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On a Derivation of the Boltzmann Equation in Quantum Field Theory

by

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Thesis submitted for the degree of
Doctor of Philosophy in Mathematics

Cardiff University
School of Mathematics

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Hardly any equation in theoretical physics has evoked as much discussion and controversy as the Boltzmann equation.

N. M. Hugenholtz

Abstract

The Boltzmann equation (BE) is a commonly used tool for the study of non-equilibrium many particle systems. It has been introduced in 1872 by Ludwig Boltzmann and has been widely generalized throughout the years. Today it is commonly used in physical applications, from the study of ordinary fluids to problems in particle Cosmology where Quantum Field Theoretical techniques are essential. Despite its numerous experimental successes, the conceptual basis of the BE is not entirely clear. For instance, it is well known that it is not a fundamental equation of physics like, say, the Heisenberg equation (HE). A natural question then arises whether it is possible to derive the BE from physical first principles, i.e. the Heisenberg equation in Quantum Field Theory.

In this work we attempted to answer this question and succeeded in deriving the BE from the HE, thus further clarifying its conceptual status. In particular, the results we have obtained are as follows. Firstly, we establish the non-perturbative validity of what we call the “pre-Boltzmann equation”. The crucial point here is that this latter equation is equivalent to the Heisenberg equation. Secondly, we proceed to consider various limits of the pre-Boltzmann equation, namely the “low density” and the “weak coupling” limits, to obtain two equations that can be considered as generalizations of the BE. These limits are always taken together with the “long time” limit, which allows us to interpret the BE as an appropriate long time limit of the HE. The generalization we obtain consists in additional “correction” terms to the usual Boltzmann collision factor, and can be associated to multiple particle scattering. Unlike the pre-Boltzmann equation, these latter results are only valid perturbatively. Finally, we briefly consider the possibility to extend these results beyond said limits and outline some important aspects in this case.

Verba volant, scripta manent.

Acknowledgments

This is the space which the author of a written work commonly uses to thank those who, in one way or another, have helped him or her complete the work, and this is indeed what I will do. But before coming to that I want to reclaim the ownership of a space that is too frequently too “academically correct” to feel authentic. First and foremost I want to use this space to express my own pride and satisfaction with this work in a way which will outlast a polite conversation with someone “inteterested”.

I started my doctoral studies with a specific goal: To obtain a solid and as broad as possible understanding of Quantum Field Theory rather than the focused and specific knowledge which is inevitably associated with a research project. And this had to be from the viewpoint of physical applications, while still catching at least a glimpse of the many mathematical problems behind it. After all has been set and done I can genuinely say that I have obtained exactly what I had wanted. Working on the Bltzmann equation allowed me to get a feel for the breadth and scope of applications of QFT, which goes way beyond the “Shut up, renormalize and calculate Feynman diagrams!” approach that subtly underlies much of the physics literature on the subject.

Of course, little did I know about what I was really embarking on. I had no idea about how hard and frustrating it would turn out to be, a feeling I found particularly common amongst fellow QFT students. I remeber reading about “QFT being the hardest subject in physics” and thinking it surely cannot be particularly harder than General Relativity. It is. By far. Had I known all of this before, I would have probably not started it. Which is why, despite the inevitable confilcts two intellectually proud persons are bound to have, I want to thank my research advisor, Stefan Hollands, for what he did. It is not an understatement to say that without his support I would have not been able to complete my studies. At least not as successfully as I did. And my understanding of QFT (physical and mathematical) would have not increased by “at least an order of magnitude”, as a past lecturer remarked after I presented my work at a seminar. And I also want to thank Christian Jäkel for contributing to my QFT understanding, and for the numerous “off-topic” conversations which we had.

But my studies were not research only. Without mentioning anyone but my last year’s flatmate and fellow QFT student Heiner Olbermann, I want to thank all my friends—undergrads, postgrad students and postdocs—you know who you are, for the “non-academic” part of the last few years. It was a lot of fun.

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Chapter 1

Introduction

The study and description of non-equilibrium phenomena in physics has always represented a big challenge for physicists and mathematicians alike. This is particularly true for thermodynamical systems which at equilibrium can be completely described by a few macroscopic parameters. For instance, a sufficiently dilute gas in thermal equilibrium is completely described by just three parameters: The pressure, the volume and the temperature of the system. This simplicity is reflected in the mathematical description of the system. If P denotes the pressure of the gas, V its volume and T the temperature, then its equation of state is given by

$$PV = Nk_B T ,$$

where k_B is a universal constant known as the “Boltzmann constant” and N is the number of molecules of a specific gas. Not only is the above relation extremely simple, but it also holds for any gas, irrespectively of the species of the gas, as long as it is sufficiently dilute.

This simplicity remains even if we abandon a thermodynamical description¹ of macroscopic systems. Without entering the domain of quantum mechanics (yet), and by adopting a heuristic, physical point of view, a statistical description of a gas at equilibrium is given in terms of its “equilibrium distribution function” $f_{\mathbf{p}}$

$$f_{\mathbf{p}} \propto e^{-\omega_{\mathbf{p}}/k_B T} ,$$

¹Thermodynamics is a “phenomenological” theory of matter in the sense that it draws its concepts directly from physical experiments. This is in sharp contrast to “fundamental” theories that start with a set of assumed first principles and try to explain Nature starting from the latter. Deriving a successful phenomenological theory from first principles is then one of the conceptual question that ought to be answered, and is the key idea of this work.

with $\omega_{\mathbf{p}} = \mathbf{p}^2/2$ being the energy of a single molecule. The physical meaning of the equilibrium distribution function is that of a “particle number density”. That is, $f_{\mathbf{p}}d\mathbf{p}$ represents the number of particles of a gas in thermal equilibrium with momentum in the interval $[\mathbf{p} - d\mathbf{p}/2, \mathbf{p} + d\mathbf{p}/2]$. Again, however, what is important to note is that any gas at equilibrium will have the above distribution function², which is known as the “Maxwell–Boltzmann” distribution function.

This is in stark contrast with the remarkable complexity of non-equilibrium systems which, in general, do not share any common unifying condition. In his attempt to shed some light on the problem, in 1873 Ludwig Boltzmann [1] derived an equation that now carries his name, i.e. the Boltzmann equation (BE), which allowed to study the approach to equilibrium of a certain class of systems

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} + \mathbf{F} \cdot \nabla_{\mathbf{p}}\right) f_{\mathbf{p}}(t, \mathbf{x}) = \int_{\mathbb{R}^9} d^3\mathbf{p}_2 d^3\mathbf{q}_1 d^3\mathbf{q}_2 \delta^{(4)}(\underline{p} + \underline{p}_1 - \underline{q}_1 - \underline{q}_2) \cdot \\ \cdot \left| \mathcal{M}(\mathbf{p}, \mathbf{p}_1 \leftrightarrow \mathbf{q}_1, \mathbf{q}_2) \right|^2 \left[f_{\mathbf{q}_1}(t, \mathbf{x}) f_{\mathbf{q}_2}(t, \mathbf{x}) - f_{\mathbf{p}}(t, \mathbf{x}) f_{\mathbf{p}_2}(t, \mathbf{x}) \right].$$

In the above equation the 4-dimensional “delta function” enforces energy and momentum conservation, and for simplicity we are also using a relativistic notation to denote both energy and momentum, i.e. $\underline{p} = (\omega, \mathbf{p})$.

It is a non-linear integro–differential equation for the particle distribution function $f_{\mathbf{p}}(t, \mathbf{x})$, which now depends on the position and time in addition to the momentum. From a physical point of view it relates the “thermalization” of a system to the interaction between particles as described by the “scattering amplitude” \mathcal{M} , and as we will extensively point out throughout this work, its derivation requires a number of assumptions. This latter aspect in particular entails that it cannot be used to study every possible out-of-equilibrium system. In addition, its mathematical form remarkably complicates its solution. Stated otherwise, it is a very complicated equation that can only be used to study a relatively small number of physical phenomena. This is what we meant by saying that non-equilibrium systems are many and do not share any common features.

The experimental success of the BE was such that it has been promptly generalized so it would be suited for the description of (general) relativistic effects as well as to account for the quantum nature of the phenomena. This is reflected also by its wide use to study many different physical applications that includes the simple diffusion of classical gasses, the computation of viscosity coefficients, see [2], and its application to

²The equal sign is obtained by multiplying the exponential “Boltzmann weight” by a constant that depends on the specific gas we are considering.

exotic matter states like the quark–gluon plasma, see e.g. [3, 4], as well as the study of the relic abundances of the lightest elements or the understanding of the observed matter–anti-matter asymmetry. For the latter two applications and for a generic introduction to the use of the BE in cosmology see [5].

Boltzmann’s original derivation adopted a heuristic viewpoint which could be understood as being halfway between the phenomenological one of classical thermodynamics, and a fundamental one that would start from first principles as we understand them today. As such, it is not entirely satisfactory from a conceptual point of view. Not only that, one could easily claim that it raises more questions than it answers! This is the meaning of Hugenholtz’s words [6] that open this work. Here we will not spend much time on all the problems that the BE raises and simply mention what we believe to be the most important one. Since it is not a fundamental (or “exact”) physical equation, when does it provide a valid description of the physical phenomena? In principle this question has long been answered and we now know that the BE provides a valid description when studying either a very weakly interacting system or a very dilute one. This is hence a feature we expected to play a crucial role in this work, and indeed it did.

These conclusions are the result of numerous investigations, both for classical and quantum systems. In the classical case, a lot of work has been devoted to gain a deeper understanding of the BE, and this has led to the so called BBGKY³ hierarchy of equations. The BBGKY equations are a chain of equations for the correlation functions ρ_s , which give the probability of finding s particles of our system having specified positions and momenta. It can be then shown that the BE can be obtained by an appropriate truncation of the BBGKY hierarchy. For a short introduction on the BBGKY equations we refer the reader to [7], and to [8] for a detailed exposition. In [6], Hugenholtz derives the non-relativistic quantum BE for a Fermi gas, i.e. a system of particles obeying the Pauli exclusion principle. His derivation employs a perturbative expansion in terms of multiple commutators. Central in his derivation is the “ $\lambda^2 t$ limit”, in which the coupling constant λ is scaled to zero and the time t to infinity in such a way that the product $\lambda^2 t$ remains finite. That is, he obtains the QBE as a “long-time and weak coupling” approximation to the full two point function. A somewhat different derivation is given in [9], where the perturbative expansion is replaced by the assumption of “restricted quasifreeness”, i.e. the assumption that in the weak coupling limit the four-point function and the eight-point function can be factorized in terms of two-point functions. Again the weak coupling and long time limits are crucial to obtain the Boltzmann equation. The review paper by Rau and Müller [10] was also particularly illuminating as it introduced us to

³BBGKY stands for Bogoliubov, Born, Green, Kirkwood and Yvon.

the projection operator technique by employing it to derive various transport equations and the QBE in particular. For further references on the derivation of the BE we refer the reader to this latter paper and the numerous references in it which, however, were not of direct relevance for us.

We also mention the work by Erdős and Yau [11] where they study the time evolution of a quantum particle in a Gaussian random environment. What they show is that in the weak coupling limit the Wigner distribution of the wave function converges to a solution of a linear Boltzmann equation globally in time. This investigation is not particularly relevant for this work, but it shows again that the weak coupling limit is essential to establish a QBE. A common feature shared by all these derivations is that the scattering amplitude in the resulting QBE appears in the Born approximation, that is to the lowest perturbative approximation, a result we improve by obtaining sub-leading corrections in the form of the full scattering matrix. Closely related to this last work is also the work of Eng and Erdős, see [12], where they study the same quantum particle in a Gaussian random environment but consider a different scaling limit, namely the low density limit. The difference between the two scaling limits is that now the collision kernel is given by the full scattering cross section of the obstacle potential. The relevance for our work in this case is the expectation of the emergence of the full scattering matrix when considering the low density limit, which is indeed what we obtain.

As a last note in this brief review of the literature we want to emphasize that this is in no way an extensive review. These are merely the studies that most affected this work. On the other hand, even this short a review would not be complete without mentioning the Kadanoff–Baym equations [13], which are considered the quantum mechanical generalization of the BE, and rely on Green’s function techniques. For a detailed exposition of these equations we recommend the textbook [14] by the same authors. The Kadanoff–Baym equations can be used to derive the QBE after some additional assumptions are made, see e.g. [15], and have even been used to derive a BE on curved spaces in [16]. Finally, for a quantitative comparison between the use of the BE and the Kadanoff–Baym equations for an interacting scalar quantum field see [17].

In this work we have asked the following question: “Is it possible to relate the Boltzmann equation to the physically fundamental Heisenberg equation?”. We have been able to give a positive answer to this question, and to the best of our knowledge this is the first time that a “direct link” between the two is established for an interacting matter model in quantum field theory. To achieve this result we have first of all appropriately modified the so called “projection operator technique”, see [10, 18]. With this tool we were able to decompose the Heisenberg equation into a part which is “parallel”

to a subspace of the algebra of observables, and the complementary, “orthogonal” part. Applying this technique to the Hamiltonian of the $P(\varphi)_2$ quantum field model we then obtain what we called the “pre-Boltzmann equation”, which is one of the main results of this work. The importance of this equation lies in the fact that even though it appears to be completely different from the Heisenberg equation, the two are in fact equivalent. One crucial difference between the two equations is that the pre-Boltzmann equation is not local in time anymore, i.e. the future evolution of an observable now depends not only on its present value but on its entire history from some initial time⁴. This equation is hardly any simpler to solve than the original Heisenberg equation, but it is a much better starting point for approximations. In particular, since we expect the BE to emerge in some appropriate long time limit, the non locality in time actually represents an advantage.

After the derivation of the pre-Boltzmann equation we find that to obtain the BE we must consider either a “long-time and dilute medium” and/or a “long-time and weakly interacting” system limit. Common to both limits is the “long-time limit”, which is what allows us to interpret the BE as a long-time approximation to the Heisenberg equation, thus clearing the conceptual status of the Boltzmann equation. The results we obtain also shed new light on the BE and we obtain some non-trivial and unexpected, albeit physically reasonable results. For instance, we find that the matrix element that appears in the BE appropriate for weakly interacting systems should be computed with the usual Feynman rules but employing a modified propagator, which accounts for the fact that we are not considering a system in the vacuum but in a generic (finite density) state⁵. In addition, we gain a systematic understanding of (non-Markovian) rescattering corrections terms one ought to consider when rescattering plays a crucial role in the physical situation.

Last but not least, we also mention that we were able to adapt the projection method for applications to a time dependent Hamiltonian, as one would have in the presence of an external time varying field. A prominent scenario for the application of this formalism is that of a quantum field propagating on a Robertson–Walker spacetime. In this case we expect additional (curvature related) corrections to the “usual” BE, i.e. to the BE one could find in standard reference textbooks like [5]. In this work we have not done that, but this is certainly relevant as the BE is frequently employed in early Universe cosmology where curvature was non-negligible, and hence curvature corrections might help give additional insight when studying some cosmological puzzles like the previously

⁴Such equations are also known as non-Markovian equations in the literature,

⁵When considering the low density limit, on the other hand, the propagators reduce to the usual vacuum ones.

mentioned matter–anti-matter asymmetry.

This thesis is organized as follows. In the first two chapters after this introductory one we collect some necessary background material that is essential for the proper understanding of this work. In particular, in chapter 2 we give a heuristic derivation of the Boltzmann equation and interpret it as a continuity equation. The exposition in this chapter is the type of exposition one may find in the physics literature, that is emphasis is placed on the intuitive meaning of the concepts rather than on mathematical rigour. Concepts like “particle” and “particle distribution function” are simply understood in a very naive and most intuitive sense. We then generalize the BE from the classical one to a BE as it is used in QFT on curved spaces, and eventually discuss some of the conceptual issues that such a “derivation/generalization” inevitably raises.

In the following chapter 3 we briefly present the second key topic of this work, Quantum Field Theory on the circle, i.e. on the $1 + 1$ dimensional spacetime $\mathbb{R} \times \mathbb{S}$ with metric $ds^2 = -dt^2 + Ldx^2$, and L is the radius of the circle. To this end we first recall some results from the Hamiltonian formulation of the theory of classical fields, and then proceed to “quantize” it. This is first done in a most naive way, which as it will be argued does not result in a mathematically well defined theory, and then rigorously starting from the so-called “Wightman axioms”.

The two following chapters, i.e. chapters 4 and 5, contain the results of this work. Specifically, in chapter 4 we establish a key result, the non-perturbative validity of the “pre-Boltzmann equation”. The first step to establish this result is the appropriate adaptation of the “projection operator method” (see [10] for a review or [18] for a specialized treatise of the method) for our needs and “decompose” the equation of motion. In doing so we also adapt it for a general situation with a time dependent Hamiltonian. We then proceed to obtain⁶ various estimates on the particle number densities, i.e. the expectation values of the number operator densities. These are crucial to establish the rigorous non-perturbative existence of the pre-Boltzmann equation, which we present afterwards.

In the last chapter before the conclusions we then consider various scaling limits of the pre-Boltzmann equation. We firstly give a perturbative expansion of the pre-Boltzmann equation and then consider the thermodynamic limit in which we take the radius of the cylinder to infinity to (formally) “extend” some results to Minkowski space. At this point we can take the “low density” limit first, and “weak coupling” limit later. The two are independent of each other and offer different perspectives on the BE. However, from a physical point of view, the weak coupling limit is richer to discuss. The main reason

⁶These estimates have been obtained by my research advisor. They are included in this work for completeness.

for this is that in order to make sense of the perturbative expansion of the S -matrix in this case we must make a working assumption about the energy of the particles, which will in general be different from the vacuum energy. This fundamentally affects the mass renormalization of the scattering amplitudes and the end result consists in a scattering amplitude that depends on the particle number density, i.e. on the very quantity we are solving the BE for. Needless to say, this additionally complicates the structure of the BE. This thesis ends with chapter 6, where we draw the appropriate conclusions.

Chapter 2

The Boltzmann equation

As already noted in the introduction, the Boltzmann equation (BE) is a standard tool for studying non equilibrium phenomena. In this chapter we heuristically¹ derive it and emphasize its meaning as a “continuity” equation, that is as an equation that states the conservation of a quantity. We then generalize it so it will be appropriate to deal with quantum particles that obey the Bose–Einstein or Fermi–Dirac statistics. As a next step we will present its (special) relativistic generalization first and subsequently extend it to curved spaces. The resulting equation is then appropriate to study non-equilibrium phenomena for relativistic gasses on arbitrary curved spacetimes, i.e. we will present a “general relativistic and quantum” Boltzmann equation. After this heuristic derivation and generalization we will finally present some of the conceptual problems associated with the Boltzmann equation that are, essentially, the core motivation for this work.

2.1 The Boltzmann equation as a continuity equation

Let us consider the description of a gas in “ \mathbf{x} - \mathbf{p} space”: i.e. the six dimensional space \mathbb{R}^6 . We emphasize that \mathbf{x} - \mathbf{p} space is *not* phase space for the latter is, in general, $2n$ -dimensional, with n being the number of degrees of freedom (cf. [19]). For example, the phase space for a system consisting of N molecules, say a gas, will be $6N$ -dimensional. The state of the system then corresponds to a single point in phase space. In our description, on the other hand, the configuration of the system will be represented by N points, i.e. one for each molecule. An immediate consequence of this difference is that the time evolution of the system will trace a single curve in phase space whereas it will

¹This means that we will give a “physicist’s derivation” of the Boltzmann equation.

consist of N curves in \mathbf{x} - \mathbf{p} space. These curves will, in general, exhibit several “collision points”, i.e. points at which they are not differentiable, corresponding to the collision between particles. We will identify a point in \mathbf{x} - \mathbf{p} space by $(\mathbf{x}, \mathbf{p}) := (x_1, \dots, p_3)$, where it is understood that both \mathbf{x} and \mathbf{p} have three components each as we are considering a gas propagating in 3-dimensional space.

As a first step, we consider a collisionless gas and we would like to understand how does the number of particles in a given volume in \mathbf{x} - \mathbf{p} space evolve with time for such a system. In other words, we are interested to trace the number of particles as their position changes but also as their velocities do. To do so, let N be the number of particle in a 6-dimensional volume V in \mathbf{x} - \mathbf{p} space. We would like to find an equation for dN/dt . It is clear that this change will not be identically zero unless we are at equilibrium since the particles will freely move around and, in the presence of an external force field, their velocities will be modified as well. The main idea is now that the number of particles will change as a result of particles flowing in and out of the volume V and we write this fact as

$$\frac{d}{dt}N(t) = N_{in}(t) - N_{out}(t) , \quad (2.1)$$

where it is clear what the symbols on the right hand side (RHS) above are. We now introduce a “particle number density” (or particle distribution function) $f_{\mathbf{p}}(t, \mathbf{x})$ so “defined” that

$$\int_V f_{\mathbf{p}}(t, \mathbf{x}) d^3\mathbf{x} d^3\mathbf{p} = N(t) .$$

Using $f_{\mathbf{p}}(t, \mathbf{x})$ we rewrite (2.1) as

$$\int_V \frac{\partial}{\partial t} f_{\mathbf{p}}(t, \mathbf{x}) d^3\mathbf{x} d^3\mathbf{p} = N_{in}(t) - N_{out}(t) . \quad (2.2)$$

The problem is now to compute the net number of particles that leaves (or enters) the volume V , i.e. to compute the RHS in the above equation.

To do so, we consider the net “flux of particles” flowing out of (or into) the volume V . Intuitively, if such a flux is given by the number of particles per unit time and per unit (5-dimensional) “surface”, then we could compute the above RHS by integrating the flux on the boundary ∂V of the volume V we are considering. The appropriate definition for such a flux in our case is

$$\text{“flux”} := [\partial_t(\mathbf{x}, \mathbf{p})] f_{\mathbf{p}}(t, \mathbf{x}) = (\mathbf{v}, \mathbf{F}) f_{\mathbf{p}}(t, \mathbf{x}) ,$$

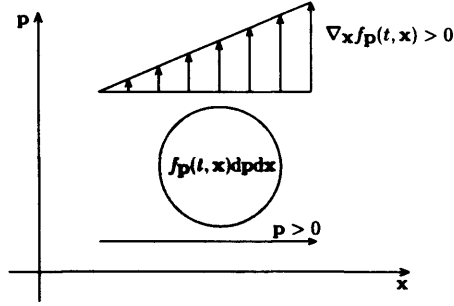


Figure 2.1: In the presence of a positive gradient in the particle density and for particles with positive velocity, more particles will leave a given volume V (the shaded circle) than enter it, hence resulting in a net decrease of the particle number within V .

where we are denoting $\mathbf{v} := \partial_t \mathbf{x}$ and $\mathbf{F} := \partial_t \mathbf{p}$. With this we then write

$$N_{in}(t) - N_{out}(t) = - \int_{\partial V} [(\mathbf{v}, \mathbf{F}) f_{\mathbf{p}}(t, \mathbf{x})] \cdot \mathbf{n} \, d^5 \Sigma, \quad (2.3)$$

where \mathbf{n} is the unit outward² vector normal to the surface at any given point, and $d^5 \Sigma$ is the appropriate integration measure on the surface ∂V . We now use Gauss' theorem and rewrite the previous integral as an integral over the volume V to obtain

$$\begin{aligned} N_{in}(t) - N_{out}(t) &= - \int_V \nabla \cdot [(\mathbf{v}, \mathbf{F}) f_{\mathbf{p}}(t, \mathbf{x})] \, d^3 \mathbf{x} \, d^3 \mathbf{p} \\ &= - \int_V \left[\mathbf{v} \cdot \nabla_{\mathbf{x}} f_{\mathbf{p}}(t, \mathbf{x}) + \mathbf{F} \cdot \nabla_{\mathbf{p}} f_{\mathbf{p}}(t, \mathbf{x}) \right] \, d^3 \mathbf{x} \, d^3 \mathbf{p}, \end{aligned} \quad (2.4)$$

where, with a slight abuse of notation we write

$$\nabla := \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial p_3} \right) = (\nabla_{\mathbf{x}}, \nabla_{\mathbf{p}}).$$

The collision-less Boltzmann equation

$$\frac{\partial}{\partial t} f_{\mathbf{p}}(t, \mathbf{x}) = - \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} f_{\mathbf{p}}(t, \mathbf{x}) - \mathbf{F} \cdot \nabla_{\mathbf{p}} f_{\mathbf{p}}(t, \mathbf{x}), \quad (2.5)$$

then follows by combining equation (2.2) with (2.4) and arguing that since the volume V was completely arbitrary, the resulting identity between integrals can only hold if

²This is a convention and we might as well have chosen a normal unit vector pointing towards the inside of the volume. The end result, however, would not change as in that case we would not be computing the change in $N(t)$ due to particle *leaving* the volume V but with particles *entering* it. We would hence need to replace the minus in (2.3) with a plus, thus giving the same result.

the corresponding identity (2.5) between integrands holds, and we have also identified $\mathbf{p} =: m\mathbf{v}$.

This is the collision-less Boltzmann equation. From the above derivation it is manifest that it has the form of a “continuity equation”, i.e. an equation akin to the, say, mass (charge) continuity equation. The physical interpretation of the latter is mass (charge) conservation and we can hence interpret the BE as a particle number continuity equation. Indeed, this is precisely how we derive it and its meaning is that of *local conservation of the number of particles*, where “local” now refers to locality both in position and momentum (velocity). A quick inspection of Figure 2.1 reveals that the minus signs on the RHS of the above equation are indeed correct. Moreover, we physically interpret the two terms on the RHS as follows. The first term corresponds to freely moving particles (in the presence of a gradient), whereas upon identifying the quantity \mathbf{F} as an “external” force field, we identify the second term as modifying the particle velocities because of the external field \mathbf{F} .

Once the collisionless BE has been established it does not take much to realize that the final momenta of two colliding particles will be different from their pre-collision momenta. This means we ought to modify the above equation by including a collision term as

$$\frac{\partial}{\partial t} f_{\mathbf{p}}(t, \mathbf{x}) = -\frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} f_{\mathbf{p}}(t, \mathbf{x}) - \mathbf{F} \cdot \nabla_{\mathbf{p}} f_{\mathbf{p}}(t, \mathbf{x}) + \left(\frac{\partial f_{\mathbf{p}}(t, \mathbf{x})}{\partial t} \right)_{\text{coll}}.$$

The collision term will clearly have to include all the details of the specific model (dynamics) as the collision-less equation has been derived on purely kinematical grounds. And it was Boltzmann’s crucial contribution to the problem of *approach to equilibrium* to give an explicit expression for the collision term, see [1]. We will not derive such a collision term here but only state the end result and, most importantly, clearly point out the assumptions that are crucial for the derivation. The end result is

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} + \mathbf{F} \cdot \nabla_{\mathbf{p}} \right) f_{\mathbf{p}}(t, \mathbf{x}) = \int_{\mathbf{R}} d^3\mathbf{p}_2 d^3\mathbf{q}_1 d^3\mathbf{q}_2 \delta^{(4)}(\underline{p} + \underline{p}_2 - \underline{q}_1 - \underline{q}_2) \cdot \left| \mathcal{M}(\mathbf{p}, \mathbf{p}_1 \leftrightarrow \mathbf{q}_1, \mathbf{q}_2) \right|^2 \left[f_{\mathbf{q}_1}(t, \mathbf{x}) f_{\mathbf{q}_2}(t, \mathbf{x}) - f_{\mathbf{p}}(t, \mathbf{x}) f_{\mathbf{p}_2}(t, \mathbf{x}) \right], \quad (2.6)$$

where the 4-dimensional delta enforces both energy and momentum conservation (and we are using a relativistic notation for the on-shell momenta \underline{p} to denote, separately, both energy and momentum) and $\mathcal{M}(\mathbf{p}, \mathbf{p}_1 \leftrightarrow \mathbf{q}_1, \mathbf{q}_2)$ is the scattering amplitude for the process $\mathbf{p} + \mathbf{p}_2 \leftrightarrow \mathbf{q}_1 + \mathbf{q}_2$.

It is important at this point that we note what are the assumptions that go into the

derivation of the above collision term. They are:

1. A process and its inverse have the same scattering amplitude, that is

$$\mathcal{M}(\mathbf{p}, \mathbf{p}_2 \rightarrow \mathbf{q}_1, \mathbf{q}_2) = \mathcal{M}(\mathbf{q}_1, \mathbf{q}_2 \rightarrow \mathbf{p}, \mathbf{p}_2) := \mathcal{M}(\mathbf{p}, \mathbf{p}_2 \leftrightarrow \mathbf{q}_1, \mathbf{q}_2) .$$

2. Only *binary* collisions are taken into account—the gas is sufficiently dilute.
3. The effect of external forces on the collision is negligible if compared to the interparticle forces.
4. The momenta of the particles are uncorrelated at *all the times*—“molecular chaos” assumption or “Stosszahlansatz”.

The first assumption above is justified (classically) by appealing to the electromagnetic origin of the interparticle forces, which is “PT” invariant³, hence implying the above assumption. The second assumption is a reasonable one for a classical gas and the third assumption is verified in practical situations where the external fields are not strong enough to influence the interparticle dynamics. Finally, the really fundamental assumption, and the most difficult to justify as well, is the last one, i.e. the assumption of molecular chaos. We will leave a critical evaluation of this assumption until Section §2.4 as it is always assumed in all the generalizations of the Boltzmann equation that we will present below, whereas the remaining assumptions will be modified⁴ leading to appropriate modifications of equation (2.6). Finally, we remark that we have only considered one species of particle without spin here. If one considers a gas of different species of molecules then it is necessary to introduce a particle number distribution function f_i , $i = 1, \dots, N$ for each of the N particle species present. In that case, the collision term will couple the different distribution functions if different molecule can scatter off each other.

2.2 Equilibrium distributions as link to the quantum Boltzmann equation

Without going into details, we note that the BE allows us to find the “equilibrium” distribution function. Intuitively, we understand equilibrium as a state where no macro-

³PT invariance of a theory means that it is invariant under the action of space and time reflections $\mathbf{x} \rightarrow -\mathbf{x}$ and $t \rightarrow -t$.

⁴The third assumption above will still be understood. Except for a physical situation where quantum gravitational effects become relevant, we are not aware of any case where it would not hold.

scopically relevant quantity changes anymore, in the sense that they do not change with time. At equilibrium, hence, there can be no gradients in the particle density as a higher density would, in the absence of some barrier, lead to the diffusion of the gas towards the lower density regions. That is, a non-equilibrium situation due to a change of density of the gas. Because of this, the equilibrium distribution function cannot depend on \mathbf{x} . Similarly, an external force would cause acceleration of the particles, which is clearly incompatible with the heuristic understanding of equilibrium we mentioned above. It follows that in equilibrium the Boltzmann equation becomes

$$\frac{\partial f_{\mathbf{p}}(t, \mathbf{x})}{\partial t} = \frac{\partial f_{\mathbf{p}}(t)}{\partial t} = \left(\frac{\partial f_{\mathbf{p}}(t)}{\partial t} \right)_{coll} = 0, \quad (2.7)$$

where the last equality follows because we understand equilibrium as time translation invariant state, i.e. a state where quantities do not depend on time. In Chapter 4 of [7] it is possible to find the proof to the following

Proposition 2.1. *Let $f_{\mathbf{p}}^{(eq)}$ be the equilibrium solution to the Boltzmann equation (2.6), i.e. a solution of (2.7). A necessary and sufficient condition satisfied by $f_{\mathbf{p}}^{(eq)}$ is*

$$f_{\mathbf{q}_1}^{(eq)} f_{\mathbf{q}_2}^{(eq)} - f_{\mathbf{p}}^{(eq)} f_{\mathbf{p}_2}^{(eq)} = 0, \quad (2.8)$$

for all kinematically allowed momenta \mathbf{p}_2 , \mathbf{q}_1 and \mathbf{q}_2 .

The proof of this claim is a direct consequence of the proof of Boltzmann's "H theorem", which can be found in [7] as well. We do not state or prove it here as we are only interested in the consequences of (2.8). In fact, taking the logarithm of that condition we immediately obtain

$$\log f_{\mathbf{p}}^{(eq)} + \log f_{\mathbf{p}_2}^{(eq)} = \log f_{\mathbf{q}_1}^{(eq)} + \log f_{\mathbf{q}_2}^{(eq)}.$$

which has the form of a "conservation law". That is, we have that the sum of two quantities before the collision equals the sum of the corresponding quantities after the collision. This means that $\log f_{\mathbf{p}}^{(eq)}$ is a conserved quantity. For a spin-less particle the conserved quantities are the energy, the three components of the momenta and a constant. Hence $\log f_{\mathbf{p}}^{(eq)}$ is a linear combination of \mathbf{p}^2 , the three components of \mathbf{p} and the arbitrary constant:

$$\log f_{\mathbf{p}}^{(eq)} = A(\mathbf{p} - \mathbf{p}_0)^2 + \log C,$$

from which one can argue that (see [7])

$$f_{\mathbf{p}}^{(eq)} \propto e^{-\omega_{\mathbf{p}}\beta} ,$$

where $\omega_{\mathbf{p}} = \mathbf{p}^2/2m$ is the energy of the particles, $\beta = 1/T$ is the inverse temperature, and we have set $k_B = 1$. This is known as the “Maxwell–Boltzmann” distribution function and it is a well known expression.

Above we have not specified the exact proportionality constants as they are not important for our goal, which is the generalization of the Boltzmann equation to the quantum case. In fact, in the above “derivation”, it was implicitly assumed that the particles were classical (distinguishable) particles. We would like to understand, however, what a corresponding *quantum* Boltzmann equation would look like. That is, how should we modify the above derivation if we wanted to derive it for *indistinguishable* (Bose or Fermi⁵) particles. From the derivation we have given above it is not at all obvious how one should proceed in order to obtain the corresponding quantum equation. A heuristic and pragmatic way to do that is then obtained by “reversing” the above argument that allowed us to obtain the Maxwell–Boltzmann distribution in order to find out what the collision term looks like.

The equilibrium distribution functions for Bose and Fermi particles are known (see [7]). They are

$$f_{\mathbf{p}}^{(eq)} = \frac{1}{e^{\beta\omega_{\mathbf{p}}} \pm 1} ,$$

where the + refers to fermions and the – to bosons. Proceeding now in a reversed order we first find

$$\beta\omega_{\mathbf{p}} = \log \left(\frac{1 \pm f_{\mathbf{p}}^{(eq)}}{f_{\mathbf{p}}^{(eq)}} \right) ,$$

where the + now refers to *bosons* and the – to *fermions*. This trivially follows from the explicit computation that leads to the above expression, but we emphasize it in order to avoid confusion later. We then note that $\omega_{\mathbf{p}}$ is a conserved quantity, and hence write

$$\log \left(\frac{1 \pm f_{\mathbf{p}_1}^{(eq)}}{f_{\mathbf{p}_1}^{(eq)}} \right) + \log \left(\frac{1 \pm f_{\mathbf{p}_2}^{(eq)}}{f_{\mathbf{p}_2}^{(eq)}} \right) = \log \left(\frac{1 \pm f_{\mathbf{q}_1}^{(eq)}}{f_{\mathbf{q}_1}^{(eq)}} \right) + \log \left(\frac{1 \pm f_{\mathbf{q}_2}^{(eq)}}{f_{\mathbf{q}_2}^{(eq)}} \right) ,$$

⁵The distinction between Bose and Fermi particles – bosons and fermions — can be stated by saying that Fermi particles obey “Pauli exclusion principle” whereas Bose particles do not. The Pauli exclusion principle is discussed in essentially all the books on quantum mechanics, see e.g. [20].

which we use to arrive at the desired result

$$f_{\mathbf{q}_1}^{(eq)} f_{\mathbf{q}_2}^{(eq)} (1 \pm f_{\mathbf{p}_1}^{(eq)}) (1 \pm f_{\mathbf{p}_2}^{(eq)}) = f_{\mathbf{p}_1}^{(eq)} f_{\mathbf{p}_2}^{(eq)} (1 \pm f_{\mathbf{q}_1}^{(eq)}) (1 \pm f_{\mathbf{q}_2}^{(eq)}) .$$

This result then immediately suggests how to modify the RHS of equation (2.6), i.e. the Boltzmann collision factor. Using the shorthand notation $f_{\mathbf{p}_i} = f(t, \mathbf{x}, \mathbf{p}_i)$, the modified (quantum) Boltzmann equation is then

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} + \mathbf{F} \cdot \nabla_{\mathbf{p}} \right) f_{\mathbf{p}} = \int_{\mathbf{R}} d^3 \mathbf{p}_2 d^3 \mathbf{q}_1 d^3 \mathbf{q}_2 \delta^{(4)}(\underline{p} + \underline{p}_1 - \underline{q}_1 - \underline{q}_2) \cdot \left| \mathcal{M}(\mathbf{p}, \mathbf{p}_1 \leftrightarrow \mathbf{q}_1, \mathbf{q}_2) \right|^2 \left[f_{\mathbf{q}_1} f_{\mathbf{q}_2} (1 \pm f_{\mathbf{p}}) (1 \pm f_{\mathbf{p}_2}) - f_{\mathbf{p}} f_{\mathbf{p}_2} (1 \pm f_{\mathbf{q}_1}) (1 \pm f_{\mathbf{q}_2}) \right] . \quad (2.9)$$

As an aside we note that the above equation reduces to the classical one (2.6) in the case where the particle densities are very small (for very dilute systems), i.e. when we have $f_{\mathbf{p}} \ll 1$. Incidentally, a very dilute system was one of the assumptions that led to equation (2.6), albeit dropping this assumption in the above derivation would have *not* led to the above equation.

Before considering the generalization of the above equation to the relativistic case, we want to stress that the “argument” that led to the QBE is nothing but a heuristic and pragmatic way of arguing why the collision factor should take that form. The QBE was introduced in [21] by Uehling and Uhlenbeck but it was given with no proper explanation, and an argument as the one we have presented here seems indeed plausible. That the above one is indeed the appropriate collision factor for the QBE has later been derived at various levels of mathematical rigour—for a formal derivation of the QBE for a Fermi gas we refer the reader to, say, [9] while a rigorous derivation for the same model but on a lattice can be found in [6].

2.3 Relativistic generalization

So far we have always been dealing with slowly moving particles but we would like to understand how to deal with relativistic particles as well. Furthermore, we would like to be able to include the creation and annihilation of particles. Clearly, the appropriate framework is then (relativistic) quantum field theory.

The first step consists in an almost trivial generalization of the LHS to the relativistic case. The particle distribution function will now be a function of the position and 4-

vectors x and p and the generalization of the LHS of the BE is simply

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m} \cdot \nabla_{\mathbf{x}} + \mathbf{F} \cdot \nabla_{\mathbf{p}} \right) f_{\mathbf{p}}(t, \mathbf{x}) \quad \longrightarrow \quad \left(p^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu} \right) f(x, p) ,$$

and where F^μ is the 4-force. More importantly, however, we want to generalize the collision factor, and we do so in two ways. First, we want to allow for scattering processes other than $2 \rightarrow 2$ scattering so as to allow particle creation, annihilation and particle decay. And since we want to consider particle decay, we are forced to introduce distribution functions for all the particle species that we are considering. Secondly, while the electromagnetic field is PT invariant, we also want to account for situations where a PT-violating interaction acts. That is, we would like to consider phenomena where a process is different from its inverse. Denoting the distribution function of a particle species a with f_a and considering a *specific* scattering process $\psi + a + b + \dots \rightarrow i + j + \dots$, the appropriate relativistic BE for the particle species ψ is

$$\begin{aligned} \left(p^\mu \frac{\partial}{\partial x^\mu} + F^\mu \frac{\partial}{\partial p^\mu} \right) f_\psi(x, p) = \int_{\mathbf{R}} d\Pi(\psi, a, b, \dots, i, j, \dots) (2\pi)^4 \delta^{(4)}(P_{in} - P_{out}) \\ \left\{ \left| \mathcal{M}(i, j, \dots \rightarrow \psi, a, b, \dots) \right|^2 f_i f_j \dots (1 \pm f_\psi) (1 \pm f_a) (1 \pm f_b) \dots + \right. \\ \left. - \left| \mathcal{M}(\psi, a, b, \dots \rightarrow i, j, \dots) \right|^2 f_\psi f_a f_b \dots (1 \pm f_i) (1 \pm f_j) \dots \right\} , \end{aligned} \quad (2.10)$$

where

$$d\Pi(a, b, c, \dots) = d\Pi_a d\Pi_b d\Pi_c \dots , \quad d\Pi_a = \frac{d^3 \mathbf{p}_a}{2\omega(\mathbf{p}_a)} ,$$

is the appropriate Lorentz invariant phase space measure and where the 4-dimensional delta again enforces the total 4-momentum conservation, and $\omega(\mathbf{p}_a) = (\mathbf{p}_a^2 + m_a^2)^{1/2}$ is the relativistic energy of a particle of the particle species a . In the above equation it is understood that the scattering amplitudes \mathcal{M} are to be computed using the appropriate quantum field theoretical rules pertaining to the specific model one is dealing with. We finally note that the above equation is only valid as a first approximation because, in principle, the RHS should contain a “sum over all the possible interactions” that could change the particle number. In practice, however, physicists most often ignore all but one process, thus using the BE as displayed above. The reason for this is that it can be often argued on physical grounds that the dominant (quantitative) contribution to some physical phenomenon comes from a single scattering process. Clearly, the same comment also applies to the classical and quantum Boltzmann equations (2.6) and (2.9) respectively.

The generalization to curved spaces is finally obtained by arguing that the only background field that influences the particle motion is the spacetime metric and that the particles move on the geodesics of spacetime hence leading to the following replacement

$$F^\mu \frac{\partial}{\partial p^\mu} \quad \longrightarrow \quad -\Gamma_{\alpha\beta}^\mu p^\alpha p^\beta \frac{\partial}{\partial p^\mu}$$

where $\Gamma_{\alpha\beta}^\mu$ are the Christoffel coefficients associated with the background metric of the spacetime. Above we use Einstein's "summation convention" and accordingly consider a sum the repeated indices, which denote the components of the corresponding tensorial quantities in a coordinate basis. We note that the above term is *not* generally covariant, but the sum

$$p^\mu \frac{\partial}{\partial x^\mu} f(x, p) - \Gamma_{\alpha\beta}^\mu p^\alpha p^\beta \frac{\partial}{\partial p^\mu} f(x, p)$$

is. The reason for this is that under a change of coordinates the "non-tensorial" term arising due to the Christoffel symbol will be cancelled by an identical contribution coming from the first term. In fact we have

$$\begin{aligned} p^\alpha \frac{\partial}{\partial x^\alpha} f(x, p) &= p^\alpha \frac{\partial}{\partial x^\alpha} \tilde{f}(\tilde{x}, \tilde{p}) = p^\alpha \left[\frac{\partial \tilde{x}^\mu}{\partial x^\alpha} \frac{\partial}{\partial \tilde{x}^\mu} + \frac{\partial \tilde{p}^\mu}{\partial x^\alpha} \frac{\partial}{\partial \tilde{p}^\mu} \right] \tilde{f}(\tilde{x}, \tilde{p}) = \\ &= \tilde{p}^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \tilde{f}(\tilde{x}, \tilde{p}) + \frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial x^\beta} p^\alpha p^\beta \frac{\partial}{\partial \tilde{p}^\mu} \tilde{f}(\tilde{x}, \tilde{p}) = \\ &= \tilde{p}^\alpha \frac{\partial}{\partial \tilde{x}^\alpha} \tilde{f}(\tilde{x}, \tilde{p}) + \frac{\partial x^\nu}{\partial \tilde{x}^\mu} \frac{\partial^2 \tilde{x}^\mu}{\partial x^\alpha \partial x^\beta} p^\alpha p^\beta \frac{\partial}{\partial p^\nu} f(x, p), \end{aligned}$$

where the "tilde" quantities refer to the quantities expressed in the new (tilde) coordinates, and $f(x, p) = \tilde{f}(\tilde{x}, \tilde{p})$ as it is a scalar quantity. The last "natural" generalization should concern the scattering amplitude in that one should use the principles of quantum field theory on curved spaces to compute them. We note, however, that "in practice", when the Boltzmann equation is used in a cosmological setting the scattering amplitudes are computed using the usual Feynman rules (see e.g. [5]).

The "general relativistic and quantum" BE as presented in this section is one of the tools to obtain quantitative prediction of Early Universe theories like Baryogenesis and Nucleosynthesis, see e.g. [5]. In the above form, the BE barely resembles the original one from equation (2.6). Its meaning, however, remains the same, i.e. the local conservation of the particle number. It might appear puzzling that the meaning is still that of particle number *conservation* as we are explicitly allowing the creation and annihilation of particles. This issue is quickly resolved by allowing for a slight generalization of our initial formulation of the problem, i.e. particle number conservation in some volume

V , where we would now have to include “sources” and “sinks” of particles to account for particle creation and annihilation due to the relativistic equivalence between mass and energy. We thus still have a (now generalized) continuity equation. Despite the (physically motivated and) reasonable arguments that lead us to the above equation and the experimental confirmation of its validity, the conceptual status of this equation is far from clear. In the next section we will address precisely the many conceptual problems associated with it.

2.4 Conceptual problems

The derivation of the BE immediately raises a number of non trivial questions. One highly unsatisfactory aspect is the Stosszahlansatz, i.e. the assumption of molecular chaos. Specifically, it raises a key conceptual issue as it is assumed to be valid at *all times* and not just, say, at some initial time t_0 . While it is very natural to assume for a system to be “chaotic” at some initial time, the question whether it will remain such is not, in principle, for us to argue about but is fixed by the underlying dynamics of the system. A more satisfactory approach then consists in assuming that particle velocities are uncorrelated at t_0 and study under what conditions would they *remain* uncorrelated at later times. This will clearly require the detailed study of a specific model, and in this sense the molecular chaos assumption was a “necessary” one given that the BE was derived in a model independent way. We are then interested whether it can be justified or not for specific models.

Another issue that even the above heuristic derivation raises lies precisely in the necessity of making a number of assumptions, of which the Stosszahlansatz is but one. This means that the BE is not an *exact*⁶ equation, so the question immediately arises about when does it provide an accurate description of some physical system? For instance, one of the assumptions in deriving equation (2.6) was that all the collisions were binary hence making it inapplicable in those cases where, say, three-body collisions are the core physical mechanism in action. This assumption has in fact been dropped later on when we obtained the relativistic BE (2.10), but in doing so it was not clear whether such a modification is consistent with the original derivation. In fact, we have not even provided a heuristic derivation of the equation (2.10) but merely “generalized” the previous results.

Far from being the first to be interested in the conceptual problems arising from

⁶In this work we will refer to an equation as being “exact” if it is assumed to be a first principle of the theory. Examples of exact equations are the Newton’s equation in classical mechanics, the Heisenberg or Schrödinger equations in quantum mechanics and Einstein’s field equations in general relativity.

the BE (see e.g. [6, 9, 22, 23], the review [10] and references therein) the issue of its validity has in principle long been settled, at least from a physical point of view. In the kinetic description of a gas there are two fundamental “time scales”⁷ associated with the collisions between particles. The first one is the average time between two collisions, τ_{coll} , while the second one is the typical time it takes for “the collision to happen”, that is, the time after which the two particles don’t “feel” each other’s interaction anymore. We denote this time scale with τ_{int} . It is a well known fact that the BE holds when the single collision can be considered *instantaneous* when compared to the average time between two subsequent collisions. That is, when $\tau_{int} \ll \tau_{coll}$, [24, 8]. Physically, this is realized when we are dealing with a very dilute system or when the interaction between particles is very weak. (These two facts will be the motivation for us to consider the low density and weak coupling limits later.) It is however clear that to give precise quantitative conditions on the validity of the BE one should consider the specific dynamics of a model.

From a classical point of view, where particles are considered as a “tiny unbreakable and localized lumps of matter”, the definition of a particle distribution function $f_{\mathbf{p}}(t, \mathbf{x})$ poses no particular problems, see [7]. What is needed to assume the existence of such a quantity is that the physical system we would like to study allows us to ignore the atomic structure of matter, as is the case when dealing with macroscopic systems. The concept of “locality in space” should then not be understood as localization in an infinitesimally small region of space, but rather localization within a tiny volume of space that can be considered infinitesimal “for all practical purposes”. The continuous description is then the appropriate one and it is also meaningful to talk about “densities” at a space point. This brief discussion may then be summarized by saying that *classically* the particle distribution function $f_{\mathbf{p}}(t, \mathbf{x})$ can be meaningfully defined.

The situation is very different in quantum mechanics. The Heisenberg uncertainty principle makes it impossible to measure a particle’s momentum and position with arbitrary precision at the same time. This in turn makes it impossible to understand a particle a small localized lump of matter that we could observe at any time, and hence the idea of “particle density” becomes unclear. While one might adopt a pragmatic attitude and simply *use* the BE in those cases where it seems appropriate to do so, the conceptual problem of justifying it remains. And we also face the additional problem of justifying, and hence understanding, the presence of the quantum correction factors $1 \pm f$ in the collision factor. As is apparent from this work, and likewise as it could be

⁷A time scale is understood to be the average time lapse over which there is a “significant” change in the system. We will not give any more precise definition of “significant” and rely on the intuitive understanding of the term.

found, say, in [6], the quantum correction factors can be understood as a consequence of the canonical commutation relation.

The problem of an appropriate way of understanding a particle gets only worse in quantum field theory where the “particle interpretation” problem of a model is not a trivial one and the question: “When does a field theory describe particles?” becomes a relevant one, see [25]. Let us simply briefly mention that the “usual” interpretation⁸ of a particle in Minkowski space is closely related to the existence of a preferred “vacuum” state, i.e. a “lowest energy” state. On the other hand, the BE is frequently employed on curved spaces of relevance for cosmology, like the Robertson–Walker spacetime. And a curved spacetime will, in general, *not possess* a vacuum state, or any preferred state for that matter. Not only does this make the problem of defining a particle in curved spaces a very difficult (and unsolved) one, but it has the additional complication of unambiguously specifying the particle content of a specific state, see e.g. [27, 28]. And unfortunately the absence of a preferred lower energy state is a characteristic of the Robertson–Walker metric, so it is not just an “academic” problem. In lieu of all these remarks, it is legitimate to question the validity of the Boltzmann equation in curved spaces at all.

Finally, but perhaps most importantly, is the problem of understanding whether the BE be justified starting from first principles. Given the many accurate experimental predictions one would indeed expect for this to be the case. But the fundamental question remains: Can we derive the BE from *first principles* and thus conceptually *justify* it? And if so, what can we learn from such a derivation? As we show in this work, it is indeed possible to derive the BE starting from the Heisenberg equation, but it is highly non-trivial task to establish this link. And once we have done that, we will understand the Boltzmann equation as the long time and dilute medium (or long time and weakly interacting) approximation to the Heisenberg equation.

⁸Even in Minkowski space there are ambiguities on the interpretation of the particle content of a theory, see [26] for more details.

Chapter 3

Classical and quantum field theory

In this chapter we review some basic features about the Hamiltonian formulation of quantum field theory. We begin with a short introduction to the Hamiltonian formulation of the theory of a classical real scalar field with polynomial (self)-interaction. We then proceed to explain how the quantum theory is obtained by means of a “quantization” procedure of the classical theory. Such a procedure has turned out to be very successful in the quantization of *free* theories, but not for physically more interesting interacting ones. To overcome this difficulties the so called “axiomatic” approach has been developed where quantum fields are understood as “operator valued distributions” [29], i.e. they are distributions which take values in the set of operators on some Hilbert space. The starting point of this framework is a set of (physically) reasonable axioms, the so called Wightman axioms (see [30] for a comprehensive introduction on the subject) and we review and discuss their physical significance. Once the axioms have been laid down, the emphasis is not on “taking the existence of interacting quantum field models” for granted but to mathematically *construct* them and prove the consistency of the construction with the axioms. A lot of effort has been put into the “constructive approach” to QFT and despite the many successes, a non-trivial interacting model in four spacetime dimensions has not yet been constructed. In this work however, we will be concerned with a two dimensional spacetime, for which the constructive approach has proved successful and we briefly review some of the ideas behind the construction of the so called $P(\varphi)$ quantum field theory in two spacetime dimensions.

