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State space investigation of bullwhip with arbitrary demand processes

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Abstract

Using state space techniques we study a "myopic" order-up-to policy. The policy is myopic because is it is optimal at minimising local inventory holding and shortage costs. In particular we study the bullwhip effect produced by the replenishment policy reacting arbitrary stochastic demand processes. We reveal that bullwhip is fundamentally caused by the co-variance between the inventory level and the demand forecast. We go on to highlight the impact of a simple control engineering inspired bullwhip reduction technique, a proportional controller in the inventory feedback loop. Although it can be shown this approach is always able to remove bullwhip, we expose that is not possible to arbitrary "tune" the proportional controller, without knowing in advance the likely structure of the demand process.

Key words

Order-up-to policy, matrix systems, minimum variance forecasts, bullwhip

1. Introduction

Recently, a number of papers have appeared that investigate the "bullwhip effect" (the variance amplification of ordering decisions in the supply chain) produced by the Order-Up-To (OUT) replenishment policy after the seminal papers of Chen, Drezner, Ryan and Simchi-Levi (2000) and Lee, Padmanabhan and Whang (1997). The bullwhip problem is especially important as it results in unnecessary costs in supply chains (see for example, Metters (1997) and Carlsson and Fullér (2000)). This includes costs such as inefficient use of production, distribution and storage capacity, recruitment and training costs, increased inventory and poor customer service levels.

The bullwhip problem is widespread throughout industry. Figure 1 shows 6 months of daily data from a real supply chain. Here, "retail sales" represents the movement of a particular (high volume, own label) product though the supermarkets tills of a major retailer. "Production" represents the manufacturer's production of the product to satisfy this demand. In between the retailer and the manufacturer two orders have been placed, one to replenish (the several hundred) individual supermarkets, and one to replenish the (half a dozen) distribution centres. There is a 5 to 1 increase in the variance in this real-life supply chain.

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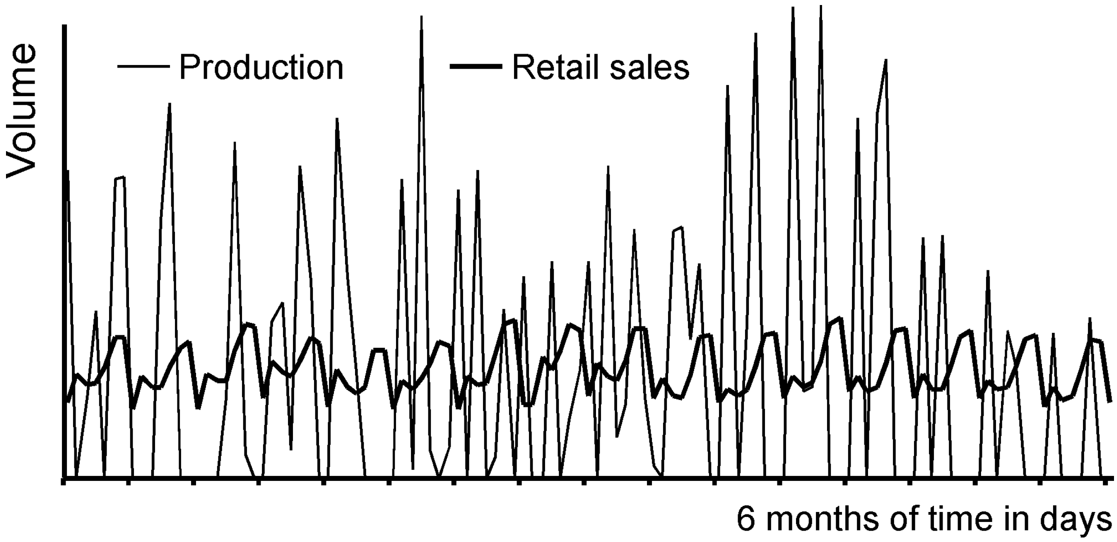

Figure 1. Bullwhip observed in the real world

The OUT policy has become a popular replenishment policy to study in bullwhip literature. This is due to its wide applicability at representing industrial ordering policies and it mathematical tractability. Indeed, a variant of the OUT policy was actually used in the supply chain of Figure 1. Of course, the industrial strength version, had a lot of "bells and whistles", but its fundamental structure is an OUT policy.

Statistical approaches are popular mechanism to study the bullwhip problem. Lee, Padmanabhan, and Whang (1997), explored different sources of the bullwhip problem in an OUT policy. They highlighted five sources; lead-times, forecasting mechanisms, gaming, batching and promotions. Chen Drezner, Ryan and Simchi-Levi (2000) showed that the classical OUT policy with simple forecasting techniques always results in bullwhip for the class of AR demands.

Control theory and transform techniques have also been applied to the bullwhip problem. The field was initiated by the Nobel Laureate, Herbert Simon (1952) who used the Laplace transform to study a continuous time ordering policy. This was quickly replicated in discrete time by Vassian (1954), Magee (1956) and Brown (1962). Adelson (1966) appears to be the first to explicitly study what we would now call bullwhip. Deziel and Eilon (1967) provide an insightful study of bullwhip and inventory variance an OUT policy. Schneeweiβ (1971, 1975, 1977) and Inderfurth (1977) used z-transforms and Wiener filtering theory to find optimal linear production or replenishment decisions for production systems with non-linear costs. Gaalman (1976, 1978 and 1979) considered the optimal linear control for a general multi dimensional production inventory systems, using the so called modern control theoretic principles.

Innovative supply chain structures have also been investigated for bullwhip effects. Lee, So and Tang (2000) take statistical approach to study a Vendor Managed Inventory (VMI) scenario. Disney and Towill (2002) investigate stability issues in

VMI supply chains. Dejonckheere, Disney, Lambrecht and Towill (2004) study a scenario where end consumer demand is shared throughout a four-echelon supply chain with z-transform techniques.

It is often supposed that the bullwhip problem cannot be avoided. Indeed it has been shown by Dejonckheere, Disney, Lambrecht and Towill (2003) that the classical OUT policy, with exponential smoothing (Brown (1963)), moving average or demand signal process (Lee, Padmanabhan and Whang, 1997) forecasting, will always produce bullwhip, for all possible demand patterns. However, recently, it has been shown (Gilbert (2002) and Chen and Disney (2003)) that the use of conditional expectation forecasting can allow the OUT policy to smooth demand (i.e. avoid the bullwhip problem) for some classes of stochastic demand patterns.

Modifying the classical OUT policy, by incorporating a gain in the inventory feed-back loop, 'bullwhip' reduction can be observed in the OUT policy. Indeed Dejonckheere, Disney, Lambrecht and Towill (2003), have shown, that it is always possible to avoid the bullwhip problem with such a technique for all classes of demand processes. At first sight, this bullwhip avoidance technique will come at the cost of holding extra inventory to achieve target customer service levels, and this is true for i.i.d stochastic demands, Magee (1956). Dejonckheere, Disney, Farasyn, Janssen, Lambrecht, Towill and van de Velde (2002) investigate this scenario explicitly.

