathbb{T} \times SU(2)$ and $U(2)$, with $a_\Gamma = 1$ in each case. However, for $SO(4)$, considered in Section 8, the joint spectral measure for Δ_{ρ_i} (over \mathfrak{D}) is given by $(1 + y)^{-2} |J_\Gamma(x, y)| / 16\pi^2 dx dy$.

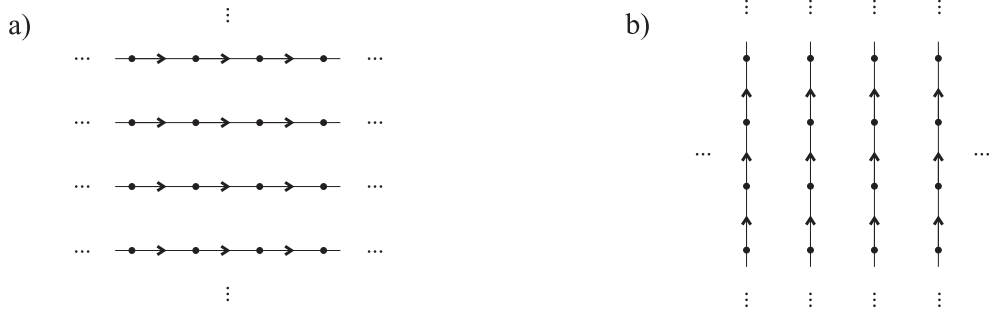


Figure 2: a) Infinite graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_0}$ for \mathbb{T} ; b) Infinite graph $\mathcal{G}_{\rho_2}^{\mathbb{Z}_0}$ for \mathbb{T}

This leads to the question of why there is a difference for $SO(4)$ (and indeed for $SU(3)$ where $a_\Gamma = 2$) and whether there is a consistent description for the spectral measure for Δ_{ρ_i} (over \mathbb{T}^2 or \mathfrak{D}) for any rank 2 Lie groups in terms of some object which is naturally associated to the group. It turns out that computing the orbit function $S_\varrho(\theta)$ for each group we obtain that $4\pi^2|S_\varrho(\theta)| = a_\Gamma|J_\Gamma(\theta_1, \theta_2)|$ for all the Lie groups G discussed in the preceding paragraph, except for $SO(4)$ where we have $4\pi^2|S_\varrho(\theta)| = (1+y)^{-1}|J_\Gamma(\theta_1, \theta_2)|$. Thus, for all the rank two Lie groups discussed in the preceding paragraph, the spectral measure for Δ_{ρ_i} over \mathbb{T}^2 is given by $d\varepsilon(\omega_1, \omega_2) = |\Gamma|^{-1}|S_\varrho(\theta)|^2 d\omega_1 d\omega_2$. The spectral measure over \mathfrak{D} is $d\tilde{\nu}(x, y) = |S_\varrho(\theta)|^2 |J_\Gamma(x, y)|^{-1} dx dy$, where we now write $S_\varrho(\theta)$ in terms of the Γ -invariant variables x, y . This leads naturally to the following conjecture:

Conjecture 3.3. The joint spectral measure ε (over \mathbb{T}^2) for the pair of graphs $(\mathcal{G}_{\rho_1}^\Gamma, \mathcal{G}_{\rho_2}^\Gamma)$ is

$$d\varepsilon(\omega_1, \omega_2) = \frac{1}{|\Gamma|} |S_\varrho(\theta)|^2 d\omega_1 d\omega_2.$$

The corresponding joint spectral measure $\tilde{\nu}$ (over \mathfrak{D}) for the pair of graphs $(\mathcal{G}_{\rho_1}^\Gamma, \mathcal{G}_{\rho_2}^\Gamma)$ is thus

$$d\tilde{\nu}(x, y) = \frac{|S_\varrho(\theta)|^2}{|J_\Gamma(x, y)|} dx dy.$$

4 \mathbb{Z}_0 : $\mathbb{T}^2 = U(1) \times U(1)$

The trivial subgroup $\mathbb{Z}_0 \in GL(2, \mathbb{Z})$ is generated by the identity matrix. We choose fundamental generators $\rho_1 = (1, 0)$, $\rho_2 = (0, 1)$. Since the action of \mathbb{Z}_0 on \mathbb{T}^2 is trivial, the graphs $\mathcal{G}_\rho^{\mathbb{Z}_0}$ and $\mathcal{H}_\rho^{\mathbb{Z}_0}$ are the same. The graphs $\mathcal{G}_\rho^{\mathbb{Z}_0}$ for $\rho = \rho_1, \rho_2$ are illustrated in Figure 2. Let $x := x_{\rho_1} = \omega_1$, $y := x_{\rho_2} = \omega_2$, and denote by Ψ be the map $\Psi_{\rho_1, \rho_2} : (\omega_1, \omega_2) \mapsto (x, y)$. Then $\mathfrak{D} := \mathfrak{D}_{\rho_1, \rho_2} = \mathbb{T}^2$ is the joint spectrum $\sigma(\Delta_{\rho_1}, \Delta_{\rho_2})$ of the commuting normal operators $\Delta_{\rho_1}, \Delta_{\rho_2}$. Under the identification $x = \omega_1 = e^{2\pi i \theta_1}$, $y = \omega_2 = e^{2\pi i \theta_2}$, the Jacobian is $J_{\mathbb{Z}_0} = 4\pi^2 e^{2\pi i(\theta_1 + \theta_2)} = 4\pi^2 xy$. By integrating $|\Gamma| |J(x, y)_{\mathbb{Z}_0}^{-1}| = 1/4\pi^2$ over y, x respectively we obtain the spectral measure $\nu_{\rho_1}, \nu_{\rho_2}$ respectively for $\Delta_{\rho_1}, \Delta_{\rho_2}$. Since $\int_{\mathbb{T}} x^m dx = 2\pi \delta_{m,0}$, we obtain that the spectral measure ν_{ρ_j} (over $\chi_{\rho_j}(\mathbb{T}^2) = \mathbb{T}$) for the graph $\mathcal{G}_{\rho_j}^{\mathbb{Z}_0}$, $j = 1, 2$, is given by $d\nu_{\rho_j}(x) = (2\pi)^{-1} dx$.

The graph $\mathcal{G}_{\rho_j}^{\mathbb{Z}_0}$, $j = 1, 2$, is given by an infinite number of copies of the representation graph $\mathcal{G}_\rho^0 = \mathcal{H}_\rho^0$ for \mathbb{T} (which has Weyl group 0), where ρ is the fundamental representation



Figure 3: a) Infinite graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$; b) Infinite graph $\mathcal{G}_{\rho_2}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$

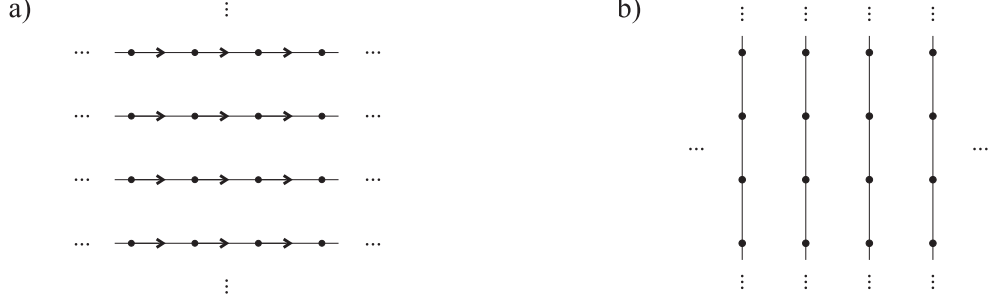


Figure 4: a) Infinite graph $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$; b) Infinite graph $\mathcal{H}_{\rho_2}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$

of \mathbb{T} . In particular the connected component of the distinguished vertex $(0, 0)$ of $\mathcal{G}_{\rho_j}^{\mathbb{Z}_0}$ is the representation graph \mathcal{G}_{ρ}^0 for \mathbb{T} .

5 $\mathbb{Z}_2^{(2)}$: $\mathbb{T} \times SU(2)$

The subgroup $\mathbb{Z}_2^{(2)} \in GL(2, \mathbb{Z})$ is generated by the matrix T_2 of Section 1. It is the Weyl group of the connected compact Lie group $\mathbb{T} \times SU(2) = U(1) \times SU(2)$. The spectral measures for $SU(2)$ were studied in [1, 10]. We choose fundamental generators $\rho_1 = (1, 0)$, $\rho_2 = (0, 1)$. The graphs $\mathcal{G}_{\rho}^{\mathbb{Z}_2^{(2)}}$, $\mathcal{H}_{\rho}^{\mathbb{Z}_2^{(2)}}$ for $\rho = \rho_1, \rho_2$ are illustrated in Figures 3-4. Let

$$x := x_{\rho_1} = \omega_1, \quad y := x_{\rho_2} = \omega_2 + \omega_2^{-1}, \quad (12)$$

and denote by Ψ be the map $\Psi_{\rho_1, \rho_2} : (\omega_1, \omega_2) \mapsto (x, y)$. A fundamental domain of $\mathbb{T}^2 / \mathbb{Z}_2^{(2)}$ is illustrated in Figure 5(a). Then $\mathfrak{D} := \mathfrak{D}_{\rho_1, \rho_2} = \mathbb{T} \times [-2, 2]$ is the joint spectrum $\sigma(\Delta_{\rho_1}, \Delta_{\rho_2})$ of the commuting normal operators $\Delta_{\rho_1}, \Delta_{\rho_2}$. Similarly, \mathfrak{D} is also the joint spectrum of the commuting normal operators $\mathbb{Z}_2^{(2)} \Delta_{\rho_1}, \mathbb{Z}_2^{(2)} \Delta_{\rho_2}$. Note that $\Delta_{\rho_2}, \mathbb{Z}_2^{(2)} \Delta_{\rho_2}$ are in fact self-adjoint. Under the change of variables $x = \omega_1, y = \omega_2 + \omega_2^{-1}$, the Jacobian is given by $J_{\mathbb{Z}_2^{(2)}} = -8\pi^2 i e^{2\pi i \theta_1} \sin(2\pi \theta_2) = -4\pi^2 \omega_1 (\omega_2 - \overline{\omega_2})$. The Jacobian is complex-valued and vanishes in \mathbb{T}^2 only on the boundaries of the images of the fundamental domain C under $\mathbb{Z}_2^{(2)}$. Now $J_{\mathbb{Z}_2^{(2)}}^2 = 16\pi^4 \omega_1^2 (\omega_2^2 - 2 + \overline{\omega_2}^2)$, which is invariant under the action of $\mathbb{Z}_2^{(2)}$ on \mathbb{T}^2 .

We write $J_{\mathbb{Z}_2^{(2)}}^2$ in terms of the $\mathbb{Z}_2^{(2)}$ -invariant elements x, y as $J_{\mathbb{Z}_2^{(2)}}^2 = -16\pi^4 x^2 (4 - y^2)$. Since $4 - y^2 \geq 0$ for all $y \in [-2, 2]$, we can write $J_{\mathbb{Z}_2^{(2)}}$ in terms of x, y as $J_{\mathbb{Z}_2^{(2)}} = 4\pi^2 i x \sqrt{4 - y^2}$.

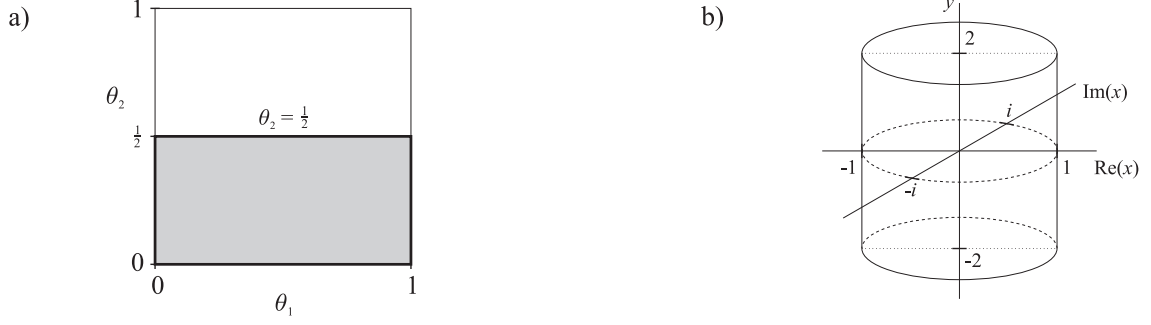


Figure 5: a) A fundamental domain of $\mathbb{T}^2/\mathbb{Z}_2^{(2)}$ for $\mathbb{T} \times SU(2)$.; b) The domain $\mathfrak{D} = \Psi(C)$ for $\mathbb{T} \times SU(2)$.

5.1 Spectral measure for $\mathcal{H}_\rho^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$

By integrating $|\Gamma| |J_{\mathbb{Z}_2^{(2)}}(x, y)^{-1}| = (2\pi^2 \sqrt{4 - y^2})^{-1}$ over y, x respectively we obtain the spectral measure $\nu_{\rho_1}, \nu_{\rho_2}$ respectively for $\mathbb{Z}_2^{(2)} \Delta_{\rho_1}, \mathbb{Z}_2^{(2)} \Delta_{\rho_2}$. Since $\int_{-2}^2 \sqrt{4 - y^2}^{-1} dy = \pi$, the spectral measure ν_{ρ_1} (over $\chi_{\rho_1}(\mathbb{T}^2) = \mathbb{T}$) for the graph $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$ is given by $d\nu_{\rho_1}(x) = (2\pi)^{-1} dx$. Since $\int_{\mathbb{T}} 1 dy = 2\pi$, the spectral measure ν_{ρ_2} (over $\chi_{\rho_2}(\mathbb{T}^2) = [-2, 2]$) for the graph $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$ is given by $d\nu_{\rho_2}(y) = (\pi \sqrt{4 - y^2})^{-1} dy$. The graph $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ is given by an infinite number of copies of the representation graph $\mathcal{G}_\rho^0 = \mathcal{H}_\rho^0$ for \mathbb{T} .

5.2 Spectral measure for $\mathcal{G}_\rho^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$

The graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ is also given by an infinite number of copies of the representation graph \mathcal{G}_ρ^0 for \mathbb{T} , thus the spectral measure ν_{ρ_1} (over $\chi_{\rho_1}(\mathbb{T}^2) = \mathbb{T}$) for the graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$ is given by $d\nu_{\rho_1}(x) = (2\pi)^{-1} dx$. The graph $\mathcal{G}_{\rho_2}^{\mathbb{Z}_2^{(2)}}$ is given by an infinite number of copies of the representation graph $\mathcal{G}_\rho^{(-1)}$ for $SU(2)$ (which has Weyl group $\mathbb{Z}_2 = \langle -1 \rangle \subset GL(1, \mathbb{Z})$). In particular the connected component of the distinguished vertex $*$ of $\mathcal{G}_{\rho_2}^{\mathbb{Z}_2^{(2)}}$, the vertex with lowest Perron-Frobenius weight (which in this case is the apex vertex, i.e. the vertex in the bottom left corner of the graphs in Figure 3), is the representation graph $\mathcal{G}_\rho^{(-1)}$ for $SU(2)$. Thus the spectral measure ν_{ρ_2} (over $\chi_{\rho_2}(\mathbb{T}^2) = [-2, 2]$) for the graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(2)}}$ for $\mathbb{T} \times SU(2)$ is given [27] by $d\nu_{\rho_2}(y) = \frac{1}{2\pi} \sqrt{4 - y^2} dy$.