3.1 Classical Hamiltonian field theory

Here we consider the classical theory of a real scalar field with polynomial interaction on $D = d + 1$ dimensional Minkowski space M . The central object defining the classical

theory is the “action” S , defined as

$$S := \int_M d^Dx \mathcal{L} = \frac{1}{2} \int_M d^Dx \left[-\partial_\mu \varphi(x) \partial^\mu \varphi(x) - m^2 \varphi^2(x) - \sum_{n=3}^{2N} b_n \varphi^n(x) \right],$$

where we introduced the “Lagrangian density”

$$\mathcal{L} := -\partial_\mu \varphi(x) \partial^\mu \varphi(x) - m^2 \varphi^2(x) - \sum_{n=3}^{2N} \frac{b_n}{n!} \varphi^n(x),$$

In classical physics, Hamilton’s principle of “least action” [19] states that the system will evolve along a “path” where the action is “minimal”. This means that we ought to “minimize” the action by considering its variations δS in order to obtain the equations of motion for the system we are considering. If we define the “functional derivative” $\delta F[\varphi]/\delta \varphi$ of a functional $F(\varphi)$ so that for each compactly supported and smooth (test) function $f(x)$ we have

$$\left. \frac{d}{d\epsilon} F(\varphi + \epsilon f) \right|_{\epsilon=0} =: \int_M d^Dx \frac{\delta F[\varphi]}{\delta \varphi(x)} f(x), \quad (3.1)$$

then according to the principle of least action one obtains the usual Euler–Lagrange equation

$$-\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi(x))} + \frac{\partial \mathcal{L}}{\partial \varphi(x)} = \partial_\mu \partial^\mu \varphi(x) - m^2 \varphi(x) - \sum_{n=3}^{2N} \frac{b_n}{(n-1)!} \varphi^{n-1}(x) = 0,$$

which is more conveniently rewritten as

$$\left(\partial_\mu \partial^\mu - m^2 \right) \varphi(x) = \sum_{n=3}^{2N} \frac{b_n}{(n-1)!} \varphi^{n-1}(x). \quad (3.2)$$

In order to formulate the theory in the Hamiltonian approach, we need to define the momentum $\pi(t, \mathbf{x})$ canonically conjugate to the field $\varphi(t, \mathbf{x})$ ¹. We define $\pi(t, \mathbf{x})$ as

$$\pi(t, \mathbf{x}) := \frac{\partial \mathcal{L}}{\partial(\partial_t \varphi(t, \mathbf{x}))} = \partial_t \varphi(t, \mathbf{x}). \quad (3.3)$$

We can now compute/define the Hamiltonian of the system through the “Legendre

¹As is customary in the Hamiltonian formulation of both classical and quantum field theory, we now separate the “time” and the “spatial” parts of the argument of φ and π

transformation”²

$$\begin{aligned}
 H &:= \int_{\mathbb{R}^d} d^d \mathbf{x} \left\{ \pi(t, \mathbf{x}) \partial_t \varphi(t, \mathbf{x}) - \mathcal{L} \right\} = \\
 &= \int_{\mathbb{R}^d} d^d \mathbf{x} \left\{ \pi^2(t, \mathbf{x}) + \left(\partial_{\mathbf{x}} \varphi(t, \mathbf{x}) \right)^2 + m^2 \varphi^2(t, \mathbf{x}) + \sum_{n=3}^{2N} \frac{\lambda_n}{n!} \varphi^n(t, \mathbf{x}) \right\}.
 \end{aligned} \tag{3.4}$$

Physically, the Hamiltonian H specifies the “energy content” of the theory, and is needed to generate the equations of motion. To specify the latter, however, we need an additional ingredient—the “Poisson bracket”—which we will use to define the time evolution of a time dependent functional F . If $F[\pi, \varphi]$ and $G[\pi, \varphi]$ are two functionals of φ and π , we define their Poisson bracket as

$$\{F, G\} := \int_{\mathbb{R}^d} d^d \mathbf{x} \left[\frac{\delta F}{\delta \varphi(t, \mathbf{x})} \frac{\delta G}{\delta \pi(t, \mathbf{x})} - \frac{\delta F}{\delta \pi(t, \mathbf{x})} \frac{\delta G}{\delta \varphi(t, \mathbf{x})} \right].$$

With the Poisson bracket we immediately find

$$\{\varphi(t, \mathbf{x}), \pi(t, \mathbf{y})\} = \delta^{(d)}(\mathbf{x} - \mathbf{y}). \tag{3.5}$$

We can now define the time evolution of a functional F by its Poisson bracket with the Hamiltonian

$$\partial_t F(t) := \{F, H\}.$$

Once we have defined the time evolution of a generic functional F , we can compute

$$\partial_t \varphi(t, \mathbf{x}) = \frac{\delta H}{\delta \pi(t, \mathbf{x})}, \quad \partial_t \pi(t, \mathbf{x}) = -\frac{\delta H}{\delta \varphi(t, \mathbf{x})},$$

which we recognize as Hamilton’s equations of motion. Using the above equations and (3.4) we straightforwardly obtain equation (3.3) alongside the field equation (3.2). This is the usual starting point for the so called “canonical quantization” of the system.

3.2 Quantization of a Hamiltonian field theory

The Hamiltonian approach to classical field theory that we have just outlined will be also used in the rest of this chapter, where we will “quantize” the theory. In particular,

²We note that the Legendre transform is only well defined if the interaction polynomial does not contain derivatives. This is the case for us, but in general it is ill defined.

we will be able to give H , the quantum Hamiltonian, a rigorous mathematical meaning. Before that, however, we present the heuristic quantization of the classical theory defined by the Hamiltonian (3.4). The treatment of the subject as we will present it here may be found in standard physics texts on the subject like [31]. As is well known, mathematical rigour should not be looked for in such a naive and heuristic approach and we briefly outline many of its flaws. The following section is then devoted to a brief exposition of the constructive approach to quantum field theory and we sketch the (rigorous) construction of a scalar theory on the circle.

3.2.1 Heuristic quantization of free fields

The quantization of the theory is obtained by “promoting” the canonically conjugate pair φ and π to operators³ on some Hilbert space. The Poisson bracket is then replaced by the commutator of the two operators, and crucially for a “quantum” theory, canonical (time zero) commutation relations (CCRs) are imposed. In analogy with (3.5) we assume that

$$[\varphi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^{(d)}(\mathbf{x} - \mathbf{y}) \mathbf{1}, \quad [\varphi(\mathbf{x}), \varphi(\mathbf{y})] = [\pi(\mathbf{x}), \pi(\mathbf{y})] = 0, \quad (3.6)$$

where $\varphi(\mathbf{x}) =: \varphi(0, \mathbf{x})$ and $\pi(\mathbf{x}) =: \pi(0, \mathbf{x})$ are the “time zero” fields, and where the bracket $[\ , \]$ denotes the commutator of two operators, i.e. $[A, B] = AB - BA$.

The time evolution of the field operators is again generated by the (quantum) Hamiltonian, which is simply *assumed* to be a self-adjoint operator on our Hilbert space and it is formally the same as the corresponding classical Hamiltonian (3.4). The evolution equations for the fields $\varphi(t, \mathbf{x})$ and $\pi(t, \mathbf{x})$ are now given by the Heisenberg equations of motion

$$\partial_t \varphi(t, \mathbf{x}) = i[H, \varphi(t, \mathbf{x})], \quad \partial_t \pi(t, \mathbf{x}) = i[H, \pi(t, \mathbf{x})].$$

The evolution equations are solved by

$$\varphi(t, \mathbf{x}) := e^{itH} \varphi(0, \mathbf{x}) e^{-itH},$$

where $\exp(\pm itH)$ are well defined unitary operators as it was assumed that H is self-adjoint. An immediate consequence of these definitions is that the CCRs (3.6) are found to hold for arbitrary times t .

A very important consequence of the quantization can be seen by considering the free theory, i.e. if $b_n = 0$ for all n in (3.4). Then we can solve the equations of motion

³To avoid cluttering with the notation we will denote the quantum fields with the same symbols as the classical fields.

and the solution can be written as

$$\varphi_0(t, \mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}} \frac{d^d \mathbf{p}}{(2\omega(\mathbf{p}))^{1/2}} \left[a(\mathbf{p}) e^{-i\omega(\mathbf{p})t + i\mathbf{x}\mathbf{p}} + a(\mathbf{p})^* e^{i\omega(\mathbf{p})t - i\mathbf{x}\mathbf{p}} \right], \quad (3.7)$$

with $\omega(\mathbf{p}) = (\mathbf{p}^2 + m^2)^{1/2}$. That the classical field is real suggests that the corresponding quantum field ought to be hermitian, which implies that $a(\mathbf{p})^*$ is the hermitian adjoint of $a(\mathbf{p})$. We also note that because we are considering a free theory, the operators $a(\mathbf{p})$ and $a(\mathbf{p})^*$ are time independent. These “creation” ($a(\mathbf{p})^*$) and “annihilation” ($a(\mathbf{p})$) operators must satisfy the commutation relations

$$[a(\mathbf{p}), a(\mathbf{k})^*] = \delta^{(d)}(\mathbf{p} - \mathbf{k}), \quad [a(\mathbf{p}), a(\mathbf{k})] = [a(\mathbf{p})^*, a(\mathbf{k})^*] = 0, \quad (3.8)$$

in order to be consistent with the CCRs (3.6). These operators can be realized on our Hilbert space as follows. We formally assign to each $\mathbf{p} \in \mathbb{R}^d$ a “1-particle vector”, $|\mathbf{p}\rangle$, and the inner product is defined formally by $\langle \mathbf{k} | \mathbf{p} \rangle = \delta^{(d)}(\mathbf{k} - \mathbf{p})$. Furthermore, we introduce a “vacuum” vector $|0\rangle$ and “ n -particle” vectors

$$|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle = \frac{1}{n!} \sum_{\pi} |\mathbf{p}_{\pi_1}\rangle \otimes |\mathbf{p}_{\pi_2}\rangle \otimes \dots \otimes |\mathbf{p}_{\pi_n}\rangle,$$

where the sum is over the set of all permutations of n elements. The inner product between these vectors is the one naturally inherited from the single particle vector inner product defined above. We then define the action of $a^*(\mathbf{p})$ on such vectors by

$$a(\mathbf{p})^* |0\rangle := |\mathbf{p}\rangle, \quad a(\mathbf{p})^* |\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle := |\mathbf{p}, \mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle,$$

which implies that the action of the hermitian conjugate operator $a(\mathbf{p})$ is

$$a(\mathbf{p}) |0\rangle = 0, \quad a(\mathbf{p}) |\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle = \sum_{i=1}^n \delta^{(d)}(\mathbf{p} - \mathbf{p}_i) |\mathbf{p}_1, \dots, \mathbf{p}_{\cancel{i}}, \dots, \mathbf{p}_n\rangle, \quad (3.9)$$

where $\mathbf{p}_{\cancel{i}}$ means that the corresponding element should be omitted. It is the action of the creation and annihilation operators on the vacuum that motivates their name: they respectively create and annihilate a particle of momentum \mathbf{p} .

From this discussion it is clear that the quantization of the free field essentially corresponds to the quantization of an infinite collection of harmonic oscillators, one for each momentum \mathbf{p} . Stated differently, the quantum theory of a free field is the theory of arbitrarily many free relativistic (non interacting) spinless particles that obey the Bose

statistics.

3.2.2 Problems with the heuristic quantization

The quantization prescription we have outlined in the previous section is plagued with mathematical problems. For instance, we have evidently not given a proper definition of the Hilbert space, the scalar product, the Hamiltonian etc. but only a formal one (as is done in much of the physics literature, see e.g. [31, 32, 33]). This is not merely an aesthetic point, but it conceals some genuine problems that arise when we take the formal expressions too literally. One of the most important issues has to do with the definition of composite fields like arbitrary powers of the field $\varphi^p(\mathbf{x})$, which are needed e.g. to give mathematical sense to the Hamiltonian H . In fact, naively computing, say, the square of a field yields a divergent expression. The reason for this pathological behaviour is that quantum fields are distributional in nature, and hence a careful definition is required.

In the case of a free theory, there is a simple solution to the problem of pointwise multiplication of fields. In this case, we can split the (free) field into a creation and an annihilation part

$$\varphi_0(t, \mathbf{x}) = a(t, \mathbf{x}) + a^*(t, \mathbf{x}) , \quad (3.10)$$

where it is clear how the two parts are to be defined, see equation (3.7). One then observes [34] that any product of the form $(a^*((\mathbf{p}_i))^m (a(\mathbf{q}_j))^n)$ is a well defined distribution valued in the quadratic forms on a suitable domain. If F and G are two functionals of the field $\varphi(t, \mathbf{x})$, we define the “normal ordered product” (or “Wick product”) of F and G by the prescription that each field operator appearing in F and G be expanded as in (3.10) and in the resulting expression for the product FG the order of creation and annihilation operators is changed so that all creation operators stand on the left of the annihilation operators. We will denote such a normal ordered product by $:FG:$ and for the simple product of two fields (the “Wick square”) we have

$$:\varphi_0^2(t, \mathbf{x}): = a^*(t, \mathbf{x})a^*(t, \mathbf{x}) + 2a^*(t, \mathbf{x})a(t, \mathbf{x}) + a(t, \mathbf{x})a(t, \mathbf{x}) . \quad (3.11)$$

One can then obtain the quantum Hamiltonian from its classical counterpart but with the normal ordering prescription just defined. One obtains

$$H_0 = \int_{\mathbf{R}} d^d \mathbf{p} \omega(\mathbf{p}) a(\mathbf{p})^* a(\mathbf{p}) . \quad (3.12)$$

We also note that one property of the normal ordering is that the vacuum expectation

value of any normal ordered quantity is zero, as a result of the action of the annihilation operator on the vacuum (3.9). One can similarly define $H = H_0 + \lambda V$ by normal ordering, see Section §3.2.4. However, a major task is to show that expressions like e^{itH} are mathematically meaningful, and this will require a much more involved analysis, which we outline in Section §3.2.4. To make things work, and for the stability of the model, it is essential that H is bounded as an operator from below by a constant $-O(\lambda)L \cdot I$ when $\lambda \geq 0$. This is a highly non-obvious fact, because, although we have that $\lambda P(\xi) \geq 0$ for any $\xi \in \mathbb{R}$, the potential V is *not* positive definite as an operator but in fact even unbounded from below. The latter is an unavoidable consequence of the above normal ordering prescription without which the expression for V would be ill-defined. The point is, however that the sum $H = H_0 + \lambda V$ is bounded from below by a constant $-O(\lambda)L$ (times the identity operator I) [35], i.e. we have that $H \geq -O(\lambda)L \cdot I$ in the sense of operators, see theorem (3.12). This is essentially because one can show that for states $\Psi \in \mathcal{H}$ for which $(\Psi, V\Psi)$ becomes very negative, the contribution $(\Psi, H_0\Psi)$ becomes very positive, and in effect overcompensates the negative contribution from the potential V .

3.2.3 The axiomatic approach

A proper mathematical construction of the interacting theory described by the Hamiltonian (3.4) will be given in the next subsections. But before we do that, we want to specify some general properties such a construction should satisfy. In particular, we briefly present the axiomatic approach to QFT within the setting of the “Wightman axioms” [25, 30]. These are a set of axioms that were introduced in an attempt to separate the essential features of any (rigorous) quantum field theory, which we want to state in mathematically precise terms, from the specific properties pertaining to the various models. Rigorous QFT then amounts to an exploration of the consequences of these axioms⁴. They are:

A. Hilbert space and Poincaré group. We have a Hilbert space \mathcal{H} which carries a unitary representation of the covering group of the Poincaré group $\overline{\mathfrak{P}}$. In \mathcal{H} there is precisely one state Ω , the physical vacuum, which is invariant under all $U(g)$, $g \in \mathfrak{P}$, i.e. the vacuum state is invariant under the action of the Poincaré group \mathfrak{P} . Finally, the spectrum of the energy-momentum operators P^μ is confined

⁴There are two alternative formulation of the axioms for QFT, the “Haag–Kastler” axioms [36] for local algebras of field operators, and the “Osterwalder–Schrader” axioms [37, 38] for the Schwinger functions of Euclidean Quantum Field Theory. We present the Wightman axioms only as they are closer to the approach adopted in this work.

to the (closed) forward light cone. This is the *spectrum condition*.

B. Fields. Fields are “operator valued distributions” [29] over Minkowski space. This means that $\varphi(f) = \int \varphi(x)f(x) d^4x$, with f in Schwartz space $\mathcal{S}(\mathbb{R}^4)$, is an (unbounded) operator, defined on a dense set $\mathcal{D} \subset \mathcal{H}$, which contains the vacuum Ω . The domain \mathcal{D} should also be invariant under the application of the operators $U(g)$, $g \in \mathfrak{P}$. In general there will be several fields in a theory, each of which may have several tensor or spinor components. Correspondingly we must take test functions for each type of field (index i) and each component (index μ) and understand $\varphi(f)$ generically as

$$\varphi(f) = \sum_{i,\mu} \int \varphi_{\mu}^i(x) f^{i\mu}(x) d^4x .$$

The set of all fields contains with each φ also its hermitian conjugate φ^* , defined as a sesquilinear form on \mathcal{D} through $\langle \psi_1 | \varphi^*(x) | \psi_2 \rangle = \overline{\langle \psi_1 | \varphi(x) | \psi_2 \rangle}$.

C. Transformation Properties. The fields transform under $\overline{\mathfrak{P}}$ as

$$U(a, \alpha) \varphi_{\mu}^i(x) U^{-1}(a, \alpha) = M_{\mu}^{(i)\nu}(\alpha^{-1}) \varphi_{\nu}^i(\Lambda(\alpha)x + a) ,$$

in the sense of distributions. Here $M_{\mu}^{(i)\nu}(\alpha)$ is a finite dimensional representation matrix of $\alpha \in \overline{\mathfrak{L}}$.

D. Causality. The fields should satisfy causal commutation relations of either bosonic or fermionic type. If the supports of the test functions f and h are spacelike to each other then

$$[\varphi^i(f), \varphi^j(h)]_{\mp} = 0, \quad [\varphi^i(f), \varphi^j(h)]_{\mp} = \begin{cases} \varphi^i(f)\varphi^j(h) - \varphi^j(h)\varphi^i(f) & \text{“(-)”} \\ \varphi^i(f)\varphi^j(h) + \varphi^j(h)\varphi^i(f) & \text{“(+)”} \end{cases} ,$$

where the “-” sign refers to bosonic fields and the “+” refers to fermionic fields.

E. Completeness of the framework. By taking linear combinations of products of the (smeared) field operators $\varphi(f)$ we should be able to approximate any operator acting on \mathcal{H} to arbitrary precision. This may be expressed by saying that \mathcal{D} contains no subspace which is invariant under all $\varphi(f)$ and whose closure is a proper subspace of \mathcal{H} .

Remark. The study of the consequences of these axioms has frequently been referred to as “Axiomatic QFT”. This is appropriate insofar as it clearly outlines the starting point of the investigation for a rigorous construction of QFT models. On the other hand, the above axioms should not be understood as “cast in stone”. They are to be regarded as working assumptions that might have to be modified as deeper understanding is gained.

The Wightman axioms aim at formulating precisely the essential features of quantum mechanics and relativity. Some of them are rather technical in nature while others have a more clear cut physical interpretation. More specifically, the meaning is as follows. Axiom **A** states that the theory ought to be formulated on a Hilbert space appropriate to account for Poincaré transformations. This is a fairly obvious requirement as Poincaré invariance is an accepted fundamental symmetry of physical laws in flat spaces. That such a Hilbert space should possess a vacuum state amounts to require that there be a state which corresponds to the intuitive idea of “emptiness”, i.e. a state in which “nothing interesting happens”. Such a state also ought to be of “minimum energy” so it is impossible to extract energy from it. In other words, the Hamiltonian should be bounded from below. This is the content of the “spectrum condition”. Axioms **B** and **C** are technical and state the mathematical nature of the quantum fields and how they transform under the action of the Poincaré group. Causality as expressed by axiom **D** has a profound yet simple physical interpretation. We know that in quantum mechanics if two operators commute then we could simultaneously determine their expectation values. On the other hand, the finite speed of light in special relativity implies that events that are spacelike separated cannot influence each other. Having the two field operators commute at spacelike separations is then a quantum mechanical implementation of Einstein’s causality, in the sense that measurements at spacelike points “cannot influence each other”. Finally, axiom **E** is another technical axiom and it asserts that it should be possible to approximate all the observables of a theory by means of field operators. In other words, fields should be the “building blocks” of the theory.

It is clear from this discussion that the Wightman axioms comprise a set of really “mild” conditions that a quantum field theory ought to satisfy. It is not surprising then that they are in fact not vacuous: Free quantum field satisfy them. The constructive approach to QFT has also provided examples of interacting quantum field theories in 2 and 3 spacetime dimensions. In this work we will build on a construction by Glimm and Jaffe for the two dimensional case of the model (3.4) (cf. [39, 40, 41, 42]), sometimes called the $P(\varphi)_2$ model⁵. The central idea of this construction is to consider, as an

⁵It has become common to denote the dimensionality of the spacetime by a subscript. This means

intermediate step, a regularized Hamiltonian, which is defined on the usual bosonic Fock space. This can be achieved by truncating the Fourier representation of the field for momenta larger than a “UV-cutoff” parameter Λ and by restricting the interaction to a finite volume only, i.e. introducing an “IR cutoff”—see next subsection for more details on these cutoffs. One then defines a (quantum) version of the Hamiltonian (3.4) with these cutoffs, and the point is to show that when the cutoffs are removed, the resulting Hamiltonian is a well defined self-adjoint operator (in a rigorous sense) on some Hilbert space \mathcal{H}^6 , which is bounded from below [cf. discussion after equation (3.12)]. In this work we will for the best part be concerned with an interacting scalar quantum field on a (1+1)-dimensional spacetime with compact spatial slices (a “cylinder”) so that the IR cutoff is intrinsically present and we will not need to impose an additional one. The methods that we will outline can establish that the model satisfies the Wightman axioms⁷, with obvious modifications of axioms A and C.

The approach and the methods we will present is not the only possible one: A completely different approach is the so called “path integral” approach, see e.g. [43, 44]. We will not use it in this work and we only mention it for completeness. The central issue of that approach is to meaningfully define the integration measure of the path integral, which also ought to satisfy some additional properties, most notably “reflection positivity”. This is done in the Euclidean setting and the quantum field theory on Minkowski space is then recovered by appealing to the previously mentioned Osterwalder–Schrader axioms.

We also want to remark that the constructive approach was successfully applied in 3 spacetime dimensions as well, albeit the results are technically much more complicated to obtain, and the work of Glimm and Jaffe [45] and by Feldman with Osterwalder [46], concluded the construction of the $\lambda\varphi_3^4$ theory in 3 spacetime dimensions. The reason for the increased level of technical complication is related to the fact that in 2 dimensions the fields are dimensionless while in 3 dimensions they have mass dimension $[1/2]$. This entails, e.g. that field monomials of higher and higher power are correspondingly more and more singular, which is the reason for the construction of $\lambda\varphi_3^4$ theory “only” as opposed to theories with arbitrary polynomial interactions like in the 2 dimensional case. As for the 4 dimensional case, we already noted that a rigorous construction of a

that $\lambda\varphi_2^4$ refers to an interacting model with quartic (self-)interaction in two spacetime dimensions. In general, the interacting scalar quantum field in two dimensions with polynomial interaction is denoted by $P(\varphi)_2$.

⁶It turns out that the Hilbert space on which the interacting theory is meaningfully defined non-perturbatively is *not* Fock space. On the other hand, if we content ourselves with the definition of the theory on a finite volume, than the resulting interacting Hilbert space can be taken to be Fock space.

⁷Not only the Wightman axioms, it has been showed that it satisfies the Haag–Kastler and the Osterwalder–Schrader axioms as well.

non-trivial, interacting quantum field theory has still not been achieved.

3.2.4 Basic features of the $P(\varphi)_2$ model

As we indicated, the construction of a quantum field theory is “plagued” with two big problems. The first one is the so called “ultraviolet or UV problem”, and is associated with the impossibility of pointwise multiplication of quantum fields, which intrinsically are operator valued distributions. Because of this it is also said that the UV problem is associated with the “short distance” behaviour of quantum fields. The second one, on the other hand, which is commonly referred to as the “infrared⁸ or IR problem”, is associated to the fact that Minkowski space is not spatially compact. This leads to complications when constructing the Hamiltonian, the Hilbert space on which it acts, the vacuum vector and so on. The problem is associated to the infinite extension of the spatial section, so the IR problem is also known as “infinite volume” problem.

The general approach to the rigorous construction of a model follows Wightman’s suggestion [47] to define a theory as limit of “cutoff” models, i.e. of models that are restricted both in momentum space, for the UV problem, and in configuration space for the IR problem. In this work, however, we will only be concerned with the non-perturbative existence of a quantum field theory on the cylinder, i.e. a spatially compact spacetime. As a consequence the construction is simpler because we do not have to worry about the IR problem as we have an intrinsic spatial cutoff. In addition, as we already indicated, to deal with the UV problem we only need to normal order the Wick powers in order to obtain a UV finite theory, which is a specific property of 1 + 1-dimensional spacetime. Finally, we also note that we will merely outline the construction of the model referring the interested reader to the books by Glimm and Jaffe [35] and Reed and Simon [48] for further details.

The spacetime we consider is the 1 + 1-dimensional cylinder $\mathbb{R} \times \mathbb{S}$ with metric

$$ds^2 = -dt^2 + L^2 dx^2 .$$

Here L is the radius of the circle and \mathbf{x} is a 2π periodic coordinate parametrizing the circle so that the “volume” of the spatial section is $2\pi L$. We will equivalently denote a generic element of the spacetime as $x = (x^0, x^1) = (t, \mathbf{x})$. Similarly, the (vector) 2-momentum will be denoted by $p = (p^0, p^1) = (\omega, \mathbf{p})$, while “on-shell” momenta will be denoted as $\underline{p} = (\omega_{\mathbf{p}}, \mathbf{p})$, with $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2/L^2 + m^2}$ and $\mathbf{p} \in \mathbb{Z}$. Finally, we define the

⁸Not to be confused with what is typically meant with IR divergencies when dealing with theories with massless particles.

Fourier transform on the cylinder as

$$f(\omega, \mathbf{p}) := \int_{\mathbb{R}} \frac{dt}{\sqrt{2\pi}} \int_0^{2\pi} \frac{dx}{\sqrt{2\pi}} f(t, \mathbf{x}) e^{i\langle p, x \rangle}, \quad f(t, \mathbf{x}) := \int_{\mathbb{R}} \frac{d\omega}{\sqrt{2\pi}} \sum_{\mathbf{p} \in \mathbb{Z}} \frac{f(\omega, \mathbf{p})}{\sqrt{2\pi}} e^{-i\langle p, x \rangle},$$

where by another slight abuse of notation we use the same symbol f to denote both a function and its Fourier transform. No confusion should arise as it is easily seen which one is meant from the argument. We also use

$$\langle p, x \rangle := -t\omega + \mathbf{p}\mathbf{x}, \quad \mathbf{p} \in \mathbb{Z}.$$

The Hilbert space we consider is the bosonic Fock space

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}_n = \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathbb{P}_n \left[\underbrace{\ell^2(\mathbb{Z}) \otimes \cdots \otimes \ell^2(\mathbb{Z})}_n \right], \quad (3.13)$$

where the summand $\mathcal{H}_0 = \mathbb{C}$ represents the “no particle” subspace and contains the (free theory “vacuum”) vector $\Omega_0 = (1, 0, 0, \dots)$ whereas $\mathcal{H}_n = \mathbb{P}_n[\ell^2(\mathbb{Z}) \otimes \cdots \otimes \ell^2(\mathbb{Z})]$ is the “ n -particle” subspace. We note that Ω_0 does *not* represent the physical vacuum of the interacting theory, i.e. the ground state of the interacting Hamiltonian H , but is an auxiliary quantity in our construction. It would be the true vacuum only if we were trying to construct the free theory. \mathbb{P}_n projects onto the subspace of totally symmetric rank n tensors over $\ell^2(\mathbb{Z})$, the 1-particle Hilbert space of square summable sequences. That is, \mathcal{H}_n is spanned by vectors Ψ_n of the form

$$\Psi_n = \mathbb{P}_n[\psi^{(1)} \otimes \cdots \otimes \psi^{(n)}] = \frac{1}{\sqrt{n!}} \sum_{\pi} \psi^{(\pi_1)} \otimes \cdots \otimes \psi^{(\pi_n)}, \quad (3.14)$$

where the sum above is over all permutations of the indices π_i , and $\psi^{(i)} \in \mathcal{H}_1 = \ell^2(\mathbb{Z})$. The scalar product on \mathcal{H} is the one inherited from $\ell^2(\mathbb{Z})$. Let $\psi_{\mathbf{p}}, \phi_{\mathbf{p}} \in \ell^2(\mathbb{Z})$. We find it convenient to define their scalar product as

$$\langle \psi | \phi \rangle_{\ell^2} = \frac{1}{L} \sum_{\mathbf{p} \in \mathbb{Z}} \bar{\psi}_{\mathbf{p}} \phi_{\mathbf{p}}, \quad (3.15)$$

where $\bar{\psi}_{\mathbf{p}}$ denotes the complex conjugate of $\psi_{\mathbf{p}}$. Furthermore, if $\Psi = (\Psi_0, \Psi_1, \dots)$ and

$\Phi = (\Phi_0, \Phi_1, \dots)$ are in \mathcal{H} , their scalar product is simply

$$\langle \Psi | \Phi \rangle_{\mathcal{H}} := \sum_{n=1}^{\infty} \langle \Psi_n | \Phi_n \rangle_{\mathcal{H}_n} ,$$

where $\langle \Psi_n | \Phi_n \rangle_{\mathcal{H}_n}$ can be obtained by combining

$$\langle \psi_1 \otimes \dots \otimes \psi_n | \phi_1 \otimes \dots \otimes \phi_n \rangle := \prod_{i=1}^n \langle \psi_i | \phi_i \rangle_{\ell^2}$$

with equation (3.14). Finally, we also find it convenient to introduce an orthonormal basis on \mathcal{H}_1 , the “momentum eigenbasis”, consisting of sequences

$$|\mathbf{p}\rangle = (\dots, 0, \dots, 0, 1, 0, \dots, 0, \dots) ,$$

where the 1 appears exactly in the \mathbf{p} -th place. We then see that we can write

$$\psi_{\mathbf{p}} = \langle \mathbf{p} | \psi \rangle_{\ell^2} .$$

Equation (3.14) can equivalently be written as

$$\Psi_n(\mathbf{p}_1, \dots, \mathbf{p}_n) = \frac{1}{\sqrt{n!}} \sum_{\pi} \psi_{\mathbf{p}_1}^{(\pi_1)} \dots \psi_{\mathbf{p}_n}^{(\pi_n)} ,$$

which we will use to emphasize the momentum dependence of an n -particle vector.

We next define the creation, $a^*(\eta)$, and annihilation, $a(\eta)$, operators for a sequence $\eta \in \ell^2(\mathbf{Z})$ as follows:

$$\begin{aligned} a^*(\eta)\Psi_n &:= \mathbb{P}_{n+1}[\eta \otimes \psi_1 \otimes \dots \otimes \psi_n] , \\ a(f)\Psi_n &:= \sum_{i=1}^n \langle \eta, \psi_i \rangle_{\ell^2} \mathbb{P}_{n-1}[\psi_1 \otimes \dots \otimes \cancel{\psi_i} \otimes \dots \otimes \psi_n] , \end{aligned} \tag{3.16}$$

where $\cancel{\psi_i}$ means that the corresponding element is to be omitted. It is easy to check that the above definition is such that creation and annihilation operators are each other’s adjoint. The names of the above operators come from the physical interpretation of their action: The creation operator acts on an n -particle “state” (vector) to create an $(n+1)$ -particle state, with the new “particle” having “wave function” η . A similar interpretation pertains to the annihilation operator, i.e. it decreases the number of particles in a given state by 1. An immediate consequence of the above definition is $a(\eta)\Omega_0 = 0$. Finally, it is a matter of straightforward computation to evaluate the action of the commutator

of the annihilation and the creation operator on a vector Ψ_n . In operator form, the commutator is

$$[a(\eta), a^*(\xi)] = \langle \eta | \xi \rangle e^2, \quad [a(\eta), a(\xi)] = [a^*(\eta), a^*(\xi)] = 0. \quad (3.17)$$

To make contact with the definitions typically found in the physics literature and with our heuristic quantization from Section §3.2.1, we introduce creation and annihilation “densities” $a_{\mathbf{p}}^*$ and $a_{\mathbf{p}}$ such that the creation and annihilation operators can be expressed as

$$a^*(\eta) = \frac{1}{L} \sum_{\mathbf{p} \in \mathbb{Z}} a_{\mathbf{p}}^* \eta_{\mathbf{p}}, \quad a(\eta) = \frac{1}{L} \sum_{\mathbf{p} \in \mathbb{Z}} a_{\mathbf{p}} \bar{\eta}_{\mathbf{p}}. \quad (3.18)$$

Their action is given by

$$(a_{\mathbf{p}}^* \Psi_n)_{n+1}(\mathbf{p}_1, \dots, \mathbf{p}_{n+1}) := \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} L \delta(\mathbf{p}, \mathbf{p}_i) \Psi_n(\mathbf{p}_1, \dots, \mathbf{p}_{i-1}, \dots, \mathbf{p}_{n+1}), \quad (3.19)$$

$$(a_{\mathbf{p}} \Psi_n)_{n-1}(\mathbf{p}_1, \dots, \mathbf{p}_{n-1}) := \sqrt{n} \Psi_n(\mathbf{p}, \mathbf{p}_1, \dots, \mathbf{p}_{n-1}).$$

On a spacetime with compact spatial section—in our case this is S^1 —this form of the creation and annihilation operators is equivalent to equation (3.16). On the other hand, for spacetimes with non-compact spatial section (like \mathbb{R} , see section §3.2.1), it is only the “smeared”⁹ form (3.16) that makes mathematical sense. Using the Definitions (3.19), the commutation relations (3.17) are then rewritten as

$$[a_{\mathbf{p}}, a_{\mathbf{k}}^*] = L \delta(\mathbf{p}, \mathbf{k}), \quad [a_{\mathbf{p}}, a_{\mathbf{k}}] = [a_{\mathbf{p}}^*, a_{\mathbf{k}}^*] = 0,$$

as can be seen by (careful) direct computation using (3.19) or, more simply, by using the definition of the scalar product (3.15) in the commutator (3.17) while using the densities (3.18). The above expression is valid in the sense of quadratic forms on their natural domain. This form of the canonical commutation relations for the creation and annihilation operator is commonly found in standard books on quantum field theory and we will often use it in the main body of this work.

With the creation and annihilation operators we can now construct the time zero (free) field operators $\varphi_0(\eta)$ and $\pi_0(\eta)$ —they are the starting point for the construction of the interacting quantum field theory. For a positive value of the “mass” $m > 0$, we

⁹With a slight abuse of language we use the terminology appropriate for the continuum case also in the present context.

set

$$\begin{aligned}\varphi_0(\eta) &= \frac{1}{\sqrt{2}} \left[a(\omega^{-1/2}\eta) + a^*(\omega^{-1/2}\eta) \right], \\ \pi_0(\eta) &= -i \frac{L}{\sqrt{2}} \left[a(\omega^{1/2}\eta) - a^*(\omega^{1/2}\eta) \right],\end{aligned}$$

where $\omega : \ell^2(\mathbf{Z}) \rightarrow \ell^2(\mathbf{Z})$ is the multiplication operator

$$(\omega\eta)(\mathbf{p}) = \omega_{\mathbf{p}}\eta_{\mathbf{p}},$$

and $\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2/L^2 + m^2}$. As one can show, $\varphi_0(\eta)$ and $\pi_0(\eta)$ are essentially self-adjoint on $\mathcal{F}_0 \subset \mathcal{H}$, the set of “finite particle number” vectors, i.e. the vectors that have at most a finite number of non-zero components in the direct sum (3.13). We refer the reader to section X.7 of [48] for a proof of the following fundamental

Theorem 3.1. *The operators $\varphi_0(\eta)$ and $\pi_0(\eta)$ are essentially self-adjoint on \mathcal{F}_0 , and for real $\eta, \xi \in \ell^2(\mathbf{Z})$ they satisfy the canonical commutation relations*

$$[\varphi_0(\eta), \pi_0(\xi)] = iL\langle\eta|\xi\rangle_{\ell^2}, \quad [\varphi_0(\eta), \varphi_0(\xi)] = [\pi_0(\eta), \pi_0(\xi)] = 0,$$

in the sense of quadratic forms on their natural domains.

Alternatively we could construct the free fields using the creation and annihilation operators in their “density” form (3.19). Rather than expressing them in momentum space, in position space they have the familiar form

$$\begin{aligned}\varphi_0(\mathbf{x}) &= \frac{1}{L(2\pi)^{1/2}} \sum_{\mathbf{p} \in \mathbf{Z}} \frac{1}{(2\omega_{\mathbf{p}})^{1/2}} \left[a_{\mathbf{p}} e^{i\mathbf{p}\mathbf{x}} + a_{\mathbf{p}}^* e^{-i\mathbf{p}\mathbf{x}} \right], \\ \pi_0(\mathbf{x}) &= \frac{-i}{(2\pi)^{1/2}} \sum_{\mathbf{p} \in \mathbf{Z}} \left(\frac{\omega_{\mathbf{p}}}{2} \right)^{1/2} \left[a_{\mathbf{p}} e^{i\mathbf{p}\mathbf{x}} - a_{\mathbf{p}}^* e^{-i\mathbf{p}\mathbf{x}} \right].\end{aligned}\tag{3.20}$$

More precisely, the time zero fields are operator valued distributions (OVDs): For a real valued test function η

$$\begin{aligned}\int_0^{2\pi} d\mathbf{x} \varphi_0(\mathbf{x})\eta(\mathbf{x}) &= \frac{1}{L\sqrt{2}} \sum_{\mathbf{p} \in \mathbf{Z}} \omega_{\mathbf{p}}^{-1/2} \left[a_{\mathbf{p}} \bar{\eta}_{\mathbf{p}} + a_{\mathbf{p}}^* \eta_{\mathbf{p}} \right] = \\ &= \frac{1}{\sqrt{2}} \left[a(\omega^{-1/2}\eta) + a^*(\omega^{1/2}\eta) \right] = \varphi_0(\eta).\end{aligned}$$

By using the results of Theorem 3.1 it is also easy to see that the CCRs (3.6) hold, a

result which has been rigorously established and not just assumed.

Our ultimate goal is to construct the Hamiltonian H , formally given by (3.4), as a self-adjoint operator on some suitable domain $\mathcal{D} \subset \mathcal{H}$. To do this, we proceed in three steps. We will want to construct our Hamiltonian by first of all splitting it in a “free” part H_0 and an interaction potential λV . These are formally given by

$$H_0 = \frac{L}{2} \int_0^{2\pi} dx \left[L^{-2} \pi_0^2(\mathbf{x}) + L^{-2} (\partial_{\mathbf{x}} \varphi_0(\mathbf{x}))^2 + m^2 \varphi_0^2(\mathbf{x}) \right],$$

and

$$V = \int_0^{2\pi} dx P(\varphi),$$

where $P(\xi)$ is an arbitrary positive polynomial of degree higher than two. The three steps are

1. Define H_0 as a self-adjoint operator on $\mathcal{D}(H_0) \subset \mathcal{H}$.
2. Define a cutoff potential V^Λ and define $V = \lim_{\Lambda \rightarrow \infty} V^\Lambda$ as a self-adjoint operator on $\mathcal{D}(V) \subset \mathcal{H}$.
3. Prove that $H = H_0 + \lambda V$ is essentially self-adjoint on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$ and bounded below.

Because we can show that H is essentially self adjoint, we can define in a mathematically unambiguous way the time evolution operators e^{itH} . Using the “ φ -bounds” of Glimm and Jaffe (see e.g. [42, 49, 50]) we can moreover define the spacetime field operators $\varphi(t, \mathbf{x}) = e^{itH} \varphi(0, \mathbf{x}) e^{-itH}$ as operator valued distributions on \mathcal{H} , i.e., if $f \in C_0^\infty(\mathbb{R} \times \mathbb{S})$, then $\varphi(f) = \int \varphi(t, \mathbf{x}) f(t, \mathbf{x})$ is essentially self-adjoint e.g. on the domain $\cap_j \mathcal{D}(H^j) \subset \mathcal{H}$.

All the operators we want to construct, H_0 , V and H , contain products of $t = 0$ fields at coinciding spatial points. We hence employ the “normal ordering” prescription, i.e. the prescription according to which in any product of fields all the creation operators should be moved to the left of the annihilation operators¹⁰. We have the following formula for a normal ordered product of fields, understood in the sense of quadratic forms on $\mathcal{F}_0 \times \mathcal{F}_0$

$$:\varphi_0(\mathbf{x}_1) \cdots \varphi_0(\mathbf{x}_n): := \sum_{l=0}^n \sum_{\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbb{Z}} C_n^l(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_n, \mathbf{p}_n) \left(\prod_{j \in X_l} a_{\mathbf{p}_j}^* \right) \left(\prod_{j \in Y_l} a_{\mathbf{p}_j} \right), \quad (3.21)$$

¹⁰In Appendix A we give an alternative but equivalent formulation of normal ordering by means of subtraction of the (free theory) vacuum expectation value of a product of fields.

where $X_l = \{1, \dots, l\}$ for $l \neq 0$ and $X_0 = \emptyset$, $Y_l = \{1, \dots, n\} \setminus X_l$, and the kernel is

$$C_n^l(\mathbf{x}_1, \mathbf{p}_1, \dots, \mathbf{x}_n, \mathbf{p}_n) = \left(\prod_{j=1}^n \frac{1}{2(\pi\omega_{\mathbf{p}_j})^{1/2}} \right) \exp\left(-i \sum_{j \in X_l} \mathbf{p}_j \mathbf{x}_j + i \sum_{j \in Y_l} \mathbf{p}_j \mathbf{x}_j\right). \quad (3.22)$$

This definition is easily understood by recalling the field expansion (3.20). Similarly one could extend the definition to include the conjugate momentum field $\pi(\mathbf{x})$ and derivatives of the field $\partial_{\mathbf{x}}\varphi(\mathbf{x})$. The important thing to keep in mind is to move the creation operators to the left of the annihilation operators. The normal ordering prescription defines the pointwise product of fields as a quadratic form on $\mathcal{F}_0 \times \mathcal{F}_0$. This quadratic form is actually smooth in the variables \mathbf{x}_i , meaning that the map

$$(\mathbf{x}_1, \dots, \mathbf{x}_n) \mapsto \langle \Psi_1 | : \varphi_0(\mathbf{x}_1) \dots \varphi_0(\mathbf{x}_n) : \Psi_2 \rangle,$$

is a smooth map in the \mathbf{x}_i for arbitrary but fixed Ψ_1 and Ψ_2 . We can therefore consider $\mathbf{x}_1, \dots, \mathbf{x}_n \rightarrow \mathbf{x}$ in (3.21) and construct powers of the field. In addition, the expectation value of any normal ordered product of field in the state Ω_0 (the free theory vacuum state) is automatically zero because of the action of the annihilation operator.

The free Hamiltonian We are now ready to define the (free) hamiltonian H_0 . According to the correspondence principle we ought to replace the classical expression defining the dynamics of the classical theory with a corresponding quantum expression. Using normal ordering we can now meaningfully define H_0 as a quadratic form on $\mathcal{F}_0 \times \mathcal{F}_0$ as follows, see [35]:

$$H_0 := \frac{L}{2} \int_0^{2\pi} dx : L^{-2} \pi_0^2(\mathbf{x}) + L^{-2} (\partial_{\mathbf{x}} \varphi_0(\mathbf{x}))^2 + m^2 \varphi_0^2(\mathbf{x}) :. \quad (3.23)$$

In fact we have(see [35])

Theorem 3.2. *As a quadratic form on $\mathcal{F}_0 \times \mathcal{F}_0$ we have*

$$H_0 = \frac{1}{L} \sum_{\mathbf{p} \in \mathbb{Z}} \omega_{\mathbf{p}} a_{\mathbf{p}}^* a_{\mathbf{p}}. \quad (3.24)$$

Moreover, H_0 leaves each subspace $\mathcal{H}_n \subset \mathcal{H}$ invariant, and on \mathcal{H}_n is the multiplication operator

$$(H_0 \Psi_n)_n(\mathbf{p}_1, \dots, \mathbf{p}_n) = \left(\sum_{i=1}^n \omega_{\mathbf{p}_i} \right) \Psi_n(\mathbf{p}_1, \dots, \mathbf{p}_n).$$

Its action on the (free theory vacuum) vector Ω_0 is

$$H_0 \Omega_0 = 0 ,$$

and H_0 is essentially self-adjoint on \mathcal{F}_0 .

From now on, we let H_0 denote the self-adjoint operator with domain

$$\mathcal{D}(H_0) = \left\{ \Psi = (\Psi_0, \Psi_1, \dots) \in \mathcal{H} : \sum_{n=1}^{\infty} \left\| (\omega_{\mathbf{p}_1} + \dots + \omega_{\mathbf{p}_n}) \Psi_n(\mathbf{p}_1, \dots, \mathbf{p}_n) \right\|_{\mathcal{H}_n}^2 < \infty \right\} .$$

With the free Hamiltonian as a self-adjoint operator we can now define the “time evolved” free fields $\varphi_0(t, \mathbf{x})$ as

$$\varphi_0(t, \mathbf{x}) := e^{itH_0} \varphi_0(\mathbf{x}) e^{-itH_0} ,$$

where the exponentials are well defined unitary operators.

The above discussion essentially completes the mathematically rigorous construction of the free scalar quantum field theory on the circle. One important aspect of the construction is the proof of the consistency with the Wightman axioms. We will not detail this and merely note that it can be done, referring the interested reader to Section X.7 (Theorem X.42) of [48] for further details. For us it is more important to proceed and outline the construction of the fully interacting theory, and we now turn our attention to the analysis of the interaction potential.

The interaction potential Let $P(\xi) = \sum_{n=4}^p b_n \xi^n$ be an arbitrary but positive polynomial¹¹. We define our interaction potential to be

$$\lambda V = \lambda L \int_0^{2\pi} dx :P(\varphi_0): , \tag{3.25}$$

in the sense of quadratic forms on $\mathcal{F}_0 \times \mathcal{F}_0$. Strictly speaking, the parameter λ above is redundant and could be omitted. However, we will keep it as in the main body of this thesis we will consider an expansion in terms of a “small” parameter, and it is simpler to have just one such parameter rather than many. We would like to understand V better, and in particular we will need to show that it is essentially self-adjoint on a suitable

¹¹We only consider polynomials of degree 4 or higher because a cubic term spoils the positivity requirement for the polynomial whereas lower powers would not lead to an interacting theory.

domain. For this we introduce the momentum cutoff field $\varphi_\Lambda(\mathbf{x})$

$$\varphi_\Lambda(\mathbf{x}) = \frac{1}{L(2\pi)^{1/2}} \sum_{|\mathbf{p}| < \Lambda} (2\omega_{\mathbf{p}})^{-1/2} \left[a_{\mathbf{p}} e^{i\mathbf{p}\mathbf{x}} + a_{\mathbf{p}}^* e^{-i\mathbf{p}\mathbf{x}} \right],$$

and the corresponding cutoff potential

$$V^\Lambda = L \int_0^{2\pi} d\mathbf{x} : P(\varphi_\Lambda) : . \quad (3.26)$$

It is not difficult to see that V^Λ has kernel [cf. equation (3.22)]

$$v^\Lambda(\mathbf{k}_1, \dots, \mathbf{k}_n) = b_n \frac{2\pi}{L^{-n+1}} (4\pi)^{-n/2} (\omega_{\mathbf{k}_1} \cdots \omega_{\mathbf{k}_n})^{-1/2} \delta\left(0, \sum_{i=1}^n \mathbf{k}_i\right).$$

For later we note (Theorem 1.2.5 of [35]):

Theorem 3.3. *Let $\epsilon > 0$. Then*

$$\sup_i \left(\omega_{\mathbf{k}_i}^{-\epsilon+1/2} v^\Lambda \right) \in \ell^2(\mathbf{Z}^n).$$

Intuitively, one would expect $\lim_{\Lambda \rightarrow \infty} V^\Lambda = V$. And indeed, in the proof of Theorem X.45 of [48], it is shown that $V^\Lambda \Psi \rightarrow V\Psi$ for $\Psi \in \mathcal{F}_0$.

Introduce the total number operator N , which we define for all $\Psi \in \mathcal{H}$ as

$$N\Psi = (0, \Psi_1, 2\Psi_2, \dots, n\Psi_n, \dots).$$

It is easy to see that

$$\sum_{\mathbf{p} \in \mathbf{Z}} a_{\mathbf{p}}^* a_{\mathbf{p}} = N.$$

Also, following Glimm and Jaffe [35], let us introduce the operators N_τ as

$$N_\tau = \frac{1}{L^\tau} \sum_{\mathbf{p} \in \mathbf{Z}} \omega_{\mathbf{p}}^\tau a_{\mathbf{p}}^* a_{\mathbf{p}}.$$

We immediately see that for $\tau = 0$ this is just the number operator N , and for $\tau = 1$ it is simply the free Hamiltonian H_0 . For a kernel $B \in \ell^2(\mathbf{Z}^{r+s})$, we define a “generalized Wick monomial”

$$T_B := \frac{1}{L^{r+s}} \sum_{\mathbf{p}_i, \mathbf{k}_j \in \mathbf{Z}} B(\mathbf{p}_1, \dots, \mathbf{p}_r, \mathbf{k}_1, \dots, \mathbf{k}_s) \left(\prod_{i \in X} a_{\mathbf{p}_i}^* \right) \left(\prod_{j \in Y} a_{\mathbf{k}_j} \right), \quad (3.27)$$

where $X = \{1, \dots, r\}$ and $Y = \{1, \dots, s\}$. The Wick monomials T_B are to be understood in the sense of quadratic forms on $\mathcal{F}_0 \times \mathcal{F}_0$. The proof of the following bounds involving the Wick monomials can be found in the literature, see Theorems 1.2.2 and 1.2.3 of [35], and we only state the result as a

Theorem 3.4 (N_τ estimates). *The following bounds hold*

$$1. \quad \left\| \left(\prod_{i=1}^r N_{\tau_i}^{-1/2} \right) T_B \left(\prod_{j=1}^s N_{\tau_j}^{-1/2} \right) \right\| \leq \left\| \prod_{i=1}^{r+s} \omega_{p_i}^{-\tau_i/2} B \right\|_{\ell^2(\mathbf{Z}^s) \rightarrow \ell^2(\mathbf{Z}^{2r+s})},$$

and in particular

$$\left\| N_\tau^{-r/2} T_B N_\tau^{-s/2} \right\| \leq \left\| \prod_{i=1}^{r+s} \omega_{p_i}^{-\tau/2} B \right\|_{\ell^2(\mathbf{Z}^s) \rightarrow \ell^2(\mathbf{Z}^{2r+s})}.$$

Here $N_\tau^{-1/2}$ is the operator that gives zero on Ω_0 and is the inverse to $N_\tau^{1/2}$ on Ω_0^\perp .