A characteristic of recent papers is the use of slightly more realistic stochastic demand models such as the Auto Regressive Moving Average demand model (Box and Jenkins 1970). Commonly, a classical control engineering approach of is taken to analyse the system's dynamic behaviour. For example, Disney, Farasyn, Lambrecht, Towill, and Van de Velde, (2003) study the ARMA model and show there are win-win scenarios resulting from the proportional controller, $1/Ti$ (here called f). They show it is actually possible to avoid bullwhip, reduce inventory requirements and achieve a desired fill-rate simultaneously.

An expected cost approach may also be applied to the OUT policy. For example, Kim and Ryan (2003) assume the demand process is Guassian and study the probability density function that describes the inventory levels over time; assigning costs to expected inventory holding and backlog positions. Disney and Grubbström (2003) applied the same technique to the probability density function of the orders and assigned costs to the fraction of production produced in normal working hours, and those produced in a capacitated position with subcontracting or over-time working. There the order policy was the OUT with exponential smoothing and the demand was assumed to be Auto Regressive. Chen and Disney (2003) studied expected inventory and order related costs in the myopic OUT policy with conditional expectation and ARMA(1,1) demand. They show that the proportional controller ($f=1/Ti$) always results in more economical performance than the classical OUT policy (where $f=Ti=1$).

Herein we introduce a general state space demand model and apply minimum variance forecasts. The demand model covers a large number of demand models, for instance; the ARIMA models of Box and Jenkins, the Brown-Meyer adapted smoothing models and the General Polynomial Growth-model of Harrison. We derived directly, by means of matrix calculations, the variance of the states of the system (the inventory and the forecast). This results in a matrix difference equation that under stability conditions converges to stationary values. In general this stationary variance matrix equation can be solved numerically. However, in this particular case, a closed form analytical expression for the variances of the state variables can be derived. From this the variance amplification of the ordering rule can be easily found.

Analysing the ordering rule variance expression in this way highlights the essential role of the co-variance between inventory and the state of the demand forecast model on the bullwhip effect. This is directly related to the eigenvalues of the system. In addition, given the expression of the ordering variance a polynomial equation can be derived that surrenders the value of the feedback parameter for which the variance amplification is 1.

The structure of our paper is as follows. In section 2 a matrix model of the myopic OUT policy is introduced. In section 3, we add the proportional controller into the classical OUT policy. Section 4 presents a state space model of a generalised stochastic demand process. Section 5 investigates the matrix variance equation of our modified OUT policy. Section 6 further analyses the order variance. We conclude in section 7.

2. The myopic order-up-to policy

The order-up-to policy is often used in inventory theory as it minimises the long-run expected inventory related costs consisting of piece-wise linear holding and stock-out costs. The inventory is reviewed and the ordering decision is made at the beginning of the period. During the period the customer orders are received and fulfilled before or at the end of the period. The inventory balance equation is given by

$$i_{t+1} = i_t + o_t - z_{t+1}, \qquad (1)$$

where; i_t is the inventory at time t, o_t is the order placed to replenish the inventory and z_{t+1} the demand during the period (t, t+1).

The (classical) order-up-to policy is completed by replenishment decision as

$$o_t = -(i_t - i_n) + \hat{z}_{t+1,t} \qquad (2)$$

where;

- i_n is the inventory norm value (a time invariant constant used to achieve a given customer service metric, such as availability, fill-rate or economic criteria, often solved via the “newsboy” technique). We may set i_n to zero without loss of generality here.
- and $\hat{z}_{t+1,t}$ is the forecast of demand at time t for the next period.

Note that the lead-time consists of a review period only and the demand forecast mechanism is based on the minimum mean squared error criterion. Thus, we are considering a “myopic” OUT policy, Chen and Disney (2003). Johnson and

Thompson (1975) proved that this policy, with $o_t \geq 0$, is optimal at minimising inventory costs. Usually it is assumed that the demand is a normally distributed variable with a constant mean and random uncorrelated error component with constant variance:

$$z_{t+1} = \overline{z} + \varepsilon_{t+1} \tag{3}$$

Thus this "myopic" OUT policy minimizes the inventory fluctuations around the average value i_n. Little attention has traditionally been given to the fluctuations in the ordering decision until the recent interest in the bullwhip problem. These fluctuations might become large for some demand characteristics

In this paper we want to analyse the effects on the inventory and the ordering decision relaxing the restrictive assumption in (3). That is, we wish to gain insight into the behaviour of the OUT policy for a general demand process that is auto-correlated. This is important as real life demands are not always i.i.d. as implied by (3). For example, see Figure 1 where there is clearly a weekly cycle in the demand pattern. In particular we will model the demand characteristics as a linear state space model of arbitrary dimension. In order to perform a linear dynamic analysis we assume $o_t \geq 0$. This seems reasonable when the average demand is sufficient large and indeed was the case in the real life scenario in Figure 1.

Eliminating o_t in the balance equation (1) using the replenishment equation (2) gives:

$$i_{t+1} = i_n - (\eta_{t+1}) \tag{4}$$

where $z_{t+1} - \hat{z}_{t+1,t} = \eta_{t+1}$ is the forecast error. Note that if $\eta_{t+1} = \varepsilon_{t+1}$, the demand forecast minimizes the variance of the forecast error and the inventory variance will be minimal. This can be easily seen by squaring equation (4) and taking the expected values:

$$\Sigma_{ii,\,t+1} = \Sigma_{\eta\eta} \tag{5}$$

where $\Sigma_{ii,\,t+1}, \Sigma_{\eta\eta}$ are the variances of the inventory at t+1 and the forecast error. Note that the inventory variance is time independent, so $\Sigma_{ii,\,t+1} = \Sigma_{ii}$. From a control engineering viewpoint the explanation is simple; the ordering rule feeds back the inventory deviation from the norm value completely and at the same time anticipates future demand as best as can possibly be achieved.

From in (4) it can be shown that:

$$o_t = (\eta_t) + \hat{z}_{t+1,\,t} \tag{6}$$

and the variance of the ordering decision is,

$$\Sigma_{oo,\,t} = \Sigma_{\eta\eta} - 2\Sigma_{\eta\hat{z},\,t} + \Sigma_{\hat{z}\hat{z},\,t} \tag{7}$$

with $\Sigma_{oo,t}, \Sigma_{\hat{z}\hat{z},t}$ the variances of o_t and $\hat{z}_{t+1,t}$ respectively, $\Sigma_{\eta\hat{z},t}$ the co-variance between the demand forecast $\hat{z}_{t+1,t}$ and the forecast error η_t. Except for i.i.d random demand processes as described in equation (3), the co-variance will generally not be zero. If the co-variance is negative the variance of the ordering decision may even become very large. Without a further description of the demand characteristics the precise bullwhip effect cannot be further analysed.