6 $\mathbb{Z}_2^{(3)}: U(2)$

The subgroup $\mathbb{Z}_2^{(3)} \in GL(2, \mathbb{Z})$ is generated by the matrix T_2' of Section 1. It is the Weyl group of the connected compact Lie group $U(2)$. We choose fundamental generators $\rho_1 = (1, 0), \rho_2 = (1, 1)$. The graphs $\mathcal{G}_\rho^{\mathbb{Z}_2^{(3)}}, \mathcal{H}_\rho^{\mathbb{Z}_2^{(3)}}$ for $\rho = \rho_1, \rho_2$ are illustrated in Figures

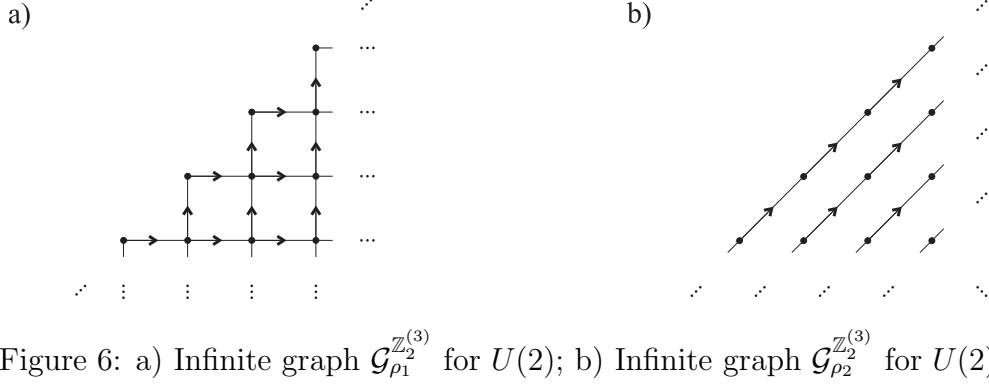


Figure 6: a) Infinite graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$; b) Infinite graph $\mathcal{G}_{\rho_2}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$

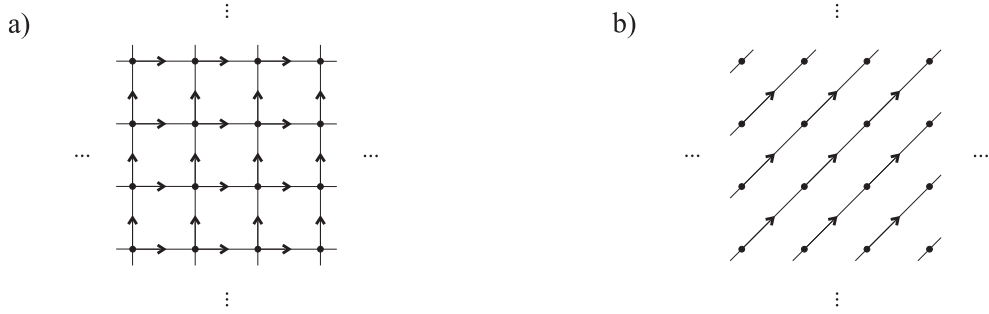


Figure 7: Infinite graph $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$; b) Infinite graph $\mathcal{H}_{\rho_2}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$

6-7. Let

$$x := x_{\rho_1} = \omega_1 + \omega_2, \quad y := x_{\rho_2} = \omega_1\omega_2, \quad (13)$$

and denote by Ψ be the map $\Psi_{\rho_1, \rho_2} : (\omega_1, \omega_2) \mapsto (x, y)$. A fundamental domain of $\mathbb{T}^2/\mathbb{Z}_2^{(3)}$ is illustrated in Figure 8(a), where the boundaries marked by arrows are identified. Then $\mathbb{T}^2/\mathbb{Z}_2^{(3)}$ is the Möbius strip, as in the figure on the left in Figure 8(b), where the dashed line $\theta_1 = \theta_2 + 1/2$ in Figure 8(a) is identified with the dashed line around the centre of the Möbius strip, and $\mathfrak{D} := \Psi(C)$ is an embedding of the Möbius strip in \mathbb{C}^2 . Under the change of variables $x = \omega_1 + \omega_2$, $y = \omega_1\omega_2$, the Jacobian is given by $J_{\mathbb{Z}_2^{(3)}} = 4\pi^2\omega_1\omega_2(\omega_2 - \omega_1)$. The Jacobian is complex-valued and vanishes in \mathbb{T}^2 only on the boundary $\theta_1 = \theta_2$ of the fundamental domain C . Now $J_{\mathbb{Z}_2^{(3)}}^2 = 16\pi^4\omega_1^2\omega_2^2(\omega_1^2 - 2\omega_1\omega_2 + \omega_2^2)$ is invariant under the action of $\mathbb{Z}_2^{(3)}$ on \mathbb{T}^2 , and we write $J_{\mathbb{Z}_2^{(3)}}^2$ in terms of the \mathbb{Z}_2^3 -invariant elements x, y as $J_{\mathbb{Z}_2^{(3)}}^2 = 16\pi^4y^2(x^2 - 4y)$. It is easy to check that $|J_{\mathbb{Z}_2^{(3)}}| = 4\pi^2\sqrt{4 - |x|^2}$.

Remark 6.1. Although the groups $\mathbb{Z}_2^{(2)}$ and $\mathbb{Z}_2^{(3)}$ are not conjugate in $GL(2, \mathbb{Z})$, they are however conjugate in $GL(2, \mathbb{R})$, where the conjugating matrix is $H = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ which has inverse $H^{-1} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} = \frac{1}{2}H$. Thus H intertwines $\mathbb{Z}_2^{(2)}$ and $\mathbb{Z}_2^{(3)}$ with $H\mathbb{Z}_2^{(2+i)} = \mathbb{Z}_2^{(3-i)}H$ for both $i = 0, 1$. The geometric effect of the action of H on \mathbb{Z}^2 is to reflect about the line $2y = x$ (or equivalently, to rotate the plane clockwise by $\pi/4$ and

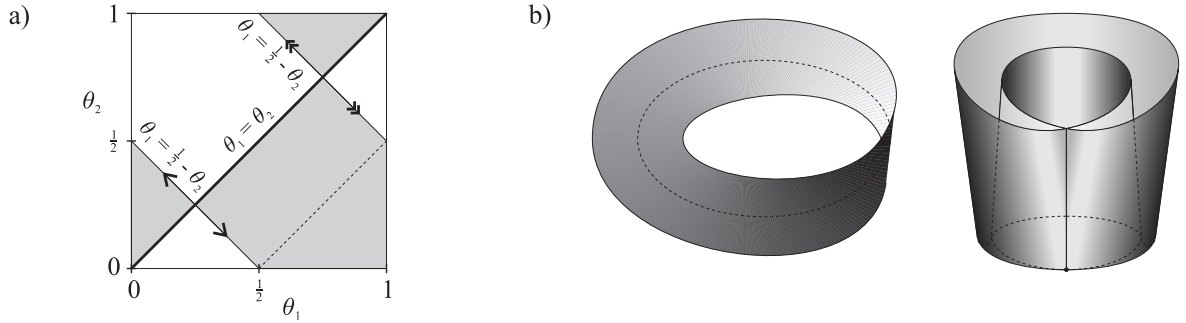


Figure 8: a) A fundamental domain C of $\mathbb{T}^2/\mathbb{Z}_2^{(3)}$ for $U(2)$.; b) The surface $\mathbb{T}^2/\mathbb{Z}_2^{(3)}$ for $U(2)$.

then reflect about the x -axis) and scale by $\sqrt{2}$. Applying this action twice simply has the effect of scaling \mathbb{Z}^2 by 2, as $H^2 = 2I$.

The origin of this relationship is the fact that the compact, connected Lie group $U(1) \times SU(2)$ (which has Weyl group $\mathbb{Z}_2^{(2)}$) is a double cover of the compact, non-connected Lie group $U(2)$ (which has Weyl group $\mathbb{Z}_2^{(3)}$). Thus not all irreducible representations of $U(1) \times SU(2)$ are irreducible representations of $U(2)$, but only the “even” ones, that is, those indexed by $\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $\lambda_1 + \lambda_2 \equiv 0 \pmod{2}$, that is, the irreducible representations μ of $U(2)$ yield the “even” irreducible representations $H\mu$ of $U(1) \times SU(2)$.

Re-drawing the Möbius strip in Figure 8(b) by “folding” it about the dashed line around its centre we obtain the figure on the right in Figure 8(b), which should now be regarded as a surface in \mathbb{R}^4 . It has a line of self-intersection when drawn in \mathbb{R}^3 (given by the vertical line down the centre of the surface), however in \mathbb{R}^4 the surface only intersects along this line at the point \bullet at the bottom of the figure. The boundary of the surface is the top edge which is isomorphic to the circle (note that this circle does not self-intersect in \mathbb{R}^4 even though it appears to when drawn in \mathbb{R}^3).

For new variables $x_{H(0,1)} = x_{(1,-1)} = \omega_1 \omega_2^{-1} + \omega_1^{-1} \omega_2$, $y_1 = y$ ($= x_{H(1,0)} = x_{(1,1)}$), in the joint spectrum $\mathfrak{D}_{(1,1),(1,-1)}$ the “inner wall” and “outer wall” in the figure on the right of Figure 8(b) are identified, thus $\mathfrak{D}_{(1,1),(1,-1)}$ is the cylinder $\mathbb{T} \times [-2, 2]$, which is the joint spectrum $\mathfrak{D}_{\mathbb{Z}_2^{(2)}}$ for $\mathbb{Z}_2^{(2)}$ in Section 5.

6.1 Spectral measure for $\mathcal{H}_\rho^{\mathbb{Z}_2^{(3)}}$ for $U(2)$

By integrating $|\Gamma| |J_{\mathbb{Z}_2^{(3)}}(x, y)^{-1}| = (4\pi^2 \sqrt{4 - |x|^2})^{-1}$ over y , x respectively we obtain the spectral measure ν_{ρ_1} , ν_{ρ_2} respectively for $\mathbb{Z}_2^{(2)} \Delta_{\rho_1}$, $\mathbb{Z}_2^{(2)} \Delta_{\rho_2}$. Alternatively, we can determine these measures independently as follows.

We determine first the spectral measure for $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ over the spectrum $\chi_{\rho_1}(\mathbb{T}^2)$, which is uniquely determined by the moments $\varphi(\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^m (\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^*)^n)$ of $\mathbb{Z}_2^{(3)} \Delta_{\rho_1}$. The spectrum $\sigma(\mathbb{Z}_2^{(3)} \Delta_{\rho_1}) = \chi_{\rho_1}(\mathbb{T}^2)$ of $\mathbb{Z}_2^{(3)} \Delta_{\rho_1}$ is given by the disc $2\mathbb{D} = \{z \in \mathbb{C} \mid |z| \leq 2\}$ with radius 2.

The m, n^{th} moment $\varphi(\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^m (\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^*)^n)$ counts the number of paths of length $m + n$

on $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ and its opposite graph $(\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}})^{\text{op}}$ (that is, the graph $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ with the orientation of all edges reversed) which start at any choice of distinguished vertex $*$, where the first m edges are on $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ and the next n edges are on its opposite graph. Due to the orientation of the edges of $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$, the paths of length m on $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ are given by the number of paths from $*$ to the m^{th} level of the graph in Figure 9.

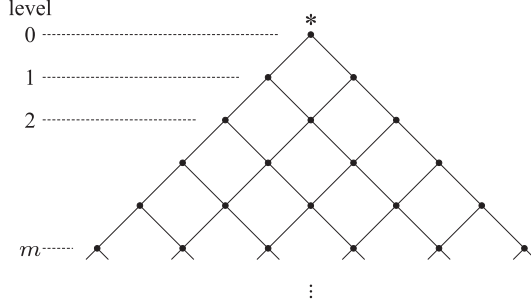


Figure 9: The Bratteli diagram for Dynkin diagram $A_{\infty, \infty}$

It is thus easy to see that $\varphi(\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^m (\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^*)^n)$ will be zero unless $m = n$. When $m = n$, $\varphi(\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^m (\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^*)^n)$ counts all pairs of paths which start at $*$ and which both end at the same vertex at the m^{th} level of the graph in Figure 9, that is, $\varphi(\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^m (\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^*)^n)$ is the dimension of the finite-dimensional algebra given by the m^{th} level of the Bratteli diagram in Figure 9. These dimensions are given by the central binomial coefficient C_m^{2m} (c.f. [10, §2.1]), thus $\varphi(\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^m (\mathbb{Z}_2^{(3)} \Delta_{\rho_1}^*)^n) = \delta_{m,n} C_m^{2m}$. The spectral measure (over $[-2, 2]$) for the Dynkin diagram $A_{\infty, \infty}$ is given by $(\pi \sqrt{4 - r^2})^{-1} dr$. Thus the spectral measure (over $2\mathbb{D}$) for $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ is given by $(\pi^2 \sqrt{4 - |x|^2})^{-1} dx$, since

$$\begin{aligned} \int_{2\mathbb{D}} \frac{x^m \bar{x}^n}{\pi^2 \sqrt{4 - |x|^2}} dx &:= \int_0^2 \int_0^{2\pi} \frac{r^{m+n} e^{i\theta(m-n)}}{\pi^2 \sqrt{4 - r^2}} d\theta dr = \int_0^2 \frac{r^{m+n}}{\pi^2 \sqrt{4 - r^2}} dr \int_0^{2\pi} e^{i\theta(m-n)} d\theta \\ &= \frac{1}{2} \int_{-2}^2 \frac{r^{2m}}{\pi^2 \sqrt{4 - r^2}} dr \delta_{m,n} 2\pi = C_m^{2m} \delta_{m,n}, \end{aligned}$$

where the penultimate equality follows since the product is zero for $m \neq n$, whilst for $m = n$ we have the integral of $r^{2m} / \sqrt{4 - r^2}$ which is an even function.

The graph $\mathcal{H}_{\rho_2}^{\mathbb{Z}_2^{(3)}}$, illustrated in Figure 7(b), is given by an infinite number of copies of the representation graph \mathcal{G}_ρ^0 for \mathbb{T} , and in particular the connected component of the distinguished vertex $(0, 0)$ of $\mathcal{H}_{\rho_2}^{\mathbb{Z}_2^{(3)}}$ is the representation graph for \mathbb{T} . Thus we have

Theorem 6.2. The spectral measure ν_{ρ_1} (over $2\mathbb{D}$) for the graph $\mathcal{H}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$ is given by

$$d\nu_{\rho_1}(x) = \frac{1}{\pi^2 \sqrt{4 - |x|^2}} dx, \quad x \in 2\mathbb{D},$$

whilst the spectral measure ν_{ρ_2} (over \mathbb{T}) for the graph $\mathcal{H}_{\rho_2}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$ is given by $d\nu_{\rho_2}(y) = (2\pi)^{-1} dy$, $y \in \mathbb{T}$.

6.2 Spectral measure for $\mathcal{G}_\rho^{\mathbb{Z}_2^{(3)}}$ for $U(2)$

We turn now to the graphs $\mathcal{G}_{\rho_j}^{\mathbb{Z}_2^{(3)}}$, $j = 1, 2$. The adjacency matrix Δ_{ρ_1} for the graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ is normal, and thus its spectral measure (over $\chi_{\rho_1}(\mathbb{T}^2)$) is uniquely determined by its moments $\varphi(\Delta_{\rho_1}^m (\Delta_{\rho_1}^*)^n)$. Its spectrum $\sigma(\Delta_{\rho_1}) = \chi_{\rho_1}(\mathbb{T}^2)$ is again given by the disc $2\mathbb{D}$ of radius 2. We define a state φ by $\varphi(\cdot) = \langle \cdot \Omega, \Omega \rangle$, where Ω is vector in $\ell^2(\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}})$ corresponding to a distinguished vertex $*$, which is chosen to be one of the vertices which is only the source (or range) of one edge. Thus the m, n^{th} moment $\varphi(\Delta_{\rho_1}^m (\Delta_{\rho_1}^*)^n)$ counts the number of paths of length $m + n$ on $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ and its opposite graph $(\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}})^{\text{op}}$, where the first m edges are on $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ and the next n edges are on its opposite graph. Due to the orientation of the edges of $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$, the paths of length m on $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ are given by the number of paths from $*$ to the m^{th} level of the graph in Figure 10.