2. In addition, we have

$$\left\| T_B (N + I)^{-p/2} \right\| \leq \|B\|_{\ell^2(\mathbf{Z}^s) \rightarrow \ell^2(\mathbf{Z}^r)}.$$

The relevance of Wick monomials lies in the fact that the important operators in our model can be expressed in terms of (linear combinations of) the former. In fact, using equation (3.24) together with the cutoff potential (3.26) and the normal ordering prescription (3.21), it is not difficult to see that the formula for the (full unshifted) Hamiltonian $H = H_0 + \lambda V$ is given by

$$\begin{aligned} H &= \frac{1}{L} \sum_{\mathbf{k} \in \mathbf{Z}} \omega_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}} + \\ &+ \pi \lambda \sum_{n=3}^p \frac{b_n}{(2\pi)^{n/2}} \sum_{\substack{i \in X, j \in Y \\ X \cup Y = \{1, \dots, n\}}} \frac{1}{L^n} \sum_{\substack{\mathbf{k}_i \in \mathbf{Z} \\ \mathbf{q}_j \in \mathbf{Z}}} \delta\left(\sum_{i \in X} \mathbf{k}_i, \sum_{j \in Y} \mathbf{k}_j\right) \prod_{j \in Y} \frac{a_{\mathbf{k}_j}^*}{\sqrt{2\omega_{\mathbf{k}_j}}} \prod_{i \in X} \frac{a_{\mathbf{k}_i}}{\sqrt{2\omega_{\mathbf{k}_i}}}, \end{aligned} \quad (3.28)$$

clearly in the sense of quadratic forms on $\mathcal{F}_0 \times \mathcal{F}_0$. The central issue in the construction of the $P_2(\varphi)$ model is to prove that the above is an essentially self-adjoint operator on a suitable domain which is also bounded from below.

To further analyze V and to prove the main theorem in the construction of the interacting model, Theorem 3.5, we introduce yet another Hilbert space representation of the Fock space structures presented above. We outline the construction of the so-called ‘‘Q-space’’ and the associated spaces $L^p(\mathbf{Q}, d\nu)$ —see [35] or the end of section

§X.7 of [48] for more details. $(\mathbf{Q}, d\nu)$ is the measure space given (formally) by infinitely many Cartesian copies of $\mathbf{C} \cong \mathbf{R}^2$ and Gaussian probability measure of total weight one,

$$d\nu = \prod_{j \geq 0} \pi^{-1} e^{-|q_j|^2} dq_j d\bar{q}_j .$$

The L^p -spaces mentioned above are nothing but the usual Hölder spaces $L^p(\mathbf{Q}, d\nu)$, i.e. the Banach spaces with the norm

$$\|\Psi\|_p = \left(\int_{\mathbf{Q}} |\Psi|^p d\nu \right)^{1/p} .$$

On \mathbf{Q} -space we have canonical (complex) multiplication and differentiation operators

$$P_{\mathbf{k}}\Psi = -i e^{|\mathbf{q}_{\mathbf{k}}|^2/2} \frac{\partial}{\partial q_{\mathbf{k}}} (e^{-|\mathbf{q}_{\mathbf{k}}|^2/2} \Psi) , \quad Q_{\mathbf{k}}\Psi = q_{\mathbf{k}}\Psi ,$$

where $q_{\mathbf{k}} \in \mathbf{C}, \mathbf{k} \geq 0$. They satisfy the standard canonical commutation relations $[Q_{\mathbf{p}}, P_{\mathbf{k}}] = i \delta(\mathbf{p}, \mathbf{k})$. The Hilbert space \mathcal{H} is related to the space of square integrable functions on \mathbf{Q} by an isometry $W : \mathcal{H} \rightarrow L^2(\mathbf{Q}, d\nu)$. This isometry relates the operators $P_{\mathbf{k}}, Q_{\mathbf{k}}$ to the creation and annihilation operators introduced above by

$$W^* Q_{\mathbf{k}} W = \frac{1}{L\sqrt{2\pi}} (a_{\mathbf{k}}^* + a_{-\mathbf{k}}) , \quad W^* P_{\mathbf{k}} W = \frac{i}{L\sqrt{2\pi}} (a_{\mathbf{k}}^* - a_{-\mathbf{k}}) .$$

The vector $\Omega_0 \in \mathcal{H}$ is mapped to $W\Omega_0 = 1$, the identity function in L^2 . This together with the relations just given uniquely determines W : Its action on states with higher particle number gives products of Hermite polynomials in the variables $q_{\mathbf{k}}$. The advantage of the \mathbf{Q} -space representation is that the interaction V turns into a multiplication operator. In fact, the Hamiltonian reads in this representation

$$W^* H W = \frac{1}{L} \sum_{\mathbf{k} \in \mathbf{Z}} \omega_{\mathbf{k}} (|P_{\mathbf{k}}|^2 + |Q_{\mathbf{k}}|^2 - 1) + \lambda V(Q_0, Q_1, \dots) . \quad (3.29)$$

Here we are again understanding V as $V = \lim_{\Lambda \rightarrow \infty} V^{\Lambda}$, we have set $Q_{-\mathbf{k}} = Q_{\mathbf{k}}^* = \bar{q}_{\mathbf{k}}$ for $\mathbf{k} \geq 0$ (and similarly for $P_{-\mathbf{k}}$), and where

$$V^{\Lambda}(q_0, q_1, \dots) = \pi \sum_{n=3}^p b_n^{\Lambda} \sum_{|\mathbf{k}_1|, \dots, |\mathbf{k}_n| \leq \Lambda} \delta\left(0, \sum_j \mathbf{k}_j\right) \frac{q_{\mathbf{k}_1} \cdots q_{\mathbf{k}_n}}{(2\pi L \omega_{\mathbf{k}_1})^{1/2} \cdots (2\pi L \omega_{\mathbf{k}_n})^{1/2}} . \quad (3.30)$$

Above we have $b_j^{\Lambda} = \sum_i (-1)^i \frac{(j+2i)!}{j!i!} c_{\Lambda}^i b_{j+2i}$, and $c_{\Lambda} = (2\pi L)^{-1} \sum_{|\mathbf{k}| \leq \Lambda} (2\omega_{\mathbf{k}})^{-1} \sim \log \Lambda$.

We now have the necessary tools to state and prove the main auxiliary theorem needed to construct the $P(\varphi)_2$ model, see Theorem X.58 of [48].

Theorem 3.5. *Let (\mathbb{Q}, ν) be a measure space with $\nu(\mathbb{Q}) = 1$ and let H_0 be the generator of a hypercontractive semigroup on $L^2(\mathbb{Q}, d\nu)$. Let V be a real-valued measurable function on (\mathbb{Q}, ν) such that $V \in L^p(\mathbb{Q}, d\nu)$ for all $p \in [1, \infty)$ and $e^{-tV} \in L^1(\mathbb{Q}, d\nu)$ for all $t > 0$. Then $H_0 + V$ is essentially self-adjoint on $\mathcal{D}(H_0) \cap \mathcal{D}(V)$ and is bounded below.*

Remarks. We remind the reader that a “semigroup” $T(t)$ can, loosely speaking, be considered as the “exponential” of a closed and densely defined operator A . They thus provide a generalization of the relationship between unitary groups and self-adjoint operators. We also point out that a “hypercontractive” semigroup has by definition the following properties

- e^{-tA} is L^p -contractive, i.e. $\|e^{-tA}\psi\|_p \leq \|\psi\|_p$ for every $\psi \in L^p \cap L^2$, all $p \in [1, \infty)$, and all $t > 0$;
- for some $b > 2$ and some constant C_b , there is $T > 0$ so that $\|e^{-tA}\psi\|_b \leq C_b\|\psi\|_2$ for all $\psi \in L^2(\mathbb{Q}, d\nu)$.

One can show (see section §X.9 of [48]) that in the above definition b plays no special role and that it is possible to replace b and 2 with arbitrary p and q , provided of course that the constant will now depend on both p and q .

The proof of Theorem 3.5 is fairly long so in addition to merely outlining it, we will break it into steps. The central idea, however, is as follows. Instead of dealing with the operators H_0 and V directly, one considers the semigroups generated by these operators, i.e. one considers their “exponentiated” form. It is then possible to prove that e^{-tH} is exponentially bounded, and hence that $H = H_0 + \lambda V$ is bounded from below. This is then used to prove that H is essentially self-adjoint. For this argument to work, it is crucial that e^{-tH_0} is a “smoothing” map, which is encoded in the requirement that e^{-tH_0} be a hypercontractive semigroup. On the other hand, e^{-tV} is a function in L^p and it increases the singularities of the vector it acts on. This latter fact can be seen using the Hölder inequality: say $\eta \in L^p$, we then have

$$\|e^{-tV}\eta\|_r \leq \|e^{-tV}\|_q \|\eta\|_p,$$

with $r^{-1} = p^{-1} + q^{-1}$, or $r = pq/(p+q) < p$ because $q/(p+q) < 1$ for $p, q > 0$. This is to say, as a map $e^{-tV} : L^p \rightarrow L^r$, with $r < p$.

Outline of the proof of Theorem 3.5. Cut the potential off in “field space” and define V_n ¹² as

$$V_n(q) := \begin{cases} V(q) & \text{if } |V(q)| \leq n \\ 0 & \text{otherwise} \end{cases} .$$

Then $H_n = H_0 + \lambda V_n$ is self-adjoint on $\mathcal{D}(H_0)$ by the Kato–Rellich theorem (see Theorem X.12 of [48]). Clearly, $V_n\Psi \rightarrow V\Psi$ as a function in $L^p(\mathbb{Q}, d\nu)$ -space for all $p \in [1, \infty)$. Because H_n is self-adjoint and bounded from below, one can define its exponential e^{-tH_n} . The first step is then to obtain various bounds for e^{-tH_n} .

Step 1: We have the following bounds:

1. For any $t > 0$,

$$\sup_n \|e^{-tV_n}\|_1 < \infty ,$$

and is uniformly bounded in t in any compact subinterval of $[0, \infty)$.

2. If $p < q$ then for each t there is a constant C_t , depending on p, q and t but independent of n , so that for all $\psi \in L^q$ it is

$$\|e^{-tH_n}\psi\|_p \leq C_t \|\psi\|_q .$$

For fixed p and q , C_t is uniformly bounded for t in a compact subinterval of $[0, \infty)$.

3. There is a constant E , independent of n , so that

$$\|e^{-tH_n}\psi\|_2 \leq e^{Et} \|\psi\|_2 . \tag{3.31}$$

What we want to emphasize here is that the last bound shows that $H_n = H_0 + V_n$ is bounded below. This is important and together with Duhamel’s formula, see below, allows us to conclude that the same will hold in the $n \rightarrow \infty$ limit, which is the content of the next step.

Step 2: Let $\psi \in L^2(\mathbb{Q}, d\nu)$. Then $T(t)\psi := \lim_{n \rightarrow \infty} e^{-tH_n}\psi$ exists and $T(t)$ is a strongly continuous semigroup of self-adjoint operators satisfying $\|T(t)\| \leq e^{Et}$. Further, there is a unique self-adjoint operator H satisfying $H \geq -E$ so that $T(t) = e^{-tH}$.

¹²One should not confuse V_n with V^Λ : They are both “cutoff potentials”, but V_n is identically zero if $V > n$ while its Fourier representation is not restricted. On the other hand V^Λ will, in general, never vanish but its Fourier modes are restricted.

The proof of this step relies on Duhamel's formula

$$e^{-tH_n}\psi = e^{-tH_m}\psi + \int_0^t e^{-(t-u)H_n}(V_m - V_n)e^{-uH_m}\psi \, du ,$$

which holds because both sides applied to a vector in $\mathcal{D}(H_0)$ solve the same first order differential equation. Specifically, this formula is used to show that for vectors $\psi \in L^2(\mathbb{Q}, d\nu)$, the sequence $e^{-tH_n}\psi$ is Cauchy in L^2 . We can hence take the limit and meaningfully define $T(t)\psi := \lim_n e^{-tH_n}\psi$. The resulting $T(t)$ is then a strongly continuous semigroup and we denote by H its (symmetric) generator. Finally, since H is bounded below, which also follows from (3.31), it is self-adjoint.

Step 3: $H_0 + V$ is essentially self-adjoint on $\mathcal{D} = \mathcal{D}(H_0) \cap \mathcal{D}(V)$ and for $\eta \in \mathcal{D}$, $H\eta = H_0\eta + V\eta$.

This last step establishes the domain \mathcal{D} of essential self-adjointness for $H_0 + V$. In addition, it also shows that on \mathcal{D} everything is “as it should be”, that is, H is indeed the sum of H_0 and V . \square

Clearly, the above theorem is not yet sufficient to claim that we have in fact constructed a quantum field theory on the cylinder, for in the above theorem H_0 is nothing but the generator of a hypercontractive semigroup and not the “free Hamiltonian”. Likewise, V is just some operator that is assumed to satisfy some hypothesis, and not necessarily the “interaction potential”. To complete the construction we are thus left with the task to show that the free Hamiltonian (3.23) and the interaction potential (3.25) do satisfy the requirements of the above theorem. It is a result of the work of Glimm and Jaffe that this is indeed the case. In particular, for the interaction potential (3.25) considered as a function on \mathbb{Q} space, we have a stronger result than $e^{-\lambda V} \in L^1$. We in fact have (Theorem 2.1.4 of [35])

Theorem 3.6. *As a function on \mathbb{Q} , we have $e^{-\lambda V} \in L^p$ for all $p < \infty$.*

Remark: The proof relies on the decomposition

$$V = V^\Lambda - (V - V^\Lambda) = V^\Lambda - \tilde{V}^\Lambda ,$$

and on the separate bounds on each of the two quantities, which are given in the following two lemmas. For the full proof of this theorem and of the subsequent lemmas we refer the reader to the literature, but we also state the two lemmas as we will need them in Section §4.2.1. The first one is (Lemma 2.1.5 of [35])

Lemma 3.7. *Let $P(\xi)$, the positive polynomial used to construct the interaction potential, be of degree p . Then*

$$-\mathcal{O}(\log^{p/2} \Lambda) \leq V^\Lambda .$$

Proof. The proof is based on the commutation relations of the creation and annihilation operators, while additionally noting that since the potential is cut off, it is necessarily bounded. The bound then follows from a simple calculation using the expansion (3.30), and the note that follows it in particular. \square

The next result is (Lemma 2.1.6 of [35])

Lemma 3.8. *Let $P(\xi)$ have degree p and let $\epsilon > 0$. Regarding \tilde{V}^Λ as a function on \mathbb{Q} and with a constant K independent of Λ and p we have*

$$\int |\tilde{V}^\Lambda(q)|^{2j} dq \leq j!^p (K\Lambda^{-1/2+\epsilon})^{2j} .$$

Proof. The proof is based on the fact that the integral in the above estimate is the square of the \mathbb{Q} -space norm of $(\tilde{V}^\Lambda)^j \Omega_0$, that is we have

$$\int |\tilde{V}^\Lambda(q)|^{2j} dq = \|(\tilde{V}^\Lambda)^j \Omega_0\|^2 .$$

We also note that \tilde{V}^Λ is a sum of (generalized Wick) monomials (3.27) whose kernels satisfy the bound $\|B^\Lambda\| \leq \mathcal{O}(\Lambda^{-1/2+\epsilon})$ by Theorem 3.3. And since $(\tilde{V}^\Lambda)^j \Omega_0$ is a state with at most pj particles it is possible, using Theorem 3.4, to obtain an estimate involving the norm of $\tilde{V}^{j-1} \Omega_0$. An induction then gives the result. \square

Finally, that H_0 is the generator of a hypercontractive semigroup is the content of the following (Proposition 2.2.3 of [35])

Theorem 3.9. *e^{-tH_0} is a contraction on $L^1(\mathbb{Q}, d\nu)$ for $t \geq 0$, and for sufficiently large t it is a contraction from $L^2(\mathbb{Q}, d\nu)$ to $L^4(\mathbb{Q}, d\nu)$.*

Remark: It might seem that this theorem does not prove that e^{-tH_0} is a hypercontractive semigroup, as per the definition given above. The two are, however, equivalent, and it should be noted that in [51] the term ‘‘hypercontractive’’ was introduced for a semigroup having the properties from the above Theorem 3.9.

At this point we have verified that the free Hamiltonian H_0 and the potential V satisfy the the assumptions of Theorem 3.5. In other words, we have in fact constructed

the $P(\varphi)_2$ model on the circle. Before concluding, however, we state three more results that we will use in what follows. The first one is (Theorem 2.2.5 of [35]) a stronger version of Theorem 3.9.

Theorem 3.10. *For some $M > 0$, e^{-tH_0} is a contraction from L^p to L^q , if*

$$\frac{1}{p} \leq (1 + Mt) \frac{1}{q}$$

and

$$\left(1 - \frac{1}{q}\right) \leq (1 + Mt) \left(1 - \frac{1}{p}\right).$$

The second one, is the well known fact that $e^{-\beta H}$ is a trace class operator, with β the inverse temperature. Since we will occasionally appeal to this result, we state it as a

Theorem 3.11. *We have*

$$\text{Tr}(e^{-\beta H}) \leq e^{K\beta^{-1}} < \infty,$$

for any $\beta > 0$. The same applies to

$$\exp\left(-\beta H - \sum_{\mathbf{k} \in \mathbb{Z}} \mu_{\mathbf{k}} a_{\mathbf{k}}^* a_{\mathbf{k}}\right),$$

where $\mu_{\mathbf{k}} \geq 0$.

Remark: A careful look at the estimates in the proof shows that the constant K is of order L , so the free energy goes as $\beta^{-1}L$, as one expects.

Proof. We give the proof without the “chemical potentials” $\mu_{\mathbf{k}}$ for simplicity. The general case is the same because the $\mu_{\mathbf{k}}$ can be absorbed into the $\omega_{\mathbf{k}}$ in the free Hamiltonian, and this only makes things better.

The Golden–Thompson inequality [52, 53] states that $\text{Tr}(e^{A+B}) \leq \text{Tr}(e^A e^B)$ for any hermitian matrices A, B . The inequality can be proved e.g. using standard properties of the trace and the Lie–Trotter product formula

$$\lim_{n \rightarrow \infty} (e^{A/n} e^{B/n})^n = e^{A+B}.$$

It is possible to apply this kind of reasoning also in the infinite dimensional context to $A = H_0, B = \lambda V$. Indeed, an appropriate version of the Lie–Trotter formula (see e.g. [48]) then holds, because H is essentially self-adjoint on the domain $\mathcal{D}(H_0) \cap \mathcal{D}(V)$,

see below. An appropriate version of the Golden–Thompson inequality then holds (see [54, 55]), which together with the operator inequality $\text{Tr}(XY) \leq \|X\|\text{Tr}|Y|$, and denoting by $\|\cdot\|_{p,q}$ the norm of an operator $L^q \rightarrow L^p$, gives:

$$\begin{aligned} \text{Tr} e^{-\beta H} &\leq \text{Tr}(e^{-\beta H_0} e^{-\beta \lambda V}) \leq \|e^{-\beta \lambda V} e^{-(\beta/2)H_0}\| \cdot \text{Tr}(e^{-(\beta/2)H_0}) \\ &\leq e^{K\beta^{-1}} \|e^{-\lambda \beta V}\|_{(2+2K\beta)/(K\beta)} \|e^{-(\beta/2)H_0}\|_{2+2K\beta,2} \\ &\leq e^{K\beta^{-1}} \|e^{-[2\lambda(1+K\beta)/K]V}\|_1^{K\beta/(2+2K\beta)} \leq e^{K\beta^{-1}}. \end{aligned}$$

Here we have used that e^{-tH_0} is a contraction between $L^2 \rightarrow L^{2+2Kt}$ for some constant $K > 0$ when $m > 0$, which can in principle be seen from the well known explicit formula in Q-space, given by

$$(e^{-tH_0}\Psi)(q) = \int_{\mathcal{Q}} \prod_{j \in \mathbb{Z}} (1 - e^{-t\omega_j})^{-1/2} \exp\left(-\frac{|q'_j - e^{-\omega_j t} q_j|^2}{2(1 - e^{-2\omega_j t})} + \frac{1}{2}|q'_j|^2\right) \Psi(q') d\nu(q'),$$

where $q_{-k} = \bar{q}_k$ for $k \geq 0$. We are also using that $e^{-\beta V}$ is a multiplication operator in Q-space whose L^p -norms are all finite for $1 < p < \infty$ (which follows by writing $V = V^\Lambda + \tilde{V}^\Lambda$, using Theorem 3.8 and the fact that V^Λ is semi-bounded from below), and we have used the inequality $\text{Tr}(e^{-(\beta/2)H_0}) \leq e^{K\beta^{-1}}$. (This latter inequality is obvious for $\beta \rightarrow \infty$. For $\beta \rightarrow 0$ it follows from, say, equation (1.2) of [56], while for finite β it holds because both functions are monotonically decreasing.) \square

Later we will also need to appeal to the so called ‘‘Rosen bounds’’ (see [57] for the original work and Theorem 3.1.3 of [35] for the specific version written below) or ‘‘Higher order estimates’’, and to a result by Glimm and Jaffe [39], so we state the results here as a

Theorem 3.12 (Rosen bounds). *Let i be a positive integer and p the degree of the polynomial P . For $\epsilon > 0$, $i \geq 3$ and $j = j(i, p, \epsilon)$ sufficiently large, there is a constant such that*

$$H_0^{3-\epsilon} N^{i+\epsilon-3} \leq K(H + \mathcal{O}(\lambda)L \cdot I)^j, \quad (3.32)$$

and the constant K does not depend on λ or j . If $\epsilon > 2$, we may take $j = i$. For $p \leq 4$ and some K' we have [39]

$$H_0^2 \leq K'(H + \mathcal{O}(\lambda)L \cdot I)^2. \quad (3.33)$$

Proof. We merely give an argument for equation (3.33). In [39] One can find the proof

of the following statement, valid for φ^4 theory,

$$- [H_0^{1/2}, [H_0^{1/2}, H]] \leq \epsilon H_0^2 + b(\epsilon, \lambda),$$

for any positive number ϵ and suitably large b . We can use this result as follows

$$\begin{aligned} H_0^2 &= H_0^{1/2} H_0 H_0^{1/2} \leq H_0^{1/2} H H_0^{1/2} = - [H_0^{1/2}, [H_0^{1/2}, H]] + H_0 H + H H_0 \leq \\ &\leq - [H_0^{1/2}, [H_0^{1/2}, H]] + (H_0 + H)^2 \leq \epsilon H_0^2 + bI + (H_0 + H)^2 \leq \\ &\leq \epsilon H_0^2 + bI + (2H_0 + \lambda V)^2 = \epsilon H_0^2 + bI + 4H_{\lambda/2}^2 \leq \epsilon H_0^2 + (H_{\lambda/2} + \mathcal{O}(\lambda)L \cdot I)^2, \end{aligned}$$

where $H_{\lambda/2} = H_0 + \lambda/2V$. Noting now that the above result holds for arbitrary λ , we could use it with $\lambda \rightarrow 2\lambda$ and hence immediately obtain equation (3.33). \square

We will use these inequalities in many places below and we also note that it implies that H is bounded below as a special case. For simplicity we will absorb the additive constant $\mathcal{O}(\lambda)L \cdot I$ into the Hamiltonian H —this does not affect the definition of the time evolution $\alpha_t(A) = e^{itH} A e^{-itH}$ of an observable A , but it has the advantage of turning H into a non-negative operator. The fact that $m > 0$ implies [35] that in the thermodynamic limit $L \rightarrow \infty$, i.e. in QFT on 2-dimensional Minkowski space, H has a state of lowest energy, the “vacuum state”, followed by a mass gap, corresponding to physical particles in the sense of scattering theory. However, we note that the value of this mass gap is *not* m , and hence this parameter must therefore *not* be confused with the physical mass. Finally, as for the free theory, consistency with the Wightman axioms should be checked. We again omit the proof and note that the above construction indeed results in an interacting field theory according to both the Wightman and Haag–Kastler axioms.

Later in Section §4.2.1 it will be important for us that the inequality (3.32) equally applies to the time evolved free Hamiltonian and time evolved number operator because H is a constant of the motion. In this way, the higher order estimates will allow us to transfer information on the number operator densities at the initial time to later times.

Chapter 4

Non-perturbative pre-Boltzmann equation

In this chapter we take the first step in the problem of deriving the Boltzmann equation from the Heisenberg equation. The result we present in this chapter is also one of the main results of this work, i.e. what we call the “pre-Boltzmann equation” (4.46). Although it may appear to be very different from the Heisenberg equation it is, in fact, equivalent to it. To show this, we first decompose the Heisenberg equation by appropriately modifying the so called “projection operator” technique [18] to fit our needs. The essential idea behind the method is to project an observable onto a subspace of the algebra of observables spanned by a set of “relevant” observables. The price we have to pay for such a projection is that the resulting equation is not local in time anymore. It is, however, a much better starting point for certain approximations than the original Heisenberg equation. Next we introduce the main observables of this work, i.e. the number operators, in the context of a hermitian scalar quantum field with polynomial (self-)interaction on the cylinder and prove various estimates satisfied by the particle number operators. Finally, we derive the pre-Boltzmann equation for our model, and we give two alternative formulations of it.

Before proceeding we want to remark on the author’s contribution to this chapter. To establish the non-perturbative validity of the pre-Boltzmann equation (4.35), two ingredients are needed: A formal derivation and the rigorous proof of the validity of the formal results. In what follows the two are tightly intertwined, but this distinction essentially corresponds to the work done by the author (the formal derivation) and his research advisor (the rigorous proof of such results). In particular, Theorems 4.4, 4.5, 4.6, and the proof of Theorem 4.7 are the work of the author’s research advisor, see also [58]. They are included in this work for completeness.

4.1 General framework

Here we set up the projection operator method [10, 18], for the case in which the Hamiltonian is time independent. After this has been accomplished, it will be straightforward to generalize the results to the case of a time dependent Hamiltonian. Such a generalization is useful for the application of the method to situations when there is a time dependent background field. This could be the case for charged particles in a varying electric field or, for cosmologically relevant applications, quantum fields propagating on a curved background like e.g. Robertson–Walker spaces. In this work, however, we will be only interested in the time independent case.

The projection method is well known in the literature and our contribution is to adapt it to our needs and to generalize it to the time dependent case. The basic framework is very general and we will explain it, for the sake of simplicity, in a finite dimensional setting. This has the definite advantage that all the steps are completely well defined. In the infinite dimensional case the calculations are formally identical, but the convergence of the various series below cannot be taken for granted.

Let $H \in M_n(\mathbb{C})$ be a Hamiltonian, self-adjoint with respect to the standard scalar product $\langle \cdot | \cdot \rangle$ on $\mathcal{H} = \mathbb{C}^n$. Given an observable $G \in M_n(\mathbb{C})$, we define its time evolution as

$$\alpha_t : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) , \quad \alpha_t(G) = e^{itH} G e^{-itH} \equiv G(t) .$$

The time evolution clearly satisfies the group law $\alpha_{t+s} = \alpha_t \circ \alpha_s$, i.e. it is an automorphism, and it is also a homomorphism of the algebra $M_n(\mathbb{C})$ for each t , that is $\alpha_t(AB) = \alpha_t(A)\alpha_t(B)$. In the context of the Boltzmann equation, we have a density matrix state ρ , i.e. a self-adjoint, positive semi-definite operator satisfying $\text{Tr}\rho = 1$, and a family of observables G_j , $j = 1, \dots, N$, and we want to study the time evolution of their expectation values,

$$g_j(t) := \text{Tr}(\rho G_j(t)) .$$

The observables G_j , $j = 1, \dots, N$ that will be of interest for us later are number operators, with j corresponding to the mode number, and H will be the Hamiltonian of the $P(\varphi)_2$ model. Of course, in that context the Hilbert space \mathcal{H} is infinite dimensional, H is unbounded etc., but for the moment we will disregard these issues and simply work in the matrix context.

We would like to derive a differential equation for the complex valued functions $g_j(t)$. This would be straightforward in principle if the set of observables (matrices) G_j was a basis of $M_n(\mathbb{C})$, i.e. when $N = n^2$ and all the G_j linearly independent. Indeed, we could then simply express the linear operator $\alpha_t : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ as a matrix in this

basis as $\alpha_t(G_j) = \sum_{k=1}^N m_{jk}(t)G_k$ for a 1-parameter group of matrices $(m_{ij}(t))$, and the desired differential equation would then simply follow by taking the expectation value of this expression and differentiating with respect to t . This procedure is of course not very practical nor actually different from solving the full Schrödinger equation, because we would need to know $m_{ij}(t)$, and this means in practice that we have to diagonalize H . At any rate, we will be interested in the generic case where the family G_j does not form a basis of $M_n(\mathbb{C})$ but only forms a relatively “small” selected family of “relevant” observables. We are not interested in diagonalizing H but only in the dynamical evolution of the relevant observables. To “discard the irrelevant information” we will employ the projection operator method and obtain an equation [cf. equation (4.7)] for the (expected values of our) set of relevant observables $g_j(t)$ that is both non-local in time and non-linear. The main advantage of such an equation is that it is more amenable to be treated with approximation techniques.

The main idea is to introduce a family of linear maps $\mathcal{P}_t : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ that are smooth in $t \in \mathbb{R}$ and with the following general properties:

1. We have, with I the unit matrix:

$$\text{range } \mathcal{P}_t = \alpha_t[\text{span}\{I, G_1, \dots, G_N\}] \subset M_n(\mathbb{C}).$$

2. We have

$$\mathcal{P}_t \circ \alpha_{t-s} \circ \mathcal{P}_s = \alpha_{t-s} \circ \mathcal{P}_s$$

for any $t \geq s$.

The family of maps \mathcal{P}_t , $t \in \mathbb{R}$, is referred to as a family of “projections” onto the space of time- t observables because $\mathcal{P}_t^2 = \mathcal{P}_t$, and we observe, however, that they are not orthogonal. We neither require the maps \mathcal{P}_t to be algebra homomorphisms. The projections will serve us to break up the time evolution of an observable into a part “parallel” to \mathcal{P}_t , and a part “parallel” to the complementary projection $\mathcal{Q}_t := id - \mathcal{P}_t$, where $id : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is the identity. Later, \mathcal{P}_t will be chosen in such a way that the latter part becomes small in a suitable sense, and this will then be treated as a perturbation.

4.1.1 Decomposition of the equations of motion

To start, and to simplify our notation, we note that the Heisenberg evolution equation for a matrix $X \in M_n(\mathbb{C})$ that comes from H can be written simply as

$$\frac{d}{dt}\alpha_t(X) = i\delta \circ \alpha_t(X), \quad \delta(X) = [H, X] = HX - XH. \quad (4.1)$$

We now decompose

$$\frac{d}{dt}\alpha_t = i\alpha_t \circ \delta = i\mathcal{P}_t \circ \alpha_t \circ \delta + i\mathcal{Q}_t \circ \alpha_t \circ \delta = i\alpha_t \circ \tilde{\mathcal{P}}_t \circ \delta + i\alpha_t \circ \tilde{\mathcal{Q}}_t \circ \delta, \quad (4.2)$$

where we have found it convenient to introduce the ‘‘Schrödinger picture’’ operators $\tilde{\mathcal{P}}_t = \alpha_{-t} \circ \mathcal{P}_t \circ \alpha_t$. The goal is now to replace $\alpha_t \circ \tilde{\mathcal{Q}}_t \circ \delta$ with an expression that involves only terms with the map $\tilde{\mathcal{P}}_t$. To this end one notes the following differential equation

$$\begin{aligned} \frac{d}{dt} [\alpha_t \circ \tilde{\mathcal{Q}}_t] &= \alpha_t \circ i\delta \circ \tilde{\mathcal{Q}}_t + \alpha_t \circ \frac{d}{dt} \tilde{\mathcal{Q}}_t = \\ &= \alpha_t \circ \tilde{\mathcal{P}}_t \circ i\delta \circ \tilde{\mathcal{Q}}_t + \alpha_t \circ \tilde{\mathcal{Q}}_t \circ i\delta \circ \tilde{\mathcal{Q}}_t + \alpha_t \circ \frac{d}{dt} \tilde{\mathcal{Q}}_t. \end{aligned} \quad (4.3)$$

This is an inhomogeneous linear differential equation which we can integrate straightforwardly [cf. appendix B], and the result is

$$\alpha_t \circ \tilde{\mathcal{Q}}_t = \tilde{\mathcal{Q}}_0 \circ \tilde{Y}_{0,t} + \int_0^t \alpha_s \circ \left[\tilde{\mathcal{P}}_s \circ i\delta \circ \tilde{\mathcal{Q}}_s + \frac{d}{ds} \tilde{\mathcal{Q}}_s \right] \circ \tilde{Y}_{s,t} ds, \quad (4.4)$$

where $\tilde{Y}_{s,t} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ is the cocycle (meaning that $\tilde{Y}_{t_1,t_2} \circ \tilde{Y}_{t_2,t_3} = \tilde{Y}_{t_1,t_3}$) defined to be the solution to the differential equation

$$\frac{d}{dt} \tilde{Y}_{s,t}(X) = \tilde{Y}_{s,t} \circ i\delta \circ \tilde{\mathcal{Q}}_t(X), \quad \tilde{Y}_{s,s}(X) = X, \quad (4.5)$$

for all $X \in M_n(\mathbb{C})$. To solve the above equation we look for a solution of the form $\tilde{Y}_{s,t} = \alpha_{-s} \circ Y_{s,t} \circ \alpha_t$, where $Y_{s,t}$ is the ‘‘Heisenberg-picture cocycle’’. Combining equations (4.1) and (4.5) we immediately get the differential equation for $Y_{s,t}$

$$\frac{d}{dt} Y_{s,t}(X) = -Y_{s,t} \circ i\delta \circ \mathcal{P}_t(X), \quad Y_{s,s}(X) = X,$$

the solution of which is given by the summation formula

$$Y_{s,t}(X) = \sum_{k=0}^{\infty} (-i)^k \int_{s < \sigma_1 < \dots < \sigma_k < t} \delta \circ \mathcal{P}_{\sigma_1} \circ \dots \circ \delta \circ \mathcal{P}_{\sigma_k}(X) d^k \sigma. \quad (4.6)$$

Note that this sum trivially converges as we can estimate it by

$$\|Y_{s,t}(X)\| \leq \|X\| \sum_{k=0}^{\infty} \frac{|t-s|^k}{k!} \|\delta\|^k (\sup_{\sigma} \|\mathcal{P}_{\sigma}\|)^k \leq e^{K|t-s|} \|X\|,$$

since the volume of the set $\{s < \sigma_1 < \dots < \sigma_k < t\}$ is given by the first term under the summation sign. Switching from the “tilde” projectors back to the original ones, equation (4.2) now takes the form

$$\begin{aligned} \frac{d}{dt} \alpha_t(X) &= \mathcal{P}_t \circ \alpha_t \circ i \delta(X) + \mathcal{Q}_0 \circ Y_{0,t} \circ \alpha_t \circ i \delta(X) + \\ &+ \int_0^t \left[\mathcal{P}_s \circ i \delta \circ \mathcal{Q}_s - \alpha_s \circ \frac{d}{ds} (\alpha_{-s} \circ \mathcal{P}_s \circ \alpha_s) \circ \alpha_{-s} \right] \circ Y_{s,t} \circ \alpha_t \circ \delta(X) ds. \end{aligned} \quad (4.7)$$

This is the equation we were looking for. It is still an exact equation in that it is equivalent to the Heisenberg equation of motion (4.1), and, if anything, it is at least as difficult to solve. But the crucial point for us is that we have been able to separate the “relevant” and “irrelevant” degrees of freedom by employing the maps \mathcal{P}_t and \mathcal{Q}_t . (This point will become more clear in what follows.) In particular it is an integro-differential equation rather than an (operator) differential equation, and it is not local in time anymore—the future evolution of the system now depends on its past history—as can be deduced by the presence of the integral. This type of equation is also said to be a “non-Markovian” equation as opposed to a “Markovian” equation which is local in time. (The Boltzmann equation, for instance, is a Markovian equation.) Even though it is non-Markovian, the above equation will be the starting point for our analysis as it is much better suited for approximations than the Heisenberg equation of motion. Finally, one should also keep in mind that we have not yet specified the \mathcal{P}_t and we will be able to obtain considerable simplifications by making an appropriate choice.

4.1.2 A specific projector

A particularly useful choice for the projectors \mathcal{P}_t is available if one is working with a set of mutually commuting hermitian observables G_j , $j = 1, \dots, N$, that is when $[G_i, G_j] = 0$ for all i, j and $G_j = G_j^*$ for all j . This is of relevance for us as we will

eventually be working with “number operators” which do satisfy these conditions and we will hence from now on assume that this is the case. We now consider, for fixed $t \in \mathbb{R}$, a reference state $w_t : M_n(\mathbb{C}) \rightarrow \mathbb{C}$ which reproduces the expectation values of the observables $G_j(t)$ in a given state ρ . That is to say, w_t ought to be a normalized ($w_t(I) = 1$) and positive ($w_t(X^*X) \geq 0$ for all $X \in M_n(\mathbb{C})$) linear functional that in addition satisfies the following property:

$$w_t(G_j(t)) = g_j(t) , \quad (4.8)$$

for all j . The above condition does not specify the state w_t uniquely. Indeed, dropping the dependence on t for notational simplicity, let $\mathcal{A} = \text{alg}\{G_j, j = 1, \dots, N\}$ be the abelian $*$ -algebra generated by the observables. A joint spectral decomposition $G_j = \sum_{\alpha=1}^n f_j(\alpha)p(\alpha)$, with $p(\alpha)$ rank-1 projections, allows us to identify \mathcal{A} with a subalgebra of the abelian $*$ -algebra of functions $\text{Fun}(\{1, \dots, n\} \rightarrow \mathbb{C})$, and we can also identify $\text{Tr}(\rho \cdot)$ with a positive linear functional on this algebra. It follows by standard arguments then, that there is a non-negative function $m : \{1, \dots, n\} \rightarrow \mathbb{R}$ of total weight one such that $g_j = \sum_{\alpha=1}^n f_j(\alpha)m(\alpha)$. We could, e.g. define our state (and reintroducing the t dependence) by

$$w_t(X) = \text{Tr} \left[X \sum_{\alpha=1}^n p_t(\alpha) m_t(\alpha) \right] . \quad (4.9)$$

However, this is generally neither the unique, nor the most useful solution to equation (4.8). Instead we consider the unique solution that maximizes the “entropy”, i.e. we pick

$$w_t(X) = \text{Tr}(\rho_t X) , \quad \text{Tr} \rho_t = 1 , \quad \rho_t \geq 0 ,$$

in such a way that the functional

$$S[\rho_t] = -\text{Tr}[\rho_t \log \rho_t]$$

is maximized *and* such that (4.8) holds for all $j = 1, \dots, N$. A standard argument involving Lagrange multipliers, see e.g. Section V.1.3 of [25], shows that, in the generic case¹, this maximizer must be of the form

$$\rho_t = \frac{1}{Z(t)} \exp \left(- \sum_{j=1}^N \mu_j(t) G_j(t) \right) .$$

¹Generic here means that the maximizer is strictly inside the convex set of all positive linear functionals on $M_n(\mathbb{C})$.

We now assume that the functions $\mu_j(t)$ have been chosen in this way, i.e. we are in the “generic case” for all t . In our applications below, the nature of the observables G_j implies that the functionals w_t always exist, and we will simply assume the same here.

We can now use the reference states w_t to construct the maps \mathcal{P}_t . First we note the following relation

$$\begin{aligned} \frac{\partial}{\partial \mu_i(t)} w_t[X] &= \frac{\partial}{\partial \mu_i(t)} \frac{\text{Tr} [e^{-\sum_j \mu_j(t) G_j(t)} X]}{Z(t)} = \\ &= \frac{\partial_{\mu_i(t)} \text{Tr} [e^{-\sum_j \mu_j(t) G_j(t)} X]}{Z(t)} - \frac{w_t[X] \partial_{\mu_i(t)} Z(t)}{Z(t)} = \\ &= -w_t[G_i(t)X] + w_t[X] g_i(t) = -w_t[\Delta G_i(t)X] , \end{aligned}$$

where $\Delta G_i(t) := G_i(t) - g_i(t)I$. If $X = G_j(t)$, we define the “correlation matrix” by

$$C_{ij}(t) := -\frac{\partial g_j(t)}{\partial \mu_i(t)} = w_t[\Delta G_i(t) \Delta G_j(t)] . \quad (4.10)$$

This matrix is positive semi-definite and we assume it to be also invertible. We denote the inverse as $C^{ij}(t)$ and remark that it can also be written as

$$C^{ij}(t) = -\frac{\partial \mu_i(t)}{\partial g_j(t)} , \quad (4.11)$$

with $g_j(t) = w_t[G_j(t)]$ as before. The projection operator is now defined as

$$\begin{aligned} \mathcal{P}_t(X) &= w_t[X] I + \sum_{i,j}^N C^{ij}(t) w_t[\Delta G_j(t)X] \Delta G_i(t) = \\ &= w_t[X] I + \sum_{j=1}^N \frac{\partial w_t[X]}{\partial g_j(t)} \Delta G_j(t) . \end{aligned} \quad (4.12)$$

This projector is known in the literature as the “Kawasaki–Gunton projector” [10, 59]. We collect a number of immediate properties of the above projector in the following

Lemma 4.1. *For any $X \in M_n(\mathbb{C})$ and any $t \in \mathbb{R}$, we have*

$$\alpha_t \circ \frac{d}{dt} (\alpha_{-t} \circ \mathcal{P}_t \circ \alpha_t(X)) \in \text{span} \{G_j(t) - g_j(t)I \mid j = 1, \dots, N\} .$$

Moreover, for any $Y \in \text{span} \{G_1(t), \dots, G_N(t)\}$, we have $\mathcal{P}_t \circ \delta(Y) = 0$, and we also have $w_t[X] = \text{Tr}[\rho \mathcal{P}_t(X)]$ for any $X \in M_n(\mathbb{C})$.

Proof. To prove the first statement, we introduce $\tilde{\mathcal{P}}_t = \alpha_{-t} \circ \mathcal{P}_t \circ \alpha_t$ as above, and we

also introduce $\tilde{w}_t = w_t \circ \alpha_t$. The first statement is then seen to be equivalent to the statement that $\frac{d}{dt}\tilde{\mathcal{P}}_t(X) \in \text{span}\{G_j - g_j(t)I \mid j = 1, \dots, N\}$. Now we have, using the “summation convention”:

$$\tilde{\mathcal{P}}_t(X) = \tilde{w}_t(X) + C^{ij}(t) \tilde{w}_t[X(G_i - g_i(t)I)](G_j - g_j(t)I).$$

When taking the t -derivative of this expression, we recall equations (4.10) and (4.11) from which it immediately follows that

$$\frac{d}{dt}\tilde{w}_t(X) = \tilde{w}_t[X(G_i - g_i(t)I)] \frac{\partial \mu_i}{\partial g_j}(t) \dot{g}_j(t) = C^{ij}(t) \tilde{w}_t[X(G_i - g_i(t)I)] \dot{g}_j(t).$$

This term cancels precisely the derivative of the second term in $\frac{d}{dt}\tilde{\mathcal{P}}_t$ when the derivative hits $g_j(t)$ in that term. The remaining terms are given by a linear combination of $G_j - g_j(t)I$, as claimed.

In order to prove the second statement, we just follow the definitions and use the cyclicity of the trace as well as the fact that each $G_j(t)$ commutes with the density matrix $\rho = Z(t)^{-1} \exp[-\sum \mu_j(t)G_j(t)]$. The last statement is again a straightforward consequence of the definitions. \square

As we will see momentarily, the properties stated in Lemma 4.1 lead to significant simplifications to equation (4.7). In fact, the last statement of the lemma says that we can evaluate the expectation value of *any* projected observable in the *arbitrary* (but fixed) density matrix state $\text{Tr}(\rho \cdot)$ by computing the expectation value of the non-projected observable in the state w_t for any fixed time t .

Let us now derive the desired integro-differential equation for the $g_j(t)$. To obtain a particularly simple form, we shall make the *initial state assumption* that ρ is such that

$$w_0(X) = \text{Tr}(\rho X), \quad \text{for any } X \in M_n(\mathbb{C}); \quad (4.13)$$

in other words that $\rho = Z^{-1} \exp[-\sum \mu_j G_j]$ for some $\mu_j \in \mathbb{R}$. The physical meaning of this hypothesis will be explained below. With this assumption in place, we proceed as follows: We take the expectation value in our density matrix state $\text{Tr}(\rho \cdot)$ for $X = G_j$ in equation(4.7). Then the first term on the RHS is seen to disappear using the last two statements of the previous lemma. The second term on the RHS is seen to disappear when acted upon by $\text{Tr}(\rho \cdot)$ using the last statement in the lemma, and the assumption (4.13) on the initial condition. For the first term under the integral in (4.7) we use

Lemma 4.1 repeatedly to write

$$\begin{aligned} \text{Tr}[\rho \mathcal{P}_s \circ \delta \circ \mathcal{Q}_s \circ Y_{s,t} \circ \alpha_t \circ \delta(G_j)] &= w_s [\delta \circ \mathcal{Q}_s \circ Y_{s,t} \circ \delta(G_j(t))] = \\ &= w_s [\delta \circ Y_{s,t} \circ \delta(G_j(t))] . \end{aligned}$$

Finally, the last term under the integral disappears when acted upon by $\text{Tr}(\rho \cdot)$, because it is in the span of $G_j(s) - g_j(s)I$, again by the preceding lemma, and this is annihilated by $\text{Tr}(\rho \cdot)$. Thus we arrive at the following:

Theorem 4.2 (Robertson equation). *Let $G_j, j = 1, \dots, N$ be a set of hermitian, mutually commuting complex $n \times n$ matrices, let ρ be a density matrix (self-adjoint, positive definite matrix of unit trace) of the general form $Z^{-1} \exp(-\sum \mu_j G_j)$. Let $g_j(t) = \text{Tr}(\rho G_j(t))$, where $G_j(t) = e^{-itH} G_j e^{itH}$ is the time evolved observable with respect to a self-adjoint Hamiltonian H , and let $Y_{t,s} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be the cocycles defined as above in equation (4.6). Then the equation*

$$\frac{d}{dt} g_j(t) = \int_0^t w_s [i\delta \circ Y_{s,t} \circ i\delta(G_j(t))] ds \quad (4.14)$$

holds, where $\delta(X) = HX - XH$. For an arbitrary density matrix state ρ , i.e. if we do not make the initial state assumption (4.13), the equation takes the form

$$\frac{d}{dt} g_j(t) = (\rho - w_0) [Y_{0,t} \circ i\delta(G_j(t))] + \int_0^t w_s [i\delta \circ Y_{s,t} \circ i\delta(G_j(t))] ds . \quad (4.15)$$

Here, we are using the shorthand $(\rho - w_0)(X) = \text{Tr}(\rho X) - w_0(X)$ for any X .

The equation in the above theorem is known in the literature as ‘‘Robertson equation’’, see [60] for the original work and [10] for a review of its use. The initial state assumption (4.13) on the density matrix state ρ is made for convenience. It can be dropped at the expense of another term in the Robertson equation which encodes the corresponding initial state, see eq. (4.15). This term describes the way in which the influence of the chosen initial state persists to later times. In the model studied below, G_j will be the number operators at the initial time, with j a mode number. In that case, the initial condition can be viewed as saying that the initial state is quasifree, or in a sense, as uncorrelated as possible. This is a physically reasonable assumption, since what one wants to study is not the effect of strong correlations persisting from the initial state to later times, but the process of approach to equilibrium.

Another heuristic argument supporting the “initial state assumption” is the following. Suppose the only information we have about a state is the knowledge of the expectation values of a restricted set of observables, as is generally the case when measuring the state of the system experimentally, since it is impossible to measure all the (infinite) correlations between various observables. Then we would like to know what is the state that satisfies the “maximal ignorance hypothesis”, i.e. the state that only contains information about the restricted set of observables *and* at the same time maximized the entropy for these observables. Such a state is a “minimum bias” state and is by definition given precisely by w_t . And by assuming that our initial state is w_0 we are stating that what we are really interested in is the time evolution of our selected observables and not in their possible correlations. This can be either because the initial state really was quasifree, or because we can neglect the effect of the initial correlations in the dynamics we are interested in. Clearly, physically it will not always be possible to make this assumption, but for us it is meaningful to do so.

4.1.3 Generalization for a time dependent Hamiltonian

Our derivation is somewhat different from the one in, say, [10], and it has the advantage that it can be transferred, relatively straightforwardly, to the case when the dynamics is given by a time dependent Hamiltonian $H(t)$ which is smooth in t^2 . Let us explain briefly the changes that have to be made to the statement and proof of Theorem 4.2 in that situation. The time evolution of a matrix $X_{t_0} \in M_n(\mathbb{C})$, with t_0 some initial time, is now

$$\alpha_{t,t_0} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C}) , \quad \alpha_{t,t_0}(X_{t_0}) := E_{t,t_0} X_{t_0} E_{t,t_0}^* ,$$

where

$$E_{t,t_0} = \sum_{n=0}^{\infty} i^n \int_{t_0 \leq \tau_1 \leq \dots \leq \tau_n \leq t} H(\tau_n) \cdots H(\tau_1) d\tau_1 \cdots d\tau_n . \quad (4.16)$$

The Heisenberg equation in this case is

$$\frac{d}{dt} \alpha_{t,t_0}(X_{t_0}) = i\delta_t \circ \alpha_{t,t_0}(X_{t_0}) , \quad \delta_t(X) := [H(t), X] , \quad (4.17)$$

and the square brackets $[,]$ denote the commutator of two matrices, exactly as in the time independent case, see equation (4.1). Here $\alpha_{t,s}$ is again an algebra automorphism and a cocycle, i.e. we have $\alpha_{t,s} \circ \alpha_{s,u} = \alpha_{t,u}$. Note that now, unlike in the time independent

²This situation is of interest because we would eventually like to apply the formalism to field theory in curved (time dependent) backgrounds, even though this will not be done in this work.

case, the time evolution automorphism $\alpha_{t,s}$ depends on two parameters t and s rather than one. Additionally, the derivation δ_t now depends on the time parameter t . The main consequence of these two facts is that $\alpha_{t,s}$ and δ_t do not commute in this case, i.e. $\alpha_{t,t_0} \circ \delta_t \neq \delta_t \circ \alpha_{t,t_0}$.

If we now introduce the ‘‘Schrödinger picture’’ derivations $\tilde{\delta}_{t,t_0} := \alpha_{t_0,t} \circ \delta_t \circ \alpha_{t,t_0}$ in addition to the ‘‘Schrödinger picture’’ projectors $\tilde{\mathcal{P}}_{t,t_0} = \alpha_{t_0,t} \circ \mathcal{P}_t \circ \alpha_{t,t_0}$, the decomposition of the equation of motion now proceeds much like in the time independent case. We proceed to decompose the Heisenberg equation of motion (4.17) as

$$\frac{\partial}{\partial t} \alpha_{t,t_0} = i\delta_t \circ \alpha_{t,t_0} = \alpha_{t,t_0} \circ i\tilde{\delta}_{t,t_0} = \alpha_{t,t_0} \circ \tilde{\mathcal{P}}_{t,t_0} \circ i\tilde{\delta}_{t,t_0} + \alpha_{t,t_0} \circ \tilde{\mathcal{Q}}_{t,t_0} \circ i\tilde{\delta}_{t,t_0}. \quad (4.18)$$

In complete analogy with (4.4) we now have

$$\alpha_{t,t_0} \circ \tilde{\mathcal{Q}}_{t,t_0} = \tilde{\mathcal{Q}}_{t_0,t_0} \circ \tilde{Y}_{t_0,t} + \int_{t_0}^t \alpha_{s,t_0} \circ \left[\tilde{\mathcal{P}}_{s,t_0} \circ i\tilde{\delta}_{s,t_0} \circ \tilde{\mathcal{Q}}_{s,t_0} + \frac{d}{ds} \tilde{\mathcal{Q}}_{s,t_0} \right] \circ \tilde{Y}_{s,t} ds,$$

so that equation (4.18) now becomes

$$\begin{aligned} \frac{\partial}{\partial t} \alpha_{t,t_0} &= \mathcal{P}_t \circ i\delta_t \circ \alpha_{t,t_0} + \mathcal{Q}_{t_0} \circ Y_{t_0,t} \circ i\delta_t \circ \alpha_{t,t_0} + \\ &+ \int_{t_0}^t \left[\mathcal{P}_s \circ i\delta_s \circ \mathcal{Q}_s - \alpha_{s,t_0} \frac{\partial}{\partial s} (\alpha_{t_0,s} \circ \mathcal{P}_s \circ \alpha_{s,t_0}) \circ \alpha_{t_0,s} \right] \circ Y_{s,t} \circ i\delta_t \circ \alpha_{t,t_0} ds. \end{aligned}$$

To obtain the above equation we have written $\tilde{Y}_{s,t} = \alpha_{t_0,s} \circ Y_{s,t} \circ \alpha_{t,t_0}$, with $Y_{s,t}$ being now given by

$$Y_{s,t}(X) = \sum_{k=0}^{\infty} (-i)^k \int_{s < \sigma_1 < \dots < \sigma_k < t} \delta_{\sigma_1} \circ \mathcal{P}_{\sigma_1} \circ \dots \circ \delta_{\sigma_k} \circ \mathcal{P}_{\sigma_k}(X) d^k \sigma. \quad (4.19)$$

And this sum is again convergent by the same argument for the convergence of (4.6). Also, by appropriately modifying the Kawasaki–Guntton projector (4.12) we could obtain the correspondingly modified version of Lemma 4.1, and we can hence immediately obtain the Robertson equation in the time dependent case:

Theorem 4.3 (Robertson equation—time dependent case). *Let $G_j(t_0)$, $j = 1, \dots, N$ be a set of hermitian, mutually commuting complex $n \times n$ matrices and let ρ be a density matrix. Let $g_j(t) = \text{Tr}(\rho G_j(t))$, where $G_j(t) = E_{t,t_0} G_j(t_0) E_{t,t_0}^*$, with E_{t,t_0} given by equation (4.16), is the time evolved observable with respect to a self-adjoint Hamiltonian $H(t)$, and let $Y_{t,s} : M_n(\mathbb{C}) \rightarrow M_n(\mathbb{C})$ be the cocycles defined as above in equation (4.19).*

Then the equation

$$\frac{d}{dt}g_j(t) = (\rho - w_0) [Y_{0,t} \circ i \delta_t(G_j(t))] + \int_{t_0}^t w_s [i \delta_s \circ Y_{s,t} \circ i \delta_t(G_j(t))] ds$$

holds. Here, we are using the shorthand $(\rho - w_0)(X) = \text{Tr}(\rho X) - w_0(X)$ for any X .