3. Adding a proportional controller to the myopic OUT policy

The introduction of a proportional controller into the inventory feedback loop might improve the dynamic behaviour of the replenishment policy. Indeed, this would be an obvious suggestion from basic control theory and actually has a long history in the field of production and inventory control, see Simon (1952), Magee (1956), Deziel and Eilon (1967), Towill (1982), Matsuyama (1997), Dejonckheere et al (2003), Chen and Disney (2003). So let's consider it here. Incorporating the proportional controller, *f*, into the inventory feedback loop in the replenishment decision we have:

$$o_t = -f(i_t - i_n) + \hat{z}_{t+1,t} \tag{8}$$

Using equation (8) in the inventory balance equation (1) yields:

$$i_{t+1} = i_n + (1-f)(i_t - i_n) - (\eta_{t+1}) \tag{9}$$

From this it can be seen that the system will be stable within the interval $0 < f < 2$. From equation (9) the inventory variance may be derived as:

$$\Sigma_{ii,t+1} = (1-f)^2 \Sigma_{ii,t} + \Sigma_{\eta\eta} \tag{10}$$

Within the stability interval equation (10) will converge to a stationary linear expression from which the inventory variance can be calculated as:

$$\Sigma_{ii} = \{1/[1-(1-f)^2]\}\Sigma_{\eta\eta} \tag{11}$$

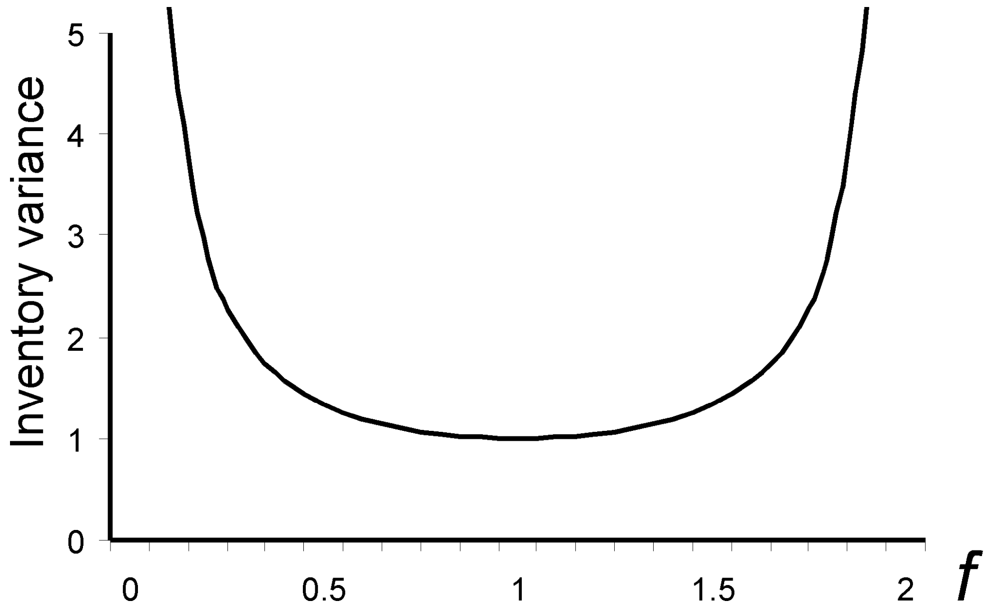

Figure 2. Inventory variance as a function of the inventory feedback gain, f

Figure 2 shows the inventory variance as function of f. As expected the variance is minimal for f=1. For $0 < f \leq 1$ the variance is a decreasing function of f and for $1 \leq f < 2$ an increasing function of f. Happily (10) also shows that the introduction of f does not introduce a co-variance relationship between the inventory and the demand forecast. The variance of the ordering rule (8) becomes:

$$\Sigma_{oo,t} = f^2 \Sigma_{ii,t} - 2f \Sigma_{i\hat{z},t} + \Sigma_{\hat{z}\hat{z},t} \tag{12}$$

or in the stationary situation:

$$\Sigma_{oo} = f^2 \Sigma_{ii} - 2f \Sigma_{i\hat{z}} + \Sigma_{\hat{z}\hat{z}} \tag{13}$$

We observe three terms, one related to the inventory variance, the second to the co-variance of the inventory and the demand forecast and finally the variance of the forecast error. Using equation (11) the influence f on the first term can be calculated. The third term is independent of f. When the demand satisfies (3) the co-variance $\Sigma_{i\hat{z}}$ is zero. Then Σ_{oo} is an increasing function of f in the interval $0 < f < 2$. As Σ_{oo} goes to infinity for $f \to 2$ for some f the variance will be larger than the variance of the demand (the so called bullwhip effect). The critical point lies at $f = 1$. For $0 < f \leq 1$, we also can observe that the inventory variance is decreasing and the ordering variance is increasing. So we are able to find in this interval a good balance between the inventory variance and the ordering variance, meaning that a weighted sum of both variances is minimal.

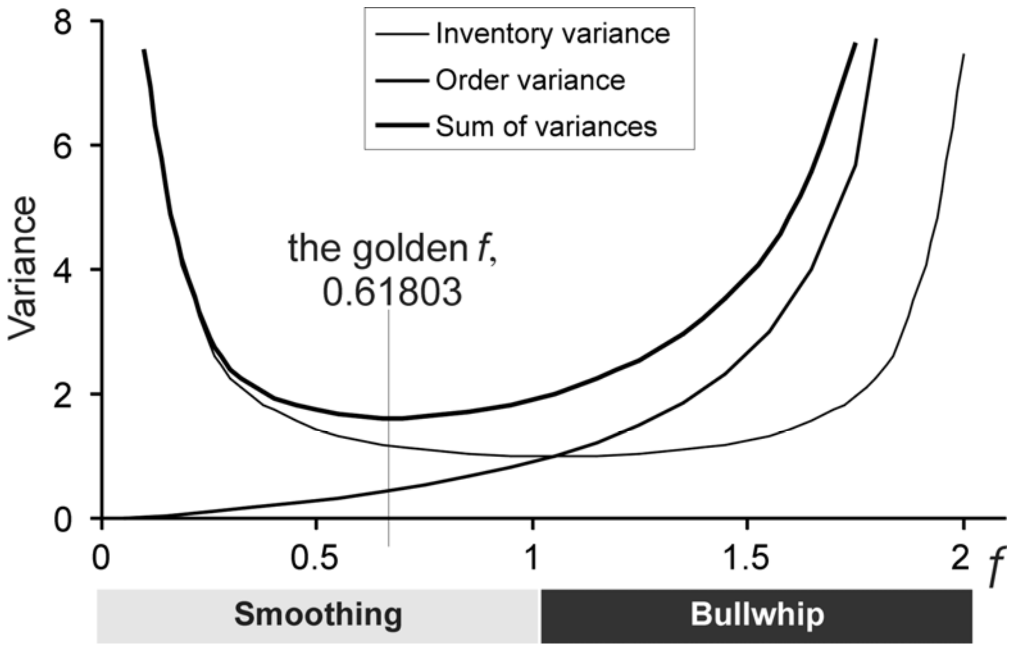

Figure 3. Inventory and order variance when demand is i.i.d.