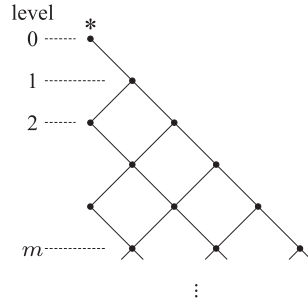


Figure 10: The Bratteli diagram for Dynkin diagram A_∞

It is thus easy to see again that $\varphi(\Delta_{\rho_1}^m (\Delta_{\rho_1}^*)^n)$ will be zero unless $m = n$. When $m = n$, $\varphi(\Delta_{\rho_1}^m (\Delta_{\rho_1}^*)^m)$ counts all pairs of paths which start at $*$ and which both end at the same vertex at the m^{th} level of the graph in Figure 10, that is, $\varphi(\Delta_{\rho_1}^m (\Delta_{\rho_1}^*)^m)$ is the dimension of the finite-dimensional algebra given by the m^{th} level of the Bratteli diagram in Figure 10. These dimensions are well known to be given by the Catalan numbers $c_m = C_m^{2m}/(m+1)$ [18, Aside 5.1.1], thus $\varphi(\Delta_{\rho_1}^m (\Delta_{\rho_1}^*)^n) = \delta_{m,n} c_m$. The spectral measure (over $[-2,2]$) for the Dynkin diagram A_∞ is the semi-circle measure $\sqrt{4-r^2} dr / 2\pi$. Thus the spectral measure (over $2\mathbb{D}$) for $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ is given by $\sqrt{4-|x|^2} dx / 2\pi^2$, since

$$\begin{aligned} \int_{2\mathbb{D}} x^m \bar{x}^n \sqrt{4-|x|^2} dx &:= \int_0^2 \int_0^{2\pi} r^{m+n} e^{i\theta(m-n)} \sqrt{4-r^2} d\theta dr \\ &= \int_0^2 r^{m+n} \sqrt{4-r^2} dr \int_0^{2\pi} e^{i\theta(m-n)} d\theta \\ &= \frac{1}{2} \int_{-2}^2 r^{2m} \sqrt{4-r^2} dr \delta_{m,n} 2\pi = 2\pi^2 c_m \delta_{m,n}. \end{aligned}$$

The graph $\mathcal{G}_{\rho_2}^{\mathbb{Z}_2^{(3)}}$ is again given by an infinite number of copies of the representation graph \mathcal{G}_ρ^0 for \mathbb{T} , thus we have

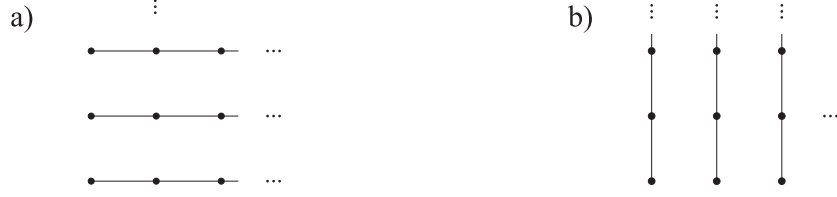


Figure 11: a) Infinite graph $\mathcal{G}_{\rho_1}^{D_4^{(1)}}$ for $SU(2) \times SU(2)$; b) Infinite graph $\mathcal{G}_{\rho_2}^{D_4^{(1)}}$ for $SU(2) \times SU(2)$

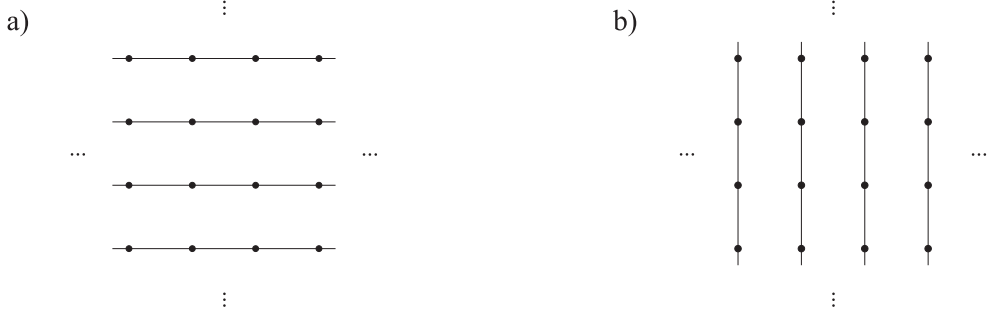


Figure 12: a) Infinite graph $\mathcal{H}_{\rho_1}^{D_4^{(1)}}$ for $SU(2) \times SU(2)$; b) Infinite graph $\mathcal{H}_{\rho_2}^{D_4^{(1)}}$ for $SU(2) \times SU(2)$

Theorem 6.3. The spectral measure ν_{ρ_1} (over $2\mathbb{D}$) for the graph $\mathcal{G}_{\rho_1}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$ is given by

$$d\nu_{\rho_1}(x) = \frac{1}{2\pi^2} \sqrt{4 - |x|^2} dx, \quad x \in 2\mathbb{D},$$

whilst the spectral measure ν_{ρ_2} (over \mathbb{T}) for the graph $\mathcal{G}_{\rho_2}^{\mathbb{Z}_2^{(3)}}$ for $U(2)$ is given by $d\nu_{\rho_2}(y) = (2\pi)^{-1} dy$, $y \in \mathbb{T}$.

7 $D_4^{(1)}$: $SU(2) \times SU(2)$

The subgroup $D_4^{(1)} \in GL(2, \mathbb{Z})$ is generated by $-I$ and the matrix T_2 of Section 1. It is the Weyl group for the connected, simply-connected, semisimple compact Lie group $SU(2) \times SU(2)$ (c.f. Section 5). We choose fundamental generators $\rho_1 = (1, 0)$, $\rho_2 = (0, 1)$. The graphs $\mathcal{G}_{\rho}^{D_4^{(1)}}$, $\mathcal{H}_{\rho}^{D_4^{(1)}}$ for $\rho = \rho_1, \rho_2$ are illustrated in Figures 11-12. Let

$$x := x_{\rho_1} = \omega_1 + \omega_1^{-1} = \cos(2\pi\theta_1), \quad y := x_{\rho_2} = \omega_2 + \omega_2^{-1} = \cos(2\pi\theta_2), \quad (14)$$

and denote by Ψ be the map $\Psi_{\rho_1, \rho_2} : (\omega_1, \omega_2) \mapsto (x, y)$. A fundamental domain C of $\mathbb{T}^2/D_4^{(1)}$ is illustrated in Figure 13(a). Under the change of variables $x = \omega_1 + \omega_1^{-1}$, $y = \omega_2 + \omega_2^{-1}$, the Jacobian is given by $J_{D_4^{(1)}} = 16\pi^2 \sin(2\pi\theta_1) \sin(2\pi\theta_2) = 4\pi^2(\omega_1\bar{\omega}_2 + \bar{\omega}_1\omega_2 - \omega_1\omega_2 - \bar{\omega}_1\bar{\omega}_2)$. The Jacobian is real and vanishes in \mathbb{T}^2 only on the boundaries of the images of the fundamental domain C under $D_4^{(1)}$. Again, $J_{D_4^{(1)}}^2$ is invariant under the

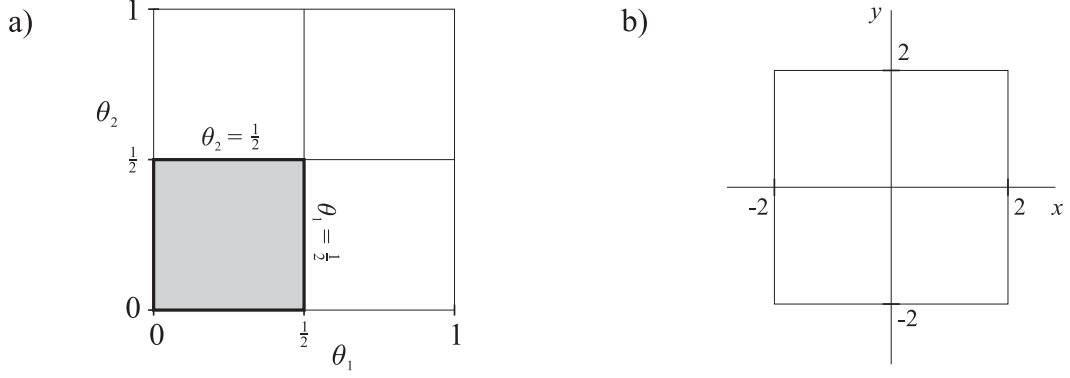


Figure 13: a) A fundamental domain C of $\mathbb{T}^2/D_4^{(1)}$ for $SU(2) \times SU(2)$; b) The domain $\mathfrak{D} = \Psi(C)$ for $SU(2) \times SU(2)$.

action of $D_4^{(1)}$ on \mathbb{T}^2 , and $J_{D_4^{(1)}}^2$ can be written in terms of the $D_4^{(1)}$ -invariant elements x, y as $J_{D_4^{(1)}}^2 = 16\pi^4(4 - x^2)(4 - y^2)$, where $4 - p^2 \geq 0$ for all $p \in [-2, 2]$. Thus we write $J_{D_4^{(1)}}$ in terms of x, y as $J_{D_4^{(1)}} = 4\pi^2\sqrt{(4 - x^2)(4 - y^2)}$.

Remark 7.1. The group $\mathbb{Z}_2^{(2)}$ is a normal subgroup of $D_4^{(1)}$. The fundamental domain C for $D_4^{(1)} = \mathbb{Z}_2^{(2)} \rtimes \mathbb{Z}_2^{(1)}$ in Figure 13(a) is obtained from the fundamental domain for $\mathbb{Z}_2^{(2)}$ in Figure 5(a) by imposing one extra symmetry which comes from the additional $\mathbb{Z}_2^{(1)}$ action. If we denote x, y for $\mathbb{Z}_2^{(2)}$ given in (12) by x_1, y_1 , and x, y for $D_4^{(1)}$ given in (14) by x_2, y_2 , then we see that $y_1 = y_2$ whilst $x_2 = 2\text{Re}(x_1)$. Then there is a homomorphism $\xi : \mathfrak{D}_{\mathbb{Z}_2^{(2)}} \rightarrow \mathfrak{D}_{D_4^{(1)}}$ from the domain $\mathfrak{D}_{\mathbb{Z}_2^{(2)}}$ of $\mathbb{Z}_2^{(2)}$ to the domain $\mathfrak{D}_{D_4^{(1)}}$ of $D_4^{(1)}$ such that $\xi(y_1) = y_2$ and $\xi(x_1) = 2\text{Re}(x_1)$, which maps the boundary of $\mathfrak{D}_{\mathbb{Z}_2^{(2)}}$ to part of the boundary of $\mathfrak{D}_{D_4^{(1)}}$. The rest of the boundary of $\mathfrak{D}_{D_4^{(1)}}$ is given by $\xi(v)$, where $v = (x, y) \in \mathfrak{D}_{\mathbb{Z}_2^{(2)}}$ such that $x \in \mathbb{R}$, i.e. $v = \Phi(t)$ for $t \in \mathbb{T}^2$ such that t is fixed under the additional \mathbb{Z}_2 action in $D_4^{(1)}$. Moreover, we see that $J_{D_4^{(1)}} = 2\text{Re}(J_{\mathbb{Z}_2^{(2)}}) = \xi(J_{\mathbb{Z}_2^{(2)}})$.

Remark 7.2. As noted in the introduction, the subgroup $\mathbb{Z}_2^{(1)}$ of $SL(2, \mathbb{Z})$ is also a normal subgroup of $D_4^{(1)} \cong \mathbb{Z}_2^{(1)} \rtimes \mathbb{Z}_2^{(2)}$. Since P_+ is not uniquely defined, the definition of formal character given in (3) does not make sense. However, for a fixed choice P_+ of fundamental domain of $\mathbb{T}^2/\mathbb{Z}_2^{(1)}$ one can define a formal character for all $\lambda \in P_+$ simply by $\chi_\lambda(\omega_1, \omega_2) = S_\lambda(\theta_1, \theta_2)$ ($= C_\lambda(\theta_1, \theta_2)$ since both elements of $\mathbb{Z}_2^{(1)}$ have determinant one), where $\omega_j = e^{2\pi i\theta_j} \in \mathbb{T}$, $j = 1, 2$. Then one can construct representation graphs $\mathcal{G}_{\rho_j}^{\mathbb{Z}_2^{(1)}}$, $\mathcal{H}_{\rho_j}^{\mathbb{Z}_2^{(1)}}$ for $j = 1, 2$, where $\rho_1 = (1, 0)$, $\rho_2 = (0, 1)$ play the role of the fundamental generators (although one also needs the fact that $\chi_{(-\lambda_1, \lambda_2)}(\omega_1, \omega_2) = \chi_{(\lambda_1, \lambda_2)}(\overline{\omega_1}, \omega_2)$ and/or $\chi_{(\lambda_1, -\lambda_2)}(\omega_1, \omega_2) = \chi_{(\lambda_1, \lambda_2)}(\omega_1, \overline{\omega_2})$ in order to generate all characters). Then $x := x_{\rho_1} = \omega_1 + \overline{\omega_1}$ and $y := x_{\rho_2} = \omega_2 + \overline{\omega_2}$, and the map $\Psi : (\omega_1, \omega_2) \mapsto (x, y)$ is a map from \mathbb{T}^2 to the joint spectrum $\mathfrak{D}^{D_4^{(1)}} = [-2, 2] \times [-2, 2]$ for $D_4^{(1)}$ described above. In this case, Ψ is a two-to-one map from P_+ to $\mathfrak{D}^{D_4^{(1)}}$. The Jacobian for the change of variables $\omega_1, \omega_2 \rightarrow x, y$ is again given by $J_{D_4^{(1)}}$, and the spectral measures (over $\mathfrak{D}^{D_4^{(1)}}$) for ρ_1, ρ_2 are given by

twice the spectral measures (over $\mathfrak{D}^{D_4^{(1)}}$) for $D_4^{(1)}$ in (15), (16) below (the factor of two comes from the fact that Ψ is now a two-to-one map from P_+ to $\mathfrak{D}^{D_4^{(1)}}$).