4.2 Number operator densities for the $P(\varphi)_2$ model

In this section we introduce the number operator densities for the $P(\varphi)_2$ model, and derive various bounds on these quantities that are needed in the following sections³. Here we rely to a considerable extent on established techniques in the study of this model, namely the “higher order estimates”—see (Rosen’s) Theorem 3.12—and the “ N -estimates”—see Theorem 3.3.

The Hamiltonian H for the model has been given in three different forms in the previous chapter: The “canonical form” (3.4), in creation and annihilation operator form (3.28), and in Q -space (3.29), and we again decompose it as $H = H_0 + \lambda V$. If $\varphi(t, \mathbf{x}) : \mathcal{D}(H) \rightarrow \mathcal{D}(H)$ is the *time- t* (interacting quantum) field of the $P(\varphi)_2$ model, which is defined as $\varphi(t, \mathbf{x}) := e^{itH} \varphi(\mathbf{x}) e^{-itH}$, we define the corresponding “annihilation”, $a_{\mathbf{p}}(t)$, and “creation”, $a_{\mathbf{p}}^*(t)$, operators at time t as:

$$\begin{aligned} a_{\mathbf{p}}(t) : \mathcal{D}(H) \rightarrow \mathcal{D}(H) , \quad a_{\mathbf{p}}(t) &:= i L^{3/2} \int_0^{2\pi} d\mathbf{x} \left[u_{\mathbf{p}}(t) \overleftrightarrow{\partial}_t \varphi(t, \mathbf{x}) \right] e^{-i\mathbf{p}\mathbf{x}} , \\ a_{\mathbf{p}}^*(t) &:= \left(a_{\mathbf{p}}(t) \right)^* , \end{aligned} \quad (4.20)$$

where $f \overleftrightarrow{\partial} g := f(\partial g) - (\partial f)g$, the star “ $*$ ” denotes hermitian conjugation and

$$\overline{u_{\mathbf{p}}(t)} = \frac{1}{\sqrt{2\pi L}} \frac{e^{-i\omega_{\mathbf{p}}t}}{\sqrt{2\omega_{\mathbf{p}}}} , \quad (4.21)$$

are the “positive frequency” mode functions. Above we have $\omega_{\mathbf{p}} = (\mathbf{p}^2/L^2 + m^2)^{1/2}$, and the mode functions are defined so that the functions $u_{\mathbf{p}}(t) \exp(-i\mathbf{p}\mathbf{x})$ are correctly

³These estimates have been obtained by my research advisor.

normalized in the “Klein–Gordon inner product”

$$(f(t, \mathbf{x}), g(t, \mathbf{x})) := iL \int_0^{2\pi} \overline{f(t, \mathbf{x})} \overleftarrow{\partial}_t g(t, \mathbf{x}) .$$

In other words, we have

$$(u_{\mathbf{p}}(t)e^{-i\mathbf{p}\mathbf{x}}, u_{\mathbf{k}}(t)e^{-i\mathbf{k}\mathbf{x}}) = \delta(\mathbf{k}, \mathbf{p}) ,$$

where $\delta(\mathbf{k}, \mathbf{p})$ is the Kronecker delta. Note that both $a_{\mathbf{p}}(t)$ and $a_{\mathbf{p}}^*(t)$ depend on time in lieu of the equations of motion. In fact

$$a_{\mathbf{p}}^{\#}(t) = e^{\pm i\omega_{\mathbf{p}}t} e^{iHt} a_{\mathbf{p}}^{\#} e^{-iHt} , \quad (4.22)$$

and $a_{\mathbf{p}}^{\#} = a_{\mathbf{p}}^{\#}(t=0)$ in the notation used in Chapter 3. Because H is a constant of the motion ($e^{itH} H e^{-itH} = H$), we may write the creation and annihilation operator expression for H [cf. equation (3.28)] in terms of $a_{\mathbf{p}}^{\#}(t)$ rather than $a_{\mathbf{p}}^{\#}(t=0) = a_{\mathbf{p}}^{\#}$ for any t . This will be extremely useful below in Section §4.2.1 when we present the estimates of the number densities at some time t . Note however, that we *cannot* separately write H_0 or V in terms of $a_{\mathbf{p}}^{\#}(t)$ at arbitrary times t because H_0 and V are *not*, separately, conserved ($[H, H_0] \neq 0 \neq [H, V]$). Using (4.20) we can also express the field at time t in terms of creation and annihilation operators at time t . After a standard computation, which is easily performed in momentum space, we obtain

$$\varphi(t, \mathbf{x}) = \frac{1}{L\sqrt{2\pi}} \sum_{\mathbf{p} \in \mathbb{Z}} \frac{1}{(2\omega_{\mathbf{p}})^{1/2}} \left[a_{\mathbf{p}}(t) e^{i\langle \mathbf{p}, \mathbf{x} \rangle} + a_{\mathbf{p}}^*(t) e^{-i\langle \mathbf{p}, \mathbf{x} \rangle} \right] . \quad (4.23)$$

The commutation relations for $a_{\mathbf{p}}(t)$ and $a_{\mathbf{p}}^*(t)$ are the same as the time-0 ones, see equation (3.8). We now define the “number operator densities”⁴

$$N_{\mathbf{p}}(t) : \mathcal{D}(H) \rightarrow \mathcal{D}(H) , \quad N_{\mathbf{p}}(t) := \frac{a_{\mathbf{p}}^*(t) a_{\mathbf{p}}(t)}{L} , \quad (4.24)$$

which are the quantities of main interest for this work. It also follows from (4.22) that the time-evolved number densities are given by

$$N_{\mathbf{k}}(t) = e^{itH} N_{\mathbf{k}} e^{-itH} = \frac{1}{L} a_{\mathbf{k}}(t)^* a_{\mathbf{k}}(t) ,$$

⁴This definition is motivated by our desire to consider the $L \rightarrow \infty$ limit, i.e. the thermodynamic limit.

where the second equality follows by a simple and straightforward calculation noting e.g. that $\frac{d}{dt}a_{\mathbf{k}}(t) = i[H, a_{\mathbf{k}}(t)] - i\omega_{\mathbf{k}}a_{\mathbf{k}}$. Using the relation (3.8) we immediately find $[N_{\mathbf{k}}(t), N_{\mathbf{p}}(t)] = 0^5$. Finally, by splitting the Hamiltonian as $H = H_0(t) + \lambda V(t)$ it is easy to see that

$$\frac{dN_{\mathbf{p}}(t)}{dt} = i[H, N_{\mathbf{p}}(t)] = i\lambda[V(t), N_{\mathbf{p}}(t)] .$$

4.2.1 Estimates on the number densities

We now discuss properties of the expected number densities in suitable states. The expectation values of our number observables for mode numbers $\mathbf{k} \in \mathbf{Z}$ in a density matrix state ρ are denoted

$$n_{\mathbf{k}}(t) = \text{Tr}[\rho N_{\mathbf{k}}(t)] ,$$

where we must require at least that $\rho \in \mathcal{S}_1(\mathcal{H})$, the space of all trace class operators on \mathcal{H} . These quantities depend on λ because the Hamiltonian $H(\lambda) = H_0 + \lambda V$ does. We will give estimates concerning the magnitude of $n_{\mathbf{k}}(t)$ as a function of the mode number, time, and the coupling constant λ . These will be used later when we investigate the “pre-Boltzmann equation”. Our main estimates are contained in the following theorem:

Theorem 4.4. *For a quartic interaction polynomial, i.e. with $p = \deg(P(\xi)) = 4$, let $n_{\mathbf{k}}(t, \lambda) = \text{Tr}(N_{\mathbf{k}}(t)\rho)$ as above, with ρ independent of λ . Additionally suppose that $\rho H^j \in \mathcal{S}_1(\mathcal{H})$ up to sufficiently large j . Then for $\frac{1}{2} > \epsilon > 0$ we have the bounds*

$$|n_{\mathbf{k}}(t, \lambda) - n_{\mathbf{k}}(s, \lambda)| \leq K\lambda|t - s|\omega_{\mathbf{k}}^{-1-\epsilon} ,$$

and

$$|n_{\mathbf{k}}(t, \lambda)| \leq K\omega_{\mathbf{k}}^{-4+\epsilon} ,$$

with a constant depending on ϵ and L .

Remark. The proof shows that the constants are of the order of $\text{Tr}(\rho H^j)$ where j is some positive number depending on ϵ . This quantity is not uniformly bounded in L (in fact typically it is $\propto L^j$), hence the estimates are not guaranteed to be preserved in

⁵The number operators hence form a set of mutually commuting operators, which was a key assumption to obtain the Robertson equation (4.14). The second fundamental ingredient for the application of the framework of Section §4.1 is that the time derivative of the “relevant” observables be given in terms of a commutator. By direct computation of $dN_{\mathbf{p}}(t)/dt$ and $i[H, N_{\mathbf{p}}(t)]$ we see that they are indeed the same, as one would expect. We do not write the resulting expression as it gives no insight whatsoever for the purposes of this work, but merely note that the equation of motion—the non-linear Klein–Gordon equation—is essential to establish such a result.

the thermodynamic limit $L \rightarrow \infty$. If ρ is so that $e^{\beta H/2} \rho e^{\beta H/2}$ is still bounded, then the constant K is of the order $(L/\beta)^j$.

Proof. The proofs of both estimates are rather similar, but the first estimate is somewhat more complicated to prove, so we only give the proof of the second one here. We start with the operator inequalities

$$N_{\mathbf{k}}(t) \leq N_{\mathbf{k}}(t)^4 \leq \omega_{\mathbf{k}}^{-2} N_{\mathbf{k}}(t) H_0(t)^2 N_{\mathbf{k}}(t) \leq K \omega_{\mathbf{k}}^{-2} N_{\mathbf{k}}(t) H^2 N_{\mathbf{k}}(t), \quad (4.25)$$

where in the first step we used that $\text{spec } N_{\mathbf{k}}(t) = \mathbb{N}$, and where in the second step we have used the obvious relations

$$H_0(t) = \sum_{\mathbf{p} \in \mathbb{Z}} \omega_{\mathbf{p}} N_{\mathbf{p}}(t) \geq \omega_{\mathbf{k}} N_{\mathbf{k}}(t).$$

In the last step of equation (4.25), we use equation (3.33), i.e. $H_0^2(t) \leq K H^2$. As usual, K denotes a constant, and we adopt the standard *constant convention* to denote all constants that may appear in the various inequalities in this paper by the same letter, K , even though, of course, they might be numerically different and/or depend on different parameters. We write $\rho(A) = \text{Tr}(\rho A)$, and we apply this state to the above operator inequality, recalling that $\rho(N_{\mathbf{k}}(t)) = n_{\mathbf{k}}(t)$. We get

$$\begin{aligned} n_{\mathbf{k}}(t) \leq K \omega_{\mathbf{k}}^{-2} \rho(N_{\mathbf{k}}(t) H^2 N_{\mathbf{k}}(t)) &= K \omega_{\mathbf{k}}^{-2} \left(\rho \left[\delta(N_{\mathbf{k}}(t))^* \delta(N_{\mathbf{k}}(t)) \right] + \right. \\ &\quad \left. + \rho[H N_{\mathbf{k}}(t)^2 H] + 2 \text{Re} \rho[H N_{\mathbf{k}}(t) \delta(N_{\mathbf{k}}(t))] \right). \end{aligned}$$

We now introduce the shorthand $X(t) = \delta(N_{\mathbf{k}}(t))(I + N(t))^{-p/2}$, with p the degree of the interaction polynomial and $N(t) = e^{itH} N e^{-itH}$ the total number operator at time t . With this shorthand, we then have the estimate

$$\rho \left(\delta(N_{\mathbf{k}}(t))^* \delta(N_{\mathbf{k}}(t)) \right) \leq \|X(t)\|^2 \rho((I + N(t))^p) \leq K \|X(t)\|^2 \rho((I + H)^p).$$

In the first step, we used that $\rho(BAA^*B^*) \leq \|A\|^2 \rho(BB^*)$, and in the second step we use (3.32) with $\epsilon = 3$. Using the Cauchy-Schwarz inequality and Rosen's inequality again,

now with $\epsilon = 1$ and $i = p + 2$, we also have

$$\begin{aligned} \left| \operatorname{Re} \rho(HN_{\mathbf{k}}(t)\delta(N_{\mathbf{k}}(t))) \right| &\leq \left(\rho(H(I + N(t))^{\frac{p}{2}} N_{\mathbf{k}}(t)^2 (I + N(t))^{\frac{p}{2}} H) \right)^{\frac{1}{2}} \left(\rho(X(t)X(t)^*) \right)^{\frac{1}{2}} \\ &\leq K\omega_{\mathbf{k}}^{-1} \|X(t)\| \left(\rho(H(I + N(t))^{\frac{p}{2}} H_0(t)^2 (I + N(t))^{\frac{p}{2}} H) \right)^{1/2} \\ &\leq K\omega_{\mathbf{k}}^{-1} \|X(t)\| \left(\rho((I + H)^j) \right)^{\frac{1}{2}}, \end{aligned}$$

for some j . In the second line, we have used that N and H_0 commute together with the fact that $N_{\mathbf{k}} \leq \omega_{\mathbf{k}}^{-1} H_0$. We also have

$$\rho(HN_{\mathbf{k}}(t)^2 H) \leq \omega_{\mathbf{k}}^{-2} \rho(HH_0(t)^2 H) \leq K\omega_{\mathbf{k}}^{-2} \rho(H^4),$$

again by (3.33). Using now the assumption that $\rho(H^j) < \infty$ for any j , we have altogether shown that

$$n_{\mathbf{k}}(t) \leq K\omega_{\mathbf{k}}^{-2} (\omega_{\mathbf{k}}^{-1} + \|X(t)\|)^2.$$

Thus the proof is complete if we can demonstrate the following

Lemma 4.5. *For each $\epsilon > 0$ there is a constant K so that*

$$\|X(t)\| = \|[N_{\mathbf{k}}(t), H](I + N(t))^{-p/2}\| \leq K\omega_{\mathbf{k}}^{-1+\epsilon}.$$

Proof. We calculate that $[N_{\mathbf{k}}(t), H] = \lambda[N_{\mathbf{k}}(t), V(t)]$ is a finite sum of operators W of the form

$$W(t) = \omega_{\mathbf{k}}^{-1/2} \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n \in \mathbf{Z}} \frac{\delta(\mathbf{k}, \mathbf{q}_1) \delta(\sum_{i \in X} \mathbf{q}_i, \sum_{l \in Y} \mathbf{q}_l)}{\prod_{i \neq j} \omega_{\mathbf{q}_i}^{1/2}} \prod_{i \in X} a_{\mathbf{q}_i}(t)^* \prod_{l \in Y} a_{\mathbf{q}_l}(t),$$

where $n \leq p$, and where $X \cup Y = \{1, \dots, n\}$. By Theorem 3.4, we have

$$\|W(t)(I + N(t))^{-p/2}\| \leq \omega_{\mathbf{k}}^{-1/2} \left\| \frac{\delta(\mathbf{k}, \mathbf{q}_1) \delta(\sum_{i \in X} \mathbf{q}_i, \sum_{l \in Y} \mathbf{q}_l)}{\prod_{i \neq j} \omega_{\mathbf{q}_i}^{1/2}} \right\|_{\ell^2},$$

where on the right side we mean the ℓ^2 norm of a function in the variables $\mathbf{q}_i \in \mathbf{Z}$. We are now going to show that $\|\dots\|_{\ell^2} \leq K\omega_{\mathbf{k}}^{-1/2+\epsilon}$, which implies the statement of the lemma. We prove this estimate for simplicity of notation in the case when $Y = \emptyset, j = 1$. Then the ℓ^2 norm is

$$\|\dots\|_{\ell^2}^2 = \sum_{\mathbf{q}_1, \dots, \mathbf{q}_n \in \mathbf{Z}} \frac{1}{\omega_{\mathbf{k}-\mathbf{q}_1} \omega_{\mathbf{q}_1-\mathbf{q}_2} \dots \omega_{\mathbf{q}_{n-1}-\mathbf{q}_n}} \leq K\omega_{\mathbf{k}}^{-1+\epsilon},$$

which we can estimate by successively reducing it to the estimate $\sum_{\mathbf{q} \in \mathbb{Z}} \omega_{\mathbf{k}-\mathbf{q}}^{-1} \omega_{\mathbf{q}}^{-1+\delta} \leq K \omega_{\mathbf{k}}^{-1+2\delta}$, where δ is small and positive. To show the last estimate, we can argue e.g. as follows for large $|\mathbf{k}|$:

$$\begin{aligned} \sum_{\mathbf{q} \in \mathbb{Z}} \omega_{\mathbf{k}-\mathbf{q}}^{-1} \omega_{\mathbf{q}}^{-1+\delta} &= \left(\sum_{|\mathbf{q}| \leq \frac{1}{2}|\mathbf{k}|} + \sum_{|\mathbf{q}| > \frac{1}{2}|\mathbf{k}|} \right) \omega_{\mathbf{k}-\mathbf{q}}^{-1} \omega_{\mathbf{q}}^{-1+\delta} \\ &\leq K \left(|\mathbf{k}|^{-1+\delta} \log |\mathbf{k}| + |\mathbf{k}|^{-1+2\delta} \sum_{\mathbf{q} \in \mathbb{Z}} \omega_{\mathbf{q}}^{-2\delta} \omega_{\mathbf{k}-\mathbf{q}}^{-1+\delta} \right) \\ &\leq K |\mathbf{k}|^{-1+2\delta} \left(1 + \sum_{\mathbf{q} \in \mathbb{Z}} \omega_{\mathbf{q}}^{-1-\delta} \right) \leq K \omega_{\mathbf{k}}^{-1+2\delta}, \end{aligned}$$

where in the last line we have used Hölder's inequality and that $\log x \leq Kx^\delta$ for large $x \geq 0$ and $\delta > 0$. \square

4.2.2 Perturbative expansion estimates

Later we will also consider a perturbative expansion of the quantities $n_{\mathbf{k}}(t, \lambda)$ in the coupling constant λ . Such perturbation expansions are known not to converge, but it is still of interest to know to what extent they can be trusted as asymptotic series. Unfortunately, we have been unable to get reliable estimates on quantities like the remainder term in the perturbative expansion of $n_{\mathbf{p}}(t)$ up to a given order. But it is possible to get, without too much difficulty, estimates on related quantities, e.g. if we let the time parameter t be imaginary. The same type of arguments also provide estimates on the error in the perturbative expansion of $n_{\mathbf{k}}(t, \lambda)$ for small t , essentially because the function in question is analytic in t for suitable states ρ .

Despite the fact that our estimates on the error term in the perturbative expansion are not satisfactory for the main purpose of this work, the development of the Boltzmann equation, we will nevertheless present our results here as they provide at least heuristic support of the use of the perturbative expansion. Also, it could be seen as a general illustration of our method, which should be applicable in other contexts. The reader only interested in the main line of the argument can skip this section. We first describe the states that we consider, which are the density matrix states of the form ($\beta > 0$)

$$\rho = e^{-\beta H/2} \sigma e^{-\beta H/2}.$$

We are allowing both ρ and $\sigma = \sigma^*$ to depend on λ , and we postulate that

$$\left\| \frac{d^n}{d\lambda^n} \sigma \right\| \leq K^n$$

for some constant and all $n = 0, 1, 2, \dots$. Under this assumption, $n_{\mathbf{k}}(t)$ can be continued analytically to complex t as long as $\text{Im } t < \beta/2$. Our result in particular covers the case $\sigma = I$, i.e. $\rho = e^{-\beta H}$. Our result is now the following:

Theorem 4.6. *Let ρ_λ be a density matrix for each $\lambda \geq 0$ satisfying the hypothesis above for some $\beta > 0$, and let $n_{\mathbf{k}}(t, \lambda) = \text{Tr}(\rho_\lambda e^{itH(\lambda)} N_{\mathbf{k}} e^{-itH(\lambda)})$. Let $r_N(t)$ be the remainder in the Taylor expansion of $n_{\mathbf{k}}(it)$ up to order N . Then for $t < \beta/2$, we have the estimate*

$$|r_N| \leq \frac{(\lambda K)^N (Np)!}{N! \omega_{\mathbf{k}}}.$$

Proof. We write $n_{\mathbf{k}}(it, \lambda)$ as a Taylor series to N -th order. The remainder in this series is given by the Schlömilch formula

$$r_N(\lambda, t) = \frac{\lambda^N}{N!} \int_0^1 (1-s)^N (\partial_\lambda^N n_{\mathbf{k}})(it, s\lambda) ds.$$

Thus, we have to estimate the N -th λ -derivative

$$\frac{d^N}{d\lambda^N} n_{\mathbf{k}}(it, \lambda) = \frac{d^N}{d\lambda^N} \text{Tr}(\sigma e^{-(\beta/2-t)H} N_{\mathbf{k}} e^{-(\beta/2+t)H}).$$

When we carry out these derivatives, they get distributed over the factors inside the trace. When derivatives hit σ , then we use our assumption that this is estimated by the factor K raised to the number of derivatives. When j derivatives hit one of the exponential factors, we use the iterated Duhamel formula

$$B_j = \frac{d^j}{d\lambda^j} e^{-s_0 H} = \int_{s_0 > s_1 > \dots > s_j > 0} e^{-(s_0-s_1)H} V e^{-(s_1-s_2)H} V \dots e^{-(s_{j-1}-s_j)H} V e^{-s_j H} d^j s.$$

We will use this for $s_0 = \beta/2 - t > 0$. Using the inequalities $\text{Tr}(AB) \leq \|A\| \text{Tr}|B|$ and $|\text{Tr}(AB)| \leq (\text{Tr}|A|^2)^{1/2} (\text{Tr}|B|^2)^{1/2}$, it is straightforward to see that the desired estimate will follow if we can show that

$$\left(\text{Tr} |B_j N_{\mathbf{k}}^{1/2}|^2 \right)^{1/2} \leq \frac{K^j (jp)!}{\omega_{\mathbf{k}}^{1/2}}.$$

Using that $N_{\mathbf{k}} \leq \omega_{\mathbf{k}}^{-1} H_0 \leq K \omega_{\mathbf{k}}^{-1} H$ by the first order estimates $H_0(t) \leq KH$ (see [57]),

it hence suffices to show

$$\int_{s_0 > s_1 > \dots > s_j > 0} \left(\text{Tr} |e^{-(s_0-s_1)H} V e^{-(s_1-s_2)H} V \dots V e^{-s_j H} H^{1/2}|^2 \right)^{1/2} d^j s \leq K^j (jp)! . \quad (4.26)$$

In order to continue, it is convenient to work in the \mathcal{Q} -space representation. We introduce the Hölder space $L^q(\mathcal{Q}, d\nu)$, with $1 \leq q \leq \infty$, as the Banach space with the norm $\|\Psi\|_q = (\int |\Psi|^q d\nu)^{1/q}$, noting that $\mathcal{H} = L^2$. In these spaces we have the usual Hölder inequality which states that $|(\Phi, \Psi)| \leq \|\Phi\|_p \|\Psi\|_q$, when $1 = \frac{1}{p} + \frac{1}{q}$. This implies in particular that $\|\Psi\|_p \leq \|\Psi\|_q$ if $q \geq p$, since $d\nu$ is a probability measure of total weight 1. If $T : L^r \rightarrow L^q$ is a linear operator, then we let $\|T\|_{q,r}$ be the operator norm; the ordinary operator norm on $\mathcal{H} = L^2$ is a special case of this. The Hölder norm of V can be estimated as follows for any j , see Theorem 3.8

$$\begin{aligned} \|V\|_{2j} &= \left(\int_{\mathcal{Q}} |V(q)|^{2j} d\nu \right)^{1/2j} = (V^j \Omega_0, V^j \Omega_0)^{1/2j} \\ &\leq (\|V(N+I)^{-\frac{p}{2}}\| \cdot \|(N+I)^{\frac{p}{2}} V^{j-1} \Omega_0\|)^{1/j} \\ &\leq K(1 + (j-1)p/2)^{p/2j} \|V^{j-1} \Omega_0\|^{1/j} . \end{aligned}$$

In the last step, we use Theorems 3.4 and 3.3, $\|V(N+I)^{-p}\| \leq K$, as well as the fact that the $V^{j-1} \Omega_0$ contains at most $p(j-1)$ ‘‘particles’’. An induction then shows

$$\|V\|_{2j} \leq K(j!)^{p/2j} \leq K j^{p/2} ,$$

where in the last step we have used Stirling’s formula. Thus, as a multiplication operator, V is an operator $L^r \rightarrow L^q$ for $q = r - 1/(Kj)$ with norm $(Kj)^p$. The second ingredient is Theorem 3.10, i.e. the fact that e^{-sH} is a contraction from L^r to L^q as long as $\frac{1}{q} \leq (1 + Ks)\frac{1}{r}$. Therefore (by duality),

$$\|e^{-sH}\|_{q,r} \leq 1, \quad \text{if } \frac{1}{q} \leq (1 + Ks)\frac{1}{r}, \quad \text{and} \quad \left(1 - \frac{1}{q}\right) \leq (1 + Ks) \left(1 - \frac{1}{r}\right) .$$

These facts are now put together to estimate the integrand of equation (4.26).

We first note that of the j interval lengths $s_0 - s_1, \dots, s_{j-1} - s_j, s_j$, at least one will be greater than or equal to s_0/j . We consider two cases. Case (1) occurs if the interval in question is the last one, i.e. when $s_j > \tau/j$. Case (2) covers the rest. Both cases are

dealt with in a similar fashion, so we will for brevity only deal with, say, case (2). Let i be the interval in question, $s_i - s_{i+1} > s_0/j$. We first estimate

$$e^{-s_j H} H^{1/2} \leq K s_j^{-1/2},$$

which leads to

$$\begin{aligned} \left| \text{Tr} \left[e^{-(s_0-s_1)H} V e^{-(s_1-s_2)H} V \dots V e^{-s_j H} H^{1/2} \right]^2 \right| &\leq K s_j^{-1} \text{Tr} \left| \prod_{k=1}^j e^{-(s_{k-1}-s_k)H} V \right|^2 \leq \\ &\leq K s_j^{-1} \text{Tr} (e^{-(s_i-s_{i+1})H} V) \left\| X X^* V e^{-(s_i-s_{i+1})H} Y^* Y \right\|, \end{aligned}$$

where we are using the shorthand notations

$$X = \prod_{k=0}^{i-1} e^{-(s_k-s_{k+1})H} V, \quad Y = \prod_{k=i+1}^{j-1} e^{-(s_k-s_{k+1})H} V.$$

The trace term on the right side is now estimated using that $s_i - s_{i+1} > s_0/j$. In order to tame the factor of V under the trace, we write $V = V(I+N)^{-p}(I+N)^p$, and we use the Rosen inequality with $\epsilon = 3$ to estimate $(I+N)^p \leq K H^p$, as well as the identity $\|V(I+N)^{-p}\| \leq K$. This gives

$$\text{Tr}(e^{-(s_i-s_{i+1})H} V) \leq \|e^{-s_0 H/(2j)} V\| \cdot \text{Tr}(e^{-s_0 H/(2j)}) \leq e^{Kj/s_0} \|H^p e^{-s_0 H/(2j)}\| \leq e^{Kj/s_0},$$

with a constant K . In order to estimate $\|X X^* V e^{-(s_i-s_{i+1})H} Y^* Y\|$, we use the mapping properties of the multiplication operator V and the contractions $e^{-\tau H}$ between the Hölder spaces L^p . We then get

$$\|X X^* V e^{-(s_i-s_{i+1})H} Y^* Y\| \leq \prod_{k=1}^{2j+1} (Kj)^p \leq [K^j (jp)!]^2 \quad (4.27)$$

because each of the $2j+1$ factors of V has norm $(Kj)^p$ as an operator from $L^r \rightarrow L^q$, where $q = r - 1/(Kj)$. This decrease in the Hölder index for each factor of V is compensated by the increase in the Hölder index caused by the $2j+1$ heat kernels $e^{-(s_k-s_{k+1})H}$ in the X and Y in (4.27). In the last step we have used Stirling's formula. Thus, in total we have shown that in case (2), we have

$$\left(\text{Tr} \left| e^{-(s_0-s_1)H} V e^{-(s_1-s_2)H} V \dots V e^{-s_j H} H^{1/2} \right|^2 \right)^{1/2} \leq s_j^{-1/2} K^j (jp)!$$

and upon integration over the $s_{\mathbf{k}}$, this gives the desired bound eq. (4.26). Case (1) is dealt with similarly. \square

4.3 The pre-Boltzmann equation

The projection operator technique from Section §4.1.1 is now combined with the results of Section 4.2.1 to derive what we called the “pre-Boltzmann equation”. This is an equation for the expected number densities $n_{\mathbf{k}}(t) = \text{Tr}[\rho N_{\mathbf{k}}(t)]$ and is, in a sense, a preliminary form of the Boltzmann equation, with which it shares some features. The number estimates from Section §4.2.1 are used to establish that the various ingredients (infinite sums) in the pre-Boltzmann equation make sense, even non-perturbatively. Despite some common features, it is also very different from the BE [cf. equation (2.10)] as it is still an exact equation. The Boltzmann equation, on the other hand, only holds when we consider appropriate limits like the thermodynamic limit $L \rightarrow \infty$, and the long time limit $t \rightarrow \infty$. In this chapter we will not yet take these limits.

We first need to define the reference states w_t [cf. equation (4.9)] for our model and the set of observables $\{N_{\mathbf{k}}\}$ where $\mathbf{k} \in \mathbf{Z}$. In accordance with our constructions in Section §4.1.2, we let w_t be the density matrix state

$$w_t(X) = \text{Tr}(\rho_t X), \quad \rho_t = Z(t)^{-1} \exp\left(-\sum_{\mathbf{k} \in \mathbf{Z}} \mu_{\mathbf{k}}(t) N_{\mathbf{k}}(t)\right),$$

where X is, say, a bounded operator, $Z(t) = \text{Tr}[\exp(-\sum_{\mathbf{p}} \mu_{\mathbf{p}}(t) N_{\mathbf{p}}(t))]$ and where the quantities $\mu_{\mathbf{k}}$ are defined through the formula

$$\mu_{\mathbf{k}}(t) = -\log \frac{n_{\mathbf{k}}(t)}{n_{\mathbf{k}}(t) + 1} > 0. \quad (4.28)$$

We now first discuss the rationale behind this definition and then argue that it is also mathematically meaningful. Recalling the discussion in Section §4.1.2, we anticipate that the above choice for the $\mu_{\mathbf{k}}(t)$ is precisely the one we need to make in order to satisfy the requirement $w_t(N_{\mathbf{k}}(t)) = \text{Tr}(\rho N_{\mathbf{k}}(t)) = n_{\mathbf{k}}(t)$, which is a crucial ingredient in the construction of the Kawasaki–Gunton projector (4.12). Indeed, it is easy to see that

$$\begin{aligned} n_{\mathbf{k}}(t) &= \text{Tr}[\rho N_{\mathbf{k}}(t)] = \text{Tr}[\rho_t N_{\mathbf{k}}(t)] = \frac{1}{Z(t)} \text{Tr}\left[e^{-\sum_{\mathbf{p}} \mu_{\mathbf{p}}(t) N_{\mathbf{p}}(t)} N_{\mathbf{k}}(t)\right] = \\ &= -\frac{\partial_{\mu_{\mathbf{k}}(t)} \text{Tr}\left[e^{-\sum_{\mathbf{p}} \mu_{\mathbf{p}}(t) N_{\mathbf{p}}(t)}\right]}{Z(t)} = -\frac{\partial \log Z(t)}{\partial \mu_{\mathbf{k}}(t)}. \end{aligned}$$

Formally, the “partition function” $Z(t)$ is given by (see e.g. Section 18.2 of [15])

$$Z(t) = \prod_{\mathbf{k} \in \mathbf{Z}} \left(1 - e^{-\mu_{\mathbf{k}}(t)}\right)^{-1}, \quad (4.29)$$

from which it straightforwardly follows that $\text{Tr}[\rho_t N_{\mathbf{k}}(t)] = n_{\mathbf{k}}(t)$ if the $\mu_{\mathbf{k}}$ are given by equation (4.28).

To see that equation (4.28) makes sense, let us assume first $n_{\mathbf{k}}(t) > 0$ for all $\mathbf{k} \in \mathbf{Z}$. As shown in Theorem 4.4 in Section §4.2.1, when the initial density matrix state $\rho \in \mathcal{S}_1(\mathcal{H})$ is such that also $\rho H^j \in \mathcal{S}_1(\mathcal{H})$ for all j , then we have $n_{\mathbf{k}}(t) \leq K\omega_{\mathbf{k}}^{-4+\epsilon}$. It then easily follows that the partition function is convergent:

$$Z(t) = \exp\left(-\sum_{\mathbf{k} \in \mathbf{Z}} \log(1 - e^{-\mu_{\mathbf{k}}(t)})\right) \leq \exp\left(K \sum_{\mathbf{k} \in \mathbf{Z}} e^{-\mu_{\mathbf{k}}(t)}\right) \leq \exp\left(K \sum_{\mathbf{k} \in \mathbf{Z}} \omega_{\mathbf{k}}^{-4+\epsilon}\right) < \infty.$$

Thus, ρ_t is indeed a trace class operator for any $t \in \mathbb{R}$, and the state w_t is well defined. The situation is the same when some $n_{\mathbf{k}}(t) = 0$, essentially because this means that $\mu_{\mathbf{k}}(t) = +\infty$, and this only improves the convergence properties. Thus, w_t is a well-defined state if the initial state of the system ρ is such that ρH^j has a finite trace for all $j \geq 0$.

A different way to characterize the state w_t is to say that it is the unique quasifree state with respect to the time t -observables $a_{\mathbf{k}}^{\#}(t)$, whose 2-point function is

$$w_t(a_{\mathbf{k}}(t)^* a_{\mathbf{p}}(t)) = L n_{\mathbf{p}}(t) \delta(\mathbf{k}, \mathbf{p}), \quad (4.30)$$

and whose n -point functions are zero for an odd number of creation/annihilation operators, and factorize into 2-point functions for an even number. More precisely, for $X, Y \subset \mathbf{Z}$ we have

$$w_t \left(\prod_{i \in X} a_{\mathbf{k}_i}(t)^* \prod_{j \in Y} a_{\mathbf{p}_j}(t) \right) = \delta_{|X|, |Y|} \sum_{f: X \rightarrow Y \text{ bijective}} \prod_{j \in X} w_t(a_{\mathbf{p}_j}(t)^* a_{f(\mathbf{p}_j)}(t)). \quad (4.31)$$

It is important to realize that for fixed t , this factorization formula for w_t will not hold for the creation and annihilation operators $a_{\mathbf{k}}(s)$ at another time $s \neq t$ unless the model is free, $\lambda = 0$. The above factorization formula also demonstrates once again that the state is well-defined also when one or more $n_{\mathbf{k}}(t)$'s happen to be equal to zero, and we have, as desired

$$n_{\mathbf{k}}(t) = w_t(N_{\mathbf{k}}(t)), \quad \text{for all } t \in \mathbb{R}, \mathbf{k} \in \mathbf{Z}. \quad (4.32)$$

The correlation matrix [cf. equation (4.10)] is found to be diagonal,

$$C_{\mathbf{p}\mathbf{k}}(t) = w_t[(N_{\mathbf{p}}(t) - n_{\mathbf{p}}(t)I)(N_{\mathbf{k}}(t) - n_{\mathbf{k}}(t)I)] = \frac{1}{L}n_{\mathbf{k}}(t)(n_{\mathbf{k}}(t) + 1)\delta(\mathbf{k}, \mathbf{p}) .$$

We now define our projector according to the general recipe laid out in Section §4.1.1. To be on the safe side, we first consider only a subset $\{N_{\mathbf{k}} : \mathbf{k} \in \mathbf{Z}, |\mathbf{k}| \leq \Lambda\}$ of observables where $\Lambda < \infty$, and we put $\mu_{\mathbf{k}} = +\infty$ for $|\mathbf{k}| > \Lambda$ in w_t and denote the correspondingly changed state as w_t^Λ . This change has the effect that equation (4.32) is valid only for $|\mathbf{k}| \leq \Lambda$, and that eq. (4.30) returns zero for $|\mathbf{k}| > \Lambda$. Also, Λ is *not* a cutoff of the theory, but merely a restriction on the set of momenta \mathbf{k} in $n_{\mathbf{k}}(t)$ that we monitor.

The projector as in equation (4.12) is then:

$$\begin{aligned} \mathcal{P}_t^\Lambda(A) &= w_t^\Lambda(A)I + \sum_{|\mathbf{p}| \leq \Lambda} \frac{w_t^\Lambda((N_{\mathbf{p}}(t) - n_{\mathbf{p}}(t)I)A)}{n_{\mathbf{p}}(t)(n_{\mathbf{p}}(t) + 1)} (N_{\mathbf{p}}(t) - n_{\mathbf{p}}(t)I) \\ &= w_t^\Lambda(A)I + \sum_{|\mathbf{p}| \leq \Lambda} \frac{\partial w_t^\Lambda(A)}{\partial n_{\mathbf{p}}(t)} (N_{\mathbf{p}}(t) - n_{\mathbf{p}}(t)I) . \end{aligned} \quad (4.33)$$

To obtain the formula in the second line, we used the analogue of equation (4.11) for $G_j \rightarrow N_{\mathbf{p}}$. In writing that expression, we have also anticipated that, in the expressions below, A will be a power series in $a_{\mathbf{p}}^\#(t)$, and $w_t(A)$ can then be written as a corresponding power series in $n_{\mathbf{p}}(t)$ by equation (4.31). The operator $\partial/\partial n_{\mathbf{k}}(t)$ then acts as the usual partial derivative operator on such an expression. In particular, it is clear from the last line that the $n_{\mathbf{p}}(t)$'s in the denominator in the first line will always cancel, and so the case $n_{\mathbf{p}}(t) = 0$ will never cause any problems.

We also need to choose our initial conditions, i.e. the quantum state ρ that we would like to investigate. As we have just explained, in order for the states w_t to be well defined, we require ρH^j to have finite trace for sufficiently large non-negative j . Furthermore, we would like to have an initial state so that the Robertson equation is valid without a “memory term”, cf. Section §4.1.1, Theorem 4.2. Thus, we would ideally like to choose as our state as $\rho = \rho_0$, where

$$\rho_0 = \frac{1}{Z} \exp\left(-\sum_{\mathbf{k} \in \mathbf{Z}} \mu_{\mathbf{k}} N_{\mathbf{k}}\right) . \quad (4.34)$$

for some $\mu_{\mathbf{k}}$. In other words, we would like to choose our state to be quasifree (with respect to the time-0 creation/annihilation operators $a_{\mathbf{k}}^\#(0)$!), and we would also like our initial state to be translation invariant⁶. In the finite dimensional context, we were free to

⁶That is, invariant under the 1-parameter group generated by the momentum operator $P =$

make this assumption. Unfortunately, in the present model with infinitely many degrees of freedom, a technical difficulty arises because we also need the initial state ρ to satisfy the condition that ρH^j be trace class for sufficiently large j . This condition is needed not only in order to guarantee that the w_t are well defined for all times $t \in \mathbb{R}$, but it turns out to be essential also in order to give sense to the other ingredients in the Robertson equation in the present infinite dimensional context, see below. Unfortunately, there seems to be a conflict between demanding that $\rho H^j \in \mathcal{S}_1(\mathcal{H})$, and that ρ be quasifree, i.e. equal to (4.34) for some $\mu_{\mathbf{k}}$. The reason for this conflict seems to be the presence of the non-trivial interaction λV in the Hamiltonian. The problem disappears if we only interpret the pre-Boltzmann equation in a perturbative sense (see Section §5.3), but here we wish to have formulae that hold non-perturbatively. So we are forced to introduce e.g. a damping factor into equation (4.34), such as taking⁷

$$\rho = Z^{-1} \exp\left(-\beta H - \sum_{\mathbf{k} \in \mathbb{Z}} \mu_{\mathbf{k}} N_{\mathbf{k}}\right)$$

where β is arbitrarily small but positive [cf. Theorem 3.11]. This has the effect of creating a memory term in the Robertson equation, whose form is given by equation (4.15), with $G_j \rightarrow N_{\mathbf{p}}$. The memory term will clearly be of order β . We will not bother much about the memory term, since this depends on the precise choice of the initial state. Also, when we pass to the perturbative expansion and consider various scaling limits such as the long-time/dilute-medium or long-time/weakly-interacting-system limits in the next chapter, we can take the initial state as quasifree, and in that case the memory term vanishes.

4.3.1 Derivation

With our definition of the projection operators and everything else in place, we can now formally appeal to the result obtained above in Theorem 4.2, see equation (4.15). This suggests:

Theorem 4.7 (“Pre-Boltzmann equation”). *Let $|\mathbf{p}| \leq \Lambda$, where $\Lambda < \infty$, and let the state ρ satisfy the “initial condition” (4.34). Then, if $N_{\mathbf{p}}(t)$ are the number operators for the φ^4 theory, we have the following equation for the expected number densities $n_{\mathbf{p}}(t) =$*

$$\sum_{\mathbf{k}} (\mathbf{k}/L) a_{\mathbf{k}}^* a_{\mathbf{k}}.$$

⁷In order to see that this state satisfies $H^j \rho \in \mathcal{S}_1$, one can e.g. use the Rosen estimates, Theorem 3.12, noting that on the right side we can replace H by $H + \sum_{\mathbf{k}} \mu_{\mathbf{k}} N_{\mathbf{k}}$.

$\text{Tr}(\rho N_{\mathbf{p}}(t))$, and dropping the reference to Λ on w_t :

$$\begin{aligned} \frac{d}{dt} n_{\mathbf{p}}(t) = & \text{“memory term”} - \lambda^2 \int_0^t ds w_s \left([V(s), [V(t), N_{\mathbf{p}}(t)]] \right) + \\ & + \sum_{n=1}^{\infty} (-i\lambda)^{r+2} \sum_{|k_1|, \dots, |k_n| \leq \Lambda} \int_0^t ds \int_{s \leq \sigma_1 \leq \dots \leq \sigma_n \leq t} w_s \left([V(s), [V(\sigma_1), N_{k_1}(\sigma_1)]] \right) \\ & \prod_{j=1}^n \frac{\partial}{\partial n_{k_j}(\sigma_j)} w_{\sigma_j} \left([V(\sigma_{j+1}), N_{k_{j+1}}(\sigma_{j+1})] \right) d^r \sigma \end{aligned} \quad (4.35)$$

where $k_{r+1} = \mathbf{p}$, $\sigma_{r+1} = t$ in the last factor in the product. The infinite sum on the right side converges absolutely. The derivatives $\partial/\partial n_{\mathbf{k}}(\sigma_j)$ are understood as explained below equation (4.33). The “memory term” is given by $\text{Tr}[(\rho - \rho_0) Y_{0,t}(\delta N_{\mathbf{p}}(t))]$, and it can be expanded in a similar absolutely convergent series.

Remark: Before we come to the proof of this theorem, we emphasize that the sum over r in equation (4.35) is *not* a perturbation series in λ , which is known *not* to converge. This is because the order m term contains terms that are themselves functions of λ , for example a term like $w_s([N_{\mathbf{k}}(t), V(t)])$ is a function of λ , the Taylor series for which would not converge.

Proof. We formally take $G_j = N_{k_j}$ in Theorem 4.2, which gives

$$\frac{d}{dt} n_{\mathbf{p}}(t) = \text{“memory term”} - \int_0^t w_s [\delta \circ Y_{s,t} \circ \delta(N_{\mathbf{p}}(t))] ds .$$

Then, if we formally substitute the series expression (4.6) for $Y_{t,s}$ with our choice (4.33) of projectors \mathcal{P}_t^Λ , $t \in \mathbb{R}$, we arrive at the expression given in the theorem. Of course this does not conclude the proof, because equation (4.14) was originally derived only in the context of matrices and it is not a priori clear to what extent it makes sense in the infinite dimensional context considered now. The main question is whether the cocycle $Y_{s,t}$ can be defined in the infinite dimensional setting, and the second question is to what extent the above compositions make sense, i.e. whether the domains match up.

We first establish that the evolution cocycle $Y_{t,s}$ is well defined. We define it by the series expression (4.6), but of course we cannot use the proof given there to show that the series is also convergent in the present setting. Instead, we need to give a new proof, the result of which we state as a

Lemma 4.8. *The series for $Y_{s,t}(A)$ converges for any A with finite “Sobolev”-norm $\|A\|_1 = \|(I + H)^{-1}A(I + H)^{-1}\|$, and for φ^4 theory we in fact have*

$$\|Y_{s,t}(A)\|_1 \leq e^{O(\lambda)|t-s|} \|A\|_1. \quad (4.36)$$

In other words $Y_{s,t} : HB(\mathcal{H})H \rightarrow HB(\mathcal{H})H$ is a bounded operator with exponentially bounded norm on the closure $HB(\mathcal{H})H$ of $B(\mathcal{H})$ under $\|\cdot\|_1$.

The lemma proves that the domains in the composition $Y_{t_1,t_2} \circ Y_{t_2,t_3}$ match up, and the series formula (4.6) for the evolution cocycle then also shows that the cocycle condition holds,

$$Y_{t_1,t_2} \circ Y_{t_2,t_3}(A) = Y_{t_1,t_3}(A) \quad \text{for all } A \in HB(\mathcal{H})H.$$

From this, it is now simple to demonstrate that it satisfies the desired differential equation.

Proof of Lemma 4.8: Let A be a bounded operator. From the series expression for $Y_{t,s}$, we can estimate, dropping the superscript “ Λ ” on the projectors and w_t for notational simplicity:

$$\begin{aligned} \|Y_{s,t}(A)\|_1 &\leq \sum_{k=0}^{\infty} \int_{s \leq \sigma_1 \leq \dots \leq \sigma_k \leq t} \|\delta \circ \mathcal{P}_{\sigma_1} \circ \dots \circ \delta \circ \mathcal{P}_{\sigma_k}(A)\|_1 d^k \sigma \\ &\leq \sum_{k=0}^{\infty} \lambda^k \int_{s \leq \sigma_1 \leq \dots \leq \sigma_k \leq t} d^k \sigma \sum_{|\mathbf{p}_1|, \dots, |\mathbf{p}_k| \leq \Lambda} \left\| [H, N_{j_1}(\sigma_1)] \right\|_1 \frac{|w_{\sigma_k}((N_{\mathbf{p}_k}(\sigma_k) - n_{\mathbf{p}_k}(\sigma_k)I)A)|}{n_{\mathbf{p}_k}(\sigma_k)} \\ &\quad \cdot \prod_{l=1}^{k-1} \frac{|w_{\sigma_l}((N_{\mathbf{p}_l}(\sigma_l) - n_{\mathbf{p}_l}(\sigma_l)I)[V(\sigma_{l+1}), N_{\mathbf{p}_{l+1}}(\sigma_{l+1})])|}{n_{\mathbf{p}_l}(\sigma_l)}. \end{aligned} \quad (4.37)$$

Our aim is to show that each term under the sum on the right side can be bounded by $\|A\|_1 (|t-s|\lambda K\Lambda)^k/k!$, which will imply that the series converges absolutely for any t , and the inequality (4.36).

In order to estimate the integrand, we use another lemma, which is at the heart of our analysis, and which makes crucial use of the estimates $n_{\mathbf{k}}(t) \leq K\omega_{\mathbf{k}}^{-4+\epsilon}$ that were derived above in Theorem 4.4.