For example when demand is i.i.d. (that is, given by (3)) the inventory and order variance and their sum is as shown in Figure 3. This case has been studied by Disney, Farasyn, Lambrecht, Towill, and van de Velde (2004) with regard to the customer service metric, the fill-rate. They also note, that in the case, the optimum value of the proportional controller to minimise the sum of the inventory and order variance is the golden ratio, 0.61803 for all lead-times.

If the co-variance term in equation (13) is non-zero the effect of f on the ordering variance can only be shown if we know $\Sigma_{i\hat{z}}$ as function of f. However some general observations can still be made. If $\Sigma_{i\hat{z}}$ is negative over the interval $0 < f < 2$ then the second term in the ordering variance is positive. Compared with the random demand model the ordering variance will increase. So bullwhip may still arise in the interval $0 < f \leq 1$. But, since the inventory variance is decreasing and the ordering variance is still increasing a good balance can be found. If $\Sigma_{i\hat{z}}$ is positive the ordering variance might decrease. However if $\Sigma_{i\hat{z}}$ is bounded the first term in ordering variance becomes dominant for $f \to 2$. This indicates that though the ordering variance is decreasing for some values of f it will be increasing for larger values of f. However, bullwhip may still arise.

Unfortunately the effects of the co-variance between the inventory and the demand forecast can only be studied further if a demand model is specified. In the next section we will introduce a general demand model that forms the basis of subsequent analysis. The policy is formulated as a state space model in order to derive a general expression for the co-variance.

4. A general linear state space demand and forecast model

Assume the demand can be described by the sum of a constant mean term and a normally distributed random variable:

$$z_{t+1} = \bar{z} + \tilde{z}_{t+1} \tag{14}$$

Since, as shown in section 2, we are only interested in the fluctuation around a constant mean we may ignore the constant term and will model only the stochastic process. Let this term be described by the linear vector difference equation:

$$z_{t+1} = My_{t+1} + \varepsilon_{t+1} \tag{15}$$

$$y_{t+1} = Dy_t + E\xi_t \tag{16}$$

where y_t is an m-dimensional state vector, M a $1 \times m$ matrix, D an $m \times m$ matrix and E an $m \times k$ matrix. The variable ε_t and the m dimensional vector ξ_t form sequences of zero mean uncorrelated normal distributed random variables with positive definite variance matrices $\Sigma_{\varepsilon\varepsilon}, \Sigma_{\xi\xi}$ and co-variance matrix $\Sigma_{\varepsilon\xi}$. If we consider the random variables as input variables, this model is a multi input single output model of order MISO (*k*+1, 1). In contrast with the model introduced in Gaalman (1976) the matrix E is introduced to explicitly demonstrate that E need not necessary be the unit matrix.

This model is rather general in the sense that a number of well-known demand models can be described in these terms. For instance the 'General Polynomial Model" of Harrison (1967) and the polynomial, exponential and seasonal models of Brown-Meyer (Brown 1963). Furthermore, Box and Jenkins (1970) single input single output ARIMA(p, I, q) model (when $q \le p + I$) can be studied with this state space approach.

Demand models can be approached from both a theoretical and a practical perspective. From the theoretical viewpoint there are, like in all physical sciences, arguments for the assuming a certain structure, here (M, D). For example a simple though frequently used demand model is given in equation (3). If the constant term $\bar{z}$ changes slowly over time this may be modelled as a random walk, resulting in a model with M = 1, D=1. An example of a practice-oriented approach is the Box & Jenkins methodology. Using historical demand data an ARIMA(p, I, q) model may be identified.

Given that the demand realizations over a period of time D and M are not unique representations of the demand process. A (similarity) transformation $w_t = Ty_t$, with T a non-singular $m \times m$ matrix, can be applied that change D and M without changing the demand realisations. Thus it provides an equivalent system representation. One important transformation leads to D in model form. In this case D is diagonal with the eigenvalues on the diagonal. More generally with some equal eigenvalues D will have a Jordan canonical representation. Due to the single dimensional output of the demand model another important transformation converts D and M into the observable canonical form as follows:

$$D = \begin{pmatrix} -d_1 & 1 & 0 & \cdots & 0 \\ -d_2 & 0 & 1 & \cdots & \\ \vdots & & 0 & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ -d_m & & & \cdots & 0 \end{pmatrix}, \; M = (1,0,...,0) \tag{17}$$

Here D is called the (left) companion matrix because the elements d_i are the coefficients of the characteristic polynomial:

$$CP(\lambda) = \det(\lambda I - D) = \lambda^m + d_1\lambda^{m-1} + \cdots d_m \tag{18}$$

By a suitable choice of the state variables y_t the ARMA$(p+I,q)$ model,

$$z_{t+1} - \phi_1 z_t \cdots - \phi_{I+p} z_{t+1-I-p} = \varepsilon_{t+1} - \theta_1\varepsilon_t \cdots - \theta_q \varepsilon_{t+1-q} \tag{19}$$

can directly be transformed into an observable canonical form. Because there is no fundamental difference between the Auto Regressive (AR) part and the Integrated (I) part of the Box & Jenkins model we might consider the ARMA$(p+I,q)$ model in equation (19) as an ARIMA(p,I,q). The elements of D are now equal to $d_j = -\phi_j$, $j = 1,\cdots,m = p+I$ and:

$$E = (\phi_1 - \theta_1, \phi_2 - \theta_2, \cdots \phi_{p+I} - \theta_{p+I})', \; \xi_t = \varepsilon_t \tag{20}$$

where $\theta_j = 0$ *for* $j > p+1$ and the transpose of a vector or matrix is denoted by $(\cdots)'$. Note as the ARMA model is a SISO system and thus the state space is now also a SISO system. The case for which $q > p+I$ (i.e. the number of moving average parameters is larger than the number of autoregressive and integration parameters) cannot be modelled in the strict sense.

Having explained the basic characteristics of our state space demand model we will now introduce the forecast model. The optimal forecast of z_{t+1} at time *t*, which is equal to the minimum variance estimation, can be obtained by the Kalman filter (Jazwinski 1970):

$$\hat{z}_{t+1,t} = M\hat{y}_{t+1,t} \tag{21}$$

$$\hat{y}_{t+1,t} = D\hat{y}_{t,t-1} + G_t(z_t - \hat{z}_{t,t-1}) \tag{22}$$

where $\hat{z}_{t+1,t}, \hat{y}_{t+1,t}$ are conditional expectations at time *t* given all the observed realisations of demand and G_t is the so called Kalman gain that can be calculated from the matrix Riccati equation. It can be shown that the forecast errors $(z_t - \hat{z}_{t,t-1}) = \eta_t$ form a sequence of zero mean uncorrelated normally distributed random variables when the forecast is optimal.