7.1 Spectral measure for $\mathcal{H}_\rho^{D_4^{(1)}}$ for $SU(2) \times SU(2)$

Here $\mathfrak{D} = [-2, 2] \times [-2, 2]$ is the joint spectrum $\sigma(D_4^{(1)}\Delta_{\rho_1}, D_4^{(1)}\Delta_{\rho_2})$ of the commuting self-adjoint operators $D_4^{(1)}\Delta_{\rho_1}, D_4^{(1)}\Delta_{\rho_2}$. Then by integrating $|\Gamma| J_{D_4^{(1)}}(x, y)^{-1}$ over y we obtain the spectral measure ν_{ρ_1} for $D_4^{(1)}\Delta_{\rho_1}$. Integrating $|\Gamma| J_{D_4^{(1)}}(x, y)^{-1}$ over x gives the same result, and thus $\nu_{\rho_1} = \nu_{\rho_2}$. Since $\int_{-2}^2 \sqrt{4 - y^2}^{-1} dy = \pi$, the spectral measures ν_{ρ_j} (over $\chi_{\rho_j}(\mathbb{T}^2) = [-2, 2]$) for the graphs $\mathcal{H}_{\rho_j}^{D_4^{(1)}}$, $j = 1, 2$, for $SU(2) \times SU(2)$ are both given by

$$d\nu_{\rho_1}(x) = d\nu_{\rho_2}(x) = (\pi\sqrt{4 - x^2})^{-1} dx, \quad x \in [-2, 2] \quad (15)$$

7.2 Spectral measure for $\mathcal{G}_\rho^{D_4^{(1)}}$ for $SU(2) \times SU(2)$

The graphs $\mathcal{G}_{\rho_j}^{D_4^{(1)}}$ are both given by an infinite number of copies of the representation graph $\mathcal{G}_\rho^{(-1)}$ for $SU(2)$, and in particular the connected component of the distinguished vertex $*$, the vertex with lowest Perron-Frobenius weight which in this case is the apex vertex, i.e. the vertex in the bottom left corner of the graphs in Figure 11. Thus the spectral measures ν_{ρ_j} (over $\chi_{\rho_j}(\mathbb{T}^2) = [-2, 2]$) for the graphs $\mathcal{G}_{\rho_j}^{D_4^{(1)}}$, $j = 1, 2$, for $SU(2) \times SU(2)$ are given by

$$d\nu_{\rho_1}(x) = d\nu_{\rho_2}(x) = \frac{1}{2\pi} \sqrt{4 - x^2} dx, \quad x \in [-2, 2] \quad (16)$$

8 $D_4^{(2)}$: $SO(4)$

The subgroup $D_4^{(2)} \in GL(2, \mathbb{Z})$ is generated by $-I$ and the matrix T_2' of Section 1. It is the Weyl group of the connected compact Lie group $SO(4)$. We choose fundamental generators $\rho_1 = (1, 0)$, $\rho_2 = (1, 1)$. The graphs $\mathcal{G}_\rho^{D_4^{(2)}}$, $\mathcal{H}_\rho^{D_4^{(2)}}$ for $\rho = \rho_1, \rho_2$ are illustrated in Figures 14-15. Let

$$x := x_{\rho_1} = \omega_1 + \omega_1^{-1} + \omega_2 + \omega_2^{-1} = 2 \cos(2\pi\theta_1) + 2 \cos(2\pi\theta_2), \quad (17)$$

$$y := x_{\rho_2} = 1 + \omega_1\omega_2 + \omega_1^{-1}\omega_2^{-1} = 1 + 2 \cos(2\pi(\theta_1 + \theta_2)), \quad (18)$$

and denote by Ψ be the map $\Psi_{\rho_1, \rho_2} : (\omega_1, \omega_2) \mapsto (x, y)$. A fundamental domain C of $\mathbb{T}^2/D_4^{(2)}$ is illustrated in Figure 16(a). Then $\mathfrak{D} = \Psi(C)$ is illustrated in Figure 16(b). Now $(\theta_1, 1/2 - \theta_1)$ maps to $\Psi(\omega_1, -\overline{\omega_1}) = (0, -1)$ for all $\theta_1 \in \mathbb{T}$, the dashed line $\theta_2 = 1/2 - \theta_1$ in Figure 16(a) contracts to the single point $(0, -1)$ in \mathfrak{D} . The boundary of C given by $\theta_1 = \theta_2$, where $\theta_1 \in [1/4, 1/2]$, yields the curve c_1 given by the parametric equations $x = 4 \cos(2\pi\theta_1)$, $y = 2 \cos(4\pi\theta_1) + 1 = 4 \cos^2(2\pi\theta_1) - 1$. Similarly, the boundary of C given by $\theta_1 = \theta_2$, where $\theta_1 \in [3/4, 1]$, yields the curve c_2 given by the same parametric equations. The boundary of C given by $\theta_1 = -\theta_2$ yields the curve c_3 given by the parametric

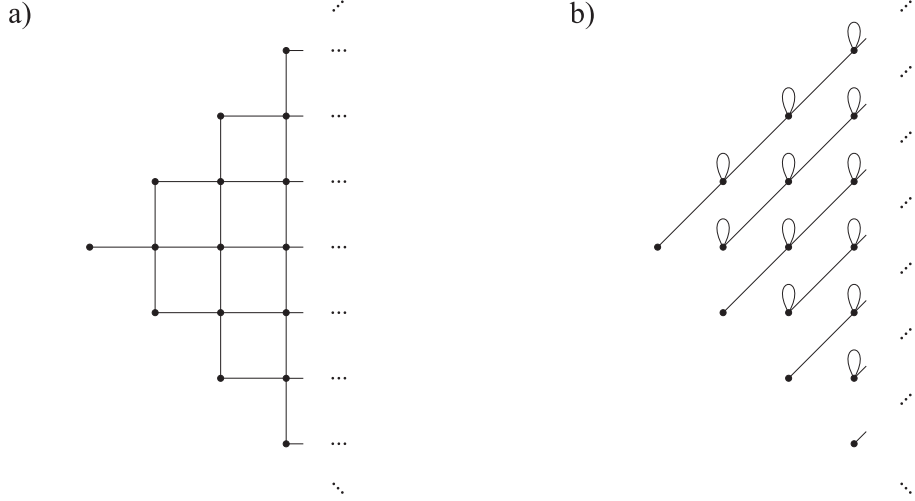


Figure 14: a) Infinite graph $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$ for $SO(4)$; b) Infinite graph $\mathcal{G}_{\rho_2}^{D_4^{(2)}}$ for $SO(4)$

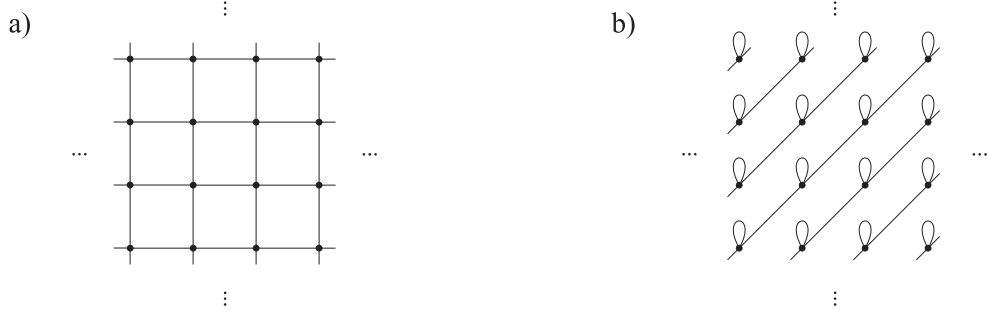


Figure 15: a) Infinite graph $\mathcal{H}_{\rho_1}^{D_4^{(2)}}$ for $SO(4)$; b) Infinite graph $\mathcal{H}_{\rho_2}^{D_4^{(2)}}$ for $SO(4)$

equations $x = 4 \cos(2\pi\theta_1)$, $y = 3$, where $\theta_1 \in [1/2, 1]$. As functions of $x \in [-4, 4]$, the boundaries c_1 , c_2 of \mathfrak{D} are thus given by $y = x^2/4 - 1$ whilst c_3 is given by $y = 3$. As functions of y , the boundaries c_1 , c_2 are given by $x = -2\sqrt{y+1}$, $x = 2\sqrt{y+1}$ respectively. Under the change of variables (17), (18), the Jacobian is given by $J_{D_4^{(2)}} = 16\pi^2(\cos(2\pi\theta_2) + \cos(2\pi(\theta_1 + 2\theta_2)) - \cos(2\pi\theta_1) - \cos(2\pi(2\theta_1 + \theta_2)))$. The Jacobian is real and vanishes in \mathbb{T}^2 only on the boundaries of the images of the fundamental domain C under $D_4^{(2)}$. Again, $J_{D_4^{(2)}}$ is invariant under the action of $D_4^{(2)}$ on \mathbb{T}^2 , and can be written in terms of the $D_4^{(2)}$ -invariant elements x , y as $J_{D_4^{(2)}}^2 = 16\pi^4(y+1)(3-y)(4y-x^2+4)$, which is non-negative since $J_{D_4^{(2)}}$ is real. Thus we can write $J_{D_4^{(2)}}$ in terms of x , y as $J_{D_4^{(2)}} = 4\pi^2\sqrt{(y+1)(3-y)(4y-x^2+4)}$.

Remark 8.1. The group $\mathbb{Z}_2^{(3)}$ is a normal subgroup of $D_4^{(2)}$. The fundamental domain C for $D_4^{(2)} = \mathbb{Z}_2^{(3)} \rtimes \mathbb{Z}_2^{(1)}$ in Figure 16(a) is obtained from the fundamental domain for $\mathbb{Z}_2^{(3)}$ in Figure 8(a) by imposing one extra symmetry which comes from the additional $\mathbb{Z}_2^{(1)}$ action. If we denote x, y for $\mathbb{Z}_2^{(3)}$ given in (13) by x_1, y_1 , and x, y for $D_4^{(2)}$ given in (17), (18) by x_2, y_2 , then we see that $x_2 = 2\text{Re}(x_1)$, whilst $y_2 = 2\text{Re}(y_1) + 1$, and there is

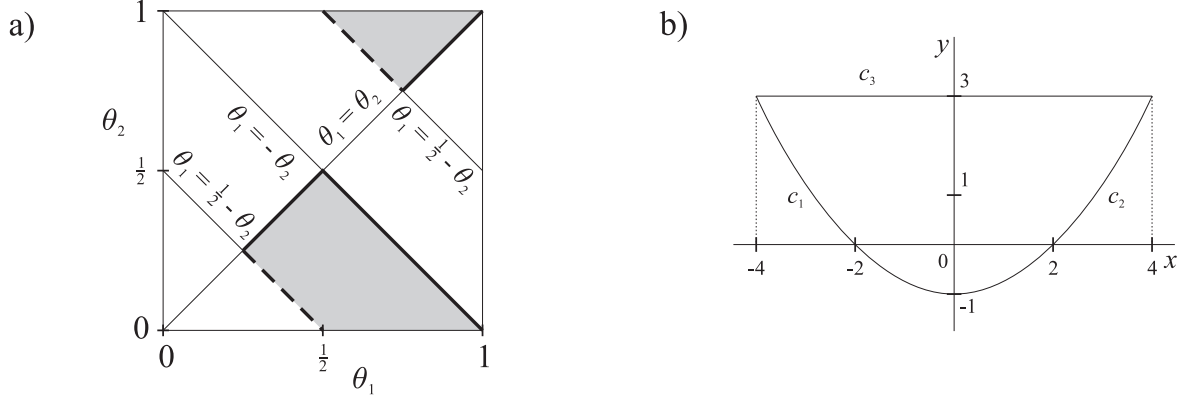


Figure 16: a) A fundamental domain C of $\mathbb{T}^2/D_4^{(2)}$ for $SO(4)$; b) The domain $\mathfrak{D} = \Psi(C)$ for $SO(4)$.

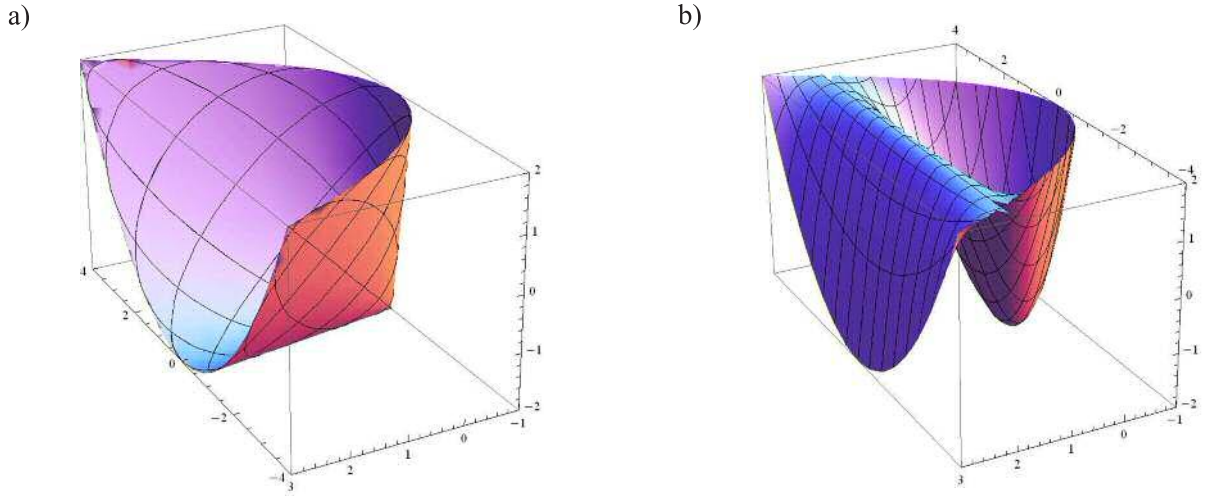


Figure 17: a) The domain $\tilde{\mathfrak{D}} = \tilde{\Phi}(C)$; b) The surface $2\text{Re}(J_{\mathbb{Z}_2^{(3)}}) = 0$.

a homomorphism $\xi : \mathfrak{D}_{\mathbb{Z}_2^{(3)}} \rightarrow \mathfrak{D}_{D_4^{(2)}}$ from the domain $\mathfrak{D}_{\mathbb{Z}_2^{(3)}}$ of $\mathbb{Z}_2^{(3)}$ to the domain $\mathfrak{D}_{D_4^{(2)}}$ of $D_4^{(2)}$ such that $\xi(x_1) = 2\text{Re}(x_1)$ and $\xi(y_1) = 2\text{Re}(y_1) + 1$. However, unlike the case of $\mathbb{Z}_2^{(2)} \triangleleft D_4^{(1)}$ described in Remark 7.1, the respective Jacobians for $\mathbb{Z}_2^{(3)} \triangleleft D_4^{(2)}$ are not related by the relation $J_{D_4^{(2)}} = \xi(J_{\mathbb{Z}_2^{(3)}})$. In fact, it is not possible to express $\text{Re}(J_{\mathbb{Z}_2^{(3)}})$ in terms of the $D_4^{(2)}$ -invariant variables x_2, y_2 . We introduce a third $D_4^{(2)}$ -invariant variable $z_2 := \omega_1\omega_2^{-1} + \omega_1^{-1}\omega_2$, and we have a map $\tilde{\Phi} : \mathbb{T}^2 \rightarrow \mathbb{R}^3$ given by $\tilde{\Phi}(\omega_1, \omega_2) = (x_2, y_2, z_2)$ whose image is a surface $\tilde{\Phi}(C) = \tilde{\mathfrak{D}} \subset \mathbb{R}^3$, illustrated in Figure 17(b). We can write $\text{Re}(J_{\mathbb{Z}_2^{(3)}})$ in terms of the three $D_4^{(2)}$ -invariant variables x_2, y_2, z_2 as

$$2\text{Re}(J_{\mathbb{Z}_2^{(3)}}) = 4\pi^2(\omega_1\omega_2^2 - \omega_1^2\omega_2) = x_2^2y_2^2 - 4x_2^2y_2 + 4x_2^2 - 4y_2^3 + 12y_2^2 - 4z_2 - 8.$$

Then setting $2\text{Re}(J_{\mathbb{Z}_2^{(3)}}) = 0$ we obtain a surface \mathfrak{J} in \mathbb{R}^3 , illustrated in Figure 17(a). Now \mathfrak{J} intersects with the surface $\tilde{\mathfrak{D}}$ only at the closed curve described by the curves

$(x_2, (x_2^2 - 4)/4, 2)$ and $(x_2, 3, (x_2^2 - 8)/4)$, and the point $(0, 0, -2)$, that is, $2\text{Re}(J_{\mathbb{Z}_2^{(3)}})$ is zero only on this closed curve, which is the boundary of $\tilde{\mathfrak{D}}$. The projection of this closed curve onto the x_2, y_2 -plane is then precisely the boundary of the joint spectrum $\mathfrak{D} = \Phi(C)$ for $D_4^{(2)}$, as in Figure 16(b).

Remark 8.2. As noted in Remark 6.1, the groups $\mathbb{Z}_2^{(2)}$ and $\mathbb{Z}_2^{(3)}$ are conjugate in $GL(2, \mathbb{R})$, with conjugating matrix H as in Remark 6.1. Since $D_4^{(i)} = \mathbb{Z}_2^{(1)} \rtimes \mathbb{Z}_2^{(i+1)}$ for $i = 1, 2$, we see that H is also the conjugating matrix for $D_4^{(1)}$ and $D_4^{(2)}$. As in Remark 6.1, the origin of this relationship is the fact that the compact, connected Lie group $SU(2) \times SU(2)$ (which has Weyl group $D_4^{(1)}$) is a double cover of the compact, connected, but not simply connected Lie group $SO(4)$ (which has Weyl group $D_4^{(2)}$).