Lemma 4.9. *For all $\mathbf{k} \in \mathbf{Z}$ we have*

$$|w_s([N_{\mathbf{k}}(t), V(t)]N_{\mathbf{p}}(s))| \leq K n_{\mathbf{p}}(s) \omega_{\mathbf{p}} \omega_{\mathbf{k}}^{-1} \quad (4.38)$$

as well as

$$|w_s([V(s), [V(t), N_k(t)])])| \leq K \omega_k^{-1} \quad (4.39)$$

uniformly in t and s . Furthermore, if A is a bounded operator, we also have

$$|w_s(N_p(s)A)| \leq K \omega_p^2 n_p(s) \|A\|_1. \quad (4.40)$$

A proof of this lemma is given below. When we now use the estimates (4.38) and (4.40) from the lemma on the terms under the last integral in (4.37), we note that the dangerous factors of $n_p(s)^{-1}$ precisely cancel out with the corresponding factor in the estimate, and the subsequent factors of ω_p and its inverse also cancel. In formulae, we have

$$\begin{aligned} & \|\delta \circ \mathcal{P}_{\sigma_1} \circ \dots \circ \delta \circ \mathcal{P}_{\sigma_k}(A)\|_1 \leq \\ & \leq (K\lambda)^k \|A\|_1 \sum_{|\mathbf{p}_1|, \dots, |\mathbf{p}_k| \leq \Lambda} \|[H, N_{\mathbf{p}_1}(\sigma_1)]\|_1 \frac{\omega_{\mathbf{p}_k}^2}{n_{\mathbf{p}_k}(\sigma_k)} n_{\mathbf{p}_k}(\sigma_k) \prod_{l=2}^k \frac{n_{\mathbf{p}_{l-1}}(\sigma_{l-1}) \omega_{\mathbf{p}_{l-1}} \omega_{\mathbf{p}_l}^{-1}}{n_{\mathbf{p}_{l-1}}(\sigma_{l-1})} \\ & \leq (K\lambda)^k \Lambda^{k+1} \|A\|_1. \end{aligned}$$

In the third line we have used $N_k(t) \leq \omega_k^{-1} H_0(t)$, and again $H_0^2 \leq KH^2$, valid for φ^4 theory [cf. equation (3.33)]

$$\begin{aligned} \|[H, N_p(t)]\|_1 &= \|R[H, N_p(t)]R\| = \|RHN_pR - RN_pHR\| \leq \\ &\leq \|RHN_pR\| + \|RN_pHR\| \leq \|N_pR\| + \|RN_p\| \leq 2\|RN_p(t)^2R\|^{1/2} \leq \\ &\leq 2\omega_p^{-1} \|RH_0(t)^2R\|^{1/2} \leq K\omega_p^{-1} \|RH^2R\|^{1/2} \leq K\omega_p^{-1}, \end{aligned}$$

with $R = (I + H)^{-1}$. If we now take into account that the volume of the integration region is $|t - s|^k k!^{-1}$, we get

$$\begin{aligned} \|Y_{s,t}(A)\|_1 &\leq \sum_{k=0}^{\infty} \int_{t > \sigma_k > \dots > \sigma_1 > s} \|\delta \circ \mathcal{P}_{\sigma_1} \circ \dots \circ \delta \circ \mathcal{P}_{\sigma_k}(A)\|_1 d^k \sigma \\ &\leq \sum_{k=0}^{\infty} \frac{(K|t - s|\lambda)^k \Lambda^{k+1}}{k!} \|A\|_1 = e^{O(\lambda)|t-s|} \|A\|_1, \end{aligned}$$

so the convergence of the series (4.37) follows for all t for any A with $\|RAR\| < \infty$. This proves Lemma 4.8. \square

To complete the proof of the theorem, the only thing left to do is to check that the combination $w_s[\delta \circ Y_{t,s} \circ \delta(N_p(t))]$ is well defined. We can estimate $w_s[\delta \circ Y_{t,s} \circ \delta(N_p(t))]$ writing down again the series expression as above, using the same type of argument as

just given, and using also eq. (4.39). Then we see that the k -th term in the sum is now dominated by $(K\Lambda\lambda|t-s|)^k/k!$, where the $|t-s|^k k!^{-1}$ again comes from the volume of the set $\{t > \sigma_k > \dots > \sigma_1 > s\}$. This shows convergence, and completes the demonstration of the theorem up to Lemma 4.9. \square

Proof of Lemma 4.9: The proof of all estimates is rather similar; we show the first estimate (4.38). We have

$$\begin{aligned} w_s\left([N_{\mathbf{k}}(t), V(t)]N_{\mathbf{p}}(s)\right) &= w_s\left([N_{\mathbf{k}}(t), V(s)]N_{\mathbf{p}}(s)\right) \\ &= w_s(N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s)) - w_s(N_{\mathbf{p}}(s)V(s)N_{\mathbf{k}}(t)) . \end{aligned}$$

The terms on the right side are estimated in exactly the same manner. We demonstrate the argument for one of them. We have, using the Cauchy-Schwarz inequality together with the fact that $w_s(N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s)) = w_s(N_{\mathbf{p}}(s)^{1/2}N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s)^{1/2})$:

$$\begin{aligned} \left|w_s(N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s))\right| &\leq \left(w_s(N_{\mathbf{p}}(s)^{1/2}N_{\mathbf{k}}(t)^2N_{\mathbf{p}}(s)^{1/2})\right)^{1/2} \\ &\quad \cdot \left(w_s(N_{\mathbf{p}}(s)^{1/2}V(s)^2N_{\mathbf{p}}(s)^{1/2})\right)^{1/2} . \end{aligned}$$

Using the inequality $(\Psi, (A+B)^2\Psi) \leq 2(\Psi, (A^2+B^2)\Psi)$ for hermitian A, B , together with equation (3.33), $H_0^2 \leq KH^2$, we can estimate

$$N_{\mathbf{k}}(t)^2 \leq K\omega_{\mathbf{k}}^{-2}H_0^2 \leq K\omega_{\mathbf{k}}^{-2}H^2 \leq 2K\omega_{\mathbf{k}}^{-2}(H_0(s)^2 + \lambda^2V(s)^2) ,$$

so we find

$$\left|w_s(N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s))\right|^2 \leq K\omega_{\mathbf{k}}^{-2} \cdot w_s\left(N_{\mathbf{p}}(s)(H_0(s)^2 + \lambda^2V(s)^2)\right) w_s\left(N_{\mathbf{p}}(s)V(s)^2\right) .$$

We continue our estimation by using Theorems 3.4 and 3.3, which imply that

$$V(s)^2 \leq \|V(s)(I+N(s))^{-p/2}\|^2 (I+N(s))^p \leq K(I+N(s))^p .$$

This then allows us to estimate

$$\begin{aligned} \left|w_s(N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s))\right| &\leq K\omega_{\mathbf{k}}^{-1} \left\{ w_s\left(N_{\mathbf{p}}(s)(I+N(s))^p\right)^2 + \right. \\ &\quad \left. + w_s(H_0^2(s)N_{\mathbf{p}}(s)) w_s\left(N_{\mathbf{p}}(s)(I+N(s))^p\right) \right\}^{\frac{1}{2}} . \end{aligned} \quad (4.41)$$

Now, we recall that the total number operator is $N = \sum_{\mathbf{k} \in \mathbf{Z}} N_{\mathbf{k}}$, and the free Hamiltonian

is $H_0 = \sum_{\mathbf{k} \in \mathcal{Z}} \omega_{\mathbf{k}} N_{\mathbf{k}}$, and similarly for the quantities at time s . Thus, in view of the inequality just given, we have reduced the problem to that of estimating quantities of the form $w_s(N_{\mathbf{k}_1}(s) \cdots N_{\mathbf{k}_r}(s))$. For this, we need a simple combinatorial formula. To derive this formula, let X be any finite subset of \mathbb{N} , and for each $i \in X$, let $\alpha_i \in \mathbb{N}$. Then we have, using elementary Fock-space algebra:

$$\begin{aligned}
w_s \left(\prod_{i \in X} N_{\mathbf{p}_i}(s)^{\alpha_i} \right) &= \prod_{i \in X} \frac{\partial^{\alpha_i}}{\partial \xi_i^{\alpha_i}} w_s \left(\exp \sum_{i \in X} \xi_i N_{\mathbf{p}_i}(s) \right) \Big|_{\xi_i=0} = \\
&= \prod_{i \in X} \frac{\partial^{\alpha_i}}{\partial \xi_i^{\alpha_i}} \frac{1}{Z(t)} \text{Tr} \left[\exp \left(\sum_{i \in X} \xi_i N_{\mathbf{p}_i}(s) - \sum_{\mathbf{k}} \mu_{\mathbf{k}}(s) N_{\mathbf{k}}(s) \right) \right] \Big|_{\xi_i=0} = \\
&= \prod_{i \in X} \frac{\partial^{\alpha_i}}{\partial \xi_i^{\alpha_i}} \frac{1}{Z(t)} \prod_{i \in X} \frac{1}{1 - e^{-\mu_{\mathbf{p}_i} + \xi_i}} \prod_{\mathbf{k} \neq \mathbf{p}_i, i \in X} \frac{1}{1 - e^{-\mu_{\mathbf{k}}}} \Big|_{\xi_i=0} = \\
&= \prod_{i \in X} \frac{\partial^{\alpha_i}}{\partial \xi_i^{\alpha_i}} \frac{1 - e^{-\mu_{\mathbf{p}_i}(s)}}{1 - e^{-\mu_{\mathbf{p}_i}(s) + \xi_i}} \Big|_{\xi_i=0} = \prod_{i \in X} \frac{\partial^{\alpha_i}}{\partial \xi_i^{\alpha_i}} \frac{1}{1 + (1 - e^{\xi_i}) n_{\mathbf{p}_i}(s)} \Big|_{\xi_i=0} = \\
&= \prod_{i \in X} n_{\mathbf{p}_i}(s) \psi_{\alpha_i}(n_{\mathbf{p}_i}(s)) .
\end{aligned}$$

In the above calculation we are using expression (4.29) for the partition function $Z(t)$ and that $n_{\mathbf{p}}(t) = e^{-\mu_{\mathbf{p}}(t)} / (1 - e^{-\mu_{\mathbf{p}}})$, see equation (4.28) and the explanation after it. The $\psi_n(x)$ are the degree $(n - 1)$ polynomials defined iteratively

$$\psi_1(x) = 1, \quad \text{and} \quad \psi_{n+1}(x) = \frac{d}{dx} [x(1+x)\psi_n(x)] .$$

This formula implies (dropping the reference to “ s ” in $n_{\mathbf{p}_i}(s)$ on the right side to lighten the notation):

$$\begin{aligned}
w_s(N_{\mathbf{k}_1}(s) \cdots N_{\mathbf{k}_p}(s)) &= \frac{1}{L^n} \sum_{X_1 \cup \dots \cup X_n = \{1, \dots, p\}} \sum_{\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathcal{Z}} \\
&\prod_{j \in X_1} \delta(\mathbf{p}_1, \mathbf{k}_j) \cdots \prod_{j \in X_n} \delta(\mathbf{p}_n, \mathbf{k}_j) n_{\mathbf{p}_1} \psi_{|X_1|}(n_{\mathbf{p}_1}) \cdots n_{\mathbf{p}_n} \psi_{|X_n|}(n_{\mathbf{p}_n}) .
\end{aligned} \tag{4.42}$$

With the help of this formula, we can now easily estimate quantities like e.g. $w_s(N(s)^p)$. In such an expression we have a p -fold iterated sum over expressions of the form (4.42). The key point is now that, after taking into account the Kronecker delta’s, we are left with iterated sums each of which is accompanied by at least one factor of $n_{\mathbf{k}}(s)$. Because we have the estimate $n_{\mathbf{k}}(s) \leq K\omega_{\mathbf{k}}^{-4+\epsilon}$ from Theorem 4.4, such a sum will converge. If we have e.g. an expression of the form $w_s(H_0(s)^2 N(s)^p)$, we can make a similar argument.

Now, after taking into account the Kronecker delta's, we are left with iterated sums which, at worst contain a factor of $\omega_{\mathbf{k}}^2$ (from the squared free Hamiltonian), and at least one factor of $n_{\mathbf{k}}(s)$. Again, because we have the estimate $n_{\mathbf{k}}(s) \leq K\omega_{\mathbf{k}}^{-4+\epsilon}$ from Theorem 4.4, such a sum will converge. By making simple arguments of this kind, we thus easily arrive at the basic estimates:

$$w_s(N_{\mathbf{p}}(s)H_0(s)^2) \leq Kn_{\mathbf{p}}(s)\omega_{\mathbf{p}}^2, \quad w_s(N_{\mathbf{p}}(s)(I+N(s))^p) \leq Kn_{\mathbf{p}}(s), \quad (4.43)$$

which allows us to continue the estimation of (4.41) as

$$\left|w_s(N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s))\right| \leq K\omega_{\mathbf{k}}^{-1}\left\{n_{\mathbf{p}}^2(s)\omega_{\mathbf{p}}^2 + K'n_{\mathbf{p}}^2(s)\right\}^{\frac{1}{2}} = Kn_{\mathbf{p}}(s)\omega_{\mathbf{k}}^{-1}\left\{\omega_{\mathbf{p}}^2 + K'\right\}^{\frac{1}{2}},$$

from which the estimate $|w_s(N_{\mathbf{k}}(t)V(s)N_{\mathbf{p}}(s))| \leq Kn_{\mathbf{p}}(s)\omega_{\mathbf{p}}\omega_{\mathbf{k}}^{-1}$ follows since for K big enough we have $K\omega_{\mathbf{p}} > K'$ because $\omega_{\mathbf{p}} > 0$. We find that this argument leads to the same estimate for $|w_s(N_{\mathbf{p}}(s)V(s)N_{\mathbf{k}}(t))|$, which concludes the proof of the first inequality (4.38). The second inequality (4.39) is dealt with in a very similar fashion. For the third inequality (4.40), we can argue e.g. by saying that

$$\begin{aligned} w_s(N_{\mathbf{k}}(s)A) &= w_s(N_{\mathbf{k}}(s)^{1/2}AN_{\mathbf{k}}(s)^{1/2}) \\ &\leq \|(I+H)^{-1}A(I+H)^{-1}\| w_s(N_{\mathbf{k}}(s)^{1/2}H^2N_{\mathbf{k}}(s)^{1/2}). \end{aligned}$$

We continue the estimation by

$$\begin{aligned} w_s(N_{\mathbf{k}}(s)^{1/2}H^2N_{\mathbf{k}}(s)^{1/2}) &= w_s(N_{\mathbf{k}}(s)H^2) \leq 2w_s(N_{\mathbf{k}}(s)H_0(s)^2) + 2\lambda^2w_s(N_{\mathbf{k}}(s)V(s)^2) \\ &\leq 2w_s(N_{\mathbf{k}}(s)H_0(s)^2) + 2K\lambda^2w_s(N_{\mathbf{k}}(s)(I+N(s))^p). \end{aligned}$$

The expressions on the right side have already been estimated in equation (4.43), and hence the desired inequality (4.40) follows. \square

4.3.2 Alternative form of the pre-Boltzmann equation

It is convenient for later purposes to write the pre-Boltzmann equation (4.35) in a way which makes more manifest the dependence of the integrands on the number densities $n_{\mathbf{p}}(s)$. For this, it is convenient, to introduce the ‘‘collision kernels’’ $B^\Lambda(E, \mathbf{p}, s)$ by the formula

$$B^\Lambda(E, \mathbf{p}, s) := \frac{\lambda E}{2\pi} \int_{\mathbb{R}} dt e^{-iE(t-s)} w_s^\Lambda([V(s), N_{\mathbf{p}}(t)]). \quad (4.44)$$

From now on we will drop the reference to Λ , for simplicity of notation. The terminology for these kernels will become clear later, where we will relate them to scattering cross sections. These kernels are distributions in E that are defined for any $s \in \mathbb{R}$ and $\mathbf{p} \in \mathbf{Z}$. We claim that the pre-Boltzmann equation can be written entirely in terms of these kernels. We will demonstrate this now for the collision term on the right side of the pre-Boltzmann equation. Similar arguments can also be applied to the memory term. However, this will later be set to zero anyway by an appropriate choice of initial state, so we will not discuss this here.

The statement is clear for the first term on the right side of the pre-Boltzmann equation, since the factor of E in front of $B(E, \mathbf{p}, s)$ can be converted to a t -derivative in the integrand, which in turn yields the resulting first term on the right side of the pre-Boltzmann equation in view of $\frac{\partial}{\partial t} N_{\mathbf{p}}(t) = i[H, N_{\mathbf{p}}(t)] = i\lambda[V(t), N_{\mathbf{p}}(t)]$, using $[H_0(t), N_{\mathbf{p}}(t)] = 0$. To cast the remaining terms in a similar form we note the following chain of equalities

$$\begin{aligned}
\lambda w_{\tau}([V(t), N_{\mathbf{p}}(t)]) &= w_{\tau}([H_0(t) + \lambda V(t), N_{\mathbf{p}}(t)]) = w_{\tau}([H(t), N_{\mathbf{p}}(t)]) = \\
&= w_{\tau}([H(\tau), N_{\mathbf{p}}(t)]) = \lambda w_{\tau}([V(\tau), N_{\mathbf{p}}(t)]) = \\
&= \lambda w_{\tau}([V(\tau), N_{\mathbf{p}}(t)]) - \lambda w_{\tau}([V(\tau), N_{\mathbf{p}}(\tau)]) = \tag{4.45} \\
&= \lambda \int_{\tau}^t d\tau' w_{\tau}([V(\tau), \partial_{\tau'} N_{\mathbf{p}}(\tau')]) = i\lambda^2 \int_{\tau}^t d\tau' w_{\tau}([V(\tau), [V(\tau'), N_{\mathbf{p}}(\tau')]]) .
\end{aligned}$$

This equality puts the terms appearing in the second half of the pre-Boltzmann equation into a form similar to the first, and so we can again express them through the collision kernel $B(E, \mathbf{p}, s)$. Above we use $[H_0(t), N_{\mathbf{p}}(t)] = 0$ in the first and fourth equality, and energy conservation, i.e. time independence of the Hamiltonian, to go from the first line to the second line. Combining equation (4.45), the definition of w_t , and that of the collision factor, equation (4.44) then leads to the following proposition:

Proposition 4.10. *The pre-Boltzmann equation can be expressed in terms of the colli-*

sion kernels $B(E, \mathbf{p}, s)$ as

$$\begin{aligned} \frac{d}{dt} n_{\mathbf{p}}(t) = & \text{“memory term”} - \int_0^t ds \int_{\mathbf{R}} dE e^{iE(t-s)} B(E, \mathbf{p}, s) + \\ & - \sum_{n=1}^{\infty} \int_0^t ds \int_{\mathbf{R}} dE \int_{\Delta_{2n}(s,t)} d^n \tau d^n \sigma \int_{\mathbf{R}^n} d^n E \sum_{|\mathbf{k}_1|, \dots, |\mathbf{k}_n| \leq \Lambda} e^{iE(\tau_1-s)} B(E, \mathbf{k}_1, s) \cdot \\ & \cdot \prod_{j=1}^n e^{iE_j(\sigma_j-\tau_j)} \frac{\partial}{\partial n_{\mathbf{k}_j}(\tau_j)} B(E_j, \mathbf{k}_{j+1}, \tau_j). \end{aligned} \quad (4.46)$$

In this expression, we are denoting by $\Delta_{2n}(s, t) = \{s < \tau_1 < \sigma_1 < \dots < \tau_n < \sigma_n < t\}$, and $\mathbf{k}_{n+1} = \mathbf{p}$ in the expression under the integral. This is an equivalent form of the pre-Boltzmann equation, and hence still valid non-perturbatively. As above, the sum over n is absolutely convergent.

Chapter 5

Scaling limits of the pre-Boltzmann equation

In this chapter we present the second main result of this work, namely we derive the Boltzmann equation (BE). Our starting point is the pre-Boltzmann equation which we derived in the previous chapter, but to obtain the BE we need to consider appropriate “scaling limits” of the former. The physical understanding of the validity of the BE suggests that we ought to consider the “long-time” limit along with either the “low density” limit and/or the “weak coupling” limit. In this chapter we consider them both. As we will see, our results differ from the usual textbook BE in more than one way. The main difference is that our equation contains other terms on the right hand side that can be interpreted as resulting from multiple collisions, or “rescattering”. There are actually infinitely many such terms corresponding to an arbitrary number of rescattering events, and they change the nature of the BE from “Markovian”, i.e. local in time, to “non-Markovian”. However, if we take both limits, then we obtain an equation that is exactly the BE as stated in the introduction.

Strictly speaking, in two spacetime dimensions we do not obtain the BE when considering both limits. The reason for this is that in two dimensions the set of outgoing and incoming momenta is the same, which results in an identically vanishing collision integral on the right hand side. However, we note that the results we have obtained in the previous chapter depend on the spacetime dimensionality “only” insofar as a rigorous construction of the number operators and the bounds on their expected values is concerned. On the other hand, if one is willing to accept the formal results of the derivation of the pre-Boltzmann equation without proper justification, then our results are *independent* of spacetime dimensionality. In other words, the same (formal) derivation could be carried over in *any* number of spacetime dimensions. Hence, in a purely

formal sense, when considering both scaling limits at the same time, we do recover the BE in higher dimensions.

A similar remark holds when discussing model dependence of our derivation. By (unjustifiably) pretending we could define number operators for arbitrary fields in any number of spacetime dimensions it is easy to see that a (formal) derivation of the pre-Boltzmann equation could be given for any quantum field theoretical model. For instance, if we wanted to derive the BE for the Yukawa theory, we should have defined three number operators: the boson number operator $N_b(t, \mathbf{p})$ along with the positively and negatively charged fermion number operators $N_+(t, \mathbf{p})$ and $N_-(t, \mathbf{p})$. We would then have to (formally) derive three equations, one for each of the number operators, thus arriving at three coupled equations. As will become clear in this chapter, spacetime dimensionality and model specific considerations do not play a significant role in this chapter either. The bottom line is that on a purely formal level, our derivation is a very general one and could easily be adapted to derive a BE for any quantum field theoretical model.

Unlike the results of the previous chapter, our main results in this chapter—the derivation of a Boltzmann-like equation in the weak coupling/low density limit—is based on perturbation theory. In this sense, and also at the level of mathematical rigour, our analysis is less complete than that leading to Theorems 4.7 and 4.10. Despite a lower level of rigour, some steps of our analysis in this chapter are largely independent of perturbation theory, and we believe that with a more careful analysis, probably all the steps could be justified non-perturbatively in some way.

Our plan for this chapter is as follows:

1. We rewrite the collision kernel $B(E, \mathbf{p}, s)$ using “local (in time) S -matrices”. This step does not involve perturbation theory, although the local S -matrices could, and later will be, expanded in a perturbation series. These arguments are given in Section §5.1.1.
2. Further, we rewrite the collision kernel in terms of the set $\{n_{\mathbf{k}}\}$, and “local scattering amplitudes”, see equation (5.21). Strictly speaking, neither in this step is perturbation theory required. We *will*, however, at this stage give the perturbative expansion of the local scattering amplitudes, [cf. equation (5.13)] as we will need it later on. This will be explained in Sections §5.1.2 and §5.1.3.
3. Before proceeding with the scaling of the pre-Boltzmann equation we present an alternative formulation of the collision kernel, namely using the formalism of “retarded products”, see Section §5.2. We do not use this formulation in this work

but nevertheless present it here as it is more suited for a derivation of the BE on curved spaces, to which this work could be adapted.

4. We will then consider the two scaling limits together with the thermodynamic limit ($L \rightarrow \infty$). Here we use some elements of perturbation theory and this will be explained in Section §5.3.
5. Finally, in Section §5.4 we present some issues that come up if we want to go beyond the weak coupling limit and consider “corrections” (in the coupling constant) to the Boltzmann-like equation we obtain in this case. The results in this section are far from complete and also highly formal. We present them because of their interest for physics, which mainly comes from the fact that the scattering amplitudes now ought to be computed with a modified propagator.

Before proceeding, however, we introduce some notation that we will be using throughout in what follows. Specifically, we will frequently encounter various sums of the following type:

$$\sum_{i \in X} \mathbf{p}_i, \quad \sum_{i \in X} \omega_{\mathbf{p}_i}, \quad \sum_{i \in X} \underline{p}_i x_i,$$

where the summation is extended to the elements of a set $X \subset \mathbb{N}$. And such sums will often appear as argument of an exponential or of a Kronecker (and later Dirac) delta. To simplify the understanding and avoid a cluttered notation, we introduce the following shorthand notation

$$\begin{aligned} X_{\mathbf{p}} &:= \{\underline{p}_i\}_{i \in X}, & p_X &:= \sum_{i \in X} \underline{p}_i, & \mathbf{p}_X &:= \sum_{i \in X} \mathbf{p}_i, \\ \omega_X &:= \sum_{i \in X} \omega_{\mathbf{p}_i}, & p_X x &:= \sum_{i \in X} \underline{p}_i x_i. \end{aligned} \tag{5.1}$$

We also recall the notation $\underline{p} = (\omega_{\mathbf{p}}, \mathbf{p})$ to denote on-shell momenta.

5.1 Collision kernel evaluation

Our aim in this section is to derive an alternative expression for the collision kernels $B(E, \mathbf{p}, s)$ in terms of (local) “Möller operators”, or local S -matrices. We define the local Möller operators as

$$\mathcal{S}(s, t) := e^{i(t-s)H_0(s)} e^{-i(t-s)H},$$

which are unitary operators on \mathcal{H} . Here, $H_0(s)$ is the free Hamiltonian defined at time s by $H_0(s) = e^{isH} H_0 e^{-isH}$. It can be written alternatively in terms of the time- s creation and annihilation operators $a_{\mathbf{k}}^{\#}(s)$, see equation (4.20). The true S -matrix of the theory (which exists for this model, see [61]) is given in terms of these local Möller operators by

$$\mathcal{S} = s\text{-}\lim_{t \rightarrow \infty} \mathcal{S}(0, t) \mathcal{S}(-t, 0) = s\text{-}\lim_{t \rightarrow \infty} \mathcal{S}(-t, t)$$

on a suitable domain of vectors in \mathcal{H} , in the thermodynamic limit $L \rightarrow \infty$. We also note the following properties

$$\frac{\partial}{\partial t} \mathcal{S}(s, t) \psi = -i\lambda e^{i(t-s)H_0(s)} V(s) e^{-i(t-s)H_0(s)} \mathcal{S}(s, t) \psi, \quad \mathcal{S}(s, s) = \mathbf{1}, \quad (5.2)$$

for suitable vectors $\psi \in \mathcal{H}$. The terminology ‘‘local Möller operators’’ arises from the fact that they are equal to the S -matrix of a theory wherein the interaction is switched on in the time interval $[s, t]$ only, i.e. they are ‘‘local in time’’.

5.1.1 Step 1

Using now the above properties and the fact that $e^{i(t-s)H} N_{\mathbf{p}}(s) e^{-i(t-s)H} = N_{\mathbf{p}}(t)$ we easily see that

$$\begin{aligned} \lambda w_s([V(s), N_{\mathbf{p}}(t)]) &= -i\partial_t w_s(N_{\mathbf{p}}(t)) = -i\partial_t w_s(e^{i(t-s)H} N_{\mathbf{p}}(s) e^{-i(t-s)H}) = \\ &= -i\partial_t w_s(\mathcal{S}^*(s, t) N_{\mathbf{p}}(s) \mathcal{S}(s, t)) = -i\partial_t w_s(\mathcal{S}^*(s, t) [N_{\mathbf{p}}(s), \mathcal{S}(s, t)]) . \end{aligned}$$

With this result we rewrite the collision kernels $B(E, \mathbf{p}, s)$ as

$$B(E, \mathbf{p}, s) := \frac{E^2}{2\pi} \int_{\mathbb{R}} dt e^{-iE(t-s)} w_s(\mathcal{S}^*(s, t) [N_{\mathbf{p}}(s), \mathcal{S}(s, t)]) . \quad (5.3)$$

We can summarize the above results in the following

Proposition 5.1. *The expected number densities $n_{\mathbf{p}}(t)$ satisfy the integro-differential equation (4.46), where the collision kernels $B(E, \mathbf{p}, s)$ are given by equation (5.3).*

Before we proceed with our derivation of the BE, we pause to note some properties of the local S -matrices just introduced. Equation (5.2) implies the following differential equation:

$$\frac{\partial}{\partial t} \mathcal{S}(s, t) \psi = -i\lambda V_0(t) \mathcal{S}(s, t) \psi, \quad \mathcal{S}(s, s) = \mathbf{1}, \quad (5.4)$$

where we have introduced the “interaction picture” potential

$$V_0(t) := e^{i(t-s)H_0(s)} V(s) e^{-i(t-s)H_0(s)} . \quad (5.5)$$

The latter can be rewritten in terms of “time- s free fields”, denoted $\varphi_0(t, \mathbf{x})$, as follows. They are defined by demanding that, at time s we have $\varphi_0(s, \mathbf{x}) = \varphi(s, \mathbf{x})$, and by defining

$$\varphi_0(t, \mathbf{x}) := e^{i(t-s)H_0(s)} \varphi(s, \mathbf{x}) e^{-i(t-s)H_0(s)} , \quad (5.6)$$

for an arbitrary time t . Such a free field will satisfy the Klein–Gordon equation and evolves according to the time- s free Hamiltonian $H_0(s)$. Clearly, this free field cannot therefore be equal to the interacting field $\varphi(t, \mathbf{x})$ apart from the time s , because the latter evolves according to the full Hamiltonian H . (We note that the free field will hence also depend on the initial time s when it coincides with the interacting field and one should, strictly speaking, somehow incorporate this information into the notation $\varphi_0(t, \mathbf{x})$, but for notational simplicity we do not do that and implicitly understand this dependence.) Our free field has a simple expression in terms of creation and annihilation operators at time s , and is given by

$$\varphi_0(t, \mathbf{x}) = \sum_{\mathbf{p} \in \mathbf{Z}} \left[\overline{U_{\mathbf{p}}(t, \mathbf{x})} a_{\mathbf{p}}(s) + U_{\mathbf{p}}(t, \mathbf{x}) a_{\mathbf{p}}^*(s) \right] , \quad (5.7)$$

for the mode functions $U_{\mathbf{p}}(t, \mathbf{x}) := u_{\mathbf{p}}(t) e^{i\mathbf{p}\mathbf{x}}$, and we recall that the functions $u_{\mathbf{p}}(t, \mathbf{x})$ have been defined in equation (4.21). The interaction picture potential is given in terms of the free field by

$$V_0(t) = L \sum_{n=2}^p b_n \int_0^{2\pi} dx : \varphi_0^n(t, \mathbf{x}) : . \quad (5.8)$$

as one can easily see by running through the definitions. Here, the normal ordering prescription indicated by double dots is defined by expanding φ_0 out into creation and annihilation operators at time s , see equation (4.20), and then moving the annihilators to the right:

$$: \varphi_0^n(t, \mathbf{x}) : = \sum_{\mathbf{p}_1, \dots, \mathbf{p}_n \in \mathbf{Z}} \prod_{i \in X} U_{\mathbf{p}_i}(t, \mathbf{x}) a_{\mathbf{p}_i}^*(s) \prod_{j \in Y} \overline{U_{\mathbf{p}_j}(t, \mathbf{x})} a_{\mathbf{p}_j}(s) , \quad (5.9)$$

where $X, Y \subset \mathbf{Z}$ and $|X \cup Y| = n$. Note that this prescription depends on the time s chosen. If we combine this with the differential equation (5.4) given for the local S -matrix, then we can express it in terms of a formal power series in the free field φ_0 . This

is obtained by integrating the differential equation (5.4) in the sense of formal power series in λ , and the solution is

$$\mathcal{S}(s, t) = \sum_{n=0}^{\infty} (-i\lambda)^n \int_{s \leq \tau_1 \leq \dots \leq \tau_n \leq t} d\tau_1 \dots d\tau_n V_0(\tau_n) \dots V_0(\tau_1) . \quad (5.10)$$

From the above expression it is clear that $\mathcal{S}(s, t)$ agrees with the “local S -matrix” of Bogoliubov [62] and Epstein and Glaser [63]. For this reason we will henceforth equivalently refer to it as either the local S -matrix or local Möller operator. From the definition of the local S -matrix we also note the following property

$$\mathcal{S}(s + a, t + a) = e^{iaH_0(s)} \mathcal{S}(s, t) e^{-iaH_0(s)} . \quad (5.11)$$

We can also give a formal series expression for the interacting field φ in terms of $\varphi_0(t, \mathbf{x})$. This series is usually given in terms of an asymptotic in-field (i.e. formally taking $s \rightarrow -\infty$), and is then called “Haag series” [64, 65]. But a similar expansion is also valid for finite s , see e.g. [66, 67, 68]; it is given below in equation (5.22).

5.1.2 Step 2

To further proceed in the derivation we will express the local Möller operators in terms of time- s creation and annihilation operators. We claim that the following formula holds

$$\begin{aligned} \mathcal{S}(s, t) = Z \mathbf{1} + Z 2\pi L \sum_{X, Y} \sum_{\substack{\mathbf{p}_1, \dots, \mathbf{p}_{|X|} \in \mathbf{Z} \\ \mathbf{q}_1, \dots, \mathbf{q}_{|Y|} \in \mathbf{Z}}} \mathcal{M}_{(s, t)}(X_p \rightarrow Y_q) \delta(\mathbf{p}_X, \mathbf{q}_Y) \\ \prod_{j \in Y} \frac{a_{\mathbf{q}_j}^*(s)}{(2L\omega_{\mathbf{q}_j})^{1/2}} \prod_{i \in X} \frac{a_{\mathbf{p}_i}(s)}{(2L\omega_{\mathbf{p}_i})^{1/2}} , \end{aligned} \quad (5.12)$$

where $X, Y \subset \mathbf{N}$ with $|X \cup Y| \geq 1$, the overall factor Z accounts for the correct normalization, and we refer the reader to the remarks preceding (5.1) for an explanation of the above notation. This formula can also be obtained in perturbation theory by using equation (5.9) in the expression for the interaction picture potential (5.8) and substituting the result in the formal power series expression of the local S -matrix equation (5.10). This will give us at the same time a perturbative formula for the quantities \mathcal{M} . We want to emphasize, however, that in order to get the above equation, we actually *do not* have to make a perturbative expansion of the local S -matrix since it is valid *non-perturbatively*. And the reason for this is that on Fock space we can expand every operator as a sum of (products of) creation and annihilation operators. We have

collected some details about this latter point in Appendix C, and refer the interested reader to it as well as to Section 6 of [69] where an expansion like ours was studied. The “momentum conservation” delta in the above expression can be understood by noting that the local Möller operators commute with the momentum operator on \mathcal{H} .

For completeness, we now give the perturbative expression for the local scattering amplitudes \mathcal{M} here in terms of (standard) position space Feynman diagrams. One can expand the time ordered products equation (5.10) into normal ordered products of the free field $\varphi_0(t, \mathbf{x})$ using an appropriate version of Wick’s theorem (see e.g. the “local Wick expansion” of [66] and [70]). One then obtains a perturbative expression for the matrix elements $\mathcal{M}_{(s,t)}(X_p \rightarrow Y_q)$ in terms of position space Feynman integrals with a “time cutoff” restricting the integration range of the time variables to the interval $[s, t]$. In the present super-renormalizable model these integrals are absolutely convergent, without the need of any sort of renormalization process beyond the normal ordering procedure which has already been carried out.¹ For a Feynman graph G , with interaction vertices as given by the polynomial in equation (3.25), let $V(G)$ be the set of vertices, $L(G)$ the set of internal lines and $E(G)$ the set of vertices connected to external lines. For each subset $X, Y \subset \mathbf{Z}$ of momenta, and $j \in E(G)$, let $X(j) \subset X$ be the ingoing momenta from X connected to the vertex j , each associated with an external line, and similarly we denote with $Y(j) \subset Y$ the outgoing momenta connected to j . The perturbative expansion of the matrix elements of the local Möller operators is then given by

$$\begin{aligned} \mathcal{M}_{(s,t)}(X_p \rightarrow Y_q) = & \sum_{n=0}^{\infty} (-i\lambda)^n \sum_{G: |V(G)|=n, s \leq \tau_1 \leq \dots \leq \tau_n \leq t} c_G L^n \int d^2x_1 \dots d^2x_n \cdot \\ & \cdot \prod_{i,j \in L(G)} \Delta_F(x_i, x_j) \prod_{j \in E(G)} \exp(ip_{X(j)}x - iq_{Y(j)}x) . \end{aligned} \quad (5.13)$$

In the above expression the τ_i are the time components of the spacetime points x_i and the space integrals are understood to be over the entire domain, i.e. $\mathbf{x}_i \in [0, 2\pi]$. The factor $c_G = |\text{Aut}(G)|^{-1} \prod_{i \in V(G)} b_{n(i)}$ is a symmetry factor associated with each diagram as is explained in standard textbooks on quantum field theory, see e.g. [31, 71, 32] (with $n(i)$ the valence of the i -th vertex). The Δ_F here are the standard Feynman propagators, i.e. the time ordered vacuum expectation values of the free fields $\varphi_0(t, \mathbf{x})$ [cf. equation (5.7) with textbook mode functions $v_k(x)$] on the cylinder spacetime $\mathbb{R} \times \mathbb{S}$, and given

¹We note, however, that later we will be considering the “full scattering amplitudes”, which one could formally obtain by considering the limit $\lim_{t \rightarrow \infty} \mathcal{M}_{(-t,t)}$. Because of this we will need to deal with mass renormalization of the scattering amplitudes, and this issue is closely related to the appropriate choice of the free dynamics $H_0(s)$, which will be different from the one we made previously.

by

$$\begin{aligned}\Delta_F(t, \mathbf{x}) &= \langle T\varphi_0(t, \mathbf{x})\varphi_0(0, 0) \rangle_0 = \frac{-i}{2\pi} \sum_{\mathbf{p} \in \mathbf{Z}} \int dE \frac{e^{-itE + i\mathbf{p}\mathbf{x}}}{-E^2 + \omega_{\mathbf{p}} - i0} = \\ &= \frac{1}{4\pi} \sum_{n \in \mathbf{Z}} K_0(\sqrt{\sigma_n + i0}), \quad \sigma_n = m^2[-t^2 + (\mathbf{x} + 2\pi n)^2].\end{aligned}\tag{5.14}$$

We note that $\langle \cdot \rangle_0 := \langle \Omega_s | \cdot | \Omega_s \rangle$, with $\Omega_s = e^{isH} \Omega_0$, is the “time- s vacuum” state which is annihilated by $a_{\mathbf{p}}(s)$ and *not* the physical vacuum. To go from the first to the second line we have used the Poisson summation formula, and K_0 is a Bessel function. The above computation basically says that the Feynman propagator on the circle arises as a “sum over images” of the Minkowski spacetime propagator. The relevant point for us here is that $K_0(\sqrt{z}) \sim \log z$, so the singularities of the Feynman propagator on the cylinder will only be present for null-related pairs of points, and are locally L^p functions for any $p < \infty$, thus implying the absolute convergence of the integral in (5.13). We conclude this step by observing that this argument is no longer valid in higher dimensions.

5.1.3 Step 3

Our goal is now to substitute our expression for the local S -matrix (5.12) into our formula for the collision kernel (5.3), and to evaluate the result. We now note that in equation (5.12) all the creation operators are on the left of the annihilation operators, that is, the expression is normal ordered. However, this normal ordering prescription is the “standard” one in that the vacuum expectation value of a normal ordered quantity vanishes identically. The collision kernels $B(E, \mathbf{p}, s)$, on the other hand, involve expectation values in the state w_s . For this reason, we find it convenient to Wick re-order the expansion of the local S -matrix (5.12). The “re-ordered expansion” of the local S -matrix is given by

$$\begin{aligned}S(s, t) &= Z \mathbf{1} + Z 2\pi L \sum_{X, Y} \sum_{\substack{\mathbf{p}_1, \dots, \mathbf{p}_{|X|} \in \mathbf{Z} \\ \mathbf{q}_1, \dots, \mathbf{q}_{|Y|} \in \mathbf{Z}}} \widetilde{\mathcal{M}}_{(s, t)}(X_p \rightarrow Y_q) \delta(\mathbf{p}_X, \mathbf{q}_Y) \\ &\quad : \prod_{j \in Y} \frac{a_{\mathbf{q}_j}^*(s)}{(2L\omega_{\mathbf{q}_j})^{1/2}} \prod_{i \in X} \frac{a_{\mathbf{p}_i}(s)}{(2L\omega_{\mathbf{p}_i})^{1/2}} :_s.\end{aligned}\tag{5.15}$$

The quantities $\widetilde{\mathcal{M}}_{(s, t)}$ are *defined* by the above equation and the double dots $: \cdot :_s$ indicate that now the creation and annihilation operators are normal ordered with respect to the state w_s and not to the “time- s vacuum” $\langle \cdot \rangle_0$ anymore, see remark immediately

after equation (5.14). For clarity, we here denote by $: \ :_0$ this latter normal ordering prescription. For two fields it is given by

$$:\varphi_0(x)\varphi_0(y):_0 := \varphi_0(x)\varphi_0(y) - \langle \varphi_0(x)\varphi_0(y) \rangle_0 ,$$

and we refer the reader to Appendix A for the corresponding formula for n fields. It is easy to see that this prescription is in fact equivalent to moving the time- s creation operators to the left. The Wick reordered normal products $: \ :_s$ are defined similarly as

$$:\varphi_0(x)\varphi_0(y):_s := \varphi_0(x)\varphi_0(y) - w_s[\varphi_0(x)\varphi_0(y)] ,$$

and we again refer to Appendix A for the generalization to n fields. In general, for Wick powers the relationship between the time- s vacuum normal ordered products $: \ :_0$ and the re-ordered normal products $: \ :_s$ is given by

$$:\varphi_0^n(x):_0 = \sum_{j=0}^{[n/2]} \frac{n!}{(n-2j)!j!2^j} [\delta m^2]^j : \varphi_0^{n-2j}(x) :_s , \quad (5.16)$$

where $\delta m^2 = \lim_{y \rightarrow x} \{w_s[\varphi_0(x)\varphi_0(y)] - \langle \varphi_0(x)\varphi_0(y) \rangle_0\}$, (cf. equation (5.43)) and $[n/2]$ denotes the integer part of $n/2$. The advantage of working with Wick-reordered products is clear: Their expectation value in the state w_s vanishes identically.

As for the \mathcal{M} , the quantities $\widetilde{\mathcal{M}}$ have a perturbative expansion. To obtain this expansion, we may Wick-reorder the interaction potential (5.8); we obtain the following expression

$$V_0(t) = L \sum_{n=1}^p b'_n \int_0^{2\pi} dx : \varphi_0^n(t, \mathbf{x}) :_s ,$$

where we have dropped a term proportional to $\mathbf{1}$ because it is irrelevant for what follows. We note, however, that in the above expression we now also have terms linear and quadratic in the fields $\varphi_0(t, \mathbf{x})$, which were not present previously. These terms, just like the term we have dropped, arise because of the re-ordering, and the expansion parameters b'_n can be related to the b_n 's by using the Wick re-ordering formula (5.16). Later on when we study the mass renormalization of the model, we will for simplicity consider only the quartic interaction and give an explicit expression for the reordered interaction potential.

The effect of the re-ordering amounts to a modification of the Feynman propagators and hence to a different expansion of the (dressed) local scattering amplitudes from that

given above in equation (5.13). In particular, the new expansion will be in terms of the “dressed” Feynman propagators

$$\tilde{\Delta}_F(t, \mathbf{x}) := w_s [T\varphi_0(t, \mathbf{x})\varphi_0(0, 0)] = \Delta_F(t, \mathbf{x}) + w_s [:\varphi_0(t, \mathbf{x})\varphi_0(0, 0):_0] . \quad (5.17)$$

A formula similar to (5.13) holds for the perturbative expansion of $\tilde{\mathcal{M}}$; we have

$$\begin{aligned} \tilde{\mathcal{M}}_{(s,t)}(X_p \rightarrow Y_q) &= \sum_{n=0}^{\infty} (-i\lambda)^n \sum_{G:|V(G)|=n} c_G \\ &\cdot L^n \int_{s \leq \tau_1 \leq \dots \leq \tau_n \leq t} d^2x_1 \dots d^2x_n \prod_{i,j \in L(G)} \tilde{\Delta}_F(x_i, x_j) \prod_{j \in E(G)} \exp(ip_{X(j)}x - iq_{Y(j)}x) . \end{aligned} \quad (5.18)$$

The formula can be again proved by simply expanding out the time ordered products in the formal expansion of the local S -matrix [cf. equation (5.10)], but now in terms of the Wick-reordered (interaction picture) potential, i.e. in terms of reordered products of creation and annihilation operators. In order to see that the integrals for the dressed propagators are still absolutely convergent, it is sufficient to show that the additional term in equation (5.17) is a sufficiently regular function. In fact, it is of class $C^3(\mathbb{R} \times \mathbb{S} \times \mathbb{R} \times \mathbb{S})$. This easily follows from the fact that $n_{\mathbf{p}}(s) \leq K\omega_{\mathbf{p}}^{-4+\epsilon}$ for any $\epsilon > 0$, by Theorem 4.4. As an aside, we can now combine equations (5.15) and (5.11) to get

$$\tilde{\mathcal{M}}_{(s+a, t+a)}(p_X \rightarrow q_Y) = \tilde{\mathcal{M}}_{(s,t)}(p_X \rightarrow q_Y) \exp\{ia(\omega_X - \omega_Y)\} . \quad (5.19)$$

Before proceeding to the evaluation of $B(E, \mathbf{p}, s)$ we need one more result, and we state it here as a

Proposition 5.2. *Let $X, Y, X', Y' \subset \mathbb{N}$, with $|X \cup Y| \geq 1$ and $|X' \cup Y'| \geq 1$. If $|X| = |X'|$ and $|Y| = |Y'|$, we have*

$$\begin{aligned} &w_s \left(: \prod_{i' \in X'} a_{\mathbf{p}_{i'}}^*(s) \prod_{j' \in Y'} a_{\mathbf{q}_{j'}}(s) :_s : \prod_{j \in Y} a_{\mathbf{q}_j}^*(s) \prod_{i \in X} a_{\mathbf{p}_i}(s) :_s \right) = \\ &= \left(\prod_{i \in X} n_{\mathbf{p}_i}(s) L\delta(\mathbf{p}_i, \mathbf{p}'_{i'}) + (i'_i \leftrightarrow i'_j) \right) \left(\prod_{j \in Y} (1 + n_{\mathbf{q}_j}(s)) L\delta(\mathbf{q}_j, \mathbf{q}'_{j'}) + (j'_i \leftrightarrow j'_j) \right) , \end{aligned} \quad (5.20)$$

and zero otherwise.

Proof. The proof relies on Wick’s theorem for the factorization of the above expression in products of expectation values of “two point functions”, i.e. expressions containing a creation and an annihilation operator only. A crucial observation here is that because the

products of creation and annihilation operators above are normal ordered with respect to the state w_s , the only non-vanishing contributions will arise when “contracting” one creation (annihilation) operator from the first normal ordered product with an annihilation (creation) operator from the second one. Let us prove, then, that when $|X| \neq |X'|$ or $|Y| \neq |Y'|$ the above expectation value vanishes.

Suppose, for the sake of the argument, that $|X| = |X'|$ but $|Y| > |Y'|$. Applying Wick’s theorem we will then be able to “fully contract” all the annihilation operators corresponding to the sets X, X' and Y' but *not* Y . The above expectation value will hence be proportional to

$$w_s \left(\prod_{j=1}^{|Y|-|Y'|} a_{\mathbf{q}_j}^*(s) \right) = 0.$$

Similarly, if $|Y'| > |Y|$, the expectation value (5.20) will then be proportional to a (vanishing) term like the above one where the creation operators are replaced by annihilation operators. It is also clear that this argument equally applies to the case when $|X| \neq |X'|$ but $|Y| = |Y'|$. We are hence left with the case $|X| \neq |X'|$ and $|Y| \neq |Y'|$.

For definiteness, let us assume first that $|X| > |X'|$ and $|Y| > |Y'|$. Applying Wick’s theorem we will be able to fully contract all the creation and annihilation operators corresponding to the sets X' and Y' but not X and Y , the result being thus proportional to

$$w_s \left(: \prod_{j=1}^{|Y|-|Y'|} a_{\mathbf{q}_j}^*(s) \prod_{i=1}^{|X|-|X'|} a_{\mathbf{p}_i}(s) :_s \right) = 0 ,$$

which vanishes because of the normal ordering prescription. If, on the other hand, $|X| > |X'|$ and $|Y| < |Y'|$ then the result will be proportional to

$$w_s \left(\prod_{j=1}^{|Y'-|Y|} a_{\mathbf{q}_j}(s) \prod_{i=1}^{|X|-|X'|} a_{\mathbf{p}_i}(s) \right) = 0 .$$

An identical argument applies to the case $|X| < |X'|$ and $|Y| \neq |Y'|$, thus showing that if $|X| \neq |X'|$ or $|Y| \neq |Y'|$ the expectation value in (5.20) is indeed zero. Finally, when $|X| = |X'|$ and $|Y| = |Y'|$, a direct application of Wick’s theorem leads to the above result, and where the single contractions have been evaluated as $w(a_{\mathbf{p}}^* a_{\mathbf{k}}) = n_{\mathbf{p}} \delta(\mathbf{p}, \mathbf{k})$ and $w(a_{\mathbf{p}} a_{\mathbf{k}}^*) = (1 + n_{\mathbf{p}}) \delta(\mathbf{p}, \mathbf{k})$. \square

Computation of $B(E, \mathbf{p}, s)$: After this preparatory work has been done we are ready to express $B(E, \mathbf{p}, s)$ in terms of the dressed local scattering amplitudes. We have

$$\begin{aligned}
B(E, \mathbf{p}, s) &= \frac{E^2}{2\pi} \int dt e^{-iE(t-s)} w_s \left(\mathcal{S}(s, t)^* [N_{\mathbf{p}}(s), \mathcal{S}(s, t)] \right) = \\
&= \frac{E^2}{2\pi} \int dt e^{-iE(t-s)} \sum_{\substack{X, Y \\ X', Y'}} \sum_{\substack{\mathbf{p}_1, \dots, \mathbf{p}_{|X|} \in \mathbb{Z} \\ \mathbf{q}_1, \dots, \mathbf{q}_{|Y|} \in \mathbb{Z}}} \sum_{\substack{\mathbf{p}'_1, \dots, \mathbf{p}'_{|X'}| \in \mathbb{Z} \\ \mathbf{q}'_1, \dots, \mathbf{q}'_{|Y'}| \in \mathbb{Z}}} L \delta(\mathbf{p}_X, \mathbf{q}_Y) L \delta(\mathbf{p}'_{X'}, \mathbf{q}'_{Y'}) \cdot \\
&\quad \cdot \widetilde{\mathcal{M}}_{(s,t)}^*(X'_{p'} \rightarrow Y'_{q'}) \widetilde{\mathcal{M}}_{(s,t)}(X_p \rightarrow Y_q) \cdot \\
&\quad \cdot w_s \left(: \prod_{i' \in X'} \frac{a_{\mathbf{p}'_{i'}}^*(s)}{(2L\omega_{\mathbf{p}'_{i'}})^{1/2}} \prod_{j' \in Y'} \frac{a_{\mathbf{q}'_{j'}}(s)}{(2L\omega_{\mathbf{q}'_{j'}})^{1/2}} :_s \left[N_{\mathbf{p}}(s), : \prod_{j \in Y} \frac{a_{\mathbf{q}_j}^*(s)}{(2L\omega_{\mathbf{q}_j})^{1/2}} \prod_{i \in X} \frac{a_{\mathbf{p}_i}(s)}{(2L\omega_{\mathbf{p}_i})^{1/2}} :_s \right] \right) = \\
&= \frac{E^2}{2\pi} \int dt e^{-iE(t-s)} \sum_{\substack{X, Y \\ X', Y'}} \sum_{\substack{\mathbf{p}_1, \dots, \mathbf{p}_{|X|} \in \mathbb{Z} \\ \mathbf{q}_1, \dots, \mathbf{q}_{|Y|} \in \mathbb{Z}}} \sum_{\substack{\mathbf{p}'_1, \dots, \mathbf{p}'_{|X'}| \in \mathbb{Z} \\ \mathbf{q}'_1, \dots, \mathbf{q}'_{|Y'}| \in \mathbb{Z}}} L \delta(\mathbf{p}_X, \mathbf{q}_Y) \delta(\mathbf{p}'_{X'}, \mathbf{q}'_{Y'}) \cdot \\
&\quad \cdot \widetilde{\mathcal{M}}_{(s,t)}^*(X'_{p'} \rightarrow Y'_{q'}) \widetilde{\mathcal{M}}_{(s,t)}(X_p \rightarrow Y_q) L \left[\sum_{j \in Y} \delta(\mathbf{p}, \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p}, \mathbf{p}_i) \right] \\
&\quad \cdot w_s \left(: \prod_{i' \in X'} \frac{a_{\mathbf{p}'_{i'}}^*(s)}{(2L\omega_{\mathbf{p}'_{i'}})^{1/2}} \prod_{j' \in Y'} \frac{a_{\mathbf{q}'_{j'}}(s)}{(2L\omega_{\mathbf{q}'_{j'}})^{1/2}} :_s : \prod_{j \in Y} \frac{a_{\mathbf{q}_j}^*(s)}{(2L\omega_{\mathbf{q}_j})^{1/2}} \prod_{i \in X} \frac{a_{\mathbf{p}_i}(s)}{(2L\omega_{\mathbf{p}_i})^{1/2}} :_s \right) = \\
&= \frac{E^2}{2\pi} \int dt e^{-iE(t-s)} \sum_{\substack{X, Y \\ X', Y'}} \sum_{\substack{\mathbf{p}_1, \dots, \mathbf{p}_{|X|} \in \mathbb{Z} \\ \mathbf{q}_1, \dots, \mathbf{q}_{|Y|} \in \mathbb{Z}}} \sum_{\substack{\mathbf{p}'_1, \dots, \mathbf{p}'_{|X'}| \in \mathbb{Z} \\ \mathbf{q}'_1, \dots, \mathbf{q}'_{|Y'}| \in \mathbb{Z}}} L \delta(\mathbf{p}_X, \mathbf{q}_Y) \delta(\mathbf{p}'_{X'}, \mathbf{q}'_{Y'}) \cdot \\
&\quad \cdot \widetilde{\mathcal{M}}_{(s,t)}^*(X'_{p'} \rightarrow Y'_{q'}) \widetilde{\mathcal{M}}_{(s,t)}(X_p \rightarrow Y_q) L \left[\sum_{j \in Y} \delta(\mathbf{p}, \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p}, \mathbf{p}_i) \right] \\
&\quad \cdot w_s \left(: \prod_{i' \in X'} \frac{a_{\mathbf{p}'_{i'}}^*(s)}{(2L\omega_{\mathbf{p}'_{i'}})^{1/2}} \prod_{j' \in Y'} \frac{a_{\mathbf{q}'_{j'}}(s)}{(2L\omega_{\mathbf{q}'_{j'}})^{1/2}} :_s : \prod_{j \in Y} \frac{a_{\mathbf{q}_j}^*(s)}{(2L\omega_{\mathbf{q}_j})^{1/2}} \prod_{i \in X} \frac{a_{\mathbf{p}_i}(s)}{(2L\omega_{\mathbf{p}_i})^{1/2}} :_s \right) \cdot
\end{aligned}$$

In the above chain of equalities the first one amounts to using the expansion (5.15) for both \mathcal{S} and \mathcal{S}^* while neglecting the terms proportional to the unity as their contributions vanish. In fact, there are two such contributions arising from the expansions of \mathcal{S} and \mathcal{S}^* . The one associated to \mathcal{S} can be dropped as the unity commutes with $N_{\mathbf{p}}$. The contributions arising from the unity in the expansion of \mathcal{S}^* , on the other hand, vanishes because it will contribute to terms of the form $w_s[: \prod a^* \prod a :_s]$, which are zero because

of the normal ordering. The second equality follows by evaluating the commutator and in the last equality we have used Proposition 5.2 without fully evaluating the expectation value, but merely restricting the sums to the sets $X' = X$ and $Y' = Y$. The concluding step of this computation consists in using Proposition 5.2 again and we obtain

$$\begin{aligned}
 B(E, \mathbf{p}, s) &= \frac{E^2}{2\pi} \int dt e^{-iE(t-s)} \sum_{\substack{i \in X \\ j \in Y}} \frac{|X|! |Y|!}{L^{|X|+|Y|}} \sum_{\substack{\mathbf{p}_i \in \mathbf{Z} \\ \mathbf{q}_j \in \mathbf{Z}}} L \left[\sum_{j \in Y} \delta(\mathbf{p}, \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p}, \mathbf{p}_i) \right] \\
 &\cdot \left| \widetilde{\mathcal{M}}_{(s,t)}(X_p \rightarrow Y_q) \right|^2 L \delta(\mathbf{p}_X, \mathbf{q}_Y) \prod_{i \in X} n_{\mathbf{p}_i}(s) \prod_{j \in Y} \left[1 + n_{\mathbf{q}_j}(s) \right], \tag{5.21}
 \end{aligned}$$

after performing the sum over the \mathbf{p}'_i and \mathbf{q}'_j , and noting that for Kronecker deltas in a finite volume we have $\delta(\mathbf{k}, \mathbf{p})\delta(\mathbf{k}, \mathbf{p}) = \delta(\mathbf{k}, \mathbf{p})$. Finally, the factors $|X|!$ and $|Y|!$ arise from the permutations in equation (5.20). This is the expression we were looking for, and we want to emphasize that it is an expression which is valid *non-perturbatively*. As we have stated above, the expansion of the S -matrix (5.15) in terms of creation and annihilation operators is valid non-perturbatively, and this is all we needed to arrive to the above result. That such an expansion can be understood also perturbatively or the fact that we have presented a perturbative expansion of the local scattering amplitudes (5.18) should not conceal the non-perturbative validity of equation (5.21). The next step will be to consider the infinite volume (thermodynamic) limit $L \rightarrow \infty$ and the scaling (weak coupling and/or low density) limit. Before that, however, we give an alternative perturbative expansion for $B(E, \mathbf{p}, s)$ in terms of retarded products.