Here we will consider only the steady state situation. So we assume that at time t an infinite number of demand observations from the past are available. If the system is observable then the Kalman gain converges to the stationary value G. Moreover the matrix $(D-GM)$, associated with the estimation error system $e_t = y_t - \hat{y}_{t,t-1}$ is stable. That is, the eigenvalues satisfy $\left|\lambda_j\right| < 1 \quad j = 1,...,m$. If the demand model is represented in the observable canonical form then:

$$D-GM = \begin{pmatrix} -d_1 - g_1 & 1 & 0 & \cdots & 0 \\ -d_2 - g_2 & 0 & 1 & \cdots & \\ \vdots & & 0 & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ -d_m - g_m & & & \cdots & 0 \end{pmatrix} \tag{23}$$

with the characteristic polynomial:

$$CP(\lambda) = \lambda^m + (d_1 + g_1)\lambda^{m-1} + \cdots (d_m + g_m) \tag{24}$$

The demand $z_t = \hat{z}_{t,t-1} + \eta_t$ and the equations (21), (22) form a SISO system with η_t as the input variable. This system is equivalent with an ARMA (m,q) model with $q \le p + I$. This means that for the demand model described in equation (15) and (16) a Box & Jenkins ARMA model is associated with parameters ϕ_j, θ_j. For this model $(d_j + g_j) = (-\phi_j + (\phi_j - \theta_j)) = -\theta_j$. So the eigenvalues of the Moving Average (MA) part satisfy $\left|\lambda_j\right| < 1 \quad j = 1,\cdots,m$ just as Box & Jenkins asserted. But contrarily to the Box & Jenkins model the eigenvalues of the AR part might be larger than one. Note that given the ARIMA model of equation (19) a state space forecast model exists with $\eta_t = \varepsilon_t$.

Assume the demand system is represented by the observable canonical form. Since the similarity with the ARIMA models we will use the parameters ϕ_j, θ_j instead of d_j, g_j. In addition for simplicity reasons we assume that the eigenvalues of D are real and distinct. This means that the characteristic polynomial of D can be written as the product of m distinct eigenvalues:

$$\det(\lambda I - D) = \lambda^m - \phi_1\lambda^{m-1} - \cdots - \phi_m = \prod_{i=1}^{m}(\lambda - \lambda_i^{\phi}) \tag{25}$$

Moreover, for the same reasons, we assume that the matrix $(D-GM)$ has real and distinct eigenvalues with $\left|\lambda_j^{\theta}\right| < 1 \quad j = 1,\cdots,m$. So the characteristic polynomial now becomes:

$$\det(\lambda I - (D-GM)) = \lambda^m - \theta_1\lambda^{m-1} - \cdots - \theta_m = \prod_{i=1}^{m}(\lambda - \lambda_i^{\theta}) \tag{26}$$

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To summarise this section we shown how a demand model may be formulated as an MISO state space model and we have introduced a minimum variance forecast system. We will not go into the calculation of the Kalman gain G by means of the matrix Riccati equation. We simply assume G (or θ_i) is known. In the next section we will integrate this forecast system into the inventory (state) model.

5. The matrix variance equation

In order to derive expressions for the variances and co-variances of the relevant variables in our system we need to match the forecast system and the inventory system. Define X_t as the state variable of the joined system, consisting of the inventory state variable i_t and $\hat{y}_{t+1,t}$ the state variable of the forecast system at time *t* (Note that since we calculate this forecast at time *t* it is part of X_t and not $\hat{y}_{t,t-1}$). The total system is now be described by the difference equation:

$$X_{t+1} = AX_t + Bu_t + C\eta_{t+1} \tag{27}$$

with:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}, B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, C = \begin{pmatrix} -1 \\ G \end{pmatrix} \tag{28}$$

where:

$$u_t = o_t - \hat{z}_{t+1,t} \tag{29}$$

Since it is not possible to influence the demand part of this system by u_t the system is not completely controllable. So if an eigenvalue of $D\ \left|\lambda_j^{\phi}\right| > 1\ \ j = 1, \cdots, m$ than the system cannot be stationary. So from here on we assume that all the eigenvalues have $\left|\lambda_j^{\phi}\right| < 1\ \ j = 1, \cdots, m$. By introducing the inventory feedback rule $u_t = -fi_t$, we are studying the OUT policy given by equation (8) and the system becomes:

$$X_{t+1} = (A - BF)X_t + C\eta_{t+1} \tag{30}$$

with $F = (-f, 0)$. The characteristic polynomial of the systems is:

$$\det(\lambda I - (A + BF)) = \det(\lambda - (1 - f))\det(\lambda I - D) \tag{31}$$

(31) consists of the eigenvalue of the inventory system and the eigenvalues of the forecast system. As mentioned in section 3 η_{t+1} forms a sequence of zero mean uncorrelated normally distributed random variables. This property makes it easy to derive a matrix variance differential equation from (30). Let the variance matrix of the state vector at time *t* be:

$$\Sigma_{XX,t} = E\{X_t X_t^{'}\} = E\begin{Bmatrix} i_t i_t & i_t(\hat{y}_{t+1,t})' \\ (\hat{y}_{t+1,t})i_t & (\hat{y}_{t+1,t})(\hat{y}_{t+1,t})' \end{Bmatrix} \tag{32}$$

then multiplying equation (30) by the transpose of X_{t+1} and taking expectation we get the variance equation:

$$\Sigma_{XX,t+1} = (A - BF)\Sigma_{XX,t}(A - BF)' + C\Sigma_{\eta\eta}C' \qquad (33)$$

Note that $\Sigma_{\eta\eta} = E\{\eta_{t+1}^2\}$ the variance of the forecast error. Under the condition $0 < f < 2$ and $\left|\lambda_j^{\phi}\right| < 1 \;\; j = 1,\cdots,m$ the matrix variance equation converges to the stationary situation, expressed by:

$$\Sigma_{XX} = (A - BF)\Sigma_{XX}(A - BF)' + C\Sigma_{\eta\eta}C' \qquad (34)$$