8.1 Spectral measure for $\mathcal{H}_\rho^{D_4^{(2)}}$ for $SO(4)$

By integrating $|\Gamma| |J_{D_4^{(2)}}(x, y)^{-1}|$ over $y, x \in \mathfrak{D}$ respectively we obtain the spectral measure $\nu_{\rho_1}, \nu_{\rho_2}$ respectively for $D_4^{(2)}\Delta_{\rho_1}, D_4^{(2)}\Delta_{\rho_2}$. More explicitly, using the expressions for the boundaries of \mathfrak{D} given in Section 8, the spectral measure ν_{ρ_1} (over $[-4, 4]$) is $d\nu_{\rho_1}(x) = \mathfrak{J}_1(x) dx$, where $\mathfrak{J}_1(x)$ is given by

$$\mathfrak{J}_1(x) = 4 \int_{(x^2-4)/4}^3 |J_{D_4^{(2)}}(x, y)|^{-1} dy.$$

The weight $\mathfrak{J}_1(x)$ is an integral of the reciprocal of the square root of a cubic in y , and thus can be written in terms of the complete elliptic integral $K(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta}^{-1} d\theta$ of the first kind. Using [7, equation 235.00], we obtain

$$\mathfrak{J}_1(x) = \frac{1}{8\pi^2} K\left(\frac{16 - x^2}{16}\right).$$

The weight $\mathfrak{J}_1(x)$ is illustrated in Figure ??(a).

The spectral measure ν_{ρ_2} (over $[-1, 3]$) is $d\nu_{\rho_2}(y) = \mathfrak{J}_2(y) dy$, where $\mathfrak{J}_2(y)$ is given by

$$\begin{aligned} \mathfrak{J}_2(y) &= 4 \int_{-2\sqrt{y+1}}^{2\sqrt{y+1}} |J_{D_4^{(2)}}(x, y)|^{-1} dx = \frac{1}{\pi^2 \sqrt{(y+1)(3-y)}} \int_{-2\sqrt{y+1}}^{2\sqrt{y+1}} \frac{1}{\sqrt{4(y+1) - x^2}} dx \\ &= \frac{1}{\pi \sqrt{(y+1)(3-y)}}. \end{aligned}$$

We observe that this is the weight which appears in Section 5.1, but shifted $y \rightarrow y - 1$. The weight $\mathfrak{J}_2(y)$ is illustrated in Figure ??(b).

Theorem 8.3. The spectral measures $\nu_{\rho_1}, \nu_{\rho_2}$ (over $\chi_{\rho_1}(\mathbb{T}^2) = [-4, 4]$, $\chi_{\rho_2}(\mathbb{T}^2) = [-1, 3]$ respectively) for the graphs $\mathcal{H}_{\rho_1}^{D_4^{(2)}}$, $\mathcal{H}_{\rho_2}^{D_4^{(2)}}$ respectively, for $SO(4)$ are given by

$$d\nu_{\rho_1}(x) = \frac{1}{8\pi^2} K\left(\frac{16 - x^2}{16}\right) dx, \quad d\nu_{\rho_2}(y) = \frac{1}{\pi \sqrt{(y+1)(3-y)}} dy.$$

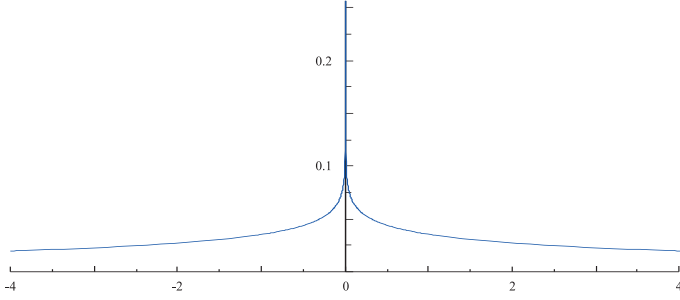


Figure 18: weight for ν_{ρ_1} for $SO(4)$.

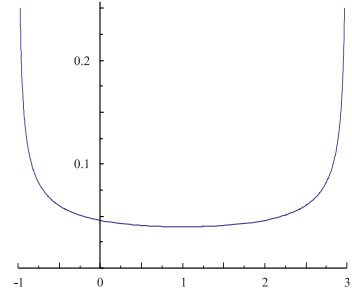


Figure 19: weight for ν_{ρ_2} for $SO(4)$.

8.2 Spectral measure for $\mathcal{G}_{\rho}^{D_4^{(2)}}$ for $SO(4)$

We now consider the spectral measure (over $\chi_{\rho_j}(\mathbb{T}^2)$) for $\mathcal{G}_{\rho_j}^{D_4^{(2)}}$, $j = 1, 2$. The adjacency matrix Δ_{ρ_1} for the graph $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$ is self-adjoint, and thus its spectral measure is uniquely determined by its moments $\varphi(\Delta_{\rho_1}^m)$. It has spectrum $\sigma(\Delta_{\rho_1}) = \chi_{\rho_1}(\mathbb{T}^2) = [-4, 4]$. We define a state φ by $\varphi(\cdot) = \langle \cdot, \Omega \rangle$, where Ω is vector in $\ell^2(\mathcal{G}_{\rho_1}^{D_4^{(2)}})$ corresponding to the distinguished vertex $*$, which is chosen to be the vertex with lowest Perron-Frobenius weight, which in this case is the apex vertex, i.e. the vertex with only one edge attached to it. Thus the m^{th} moment $\varphi(\Delta_{\rho_1}^m)$ counts the number of closed paths of length m on $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$ which start and end at $*$.

Lemma 8.4. The number α_m of closed paths of length m on $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$ which start and end at $*$ is given

$$\alpha_m = \begin{cases} c_{m/2} & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd.} \end{cases}$$

Proof: Since $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$ is bipartite it is clear that the odd moments are all zero. Any closed path of length $2n$ is given by a pair of paths which start at $*$ and which both end at the same vertex at the n^{th} level of the Bratteli diagram whose inclusion graph at each level is given by $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$, and thus $\varphi(\Delta_{\rho_1}^m)$ counts the number of such pairs of paths. Any path of length n on the Bratteli diagram is equivalent to a path in the lattice \mathbb{N}^3 from $(0, 0, 0)$ to (i, j, n) , for some $(i, j) \in \mathbb{N}^2$, with steps belonging to the set $\mathcal{S} = \{(1, 1, 1), (1, -1, 1), (-1, 1, 1), (-1, -1, 1)\}$, where here we think of \mathbb{N} as including 0. Then a closed path of length $2n$ is equivalent to a path in the lattice \mathbb{N}^3 from $(0, 0, 0)$ to $(0, 0, 2n)$ with steps belonging to the set \mathcal{S} . Any such path is given by a sequence of steps $(\epsilon_j^1, \epsilon_j^2, 1)$ from \mathcal{S} , $j = 1, 2, \dots, 2n$. Thus we obtain two sequences $\epsilon^l = (\epsilon_1^l, \epsilon_2^l, \dots, \epsilon_{2n}^l)$, for $l = 1, 2$, where $\epsilon_j^l \in \{\pm 1\}$, $\sum_j \epsilon_j^l = 0$ and $\sum_{j=1}^k \epsilon_j^l \geq 0$ for each k such that $\epsilon_k^l = -1$. Such a sequence is enumerated by the Catalan numbers c_n [25, Corollary 6.2.3(ii)], and thus the number of closed paths of length $2n$ on $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$, starting and ending at $*$, is given by the number of pairs of such sequences, which is c_n^2 . \square

Remark 8.5. This proves a conjecture from the Online Encyclopedia of Integer Sequences (OEIS) [24], namely that the number of walks on \mathbb{N}^2 starting and ending at $(0, 0)$

and consisting of $2n$ steps taken from $\{(-1, -1), (-1, 1), (1, -1), (1, 1)\}$, is given by the squared Catalan numbers A001246. Moreover, from [6, Proposition 8], we also find that the squared Catalan numbers satisfy the following relation with multinomial coefficients $(m_1, m_2, m_3, m_4)! = (\sum_{i=1}^4 m_i)! / (m_1! m_2! m_3! m_4!)$:

$$c_n^2 = \sum_{k=0}^n (k, k, n-k, n-k)! + \sum_{k=0}^{n-1} (k, k+2, n-k-1, n-k-1)! - 2 \sum_{k=0}^{n-1} (k, k+1, n-k-1, n-k)!.$$

Thus the $2n^{\text{th}}$ moment is $\varphi(\Delta_{\rho_1}^{2n}) = c_n^2$. The following result is claimed in [24], but we have not found a proof of the claim in the literature. Therefore we give a proof here.

Lemma 8.6. The generating function for the squared Catalan numbers is given by

$$M(z) = \frac{1}{\pi z} E(16z) - \frac{1-16z}{2\pi z} K(16z) - \frac{1}{4z},$$

where $E(m) = \int_0^{\pi/2} \sqrt{1 - m \sin^2 \theta} d\theta$ is the complete elliptic integral of the second kind.

Proof: The generating function $M(z)$ for the squared Catalan numbers satisfies the linear differential equation [8]

$$z^2(1-16z) \frac{d^2 M}{dz^2} + x(3-32z) \frac{dM}{dz} + (1-4z)M(z) = 1, \quad (19)$$

where it is easy to check that the coefficient of z^n satisfies the recurrence relation $(n+2)^2 c_{n+1}^2 = 4(2n+1)^2 c_n^2$ for the squared Catalan numbers. Now since (see e.g. [29, §22.736])

$$\frac{d}{dz} K(16z) = \frac{E(16z) - (1-16z)K(16z)}{2(1-16z)z}, \quad \frac{d}{dz} E(16z) = \frac{E(16z) - K(16z)}{2z},$$

we have

$$\begin{aligned} \frac{d}{dz} M(z) &= \frac{1-16z}{4\pi z^2} K(16z) - \frac{3}{4\pi z^2} E(16z) + \frac{1}{4z^2}, \\ \frac{d^2}{dz^2} M(z) &= \frac{24z-1}{4\pi z^3} K(16z) + \frac{5}{4\pi z^3} E(16z) - \frac{1}{2z^3}. \end{aligned}$$

Thus a simple calculation shows that $M(z)$ satisfies the differential equation (19) for the squared Catalan numbers. \square

A compactly supported probability measure μ on \mathbb{R} can be recovered from the generating function of its moments by $\mu = -\lim_{\varepsilon \rightarrow 0} \text{Im}(G_\mu(t+i\varepsilon))/\pi$ (see e.g. [17]), where $G_\mu(z) = \int_{\mathbb{R}} (z-t)^{-1} d\mu(t)$ is the Cauchy transform for μ , which has power series expansion about $z = \infty$ given by $G_\mu(z) = \sum_{n=0}^{\infty} m_n z^{-n-1}$, where m_n are the moments of μ . Thus $G_\mu(z)$ is related to the generating series for the moments by $G_\mu(z) = M(z^{-1})/z$.

Theorem 8.7. The spectral measure ν_{ρ_1} (over $\chi_{\rho_1}(\mathbb{T}^2) = [-4, 4]$) for the graph $\mathcal{G}_{\rho_1}^{D_4^{(2)}}$ for $SO(4)$ is given by

$$d\nu_{\rho_1}(x) = \frac{1}{2\pi^2|x|} \left(2x^2 E\left(\frac{x^2-16}{x^2}\right) - (x^2+16)K\left(\frac{x^2-16}{x^2}\right) \right) dx.$$

Proof: The generating function for the moments of ν_{ρ_1} is $M_1(z) = M(z^2)$, where $M(z)$ is given in Lemma 8.6, since the odd moments of ν_{ρ_1} are all zero. Thus the Cauchy transform for $M_1(z)$ is

$$G_{\nu_{\rho_1}}(z) = \frac{z}{\pi} E(16z^{-2}) - \frac{z^2 - 16}{2\pi z} K(16z^{-2}) - \frac{z}{4}.$$

Now, for $z \in [0, 4]$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \operatorname{Im} \left(\frac{t + i\varepsilon}{\pi} E(16(t + i\varepsilon)^{-2}) \right) &= \lim_{\varepsilon \rightarrow 0} \operatorname{Im} \left(\frac{t + i\varepsilon}{\pi} \int_0^{\pi/2} \sqrt{1 - 16(t + i\varepsilon)^{-2} \sin^2 \theta} \, d\theta \right) \\ &= \operatorname{Im} \left(\frac{t}{\pi} \int_0^{\pi/2} \sqrt{1 - 16t^{-2} \sin^2 \theta} \, d\theta \right) = \operatorname{Im} \left(\frac{ti}{\pi} \int_0^{\pi/2} \sqrt{16t^{-2} \sin^2 \theta - 1} \, d\theta \right) \\ &= \frac{t}{\pi} \int_{\sin^{-1} |t/4|}^{\pi/2} \sqrt{16t^{-2} \sin^2 \theta - 1} \, d\theta, \end{aligned}$$

since $16t^{-2} \sin^2 \theta \geq 1$ for $\theta \geq \sin^{-1} |t/4|$. Thus

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \operatorname{Im} \left(\frac{t + i\varepsilon}{\pi} E(16(t + i\varepsilon)^{-2}) \right) &= \frac{-ti}{\pi} \left(\int_0^{\pi/2} \sqrt{1 - 16t^{-2} \sin^2 \theta} \, d\theta - \int_0^{\sin^{-1} |t/4|} \sqrt{1 - 16t^{-2} \sin^2 \theta} \, d\theta \right) \\ &= \frac{-ti}{\pi} (E(16t^{-2}) - E(\sin^{-1} |t/4|, 16t^{-2})). \end{aligned}$$

Similarly,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \operatorname{Im} \left(\frac{(t + i\varepsilon)^2 - 16}{2\pi(t + i\varepsilon)} K(16(t + i\varepsilon)^{-2}) \right) &= \frac{-(t^2 - 16)i}{2\pi t} (K(16t^{-2}) - F(\sin^{-1} |t/4|, 16t^{-2})). \end{aligned}$$

Since, $\lim_{\varepsilon \rightarrow 0} \operatorname{Im}((t + i\varepsilon)/4) = 0$, we obtain for $z \in [0, 4]$ that

$$\begin{aligned} d\nu_{\rho_1}(x) &= \frac{i}{2\pi^2 x} (2x^2 (E(16x^{-2}) - E(\sin^{-1} |x/4|, 16x^{-2})) \\ &\quad - (x^2 - 16) (K(16x^{-2}) - F(\sin^{-1} |x/4|, 16x^{-2}))) \, dx, \end{aligned}$$

where $F(\phi, m)$, $E(\phi, m)$ are the elliptic integrals of the first, second kind respectively, defined by

$$F(\phi, m) = \int_0^\phi \frac{1}{\sqrt{1 - m \sin^2 \theta}} \, d\theta, \quad E(\phi, m) = \int_0^\phi \sqrt{1 - m \sin^2 \theta} \, d\theta,$$

so that $K(m) = F(\pi/2, m)$, $E(m) = E(\pi/2, m)$ are the complete elliptic integrals of the first, second kind respectively. Similarly for $z \in [-4, 0]$ we obtain the same expression for

$d\nu_{\rho_1}(x)$ but multiplied by a factor of -1 . Then the result follows from the elliptic integral identities 111.09 in [7]. \square

We now consider the graph $\mathcal{G}_{\rho_2}^{D_4^{(2)}}$ for the second fundamental generator ρ_2 . The connected component of the distinguished vertex $*$ is the infinite graph $\mathcal{A}^{(\infty_{\text{odd}})*} := \lim_{n \rightarrow \infty} \mathcal{A}^{(2n+1)*}$, whilst the connected component of the vertex $(1, 0)$ immediately to its right is the infinite graph $\mathcal{A}^{(\infty_{\text{even}})*} := \lim_{n \rightarrow \infty} \mathcal{A}^{(2n)*}$, where the graphs $\mathcal{A}^{(l)*}$ are $SU(3)$ \mathcal{ADE} graphs which classify the conjugate modular invariant partition functions for $SU(3)$ integrable statistical mechanical models, and which also yield a non-negative integer matrix representation (nimrep) of the fusion rules of $SU(3)$ at level $l - 3$ (see e.g. [15] for more details about the other $SU(3)$ \mathcal{ADE} graphs, $SU(3)$ modular invariants and nimrep theory). Spectral measures for the $\mathcal{A}^{(l)*}$ graphs were computed in [10, Theorem 7]. The distinguished vertex $*$ with lowest Perron-Frobenius weight in each case is the vertex labelled 1 in [10, Fig. 13]. However, the spectral measures determined in [10, Theorem 7] are for the case where $*$ is the vertex labelled $\lfloor (l - 1)/2 \rfloor$ in [10, Fig. 13], and not for the case where $*$ is the vertex with lowest Perron-Frobenius weight as claimed. We thus provide the correct result here for the spectral measure of $\mathcal{A}^{(l)*}$ where the distinguished vertex $*$ is chosen to be the vertex with lowest Perron-Frobenius weight:

Theorem 8.8. The spectral measure of $\mathcal{A}^{(l)*}$, $l < \infty$, (over \mathbb{T}), where the distinguished vertex $*$ is chosen to be the vertex with lowest Perron-Frobenius weight, is

$$d\varepsilon(u) = \alpha(u)d_{l/2}u, \quad (20)$$

where $d_{l/2}u$ is the uniform measure over l^{th} roots of unity, and

$$\alpha(u) = \begin{cases} 2\text{Im}(u)^2 & \text{for } l \text{ even,} \\ 1 - \text{Re}(u) & \text{for } l \text{ odd.} \end{cases}$$

Proof: From [16] the eigenvectors of $\mathcal{A}^{(l)*}$ are $\psi_a^\lambda = 2\sqrt{l^{-1}} \sin(2\pi a\lambda/l)$, where $\lambda, a = 1, 2, \dots, \lfloor (l - 1)/2 \rfloor$. For each eigenvalue $\beta^{(\lambda)}$ of $\mathcal{A}^{(l)*}$, the eigenvector entry corresponding to the distinguished vertex $*$ = $\lfloor (l - 1)/2 \rfloor$ is $\psi_*^\lambda = 2\sqrt{l^{-1}} \sin(2\pi \lfloor (l - 1)/2 \rfloor \lambda/l)$.