5.2 The collision kernel in terms of retarded products

We now give an equivalent alternative way to expand $B(E, p, s)$ in a formal power series in λ . This derivation relies on an alternative expansion of the interacting field φ in terms of the free field φ_0 . The relevant formula is well known in the literature (“Haag’s series”), see e.g. [64, 66, 72]. It is, in our notation

$$\varphi(t, x) = \varphi_0(t, x) + \sum_{n=1}^{\infty} \frac{(i\lambda)^n}{n!} \int_{[s,t]^n} \mathcal{R}_n(\varphi_0(t, x); V_0(\tau_1) \otimes \cdots \otimes V_0(\tau_n)) d^n \tau, \tag{5.22}$$

which is valid for $t \geq s$. Here, the notation \mathcal{R}_n means a “retarded product”. Mathematically, it is convenient to take the view that it is a bilinear map $\mathcal{R}_n : \mathcal{F} \times (\otimes^n \mathcal{F}) \rightarrow$

$\text{End}(\mathcal{H})$, where \mathcal{F} is the linear space of *classical* local expressions of a fictitious classical field φ_0 of the form $A = \int W[\varphi_0(x), \partial\varphi_0(x), \dots, \partial^r\varphi_0(x)]f(x) d^2x$, with W a multivariate polynomial, and $f \in C^\infty(S^1 \times \mathbb{R})$. (With a slight abuse of notation, we denote with φ_0 both the classical field which appears as argument of a retarded product and the free *quantum* field (5.7). However, from the current discussion it is clear that there is no risk of confusion.) In particular, in \mathcal{F} , no field equations are assumed. The \mathcal{R}_n take their values in a suitable space of quadratic forms in \mathcal{H} , for details see [66, 67, 70, 73]. The retarded products are distributional in nature, i.e. for $B, A_1, \dots, A_n \in \mathcal{F}$ of the above form, $\mathcal{R}_n(B; \otimes_i A_i)$ is a distribution in the test functions h, f_1, \dots, f_n implicit in B, A_1, \dots, A_n . For example, inside the retarded product, $V_0(t) = \sum b_n \int \varphi_0(t, x)^n dx$ means the *classical* expression for the potential (hence no “normal ordering”), and it is not understood that the classical field φ_0 is to satisfy a field equation when standing inside \mathcal{R}_n . For a single factor and $W(x) = W[\varphi_0(x), \partial\varphi_0(x), \dots, \partial^r\varphi_0(x)]$, we have

$$\mathcal{R}_0(W(x)) = :W(x):_0 .$$

Thus, the formula for the interacting field $\varphi(x)$ has $\varphi_0(x)$ as its lowest order term, as required. If B, A_i are smeared polynomials U, W_i in $\varphi_0(x)$ but *not* its partial derivatives (viewed again as “classical expressions”), then there is a similarly simple expression also for the corresponding retarded product $\mathcal{R}_n(B; \otimes_i A_i)$ with n factors; it is given by a sum of multiple commutators multiplied by step functions, see e.g. [67]. However, if the arguments of the retarded products contain derivatives, then this simple formula becomes ill-defined, essentially because one then has to perform renormalization². In this case, the retarded products may be thought of as defined by a combinatorial formula in terms of time-ordered and anti time-ordered products T and \bar{T} respectively, (see e.g. [66, 67]), which is

$$\mathcal{R}_n\left(B; \bigotimes_{i=1}^n A_i\right) = \sum_{X \cup Y = \{1, \dots, n\}} \bar{T}_{|X|+1}\left(B \otimes \bigotimes_{j \in X} A_j\right) T_{|Y|}\left(\bigotimes_{i \in Y} A_i\right).$$

This then leaves one with the product of defining the ordinary time ordered products, see e.g. [63, 66, 67, 70, 73]. It follows from these constructions that the time ordered/retarded products have an expression in terms of $a_k^\#(s)$, the creation/annihilation operators at time s .

The retarded products owe their name to their support properties: If B, A_i are

²When carrying out this renormalization, it turns out to be of considerable advantage to consider the arguments of the retarded products to be classical expressions, and this is why we proceed here in this way.

polynomials in φ_0 and its partial derivatives, viewed again as “classical expressions”, then we have (see [74])

$$\text{supp } \mathcal{R}_n(B; \otimes_{i=1}^n A_i) \subset \left\{ (x_1, \dots, x_n, x) \mid x_i \in \text{supp } A_i \cap J^-(\text{supp } B), \right. \\ \left. x \in \text{supp } B, \text{ for all } i = 1, \dots, n \right\},$$

where where $J^\pm(S)$ denotes the causal future/past of a set $S \subset \mathbb{R} \times S^1$, and where we define the support of an expression $A = \int W[\varphi_0(x), \partial\varphi_0(x), \dots, \partial^r\varphi_0(x)]f(x) d^2x$ to be equal to the support of the testfunction f .

We can now start with our task of expanding expressions on the right side of the pre-Boltzmann equation in perturbation theory. This is accomplished essentially by inserting a perturbative formula for the (interacting model) number operator $N_p(t)$ into eq. (4.44). In order to do this in an efficient way, we proceed as follows. First, we note that, by the Glaser-Lehmann-Zimmermann (GLZ)-formula (see [67, 65]), we have³:

$$\begin{aligned} [V(s), N_p(t)] &= \\ &= \sum_{n,m=0}^{\infty} \frac{(-i\lambda)^{n+m}}{(n+m)!} \int_{[s,t]^{n+m}} \left[\mathcal{R}_n(V_0(s); \bigotimes_{j=1}^n V_0(\sigma_j)), \mathcal{R}_m(N_p(t); \bigotimes_{l=1}^m V_0(\sigma_l)) \right] d^{n+m}\sigma = \\ &= \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \int_{[s,t]^n} \left\{ \mathcal{R}_n(V_0(s); N_p(t) \otimes \bigotimes_{j=1}^n V_0(\sigma_j)) + \right. \\ &\quad \left. - \mathcal{R}_n(N_p(t); V_0(s) \otimes \bigotimes_{j=1}^n V_0(\sigma_j)) \right\} d^n\sigma. \end{aligned}$$

Then, multiplying both sides with a step function and using the support properties of the retarded products, we get:

$$\theta(t-s)[V(s), N_p(t)] = \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \int_{\mathbb{R}^n} \mathcal{R}_n \left(N_p(t); V_0(s) \otimes \bigotimes_{j=1}^n V_0(\sigma_j) \right) d^n\sigma.$$

We now use this expression in our expression (4.44) for the collision kernel $B(E, \mathbf{p}, s)$. In this formula, we may multiply the integrand by a step function $\theta(t-s)$, because the opposite step function $\theta(s-t)$ would give a contribution to $B(E, \mathbf{p}, s)$ that is analytic for $\text{Im}E > 0$, and which would for this reason vanish when substituted back into the

³Note that $N_p(t)$ when expressed in terms of the free field φ_0 is not a local expression in \mathcal{F} . However, it is still local in time, and this is sufficient in order for the retarded (or time ordered) product to make sense in two spacetime dimensions.



pre-Boltzmann equation. The (unproved) assumption here is clearly that when we deform the integration contour in the upper half plane, the contribution arising through this deformation vanishes in the limit where the integration (semi-circular) domain is extended to an infinite radius. Therefore, assuming that these contributions are irrelevant, our collision term becomes

$$B(E, \mathbf{p}, s) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(i\lambda)^n}{n!} \int_{\mathbb{R}} dt e^{-iE(t-s)} \cdot \int_{[s,t]^n} w_s \left[\mathcal{R}_n \left(N_{\mathbf{p}}(t); V_0(s) \otimes \bigotimes_{j=1}^n V_0(\sigma_j) \right) \right] d^n \sigma .$$

This is our second expression for the collision factor. The expectation values of the retarded products in the states w_s can be evaluated in terms of Feynman integrals with “propagators” (5.17) using a version of Wick’s theorem [68, 66, 67], because the retarded products are expressible in terms of $a_{\mathbf{p}}^{\#}(s)$, and because the states w_s are quasifree, cf. (4.31). The propagators can be evaluated in terms of the factors $n_{\mathbf{p}}(s)$, but we will not show this here. In higher dimensions $d > 0$ (or for any model that is not super-renormalizable), the fully renormalized retarded product must be understood in the above expression.

In summary, our pre-Boltzmann equation (4.46) together with the perturbative expression for $B(E, \mathbf{p}, s)$ gives us a closed set of integro-differential equations for the unknown quantities $n_{\mathbf{p}}(t)$. These equations are not particularly simple, but we will see in the next section that they form a good starting point for a further expansions, namely a simultaneous expansion essentially of the inverse observation time $1/t$, and the coupling constant λ (or typical initial density $n_{\mathbf{p}}(0)$).

5.3 Weakly interacting and dilute systems

We would now like to take the long-time limit and the thermodynamic limit ($L \rightarrow \infty$) of the number densities $\{n_{\mathbf{p}}\}$. In order to have any reasonable hope that such a limit might exist, it is clearly necessary on physical grounds to take simultaneously (at least) one of the two further limits:

- (a) The low density limit (i.e. the initial densities $n_{\mathbf{p}}(0) \rightarrow 0$) and/or
- (b) The weak coupling limit (i.e. $\lambda \rightarrow 0$).

Both limits have been studied previously. For instance, the weak coupling limit has been studied in [6] and in [9] in the context of a lattice Fermi gas with $2 \rightarrow 2$ interactions

within the framework of (non relativistic) quantum mechanics. The low density limit, on the other hand, has been analyzed in, e.g. [12] for a quantum particle interacting with a random potential. For weakly interacting particles the common feature is the emergence of the scattering amplitude in the Born approximation whereas for dilute gases a full scattering amplitude emerges.

The idea in both cases is to introduce a new small parameter, ϵ , into the problem to “control” the limits. The physical meaning of this parameter is not that of an additional coupling constant, but instead it characterizes the initial density matrix state $\rho(\epsilon)$ of the system (or the strength of the interactions for weakly interacting particles), as well as the time over which we observe its evolution. Roughly speaking, the time duration over which we wish to observe the system is of order ϵ^{-1} , whereas the initial densities (coupling constants) are of order ϵ^α for suitable $\alpha > 0$. The idea is then to consider an expansion of the observable quantities in the new small parameter ϵ . There are two related issues associated with this limit:

1. Does the limit of $\{n_{\mathbf{p}}(t)\}$ exist?
2. What equation does it satisfy?

In what follows we will study what limiting equation do the (suitably rescaled) quantities $\{n_{\mathbf{p}}(t)\}$ satisfy in either (or both) the low density as well as the weak coupling limit. In doing so, we will *assume* that the answer to the question 1. above is *yes*, and we will arrive at *different* answers for dilute and weakly interacting gases. Both answers have some features in common and the BE as stated in the Introduction will arise only when both limits are considered simultaneously. These results will be obtained using the pre-Boltzmann equation and the perturbative formulae for the collision factor $B(e, \mathbf{p}, s)$ derived in sections §5.1.2 and §5.1.2. Our derivation is in part formal because we do not control the perturbation series and we will interchange several limits and integrals without proper mathematical justification, but we believe a more rigorous derivation could be given.

5.3.1 The thermodynamic limit

Before proceeding to explicitly consider the scaling details of the weak coupling and low density limits we reconsider the pre-Boltzmann equation by taking the thermodynamic limit $L \rightarrow \infty$. As already mentioned, we are going to discuss this limit only in a heuristic fashion as we believe that a rigorous treatment would be considerably more involved while leading to the same conclusion.

Since the quantities $n_{\mathbf{p}}(t)$ were defined as densities [cf. (4.24)], we expect that they will possess a well defined thermodynamic limit. Physically this means that the particle densities is finite. Assuming this to be the case, the pre-Boltzmann equation is expected to continue to hold in said limit, at the very least in the sense of formal power series in λ . To obtain the collision factors, one must then only make the following (standard) replacements

$$\begin{aligned}
n_{\mathbf{p}}(t), \quad \mathbf{p} \in \mathbf{Z} &\xrightarrow{L \rightarrow \infty} n_{\mathbf{p}}(t), \quad \mathbf{p} \in \mathbb{R} \\
L\delta/\delta n_{\mathbf{p}}, \quad \mathbf{p} \in \mathbf{Z} &\xrightarrow{L \rightarrow \infty} \delta/\delta n_{\mathbf{p}}, \quad \mathbf{p} \in \mathbb{R} \\
L \int_0^{2\pi} dx (\cdot), \quad \mathbf{x} \in [0, 2\pi] &\xrightarrow{L \rightarrow \infty} \int_{\mathbb{R}} dx (\cdot), \quad \mathbf{x} \in \mathbb{R} \\
\frac{1}{L} \sum_{\mathbf{p} \in \mathbf{Z}} f\left(\frac{\mathbf{p}}{L}\right), \quad \mathbf{p} \in \mathbf{Z} &\xrightarrow{L \rightarrow \infty} \int_{\mathbb{R}} d\mathbf{p} f(\mathbf{p}), \quad \mathbf{p} \in \mathbb{R} \\
\langle p, x \rangle = -p_0 t + \mathbf{p} \mathbf{x}, \quad \mathbf{p} \in \mathbf{Z} &\xrightarrow{L \rightarrow \infty} px = -p_0 t + \mathbf{p} \mathbf{x}, \quad \mathbf{p} \in \mathbb{R} \\
L\delta(\mathbf{p}, \mathbf{k}), \quad \mathbf{p}, \mathbf{k} \in \mathbf{Z} &\xrightarrow{L \rightarrow \infty} \delta(\mathbf{p} - \mathbf{k}), \quad \mathbf{p}, \mathbf{k} \in \mathbb{R} \\
\omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2/L + m^2}, \quad \mathbf{p} \in \mathbf{Z} &\xrightarrow{L \rightarrow \infty} \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}, \quad \mathbf{p} \in \mathbb{R}
\end{aligned}$$

In the second line $\delta/\delta n_{\mathbf{p}}$ is the functional derivative with respect to the particle number densities $n_{\mathbf{p}}(t)$ in the infinite volume case [cf. Section §3.1, equation (3.1)], and the other modifications are self-explanatory and intuitive. After the thermodynamic limit, our QFT is defined on $\mathbb{R}^{(1,1)}$ rather than $\mathbb{R} \times \mathbb{S}$. The pre-Boltzmann equation now reads

$$\begin{aligned}
\dot{n}_{\mathbf{p}}(t) &= \int_0^t ds \int_{\mathbb{R}} dE e^{iE(T-s)} B(E, \mathbf{p}, s) + \\
&+ \sum_{n=1}^{\infty} (-1)^n \int_0^T ds \int_{s \leq \tau_1 \leq \dots \leq \tau_n \leq T} d\tau_1 \dots d\tau_n \int_{\mathbb{R}} d\mathbf{k}_1 \dots d\mathbf{k}_n \int_{\mathbb{R}} dE e^{iE(\tau_1-s)} B(E, \mathbf{k}_1, s) \\
&\quad \prod_{i=1}^n \frac{\delta}{\delta n_{\mathbf{k}_i}(\tau_i)} \int_{\tau_i}^{\tau_{i+1}} d\tau'_i \int_{\mathbb{R}} dE_i e^{iE_i(\tau'_i - \tau_i)} B(E_i, \mathbf{k}_{i+1}, \tau_i),
\end{aligned}$$

and we still expect this equation to be an exact equation.

As a preparation for later, we scale the time t and introduce a rescaled “time” T by writing

$$t = T/\epsilon,$$

which in addition to a change of variables $s \rightarrow s/\epsilon$ and $E \rightarrow \epsilon E$ in the integral on the first line below and analogous changes for the remaining terms, we then use to rewrite the above equation as

$$\begin{aligned} \partial_T n_{\mathbf{p}}(T/\epsilon) &= \frac{1}{\epsilon} \int_0^T ds \int_{\mathbf{R}} dE e^{iE(T-s)} B\left(\epsilon E, \mathbf{p}, \frac{s}{\epsilon}\right) + \\ &+ \sum_{n=1}^{\infty} \frac{(-1)^n}{\epsilon^{n+1}} \int_0^T ds \int_{s \leq \tau_1 \leq \dots \leq \tau_n \leq T} d\tau_1 \dots d\tau_n \int_{\mathbf{R}} d\mathbf{k}_1 \dots d\mathbf{k}_n \int_{\mathbf{R}} dE e^{iE(\tau_1-s)} B\left(\epsilon E, \mathbf{k}_1, \frac{s}{\epsilon}\right) \\ &\quad \prod_{i=1}^n \frac{\delta}{\delta n_{\mathbf{k}_i}(\frac{\tau_i}{\epsilon})} \int_{\tau_i}^{\tau_{i+1}} d\tau'_i \int_{\mathbf{R}} dE_i e^{iE_i(\tau'_i-\tau_i)} B\left(\epsilon E_i, \mathbf{k}_{i+1}, \frac{\tau_i}{\epsilon}\right). \end{aligned} \quad (5.23)$$

We note that so far we have only performed a trivial change of variables and so the above formula (presumably) still holds exactly. In this form, it is also formally clear that one needs to consider some additional scaling if the result is to be finite in the $\epsilon \rightarrow 0$ limit. Finally, we will also consider the $\Lambda \rightarrow \infty$ limit in order to have a closed set of equations for all the momenta.

In order to write the rescaled collision factor we use the property (5.19) to shift the squared local scattering amplitude

$$\begin{aligned} |\widetilde{\mathcal{M}}_{(s,t)}(X_p \rightarrow Y_q)|^2 &= \widetilde{\mathcal{M}}_{(0,t-s)}^*(X_p \rightarrow Y_q) e^{is(\omega_X - \omega_Y)} \widetilde{\mathcal{M}}_{(0,t-s)}(X_p \rightarrow Y_q) e^{-is(\omega_X - \omega_Y)} = \\ &= |\widetilde{\mathcal{M}}_{(0,t-s)}(X_p \rightarrow Y_q)|^2, \end{aligned}$$

and rewrite the rescaled collision kernel (5.21) in the thermodynamic limit as

$$\begin{aligned} B\left(\epsilon E, \mathbf{p}, \frac{s}{\epsilon}\right) &= \sum_{\substack{i \in X \\ j \in Y}} |X|! |Y|! \int_{\mathbf{R}} d\Pi_{X_p \cup Y_q} \int_{\mathbf{R}} dt \epsilon \frac{E^2}{2\pi} e^{-iEt} \left| \widetilde{\mathcal{M}}_{(0,t/\epsilon)}(X_p \rightarrow Y_q) \right|^2 \cdot \\ &\cdot \delta(\mathbf{p}_X - \mathbf{q}_Y) \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \prod_{i \in X} n_{\mathbf{p}_i}(s/\epsilon) \prod_{j \in Y} \left(1 + n_{\mathbf{q}_j}(s/\epsilon)\right), \end{aligned} \quad (5.24)$$

and where we have performed another change of variables $t - s \rightarrow t$ and introduced

$$d\Pi_{X_p \cup Y_q} := \prod_{\substack{i \in X \\ j \in Y}} d\Pi_{\mathbf{p}_i} d\Pi_{\mathbf{q}_j}, \quad d\Pi_{\mathbf{k}} = \frac{d\mathbf{k}}{2\omega(\mathbf{k})} = \frac{d\mathbf{k}}{2(\mathbf{k}^2 + m^2)^{1/2}},$$

the Lorentz invariant measure on the mass hyperboloids \mathbb{H}_m

$$\mathbb{H}_m := \{(\omega, \mathbf{p}) \in \mathbb{R}^2 \mid \omega = \pm \sqrt{\mathbf{p}^2 + m^2}\} .$$

5.3.2 The Boltzmann equation for dilute gases

To introduce the long time–dilute medium limit we have to additionally rescale the particle number densities $n_{\mathbf{p}}(s)$ and in addition introduce the physical time T . We write

$$t \rightarrow T/\epsilon , \quad n_{\mathbf{p}}(t) = \epsilon^\alpha \nu_{\mathbf{p}}(T, \epsilon) , \quad \nu_{\mathbf{p}}(T) := \lim_{\epsilon \rightarrow 0} \nu_{\mathbf{p}}(T, \epsilon) , \quad (5.25)$$

for a suitable α . The idea of the above replacements is to take $\epsilon \rightarrow 0^+$ while keeping T fixed (so that $t \rightarrow \infty$) and at the same time keeping the initial density $\nu_{\mathbf{p}}(0, \epsilon)$ —and hence the initial state (4.13)—fixed (so that $n_{\mathbf{p}}(0) \rightarrow 0$). We claim that the limiting quantities, if they exist, satisfy an equation which is similar to the Boltzmann equation, and which reduces to the Boltzmann equation if in addition we assume that the collision time is long (e.g. when λ becomes small). Note also that the last equation/definition above is clearly an assumption in the sense that the full mathematical demonstration of this claim would require us to control the limit $\epsilon \rightarrow 0$ of $\nu_{\mathbf{p}}(T, \epsilon)$, and for this we would have to look at the full non perturbative dynamics of the model. In principle this ought to be possible using methods similar to those described previously (cf. Section §4.2.1), but we do not believe that such an analysis would necessarily offer more insight into the nature of the limit than a formal derivation. We will give such a formal derivation here by starting simply with the *assumption* that the limits in (5.25) exist in a suitably strong sense. We then use the pre-Boltzmann equation to see what equation does this limit satisfy. Above we also have an as of yet unspecified parameter α , which we will keep arbitrary and will eventually argue that if the $\epsilon \rightarrow 0$ limit of the pre-Boltzmann equation is to be finite, then we will be forced to consider $\alpha = 1$. By additionally assuming that we can exchange a limit with an integral, we will arrive at equation (5.28).

We would now like to take the limit as $\epsilon \rightarrow 0^+$ of the rescaled collision factor

$$\epsilon^{-\alpha-1} B\left(\epsilon E, \mathbf{p}, \frac{s}{\epsilon}\right) ,$$

where the $\epsilon^{-\alpha-1}$ factor comes from the LHS of the pre-Boltzmann equation after the rescalings in (5.25) have been used. Later we will argue that, if we choose $\alpha = 1$, then the limit exist, and is in fact independent of E .

To state what the limit is, we define the Boltzmann collision factor in the low density

limit $\mathcal{B}_\nu(\mathbf{p}, T)$ to be the expression

$$\begin{aligned} \mathcal{B}_\nu(\mathbf{p}, T) = & -2 \sum_{j \in Y} |Y|! \int_{\mathbf{R}} d\Pi(\mathbf{p}_1, \mathbf{q}_1, \dots, \mathbf{q}_{|Y|}) (2\pi)^2 \delta^{(2)}\left(\underline{p} + \underline{p}_1 - \sum_{j \in Y} \underline{q}_j\right) \\ & \left| \mathcal{M}(\underline{p}, \underline{p}_1 \rightarrow \underline{q}_1, \dots, \underline{q}_{|Y|}) \right|^2 \nu_{\mathbf{p}}(T) \nu_{\mathbf{p}_1}(T) + \\ & + \sum_{j \in Y} |Y|! |Y| \int_{\mathbf{R}} d\Pi(\mathbf{p}_1, \mathbf{p}_2, \mathbf{q}_2, \dots, \mathbf{q}_{|Y|}) (2\pi)^2 \delta^{(2)}\left(\underline{p}_1 + \underline{p}_2 - \underline{p} - \sum_{j=2}^{|Y|} \underline{q}_j\right) \\ & \left| \mathcal{M}(\underline{p}_1, \underline{p}_2 \rightarrow \underline{p}, \underline{q}_2, \dots, \underline{q}_{|Y|}) \right|^2 \nu_{\mathbf{p}_1}(T) \nu_{\mathbf{p}_2}(T), \end{aligned} \quad (5.26)$$

where $\mathcal{M}(X_p \rightarrow Y_q)$ now denotes the *full, not dressed* scattering matrix element of the theory (as opposed to the local dressed matrix elements in the pre-Boltzmann equation), with the energy-momentum conservation delta's taken out. Then we claim that we have, in the sense of distributional boundary values:

$$\lim_{\epsilon \rightarrow 0^+} \text{B. V.}_{\text{Im}E < 0, \text{Im}E \rightarrow 0} \epsilon^{-\alpha-1} B\left(\epsilon E, \mathbf{p}, \frac{s}{\epsilon}\right) = \frac{B(\mathbf{p}, s)}{\omega(\mathbf{p})}, \quad (5.27)$$

plus a contribution that is the boundary value of an analytic function for $\text{Im}E > 0$, but that will not contribute to the expression eq. (5.23). Indeed, for that contribution, we can deform the contour of the dE -integration in eq. (5.23) to the trivial contour within the half plane $\text{Im}E > 0$, as the exponent $e^{iE(T-s)}$ provides a damping there (note that $s < T$), and the same applies to the other dE_i -integrals in eq. (5.23). Substituting the limit (5.27) into the limit of pre-Boltzmann equation (5.23) then delivers the final result

$$\begin{aligned} \omega(\mathbf{p}) \partial_T \nu_{\mathbf{p}}(T) = & \mathcal{B}_\nu(\mathbf{p}, T) + \sum_{n=1}^{\infty} (-1)^n \int_0^T ds \int_{s \leq \tau_1 \leq \dots \leq \tau_n \leq T} d\tau_1 \dots d\tau_n \\ & \int_{\mathbf{R}^n} d\Pi(\mathbf{k}_1, \dots, \mathbf{k}_n) \mathcal{B}_\nu(\mathbf{k}_1, s) \frac{\delta}{\delta \nu_{\mathbf{k}_1}(\tau_1)} \mathcal{B}_\nu(\mathbf{k}_2, \tau_1) \dots \frac{\delta}{\delta \nu_{\mathbf{k}_n}(\tau_n)} \mathcal{B}_\nu(\mathbf{k}_{n+1}, \tau_n), \end{aligned} \quad (5.28)$$

because each dE integration in the first integral in eq. (5.23) now yields a delta function⁴, the effect of which is that the subsequent ds -integration can be performed trivially. The same remark applies to the other iterated integrals on the right side, where we denote $\mathbf{k}_{n+1} = \mathbf{p}$. This equation is the main result in the case of a dilute medium, and the derivation of (5.27) is given below.

⁴Note that when we substitute equation (5.27), we may take the ds -integration from 0 to ∞ when we take the limit, because the integral from T to ∞ does not make a contribution as the dE integration contour can then be deformed to the trivial contour within the domain $\text{Im}E < 0$.

The above derivation, which is to some extent formal and perturbative, also justifies the remarks we made in the beginning of this chapter, namely that at no point we use the fact that we are in two spacetime dimensions. Because of this, the above equation could be understood to be (formally) valid in any number of dimensions. Equation (5.28) then shows how the rescaled number densities $\nu_{\mathbf{p}}$ evolve with time in the long-time, dilute state limit.

As emphasized previously, there are several differences between our result and the standard BE, and we discuss the physical interpretation of equation (5.28) together with the collision factor (5.26) in Section §5.3.4. Here, however, let us note that the differences between our equation (5.28) and the classical Boltzmann equation (2.6) disappear if we additionally consider the case that λ , the coupling constant, is small. In that case, all processes with more than 2 outgoing particles are suppressed as well because they are intrinsically of higher order in the coupling constant, and all the higher ($n \geq 1$) “rescattering terms” in (5.28) are suppressed by powers of λ . Thus, the leading contribution will arise from the $2 \rightarrow 2$ scattering processes and a single collision factor. The corresponding leading approximation of equation (5.28) for small λ is (in d dimensions):

$$\omega(\mathbf{p})\partial_T \nu_{\mathbf{p}}(T) = \frac{1}{2} \int_{\mathbb{R}^{3d}} d\Pi_{\mathbf{p}_2} d\Pi_{\mathbf{q}_1} d\Pi_{\mathbf{q}_2} (2\pi)^d \delta^{(d)}(\underline{p} + \underline{p}_1 - \underline{q}_1 - \underline{q}_2) \cdot \quad (5.29)$$

$$\cdot \left| \mathcal{M}(\underline{p}, \underline{p}_1 \leftrightarrow \underline{q}_1, \underline{q}_2) \right|^2 \left(\nu_{\mathbf{q}_1}(T) \nu_{\mathbf{q}_2}(T) - \nu_{\mathbf{p}}(T) \nu_{\mathbf{p}_1}(T) \right),$$

where the matrix element is now denoting the *Born approximation*. This is indeed the relativistic⁵ version of the familiar Boltzmann equation. Finally, another previously mentioned feature of the above equation is now clear, namely that in two dimensions the right hand side of equation (5.29) vanishes. This is because the momentum conservation delta enforces $\{\mathbf{p}, \mathbf{p}_1\} = \{\mathbf{q}_1, \mathbf{q}_2\}$.

In order to get a somewhat better qualitative understanding when the “rescattering terms” can be neglected, let us introduce the L^1 -norm of a function $f_{\mathbf{p}}$ on the mass hyperboloid \mathbb{H}_m as $\|f\|_{L^1} = \int |f_{\mathbf{p}}| d\Pi_{\mathbf{p}}$. Then the L^1 -norm of the n -th rescattering term

⁵Note the relativistic kinematical factors implicit in $d\Pi$, as well as in the expression $\omega(\mathbf{p})\partial_T$, which is equal to $p^\mu \partial_\mu$ for the homogeneous state that we consider, because the $\nu_{\mathbf{p}}$ ’s are independent of the spatial coordinate.

in eq. (5.28) as a function of p is immediately estimated by

$$\begin{aligned} \sup_{\tau} \|\mathcal{B}_\nu(\tau)\|_{L^1} & \int_{0 < \tau_1 < \dots < \tau_n < T} d^n \tau \|\mathcal{B}_\nu(\tau_1)\|_{L^1} \|\delta \mathcal{B}_\nu(\tau_1)/\delta \nu\|_{L^1 \rightarrow L^1} \cdots \|\delta \mathcal{B}_\nu(\tau_n)/\delta \nu\|_{L^1 \rightarrow L^1} \\ & \leq \frac{1}{n!} \left(T \sup_{0 < s < T} \sup_{k \in \mathbb{H}_m} \int_{\mathbb{H}_m} \left| \frac{\delta \mathcal{B}_\nu(s)}{\delta \nu_k} \right| d\Pi_p \right)^n \equiv \frac{(T/T_0)^n}{n!}. \end{aligned}$$

Here, T_0 is defined by the last equation, and $\delta \mathcal{B}(s)/\delta \nu$ is the operator from $L^1 \rightarrow L^1$ that is defined by the kernel $\delta \mathcal{B}_\nu(s)/\delta \nu_k$. It is not difficult to see (compare the discussion on p. 93 of [7]) that T_0 is interpreted as a time of the order of the maximum collision time (i.e. the average time between two collisions) for the particles of arbitrary momentum \mathbf{k} in the medium, between time zero and time T . The estimate hence tells us that we are allowed to drop the rescattering terms if $T \ll T_0$. Now, the physical time over which the system is observed has actually been rescaled as $t = T/\epsilon$, by equation (5.25), and we have in fact even taken the limit as $\epsilon \rightarrow 0$. Therefore, in terms of the physical time t , the condition that $T \ll T_0$ would mean, for finite but very small ϵ , that $\epsilon t \ll T_0$, which would appear to be reasonable.

As an aside, we also note that the estimate tells us that if we could actually mathematically prove that T_0 was non-zero, then the series in eq. (5.28) would converge. We strongly believe this to be the case, but have not attempted to prove this. Note however, that in the case of the pre-Boltzmann equation, convergence of the corresponding series was proved, and this ought to provide a good indication here, too.

Formal argument for the validity of equation (5.27)

First of all, we observe that in the low density limit, the propagator (5.17) will no longer depend on the particle number densities and hence reduce to the standard Feynman propagator (5.14). Therefore, by equations (5.17) and (5.18), we can also replace the dressed local scattering amplitude $\widetilde{\mathcal{M}}_{(s,t)}$ in the pre-Boltzmann equation by $\mathcal{M}_{(s,t)}$. Furthermore, from the perturbative expression of \mathcal{M} given in (5.13) it is clear that, for finite ϵ , $\mathcal{M}_{(0,t/\epsilon)}$ viewed as a function of $X_p = \{p_1, \dots, p_n\} \in \mathbb{R}^{2n}$ and $Y_q = \{q_1, \dots, q_m\} \in \mathbb{R}^{2m}$ respectively, is an analytic functions of the variables p_i^0 and q_j^0 . When $\epsilon \rightarrow 0^+$, the limit, if it exists, will still be analytic for $\text{Im } p_i^0 < 0$ and $\text{Im } q_j^0 > 0$, because in the integral over the time coordinates in equation (5.13) we can safely continue the frequency arguments in the exponentials to the indicated domain. In other words, we expect that there exists a function $\mathcal{F}_{X_p \rightarrow Y_q}$ analytic in this domain and such that the following relation holds in

the sense of distributions

$$\lim_{\epsilon \rightarrow 0^+} \mathcal{M}_{(0,t/\epsilon)}(X_p \rightarrow Y_q) = \underset{\substack{p_i^0 \rightarrow \omega(\mathbf{p}_i), q_j^0 \rightarrow \omega(\mathbf{q}_j) \\ \text{Im } p_i^0 < 0, \text{Im } q_j^0 > 0}}{\text{B. V.}} \mathcal{F}_{X_p \rightarrow Y_q}(\{(p_i^0, \mathbf{p}_i)\}, \{(q_j^0, \mathbf{q}_j)\}). \quad (5.30)$$

Here ‘‘B. V.’’ means distributional boundary value of an analytic function [75]. The existence of this limit follows from the work of Epstein and Glaser [76], to arbitrary orders in perturbation theory⁶. One would also expect this to be true non-perturbatively, but we have not been able to see it. In the following, we will denote with $\widetilde{\mathcal{M}}_{(0,\infty)}(X_p \rightarrow Y_q)$ this boundary value. It corresponds to the matrix elements of the (full) scattering matrix with an interaction switched on at $t = 0$. The relation with the full matrix element is easily seen from the following formal calculation

$$\begin{aligned} \mathcal{M}(X_p \rightarrow Y_q) &:= \mathcal{M}_{(-\infty,+\infty)}(X_p \rightarrow Y_q) = \\ &= -\left(\mathcal{M}_{(+\infty,+\infty)}(X_p \rightarrow Y_q) - \mathcal{M}_{(-\infty,+\infty)}(X_p \rightarrow Y_q)\right) = \\ &= -\int_{\mathbb{R}} ds \frac{\partial}{\partial s} \mathcal{M}_{(s,+\infty)}(X_p \rightarrow Y_q) = \\ &= -\int_{\mathbb{R}} ds \frac{\partial}{\partial s} \mathcal{M}_{(0,+\infty)}(X_p \rightarrow Y_q) e^{is(\omega_Y - \omega_X)} = \\ &= i(\omega_X - \omega_Y) \mathcal{M}_{(0,+\infty)}(X_p \rightarrow Y_q) \int_{\mathbb{R}} ds e^{is(\omega_Y - \omega_X)} = \\ &= 2\pi\delta(\omega_X - \omega_Y) i(\omega_Y - \omega_X) \mathcal{M}_{(0,+\infty)}(X_p \rightarrow Y_q) = \\ &= 2\pi\delta(\omega_X - \omega_Y) \underset{\substack{p_i^0 \rightarrow \omega(\mathbf{p}_i), q_j^0 \rightarrow \omega(\mathbf{q}_j) \\ \text{Im } p_i^0 < 0, \text{Im } q_j^0 > 0}}{\text{B. V.}} i(p_X^0 - q_Y^0) \mathcal{F}_{X_p \rightarrow Y_q}(\{(p_i^0, \mathbf{p}_i)\}, \{(q_j^0, \mathbf{q}_j)\}). \end{aligned} \quad (5.31)$$

Here we have again used equation (5.19), whereas in the last line we have employed the definition given in (5.30). The non trivial assumption here is, of course, that the boundary value of $(p_X^0 - q_Y^0) \mathcal{F}_{X_p \rightarrow Y_q}$ can indeed be restricted to $\omega_X = \omega_Y$, which is certainly not obvious (see [63] for related work). Thus, up to a standard energy conservation delta function, the full matrix element is equal to the scattering matrix element with interaction switched on at time $t = 0$, multiplied by the energy. Finally, we also note

⁶Here, it is essential that one takes the parameter m in the free Hamiltonian to be the true physical mass of the theory. In other words, it is assumed that the mass has been renormalized. It should also be noted that in [76] the adiabatic limit is defined in terms of some sort of averaging procedure in momentum space around the mass hyperboloids, rather than a boundary value prescription as above. However, the two are seen to be equivalent.

the distributional equality

$$\begin{aligned} \frac{\partial}{\partial s} \mathcal{M}_{(s, \infty)}(X_p \rightarrow Y_q) \Big|_{s=0} &= i(\omega_Y - \omega_X) \mathcal{M}_{0, \infty}(X_p \rightarrow Y_q) \\ &\equiv \text{B. V.} \quad i(p_X^0 - q_Y^0) \mathcal{F}_{X_p \rightarrow Y_q} \left(\{(p_i^0, \mathbf{p}_i)\}, \{(q_j^0, \mathbf{q}_j')\} \right). \end{aligned} \quad (5.32)$$

$p_i^0 \rightarrow \omega(\mathbf{p}_i), \quad q_j^0 \rightarrow \omega(\mathbf{q}_j)$
 $\text{Im } p_i^0 < 0, \quad \text{Im } q_j^0 > 0$

With this preparation in place we are ready to take the $\epsilon \rightarrow 0^+$ limit of the local scattering element. Using the property (5.19), we can write the relevant integral [cf. (5.24)] as

$$\begin{aligned} \frac{E}{2\pi} \int_{\mathbf{R}} dt e^{-iEt} \epsilon E |\mathcal{M}_{(0, \frac{\epsilon}{2})}(X_p \rightarrow Y_q)|^2 &= -i \frac{E}{2\pi} \int_{\mathbf{R}} dt e^{-iEt} \epsilon \frac{\partial}{\partial t} |\mathcal{M}_{(0, \frac{\epsilon}{2})}(X_p \rightarrow Y_q)|^2 = \\ &= -i \frac{E}{2\pi} \int_{\mathbf{R}} dt e^{-iEt} \epsilon \frac{\partial}{\partial t} |\mathcal{M}_{(0, \frac{\epsilon}{2} - s)}(X_p \rightarrow Y_q)|^2 \Big|_{s=0} = \\ &= i \frac{E}{2\pi} \int_{\mathbf{R}} dt e^{-iEt} \frac{\partial}{\partial s} |\mathcal{M}_{(0, \frac{\epsilon}{2} - s)}(X_p \rightarrow Y_q)|^2 \Big|_{s=0} = \\ &= i \frac{E}{2\pi} \int_{\mathbf{R}} dt e^{-iEt} \frac{\partial}{\partial s} |\mathcal{M}_{(s, \frac{\epsilon}{2})}(X_p \rightarrow Y_q)|^2 \Big|_{s=0} = \\ &= i \frac{E}{\pi} \int_{\mathbf{R}} dt e^{-iEt} \text{Re} \left\{ \mathcal{M}_{(0, \frac{\epsilon}{2})}^*(X_p \rightarrow Y_q) \frac{\partial}{\partial s} \mathcal{M}_{(s, \frac{\epsilon}{2})}(X_p \rightarrow Y_q) \right\}_{s=0} = \\ &= i \frac{E}{\pi} \int_0^{\infty} dt e^{-iEt} \text{Re} \left\{ \mathcal{M}_{(0, \frac{\epsilon}{2})}^*(X_p \rightarrow Y_q) \frac{\partial}{\partial s} \mathcal{M}_{(s, \frac{\epsilon}{2})}(X_p \rightarrow Y_q) \right\}_{s=0} + \\ &\quad + i \frac{E}{\pi} \int_{-\infty}^0 dt e^{-iEt} \text{Re} \left\{ \mathcal{M}_{(0, \frac{\epsilon}{2})}^*(X_p \rightarrow Y_q) \frac{\partial}{\partial s} \mathcal{M}_{(s, \frac{\epsilon}{2})}(X_p \rightarrow Y_q) \right\}_{s=0}. \end{aligned}$$

Now, the integral in the last line above is analytic in $\text{Im} E > 0$ and will hence not contribute when we evaluate its contribution to the collision factor in the pre-Boltzmann equation as we could then deform the contour of the dE (respectively dE_i) integrations into the lower half plane and get zero. We hence need only consider the integral on the next to last line above, i.e. with integration domain $[0, \infty)$. In the $\epsilon \rightarrow 0$ limit, the integrand becomes independent of t except for the exponential $\exp(-iEt)$, and we can trivially perform the dt integration which yields simply a factor of $(E - i0)^{-1}$. Using

expressions (5.31) and (5.32) to obtain

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0^+} \epsilon \frac{E^2}{2\pi} \int_{\mathbb{R}} dt e^{-iEt} |\mathcal{M}_{(0, \frac{\epsilon}{E})}(X_p \rightarrow Y_q)|^2 &= \\
&= \left| (\omega_X - \omega_Y) \mathcal{M}_{(0, \infty)}(X_p \rightarrow Y_q) \right|^2 \operatorname{Re} \left\{ \begin{array}{c} \text{B. V.} \\ p_i^0 \rightarrow \omega(\mathbf{p}_i), q_j^0 \rightarrow \omega(\mathbf{q}_j) \\ \operatorname{Im} p_i^0 < 0, \operatorname{Im} q_j^0 > 0 \end{array} \frac{i}{E_X - E_Y} \right\} = \quad (5.33) \\
&= 4\pi^2 \delta(\omega_X - \omega_Y) |\mathcal{M}(X_p \rightarrow Y_q)|^2,
\end{aligned}$$

where $\mathcal{M}(X_p \rightarrow Y_q)$ is now the full scattering matrix element of the theory, with the energy–momentum delta's *taken out*. To obtain this expression we also used the well known distributional equality

$$\frac{1}{x \pm i0} = \mathcal{P} \frac{1}{x} \mp i\pi \delta(x),$$

where \mathcal{P} denotes the Cauchy principal value. Equation (5.33) now no longer depends on E and we will use it in order to evaluate the limit of the collision factor (5.24).

Remark. Strictly speaking, the results we have obtained in this section are only valid for the “standard” scattering amplitude \mathcal{M} and not to the dressed one $\widetilde{\mathcal{M}}$. Because of that, the above analysis only justifies the use of equation (5.33) in the low density limit where the dressed amplitude gets replaced by the standard scattering amplitude. We will, however, use equation (5.33) formally also for the dressed amplitudes and proceed to “derive” the Boltzmann collision factor for weakly interacting systems, see Section §5.3.2. Albeit mathematically not justified, the main reason for us to do so, is the fact that in the latter case we obtain some physically interesting results.

The collision factor

To obtain the desired equation (5.26) we now use the results in the previous section and the scaling assumptions (5.25) to (formally) write

$$\begin{aligned}
\lim_{\epsilon \rightarrow 0^+} \epsilon^{-\alpha-1} B\left(\epsilon E, \mathbf{p}, \frac{\mathbf{s}}{\epsilon}\right) &= \lim_{\epsilon \rightarrow 0^+} \sum_{\substack{i \in X \\ j \in Y}} |X|! |Y|! \epsilon^{\alpha(|X|-1)-1} \int_{\mathbb{R}} d\Pi_{X_p \cup Y_q} (2\pi)^2 \delta^{(2)}(\underline{p}_X - \underline{q}_Y) \cdot \\
&\cdot \left| \mathcal{M}(X_p \rightarrow Y_q) \right|^2 \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \prod_{i \in X} \nu_{\mathbf{p}_i}(\epsilon, T) + \dots
\end{aligned}$$

where the dots mean that we are omitting terms that are of higher order in ϵ and will hence vanish when we take the limit. Heuristically speaking, they come from the fact that in the low density limit $1 + \nu_{\mathbf{p}}(T) \approx 1$. From the above expression we now immediately see that the sum over all sets $Y \subset \mathbb{N}$ is unrestricted by the limit. On the other hand, for the limit to be finite, it must be that $\alpha(|X| - 1) - 1 = 0$, from which the condition $\alpha = 1/(|X| - 1)$ follows. By noting then that $|X| \geq 2$, we deduce that $\alpha \leq 1$. However, if we pick $\alpha < 1$ then all the terms in the above sum such that $|X| < 1 + 1/\alpha$ will contain an inverse power of ϵ and will diverge in the $\epsilon \rightarrow 0$ limit. (And there will always be at least one such term as $1 + 1/\alpha > 2$ when $\alpha < 1$.) It follows that we will obtain a finite limit only for $\alpha = 1$. Taking now the limit in the above equation we get

$$\lim_{\epsilon \rightarrow 0^+} \epsilon^{-\alpha-1} B\left(\epsilon E, \mathbf{p}, \frac{s}{\epsilon}\right) = 2 \sum_{j \in Y} |Y|! \int_{\mathbb{R}} d\Pi_{\{\mathbf{p}_1, \mathbf{p}_2\} \cup Y_q} (2\pi)^2 \delta^{(2)}(\underline{p}_1 + \underline{p}_2 - \underline{q}_Y) \cdot \\ \cdot \left| \mathcal{M}(X_p \rightarrow Y_q) \right|^2 \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \prod_{i \in X} \nu_{\mathbf{p}_i}(T).$$

To obtain the desired result, equation (5.26), we now simply integrate against the remaining deltas $\delta(\mathbf{p} - \mathbf{q}_j)$ and $\delta(\mathbf{p} - \mathbf{p}_i)$, and note that we are working with a PT -invariant model so that $\mathcal{M}(X_p \rightarrow Y_q) = \mathcal{M}(Y_q \rightarrow X_p)$.

5.3.3 The Boltzmann equation for weakly interacting systems

We will now consider the so called $\lambda^2 t$ scaling limit of equation (5.24), i.e. the “weak coupling” limit. This scaling limit corresponds to the physical situation of a weakly interacting system which we observe over a very long time. The way to consider this limit is to rescale the expansion parameter λ to zero in such a way that $\lambda^2 t = T$ remains finite in the limit. We also note at this point that the number densities $n_{\mathbf{p}}(t)$ depend on λ and we denote them by $n_{\mathbf{p}}^{\lambda}(t)$. Additionally, we need to make the assumption that they possess a finite limit. That is to say, in the weak coupling limit we assume

$$\lambda \rightarrow \sqrt{\epsilon} \lambda, \quad t \rightarrow T/\epsilon, \quad \lim_{\epsilon \rightarrow 0^+} n_{\mathbf{p}}^{\lambda}(t) = f_{\mathbf{p}}(T), \quad (5.34)$$

The full mathematical proof of the existence of the limit would require us to look at the full non-perturbative dynamics of the model. While this ought to be possible in principle, we have not done it. We will, again partially formally, derive an equation that the quantities $f_{\mathbf{p}}(T)$ must satisfy if the limit (5.34) exists in a sufficiently strong sense.

First, let us suppose that the adiabatic limit $\widetilde{\mathcal{M}} := \lim \widetilde{\mathcal{M}}_{(s,t)}$ [cf. equation (5.15)] exists in the sense that we can take the limit $s \rightarrow -\infty$ and $t \rightarrow \infty$ to obtain the full

(not local) dressed scattering amplitude. This very assumption actually turns out to be *wrong* for any particle distribution $n_{\mathbf{p}}(t)$, and we will explain in much more detail what has to be done to make things work (see Section §5.4). However, for pedagogical purposes, let us ignore this fact for the moment and let us pretend (incorrectly) that $\widetilde{\mathcal{M}}$ exists. Since $\widetilde{\mathcal{M}} = \mathcal{O}(\lambda)$, we may then also formally calculate its “Born approximation” as

$$\lim_{\lambda \rightarrow 0} \frac{\widetilde{\mathcal{M}}(X_p \rightarrow Y_q)}{\lambda} = i .$$

Then, let us formally define $\mathcal{B}_\lambda(\mathbf{p}, T)$, the Boltzmann collision factor in the weak coupling limit, as

$$\begin{aligned} \mathcal{B}_\lambda(\mathbf{p}, T) := & \frac{\lambda^2}{2} \sum_{\substack{i \in X, j \in Y \\ |X|+|Y|=p}} |X|! |Y|! |X| \int_{\mathbb{R}} d\Pi_{X_q^1 \cup Y_q} (2\pi)^2 \delta^{(2)}(\underline{p} + \underline{q}_{X^1} - \underline{q}_Y) \\ & \left\{ \left(\prod_{j \in Y} f_{\mathbf{q}_j}(T) \right) (1 + f_{\mathbf{p}}(T)) \prod_{i \in X^1} (1 + f_{\mathbf{p}_i}(T)) - f_{\mathbf{p}}(T) \prod_{i \in X^1} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)) \right\} , \end{aligned} \quad (5.35)$$

where p in the sum above is the degree of our interaction polynomial [cf. equation (5.8)]. We now claim that formally the quantities $f_{\mathbf{p}}(T)$ satisfy the following equation

$$\begin{aligned} \omega(\mathbf{p}) \partial_T f_{\mathbf{p}}(T) = & \mathcal{B}_\lambda(\mathbf{p}, T) + \sum_{n=1}^{\infty} (-1)^n \int_{s \leq \tau_1 \leq \dots \leq \tau_n \leq T} d\tau_1 \dots d\tau_n \\ & \int_{\mathbb{R}} d\Pi(\mathbf{k}_1, \dots, \mathbf{k}_n) \mathcal{B}_\lambda(\mathbf{k}_1, \tau_1) \frac{\delta}{\delta f_{\mathbf{k}_1}(\tau_1)} \mathcal{B}_\lambda(\mathbf{k}_2, \tau_1) \dots \frac{\delta}{\delta f_{\mathbf{k}_n}(\tau_n)} \mathcal{B}_\lambda(\mathbf{p}, \tau_n) . \end{aligned} \quad (5.36)$$

This is the main result for the weakly interacting case. This equation has formally the same structure as the one we obtained for dilute gases [cf. equation (5.28)], but the collision factor \mathcal{B} is different in the two cases.