This equation can in general be solved by a range of numerical techniques. In this particular case the structure of A and B allows further analytical investigation. Equation (32) shows that three components are relevant to ($\Sigma_{ii} = E\{i_t^2\}, \Sigma_{\hat{y}i} = E\{(\hat{y}_{t+1,t})i_t\}, \Sigma_{\hat{y}\hat{y}} = E\{(\hat{y}_{t+1,t})(\hat{y}_{t+1,t})'\}$) resulting in:

$$\Sigma_{ii} = (1 - f)^2\Sigma_{ii} + \Sigma_{\eta\eta} \qquad (35)$$

$$\Sigma_{\hat{y}i} = (1 - f)D\Sigma_{\hat{y}i} - G\Sigma_{\eta\eta} \qquad (36)$$

$$\Sigma_{\hat{y}\hat{y}} = D\Sigma_{\hat{y}\hat{y}}D' + GG'\Sigma_{\eta\eta} \qquad (37)$$

Equation (35) corresponds with (10), the inventory variance those closed form solution was shown in (11). As expected the inventory variance is not influenced by the demand but only by the (minimum) variance of the forecast error. The variance of the forecast, $\Sigma_{\hat{y}\hat{y}}$, (37) is not influenced by the feedback factor f. The co-variance between the inventory and the forecasted demand state is dependent of the feedback factor, the demand model (D) and the Kalman gain G. Equation (36) can be written as:

$$\Sigma_{\hat{y}i} = -[I - (1 - f)D]^{-1}G\Sigma_{\eta\eta} \qquad (38)$$

Since the eigenvalues $\lambda_j = [1 - (1 - f)\lambda_j^{\phi}]$ are $\neq 0$ in the stability interval $0 < f < 2$ the matrix inverse exists.

In the next section we will analyse the effects of the general demand model on the variance of the ordering policy. Here, equation (38) will play an important role.

6. Analysis of the order variance

The variance of the ordering policy was given in equation (13). Since $\Sigma_{i\hat{z}} = \Sigma_{\hat{z}i} = M\Sigma_{\hat{y}i}, \Sigma_{\hat{z}\hat{z}} = M\Sigma_{\hat{y}\hat{y}}M'$ the ordering variance becomes:

$$\Sigma_{oo} = f^2\Sigma_{ii} - 2f\Sigma_{i\hat{z}} + \Sigma_{\hat{z}\hat{z}} = f^2\Sigma_{ii} + 2fW(f)\Sigma_{\eta\eta} + M\Sigma_{\hat{y}\hat{y}}M' \qquad (39)$$

where the scalar $W(f)$ is the function of f :

$$W(f) = M[I-(1-f)D]^{-1}G \tag{40}$$

In order to evaluate the variance of the ordering rule we first concentrate on the co-variance term in (39). Due to the observable canonical form of D the inverse can be calculated as:

$$W(f) = \frac{\sum_{j=1}^{m}(\phi_j-\theta_j)(1-f)^{j-1}}{1-\sum_{j=1}^{m}\phi_i(1-f)^j} \tag{41}$$

(41) can be rewritten using the eigenvalues as:

$$W(f) = \frac{\sum_{j=1}^{m}(\phi_j-\theta_j)(1-f)^{j-1}}{\prod_{j=1}^{m}[1-(1-f)\lambda_j^{\phi}]} = \sum_{j=1}^{m}\frac{r_j}{[1-(1-f)\lambda_j^{\phi}]} \tag{42}$$

The RHS of equation (42) is obtained using partial fraction expansion with:

$$r_i = \frac{\prod_{j=1}^{m}(\lambda_i^{\phi}-\lambda_j^{\theta})}{\prod_{\substack{j=1\\ j\neq i}}^{m}(\lambda_i^{\phi}-\lambda_j^{\phi})} \tag{43}$$

We see that $W(f)$ consists of the sum of m hyperbolic functions with asymptotes at $f=(1-\lambda_j^{\phi})/\lambda_j^{\phi}$.

For $0<\lambda_j^{\phi}<1,\ (1-\lambda_j^{\phi})/\lambda_j^{\phi}<0$ and for $-1<\lambda_j^{\phi}<0,\ (1-\lambda_j^{\phi})/\lambda_j^{\phi}>2$ so $W(f)$ has no asymptote in the stability interval $0<f<2$. $W(f)$ is bounded on this interval. Three interesting points are:

$$W(f=0) = \frac{\sum_{j=1}^{m}(\phi_j-\theta_j)}{1-\sum_{j=1}^{m}\phi_j},\ W(f=1)=(\phi_1-\theta_1),\ W(f=2)=\frac{\sum_{j=1}^{m}(\phi_j-\theta_j)(-1)^{j-1}}{1-\sum_{j=1}^{m}\phi_j(-1)^j} \tag{44}$$

All three values can be negative, though the denominators are always positive. The sign of r_i depends on the relative values of the λ_j^{ϕ} and λ_j^{θ}'s. If for instance the λ_j^{ϕ} and λ_j^{θ}'s can be ordered as $\lambda_1^{\theta}<\lambda_1^{\phi}\cdots\lambda_j^{\theta}<\lambda_j^{\phi}\cdots<\lambda_m^{\theta}<\lambda_m^{\phi}$ then all r_i are positive. The sum of the r_i 's is equal to $(\phi_1-\theta_1)$.

With this insight into the characteristics of $W(f)$ we may now consider its influence on the ordering variance. Since the third term in equation (38) does not depend on f we need to only consider the first two terms. Using equation (11) we introduce $H(f)$ as the part of Σ_{oo} dependent on *f*:

$$\Sigma_{oo} = [H(f)]\Sigma_{\eta\eta} + M\Sigma_{\hat{y}\hat{y}}M' \qquad (45)$$

with:

$$H(f) = \frac{f}{2-f} + 2fW(f) \qquad (46)$$

This function for $-\infty < f < \infty$ has m+1 asymptotes, meaning that there are m+1 values for which $H(f) \to \infty$ (and also $m+1$ values, $H(f) \to -\infty$). In the interval $0 < f < 2$ the first term is a strictly increasing positive function with an asymptote for $f \to 2$. The second term is bounded on this interval but could be large for $\lambda_j^\phi \to -1$. $H(f=0)=0$ and $H(f=1) = 1 + 2(\phi_1 - \theta_1)$. The last value is remarkable since it depends only on the first parameters of the characteristic polynomial of D and of $(D-GM)$.

$W(f)$ can be an increasing function of f in the stability interval and so can be $H(f)$. A sufficient, but not necessary, condition is that all r_j are positive. Then also $(\phi_1 - \theta_1) > 0$, so we have "already" a bullwhip effect $\Sigma_{oo} > \Sigma_{zz}$ at $f = 1$. Eliminating the second term in (44) using $\Sigma_{zz} = M\Sigma_{\hat{y}\hat{y}}M' + \Sigma_{\eta\eta}$ gives:

$$\Sigma_{oo} - \Sigma_{zz} = (H(f)-1)\Sigma_{\eta\eta} \qquad (47)$$

which is larger than zero for $f=1$ if $(\phi_1 - \theta_1) > 0$. More generally when $\sum_{j=1}^{m} r_j = (\phi_j - \theta_j) > 0$ bullwhip arises in the interval $0 < f < 1$.