In the case where $l = 2n$ is even,

$$\psi_*^\lambda = \frac{2}{\sqrt{2n}} \sin\left(\frac{\pi}{n}(n - 1)\lambda\right) = (-1)^{\lambda+1} \frac{2}{\sqrt{2n}} \sin\left(\frac{2\pi}{2n}\lambda\right) = (-1)^{\lambda+1} \frac{2}{\sqrt{l}} \sin\left(\frac{2\pi}{l}\lambda\right).$$

Then $|\psi_*^\lambda|^2 = 4 \sin^2(2\pi\lambda/l)/l$ and the result follows from [10, Theorem 7].

In the case where $l = 2n + 1$ is odd,

$$\begin{aligned} \psi_*^\lambda &= \frac{2}{\sqrt{2n+1}} \sin\left(\frac{2\pi}{2n+1}n\lambda\right) = \frac{2}{\sqrt{2n+1}} \sin\left(\pi\left(1 - \frac{1}{2n+1}\right)\lambda\right) \\ &= (-1)^{\lambda+1} \frac{2}{\sqrt{2n+1}} \sin\left(\frac{\pi}{2n+1}\lambda\right) = (-1)^{\lambda+1} \frac{2}{\sqrt{l}} \sin\left(\frac{\pi}{l}\lambda\right) \\ &= (-1)^{\lambda+1} \frac{1}{\sqrt{l}} \sqrt{1 - \cos\left(\frac{2\pi}{l}\lambda\right)}. \end{aligned}$$

Then $|\psi_*^\lambda|^2 = (1 - \cos(2\pi\lambda/l))/l$ and the result follows as in [10, Theorem 7] but with the weighting $2 \sin^2(2\pi\lambda/l)$ replaced by $1 - \cos(2\pi\lambda/l)$. \square

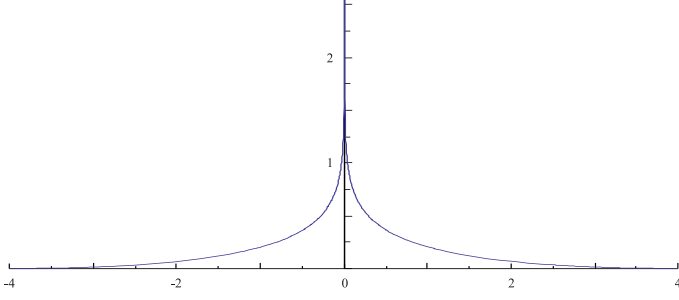


Figure 20: weight for ν_{ρ_1} for $SO(4)$.

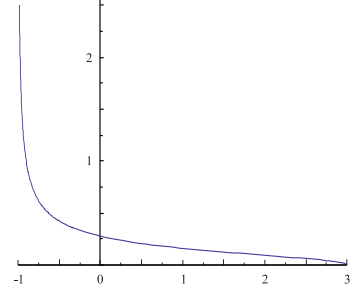


Figure 21: weight for ν_{ρ_2} for $SO(4)$.

Theorem 8.9. The spectral measure of the infinite graphs $\mathcal{A}^{(\infty\text{even})^*}$, $\mathcal{A}^{(\infty\text{odd})^*}$, (over \mathbb{T}), where the distinguished vertex $*$ is chosen to be the vertex with lowest Perron-Frobenius weight, is

$$d\varepsilon(u) = \alpha(u)du, \quad (21)$$

where du is the uniform Lebesgue measure over \mathbb{T} , and

$$\alpha(u) = \begin{cases} 2\text{Im}(u)^2 & \text{for } \mathcal{A}^{(\infty\text{even})^*}, \\ 1 - \text{Re}(u) & \text{for } \mathcal{A}^{(\infty\text{odd})^*}. \end{cases}$$

Over their spectrum $[-1, 3]$, the corresponding spectral measures ν_{even} , ν_{odd} of the infinite graphs $\mathcal{A}^{(\infty\text{even})^*}$, $\mathcal{A}^{(\infty\text{odd})^*}$ respectively, are

$$d\nu_{\text{even}}(y) = \frac{1}{2\pi} \sqrt{(3-y)(y+1)} dy, \quad d\nu_{\text{odd}}(y) = \frac{1}{2\pi} \frac{\sqrt{3-y}}{\sqrt{y+1}} dy. \quad (22)$$

Proof: For $\mathcal{A}^{(\infty\text{even})^*}$ the result is given in [10, §7.3]. For $\mathcal{A}^{(\infty\text{odd})^*}$, with the change of variables $y = 2 \cos(2\pi\theta) + 1$ as in [10, §7.3], where $u = e^{2\pi i\theta}$, we have $1 - \cos(2\pi\theta) = (3-y)/2$, and the result follows in a similar way to that for $\mathcal{A}^{(\infty\text{even})^*}$. \square

Thus we obtain the following result:

Theorem 8.10. The spectral measure ν_{ρ_2} (over $\chi_{\rho_2}(\mathbb{T}^2) = [-1, 3]$) for the graph $\mathcal{G}_{\rho_2}^{D_4^{(2)}}$ for $SO(4)$ is given by

$$d\nu_{\rho_2}(y) = \frac{1}{2\pi} \frac{\sqrt{3-y}}{\sqrt{y+1}} dy.$$

The weights for the measures ν_{ρ_i} , $i = 1, 2$, are illustrated in Figure ??.

9 $D_6^{(1)}$: $PSU(3)$

The subgroup $D_6^{(1)} \in GL(2, \mathbb{Z})$ is generated by the matrices T_3 , T'_2 of Section 1. It is the Weyl group for the connected compact Lie group $PSU(3)$. We choose fundamental generators $\rho_1 = (1, 0)$, $\rho_2 = (1, 1)$. The graphs $\mathcal{G}_\rho^{D_6^{(1)}}$, $\mathcal{H}_\rho^{D_6^{(1)}}$ for $\rho = \rho_1, \rho_2$ are illustrated

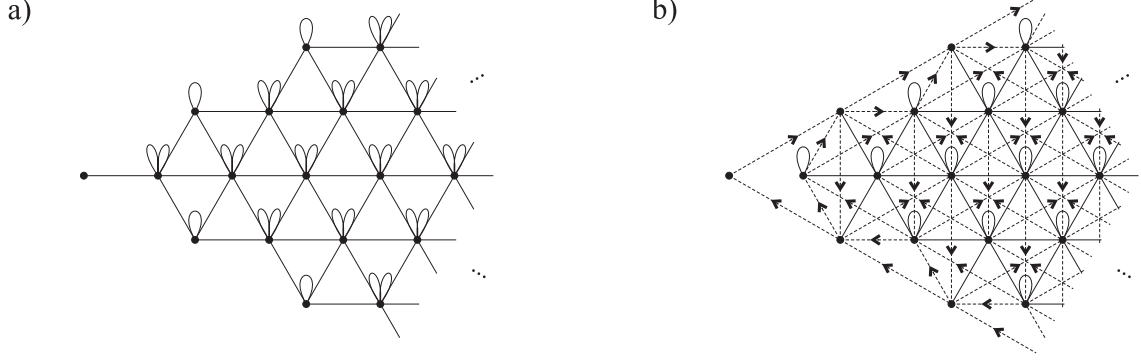


Figure 22: a) Infinite graph $\mathcal{G}_{\rho_1}^{D_6^{(1)}}$ for $PSU(3)$; b) Infinite graph $\mathcal{G}_{\rho_2}^{D_6^{(1)}}$ for $PSU(3)$

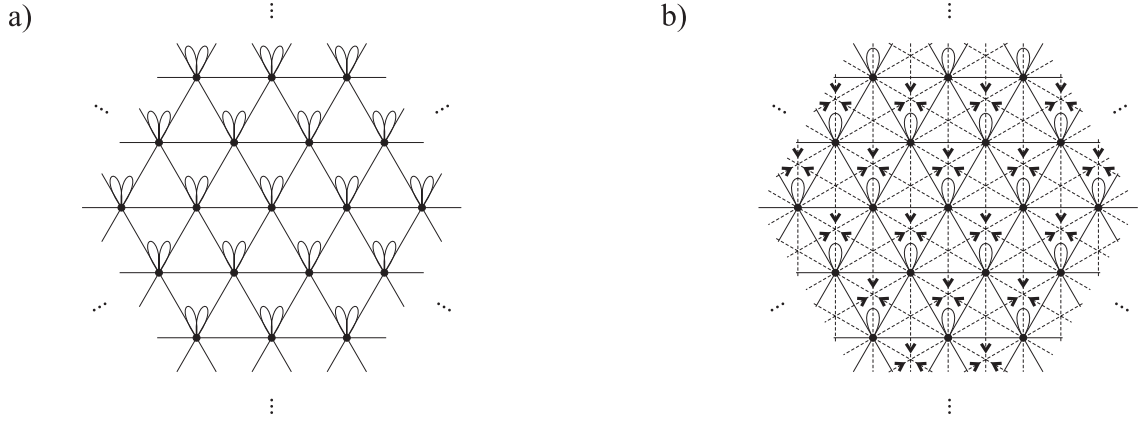


Figure 23: a) Infinite graph $\mathcal{H}_{\rho_1}^{D_6^{(1)}}$ for $PSU(3)$; b) Infinite graph $\mathcal{H}_{\rho_2}^{D_6^{(1)}}$ for $PSU(3)$

in Figures 22-23. The dashed lines in Figures 22(b) and 23(b) are directed edges, whilst the solid lines are undirected edges. For $\omega_j = e^{2\pi i \theta_j} \in \mathbb{T}$, $j = 1, 2$, let

$$\begin{aligned} x &:= x_{\rho_1} = 2 + \omega_1 + \omega_1^{-1} + \omega_2 + \omega_2^{-1} + \omega_1 \omega_2^{-1} + \omega_1^{-1} \omega_2 \\ &= 2 + 2 \cos(2\pi \theta_1) + 2 \cos(2\pi \theta_2) + 2 \cos(2\pi(\theta_1 - \theta_2)), \end{aligned} \quad (23)$$

$$\begin{aligned} y &:= x_{\rho_2} = 1 + \omega_1 \omega_2 + \omega_1 \omega_2^{-2} + \omega_1^{-2} \omega_2 + \omega_1 + \omega_1^{-1} + \omega_2 + \omega_2^{-1} + \omega_1 \omega_2^{-1} + \omega_1^{-1} \omega_2 \\ &= 1 + e^{2\pi i(\theta_1 + \theta_2)} + e^{2\pi i(\theta_1 - 2\theta_2)} + e^{2\pi i(-2\theta_1 + \theta_2)} + 2 \cos(2\pi \theta_1) + 2 \cos(2\pi \theta_2) + 2 \cos(2\pi(\theta_1 - \theta_2)), \end{aligned} \quad (24)$$

and denote by Ψ be the map $\Psi_{\rho_1, \rho_2} : (\omega_1, \omega_2) \mapsto (x, y)$.

A fundamental domain of $\mathbb{T}^2/D_6^{(1)}$ is illustrated in Figure 24(a), where the boundaries marked by arrows are identified. Then $\mathbb{T}^2/D_6^{(1)}$ is a surface, illustrated in Figure 24(b). It is the surface of a cone, with two singular points, one at the apex of the cone, and one on its boundary at the bottom, which is the only boundary of the surface. The boundary of C given by $\theta_2 = 0$, where $\theta_1 \in [0, 1/2]$, yields the curve c_1 given by the parametric equations $x = 4 \cos(2\pi \theta_1) + 4$, $y = e^{-4\pi i \theta_1} + 2e^{2\pi i \theta_1} + 4 \cos(4\pi \theta_1) + 3$, where $x \in [0, 8]$ for

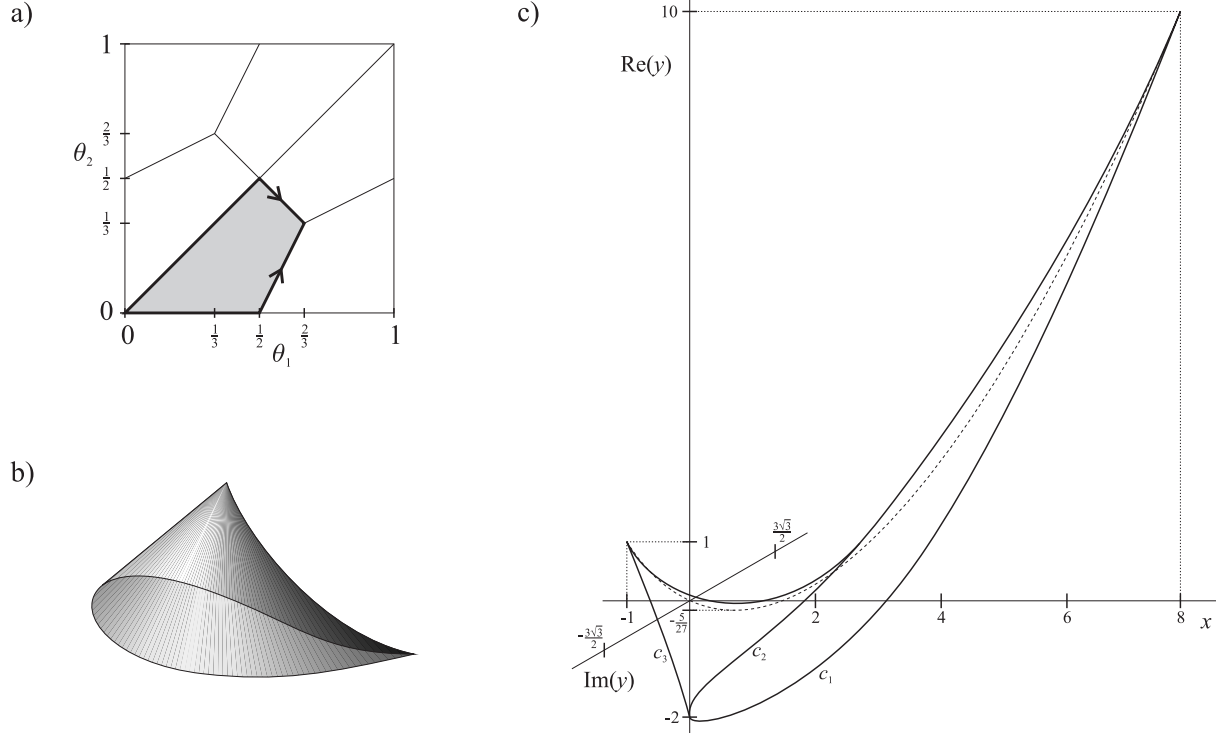


Figure 24: $PSU(3)$: a) A fundamental domain C of $\mathbb{T}^2/D_6^{(1)}$; b) The surface $\mathbb{T}^2/D_6^{(1)}$; c) The domain $\mathfrak{D} = \Psi(C)$ for $PSU(3)$.