The collision factor

By (incorrectly) pretending that $\widetilde{\mathcal{M}}$ exists, we can again (formally) “derive” equation (5.33) for $\widetilde{\mathcal{M}}$ and substitute it into (5.24). By (formally) repeating the arguments in the

previous section we obtain

$$\begin{aligned} \frac{\mathcal{B}_\lambda(\mathbf{p}, T)}{\omega(\mathbf{p})} &= \lim_{\epsilon \rightarrow 0^+} \frac{1}{\epsilon} \int_0^T ds \int_{\mathbf{R}} dE e^{iE(T-s)} B\left(\epsilon E, \mathbf{p}, \frac{s}{\epsilon}\right) = \frac{\lambda^2}{2} \sum_{\substack{i \in X \\ j \in Y}} |X|! |Y|! \\ &\int_{\mathbf{R}} d\Pi_{X_p \cup Y_q} (2\pi)^2 \delta^{(2)}(\underline{p}_X - \underline{q}_Y) \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \cdot \\ &\cdot \prod_{i \in X} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)), \end{aligned} \quad (5.37)$$

To cast \mathcal{B}_λ in a more familiar form, we note the following chain of equalities

$$\begin{aligned} &\left\{ \int_{\mathbf{R}} d\Pi_{X_p \cup Y_q} \delta^{(2)}(\underline{p}_X - \underline{q}_Y) \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \cdot \right. \\ &\quad \cdot \left. \prod_{i \in X} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)) \right\} + \left\{ \int_{\mathbf{R}} d\Pi_{Y_p \cup X_q} \delta^{(2)}(\underline{p}_Y - \underline{q}_X) \cdot \right. \\ &\quad \cdot \left. \left[\sum_{j \in X} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in Y} \delta(\mathbf{p} - \mathbf{p}_i) \right] \prod_{i \in Y} f_{\mathbf{p}_i}(T) \prod_{j \in X} (1 + f_{\mathbf{q}_j}(T)) \right\} = \\ &= \left\{ \int_{\mathbf{R}} d\Pi_{X_p \cup Y_q} \delta^{(2)}(\underline{p}_X - \underline{q}_Y) \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \cdot \right. \\ &\quad \cdot \left. \prod_{i \in X} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)) \right\} - \left\{ \int_{\mathbf{R}} d\Pi_{X_p \cup Y_q} \delta^{(2)}(\underline{p}_X - \underline{q}_Y) \cdot \right. \\ &\quad \cdot \left. \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \prod_{i \in X} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)) \right\} = \\ &= \int_{\mathbf{R}} d\Pi_{X_p \cup Y_q} \delta^{(2)}(\underline{p}_X - \underline{q}_Y) \left[\sum_{j \in Y} \delta(\mathbf{p} - \mathbf{q}_j) - \sum_{i \in X} \delta(\mathbf{p} - \mathbf{p}_i) \right] \cdot \\ &\quad \cdot \left\{ \prod_{i \in X} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)) - \prod_{j \in Y} f_{\mathbf{q}_j}(T) \prod_{i \in X} (1 + f_{\mathbf{p}_i}(T)) \right\}. \end{aligned} \quad (5.38)$$

Let us explain what we did above. From the sum over all possible $X, Y \subset \mathbb{N}$ in (5.37), we pick two terms corresponding to specific X, Y and X', Y' , and such that $X' = Y$ and $Y' = X$. If we now introduce the schematic notation ($|X| \rightarrow |Y|$) to denote the summand in (5.37) corresponding to the sets X and Y , we extract the terms ($|X| \rightarrow |Y|$) and ($|X'| \rightarrow |Y'|$) = ($|Y| \rightarrow |X|$) and we sum them. That is, above we have computed ($|X| \rightarrow |Y|$) + ($|Y| \rightarrow |X|$). The chain of equalities is then obtained by noting that in the first equality we have replaced the dummy integration variables $\mathbf{p}_i \leftrightarrow \mathbf{q}_j$ in the

second integral and then rearrange the various terms. As a last step we now integrate over the momenta corresponding to the various delta functions to obtain

$$\begin{aligned}
& \frac{|X|}{2\omega(\mathbf{p})} \int_{\mathbf{R}} d\Pi_{X_{\mathbf{p}}^1 \cup Y_{\mathbf{q}}} \delta^{(2)}(\underline{p} + \underline{p}_{X^1} - \underline{q}_Y) \left[\left(\prod_{j \in Y} f_{\mathbf{q}_j}(T) \right) (1 + f_{\mathbf{p}}(T)) \prod_{i \in X^1} (1 + f_{\mathbf{p}_i}(T)) + \right. \\
& \qquad \qquad \qquad \left. - f_{\mathbf{p}}(T) \prod_{i \in X^1} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)) \right] + \\
& + \frac{|Y|}{2\omega(\mathbf{p})} \int_{\mathbf{R}} d\Pi_{Y_{\mathbf{p}}^1 \cup X_{\mathbf{q}}} \delta^{(2)}(\underline{p} + \underline{p}_{Y^1} - \underline{q}_X) \left[\left(\prod_{j \in X} f_{\mathbf{q}_j}(T) \right) (1 + f_{\mathbf{p}}(T)) \prod_{i \in Y^1} (1 + f_{\mathbf{p}_i}(T)) + \right. \\
& \qquad \qquad \qquad \left. - f_{\mathbf{p}}(T) \prod_{i \in Y^1} f_{\mathbf{p}_i}(T) \prod_{j \in X} (1 + f_{\mathbf{q}_j}(T)) \right], \tag{5.39}
\end{aligned}$$

where $X^1 = X \setminus \{1\}$. Using the above result we can finally write $\mathcal{B}_\lambda(\mathbf{p}, T)$ as in (5.35).

Before proceeding with the discussion we now want to inspect $\mathcal{B}_\lambda(\mathbf{p}, T)$ in more detail. Specifically, we want to comment on the sum in equation (5.35). First of all we note that we can physically interpret the above sum as the sum over all the $|X| \leftrightarrow |Y|$ scattering processes, i.e. a sum over all processes in which the interaction amongst $|X|$ initial particles results in $|Y|$ final particles. Such a process occurs with an amplitude that in the Born approximation is given by $i\lambda$. In the above form the collision factor emphasizes the fact that a process and its inverse ought to be considered on the same footing. In the form of equation (5.37), on the other hand, the change in the number of particles with momentum \mathbf{p} is due to two unrelated processes: A first process in which there is a particle with momentum \mathbf{p} amongst the $|X|$ incoming ones, and another process where it is amongst the $|Y|$ outgoing ones. Clearly, the expansion (5.35) is physically more meaningful.

Given now such a (physical) interpretation and by recalling the expansion of the local S -matrix (5.15) (where $|X|, |Y| \geq 1$), one is led to think that the above sum includes all the possible processes, with arbitrary $|X|$ and $|Y|$. We will now argue that this is not so, and that in fact the above sum reduces to the sum over all “physical” processes. To do so, we split the above sum into two classes of terms: (a) (unphysical) processes that violate momentum conservation, (b) everything else, i.e. physical processes.

For processes (a), we note that the momentum conservation delta will be of the form $\delta^{(2)}(k)$ with $k \neq 0$ and hence vanishing. And it will be of this form precisely because momentum is not conserved—we cannot arrange the sum of the (on-shell) 2-momenta to vanish by respecting at the same time some elementary physical assumptions. An example of such a process is a $1 \rightarrow 3$ process, that is the “decay of one particle into three

particles with same mass as the decaying one” and it is an easy exercise of relativistic kinematics to show that such a process violates momentum conservation⁷.

The terms of type (b) are the physically allowed processes and these will, in general, contribute to the change in the particle number. However, we additionally point out that as one might intuitively imagine, the “mere propagation” of particles—a process where $|X| = |Y| = 1$ —is also to be excluded from the above sum. The reason for this is that the momentum conservation delta will enforce equality of \underline{p} and \underline{q}_1 which will in turn enforce that the term in curly brackets will vanish. We also note that a specific choice for the interaction polynomial might impose additional constraints to the above sum. For instance, if the interaction polynomial contains only even powers in the field, then all the processes with $|X| + |Y|$ odd will vanish as the corresponding scattering amplitudes would vanish identically.

Last, but certainly not least, we also want to point out a subtle point about the form of collision factor (5.35). That we have obtained a sum over all (physical) collision processes in the $\lambda^2 t$ limit is a *model dependent* feature that in fact further restricts the sum so that $|X| + |Y| = p$, where p is the highest degree in our interaction polynomial. This is easily understood by noting that at tree level, i.e. in the Born approximation, the sum of incoming and outgoing particles can be at most p . However, when we later briefly extend our analysis beyond the weak coupling limit in Section §5.4, we will obtain corrections (in the coupling constant) to equation (5.36). And one aspect of these corrections will be that the scattering amplitude will emerge in its full perturbative expansion. Because of this, the only change to the Boltzmann collision factor (5.35) will come from the fact that the sum will now have to be extended to *all* physically allowed processes. And this will hold *independently* of the specific model, the reason being that in quantum field theory an arbitrary number of particles can be created, albeit this will only occur at correspondingly high orders in perturbation theory.

5.3.4 Physical interpretation

Here we give a physical interpretation of the results we obtained in Sections §5.3.2 and §5.3.3. In general we can say that in both cases we obtain an equation that is *different* from the BE as we introduced in Chapter 2. The main difference lies in the fact that our equation is *non-Markovian* whereas the BE is local in time. Such a non locality manifests itself in the form of additional (infinite) terms on the right hand side of our equation.

⁷It should be noted that this is a model dependent feature: Had we started with a theory with more than one field and with different masses, then if the sum of the masses of the three outgoing particles was less than the mass of the decaying one, such a process would be allowed.

It is only by *truncating* (by hand) our equation that we could recover the Boltzmann equation. Alternatively, we obtain the BE by considering both the low density and weak coupling limits at the same time. The physical interpretation of these additional terms is attributed to particle rescattering, i.e. multiple scatterings between particles. Intuitively one would expect no particle rescattering at least in the low density limit, but the fact that we are really considering a long time limit in addition to any other limit clears why particle rescattering will inevitably be present. In fact, even if the density of the particles is very low, in the limit of infinite time every particle will undergo multiple (actually infinite) collisions.

Another difference is given by the presence, in our results [cf. equations (5.26) and (5.35)] of a sum over infinitely many collision processes whereas in the BE (2.10) there only appears a single collision process. This difference is easily understood by recalling that the BE gives an answer to the question: “How does the particle number (locally) change in a sufficiently dilute or weakly interacting system?”. A pragmatic, physical answer would then involve the identification of (typically) one specific process that is the principal physical reason for such a change and, eventually, arrive at the BE (2.10). In principle, however, it is intuitively clear that the particle number will not change only because of *one* specific process, but because of *all* the possible processes a particle can be involved in. And our derivation is “mathematical” rather than physical, thus answering the previous question “in principle” rather than “in practice”. We consider it a remarkable feature of our derivation, and of quantum field theory, that this latter intuitive feature emerges naturally. But let us now make some remarks which are specific to the two cases separately.

In the low density limit, the collision factor $B_\nu(\mathbf{p}, T)$ involves a sum over all $2 \rightarrow n$ scattering processes. A heuristic “explanation” of this fact can be given by arguing that in a dilute system the likelihood of more than two particles colliding is negligible and hence only two particle scattering will contribute. The two distinct terms in (5.26) are then understood as follows. Keeping in mind that we are interested in the change of the number of particles with *specific* momentum \mathbf{p} , as is clear from the LHS of (5.28), we observe that a particle with momentum \mathbf{p} can be either in the set of *outgoing* particles or it could be one of the two *incoming* (colliding) particles. In the first case, the contribution will be positive while in the second case it will be negative, as intuitively clear and as the signs in (5.26) reveal. For weakly interacting systems, on the other hand, there is no reason why only two particles should scatter at any time. And this heuristic argument is “confirmed” by our (formal) “derivation”.

By the above heuristic discussion on the physical interpretation of some aspects of

Dilute systems		Weakly interacting systems
$2 \rightarrow n$ processes	vs.	All (physical) processes
Full scattering matrix	vs.	Born approximation
Vacuum amplitudes	vs.	Dressed amplitudes*

Table 5.1: Comparison of the results in the low density and weak coupling limits.

*: This is only obtained by going beyond the weak coupling limit, see Section §5.4.

equations (5.26) and (5.35), we can say that a characterizing feature of the collision factor $B_\nu(\mathbf{p}, T)$ is the emergence of $2 \rightarrow n$ scattering processes “only”. An analogue characterization can be given for the collision factor $B_\lambda(\mathbf{p}, T)$ in the weak coupling limit as well. In this latter case, what we observe is that the scattering amplitude emerges in its Born approximation. And this too can be given a physical interpretation: If the interactions between particles is very weak, then higher order corrections will be completely negligible. This ought to be compared with the emergence of the full scattering matrix in the low density limit.

There is one more difference between the weak coupling and the low density limits that is worth pointing out. In Section §5.4 we will outline one possible way to extend our results beyond the weak coupling limit. Although very formal and not complete, our results do exhibit one particularly interesting feature, namely that if one wants to consider corrections (in the coupling constant) to the weak coupling limit equation, than the full scattering amplitude emerges again. The point here is, however, that it is not the “vacuum” scattering amplitude like the one we have in the low density limit but the “dressed” one. This last feature can be physically understood as follows. For a very dilute system, the single collisions will not “feel the effect” of the surrounding particles. On the other hand, a weakly interacting system might be very dense so that each collision will “feel” the surrounding “bath of particles”. And this feature is encoded in the dressed propagator. We summarize these conclusions in Table 5.1.

5.4 Beyond the weak coupling limit

In our considerations, an important role was played by the “dressed local amplitudes” $\widetilde{\mathcal{M}}_{(-t,t)}$. These formally represent a scattering amplitude for particles in a medium characterized by occupation numbers $n_{\mathbf{p}}(s)$, and an interaction that is switched on during the time interval $(-t, t)$. It was important to obtain the limit of this as $t \rightarrow \infty$, together with either $\lambda \rightarrow 0$ (weak coupling) or $n_{\mathbf{p}}(0) \rightarrow 0$ (low density). If one wants to go beyond these limits, one is confronted with the problem of making sense of the limit

$\widetilde{\mathcal{M}} = \lim_{t \rightarrow \infty} \widetilde{\mathcal{M}}_{(-t,t)}$ without the weak coupling/low density assumption. This limit, however, simply does not exist, unless all $n_{\mathbf{p}}(s) = 0$, which is the situation covered in the existence proof of the adiabatic limit by Epstein and Glaser, cf. [76], and which we appealed to above. To show that the dressed amplitudes $\widetilde{\mathcal{M}}_{(s,t)}$ do *not* possess such a limit is much simpler and in this section we will see why this is so.

The physical origin of this problem appears to be that in a medium characterized by a non-trivial distribution of occupation numbers (i.e. with a finite density of particles), the notion of a “free particle” is different from that in the vacuum. Because of this, one may expect that the energy of a free particle $\mathcal{E}(\mathbf{p})$ in a medium will, in general, *not* be the vacuum energy, $\mathcal{E}(\mathbf{p}) \neq \omega(\mathbf{p})$, see e.g. [77]. Consequently, one should not expand the theory around the free theory given by the free Hamiltonian H_0 as in equation (3.23), but instead use a different Hamiltonian that is based on the dispersion relation $E = \mathcal{E}(\mathbf{p})$ for the underlying free dynamics. This has been observed in the case of thermal equilibrium states e.g. by Landsman in [78], where he analyzed why “naive” thermal perturbation theory, i.e. perturbation theory in terms of the vacuum free Hamiltonian, leads to a non vanishing self-energy of the thermal propagator on the mass shell. It is also related to the well-known fact that the Feynman propagator for finite temperature equilibrium states, i.e. KMS states [79], is not Lorentz invariant, see e.g. [80, 81, 82]. Seen in this light then it is not surprising that the adiabatic limit of $\widetilde{\mathcal{M}}_{(s,t)}$ does not exist—we (implicitly) attempted a perturbative expansion of the local scattering amplitudes in terms of an unphysical notion of free particle in a medium, and consequently a “wrong” free Hamiltonian.

Unfortunately, these considerations do not tell us what we should take for $\mathcal{E}(\mathbf{p})$, i.e. what is the correct notion of free particle in a medium characterized by occupation numbers $n_{\mathbf{p}}(s)$. We will, however, explain how a “self-consistent” determination of this quantity is possible in the next subsection. The basic idea is that $\mathcal{E}(\mathbf{p})$ can be (self-consistently) obtained by a “renormalization condition” on the self-energy in perturbation theory around a free theory characterized by the dispersion law $E = \mathcal{E}(\mathbf{p})$. This renormalization condition is similar in nature to that appearing in ordinary perturbation theory in the vacuum. There, the dispersion relation is of course $E = \omega(\mathbf{p}) = \sqrt{\mathbf{p}^2 + m^2}$, and the renormalization condition concerns the value of the constant m^2 . This is the mass parameter that has to be used in the free Hamiltonian, and the renormalization condition is (cf. [31, 32, 83, 84])

$$\widetilde{\Pi}(p^2) \Big|_{p^2=-m^2} = 0, \quad \frac{\partial}{\partial p^2} \widetilde{\Pi}(p^2) \Big|_{p^2=-m^2} = 0.$$

In the above equation $\tilde{\Pi}$ is the “perturbative self-energy”, which is formally defined as the sum of all “one particle irreducible” diagrams with two external legs, see Section §5.4.1. The meaning of the above condition is that the parameter m^2 in the free Hamiltonian must be equal to the mass of the *interacting* theory. This observation will play an important role in what follows.

Before presenting our self-consistent approach to the problem, however, we want to comment on the rigour of this approach, or better, on its lack of rigour. What follows crucially relies on the assumption that in a generic state we will have a functional (dispersion) relation between a particle’s momentum \mathbf{p} and its energy $\mathcal{E}(\mathbf{p})$. As (physically) reasonable as it seems, however, this assumption hides a non-trivial consequence that necessarily invalidates the results in this section. This hidden subtlety is as follows. In [85] some consequences of the KMS condition are studied on the properties of (quasi-)particles for which it is assumed that a functional relation between their momentum and energy exists. One of the conclusions they obtain is that such an assumption leads to a unit S-matrix, which means that there is no scattering between these (quasi-)particles. In other words, the assumption of the existence of a dispersion relation implies that the model is free. Clearly, then, such an assumption cannot be made for an interacting model like ours. One might argue that we are not considering a KMS state, but as remarked in the introductory chapter to this work, KMS states are relatively “simple” (equilibrium) states. We, on the other hand, are considering a generic non-equilibrium state and hence cannot expect any “simplifications” with respect to the KMS case.

Nevertheless, there are compelling reasons for us to present our results. The first and most important one is that our “derivation” of the BE in the weak coupling limit was not justified. We are clearly referring to the fact that we assumed that we could consider the limit of a divergent quantity, namely of the full (not local) dressed scattering amplitudes. Something ought to be done to “remove” the divergencies and the results in this section are a step in that direction. They also provide a strong argument in support of the claim that a fully rigorous derivation would require *some* kind of mass renormalization. Clearly, the problem then remains as to how should such a renormalization be done. In the light of our results, a reasonable attempt appears to be to assume the existence of a “dispersion correlation” (sharply peaked around a dispersion relation) and hence circumvent the results from [85]. On physical grounds we expect such a “dispersion correlation” to exist—physicists do measure momenta and energies of particles in various states after all. We have not pursued this idea at all.

Another reason for presenting our results is that they offer some practical novelties when it comes to solve a problem using the Boltzmann equation. Specifically, we “show”

that the scattering amplitudes should be computed with a modified propagator and we also find how to (self-consistently) compute it. This, in turn, might lead to improved quantitative physical predictions. Finally, in Section 5.4.2 we also briefly argue that it is necessary to justify the use of the BE beyond the Born approximation. And in order to do so, it will again be necessary to tackle the divergencies of the (dressed) scattering amplitudes.

It appears that we are facing a formidable problem and we hope that what we did will shed some light into its possible solution.

5.4.1 Self consistent mass renormalization

As we have said, the ansatz we make is that the energy of the particles in a medium is $\mathcal{E}(\mathbf{p}) \neq \omega(\mathbf{p})$. This “simple” assumption has a number of non-trivial consequences, as it will become clear in what follows. In particular, it will allow us to (formally) avoid certain divergencies in the perturbative expansion of $\widetilde{\mathcal{M}}_{(s,t)}$, and we will obtain a prescription to compute $\mathcal{E}(\mathbf{p})$ once we push our argument to the end. Before proceeding, we note that we will make “one step back” so we can make “two steps forward” in that we will go back to the finite volume case so we could use the general framework set up previously to compute the Feynman propagator in the present case and then obtain its thermodynamic limit by (formally) letting $L \rightarrow \infty$, as done previously. (In the finite volume case we will be denoting the energy of the particles by $\mathcal{E}_{\mathbf{p}}$ rather than $\mathcal{E}(\mathbf{p})$, which we will use to denote its thermodynamic limit.) And as we also want to give an explicit example of the kind of equations that one has to solve in order to find $\mathcal{E}(\mathbf{p})$, we will be specifically working with a quartic interaction, that is, we will have $V(t) = \lambda/4! \varphi^4(t, \mathbf{x})$.

If we assume that the energy is $\mathcal{E}_{\mathbf{p}}$, then the appropriate choice for the free dynamics is

$$H_0(s) = \frac{1}{L} \sum_{\mathbf{p} \in \mathbf{Z}} N_{\mathbf{p}}(s) \mathcal{E}_{\mathbf{p}}(s) . \quad (5.40)$$

We can now immediately compute the interaction picture fields $\varphi_0(t, \mathbf{x})$ according to (5.6). Recalling the expression of the interaction field in terms of the creation and annihilation operators (4.23) we immediately get

$$\varphi_0(t, \mathbf{x}) = \frac{1}{L\sqrt{2\pi}} \sum_{\mathbf{p} \in \mathbf{Z}} \left\{ \frac{1}{(2\omega_{\mathbf{p}})^{1/2}} e^{-it\mathcal{E}_{\mathbf{p}}(s) + i\mathbf{p}\mathbf{x}} a_{\mathbf{p}}(s) e^{-is[\omega_{\mathbf{p}} - \mathcal{E}_{\mathbf{p}}(s)]} + \text{h.c.} \right\} ,$$

where “h.c.” stands for hermitian conjugate. This field satisfies an equation akin to the

Klein–Gordon equation

$$\left[\partial_t^2 + \mathcal{E}(\partial, s) \right] \varphi_0(t, \mathbf{x}) = 0, \quad \left[\partial_t^2 + \mathcal{E}_{\mathbf{p}}(s) \right] \varphi_0(t, \mathbf{p}) = 0,$$

where $\mathcal{E}(\partial, s)$ is the “position space representation of $\mathcal{E}_{\mathbf{p}}(s)$ ”, and we emphasize that it is a *very* formal object and we have written the above equation just to force a similarity with the usual free field Klein–Gordon equation. The mathematically meaningful equation is the second one above, i.e. the equation of motion satisfied by $\varphi_0(t, \mathbf{p})$.

Given the free field, we can immediately compute the Feynman propagator in the state w_s as

$$\begin{aligned} \tilde{\Delta}(t, \mathbf{x}) &:= w_s [T \varphi_0(t, \mathbf{x}) \varphi_0(0, 0)] = w_s [\varphi_0(t, \mathbf{x}) \varphi_s(0)] \theta(t) + w_s [\varphi_0(0) \varphi_s(t, \mathbf{x})] \theta(-t) = \\ &= \frac{1}{2\pi L^2} \sum_{\mathbf{p}, \mathbf{k} \in \mathbb{Z}} \frac{1}{2(\omega_{\mathbf{p}} \omega_{\mathbf{k}})^{1/2}} \left\{ \left[w_s(a_{\mathbf{p}} a_{\mathbf{k}}^*) e^{i\langle \underline{\mathbf{p}}, x \rangle} + w_s(a_{\mathbf{p}}^* a_{\mathbf{k}}) e^{-i\langle \underline{\mathbf{p}}, x \rangle} \right] \theta(t) + \right. \\ &\quad \left. + \left[w_s(a_{\mathbf{k}} a_{\mathbf{p}}^*) e^{-i\langle \underline{\mathbf{p}}, x \rangle} + w_s(a_{\mathbf{k}}^* a_{\mathbf{p}}) e^{i\langle \underline{\mathbf{p}}, x \rangle} \right] \theta(-t) \right\} = \\ &= \frac{1}{2\pi L} \sum_{\mathbf{p}, \mathbf{k} \in \mathbb{Z}} \frac{\delta_{\mathbf{k}, \mathbf{p}}}{2(\omega_{\mathbf{p}} \omega_{\mathbf{k}})^{1/2}} \left\{ \left[(1 + n_{\mathbf{p}}(s)) e^{i\langle \underline{\mathbf{p}}, x \rangle} + n_{\mathbf{p}}(s) e^{-i\langle \underline{\mathbf{p}}, x \rangle} \right] \theta(t) + \right. \\ &\quad \left. + \left[(1 + n_{\mathbf{p}}(s)) e^{-i\langle \underline{\mathbf{p}}, x \rangle} + n_{\mathbf{p}}(s) e^{i\langle \underline{\mathbf{p}}, x \rangle} \right] \theta(-t) \right\} = \\ &= \frac{1}{2\pi L} \sum_{\mathbf{p} \in \mathbb{Z}} \frac{1}{2\omega_{\mathbf{p}}} \left\{ \left[e^{i\langle \underline{\mathbf{p}}, x \rangle} \theta(t) + e^{-i\langle \underline{\mathbf{p}}, x \rangle} \theta(-t) \right] + n_{\mathbf{p}}(s) \left[e^{i\langle \underline{\mathbf{p}}, x \rangle} + e^{-i\langle \underline{\mathbf{p}}, x \rangle} \right] \right\} = \\ &= \frac{1}{2\pi L} \int_{\mathbb{R}} dE \sum_{\mathbf{p} \in \mathbb{Z}} \frac{\mathcal{E}_{\mathbf{p}}(s)}{\omega_{\mathbf{p}}} \left\{ \frac{-i}{-E^2 + \mathcal{E}_{\mathbf{p}}^2(s) + i0} + n_{\mathbf{p}}(s) \delta(\mathcal{E}_{\mathbf{p}}^2(s) - E^2) \right\} e^{-i\langle \underline{\mathbf{p}}, x \rangle}, \end{aligned}$$

where we are now denoting $p = (E, \mathbf{p})$, $\underline{p} = (\mathcal{E}_{\mathbf{p}}(s), \mathbf{p})$ and we have used $n_{\mathbf{p}}(s) = n_{-\mathbf{p}}(s)$, which is valid for a spatially homogenous state and for simplicity we assume this to be the case. Treating the general case would not add anything to our conceptual understanding of the problem while complicating some of the expressions below. Provided now that the thermodynamic limit of the above expression exists, the momentum space representation of the Feynman propagator that will be part of our Feynman rules can be straightforwardly read from the last line.

Our next task is to determine the interaction potential as required for the perturbative expansion. Clearly,

$$V(s) := H - H_0(s),$$

and what we want now is to have an expression of $V(s)$ in terms of the interacting field

$\varphi(s, x)$. If we now define

$$h_{\mathbf{p}}^2(s) := \frac{\mathcal{E}_{\mathbf{p}}(s)}{\omega_{\mathbf{p}}}, \quad g_{\mathbf{p}}^2(s) := \mathcal{E}_{\mathbf{p}}(s) \omega_{\mathbf{p}},$$

we have that

$$\begin{aligned} H_0(s) &= \frac{1}{L} \sum_{\mathbf{p} \in \mathbf{Z}} N_{\mathbf{p}}(s) \mathcal{E}_{\mathbf{p}}(s) = \\ &= \frac{L}{2} \sum_{\mathbf{p} \in \mathbf{Z}} \left\{ h_{\mathbf{p}}^2(s) \partial_s \varphi^*(s, \mathbf{p}) \partial_s \varphi(s, \mathbf{p}) + g_{\mathbf{p}}^2(s) \varphi^*(s, \mathbf{p}) \varphi(s, \mathbf{p}) \right\}, \end{aligned}$$

so that the interaction potential is

$$\begin{aligned} V(s) &= \frac{L}{2} \sum_{\mathbf{p} \in \mathbf{Z}} \left\{ [1 - h_{\mathbf{p}}^2(s)] : \partial_s \varphi^*(s, \mathbf{p}) \partial_s \varphi(s, \mathbf{p}) : + \right. \\ &\quad \left. + [\omega_{\mathbf{p}}^2 - g_{\mathbf{p}}^2(s)] : \varphi^*(s, \mathbf{p}) \varphi(s, \mathbf{p}) : \right\} + L \frac{\lambda}{4!} \int_0^{2\pi} dx : \varphi^4(s, \mathbf{x}) : , \end{aligned} \quad (5.41)$$

which we can use to compute the “interaction picture” potential. Before that, however, we want to comment on the above expression, and on the “smallness” of the potential specifically. First of all, we note that by introducing $\mathcal{E}_{\mathbf{p}}(s)$ into the Hamiltonian we have *not* modified it, as can be seen immediately by computing $H_0(s) + V(s)$. This observation gives us some clues about $\mathcal{E}_{\mathbf{p}}(s)$. In fact, in the $\lambda \rightarrow 0$ limit, the Hamiltonian reduces to the standard free Hamiltonian $\sum_{\mathbf{p}} N_{\mathbf{p}} \omega_{\mathbf{p}} / L$. We hence expect that

$$\mathcal{E}_{\mathbf{p}}(s) = \omega_{\mathbf{p}} + \mathcal{O}(\lambda), \quad (5.42)$$

as in this case $H_0(s)$ immediately reduces to what one would expect and $V(s)$ vanishes when $\lambda \rightarrow 0$. This also tells us that $V(s) = \mathcal{O}(\lambda)$, and can hence be indeed treated as a small perturbation to the free dynamics thus justifying the use of perturbation theory. At this point, however, (5.42) has not in any way been established yet, so we complete our ansatz with this additional condition.

Recalling equation (5.5) for the interaction picture potential, we see that in addition

to the interaction picture fields we need to compute

$$\begin{aligned}
\pi_0(t, \mathbf{p}) &:= e^{i(t-s)H_0(s)} \partial_s \varphi(s, \mathbf{p}) e^{-i(t-s)H_0(s)} = \\
&= (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \frac{1}{\sqrt{2\pi L}} e^{i(t-s)H_0(s)} \left[e^{-is\omega_{\mathbf{p}}} a_{-\mathbf{p}}(s) - e^{is\omega_{\mathbf{p}}} a_{\mathbf{p}}^*(s) \right] e^{-i(t-s)H_0(s)} = \\
&= \frac{\omega_{\mathbf{p}}}{\mathcal{E}_{\mathbf{p}}(s)} \frac{-i}{\sqrt{2\pi L}} \frac{\mathcal{E}_{\mathbf{p}}(s)}{(2\omega_{\mathbf{p}})^{1/2}} \left[e^{-it\mathcal{E}_{\mathbf{p}}(s) - is[\omega_{\mathbf{p}} - \mathcal{E}_{\mathbf{p}}(s)]} a_{-\mathbf{p}}(s) - \text{h. c.} \right] = \\
&= \frac{\omega_{\mathbf{p}}}{\mathcal{E}_{\mathbf{p}}(s)} \partial_t \varphi_{\mathbf{d}}(t, \mathbf{p}) = h_{\mathbf{p}}^{-2}(s) \partial_t \varphi_{\mathbf{d}}(t, \mathbf{p}) .
\end{aligned}$$

With this result we write the interaction picture potential $V_0(t)$ as

$$\begin{aligned}
V_0(t) &= \frac{L}{2} \sum_{\mathbf{p} \in \mathbf{Z}} \left\{ \frac{1 - h_{\mathbf{p}}^2(s)}{h_{\mathbf{p}}^4(s)} : \partial_t \varphi_{\mathbf{d}}^*(t, \mathbf{p}) \partial_t \varphi_0(t, \mathbf{p}) : + \right. \\
&\quad \left. + [\omega_{\mathbf{p}}^2 - g_{\mathbf{p}}^2(s)] : \varphi_{\mathbf{d}}^*(t, \mathbf{p}) \varphi_0(t, \mathbf{p}) : \right\} + L \frac{\lambda}{4!} \int_0^{2\pi} dx : \varphi_0^4(t, \mathbf{x}) : .
\end{aligned}$$

As before, we are now left the final task of re-ordering the potential with respect to the state w_s . Let us define

$$\begin{aligned}
\delta m^2(t, \mathbf{x}) &:= w_s [: \varphi_0(t, \mathbf{x}) \varphi_0(0) :_0] = \\
&= \frac{1}{2\pi L^2} \sum_{\mathbf{k}, \mathbf{p} \in \mathbf{Z}} \frac{1}{2(\omega_{\mathbf{k}} \omega_{\mathbf{p}})^{1/2}} \left\{ w_s [a_{\mathbf{k}}^*(s) a_{\mathbf{p}}(s)] e^{-i(\mathbf{k}, \mathbf{x})} + w_s [a_{\mathbf{p}}^*(s) a_{\mathbf{k}}(s)] e^{i(\mathbf{k}, \mathbf{x})} \right\} = \\
&= \frac{1}{2\pi L} \sum_{\mathbf{k}, \mathbf{p} \in \mathbf{Z}} \frac{1}{2(\omega_{\mathbf{k}} \omega_{\mathbf{p}})^{1/2}} n_{\mathbf{p}}(s) \delta(\mathbf{k}, \mathbf{p}) \left\{ e^{-i(\mathbf{k}, \mathbf{x})} + e^{i(\mathbf{k}, \mathbf{x})} \right\} = \\
&= \frac{1}{2\pi L} \sum_{\mathbf{p} \in \mathbf{Z}} \frac{n_{\mathbf{p}}(s)}{2\omega_{\mathbf{p}}} \left\{ e^{-i(\mathbf{p}, \mathbf{x})} + e^{i(\mathbf{p}, \mathbf{x})} \right\} = \\
&= \frac{1}{2\pi L} \int_{\mathbf{R}} dE \sum_{\mathbf{p} \in \mathbf{Z}} \frac{\mathcal{E}_{\mathbf{p}}(s)}{\omega_{\mathbf{p}}} n_{\mathbf{p}}(s) \delta(\mathcal{E}_{\mathbf{p}}^2(s) - E^2) e^{-i(\mathbf{p}, \mathbf{x})} ,
\end{aligned}$$

where we have used $n_{\mathbf{p}} = n_{-\mathbf{p}}$ again, we now use Wick re-ordering formula (A.1) for the quartic term in the potential and get

$$: \varphi_0^4(t, \mathbf{x}) :_0 = : \varphi_0^4(t, \mathbf{x}) :_s + 6 \delta m^2 : \varphi_0^2(t, \mathbf{x}) :_s + 3 [\delta m^2]^2 \mathbf{1} .$$

In the above expression

$$\delta m^2 := \delta m^2(0, 0) = \frac{1}{2\pi L} \sum_{\mathbf{p} \in \mathbf{Z}} \frac{n_{\mathbf{p}}(s)}{\omega_{\mathbf{p}}} . \quad (5.43)$$

We now observe that the re-ordering of the remaining (mass counter-)terms in the potential results in having additional terms proportional to the unity. Such terms would end up contributing a phase factor to the local scattering amplitudes \mathcal{S} which is cancelled in the product $\mathcal{S}^* \mathcal{S}$. Dropping all the c-number terms we hence rewrite the interaction potential as

$$\begin{aligned} V_0(t) = & \frac{L}{2} \sum_{\mathbf{p} \in \mathbf{Z}} \left\{ \frac{1 - h_{\mathbf{p}}^2(s)}{h_{\mathbf{p}}^4(s)} : \partial_t \varphi_0^*(t, \mathbf{p}) \partial_t \varphi_0(t, \mathbf{p}) :_s + \right. \\ & \left. + \left[\omega_{\mathbf{p}}^2 - g_{\mathbf{p}}^2(s) + \frac{\lambda}{4} \delta m^2 \right] : \varphi_0^*(t, \mathbf{p}) \varphi_0(t, \mathbf{p}) :_s \right\} + L \frac{\lambda}{4!} \int_0^{2\pi} d\mathbf{x} : \varphi_0^4(t, \mathbf{x}) :_s . \end{aligned} \quad (5.44)$$

From this expression it is clear that we will have two types of vertices appearing in our local S matrix, the usual 4-point interaction vertex and an additional (2-point) mass “counter term”. It is the latter term that is required in order to avoid certain divergencies in the (full) scattering amplitudes, as we now explain.

On-shell Feynman diagrams

At this point we need to assume we have taken the thermodynamic limit $L \rightarrow \infty$ and that such a limit is well defined and finite up to the considerations that we present here. In the thermodynamic limit, the momentum space representation of the Feynman propagator is

$$\tilde{\Delta}_F(E, \mathbf{p}) = \frac{\mathcal{E}(\mathbf{p}, s)}{\omega(\mathbf{p})} \frac{-i}{-E^2 + \mathcal{E}^2(\mathbf{p}, s) + i0} + \delta m^2(E, \mathbf{p}) , \quad (5.45)$$

and

$$\delta m^2(E, \mathbf{p}) = \frac{\mathcal{E}(\mathbf{p}, s)}{\omega(\mathbf{p})} n_{\mathbf{p}}(s) \delta(\mathcal{E}^2(\mathbf{p}, s) - E^2) .$$

Consider now the contribution to $\widetilde{\mathcal{M}}(2 \leftrightarrow 2)$ coming from Figure 5.1. Omitting some irrelevant numerical and phase factors, and recalling that $\underline{p} = (\mathcal{E}_{\mathbf{p}}(s), \mathbf{p})$ is the on-shell

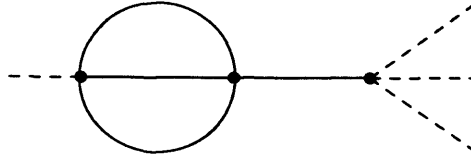


Figure 5.1: Divergent third order contribution to $\widetilde{\mathcal{M}}(2 \leftrightarrow 2)$. The dashed lines do not contribute and denote “external amputated legs”.

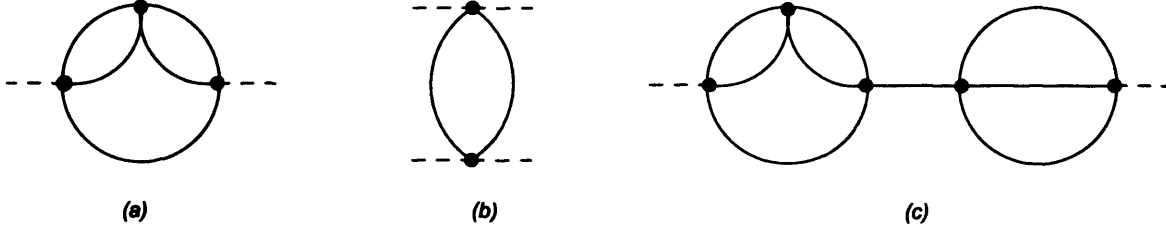


Figure 5.2: Examples of 1PI diagrams with two (a) and four (b) (amputated) external legs. The diagram in figure (c) is not 1PI.

momentum corresponding to the modified mass shell $\mathcal{E}_{\mathbf{p}}(s)$, we evaluate it as

$$\begin{aligned}
 & \int_{\mathbf{R}} d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2x d^2y d^2z e^{i\mathbf{p}_1x_1 + i\mathbf{p}_2x_2 - i\mathbf{q}_1y_1 - i\mathbf{q}_2y_2} \cdot \\
 & \quad \cdot \delta^{(2)}(x - x_1) \delta^{(2)}(y - x_2) \delta^{(2)}(y - y_1) \delta^{(2)}(y - y_2) \widetilde{\Delta}^3(x - z) \widetilde{\Delta}(y - z) = \\
 & = \int_{\mathbf{R}} d^2x d^2y d^2z \widetilde{\Delta}_F^3(x - z) \widetilde{\Delta}_F(y - z) e^{i\mathbf{p}_1x} e^{iy(\mathbf{p}_2 - \mathbf{q}_1 - \mathbf{q}_2)} = \\
 & = \delta^{(2)}(\underline{p}_1 + \underline{p}_2 - \underline{q}_1 - \underline{q}_2) \widetilde{\Delta}_F(\mathcal{E}(\mathbf{p}_1, s), \mathbf{p}_1) \int_{\mathbf{R}^2} d^2x \widetilde{\Delta}_F^3(x) e^{ix(\mathbf{p}_2 - \mathbf{q}_1 - \mathbf{q}_2)},
 \end{aligned}$$

which is divergent as can be seen by considering (5.45): Formally we have

$$\widetilde{\Delta}_F(\mathcal{E}(\mathbf{p}_1, s), \mathbf{p}_1) \propto \frac{-i}{-\mathcal{E}^2(\mathbf{p}_1, s) + \mathcal{E}^2(\mathbf{p}_1, s) + i0} + n_{\mathbf{p}}(s) \delta(\mathcal{E}^2(\mathbf{p}_1, s) - \mathcal{E}^2(\mathbf{p}_1, s)).$$

It is now clear that the divergence comes from an “on-shell Feynman propagator”. Similar divergencies occur in other diagrams. However, if we sum up all diagrams, then we show that there exists a choice of $\mathcal{E}(\mathbf{p}, s)$ such that these divergencies cancel. In fact, this $\mathcal{E}(\mathbf{p}, s)$ is defined by equation (5.47).

To do this let us introduce some terminology. We call a diagram “one particle irreducible” (or 1PI) if it is connected and cannot be disconnected by cutting through any *one* internal propagator, see Figure 5.2, and we will denote by $V_2^i(p; s)$ ($V_4^i(p; s)$) a generic 1PI diagram with two (four) external legs. Let us then introduce the 2-point vertex function, $V_2(p; s) = V_2(E, \mathbf{p}; s)$, as the sum of all 1PI diagrams with two (amputated)

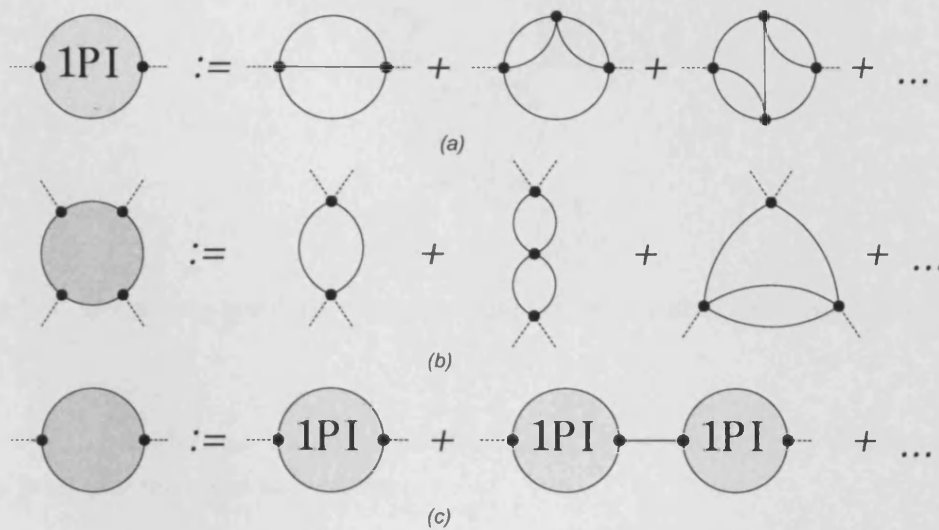


Figure 5.3: Graphical representation of (a) the 2-point vertex function $V_2(p)$, (b) the 4-point vertex function $V_4(p_1, p_2, q_1, q_2)$ and (c) of the re-summed 2-point vertex function $\mathcal{V}_2(p)$.

external legs. Similarly, we define the 4-point vertex function, $V_4(p_1, p_2, q_1, q_2; s)$, as the sum of all 1PI diagrams with 4 (amputated) external legs. Dropping all the dependencies on the time s in what follows, we schematically write

$$V_2(p) = \sum_i V_2^i(p), \quad V_4(p) = \sum_i V_4^i(p).$$

Finally we also define the “re-summed 2-point vertex function” $\mathcal{V}_2(p)$ by the geometric series

$$\mathcal{V}_2(p) := V_2(p) + V_2(p)\tilde{\Delta}_F(p)V_2(p) + \dots = V_2(p) \sum_{n=0}^{\infty} \left(V_2(p)\tilde{\Delta}_F(p) \right)^n = \frac{V_2(p)}{1 - V_2(p)\tilde{\Delta}_F(p)}.$$

All these objects can be graphically represented as in Figure 5.3.

Having identified an on-shell propagator as the origin of the divergence, we immediately see that all the terms of the form

$$V_4^i(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2)\tilde{\Delta}_F(\underline{p}_1)V_2^j(\underline{p}_1),$$

will be divergent as well. Not only, it is also clear that there will be similar divergent contributions for each of the external four momenta \underline{p}_1 , \underline{p}_2 , \underline{q}_1 and \underline{q}_2 . This also clears the statement that the problem is not specific to $\tilde{\mathcal{M}}(2 \leftrightarrow 2)$ or to some 1PI graph with two external legs in particular: It is a generic feature of $\tilde{\mathcal{M}}(|X| \leftrightarrow |Y|)$.

To have a more compact expression of the above mentioned divergent contributions

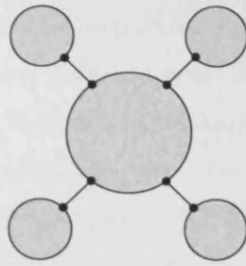


Figure 5.4: Schematic graphical representation of the on-shell propagator divergences in $\widetilde{\mathcal{M}}(2 \leftrightarrow 2)$.

to $\widetilde{\mathcal{M}}(2 \leftrightarrow 2)$ we perform a formal resummation of both all the 1PI diagrams with two and with four external legs and write

$$\sum_{i,j} V_4^i(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \widetilde{\Delta}_F(\underline{p}_1) V_2^j(\underline{p}_1) = V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \widetilde{\Delta}_F(\underline{p}_1) V_2(\underline{p}_1), \quad (5.46)$$

where the sum is over all 1PI diagrams with four and two external legs. And again there will be one such term for each of the four momenta in V_4 . Next we note that (5.46) does not account for all the divergences. Another set of divergent contributions will come from terms containing two on-shell propagators, which after a formal resummation can be written as

$$V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \widetilde{\Delta}(\underline{p}_1) V_2(\underline{p}_1) \widetilde{\Delta}(\underline{p}_1) V_2(\underline{p}_1).$$

We note that the above term formally represents infinite divergent terms, each of which has two on-shell propagators. In general, there will be similar terms containing n on-shell propagators with $n = 1, 2, \dots$. We again formally re-sum all these terms and obtain a very compact expression for the divergent contributions to $\widetilde{\mathcal{M}}(2 \leftrightarrow 2)$ that arise because of on-shell propagators and are “attached” to one of the four external momenta. We have

$$V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \sum_{n=1}^{\infty} \left(\widetilde{\Delta}(\underline{p}_1) V_2(\underline{p}_1) \right)^n = V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \widetilde{\Delta}(\underline{p}_1) \mathcal{V}_2(\underline{p}_1).$$

Considering now the contributions coming from all the four momenta we get the final expression representing *all* the divergent contributions to $\widetilde{\mathcal{M}}(2 \leftrightarrow 2)$ arising because of the quartic interaction vertex. It is

$$V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \left[\widetilde{\Delta}(\underline{p}_1) \mathcal{V}_2(\underline{p}_1) + \widetilde{\Delta}(\underline{p}_2) \mathcal{V}_2(\underline{p}_2) + \widetilde{\Delta}(\underline{q}_1) \mathcal{V}_2(\underline{q}_1) + \widetilde{\Delta}(\underline{q}_2) \mathcal{V}_2(\underline{q}_2) \right],$$

and can be graphically represented as in Figure 5.4. We again note that the above expression is a formal object in that it represents a divergent quantity, but we will

use it to better understand the overall structure of $\widetilde{\mathcal{M}}(2 \leftrightarrow 2)$ and eventually derive an equation for $\mathcal{E}(\mathbf{p}, s)$ which, once solved, will lead to the cancellation of all such divergences order by order in the coupling constant. We also observe that the above four contributions are all formally identical and from now we only concentrate on one.

Up to this point we have only been concerned with the contribution to the scattering amplitude due to the quartic term in the interaction polynomial and we now turn our attention to the remaining mass (counter) term. To this term will correspond a vertex with two propagators attached to it. The resulting matrix elements will then be constructed out of the Feynman propagators $\widetilde{\Delta}_F(E, \mathbf{p})$ in equation (5.45), but the rationale behind the construction of the diagrams will be the same as before. For instance, to the diagram in Figure 5.1 will correspond an identical one with the “bubble” replaced by a single “mass vertex”, whose contribution in momentum space can be immediately read off the thermodynamic limit of (5.44) to be

$$\delta M^2(E, \mathbf{p}) = E^2 \frac{1 - h^2(\mathbf{p}, s)}{h^4(\mathbf{p}, s)} + \omega^2(\mathbf{p}) - g^2(\mathbf{p}, s) + \frac{\lambda}{4} \delta m^2,$$

where with a minor abuse of notation we have denoted by δm^2 the thermodynamic limit of (5.43), i.e.

$$\delta m^2 = \int_{\mathbb{R}} \frac{d\mathbf{p}}{2\pi} \frac{n_{\mathbf{p}}(s)}{\omega(\mathbf{p})}.$$

We hence note that the previous discussion on divergencies arising due to on-shell Feynman diagrams can be repeated for the mass vertex too, hence (apparently) giving another divergent contribution to the matrix element. But it is precisely this term that will, in fact, be “adjusted” so that the divergencies cancel exactly.