Even though $W(f)$ is negative, $H(f)$ can still increase over the stability interval. As long as $H(f)$ increases over the interval, the partial derivative of the ordering variance is positive. So we can find a suitable balance between the ordering variance and inventory variance in the interval $0 < f < 1$. Moreover bullwhip might arise for $f < 1$ or for $f > 1$.

If $W(f)$ is negative, $H(f)$ might also be negative for some f. As $H(f) \to \infty$ for $f \to 2$, $H(f)$ will have a minimum on the stability interval. A necessary condition for this case is that the partial derivative of $H(f)$ at $f = 0$ is negative. This gives:

$$1 + 3\sum_{j=1}^{m}\phi_j - 4\sum_{j=1}^{m}\theta_j < 0 \qquad (48)$$

The minimum might occur at $f^* < 1$ or at $f^* > 1$. In the first case as $H(f)$ increases for $f > f^*$ a balance between the ordering variance and inventory variance can be found in the interval $f^* < f < 1$. In the latter case a balance can only be realized in $1 < f < f^*$. However f in the interval $1 < f < 2$ is usually not being advised because it over compensates for inventory deviations (see (9)).

The nominator of the partial derivative of $H(f)$ consists of the sum of m+1 terms. Each term is a product of m squared terms $\{2-f, 1-(1-f)\lambda_j^{\phi}, j=1,\ldots,m\}$. Except for m=1 a closed form solution has been found. If $(\phi_1-\theta_1)<0$ the two solutions are:

$$f_1=\frac{2\beta+\phi_1-1}{\beta+\phi_1};\ f_2=\frac{-2\beta+\phi_1-1}{-\beta+\phi_1} \tag{49}$$

where $\beta=\sqrt{(\theta_1-\phi_1)(1-\phi_1)}$ and f_1 gives the minimum and f_2 the maximum value. The minimum not necessarily lies in the stability interval. The necessary and sufficient condition is given by equation (48), so $f^*=f_1$ here. For $\beta>1$ the minimum satisfies $1<f^*<2$. By the way condition (48) includes $(\phi_1-\theta_1)<0$, but not vice versa. Figure 4 illustrates some typical order variances as a function of f in the class of ARMA (1,1) demands.

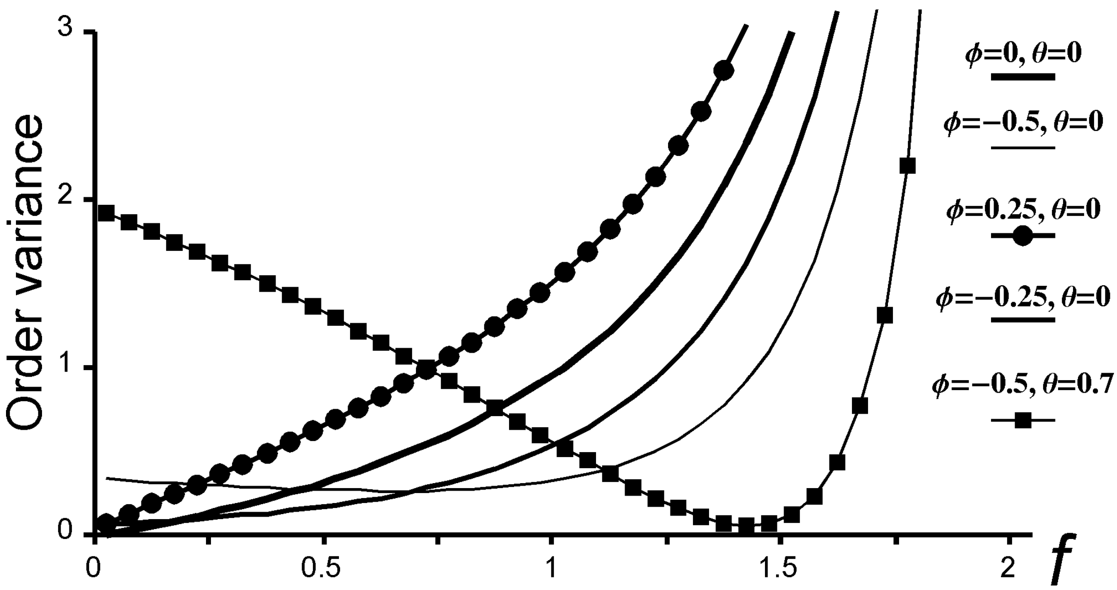

Figure 4. The order variance for different ARMA(1,1) demands

The critical condition for bullwhip is defined as $\Sigma_{oo}=\Sigma_{zz}$ which gives using equations (39) and (47):

$$2f(2-f)W(f)=1-f \tag{50}$$

Using equation (41) this can be written as a polynomial with degree m+1:

$$\sum_{j=1}^{m}\phi_j(1-f)^{j-1}-f(2-f)\sum_{j=1}^{m}\theta_j(1-f)^{j-1}=(1-f) \tag{51}$$

For each of the m+1 asymptotes there is a solution of this polynomial. Consequently in the stability interval there is one solution. Closed form solutions have not been derived for *m*>1, though relatively easy numerical techniques can be used, with the asymptotic values, for example, as starting values. For *m*=1 the solution in $0 < f < 2$ is given by:

$$f^{+} = \frac{-(2\theta_1 - 1) + \sqrt{(2\theta_1 - 1)^2 - 4(1-\phi_1)\theta_1}}{2\theta_1} \tag{52}$$

(52) corresponds to the bullwhip boundary given in Chen and Disney (2003), thus supporting our findings.

7. Conclusions

The use of a proportional controller has been has often been proposed as a solution to the bullwhip problems caused the classical order-up-to policy. The classical order-up-to policy minimizes inventory fluctuations (variance) but does not attempt to control ordering fluctuations. This may lead to the so-called bullwhip problem where the ordering variance is amplified as it moves through a production/ distribution/ replenishment decision. By a suitable choice of the feedback parameter value the proportional controller strives for a good balance between the ordering and inventory variances.

However, relatively less is known about the behavior of the proportional policy for arbitrary demand characteristics. In this paper the performance of the proportional policy has been analyzed in a general state space demand model. The crucial role of the co-variance between the inventory and the demand forecast on the bullwhip effect has been highlighted. To derive an analytical expression of the ordering variance the so-called separation principle is used. That is, a linear system disturbed by normally distributed stochastic variables can be separated into a minimum variance estimation problem and a control problem based on the estimated state variables.