$\theta_1 \in [0, 1/2]$. We have

$$\begin{aligned} \text{Re}(y) &= (y + \bar{y})/2 = \cos(4\pi\theta_1) + 6 \cos(2\pi\theta_1) + 3 = 2 \cos^2(2\pi\theta_1) + 6 \cos(2\pi\theta_1) + 3 \\ &= (x^2 + 4x - 16)/8, \end{aligned} \quad (25)$$

$$\begin{aligned} \text{Im}(y) &= (y - \bar{y})/2i = \sin(4\pi\theta_1) - 2 \sin(2\pi\theta_1) = 2 \sin(2\pi\theta_2)(\cos(2\pi\theta_1) - 1) \\ &= \pm \sqrt{x(8-x)^3}/8, \end{aligned} \quad (26)$$

where $x(8-x)^3 \geq 0$ for $x \in [0, 8]$. Since $\text{Im}(\Psi(e^{2\pi i/3}, 1)) > 0$, the boundary c_1 of \mathfrak{D} is given by $8y = x^2 + 4x - 16 + i\sqrt{x(8-x)^3}$. Similarly, the boundary of C given by $\theta_1 = \theta_2$, where $\theta_1 \in [0, 1/2]$, yields the curve c_2 given by the same parametric equations except with $y \leftrightarrow \bar{y}$. Thus the boundaries c_1, c_2 of \mathfrak{D} are given by $8y = x^2 + 4x - 16 + i\sqrt{x(8-x)^3}$, $8y = x^2 + 4x - 16 - i\sqrt{x(8-x)^3}$ respectively. The point $(\theta_1, \theta_2) = (2/3, 1/3)$ on the boundary of C maps to $\Psi(e^{-2\pi i/3}, e^{2\pi i/3}) = (-1, 1)$, which is a singular point in \mathfrak{D} . The boundaries of C given by $\theta_2 = -\theta_1$ and $\theta_2 = 2\theta_1$, for $\theta_1 \in [1/2, 2/3]$, are identified, and yield the curve c_3 given by the parametric equations $x = 2 \cos(4\pi\theta_1) + 4 \cos(2\pi\theta_1) + 2 = 4 \cos^2(2\pi\theta_1) + 4 \cos(2\pi\theta_1)$ and $y = 2 \cos(6\pi\theta_1) + 2 \cos(4\pi\theta_1) + 4 \cos(2\pi\theta_1) + 1 = 8 \cos^3(2\pi\theta_1) + 4 \cos^2(2\pi\theta_1) - 2 \cos(2\pi\theta_1)$. Then writing $\cos(2\pi\theta_1)$ in terms of x , we obtain that c_3 maps to $y = -2x - 1 \pm (x+1)^{3/2}$ in \mathfrak{D} , where \pm is taken to be $-$. On c_3 , $y \in \mathbb{R}$. The curve c_3 lies on the interior of the surface \mathfrak{D} , except at the point $(0, -2)$ on c_1, c_2 , and the singular point $(-1, 1)$. Then $\mathfrak{D} = \Psi(C)$ is the surface of the cone illustrated in

Figure 24(c), with boundary given by the curves c_1, c_2 . The equation of the surface is

$$\operatorname{Im}(y)^2 = x^3 - \operatorname{Re}(y)^2 - 4x\operatorname{Re}(y) - x^2 - 2\operatorname{Re}(y) - x. \quad (27)$$

Away from the curve c_3 , $y \in \mathbb{R}$ only for $\theta_1 = 2\theta_2$ in the fundamental domain C , where $\theta_2 \in [0, 1/3]$. Such y is given by the dashed curve in Figure 24(c). This curve is also given by $\theta_2 = -\theta_1$ for $\theta_1 \in [2/3, 1]$ (i.e. in $T_3(C)$), and thus it is given by the same equation $y = -2x - 1 \pm (x + 1)^{3/2}$, where now \pm is taken to be $+$.

We can also write the curves c_i as functions of y . For c_1 and c_2 , we have from (25) and $x \geq -1$ that $x = 2(-1 + \sqrt{5 + \operatorname{Re}(y)})$. For $\theta_2 = -\theta_1$, $\theta_2 = 2\theta_1$ and $\theta_1 = 2\theta_2$, i.e. for $y \in \mathbb{R}$, the parametric equation for y is a cubic in $\cos(2\pi\theta_1)$. Solving this cubic we obtain

$$\begin{aligned} x &= 4p_2(y) + 4p_2(y)^2, & y &\in [-2, 1], \\ x &= 4p_3(y) + 4p_3(y)^2, & y &\in [-5/27, 1], \\ x &= 4p_1(y) + 4p_1(y)^2, & y &\in [-5/27, 10], \end{aligned}$$

where p_i is given by $6p_i(y) = -1 + 2^{-1/3}\epsilon_i P + 2^{-1/3}4\bar{\epsilon}_i P^{-1}$, for $\epsilon_j = e^{2\pi i(j-1)/3}$, $P = (27y - 11 + \sqrt{3^3(27y^2 - 22y - 5)})^{1/3}$. The first equation above is the equation for the curve c_3 , whilst the latter two are the equations for the dashed curve in Figure 24(c). The surface \mathfrak{D} is the joint spectrum $\sigma(D_6^{(1)}\Delta_{\rho_1}, D_6^{(1)}\Delta_{\rho_2})$ of the commuting normal operators $D_6^{(1)}\Delta_{\rho_1}, D_6^{(1)}\Delta_{\rho_2}$. In fact, $D_6^{(1)}\Delta_{\rho_1}$ is a self-adjoint operator.

For the change of variables (23), (24), the Jacobian is given by $J_{D_6^{(1)}} = -4\pi^2\omega_1^{-3}\omega_2^{-3}(1 - \omega_1)(1 - \omega_2)(\omega_1 - \omega_2)(\omega_1 + \omega_2 + \omega_1\omega_2)^3$. The Jacobian is complex and vanishes in \mathbb{T}^2 only on the images under $D_6^{(1)}$ of the boundaries of the fundamental domain corresponding to the curves c_1 and c_2 in \mathfrak{D} , and the singular point $(2/3, 1/3)$. Note that $J_{D_6^{(1)}} \neq 0$ on the lines $\theta_2 = -\theta_1$ and $\theta_2 = 2\theta_1$, for $\theta_1 \in [1/2, 2/3]$ – which correspond to the curve c_3 in \mathfrak{D} – except at the endpoints $(1/2, 1/2)$ and $(2/3, 1/3)$, although $\operatorname{Re}(J_{D_6^{(1)}}) = -4\pi^2(\cos(2\pi(\theta_1 - 3\theta_2)) + \cos(2\pi(2\theta_1 - 3\theta_2)) - \cos(2\pi(3\theta_1 - 2\theta_2)) + \cos(2\pi(3\theta_1 - \theta_2)) - \cos(2\pi(2\theta_1 + \theta_2)) + \cos(2\pi(\theta_1 + 2\theta_2))) = 0$ on these lines. Again, $J_{D_6^{(1)}}^2$ is invariant under the action of $D_6^{(1)}$ on \mathbb{T}^2 , and can be written in terms of the $D_6^{(1)}$ -invariant elements x, y as $J_{D_6^{(1)}}^2 = -16\pi^4(y+2x+1)(16+24x-13x^2+2x^3+16y-4xy-x^2y+4y^2) = -\pi^4(y+2x+1)(8y-x^2-4x+16+i\sqrt{x(8-x)^3})(8y-x^2-4x+16-i\sqrt{x(8-x)^3})$. Thus we can write J in terms of x, y as $J_{D_6^{(1)}} = 4\pi^2 i \sqrt{(y+2x+1)(16+24x-13x^2+2x^3+16y-4xy-x^2y+4y^2)} = \pi^2 i \sqrt{(y+2x+1)(8y-x^2-4x+16+\sqrt{Q})(8y-x^2-4x+16-\sqrt{Q})}$, where $Q = x(x-8)^3$. We have $Q \geq 0$ for $x \in [-1, 0]$, whereas for $x \in [0, 8]$, $Q \leq 0$ so that \sqrt{Q} is purely imaginary. The Jacobian $J_{D_6^{(1)}}$ is a cubic in y , with two of the three roots appearing as the equations of the boundaries c_1, c_2 of \mathfrak{D} . The third root intersects with \mathfrak{D} only at the singular point $(-1, 1)$.

9.1 Spectral measure for $\mathcal{H}_\rho^{D_6^{(1)}}$ for $PSU(3)$

We first determine the spectral measure ν_{ρ_1} (over $\sigma_{\rho_1} = [-1, 8]$) for $D_6^{(1)}\Delta_{\rho_1}$. For the change of variables $\mathbb{T}^2 \ni (e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \mapsto (x, y_1)$, where $y_1 = \operatorname{Re}(y)$, the Jacobian J_{x, y_1} is given

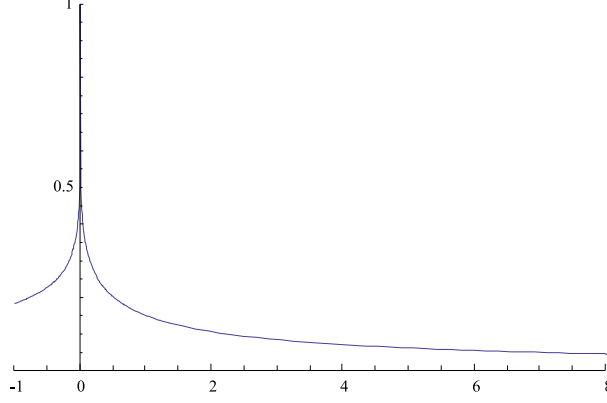


Figure 25: weight $\mathfrak{J}_1(x)$ for ν_{ρ_1} for $PSU(3)$.

by

$$\begin{aligned}
J_{x,y_1}(\theta_1, \theta_2) &= 4\pi^2 (\cos(2\pi(\theta_1 + 2\theta_2)) + \cos(2\pi(2\theta_1 - 3\theta_2)) + \cos(2\pi(3\theta_1 - \theta_2)) \\
&\quad - \cos(2\pi(2\theta_1 + \theta_2)) - \cos(2\pi(3\theta_1 - 2\theta_2)) - \cos(2\pi(\theta_1 - 3\theta_2))), \\
J_{x,y_1}(x, y_1) &= 4\pi^2 \sqrt{(x^2 + 4x - 16 - 8y_1)(y_1^2 - x^3 + x^2 + x + 2y_1 + 4xy_1)}.
\end{aligned}$$

The map $\mathbb{T}^2 \ni (\omega_1, \omega_2) \mapsto (x, y_1)$ is a twelve-to-one map, and $J_{x,y_1}(x, y_1)$ is invariant under the dihedral group D_{12} which contains $D_6^{(1)}$ as a normal subgroup. Then by integrating $|D_{12}| |J_{x,y_1}(x, y_1)|^{-1}$ over y_1 we obtain the spectral measure ν_{ρ_1} for $D_6^{(1)}\Delta_{\rho_1}$. Thus the spectral measure ν_{ρ_1} (over $[-1, 8]$) is $d\nu_{\rho_1}(x) = \mathfrak{J}_1(x) dx$, where $\mathfrak{J}_1(x)$ is given by

$$\begin{aligned}
&12 \int_{-2x-1-\sqrt{x+1}^3}^{-2x-1+\sqrt{x+1}^3} |J_{x,y_1}(x, y_1)|^{-1} dy_1 \text{ for } x \in [-1, 0], \\
&12 \int_{(x^2+4x-19)/8}^{-2x-1+\sqrt{x+1}^3} |J_{x,y_1}(x, y_1)|^{-1} dy_1 \text{ for } x \in (0, 8].
\end{aligned}$$

The weight $\mathfrak{J}_1(x)$ is an integral of the reciprocal of the square root of a cubic in y , and thus can be written in terms of the complete elliptic integral $K(m)$ of the first kind. Using [7, equation 235.00], we obtain

$$\mathfrak{J}_1(x) = \begin{cases} \frac{6}{\pi^2 \sqrt{8 - 20x - x^2 + 8\sqrt{x+1}^3}} K(v(x)) & \text{for } x \in [-1, 0], \\ \frac{3}{2\pi^2 \sqrt[4]{x+1}^3} K(v(x)^{-1}) & \text{for } x \in (0, 8], \end{cases},$$

where $v(x) = 16\sqrt{x+1}^3 / (8 - 20x - x^2 + 8\sqrt{x+1}^3)$. The weight $\mathfrak{J}_1(x)$ is illustrated in Figure 25.

We now consider the spectral measure ν_{ρ_2} (over $\chi_{\rho_2}(\mathbb{T}^2)$) for $D_6^{(1)}\Delta_{\rho_2}$. The spectrum $\sigma_{\rho_2} = \chi_{\rho_2}(\mathbb{T}^2)$ of ρ_2 is given by the region $\mathfrak{D}_{1,\rho_2} \subset \mathbb{C}$, illustrated in Figure ??(a), whose boundary can be obtained from equations (25), (26) as $\text{Im}(y) = \pm \sqrt{(a(y) - 1)(5 - a(y))^3}$,

where $a(y) = \sqrt{5 + 2\operatorname{Re}(y)}$. Let y_1, y_2 denote the real and imaginary parts of y respectively. The equation (27) of the surface \mathfrak{D} is cubic in x , thus for any pair (y_1, y_2) there are at most 3 values of x for which $(x, y_1, y_2) \in \mathfrak{D}$. Denote by \mathfrak{D}' the subregion of \mathfrak{D}_{1,ρ_2} illustrated in Figure ??(a), whose boundaries are obtained when the discriminant $-256(1 + 3y_1)^3 + (11 + 90y_1 + 27y_1^2 + 27y_2^2)^2$ of the cubic polynomial (27) in x is zero, i.e. by $y_2 = \frac{\varepsilon}{3\sqrt{3}}\sqrt{-11 - 90y_1 - 27y_1^2 + \varepsilon'16\sqrt{1 + 3y_1}^3}$ for $\varepsilon, \varepsilon' \in \{\pm 1\}$. Consider the restriction $\Upsilon = \Psi_{1,\rho_2}|_C$ to the fundamental domain C of the map $\Psi_{1,\rho_2} : \mathbb{T}^2 \rightarrow \mathfrak{D}_{1,\rho_2}$. For $y \in \mathfrak{D}'$, its pre-image $\Upsilon^{-1}(y)$ in C contains three distinct points, whilst for $y \in \mathfrak{D}_{1,\rho_2} \setminus \mathfrak{D}'$ its pre-image $\Upsilon^{-1}(y)$ is unique.