From this it follows that (5.46) should be modified to take into account the presence of quadratic term in the interaction potential. Defining the propagator “self energy” as

$$\Pi(E, \mathbf{p}) := V_2(E, \mathbf{p}) + \delta M^2(E, \mathbf{p}),$$

we have that the appropriate modification to (5.46) is simply

$$V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \widetilde{\Delta}(\underline{p}_1) \left(V_2(\underline{p}_1) + \delta M^2(\underline{p}_1) \right) = V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \widetilde{\Delta}(\underline{p}_1) \Pi(\underline{p}_1).$$

Formally, the situation is now identical to the standard vacuum case and we can again

re-sum all the self-energy contributions to obtain

$$V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \sum_{n=1}^{\infty} \left(\tilde{\Delta}(\underline{p}_1) \Pi(\underline{p}_1) \right)^n = V_4(\underline{p}_1, \underline{p}_2, \underline{q}_1, \underline{q}_2) \frac{\tilde{\Delta}(\underline{p}_1) \Pi(\underline{p}_1)}{1 - \tilde{\Delta}(\underline{p}_1) \Pi(\underline{p}_1)},$$

from which we see that if $\Pi(\underline{p}) = \Pi(\mathcal{E}(\mathbf{p}, s), \mathbf{p}) = 0$ then indeed all the on-shell propagator divergences cancel. And this is precisely our self-consistency/renormalization condition that we use to (mass) renormalize the theory, determine $\mathcal{E}(\mathbf{p}, s)$ and to complete our Feynman rules. We write it as

$$\Pi(E, \mathbf{p}) \Big|_{E=\mathcal{E}(\mathbf{p}, s)} = 0. \quad (5.47)$$

Remark. We want to emphasize at this point that we call $\Pi(E, \mathbf{p})$ “self-energy” only because of its formal similarity with the self-energy of the full propagator in the vacuum theory. That is, our “self energy” has the same *diagrammatic* expression as the perturbatively computed self energy in the vacuum theory. But we also want to point out that aside from the obvious difference between the two arising because they are to be computed with different propagators, a fundamental difference between the two is that in the vacuum theory the vanishing of the self energy is justified by the *known* analytic properties of the full non-perturbative propagator, see [83, 84], whereas in our case we are not even willing to conjecture this fact.

We now have an equation that will, in principle, allow us to find the modified mass shell $\mathcal{E}(\mathbf{p}, s)$ to all orders in the coupling constant. A quick inspection of the above condition straightforwardly gives us the “first order” equation for $\mathcal{E}(\mathbf{p}, s)$. In fact, to order $\mathcal{O}(\lambda)$ equation (5.47) is simply

$$\Pi(\mathcal{E}(\mathbf{p}, s), \mathbf{p}) = \delta M^2(\mathcal{E}(\mathbf{p}, s), \mathbf{p}) = 2\omega(\mathbf{p}) \mathcal{E}(\mathbf{p}, s) + 2\omega^2(\mathbf{p}) + \frac{\lambda}{4} \delta m^2 = 0,$$

which yields

$$\mathcal{E}(\mathbf{p}, s) = \omega(\mathbf{p}) - \frac{\lambda}{8\omega(\mathbf{p})} \delta m^2(s) + \mathcal{O}(\lambda^2) = \omega(\mathbf{p}) - \frac{\lambda}{8\omega(\mathbf{p})} \int_{\mathbb{R}} \frac{d\mathbf{k}}{2\pi} \frac{n_{\mathbf{k}}(s)}{\omega(\mathbf{k})} + \mathcal{O}(\lambda^2). \quad (5.48)$$

An immediate observation about the above result is that our “modified” ansatz, i.e. the assumption that $\mathcal{E}(\mathbf{p}, s) = \omega(\mathbf{p}) + \mathcal{O}(\lambda)$ is verified—our assumption was indeed “self consistent”. In turn, this means that our split of the Hamiltonian as summarized by equations (5.40) and (5.41) results in an interacting potential of order $\mathcal{O}(\lambda)$, so perturba-

tion theory is applicable. Also, the above result is telling us that the energy of a particle is *reduced* with respect to the vacuum case when it is propagating in a surrounding medium. Stated differently, for a given spatial momentum \mathbf{p} , the energy of a particle will be maximal in the vacuum. We find this result particularly interesting.

Additional insight into the modified mass shell $\mathcal{E}(\mathbf{p}, s)$ is given by the truncating equation (5.47) to second order. The resulting equation is

$$\begin{aligned} \mathcal{E}(\mathbf{p}, s) = & \omega(\mathbf{p}) - \frac{\lambda}{8\omega(\mathbf{p})} \int_{\mathbf{R}} \frac{d\mathbf{k}}{2\pi} \frac{n_{\mathbf{k}}(s)}{\omega(\mathbf{k})} + \\ & - \frac{\lambda^2}{2\omega(\mathbf{p})} \int_{\mathbf{R}} \frac{d^2k}{(2\pi)^2} \frac{d^2q}{(2\pi)^2} \tilde{\Delta}_F(E_k - E_q - \mathcal{E}(\mathbf{p}, s), \mathbf{k} - \mathbf{q} - \mathbf{p}) \tilde{\Delta}_F(E_k, \mathbf{k}) \tilde{\Delta}_F(E_q, \mathbf{q}) . \end{aligned}$$

What is immediately clear from this formula is that it is a highly non trivial equation. A reasonable attempt to solving it is by means of an iteration procedure, the first step of which is to use equation (5.48) on the right hand side of the above equation. At any rate, however, the full determination not only of $\mathcal{E}(\mathbf{p}, s)$ but of the Feynman propagator (5.45) itself requires the knowledge of $n_{\mathbf{p}}(t)$, that is of the very quantity we are solving the Boltzmann equation for. It is worth noting though that in principle we have a closed set of equations, and on physical grounds we expect it to have a (unique) solution. The mathematical proof of this statement, however, appears highly non-trivial and has not been addressed at all in this work.

5.4.2 A hierarchy of equations—work in progress

In many physical applications the BE is used by computing the scattering amplitudes beyond the Born approximation, see [5] for an application of the BE to baryogenesis. In this latter case this is not done to increase the accuracy of the prediction as it is a necessary feature to obtain any results at all. In other words, the standard out-of-equilibrium decay mechanism for baryogenesis is invisible at tree level. Because of this, one would like to justify, even formally, the use of the full scattering amplitude in the BE as opposed to the Born approximation “only”. And the results from the previous section show that this ought to be possible as once the existence of the dressed scattering amplitudes has been established, the corrections will come precisely in the form of the full amplitude, as in the low density case. Clearly, however, these will be sub-leading corrections to the Born approximation and here we very briefly present an idea on how one could proceed to establish such a result. Note also that this section should be considered as “work in progress”, which nevertheless offers an interesting perspective on

what we could expect to obtain.

Remember that in the weak coupling case we assumed the particle number densities $n_{\mathbf{p}}(t)$ possess a finite limit $f_{\mathbf{p}}(T)$ under the scalings (5.34). Consistently with these assumptions, we could consider an expansion of $n_{\mathbf{p}}(t)$ in terms of the coupling constant λ , i.e. we could write

$$n_{\mathbf{p}}(t) = f_{\mathbf{p}}(T) + \lambda f_{\mathbf{p}}^{(1)}(T) + \lambda^2 f_{\mathbf{p}}^{(2)}(T) + \dots .$$

This expression then ought to be used together with the pre-Boltzmann equation (4.46) to obtain a hierarchy of equations that could be used to compute sub-leading corrections to equation (5.36). We do not present the results of this calculation but emphasize on the fact that now the Boltzmann collision factor $\mathcal{B}_{\lambda}(\mathbf{p}, T)$ will be *different* from the corresponding one in equation (5.35). Specifically, we now expect the collision factor to be

$$\mathcal{B}_{\lambda}(\mathbf{p}, T) = \frac{1}{2} \sum_{\substack{i \in X \\ j \in Y}} |X|! |Y|! |X| \int_{\mathbb{R}} d\Pi_{X^1 \cup Y^q} (2\pi)^2 \delta^{(2)}(\underline{p} + \underline{q}_{X^1} - \underline{q}_Y) \left| \widetilde{\mathcal{M}}(X_p \leftrightarrow Y_q) \right|^2 \\ \left\{ \left(\prod_{j \in Y} f_{\mathbf{q}_j}(T) \right) (1 + f_{\mathbf{p}}(T)) \prod_{i \in X^1} (1 + f_{\mathbf{p}_i}(T)) - f_{\mathbf{p}}(T) \prod_{i \in X^1} f_{\mathbf{p}_i}(T) \prod_{j \in Y} (1 + f_{\mathbf{q}_j}(T)) \right\} ,$$

where we are using the same notation as previously, see equation (5.35). Comparing the above expression with equation (5.35) one could note two differences. The main one is the emergence of the full scattering amplitude $\widetilde{\mathcal{M}}$ as opposed to its Born approximation $i\lambda$. The second one is closely related to this fact and is the fact that the sum over all the scattering processes is now *not* restricted to $|X| + |Y| = p$, with p the degree of the interaction polynomial. This is easy to understand. Since the scattering amplitude will now include higher order contributions, we will have creation of particles also because of the lower degree interaction vertices. Let us be more specific. In the Born approximation, a process like $2 \rightarrow 4$, i.e. the creation of 2 (additional) particles after a collision of 2 will occur because of a 6th degree interaction vertex. The same interaction vertex will be responsible for the $3 \rightarrow 3$ and $4 \rightarrow 2$ processes. By allowing higher order corrections to the scattering amplitudes, on the other hand, these processes can, and will, occur because of the quartic interaction vertex. In particular, the $2 \rightarrow 4$, $3 \rightarrow 3$ and $4 \rightarrow 2$ are second order (in λ) processes for the φ^4 interacting term. And they do not involve any loops either. It is then clear that if we account for *all* orders in perturbation theory, the sum over the (physical) processes will not be restricted. And this remains true even if one considers the quartic interaction only! Another remark about the formal structure of

the “new” collision factor is as follows. One might wonder if the calculations (5.38) and (5.39) are still valid in this case as the scattering amplitude would now appear in this calculations. The answer is affirmative and the reason is that our model is PT-invariant, which ensures that $\widetilde{\mathcal{M}}(X_p \rightarrow Y_q) = \widetilde{\mathcal{M}}(Y_q \rightarrow X_p)$. Last but certainly not least, we should say that we also expect higher order collision factors $\mathcal{B}_\lambda^{(1)}(\mathbf{p}, T), \mathcal{B}_\lambda^{(2)}(\mathbf{p}, T), \dots$ to emerge from this approach, but we have not explored it that far.

It now seems appropriate to present the Feynman rules we expect to be part of the “practical” conclusions of the this (work in progress) analysis. They are written here for reading convenience and are a straightforward adaptation of the usual Feynman rules one could find in quantum field theory textbooks, like e.g. [32]. Without discussing the rules at all, and for the quartic interaction only, we have:

1. Draw all possible *topologically inequivalent* Feynman diagrams for a specific collision process.
2. To each *external* line associate its own 2-momentum corresponding to the specific particle.
3. Each vertex connects four lines and momentum conservation is imposed at it.
4. To each *internal* line associate a “dressed propagator”

$$\widetilde{\Delta}_F(E, \mathbf{p}) = \frac{\mathcal{E}(\mathbf{p}, s)}{\omega(\mathbf{p})} \left[\frac{-i}{-E^2 + \mathcal{E}^2(\mathbf{p}, s) + i0} + n_{\mathbf{p}}(s) \delta(\mathcal{E}^2(\mathbf{p}, s) - E^2) \right].$$

5. To each vertex associate a factor $(i\lambda)$.
6. A diagram with L loops will have L undetermined momenta k_i , $i = 1, \dots, L$. Integrate over each of these momenta with measure $d^2k_i/(2\pi)^2$.
7. A symmetry factor is associated to each diagram (see [32] for a discussion of symmetry factors).
8. The scattering amplitude is given by the sum of all diagrams computed by the previous rules.

In the above rules it is also assumed that mass renormalization has been carried over. If this is not so, one should include the “mass counter terms”. The rules for the full polynomial interaction are easily generalized, the only change being that there will now be different vertices corresponding to the various powers of the interaction polynomial. Clearly, each vertex ought to be weighted by the appropriate coupling constant.

As stated previously, these results are just the tip of the iceberg of what one might expect to obtain in pushing the study of the derivation beyond the weak coupling limit. We have not done much about it and think it would certainly be interesting to do so, not only from a conceptual point of view. As the Feynman rules above show, there are some practical novelties such an analysis would offer and it certainly is of interest to see if any other modifications to the “textbook” Boltzmann equation would emerge.

Chapter 6

Conclusions

In this thesis we have investigated various conceptual issues related with the Boltzmann equation (BE). The BE has originally been obtained in a heuristic way as an attempt to better understand non-equilibrium processes, and despite its remarkable mathematical complexity, it has been widely employed in “practical” physical applications. Because of its experimental success it has been generalized in order to account for the quantum mechanical aspects of microscopic phenomena as well as to be able to use it in a (general) relativistic setting. Currently, the BE is employed in a wide range of applications, including early Universe theories like Nucleosynthesis and Baryogenesis¹.

It is perhaps precisely this experimental success of the BE that prompted us, amongst many others, to try and understand its conceptual status better. The original derivation, in fact, was not free from dubious and non-trivial assumptions. Amongst them, most notably, the so called “Stosszahlansatz”, or assumption of “molecular chaos”. According to this assumption, every collision between particles is statistically un-correlated to any other collision, and this is assumed to hold at every single instant of time. (Mathematically this translates into a factorization of the multi-particle distribution function into a product of single particle distribution functions.) The conceptual problem with such an assumption lies in the fact that it was assumed to be valid at every instant of time. Stated otherwise, it is conceptually perfectly acceptable to assume the factorization property at some initial time t_0 , say. But whether this condition will be “replicated” at later times depends, in principle, on the underlying dynamics. Intuitively, in a very dense and strongly interacting system where a collision would occur “almost at the same time as the previous one”, one would certainly expect the two collisions to be (strongly) correlated, and the Stosszahlansatz violated. Clearly then the BE will not provide a

¹Nucleosynthesis is the theory that aims to account for the abundance of the lightest elements in the Universe, while Baryogenesis is the theory with the goal of explaining the observed matter–anti-matter asymmetry in the Universe.

valid description of the system. From this also follows another major conceptual issue with the BE, i.e. unlike the Heisenberg (or Schrödinger) equation, it is not a fundamental, or “exact”, equation of physics. Not only, its relation to the Heisenberg equation (HE) was also not entirely clear, in the sense that it was not clear to what extent could the BE be derived from the HE, and hence be understood as a particular approximation to the latter.

These, essentially, were the concerns we have raised and studied in this thesis. Specifically, with this work we have questioned the validity of the Boltzmann equation. We wanted to understand whether the BE can be obtained, rigorously, starting from physically accepted first principles, i.e. Quantum Field Theory on Minkowski space.

To attack the problem we have employed the “Projection Operator Method”. This technique allowed us to “decompose” the equations of motion, i.e. Heisenberg’s equation, and obtain what is known as the “Robertson equations” for the number operator densities. This was done for a specific model, the real scalar quantum field with polynomial (self-)interaction on the two dimensional cylinder $\mathbb{R} \times S$. The Robertson equation allowed us to obtain what we called the “pre-Boltzmann equation”. The non-perturbative existence of the latter was proved², hence establishing the equivalence of the pre-Boltzmann equation with the HE. The latter is hardly any simpler to solve exactly than the original HE, but it is a much better starting point to study some of its limits. In particular we were interested in the long-time limit of the HE.

After the pre-Boltzmann equation has been obtained, we proceeded to consider its perturbative expansion. Moreover, as we were interested in understanding the Boltzmann equation on Minkowski space, we also considered the “thermodynamic limit”, i.e. we formally considered the $L \rightarrow \infty$ limit, with L being the radius of the cylinder. The next step was to consider the long-time limit of the perturbative expansion of the pre-Boltzmann equation, now formally considered on Minkowski space.

Previous work on the Boltzmann equation suggested that the long-time limit by itself should not be enough: In order to obtain a meaningful limit, one ought to additionally consider either an increasingly dilute medium or a weakly interacting one. Specifically, for a dilute gas we rescale the time t and the particle number densities $f_{\mathbf{p}}(t)$ as $t \rightarrow T/\epsilon$ and $f_{\mathbf{p}} \rightarrow \epsilon^\alpha f_{\mathbf{p}}$, where the exponent α had to be $\alpha = 1$ if the result was to be finite. On the other hand, the weak coupling limit was considered by rescaling $t \rightarrow T/\epsilon$, and $\lambda \rightarrow \sqrt{\epsilon} \lambda$. Clearly, in both cases the non-trivial assumption was that $f_{\mathbf{p}}$ had a meaningful limit. But by making this assumption, we were able to derive the Boltzmann equation in both cases. The two cases exhibited a number of differences, and they

²The non-perturbative existence of the pre-Boltzmann equation relies heavily on the estimates for the number densities we presented in Section §4.2.1, and was established by my research advisor.

confirmed the intuitive expectations we had about the two cases. Most notably, in the low density limit only 2-particle scattering processes should be considered and the scattering amplitude naturally appears to all orders in the coupling constant. In the weakly interacting case, on the other hand, the scattering amplitude appears only in the Born approximation, i.e. at “tree level”, while all possible particle scattering processes enter the Boltzmann equation. We should remark however, that in the latter case we obtain (sub-leading) corrections to the scattering amplitude, and these come precisely in the form of the full scattering amplitude.

A physically very interesting issue arises in the case of the weak coupling limit, and it is related to the “mass renormalization” of the scattering amplitudes. From a mathematical point of view, the (perturbative) scattering amplitudes are divergent unless some “renormalization” procedure is applied. And the naive use of the usual mass renormalization one encounters in the vacuum case is not sufficient because the scattering amplitudes in this case are not Lorentz invariant. This is not a surprise as we considered a generic state, which in general is not Lorentz invariant. This problem appears to be mathematically extremely challenging and is still open. Our contribution to its solution came by means of a “self consistent” working assumption motivated by the physical understanding of the problem. Very briefly, in a generic state the dispersion relation of a particle, i.e. its energy, will not be its vacuum one. We hence made precisely the assumption that the particles’ energy $\mathcal{E}(\mathbf{p})$ was not the vacuum one, i.e. $\mathcal{E}(\mathbf{p}) \neq \omega(\mathbf{p})$. In addition, we considered a different perturbative expansion that depended on $\mathcal{E}(\mathbf{p})$. This allowed us to impose a condition, i.e. the vanishing of the “perturbative self-energy” on the (modified) mass shell, that effectively “removes” the divergencies, or better, it ensures there are no divergences in the first place. In addition, the vanishing of the perturbative self energy allows us to determine the particles’ energy, and we find that it depends on $f_{\mathbf{p}}(t)$, that is, it depends on the very quantity one would like to solve the Boltzmann equation for. This “circularity” is typical of self-consistent working assumptions.

We also want to remark that the Boltzmann collision factor, i.e. the right hand side of the textbook BE, is not the only term that we obtain on the RHS of our equation, even in the long-time limit. We do, in fact, obtain (non-Markovian) correction terms. These are “remnants” from the pre-Boltzmann equation and are associated with the rescattering of particles. Intuitively one could understand the emergence of these terms by thinking that as we consider longer and longer time intervals, particle rescattering will necessarily become a competing effect, albeit smaller. Also, when such multiple collisions are taken into account, it is not too surprising that the resulting terms are non-Markovian, i.e. that

the entire history of the system ought to be considered. Hence, strictly speaking, we do not obtain the Boltzmann equation, but a more general equation, which contains the Boltzmann collision factor as it first, Markovian, term. In this sense, our result is more general than the BE in that it holds for dilute systems and/or weakly interacting ones even when multiple scattering cannot be neglected.

Finally, in considering the two previously mentioned limits, we have cleared the conceptual status of the BE, in the sense that we now understand it as a “long-time” approximation of the Heisenberg equation. The role of the Stosszahlansatz is then “hidden” in this long-time limit, which encodes a very specific physical meaning. Together with considerations on the particle density and/or strength of the interaction between the particles, we understand the Stosszahlansatz as an assumption that forces (by hand) the system to “forget about its past” *and* “neglect multiple scattering”. To impose the Stosszahlansatz would then amount to drop the (non-Markovian) rescattering terms in our scaling limits of the pre-Boltzmann equation. We should say that these are not entirely new results as the scaling limits are known in the literature, but they have not been established in QFT as of yet.

Before concluding this thesis we want to point out some deficiencies in our treatment and open issues that certainly justify additional work on the Boltzmann equation. First of all, whereas the pre-Boltzmann equation holds in a non-perturbative sense, the Boltzmann equation as we establish it, is only valid perturbatively. This is clearly not entirely satisfactory and ideally one would like to establish such a result non-perturbatively. Furthermore, the perturbative derivation was not entirely rigorous, even by disregarding the non-convergence of the perturbative expansion of the scattering amplitudes. Some of the manipulations were only formal and although we believe that a rigorous treatment would not alter the end result, it would still be interesting to verify that this is indeed so.

The previous remarks on the validity of the perturbative expansion really only apply for the low density scaling of the pre-Boltzmann equation. In the weak coupling limit the derivation is, in fact, much more formal than the corresponding one for the low density limit. Intuitively, the reason is that in a low density limit, the physical state is “similar to the vacuum” precisely because of the extreme diluteness of the particles. On the other hand, a weakly interacting system might not be like the vacuum at all. Mathematically this is reflected in the fact that in the low density limit all the quantities are the vacuum ones, i.e. the Feynman propagator is the vacuum propagator and so are the scattering amplitudes. And (perturbative) quantum field theory has mostly been developed in the vacuum for which many results have been also rigorously justified from

non-perturbative investigations, like the singularity structure of the (full) propagator which justifies the usual mass renormalization prescription. In a generic case, on the other hand, little is known and it should always be dealt with on a case-by-case basis. In our case, these issues enter through the presence of the dressed propagator and the corresponding dressed amplitudes. The latter exhibit the same type of divergence that requires mass renormalization in the vacuum case, but now the well known procedure cannot be applied. Our self-consistent solution, which mimics the vacuum situation, although very reasonable, lacks any sort of justification other than “results justifying the means”. A particularly interesting aspect of our solution is the meaning, if any, of the vanishing of the self energy on the (modified) mass-shell. This would require a deeper, non-perturbative understanding of the full propagator of our theory.

We also want to point out here that our solution to the mass renormalization problem represents only a “guideline” of what a rigorous result could be. The main reason for this is that the very assumption of a functional relation between a particle’s momentum and its energy *cannot* be made for an interacting model. That is, by assuming such a functional relation, we are inevitably led to a free field. This is known to be the case for thermal states, which are relatively “simple” states, and it would certainly be a big surprise if in a generic state the situation would simplify. One possibility then is that the energy and momentum of a particle are strongly correlated with the correlation distribution being narrowly peaked around some dispersion relation $\mathcal{E}(\mathbf{p})$. It is, however, clear that to prove such a result a formidable amount of work would be required and we have not even attempted to do so.

Another aspect that is worth of additional investigation is the fact that our treatment does not allow for a spatial dependence of the particle number density $f_{\mathbf{p}}$. The reason for this is easily traced to our definition of the number density operator, which is in no sense a local quantity. We know, however, that the textbook Boltzmann equation postulates the existence of a particle number density that depends on space as well as on time and on momenta. So a treatment that would allow for a spatially dependent particle number density would clearly be desirable. One such result has been obtained by Buchholtz in [86] where he establishes the validity of the collisionless Boltzmann equation for a massless scalar quantum field within the framework laid down in [87] for the analysis of non-equilibrium states in QFT. So one natural attempt would be to try and employ the local thermal equilibrium states of the latter work together with the projection operator method we have so successfully employed in this work.

Finally, another topic that is certainly worth of additional investigations is the extension of our results to a cosmological setting, namely to Robertson–Walker spaces. We

have, in fact, appropriately modified the projection operator method so that it could be applied to a time dependent case, i.e. to the case where the Hamiltonian depends on time. For the case of RW spaces we expect three terms emerging as a result: the Boltzmann collision term as we have obtained it in this work (clearly appropriately modified), a term that would describe Bogoliubov transformation type creation and annihilation of particles, and an additional term to account for the expansion of the Universe. Such a result might then be applied to those cosmological investigations where the Boltzmann equation is a crucial tool, like the Nucleosynthesis and Baryogenesis theories. Given the expected modifications to the usual Boltzmann equation, it would be interesting to see if existing quantitative prediction would be modified by these results.

To conclude, we want to note how this work, despite having elucidated the status of the BE within QFT to a large extent, it is far from being a concluding answer to all the questions the BE raises. As we have outlined above, the questions left open with this work alone are many and varied. Needless to say, the BE will clearly continue to attract a lot of intellectual effort. Hugenholtz's words come to mind once again, and we hope that in the future we can further reinforce the validity of that claim.

Appendix A

Wick reordering for normal ordered products

In Chapter 3 we introduced the notion of “normal ordering”. We explained that a normal ordered product of fields is one where all the creation operators have been moved to the left of the annihilation operators. An important point of such a normal ordering prescription, we noted, is that its expectation value in the (free theory) vacuum state is identically zero. In addition, this makes the evaluation of vacuum expectation values products of (free) fields very simple. (For this reason it is more appropriate to talk about “normal ordering with respect to the vacuum state”.) On the other hand, in Chapter 5 we are confronted with the task of evaluating products of normal ordered fields in the states w_s introduced in Chapter 4. Because of this, we find it convenient to “normal re-order” (or Wick re-order) said products of fields. For this reason we give here an alternative, but equivalent, formulation of the idea of normal ordering for free fields, in terms of a “subtraction procedure”. It should be noted that this Appendix is not a complete review of the so-called “Wick calculus”, i.e. of the computational rules to deal with normal ordered products of fields. Our main theorem is Theorem A.1 below, which we present as we need it in the main body of this work.

Following [88] we define the normal ordered prescription for the product of n (free) fields as follows. First we define a “quasi-free” state ω as a state that is completely determined by its 2-point function (i.e. the expectation value of a product of two free fields). More precisely, if $\varphi_0(x)$ is a free quantum field, then a quasi-free state satisfies

(by definition)

$$\omega[\varphi_0(x_1) \cdots \varphi_0(x_n)] = \begin{cases} \sum_{\sigma} \prod_{i=1}^j \omega[\varphi_0(x_{\sigma(i)}) \varphi_0(x_{\sigma(i+j)})] & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases},$$

in the sense of distributions. In the above expression the sum is over all permutations σ of $\{1, \dots, n\}$ with $\sigma(1) < \sigma(2) < \dots < \sigma(j)$ and $\sigma(i) < \sigma(i+j)$, $i = 1, \dots, j$. In other words, for a quasi-free state, once we know its 2-point function, all the remaining n -point functions are known through the above formula.

We call $f_{\mathbf{p}}$, $\mathbf{p} \in \mathbf{Z}$ a “terminating sequence” if it consists of at most a finite number of non-zero elements. We then define $\tilde{\mathcal{F}}_0 \subset \mathcal{H}$, to be the subset of the bosonic Fock space (3.13) that contains at most a finite number of components in the direct sum (3.13), each of which is a tensor product of terminating sequences. We now have the following

Definition (Normal ordering). *Let ω be a quasi-free state. A normal ordering prescription $: \cdot :_{\omega}$ (with respect to ω) for the operator valued distribution $\varphi(x)$ is given by the following recursion relation:*

$$\begin{aligned} : \varphi_0(x) :_{\omega} &= \varphi_0(x), \\ : \varphi_0(x_1) \cdots \varphi_0(x_{n+1}) :_{\omega} &= : \varphi_0(x_1) \cdots \varphi_0(x_n) :_{\omega} \varphi_0(x_{n+1}) + \\ &\quad - \sum_{l=1}^n : \varphi_0(x_1) \cdots \widehat{\varphi_0(x_l)} \cdots \varphi_0(x_n) :_{\omega} \omega[\varphi_0(x_l) \varphi_0(x_{n+1})], \end{aligned}$$

where the $\widehat{\varphi_0(x_l)}$ symbol means omitting the corresponding element.

The first thing we want to remark now is that normal ordered products and Wick powers (see below) are well defined operator valued distributions on $\tilde{\mathcal{F}}_0$. We then note that the expectation value of a normal ordered product of fields has vanishing expectation value in the state with respect it has been normal ordered. In other words $\omega[:\Phi:_{\omega}] = 0$, where Φ denotes a generic product of fields. This easily follows from the above definition. Second, the normal ordered product $: \varphi(x_1) \cdots \varphi(x_n) :_{\omega}$ of n field is a *smooth* function of the variables x_1, \dots, x_n , see [43]. That is to say, the “Wick powers” (or “Wick monomials”)

$$:\varphi_0^n(x):_{\omega} := \lim_{x_1, \dots, x_n \rightarrow x} : \varphi_0(x_1) \cdots \varphi_0(x_n) :_{\omega},$$

are well defined operator valued distributions on $\tilde{\mathcal{F}}_0$.

As an example of the above definitions consider the normal ordering of $\varphi(x)\varphi(y)$

with respect to the vacuum state of a free scalar theory. We have

$$:\varphi_0(x)\varphi_0(y):_0 = \varphi_0(x)\varphi_0(y) - \langle \varphi_0(x)\varphi_0(y) \rangle_0 .$$

In this example it is immediate to see that the above expression is equivalent to moving all the creation operators to the left. In particular, the “Wick square” [cf. equation (3.11)] is now given by

$$:\varphi_0^2(x):_0 = \lim_{y \rightarrow x} \left[\varphi_0(x)\varphi_0(y) - \langle \varphi_0(x)\varphi_0(y) \rangle_0 \right] .$$

While the limit of the individual terms on the RHS of the above expression is ill defined, the limit of the difference is well defined. The reason for this is that both terms have “the same singularity structure”.

Elaborating the above simple example some more, we see that given two quasi-free states ω and ψ we can write

$$\begin{aligned} :\varphi_0(x)\varphi_0(y):_\omega &= \varphi_0(x)\varphi_0(y) - \omega[\varphi_0(x)\varphi_0(y)] = \\ &= \varphi_0(x)\varphi_0(y) - \omega[\varphi_0(x)\varphi_0(y)] + \psi[\varphi_0(x)\varphi_0(y)] - \psi[\varphi_0(x)\varphi_0(y)] = \\ &=: \varphi_0(x)\varphi_0(y):_\psi + \psi[: \varphi_0(x)\varphi_0(y):_\omega] , \end{aligned}$$

which is a simple example of “Wick re-ordering” (or “change of covariance” in normal ordering, see [43]). The normal re-ordered products are again well defined operator valued distributions on $\tilde{\mathcal{F}}_0$. The above formula is telling us how one can change the normal ordering prescription of a product of two fields, and we find that the normal reordered Wick square is simply

$$:\varphi_0^2(x):_\omega = : \varphi_0^2(x):_\psi + \psi[: \varphi_0^2(x):_\omega] \mathbf{1} .$$

The above formula can clearly be generalized to an arbitrary Wick power and the generic change of covariance formula is the content of the following theorem, the proof of which can be found in [43]:

Theorem A.1 (Wick reordering or change of “covariance”). *Reordering of Wick monomials is given by the formula*

$$:\varphi_0^n(x):_\omega = \sum_{j=0}^{[n/2]} \frac{n!}{(n-2j)! j! 2^j} [\delta c(x)]^j : \varphi_0^{n-2j}(x):_\psi , \quad (\text{A.1})$$

where $\delta c(x) = \lim_{y \rightarrow x} \{\psi[\varphi(x)\varphi(y)] - \omega[\varphi(x)\varphi(y)]\}$, and $[n/2]$ denotes the integer part of $n/2$.

We do not repeat the proof of this theorem and simply note that it can be extended to consider arbitrary products of fields.

Appendix B

Formal solution of an operator differential equation

In this appendix we show how could we obtain a formal solution to a non-homogeneous operator linear differential equation. Such an equation is encountered in the decomposition of Heisenberg's equation in Chapter 4, see equation (4.3), where it is also shown that in the context of a finite dimensional algebra the resulting series is convergent and hence the solution is not only a formal one.

In the decomposition of Heisenberg's equation we are confronted with a non-homogeneous operator differential equation of the form

$$\partial_t K_t = K_t \circ \delta_t + B_t . \quad (\text{B.1})$$

The first step to obtain a solution to the above equation is to solve the associated homogeneous equation, i.e.

$$\partial_t K_t = K_t \circ \delta_t .$$

The solution of the above equation is given by an (inverse) time ordered exponential of δ_t

$$K_t = K_{t_0} \circ Y_{t_0,t} ,$$

where

$$Y_{t_0,t} = \mathbf{1} + \sum_{n=1}^{\infty} \int_s^t dt_1 \int_s^{t_1} dt_2 \dots \int_s^{t_{n-1}} dt_n \delta_{t_n} \circ \dots \circ \delta_{t_2} \circ \delta_{t_1} .$$

To find the solution to the non-homogeneous equation (B.1) we assume

$$K_t = H_t \circ Y_{t_0,t} , \quad (\text{B.2})$$

which is analogous to the usual approach to the solution of an ordinary first order non-homogeneous differential equation. By taking the derivative in equation (B.2) and equating the result to the right hand side of (B.1) one obtains

$$(\partial_t H_t) Y_{t_0,t} = B_t .$$

At this point it is sufficient to multiply the latter equation from the right by Y_{t,t_0} and formally integrate to obtain

$$H_t = H_{t_0} + \int_{t_0}^t ds B_s \circ Y_{s,t_0} .$$

Finally, noting that $H_{t_0} = K_{t_0}$, from equation (B.2) we then immediately obtain

$$K_t = K_{t_0} \circ Y_{t_0,t} + \int_{t_0}^t ds B_s \circ Y_{s,t} .$$

Appendix C

Expansion of operators on Fock space

In Chapter 5 we have given an expansion of the local S -matrix (5.10) in terms of (time- s) creation and annihilation operators, see equations (5.12) and (5.15). Such an expansion *could* be understood as a perturbative expansion of the local S -matrix, but the important point is that it is valid *non-perturbatively*. Here we argue that any operator on Fock space can be expanded in terms of creation and annihilation operators.

For simplicity we will consider here only the case of a single “harmonic oscillator”, that is, we will only consider a single pair of creation and annihilation operators. The generalization of this discussion to the case of relevance in the main body of this work is then fairly straightforward, albeit more involved. Our Hilbert space is spanned by an orthonormal basis $\{|n\rangle\}$ (i.e. we have $\langle n|m\rangle = \delta(m,n)$). The creation operator a^* acts on the Hilbert space in the usual way

$$a^*|n\rangle = \sqrt{n+1}|n+1\rangle .$$

The action of its adjoint with respect to the given scalar product, the annihilation operator a , is as usual

$$a|n\rangle = \sqrt{n}|n-1\rangle .$$

Our claim is now that given a finite rank operator W on the Hilbert space \mathcal{H} , we can write it as follows

$$W = \sum_{m,n=0}^{\infty} \frac{W_{m,n}}{\sqrt{m!}\sqrt{n!}} (a^*)^m a^n . \quad (\text{C.1})$$

To prove this claim, we need to find the quantities $W_{m,n}$. Recall that all the vectors in \mathcal{H}

can be constructed by repeatedly acting on the “lowest energy” vector $|0\rangle$ with creation operators

$$|n\rangle = \frac{(a^*)^n |0\rangle}{\sqrt{n!}} .$$

We now want to compute the expectation value $\tilde{w}_{p,q} := \langle p|W|q\rangle$. Using (C.1) we have

$$\begin{aligned} \tilde{w}_{p,q} &= \langle p|W|q\rangle = \left\langle \frac{(a^*)^p}{\sqrt{p!}} 0 \left| W \frac{(a^*)^q}{\sqrt{q!}} 0 \right\rangle = \\ &= \sum_{m,n=0}^{\infty} \frac{W_{m,n}}{\sqrt{p!} \sqrt{q!} m! n!} \langle (a^*)^p 0 | (a^*)^m a^n (a^*)^q 0 \rangle = \\ &= \sum_{m,n=0}^{\infty} \frac{W_{m,n}}{\sqrt{p!} \sqrt{q!} m! n!} \langle a^m (a^*)^p 0 | a^n (a^*)^q 0 \rangle . \end{aligned} \quad (\text{C.2})$$

To push the computation further we need the following calculation

$$a^m (a^*)^p |0\rangle = \sqrt{p!} a^m |p\rangle = \sqrt{p!} \sqrt{\frac{p!}{(p-m)!}} |p-m\rangle = \frac{p!}{(p-m)!} (a^*)^{p-m} |0\rangle ,$$

which is valid for $p \geq m$. For $p < m$, on the other hand, we would simply obtain zero. By plugging this result in equation (C.2) we then get

$$\begin{aligned} \tilde{w}_{p,q} &= \sum_{m=0}^p \sum_{n=0}^q W_{m,n} \delta(p-m, q-n) \frac{\sqrt{p!}}{\sqrt{(p-m)!} m!} \frac{\sqrt{q!}}{\sqrt{(q-n)!} n!} = \\ &= \sum_{m=0}^p \sum_{n=0}^q \binom{p}{m}^{1/2} \binom{q}{n}^{1/2} W_{m,n} \delta(p-m, q-n) = \\ &= \sum_{m=p-q}^p \binom{p}{m}^{1/2} \binom{p-(p-q)}{m-(p-q)}^{1/2} W_{m,m-(p-q)} , \end{aligned}$$

where we have $p \geq m$, $q \geq n$ in addition to $p \geq q$, which is required for the last equality.

We can now solve the above system of equation for the quantities $W_{m,n}$. For instance, for $p = q$ we get

$$\tilde{w}_{p,p} = \sum_{m=0}^p \binom{p}{m} W_{m,m} ,$$

which we can use to calculate

$$\tilde{w}_{0,0} = W_{0,0} , \quad \tilde{w}_{1,1} = W_{0,0} + W_{1,1} \Rightarrow W_{1,1} = \tilde{w}_{0,0} - \tilde{w}_{1,1} ,$$

and proceeding like above, we promptly obtain

$$W_{p,p} = \sum_{m=0}^p (-1)^m \binom{p}{m} \tilde{w}_{m,m} .$$

We can similarly treat the case $p \neq q$. For instance, if $q = 0$ we get $W_{p,0} = \tilde{w}_{p,0}$, if $q = 1$ we have $W_{p,1} = \tilde{w}_{p,1} - \sqrt{p} \tilde{w}_{p,0}$ and so on. In general, to find $W_{p,q}$ we will need to find all the $W_{p,q'}$ with $q' < q$. The upshot of this discussion is that we can indeed write any finite rank operator W on \mathcal{H} as in equation (C.1). By taking limits, the same is hence true for any bounded operator on \mathcal{H} .

Now that we have dealt with the simpler case of a single harmonic oscillator, it is fairly straightforward to see that the same could be done on Fock space (3.13). Such an expansion will then require a summation over all momenta as well, and the result is just an expression like our expansions of the (local) \mathcal{S} -matrix (5.12) and (5.15) in terms of the local (dressed) scattering amplitudes \mathcal{M} and $\widetilde{\mathcal{M}}$ respectively.

Bibliography

- [1] L. Boltzmann. Weitere Studien über das Wärmegleichgewicht Gasmolekülen. *Wiener Berichte*, 66:275–370, 1872.
- [2] F. Reif. *Fundamentals of Statistical and Thermal Physics*. McGraw–Hill Science/Engineering/Math, 1965.
- [3] P. Danielewicz and M. Gyulassy. Dissipative phenomena in quark gluon plasmas. *Physical Review D*, 31:53–62, 1985.
- [4] G. Baym, H. Monien, C. J. Pethick, and D. G. Ravenhall. Transverse interactions and transport in relativistic quark–gluon and electromagnetic plasmas. *Physics Review Letters*, 64:1867–1870, 1990.
- [5] E. W. Kolb and M. S. Turner. *The Early Universe (Frontiers in Physics)*. Perseus Books, 1994.
- [6] N. M. Hugenholtz. Derivation of the Boltzmann equation for a Fermi gas. *Journal of Statistical Physics*, 32(2):231–254, 1983.
- [7] K. Huang. *Statistical Mechanics*. Wiley, Second edition, 1987.
- [8] N. N. Bogoliubov. Problems of a dynamical theory in statistical physics. In J. de Boer and G. E. Uhlenbeck, editors, *Studies in Statistical Mechanics, Vol. I*. North Holland, 1962.
- [9] L. Erdős, M. Salmhofer, and H.-T. Yau. On the quantum Boltzmann equation. *Journal of Statistical Physics*, 116(1–4):367–380, 2004.
- [10] J. Rau and B. Müller. From reversible quantum microdynamics to irreversible quantum transport. *Physics Reports*, 272:1–59, 1996.
- [11] L. Erdős and H.-T. Yau. Linear Boltzmann equation as the weak coupling limit of a random Schrödinger equation. *Communications in Pure and Applied Mathematics*, 53:667–735, 2000.

- [12] D. Eng and L. Erdős. The Linear Boltzmann equation as the low density limit of a random Schrödinger equation. arXiv:math-ph/0412044v2, 2005.
- [13] G. Baym and L. P. Kadanoff. Conservation laws and correlation functions. *Physical Review*, 124:287–299, 1961.
- [14] G. Baym and L. P. Kadanoff. *Quantum Statistical Mechanics*. Benjamin, New York, 1962.
- [15] V. P. Nair. *Quantum Field Theory, A Modern Perspective*. Springer, 2005.
- [16] A. Hohenegger, A. Kartavtsev, and M. Lindner. Deriving Boltzmann equations from Kadanoff-Baym equations in curved space-time. *Physical Review D*, 78:085027, 2008.
- [17] M. M. Müller. Comparing Boltzmann vs. Kadanoff-Baym. *Journal of Physics: Conference Series*, 35:390–397, 2006.
- [18] H. Grabert. *Projection Operator Techniques in Nonequilibrium Statistical Mechanics*. Springer-Verlag, Berlin Heidelberg New York, 1982.
- [19] V. I. Arnold. *Mathematical methods of classical mechanics*. Springer, Second edition, 1989.
- [20] L. E. Ballentine. *Quantum Mechanics, A Modern Development*. World Scientific Publishing, 1998.
- [21] E. A. Uehling and G. E. Uhlenbeck. Transport phenomena in Einstein-Bose and Fermi-Dirac gases. I. *Physical Review*, 43:552–561, 1933.
- [22] L. van Hove. *Physica*, 21:517–540, 1955.
- [23] D. Benedetto, F. Castella, R. Esposito, and M. Pulvirenti. A short review on the derivation of the nonlinear quantum boltzmann equation. *Communications in Mathematical Sciences*, 5:55–71, 2007.
- [24] H. Spohn. Kinetic equations from Hamiltonian dynamics: Markovian limits. *Reviews of Modern Physics*, 52:569–615, 1980.
- [25] R. Haag. *Local Quantum Physics (Fields, Particles, Algebras)*. Springer, Second edition, 1996.

- [26] S. A. Fulling. Nonuniqueness of canonical field quantization in Riemannian space-time. *Physical Review D*, 7(10):2850–2862, May 1973.
- [27] N. D. Birrell and P. C. W. Davies. *Quantum Fields in Curved Space*. Cambridge University Press, 1984.
- [28] S. A. Fulling. *Aspects of Quantum Field Theory in Curved Space-Time*. Cambridge University Press, 1989.
- [29] L. Gårding and A. S. Wightman. Fields as operator valued distributions in relativistic quantum theory. *Arkiv för Fysik*, 28:129–184, 1964.
- [30] R. F. Streater and A. S. Wightman. *PCT, Spin and Statistics, and All That*. Princeton University Press, New edition, 2000.
- [31] M. E. Peskin and D. V. Schroeder. *An Introduction to Quantum Field Theory*. Addison-Wesley, Reading, USA, 1995.
- [32] M. Srednicki. *Quantum Field Theory*. Cambridge University Press, 2007.
- [33] A. Zee. *Quantum Field Theory in a Nutshell*. Princeton University Press, 2003.
- [34] G. C. Wick. The evaluation of the collision matrix. *Physical Review*, 80:268–272, 1950.
- [35] J. Glimm and A. Jaffe. *Quantum Field Theory and Statistical Mechanics*. Birkhäuser, Boston, 1982.
- [36] R. Haag and D. Kastler. An algebraic approach to quantum field theory. *Journal of Mathematical Physics*, 5:848–861, 1964.
- [37] K. Osterwalder and R. Schrader. Axioms for Euclidean Green’s functions. *Communications in Mathematical Physics*, 31:83–112, 1973.
- [38] K. Osterwalder and R. Schrader. Axioms for Euclidean Green’s functions II. *Communications in Mathematical Physics*, 42:281–305, 1975.
- [39] J. Glimm and A. M. Jaffe. A $\lambda\phi^4$ quantum field theory without cutoffs. I. *Physical Review*, 176:1945–1951, 1968.
- [40] J. Glimm and A. M. Jaffe. The $\lambda\phi^4$ quantum field theory without cutoffs. II. The field operators and the approximate vacuum. *Annals of Mathematics*, 91:362–401, 1970.

- [41] J. Glimm and A. M. Jaffe. The $\lambda\phi_2^4$ quantum field theory without cutoffs. III. The physical vacuum. *Acta Mathematica*, 125:203–267, 1970.
- [42] J. Glimm and A. M. Jaffe. The $\lambda\phi_2^4$ quantum field theory without cutoffs. IV. Perturbations of the Hamiltonian. *Journal of Mathematical Physics*, 13:1568–1584, 1972.
- [43] J. Glimm and A. Jaffe. *Quantum Physics: A Functional Integral Point of View*. Springer, Berlin Heidelberg New York, Second edition, 1987.
- [44] B. Simon. *The $P(\varphi)_2$ Euclidean (Quantum) Field Theory*. Princeton University Press, 1974.
- [45] J. Glimm and A. M. Jaffe. Positivity of the $\lambda\phi_3^4$ Hamiltonian. *Fortschritte der Physik*, 21:327–376, 1973.
- [46] J. Feldman and K. Osterwalder. The Wightman axioms and the mass gap for weakly coupled $\lambda\phi_3^4$ quantum field theories. *Annals of Physics*, 97:80–135, 1976.
- [47] A. S. Wightman. Introduction to some aspects of the relativistic dynamics of quantized fields. In Lévy, M., editor, *1964 Cargèse Summer School Lectures*. Gordon and Breach, New York, 1967.
- [48] M. Reed and B. Simon. *Fourier Analysis, Self-Adjointness (Methods of Modern Mathematical Physics)*, volume 2. Academic Press, 1975.
- [49] F. Guerra, L. Rosen, and B. Simon. Nelson's symmetry and the infinite volume behaviour of the vacuum in $P(\varphi)_2$. *Communications in Mathematical Physics*, 27:10–22, 1972.
- [50] J. Frölich. Schwinger functions and their generating functionals: I. *Helvetica Physica Acta*, 47:265–306, 1974.
- [51] B. Simon and R. Hoegh-Krohn. Hypercontractive semigroups and two dimensional self-coupled Bose fields. *Journal of Functional Analysis*, 9:121–180, 1972.
- [52] S. Golden. Lower bounds for the Helmholtz function. *Physical Review*, 137:B1127–B1128, 1965.
- [53] C. J. Thompson. Inequalities with applications in statistical mechanics. *Journal of Mathematical Physics*, 6:1812–1813, 1965.

- [54] M. B. Ruskai. Inequalities for traces on von Neumann algebras. *Communications in Mathematical Physics*, 26:280–289, 1972.
- [55] H. Araki. Golden–Thompson and Peierls–Bogoliubov inequalities for a general von Neuman algebra. *Communications in Mathematical Physics*, 34:167–178, 1973.
- [56] V. V. Borzov. Upper bound for the partition function of $P(\varphi)_2$ Euclidean field theory with Dirichlet boundary conditions. *Teoreticheskaya i Matematicheskaya Fizika*, 57(3):373–381, 1983.
- [57] L. Rosen. The $(\phi^{2n})_2$ quantum field theory: Higher order estimates. *Communications on Pure and Applied Mathematics*, 24:417–457, 1971.
- [58] S. Hollands and G. Leiler. On the derivation of the Boltzmann equation in quantum field theory: Flat spacetime. *arXiv: 1003.1621*, 2010.
- [59] K. Kawasaki and J. D. Gunton. Theory of nonlinear transport processes: Nonlinear shear viscosity and normal stress effects. *Physical Review A*, 8(4):2048–2064, October 1973.
- [60] B. Robertson. Equations of motion in nonequilibrium statistical mechanics. *Physical Review*, 144:151–161, 1966.
- [61] R. Hoegh-Krohn. On the spectrum of the space cut-off $P(\phi)$ Hamiltonian in two space-time dimensions. *Communications in Mathematical Physics*, 21:256–260, 1971.
- [62] N. N. Bogoliubov and D. V. Shrikov. *Introduction to the Theory of Quantized Fields*. Interscience Publishers, Inc., New York, 1959.
- [63] H. Epstein and V. Glaser. The role of locality in perturbation theory. *Annales de l'institut Henri Poincaré (A) Physique théorique*, 19:211–295, 1973.
- [64] R. Haag. On quantum field theories. *Kong. Dan. Vid. Sel. Mat. Fys. Med.*, 29N12:1–37, 1955.
- [65] V. Glaser, H. Lehmann, and W. Zimmeramn. Field operators and retarded functions. *Il Nuovo Cimento*, 6:1122–1128, 1957.
- [66] R. Brunetti and K. Fredenhagen. Microlocal analysis and interacting quantum field theories: Renormalization on physical backgrounds. *Communications in Mathematical Physics*, 208:623–661, 2000.

- [67] M. Dütsch and K. Fredenhagen. Causal perturbation theory in terms of retarded products, and a proof of the action Ward identity. arXiv:hep-th/0403213, 2004.
- [68] S. Hollands and R. M. Wald. On the renormalization group in curved spacetime. *Communications in Mathematical Physics*, 237:123–160, 2003.
- [69] H. Araki. A lattice of von Neumann algebras associated with the quantum theory of a free Bose field. *Journal of Mathematical Physics*, 4(11):1343–1362, 1963.
- [70] S. Hollands and R. M. Wald. Local Wick polynomials and time ordered products of quantum fields in curved spacetime. *Communications in Mathematical Physics*, 223:289–326, 2001.
- [71] C. Itzykson and J. B. Zuber. *Quantum Field Theory*. McGraw-Hill, New York, USA, 1980.
- [72] O. Steinmann. Axiomatic approach to perturbative quantum field theory. *Annales de l'institut Henri Poincaré (A) Physique théorique*, 63:399–409, 1995.
- [73] S. Hollands and R. M. Wald. Existence of local covariant time ordered products of quantum fields in curved spacetime. *Communications in Mathematical Physics*, 231:309–345, 2002.
- [74] M. Dütsch and K. Fredenhagen. A local (perturbative) construction of observables in gauge theories: The example of QED. *Communications in Mathematical Physics*, 203:71–105, 1999.
- [75] L. Hörmander. *The Analysis of Linear Partial Differential Operators I*. Springer-Verlag, Berlin New York, Second edition, 1990.
- [76] H. Epstein and V. Glaser. Adiabatic limit in perturbation theory. CERN-TH-1344, 1975.
- [77] D. A. Gurnett and A. Bhattacharjee. *Introduction to Plasma Physics—With Space and Laboratory Applications*. Cambridge University Press, 2005.
- [78] N. P. Landsman. Nonshell unstable particles in thermal field theory. *Annals of Physics*, 186:141, 1988.
- [79] R. Haag, N. M. Hugenholtz, and M. Winnink. On the equilibrium states in quantum statistical mechanics. *Communications in Mathematical Physics*, 5(3):215–236, 1967.

- [80] M Le Bellac. *Thermal Field Theory*. Cambridge University Press, 1996.
- [81] N. P. Landsman and C. G. van Weert. Real and imaginary time field theory at finite temperature and density. *Physics Reports*, 145:141, 1987.
- [82] C. Gérard and C. D. Jäkel. Thermal quantum fields without cut-offs in 1+1 space-time dimensions. *Reviews in Mathematical Physics*, 17:113–174, 2005.
- [83] G. Källén. The definition of renormalization constants in quantum electrodynamics. *Helvetica Physica Acta*, page 417, 1952.
- [84] H. Lehmann. On the properties of propagation functions and renormalization constants of quantized fields. *Nuovo Cimento*, 11:342–357, 1954.
- [85] H. Narnhofer, M. Requardt, and W. Thirring. Quasi-particles at finite temperatures. *Communications in Mathematical Physics*, 92:247–268, 1983.
- [86] D. Buchholz. On hot bangs and the arrow of time in relativistic quantum field theory. *Communications in Mathematical Physics*, 237:271–288, 2003.
- [87] D. Buchholz, I. Ojima, and H. Roos. Thermodynamic properties of non-equilibrium states in quantum field theory. *Annals of Physics*, 297:219–242, 2002.
- [88] R. Brunetti, K. Fredenhagen, and M. Köhler. The microlocal spectrum condition and Wick polynomials of free fields on curved spacetimes. *Communications in Mathematical Physics*, 180:633–652, 1996.