The separation principle has allowed us to obtain analytical expressions illustrating the influence of the parameters and/or eigenvalues of the demand (forecast) model. From this expression bullwhip generating and order smoothing components can be uncovered. In general this leads to a paradoxical situation. If the co-variance is negative the ordering variance increases (compared with the inventory related component or equally the random demand case) bullwhip arises in the control interval $0 < f < 1$, but at the same time a good balance between inventory and ordering variance can be found in this interval. If the co-variance is positive the ordering variance decreases with respect to the random demand case and even could have a minimum in the interval $1 < f < 2$. In the latter case though the ordering variance is relatively low no balance can be found between inventory and ordering variance in the control interval $0 < f < 1$. However, a balance can be found in the region of $1 < f < 2$, but this not an attractive situation since this leads to an overreaction to inventory deviations.

Our overall conclusion is that the functioning of the proportional controller heavily depends on the demand model and its parameters values and as such requires considerable care when used in a real-life situation. Moreover a reasonably functioning robust controller suitable for a broad class of demand models with their parameter values is not immediately obvious. Future research will analyze the effects of ordering delays when using this general demand model. Further work will be devoted to finding better performing ordering rules.

8. Bibliography

Adelson, R.M. (1966) The dynamic behaviour of linear forecasting and scheduling rules. *Operational Research Quarterly* 17 (4) 447-462.

Box, G.E. and Jenkins, G.M. 1970. Time series analysis forecasting and control. Holden-Day, San Francisco.

Brown, R.G. (1963) Smoothing, forecasting and prediction of discrete time series, Prentice-Hall, Inc., Englewood Cliffs, New Jersey.

Carlsson, C and Fullér, R. (2000) Fuzzy Approach to The Bullwhip Effect, Proceedings of the 15th European Meeting on Cybernetics and Systems Research, Vienna, April 25-18, pp 228-233.

Chen, Y.F. and Disney, S.M., (2003) The order-up-to policy "sweet spot" - Using proportional controllers to eliminate the bullwhip problem, EUROMA POMS Conference, Como Lake, Italy, 16-18 June, Vol. 2, pp551-560.

Chen, Y.F., Drezner, Z., Ryan J.K., Simchi-Levi, D. 2000. Quantifying the Bullwhip effect in a simple supply chain: The impact of forecasting, lead-times and information. *Management Science* 46 436-443.

Dejonckheere, J., Disney, S.M., Farasyn, I., Janssen, F., Lambrecht, M., Towill, D.R. and Van de Velde, W. (2002) Production and Inventory Control; The variability trade-off, Proceedings of the 9th EUROMA Conference, June 2-4, Copenhagen, Denmark.

Dejonckheere, J., Disney, S.M., Lambrecht, M.R. and Towill, D.R., (2003), Measuring and avoiding the bullwhip effect: A control theoretic approach, *European Journal of Operational Research* 147 (3) 567-590.

Dejonckheere, J., Disney, S.M., Lambrecht, M.R. and Towill, D.R., (2004), The impact of information enrichment on the bullwhip effect in supply chains: A control engineering perspective, *European Journal of Operational Research*, 152 727-750.

Deziel, D.P. and Eilon, S. (1967) A linear production-inventory control rule. *The Production Engineer* 43 93-104.

Disney, S.M. and Towill, D.R. (2002), A discrete linear control theory model to determine the dynamic stability of Vendor Managed Inventory supply chains, *International Journal of Production Research* 40 (1) 179-204.

Disney, S.M. and Towill, D.R. (2003), On the bullwhip and inventory variance produced by an ordering policy, *OMEGA: The International Journal of Management Science* 31 (3) 157-167.

Disney, S.M., Farasyn, I., Lambrecht, M., Towill, D.R. and Van de Velde, W. (2003) Creating win-win scenarios from the bullwhip problem - By design not accident, EUROMA POMS Conference, Como Lake, Italy, 16-18 June, 2 561-570.

Disney, S.M. and Grubbström, R.W. (2003) The economic consequences of a production and inventory control policy. 17th International conference on Production Research, Virginia, USA, 3-7 August.

Disney, S.M., Farasyn, I., Lambrecht, M., Towill, D.R. and Van de Velde, W., (2004), Smooth is smart: Bullwhip, inventory, fill-rates and the golden ratio, Working Paper, Cardiff Business School and K. U. Leuven, Belgium.

Gaalman, G.J. (1976) Optimal linear control of linear production-inventory systems, *International Journal of Systems Science* 7 1005-1015.

Gaalman, G.J. (1978) On the optimal time-invariant control of linear Gaussian non-quadratic problems, *International Journal of Control* 28 635-648.

Gaalman, G.J. (1979) On the optimal time-variant control of linear production inventory systems, *Engineering and Production Economics* 4 147-163.

Gilbert, K., (2002) An ARIMA supply chain model. Working Paper, University of Tennessee, USA. Available for download from http://bus.utk.edu/mgmtsci/oldsite/gilbert/

Harrison, P.J. (1967) Exponetial smoothing and short term sales-forecasting, *Management Science*, 13, 831-842.

Inderfurth, K. (1977) Zur Gute linearer Entscheidungsregeln in Produktions-Lagerhaltungsmodellen, Westdeutschen Verlag.

Jaswinski, A., (1970) Stochastic processes and linear filtering theory, Academic Press

Johnson, G. and Thomson, H., 1975. Optimality of myopic inventory policies for certain dependent demand processes. *Management Science* 21 1303-1307.

Kim, H-K, Ryan, J.K. (2003) The cost impact of using simple forecasting techniques in a supply chain. *Navel Research Logistics* 50 388-411.

Lee, H.L., Padmanabhan, P., Whang, S. (1997) Information distortion in a supply chain: the Bullwhip Effect. *Management Science* 43 543-558.

Lee, H.L., So, K.C. and Tang, C.S. (2000) The value of information sharing in a two level supply chain. *Management Science* 46 (5) 543-558.

Magee, J.F. (1956) Guides to Inventory Control Part II, *Harvard Business Review,* Mar-April, 106-116.

Matsuyama, K. (1997) Maintaining optimal inventory level by feedbacks. *International Journal of Production Economics* 53 57-69.

Metters, R. (1997) Quantifying the bullwhip effect in supply chains. *Journal of Operations Management* 15 89-100.

Schneeweiβ, C., (1971) Smoothing production by inventory - An applicat:ion of the Wiener filtering theory, *Management Science* 17 472-483.

Schneeweiβ, C., (1975) Dynamic certainty equivalents in production smoothing theory, *International Journal of Systems Science* 6 (4) 353-365.

Schneeweiβ, C. (1977) Inventory-production theory, Springer.

Simon, H.A. (1952) On the application of servomechanism theory to the study of production control. *Econometrica* 20 247-268.

Tsypkin, Y.Z., 1964. Sampling systems theory and its application. Vol. 2 Pergamon Press, Oxford.

Towill D.R. (1982) Dynamic analysis of an inventory and order based production control system. *International Journal of Production Research* 20 369-383.

Vassian H.J. (1955) Application of discrete variable servo theory to inventory control. *Journal of the Operations Research Society of America* 3 (3) 272-282.