For the change of variables $\mathbb{T}^2 \ni (e^{2\pi i\theta_1}, e^{2\pi i\theta_2}) \mapsto (y_1, y_2)$ the Jacobian J_{y_1,y_2} is given by $J_{y_1,y_2}(\theta_1, \theta_2) = 4\pi^2(2(\sin(2\pi(\theta_1 - \theta_2)) - \sin(2\pi\theta_1) + \sin(2\pi\theta_2)) + 4(\sin(4\pi(\theta_1 - \theta_2)) - \sin(4\pi\theta_1) + \sin(4\pi\theta_2)) + 3(\sin(6\pi(\theta_1 - \theta_2)) - \sin(6\pi\theta_1) + \sin(6\pi\theta_2)) + \sin(2\pi(\theta_1 + 2\theta_2)) + \sin(2\pi(2\theta_1 - 3\theta_2)) + \sin(2\pi(3\theta_1 - 2\theta_2)) - \sin(2\pi(2\theta_1 + \theta_2)) - \sin(2\pi(\theta_1 - 3\theta_2)) + \sin(2\pi(3\theta_1 - \theta_2)))$, which can be written in terms of the three $D_6^{(1)}$ -invariant variables x, y_1, y_2 as

$$J_{y_1,y_2}(x, y_1, y_2) = 2\sqrt{2}\pi^2\sqrt{b(x, y_1, y_2)},$$

where $b(x, y_1, y_2) = 16 - 9x + 34x^2 - 2y_1 + 178xy_1 + 24x^2y_1 + 39y_1^2 - 97xy_1^2 - 10x^2y_1^2 + 4y_1^3 + 24xy_1^3 - 9y_1^4 - 69y_2^2 + 143xy_2^2 - 42x^2y_2^2 - 124y_1y_2^2 + 24xy_1y_2^2 - 18y_1^2y_2^2 - 9y_2^4$. The solutions of (27) for x are

$$6x_j = 2 - 2^{2/3}\epsilon_j P_x - 82^{1/3}\bar{\epsilon}_j(1 + 3y_1)P_x^{-1} \quad (28)$$

where $\epsilon_j = e^{2\pi i(j-1)}$, $j = 1, 2, 3$, and

$$P_x = \left(-11 - 90y_1 - 27y_1^2 - 27y_2^2 + \sqrt{-256(1 + 3y_1)^3 + (11 + 90y_1 + 27y_1^2 + 27y_2^2)^2} \right)^{1/3}.$$

Then the spectral measure ν_{ρ_2} (over $\sigma_{\rho_2} = \mathfrak{D}_{1,\rho_2}$) is $d\nu_{\rho_2}(y) = \mathfrak{J}_2(y_1, y_2) dy_1 dy_2$, where $\mathfrak{J}_2(y_1, y_2)$ is given by

$$\mathfrak{J}_2(y_1, y_2) = \begin{cases} |J_{y_1,y_2}(x_1, y_1, y_2)|^{-1} & \text{for } y \in A, \\ |J_{y_1,y_2}(x_2, y_1, y_2)|^{-1} & \text{for } y \in B, \\ (\sum_{j=1}^3 |J_{y_1,y_2}(x_j, y_1, y_2)|)^{-1} & \text{for } y \in \mathfrak{D}', \end{cases} \quad (29)$$

where region A in Figure ??(a) is given by $y \in \mathfrak{D}_{1,\rho_2}$ such that $y_1 < -1/3$, and $27y_2^2 < -11 - 90y_1 - 27y_1^2 - 16\sqrt{1 + 3y_1}^3$ for $-1/3 \leq y_1 < -5/27$; region B by $27y_2^2 > -11 - 90y_1 - 27y_1^2 + 16\sqrt{1 + 3y_1}^3$ for $-1/3 \leq y_1 \leq 1$, and $y_1 > 1$. The weight $\mathfrak{J}_2(y_1, y_2)$ is illustrated in Figure ??(b).

Theorem 9.1. The spectral measures $\nu_{\rho_1}, \nu_{\rho_2}$ (over $\chi_{\rho_1}(\mathbb{T}^2) = [-1, 8]$, $\chi_{\rho_2}(\mathbb{T}^2) = \mathfrak{D}_{1,\rho_2}$ respectively) for the graphs $\mathcal{H}_{\rho_1}^{D_6^{(1)}}$, $\mathcal{H}_{\rho_2}^{D_6^{(1)}}$ respectively, for $PSU(3)$ are given by

$$d\nu_{\rho_1}(x) = \begin{cases} \frac{6}{\pi^2\sqrt{8 - 20x - x^2 + 8\sqrt{x+1}^3}} K(v(x)) dx & \text{for } x \in [-1, 0], \\ \frac{3}{2\pi^2\sqrt[4]{x+1}^3} K(v(x)^{-1}) dx & \text{for } x \in (0, 8], \end{cases},$$

$$d\nu_{\rho_2}(y) = \mathfrak{J}_2(y_1, y_2) dy_1 dy_2,$$

where $\mathfrak{J}_2(y_1, y_2)$ is given by (29).

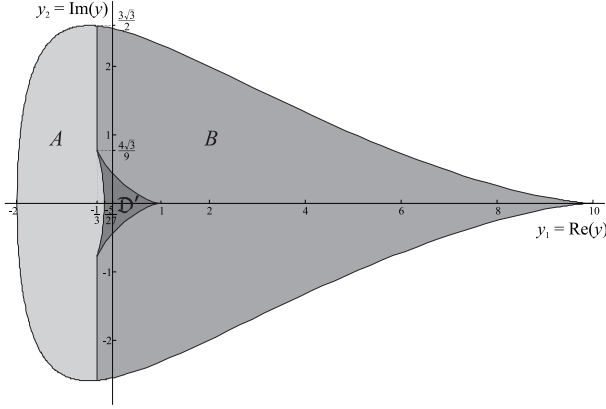


Figure 26: \mathfrak{D}_{1, ρ_2} for $PSU(3)$.

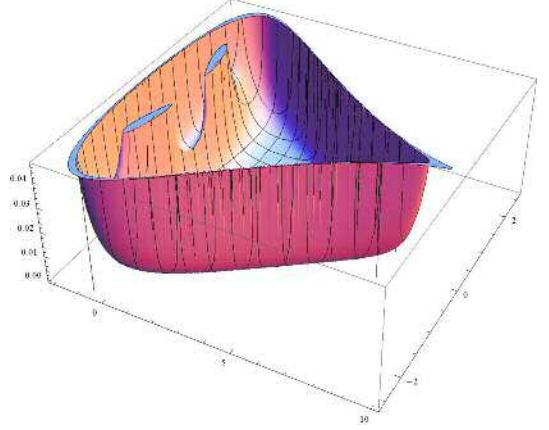


Figure 27: weight $\mathfrak{J}_2(y_1, y_2)$ for ν_{ρ_2} for $PSU(3)$.

Remark 9.2. Similar remarks may be made regarding the relationship between the groups $D_6^{(1)}$ (the Weyl group for $PSU(3)$) and $D_6^{(2)}$ (the Weyl group for $SU(3)$) as were made in Remarks 6.1, 8.2, for the groups $\mathbb{Z}_2^{(2)}$, $\mathbb{Z}_2^{(3)}$ and $D_4^{(1)}$, $D_4^{(2)}$ respectively. The groups $D_6^{(1)}$ and $D_6^{(2)}$ are conjugate in $GL(2, \mathbb{R})$, with conjugating matrix H given by $H = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$ which has inverse $H^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ -2/3 & 1/3 \end{pmatrix} = -\frac{1}{3}H$. As in Remark 6.1, the origin of this relationship is the fact that the compact, connected Lie group $SU(3)$ (which has Weyl group $D_6^{(2)}$) is a triple cover of the compact, connected, but not simply connected Lie group $PSU(3)$. Thus not all irreducible representations of $SU(3)$ are irreducible representations of $PSU(3)$, but only those of triality zero, that is, those indexed by $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$ such that $\lambda_1 - \lambda_2 \equiv 0 \pmod{3}$.

Remark 9.3. There is also a relation between the Lie groups $PSU(3)$ and G_2 , in that $D_6^{(1)}$ (the Weyl group for $PSU(3)$) is a normal subgroup of D_{12} (the Weyl group for G_2). This relation is reflected in the spectral measures for these Lie groups. The spectral measures associated to G_2 were studied in [12]. The fundamental domain F for $D_{12} = D_6^{(1)} \rtimes \mathbb{Z}_2$ in [12, Figure 8] is obtained from the fundamental domain for $D_6^{(1)}$ in Figure 24(a) by imposing one extra symmetry which comes from the additional \mathbb{Z}_2 action. Suppose we denote by x', y' the variables for D_{12} given by

$$\begin{aligned} x' &:= 1 + \omega_1 + \omega_1^{-1} + \omega_2 + \omega_2^{-1} + \omega_1\omega_2^{-1} + \omega_1^{-1}\omega_2 \\ y' &:= x' + 1 + \omega_1\omega_2 + \omega_1^{-1}\omega_2^{-1} + \omega_1^2\omega_2^{-1} + \omega_1^{-2}\omega_2 + \omega_1\omega_2^{-2} + \omega_1^{-1}\omega_2^2. \end{aligned}$$

Then we see that $x' = x - 1$ whilst $y' = 2\text{Re}(y) - x + 2$, and there is a 2-to-1 homomorphism $\xi : \mathfrak{D}_{D_6^{(1)}} \rightarrow \mathfrak{D}_{D_{12}}$ from the domain $\mathfrak{D}_{D_6^{(1)}}$ of $D_6^{(1)}$ to the domain $\mathfrak{D}_{D_{12}}$ of D_{12} such that $\xi(x) = x - 1$ and $\xi(y) = 2\text{Re}(y) - x + 2$, which maps the boundary of $\mathfrak{D}_{D_6^{(1)}}$ to part of the boundary of $\mathfrak{D}_{D_{12}}$. The rest of the boundary of $\mathfrak{D}_{D_{12}}$ is given by $\xi(v)$, where $v = (x, y) \in \mathfrak{D}_{D_6^{(1)}}$ such that $y \in \mathbb{R}$, i.e. $v = \Phi(t)$ for $t \in \mathbb{T}^2$ such that t is fixed under the additional \mathbb{Z}_2 action in D_{12} . Moreover, $J_{D_{12}} = 2\text{Re}(J_{D_6^{(1)}}) = \xi(J_{D_6^{(1)}})$ (c.f. Remark 7.1).

9.2 Spectral measure for $\mathcal{G}_\rho^{D_6^{(1)}}$ for $PSU(3)$

In this section we make the assumption that Conjecture 3.3 in Section 3.2 holds for the joint spectral measure of the pair of graphs $\mathcal{H}_{\rho_1}^{D_6^{(1)}}$, $\mathcal{H}_{\rho_2}^{D_6^{(1)}}$. Then we determine from this the spectral measures ν_{ρ_1} , ν_{ρ_2} (over $\chi_{\rho_1}(\mathbb{T}^2) = [-1, 8]$, $\chi_{\rho_2}(\mathbb{T}^2) = \mathfrak{D}_{1,\rho_2}$ respectively) for the graphs $\mathcal{H}_{\rho_1}^{D_6^{(1)}}$, $\mathcal{H}_{\rho_2}^{D_6^{(1)}}$ respectively, for $PSU(3)$. We have $S_{(1,0)}(\theta) = \omega_1 - \omega_1^{-1} - \omega_2 + \omega_2^{-1} + \omega_1^{-1}\omega_2 - \omega_1\omega_2^{-1}$.

We consider first the spectral measure ν_{ρ_1} . Now $S_\rho(\theta) = S_{(1,0)}(\theta)$ may be written in terms of the variables x , y_1 as $S_{(1,0)}(x, y_1) = \sqrt{8y_1 - x^2 - 4x + 16}$. Then the spectral measure ν_{ρ_1} (over $[-1, 8]$) is given by $d\nu_{\rho_1}(x) = \mathfrak{J}_3(x) dx$, where $\mathfrak{J}_3(x)$ is given by

$$4\sqrt{2} \int_{-2x-1-\sqrt{x+1}^3}^{-2x-1+\sqrt{x+1}^3} \frac{\sqrt{8y_1 - x^2 - 4x + 16}}{\sqrt{x^3 - x^2 - x - y_1^2 - 2y_1 - 4xy_1}} dy_1 \quad \text{for } x \in [-1, 0],$$

$$4\sqrt{2} \int_{(x^2+4x-19)/8}^{-2x-1+\sqrt{x+1}^3} \frac{\sqrt{8y_1 - x^2 - 4x + 16}}{\sqrt{x^3 - x^2 - x - y_1^2 - 2y_1 - 4xy_1}} dy_1 \quad \text{for } x \in (0, 8].$$

The weight $\mathfrak{J}_3(x)$ can be written in terms of the complete elliptic integrals $K(m)$, $E(m)$ of the first, second kind respectively. Using [7, equation 235.05, 235.06], we obtain that $\mathfrak{J}_3(x)$ is given by

$$\frac{\sqrt{8 - 20x - x^2 + 8\sqrt{x+1}^3}}{\pi^2} E(v(x)) \quad \text{for } x \in [-1, 0],$$

$$\frac{1}{4\pi^2} \left(16\sqrt[4]{x+1}^3 E(v(x)^{-1}) + \frac{\sqrt{8 - 20x - x^2 - 8\sqrt{x+1}^3}}{\sqrt[4]{x+1}^3} K(v(x)^{-1}) \right) \quad \text{for } x \in (0, 8],$$

where $v(x) = 16\sqrt{x+1}^3 / (8 - 20x - x^2 + 8\sqrt{x+1}^3)$. The weight $\mathfrak{J}_3(x)$ is illustrated in Figure ??(a). It has been verified using Mathematica that the first 10 moments of this measure are correct.

We now turn to the spectral measure ν_{ρ_2} (over $\sigma_{\rho_2} = \mathfrak{D}_{1,\rho_2}$), which is given by $d\nu_{\rho_2}(y) = \mathfrak{J}_4(y_1, y_2) dy_1 dy_2$, where $\mathfrak{J}_4(y_1, y_2)$ is given by

$$\mathfrak{J}_4(y_1, y_2) = \begin{cases} |S_{(1,0)}(x_1, y_1)|^2 / |J_{y_1, y_2}(x_1, y_1, y_2)| & \text{for } y \in A, \\ |S_{(1,0)}(x_2, y_1)|^2 / |J_{y_1, y_2}(x_2, y_1, y_2)| & \text{for } y \in B, \\ \sum_{j=1}^3 |S_{(1,0)}(x_j, y_1)|^2 / |J_{y_1, y_2}(x_j, y_1, y_2)| & \text{for } y \in \mathfrak{D}', \end{cases} \quad (30)$$

where the regions A , B and \mathfrak{D}' are as in Section 9.1, and x_j are as given by (28). The weight $\mathfrak{J}_4(y_1, y_2)$ is illustrated in Figure ??(b).

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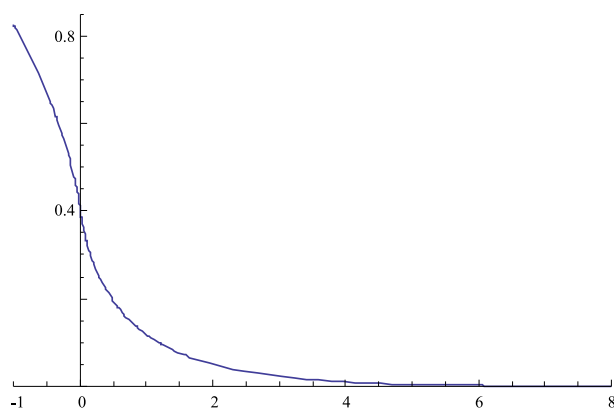


Figure 28: weight $\mathfrak{J}_3(x)$ for ν_{ρ_1} for $PSU(3)$.

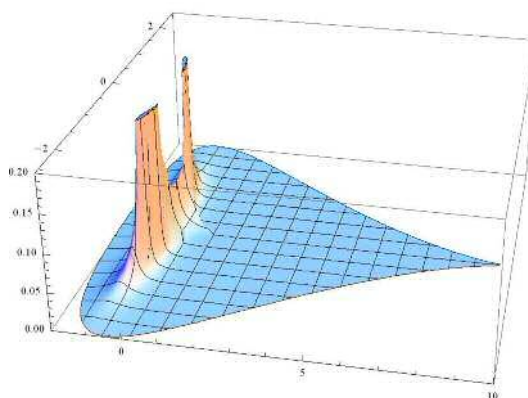


Figure 29: weight $\mathfrak{J}_4(y_1, y_2)$ for ν_{ρ_2} for $PSU(3)$.

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